System-time entanglement in a discrete-time model

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(Received 10 December 2015; published 27 June 2016)

We present a model of discrete quantum evolution based on quantum correlations between the evolving system and a reference quantum clock system. A quantum circuit for the model is provided, which in the case of a constant Hamiltonian is able to represent the evolution over 2^n time steps in terms of just n time qubits and n control gates. We then introduce the concept of system-time entanglement as a measure of distinguishable quantum evolution, based on the entanglement between the system and the reference clock. This quantity vanishes for stationary states and is maximum for systems jumping onto a new orthogonal state at each time step. In the case of a constant Hamiltonian leading to a cyclic evolution it is a measure of the spread over distinct energy eigenstates and satisfies an entropic energy-time uncertainty relation. The evolution of mixed states is also examined. Analytical expressions for the basic case of a qubit clock, as well as for the continuous limit in the evolution between two states, are provided.

DOI: 10.1103/PhysRevA.93.062127

I. INTRODUCTION

Since the establishment of the foundations of quantum mechanics, time has been mostly considered as an external classical parameter. Various attempts to incorporate time in a fully quantum framework have, nonetheless, been made, starting with the Page and Wootters mechanism [1] and other subsequent proposals [2,3]. This subject has recently received increasing attention in both quantum mechanics [4–8] and general relativity [9,10], where this problem is considered a key issue in the connection between the two theories. In the present work we introduce a simple discrete quantum model of evolution, which, on one hand, constitutes a consistent discrete version of the formalism in [1] and [9] and, on the other hand, provides a practical means to simulate quantum evolutions. We show that a quantum circuit for the model can be constructed, which in the case of a constant Hamiltonian is able to simulate the evolution over $N = 2^n$ times in terms of just n time qubits and $O(n)$ gates, providing the basis for a parallel-in-time simulation.

We then introduce and discuss the concept of system-time entanglement, which arises naturally in the present scenario, as a quantifier of the actual distinguishable evolution undergone by the system. This quantifier can be related to the minimum time necessarily elapsed by the system. For a constant Hamiltonian we show that this entanglement is bounded above by the entropy associated with the spread over energy eigenstates of the initial state, reaching this bound for a spectrum leading to a cyclic evolution, in which case it satisfies an entropic energy-time uncertainty relation. Illustrative analytical results for a qubit clock, which constitutes the basic building block in the present setting, are provided. The continuous limit for the evolution between two arbitrary states is also analyzed.

II. FORMALISM

A. History states

We consider a bipartite system $S+T$, where S represents a quantum system and T a quantum clock system with finite Hilbert-space dimension N. The whole system is assumed to be in a pure state of the form

$$|\Psi\rangle = \frac{1}{\sqrt{N}}\sum_{t=0}^{N-1}|\psi_t\rangle|t\rangle, \tag{1}$$

where $\{|t\rangle,\ t=0,\ldots,N-1\}$ is an orthonormal basis of T and $\{|\psi_t\rangle,\ t=0,\ldots,N-1\}$ a set of arbitrary pure states of S. This state can describe, for instance, the whole history of evolution of an initial pure state $|\psi_0\rangle$ of S at a discrete set of times t. The state $|\psi_t\rangle$ at time t can be recovered as the conditional state of S after a local measurement at T in the previous basis with result t,

$$|\psi_t\rangle\langle\psi_t| = \frac{\mathrm{Tr}_T\,[|\Psi\rangle\langle\Psi|\,\Pi_t]}{\langle\Psi|\Pi_t|\Psi\rangle}, \tag{2}$$

where $\Pi_t = \mathbb{1}\otimes|t\rangle\langle t|$. In shorthand notation, $|\psi_t\rangle \propto \langle t|\Psi\rangle$.

If we write

$$|\psi_t\rangle = U_t|\psi_0\rangle, \quad t=0,\ldots,N-1, \tag{3}$$

where U_t are unitary operators at S (with $U_0=\mathbb{1}$), state (1) can be generated with the schematic quantum circuit in Fig. 1. Starting from the product initial state $|\psi_0\rangle|0\rangle$, a Hadamard-like gate [11] at T turns it into the superposition $\frac{1}{\sqrt{N}}\sum_{t=0}^{N-1}|\psi_0\rangle|t\rangle$, after which a control-like gate $\sum_t U_t\otimes|t\rangle\langle t|$ will transform it into state (1). A specific example is provided in Fig. 2.

From a formal perspective, state (1) is a "static" eigenstate of the $S+T$ translation "superoperator"

$$\mathcal{U} = \sum_{t=1}^{N} U_{t,t-1}\otimes|t\rangle\langle t-1|, \tag{4}$$

where $U_{t,t-1}=U_tU_{t-1}^\dagger$ evolves the state of S from $t-1$ to t ($|\psi_t\rangle = U_{t,t-1}|\psi_{t-1}\rangle$) and the cyclic condition $|N\rangle\equiv|0\rangle$, i.e., $U_{N,N-1}=U_{N-1}^\dagger$, is imposed. Then

$$\mathcal{U}|\Psi\rangle = |\Psi\rangle, \tag{5}$$

showing that state (1) remains strictly invariant under such global translations in the $S+T$ space.

