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Simple compactifications and black p-branes in Gauss-Bonnet and Lovelock theories

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Abstract: We look for the existence of asymptotically flat simple compactifications of the form $\mathcal{M}_{D-p} \times T^p$ in D-dimensional gravity theories with higher powers of the curvature. Assuming the manifold $\mathcal{M}_{D-p}$ to be spherically symmetric, it is shown that the Einstein-Gauss-Bonnet theory admits this class of solutions only for the pure Einstein-Hilbert or Gauss-Bonnet Lagrangians, but not for an arbitrary linear combination of them. Once these special cases have been selected, the requirement of spherical symmetry is no longer relevant since actually any solution of the pure Einstein or pure Gauss-Bonnet theories can then be toroidally extended to higher dimensions. Depending on p and the spacetime dimension, the metric on $\mathcal{M}_{D-p}$ may describe a black hole or a spacetime with a conical singularity, so that the whole spacetime describes a black or a cosmic p-brane, respectively. For the purely Gauss-Bonnet theory it is shown that, if $\mathcal{M}_{D-p}$ is four-dimensional, a new exotic class of black hole solutions exists, for which spherical symmetry can be relaxed. Under the same assumptions, it is also shown that simple compactifications acquire a similar structure for a wide class of theories among the Lovelock family which accepts this toroidal extension. The thermodynamics of black p-branes is also discussed, and it is shown that a thermodynamical analogue of the Gregory-Laflamme transition always occurs regardless the spacetime dimension or the theory considered, hence not only for General Relativity.

Relaxing the asymptotically flat behavior, it is also shown that exact black brane solutions exist within a very special class of Lovelock theories.

Keywords: p-branes, Classical Theories of Gravity, Black Holes.

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Contents

1. Introduction

The metric theory of gravity consisting in second order equations of motion for the Riemann tensor and leading, besides, to a conserved stress-tensor for the matter fields is unique and is the quoted Lovelock theory of gravity [1]. As it was early pointed out by Lanczos [2], this theory does not lead to classical modifications to the four-dimensional Einstein's theory of general relativity, though actually differs from that in higher dimensions. In the generic case, such differences correspond to short-distance modifications to the Einstein theory and, even though these become negligible at large scales, actually lead to important corrections to the short scale physics. Perhaps, the black hole physics is the most celebrated example of this; for which both the thermodynamical and geometrical features turn out to be substantially modified by the inclusion of additional terms in the action of Lovelock theory.

The Lovelock Lagrangian density in D dimensions is

$$\mathcal{L} = \sum_{n=0}^{N} a_n \mathcal{L}_n \,, \tag{1.1}$$

where $2N = D - 2$ (for even dimensions D) while $2N = D - 1$ (for odd dimensions D). In (1.1), a_n are arbitrary constants which represent the coupling of the terms in the Lagrangian, and $\mathcal{L}_n$ is given by

$$\mathcal{L}_k = \frac{1}{2^k}\sqrt{-g}\,\delta^{i_1 \dots i_{2k}}_{j_1 \dots j_{2k}} R^{j_1 j_2}{}_{i_1 i_2} \dots R^{j_{2k-1} j_{2k}}{}_{i_{2k-1} i_{2k}} \,. \tag{1.2}$$

Here $R^{\mu}{}_{\nu\rho\gamma}$ is the Riemann tensor, g is the determinant of the metric $g_{\mu\nu}$ and $\delta^{i_1 \dots i_{2k}}_{j_1 \dots j_{2k}}$ is the generalized Kronecker delta of order $2k$. Then, the action reads

$$I = \int d^D x \mathcal{L} \,. \tag{1.3}$$

With this notation, the Lagrangian up to second order is given by the sum of three terms; namely

$$\begin{aligned}
\mathcal{L}_0 &= \sqrt{-g}\,, \\
\mathcal{L}_1 &= \frac{1}{2}\sqrt{-g}\,\delta^{i_1 i_2}_{j_1 j_2} R^{j_1 j_2}{}_{i_1 i_2} = \sqrt{-g} R\,, \\
\mathcal{L}_2 &= \frac{1}{4}\sqrt{-g}\,\delta^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4} R^{j_1 j_2}{}_{i_1 i_2} R^{j_3 j_4}{}_{i_3 i_4} = \sqrt{-g}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right),
\end{aligned}$$

For dimensions $D = 5$ and $D = 6$ the Lovelock Lagrangian is a linear combination of the Einstein-Hilbert term $\mathcal{L}_1$ and the often called Gauss-Bonnet term $\mathcal{L}_2$, which receives such a name because it corresponds to the Euler density in four dimensions. The theory in dimensions higher than five could also include the cubic Lagrangian $\mathcal{L}_3$, whose physical implications were early studied by Müller-Hoissen [3] and have been revisted recently in [4] and [5]. Lovelock terms can also been understood in the context of BRST cohomology [6].

The complexity of the Lovelock theory, basically due to higher order terms in the curvature tensor as well as the plethora of coupling constants, makes that the task of finding analytical exact solutions for this turns out to be a highly non-trivial problem (see, for instance, ref. [7]). However, as we will see below, for certain class of solutions to exist the coefficients a_n have to be fine-tuned in a precise form. This choice corresponds to be the same as requiring the theory to have a unique maximally symmetric vacuum with a fixed cosmological constant as in ref. [8]. It turns out that for such cases the solutions can be found in a closed form. Let us discuss an example of this: consider the case of the

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Einstein-Gauss-Bonnet theory in five dimensions, whose equations of motion take the form

$$\Lambda g_{\mu\nu} + \beta\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) + \alpha\left(2RR_{\mu\nu} - 4R_{\mu\rho}R^{\rho}{}_{\nu} - 4R_{\rho\delta}R^{\rho\delta}{}_{\mu\nu}\right.$$

$$\left. +2R_{\mu\rho\delta\gamma}R_{\nu}{}^{\rho\delta\gamma} - \frac{1}{2}g_{\mu\nu}(R_{\rho\delta\gamma\lambda}R^{\rho\delta\gamma\lambda} - 4R_{\rho\delta}