Equation (5) holds for *any* choice of initial state $|\psi_0\rangle$ in (1). The eigenvalue 1 of $\mathcal{U}$ then has a degeneracy equal to

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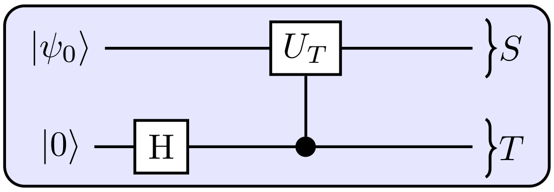

FIG. 1. Schematic circuit representing the generation of the system-time pure state, (1). The control gate performs the operation U_t on S if T is in state $|t\rangle$, while the Hadamard-type gate H creates the superposition $\propto \sum_{t=0}^{N-1}|t\rangle$.

the Hilbert-space dimension M of S, since for M orthogonal initial states $|\psi_0^j\rangle$, $\langle\psi_0^j|\psi_0^l\rangle = \delta_{jl}$, the ensuing states $|\Psi^l\rangle$ are orthogonal due to Eq. (3):

$$\langle\Psi^l|\Psi^j\rangle = \frac{1}{N}\sum_{t=0}^{N-1}\langle\psi_t^l|\psi_t^j\rangle = \langle\psi_0^l|\psi_0^j\rangle = \delta_{lj}. \qquad (6)$$

The remaining eigenstates of $\mathcal{U}$ are of the form $|\Psi_k\rangle = \frac{1}{\sqrt{N}}\sum_{t=0}^{N-1} e^{i2\pi kt/N}|\psi_t\rangle|t\rangle$ with k integer and represent the evolution associated with operators $U_t^k = e^{i2\pi kt/N}U_t$:

$$\mathcal{U}|\Psi_k\rangle = e^{-i2\pi k/N}|\Psi_k\rangle, \quad k = 0,\dots,N-1. \qquad (7)$$

All eigenvalues $\lambda_k = e^{-i2\pi k/N}$ are M-fold degenerate by the same previous arguments. The full set of N eigenvalues and a choice of MN orthogonal eigenvectors of $\mathcal{U}$ are thus obtained. We may then write, for general U_t,

$$\mathcal{U} = \exp[-i\mathcal{J}], \qquad (8)$$

with $\mathcal{J}$ Hermitian and satisfying $\mathcal{J}|\Psi_k\rangle = 2\pi\frac{k}{N}|\Psi_k\rangle$ for $k = 0,\dots,N-1$. In particular, states (1) satisfy

$$\mathcal{J}|\Psi\rangle = 0, \qquad (9)$$

which represents a discrete counterpart of the Wheeler-DeWitt equation [9,12,13] determining state $|\Psi\rangle$ in continuous-time theories [9]. In the limit where t becomes a continuous unrestricted variable, state (1) with condition (3) becomes, in fact, that considered in [9]. Note, however, that here $\mathcal{J}$ is actually defined just *modulo* N, as any $\mathcal{J}$ satisfying $\mathcal{J}|\Psi_k\rangle = 2\pi(\frac{k}{N}+n_k)|\Psi_k\rangle$ with n_k integer will also fulfill Eq. (8).

All $|\Psi_k\rangle$ are also eigenstates of the Hermitian operators $\mathcal{U}_\pm = i^{\frac{1\mp 1}{2}}(\mathcal{U}\pm\mathcal{U}^\dagger)/2$, with eigenvalues $\cos\frac{2\pi k}{N}$ and $\sin\frac{2\pi k}{N}$, respectively, i.e., 1 and 0 for states (1). The latter can then also be obtained as ground states of $-\mathcal{U}_+$. An Hermitian operator $\mathcal{H}$ similar to $-\mathcal{U}_+$ but with no cyclic condition [$\mathcal{H} = -\tilde{\mathcal{U}}_+ + I_S\otimes I_T$, with $\tilde{\mathcal{U}} = \mathcal{U} - U_{N-1}^\dagger|0\rangle\langle N-1| + \frac{1}{2}I_S\otimes(|0\rangle\langle 0| + |N-1\rangle\langle N-1|)$] was considered in [6] for deriving a variational approximation to the evolution.

B. Constant evolution operator

If $U_{t,t-1} = U\ \forall\, t$, then

$$U_t = (U)^t = \exp[-iHt], \quad t = 0,\dots,N-1, \qquad (10)$$

where H represents a constant Hamiltonian for system S. In this case state (1) can be generated with the first step of the circuit employed for phase estimation [11], depicted in Fig. 2. If $N = 2^n$, this circuit, consisting of just n time qubits and $m = \log_2 M$ system qubits, requires only n initial single-qubit Hadamard gates on the time qubits if initialized at $|0\rangle$ (such that $|0\rangle_T \equiv \otimes_{j=1}^n|0_j\rangle \to \otimes_{j=1}^n\frac{|0_j\rangle+|1_j\rangle}{\sqrt{2}} = \frac{1}{\sqrt{N}}\sum_{t=0}^{N-1}|t\rangle$ for $t = \sum_{j=1}^n t_j 2^{j-1}$), plus n control $U^{2^{j-1}}$ gates acting on the system qubits, which perform the operation $U^t|\psi_0\rangle = \prod_{j=1}^n U^{t_j 2^{j-1}}|\psi_0\rangle$. A measurement of the time qubits with result t makes S collapse to the state $|\psi_t\rangle = e^{-iHt}|\psi_0\rangle$.

In addition, if U in (10) satisfies the cyclic condition $U^N = \mathbb{1}$, which implies that H should have eigenvalues $2\pi k/N$ with k integer, Eq. (4) can be written as

$$\mathcal{U} = U\otimes V = \exp[-i(H\otimes\mathbb{1}_T + \mathbb{1}_S\otimes P)], \qquad (11)$$

where $V = \exp[-iP] = \sum_{t=1}^N|t\rangle\langle t-1|$ is the (cyclic) time translation operator. Its eigenstates are the discrete Fourier transform (FT) of the time states $|t\rangle$,

$$V|\tilde{k}\rangle = e^{-i2\pi k/N}|\tilde{k}\rangle, \quad |\tilde{k}\rangle = \frac{1}{\sqrt{N}}\sum_{t=0}^{N-1}e^{i2\pi kt/N}|t\rangle, \qquad (12)$$

for $k = 0,\dots,N-1$, such that P is the "momentum" associated with the time operator T,

$$T|t\rangle = t|t\rangle, \quad P|\tilde{k}\rangle = 2\pi\frac{k}{N}|\tilde{k}\rangle. \qquad (13)$$

Hence, in this case $\mathcal{J} = H\otimes\mathbb{1}_T + \mathbb{1}_S\otimes P$ adopts the same form as that of continuous theories [9].

C. System-time entanglement

Suppose now that one wishes to quantify consistently the "amount" of distinguishable evolution of a pure quantum state. This measure can be related to a minimum time τ_m (number or fraction of steps) necessarily elapsed by the system. If the state is stationary, $|\psi_t\rangle \propto |\psi_0\rangle\forall\, t$, the quantifier should vanish (and $\tau_m = 0$), whereas if all N states $|\psi_t\rangle$ are orthogonal to each other, the quantifier should be maximum (with $\tau = N-1$), indicating that the state has indeed evolved through N distinguishable states. We now propose the entanglement of pure state (1) (system-time entanglement) as such a quantifier,

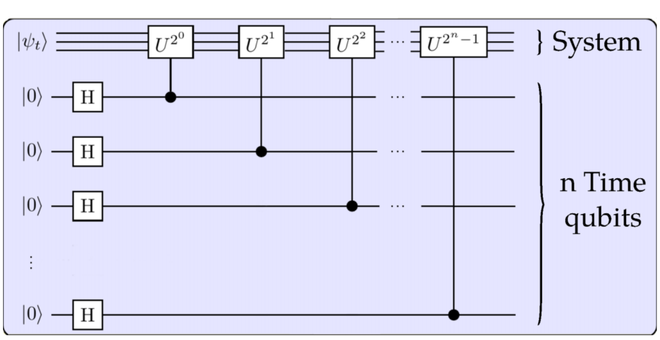

FIG. 2. Circuit representing the generation of system-time state (1) for $U_t = (U)^t$ and $N = 2^n$. The n control gates perform the operation $U^t = U^{\sum_{j=1}^n t_j 2^{j-1}}$ on the system after writing t in the binary form $t = \sum_{j=1}^{n-1} t_j 2^{j-1}$, while the n Hadamard gates lead to a coherent sum over all values of the t_j's, i.e., over all t's from 0 to $2^n - 1$.

with τ_m an increasing function of this entanglement. In Figs. 1 and 2, such entanglement is just that between the system and the time qubits, generated by the control U_t.

We first note that Eq. (1) is not, in general, the Schmidt decomposition [11] of state $|\Psi\rangle$, which is

$$|\Psi\rangle = \sum_k \sqrt{p_k}|k\rangle_S|k\rangle_T, \tag{14}$$

where $|k\rangle_{S(T)}$ are orthogonal states of S and T ($_\mu\langle k|k'\rangle_\mu = \delta_{kk'}$) and p_k the eigenvalues of the reduced states of S and T,

$$\rho_{S(T)} = \mathrm{Tr}_{T(S)}|\Psi\rangle\langle\Psi| = \sum_k p_k|k\rangle_{S(T)}\langle k|. \tag{15}$$

The entanglement entropy between S and T is then

$$E(S,T) = S(\rho_S) = S(\rho_T) = -\sum_k p_k \log_2 p_k, \tag{16}$$

where $S(\rho) = -\mathrm{Tr}\rho\log_2\rho$ is the von Neumann entropy.

Equation (16) satisfies the basic requirements of an evolution quantifier. If the state of S is stationary, $|\psi_t\rangle = e^{i\gamma_t}|\psi_0\rangle\ \forall t$, state (1) becomes separable,

$$|\Psi\rangle = |\psi_0\rangle\left(\frac{1}{\sqrt{N}}\sum_t e^{i\gamma_t}|t\rangle\right), \tag{17}$$

implying $E(S,T) = 0$. In contrast, if $|\psi_t\rangle$ evolves through N orthogonal states, then $|\Psi\rangle$ is *maximally entangled*, with Eq. (1) already its Schmidt decomposition and

$$E(S,T) = E_{\max}(S,T) = \log_2 N. \tag{18}$$

It is then natural to define the minimum time τ_m as

$$\tau_m = 2^{E(S,T)} - 1, \tag{19}$$

which takes the values 0 and $N-1$ for the previous extreme cases. The vast majority of evolutions will lie in between. For instance, a periodic evolution of period $L < N$ with N/L integer, such that $|\psi_{t+L}\rangle = e^{i\gamma}|\psi_t\rangle\ \forall\, t$, will lead to

$$|\Psi\rangle = \frac{1}{\sqrt{L}}\sum_{t=0}^{L-1}|\psi_t\rangle|t_L\rangle, \quad |t_L\rangle = \sqrt{\frac{L}{N}}\sum_{k=0}^{N/L-1} e^{i\gamma k}|t+Lk\rangle, \tag{20}$$

with $\langle t'_L|t_L\rangle = \delta_{tt'}$. Hence, its entanglement $E(S,T)$ *will be the same as that obtained with an L-dimensional effective clock*, as it should be. Its maximum value, obtained for L orthogonal states, will then be $\log_2 L$, in which case $\tau_m = L-1$.

The Schmidt decomposition, (14), represents in this context the "actual" evolution between orthogonal states, with p_k proportional to the "permanence time" in each of them. A measurement on T in the Schmidt basis would always identify orthogonal states of S for different results (and vice versa), with the probability distribution of results indicating the permanence in these states. If in Eq. (1) there are n_k times t where $|\psi_t\rangle \propto |k\rangle_S$, with $\sum_k n_k = N$ and $|k\rangle_S$ orthogonal states, then

$$|\Psi\rangle = \sum_k \sqrt{\frac{n_k}{N}}|k\rangle_S\left(\frac{1}{\sqrt{n_k}}\sum_{t/|\psi_t\rangle\propto|k\rangle_S} e^{i\gamma_t}|t\rangle\right),$$

which is the Schmidt decomposition, (14), with $p_k \propto n_k$, i.e., proportional to the total time in state $|k\rangle_S$. Note also that Eqs. (14)–(16) are essentially symmetric, so that the roles of S and T can, in principle, be interchanged.

Quadratic entanglement

A simple quantifier for the general case can be obtained through the entanglement determined by the entropy $S_2(\rho) = 2(1-\mathrm{Tr}\rho^2)$, which is just a linear function of the purity $\mathrm{Tr}\rho^2$ and does not require evaluation of the eigenvalues of ρ [14–16] (purity is also more easily accessible experimentally [17]). We obtain, using $\rho_S = \frac{1}{N}\sum_t|\psi_t\rangle\langle\psi_t|$,

$$\begin{aligned}E_2(S,T) &= S_2(\rho_T) = S_2(\rho_S) = 2\big(1-\mathrm{Tr}\,\rho_S^2\big)\\ &= 2\tfrac{N-1}{N}\left(1-\tfrac{1}{N(N-1)}\sum_{t\neq t'}|\langle\psi_t|\psi_{t'}\rangle|^2\right),\end{aligned} \tag{21}$$

which is just a decreasing function of the average pairwise squared fidelity between all visited states. If they are all proportional, $E_2(S,T) = 0$, whereas if they are all orthogonal, $E_2(S,T) = 2\frac{N-1}{N}$ is maximum. If S and T are qubits, $E_2(S,T)$ is just the squared *concurrence* [18] of $|\Psi\rangle$.

D. Relation to energy spread

In the constant case, (10), we may expand $|\psi_0\rangle$ in the eigenstates of U or H, $|\psi_0\rangle = \sum_k c_k|k\rangle$ with $H|k\rangle = E_k|k\rangle$, such that $|\psi_t\rangle = \sum_k c_k e^{-iE_kt}|k\rangle$ and

$$|\Psi\rangle = \frac{1}{\sqrt{N}}\sum_{k,t} c_k e^{-iE_kt}|k\rangle|t\rangle = \sum_k c_k|k\rangle|\tilde{k}\rangle_T, \tag{22}$$

with $|\tilde{k}\rangle_T = \frac{1}{\sqrt{N}}\sum_t e^{-iE_kt}|t\rangle$. We can always assume all E_k distinct in (22) such that $c_k|k\rangle$ is the projection of $|\psi_0\rangle$ onto the eigenspace with energy E_k. In the cyclic case $U^N = \mathbb{1}$, with $E_k = 2\pi k/N$, $k = 0,\ldots,N-1$, the states $|\tilde{k}\rangle_T$ become the orthogonal FT states $|-\tilde{k}\rangle$ (12). Equation (22) is then *the Schmidt decomposition*, (14), with $p_k = |c_k|^2$ and

$$E(S,T) = -\sum_k |c_k|^2\log_2|c_k|^2. \tag{23}$$

For this spectrum, entanglement then becomes *a measure of the spread of the initial state* $|\psi_0\rangle$ *over the eigenstates of* H with distinct energies. The same holds in the quadratic case, (21), where $E_2(E,T) = 2\sum_k|c_k|^2(1-|c_k|^2)$. If there is no dispersion, $|\psi_0\rangle$ is stationary and entanglement vanishes, while if $|\psi_0\rangle$ is uniformly spread over N eigenstates, it is maximum $[E(S,T) = \log_2 N]$.

While Eq. (23) also holds for a displaced spectrum $E_k = E_0 + 2\pi k/N$, for an arbitrary spectrum $\{E_k\}$ it will hold approximately if the overlaps $_T\langle\tilde{k}|\tilde{k}'\rangle_T = \frac{1}{N}\sum_t e^{-i(E_k-E_{k'})t}$ are sufficiently small for $k\neq k'$. In general, we actually have the strict bound

$$E(S,T) \leqslant -\sum_k |c_k|^2\log_2|c_k|^2, \tag{24}$$

since $|c_k|^2 = \sum_{k'} p_{k'}|\langle k|k'\rangle_S|^2$, with $|k\rangle$ the eigenstates of H and $|k'\rangle_S$ the Schmidt states in (14), which implies that the $|c_k|^2$'s are *majorized* [19] by the p_k's,

$$\{|c_k|^2\} \prec \{p_k\}, \tag{25}$$

where $\{|c_k|^2\}$ and $\{p_k\}$ denote the sets sorted in decreasing order. Equation (25) (meaning $\sum_{k=1}^{j}|c_k|^2 \leqslant \sum_{k=1}^{j} p_k$ for $j = 1\ldots,N-1$) implies that inequality (24) actually holds for any Schur-concave function of the probabilities [19], in particular, for any entropic form $S_f(\rho) = \mathrm{Tr}\, f(\rho)$ with $f(p)$ concave and satisfying $f(0) = f(1) = 0$ [16,20], such as the von Neumann entropy $[f(\rho) = -\rho\log_2\rho]$ and the previous S_2 entropy $[f(\rho) = 2\rho(\mathbb{1}-\rho)]$:

$$E_f(S,T) = \sum_k f(p_k) \leqslant \sum_k f(|c_k|^2), \tag{26}$$

as can be easily verified. Equations (23)–(26) then indicate that the entropy of the spread over Hamiltonian eigenstates of the initial state provides an upper bound to the corresponding system-time entanglement entropy than can be generated by *any* Hamiltonian diagonal in the states $|k\rangle$. The bound is always reached for an equally spaced spectrum $E_k = 2\pi k/N \in [0,2\pi]$ leading to a cyclic evolution, which therefore generates *the highest possible system-time entanglement for a given initial spread* $\{|c_k|^2\}$.

E. Energy-time uncertainty relations

For the aforementioned equally spaced spectrum, we may also expand the state $|\psi_0\rangle$ of S in an orthogonal set of uniformly spread states,

$$|\psi_0\rangle = \sum_{l=0}^{N} \tilde{c}_l|\tilde{l}\rangle_S, \quad |\tilde{l}\rangle_S = \frac{1}{\sqrt{N}}\sum_k e^{i2\pi kl/N}|k\rangle, \tag{27}$$

with $\tilde{c}_l = \frac{1}{\sqrt{N}}\sum_k e^{-i2\pi k/N}c_k$ the FT of the c_k's in (22). Since $U^t|\tilde{l}\rangle_S = |\widetilde{l-t}\rangle_S$, it is verified that these maximally spread states $|\tilde{l}\rangle_S$ [which, according to Eq. (23), lead to maximum system-time entanglement $E(S,T) = \log_2 N$] indeed evolve through N orthogonal states $|\widetilde{l-t}\rangle_S$. Moreover, Eq. (22) becomes

$$|\Psi\rangle = \sum_{l,t}\tilde{c}_l|\widetilde{l-t}\rangle_S|t\rangle = \sum_l |\tilde{l}\rangle_S\left(\sum_t \tilde{c}_t|t-l\rangle\right), \tag{28}$$

showing that $\tilde{c}_l$ determines the distribution of time states $|t\rangle$ assigned to each state $|\tilde{l}\rangle_S$, i.e., the uncertainty in its time location. Being related through a finite FT, $\{c_k\}$ and $\{\tilde{c}_l\}$ satisfy various uncertainty relations, such as [21–23]

$$E(S,T) + \tilde{E}(S,T) \geqslant \log_2 N, \tag{29}$$

where $\tilde{E}(S,T) = -\sum_l |\tilde{c}_l|^2\log_2|\tilde{c}_l|^2$ is the entropy characterizing the time uncertainty and $E(S,T)$ the energy uncertainty, (23). If localized in energy $[|c_k| = \delta_{kk'}, E(S,T) = 0]$, Eq. (29) implies the maximum time uncertainty $[|\tilde{c}_l| = \frac{1}{\sqrt{N}}, \tilde{E}(S,T) = \log_2 N]$, and vice versa. We also have $n(\{c_k\})\,n(\{\tilde{c}_l\}) \geqslant N$ [24], where $n(\{\alpha_k\})$ denotes the number of nonzero α_k's. Bounds for the product of variances in the discrete FT are discussed in [25].

F. Mixed states

Let us now consider that S is a bipartite system, $A + B$. By taking the partial trace of (1),

$$\rho_{BT} = \mathrm{Tr}_A\,|\Psi\rangle\langle\Psi| = \sum_j {}_A\langle j|\Psi\rangle\langle\Psi|j\rangle_A, \tag{30}$$

we see that the system-time state for a subsystem is a *mixed state*. Of course, the state of B at time t, now setting $\Pi_t = I_B\otimes|t\rangle\langle t|$, is given by the standard expression

$$\rho_{Bt} = \frac{\mathrm{Tr}_T\,\rho_{BT}\Pi_t}{\mathrm{Tr}\,\rho_{BT}\Pi_t} = \mathrm{Tr}_A|\psi_t\rangle\langle\psi_t|. \tag{31}$$

If the initial state of S is $|\psi_0\rangle = \sum_j\sqrt{q_j}|j\rangle_A|j\rangle_B$ (Schmidt decomposition), Eqs. (30) and (31) determine the evolution of an initial mixed state $\rho_{B0} = \sum_j q_j|j\rangle_B\langle j|$ of B, considered as a subsystem in a purified state undergoing unitary evolution. For instance, if just subsystem B evolves, such that $U_t = I_A\otimes U_{Bt}\,\forall\, t$, Eq. (30) leads to

$$\rho_{BT} = \sum_j q_j|\Psi_j\rangle_{BT}\langle\Psi_j|, \tag{32}$$

where $|\Psi_j\rangle_{BT} = \frac{1}{\sqrt{N}}\sum_{t=0}^{n-1}U_{Bt}|j\rangle_B|t\rangle$. Equation (31) is then the mixture of the pure $B+T$ states associated with each eigenstate of ρ_{B0} and implies the unitary evolution $\rho_{Bt} = U_{Bt}\rho_{B0}U_{Bt}^\dagger$.

Since state (30) is, in general, mixed, the correlations between T and a subsystem B can be more complex than those with the whole system S. State (30) can, in principle, exhibit distinct types of correlations, including entanglement [26,27], discord-like correlations [28–31], and classical-type correlations. The exact evaluation of the quantum correlations is also more difficult, being in general a hard problem [32,33]. We here consider just the entanglement of formation [27] $E(B,T)$ of state (30), which, if nonzero, indicates that (30) cannot be written as a convex mixture of pure product states [26] $|\Psi_\alpha\rangle_{BT} = |\psi_\alpha\rangle_B|\phi_\alpha\rangle_T$. In this context the latter represent essentially *stationary* states. Separability with time would then indicate that ρ_{BT} can be written as a convex mixture of such states, requiring no quantum interaction with the clock system for its formation.

III. EXAMPLES

A. The qubit clock

As an illustration, we examine the basic case of a qubit clock ($N = 2$). Equation (1) becomes

$$\begin{aligned}|\Psi\rangle &= (|\psi_0\rangle|0\rangle + |\psi_1\rangle|1\rangle)/\sqrt{2}\\ &= \sqrt{p_+}|{+}{+}\rangle + \sqrt{p_-}|{-}{-}\rangle,\\ p_\pm &= (1\pm|\langle\psi_0|\psi_1\rangle|)/2,\end{aligned} \tag{33}$$

where $|\psi_1\rangle = U|\psi_0\rangle$ and (33) is its Schmidt decomposition, with $|\pm\rangle_S = (|\psi_0\rangle \pm e^{-i\gamma}|\psi_1\rangle)/\sqrt{4p_\pm}$, $|\pm\rangle_T = (|0\rangle \pm e^{i\gamma}|1\rangle)/\sqrt{2}$, and $e^{i\gamma} = \frac{\langle\psi_0|\psi_1\rangle}{|\langle\psi_0|\psi_1\rangle|}$. Hence, $E(S,T) = -\sum_{\nu=\pm}p_\nu\log p_\nu$ will be fully determined by the overlap or *fidelity* $|\langle\psi_0|\psi_1\rangle|$ between the initial and the final states, decreasing as the fidelity increases and becoming maximum for orthogonal states. The quadratic entanglement entropy $E_2(S,T)$ becomes just

$$E_2(S,T) = 4p_+p_- = 1 - |\langle\psi_0|\psi_1\rangle|^2. \tag{34}$$

These results hold for arbitrary dimension M of S.

The operator, (4), becomes $\mathcal{U} = U\otimes|1\rangle\langle 0| + U^\dagger\otimes|0\rangle\langle 1|$ and is directly Hermitian, with eigenvalues $e^{i2k\pi/2} = \pm 1$ for

$k=0$ or 1, M-fold degenerate. Hence, in this case

$$\mathcal{J}=\pi(\mathcal{U}-\mathbb{1})/2, \tag{35}$$

involving coupling between S and T unless $U^\dagger \propto U$.

For $|\psi_1\rangle$ close to $|\psi_0\rangle$, Eq. (34) becomes proportional to the Fubini-Study metric [34]. If $U=\exp[-i\epsilon h]$, an expansion of $|\psi_0\rangle$ in the eigenstates of h, $|\psi_0\rangle=\sum_k c_k|k\rangle$ with $h|k\rangle=\varepsilon_k|k\rangle$, leads to

$$E_2(S,T)=1-\left|\sum_k |c_k|^2 e^{-i\epsilon\varepsilon_k}\right|^2 \approx \epsilon^2(\langle h^2\rangle-\langle h\rangle^2), \tag{36}$$

where the last expression holds up to $O(\epsilon^2)$. Hence, for a "small" evolution the system-time entanglement of a single step is determined by the energy fluctuation $\langle h^2\rangle-\langle h\rangle^2$ in $|\psi_0\rangle$ ($\langle O\rangle\equiv\langle\psi_0|O|\psi_0\rangle$), with $E_2(S,T)$ directly proportional to it. For instance, if S is also a single qubit and $\varepsilon_1-\varepsilon_0=\varepsilon$, the exact expression becomes

$$E_2(S,T)=4\sin^2\left(\frac{\epsilon\varepsilon}{2}\right)|c_0|^2|c_1|^2 \tag{37}$$

$$=4\sin^2\left(\frac{\epsilon\varepsilon}{2}\right)\frac{\langle h^2\rangle-\langle h\rangle^2}{\varepsilon^2}, \tag{38}$$

which reduces to (36) for small ϵ. It is also verified that $E_2(S,T)\leqslant S_2(|c_0|^2,|c_1|^2)=4|c_0|^2|c_1|^2$, i.e., it is upper bounded by the quadratic entropy of the energy spread [Eq. (26)], reaching the bound for $E=\epsilon\varepsilon=\pi$, in agreement with the general result, (23)–(24). Returning to the case of a general S, we also note that $E_2(S,T)$ determines the minimum time required for the evolution from $|\psi_0\rangle$ to $|\psi_1\rangle$ in standard continuous-time theories [34], which depends on the fidelity $|\langle\psi_0|\psi_1\rangle|$ and can then be expressed in terms of E_2 as $\hbar\sin^{-1}(\sqrt{E_2(S,T)})/\sqrt{\langle h^2\rangle-\langle h\rangle^2}$.

Let us now assume that $S=A+B$ is a two qubit-system, with $U=I_A\otimes U_B$. As previously stated, starting from an initial entangled pure state of $A+B$ (purification of ρ_{B0}), state (33) will determine the evolution of the reduced state of B, leading to

$$\rho_{Bt}=p|\psi_t^0\rangle\langle\psi_t^0|+q|\psi_t^1\rangle\langle\psi_t^1|,\quad t=0,1, \tag{39}$$

where $p+q=1$, $\langle\psi_0^0|\psi_0^1\rangle=0$, and $|\psi_1^j\rangle=U_B|\psi_0^j\rangle$ for $j=0,1$. The reduced state, (32), of $B+T$ becomes

$$\rho_{BT}=p|\Psi_0\rangle\langle\Psi_0|+q|\Psi_1\rangle\langle\Psi_1|, \tag{40}$$

with $|\Psi_j\rangle=\frac{1}{\sqrt{2}}(|\psi_0^j\rangle|0\rangle+|\psi_1^j\rangle|1\rangle)$. Since (40) is a two-qubit mixed state, its entanglement of formation can be obtained through the concurrence [18] $C(B,T)$, whose square is just the entanglement monotone associated with the quadratic entanglement entropy E_2 [$C^2(B,T)=E_2(B,T)$ for a pure $B+T$ state]. It adopts here the simple expression

$$C^2(B,T)=(p-q)^2\big(1-|\langle\psi_0^j|\psi_1^j\rangle|^2\big), \tag{41}$$

where $|\langle\psi_0^j|\psi_1^j\rangle|=|\langle\psi_0^j|U_B|\psi_0^j\rangle|$ is the same for $j=0$ or 1 in a qubit system if $\langle\psi_0^0|\psi_0^1\rangle=0$. Equation (41) is then the pure-state result, (34), for any of the eigenstates of ρ_{B0} diminished by the factor $(p-q)^2$, vanishing if ρ_{B0} is maximally mixed ($p=q$). Remarkably, Eq. (41) can also be written as

$$C^2(B,T)=1-F^2(\rho_{B0},\rho_{B1}), \tag{42}$$

where $F(\rho_{B0},\rho_{B1})=\mathrm{Tr}\sqrt{\rho_{B0}^{1/2}\rho_{B1}\rho_{B0}^{1/2}}$ is again the *fidelity* between the initial and the final reduced mixed states of B ($F=|\langle\psi_0|\psi_1\rangle|$ if ρ_{B0} and ρ_{B1} are pure states). Note also that the total quadratic entanglement entropy is here

$$E_2(S,T)=1-\big|p\langle\psi_1^0|\psi_0^0\rangle+q\langle\psi_1^1|\psi_0^1\rangle\big|^2,$$

satisfying $E_2(S,T)\geqslant C^2(B,T)$, in agreement with the monogamy inequalities [14,35], coinciding iff $pq=0$ (pure case).

B. The continuous limit

Let us now assume that system S is a qubit, with T of dimension N ($t=0,\dots,N-1$). This case can also represent the evolution from an initial state $|\psi_0\rangle$ to an arbitrary final state $|\psi_f\rangle$ in a general system S of Hilbert-space dimension M if all intermediate states $|\psi_t\rangle$ belong to the subspace generated by $|\psi_0\rangle$ and $|\psi_f\rangle$, such that the whole evolution is contained in a two-dimensional subspace of S. Writing the system states as

$$|\psi_t\rangle=\alpha_t|0\rangle+\beta_t|1\rangle,\quad t=0,\dots,N-1, \tag{43}$$

with $\langle 0|1\rangle=0$ and $|\alpha_t|^2+|\beta_t|^2=1$, we may rewrite state (1) as

$$\begin{aligned}|\Psi\rangle&=\frac{1}{\sqrt{N}}\left[|0\rangle\left(\sum_t\alpha_t|t\rangle\right)+|1\rangle\left(\sum_t\beta_t|t\rangle\right)\right]\\&=\alpha|0\rangle|\phi_0\rangle+\beta|1\rangle|\phi_1\rangle,\end{aligned} \tag{44}$$

where $|\phi_0\rangle=\frac{1}{\sqrt{N}\alpha}\sum_t\alpha_t|t\rangle$, $|\phi_1\rangle=\frac{1}{\sqrt{N}\beta}\sum_t\beta_t|t\rangle$ are normalized (but not necessarily orthogonal) states of T and all sums over t are from 0 to $N-1$, with

$$\alpha^2=\frac{1}{N}\sum_t|\alpha_t|^2,\quad \beta^2=\frac{1}{N}\sum_t|\beta_t|^2=1-\alpha^2. \tag{45}$$

The Schmidt coefficients of state (44) are given by

$$p_\pm=\tfrac{1}{2}\big(1\pm\sqrt{1-4\alpha^2\beta^2(1-|\langle\phi_1|\phi_0\rangle|^2)}\big). \tag{46}$$

We then obtain

$$\begin{aligned}E_2(S,T)&=4p_+p_-=4\alpha^2\beta^2(1-|\langle\phi_1|\phi_0\rangle|^2)\\&=4(\alpha^2\beta^2-\gamma^2),\quad \gamma=\frac{1}{N}\left|\sum_t\beta_t^*\alpha_t\right|,\end{aligned} \tag{47}$$

a result which also follows directly from Eq. (21).

Let us consider, for instance, the states

$$|\psi_t\rangle=\cos\left(\frac{\phi t}{N-1}\right)|0\rangle+\sin\left(\frac{\phi t}{N-1}\right)|1\rangle, \tag{48}$$

such that S evolves from $|\psi_0\rangle=|0\rangle$ to

$$|\psi_f\rangle=\cos\phi|0\rangle+\sin\phi|1\rangle$$

in $N-1$ steps through intermediate equally spaced states contained within the same plane in the Bloch sphere of S. The

S-T entanglement of this N-time evolution can be evaluated exactly with Eqs. (45)–(47), which yield

$$E_2(S,T_N) = 1 - \frac{\sin^2\left(\frac{N\phi}{N-1}\right)}{N^2\sin^2\left(\frac{\phi}{N-1}\right)}. \quad (49)$$

For $N = 2$ (single step) we recover Eq. (34) $[E_2(S,T_2) = 1 - \cos^2\phi = 1 - |\langle\psi_0|\psi_f\rangle|^2]$. If $\phi \in [0,\pi/2]$, $E_2(S,T_N)$ is a *decreasing* function of N (and an increasing function of ϕ) but rapidly saturates, approaching *a finite limit for* $N \to \infty$, namely,

$$E_2(S,T_\infty) = 1 - \frac{\sin^2\phi}{\phi^2}. \quad (50)$$

Therefore, system-time entanglement decreases as the number of steps through intermediate states between $|\psi_0\rangle$ and $|\psi_f\rangle$ is increased, reflecting the lower average distinguishability between the evolved states, but remains *finite* for $N \to \infty$. In this limit it is still an increasing function of ϕ for $\phi \in [0,\pi/2]$, reaching $1 - 4/\pi^2 \approx 0.59$ for $\phi = \pi/2$, i.e., when the system evolves to an orthogonal state ($|\psi_f\rangle = |1\rangle$), and decreasing to $\approx\phi^2/3$ for $\phi \to 0$. Hence, compared with a single-step evolution ($N = 2$), the ratio $E_2(S,T_\infty)/E_2(S,T_2)$ increases from 1/3 for $\phi \to 0$ to ≈ 0.59 for $\phi \to \pi/2$.

If ϕ is increased beyond $\pi/2$, the coefficients α_t and β_t cease to be all positive and entanglement can increase beyond ≈ 0.59 due to the decreased overlap γ, reflecting higher average distinguishability between evolved states. Entanglement $E_2(S,T_\infty)$ in fact reaches 1 at $\phi = \pi$ (and also $k\pi$, $k \geqslant 1$ integer), i.e., when the final state is proportional to the initial state after having covered the whole circle in the Bloch sphere, since for these values the time states $|\phi_0\rangle$ and $|\phi_1\rangle$ become orthogonal and with equal weights. Note also that for $\phi > \pi/2$, $E_2(S,T_N)$ is not necessarily a decreasing function of N or an increasing function of ϕ, exhibiting oscillations: $E_2(S,T_N) = 1$ for $\phi = k\pi(N-1)/N$, $k \neq lN$, and $E_2(S,T_N) \to 0$ for $\phi \to l\pi(N-1)$, l integer.

IV. CONCLUSIONS

We have proposed a parallel-in-time discrete model of quantum evolution based on a finite-dimensional clock entangled with the system. The ensuing history state satisfies a discrete Wheeler-DeWitt-like equation and can be generated through a simple circuit, which for a constant evolution operator can be efficiently implemented with just $O(n)$ qubits and control gates for 2^n time intervals.

We have then shown that the system-clock entanglement $E(S,T)$ is a measure of the actual distinguishable evolution undergone by one of the systems relative to the other. A natural interpretation of the Schmidt decomposition in terms of permanence in distinguishable evolved states is also obtained. For a constant Hamiltonian leading to a cyclic evolution, this entanglement is a measure of the energy spread of the initial state and satisfies an entropic uncertainty inequality with a conjugated entropy which measures the time spread. This Hamiltonian was rigorously shown to provide the maximum entanglement $E(S,T)$ compatible with a given distribution over Hamiltonian eigenstates. For other Hamiltonians, $E(S,T)$ [and also general entanglement entropies $E_f(S,T)$] are strictly bounded by the corresponding entropy of this distribution. We have also considered the evolution of mixed states. Although in this case the evaluation and interpretation of system-clock entanglement and correlations become more involved, in the simple yet fundamental case of a qubit clock coupled with a qubit subsystem, such entanglement was seen to be directly determined by the fidelity between the initial and the final states of the qubit. A direct relation between this entanglement and energy fluctuation was also derived for the pure case. Finally, we have also shown that $E(S,T)$ does remain finite and nonzero in the continuous limit, i.e., when the system evolves from an initial to a final state through an arbitrarily large number of closely lying, equally spaced intermediate states.

The present work opens the way to various further developments, starting from the definition of a proper time basis according to the Schmidt decomposition. It could also be possible in principle to incorporate other effects such as interaction between clocks [7] and explore possibilities of an emergent space-time or a qubit model for quantum time crystals [36]. At the very least, it provides a change of perspective, allowing us to identify a qubit clock as a fundamental "building block" of discrete-time-based quantum evolution.

ACKNOWLEDGMENTS

The authors acknowledge support from CIC (R.R.) and CONICET (A.B., N.G., M.C.) of Argentina.

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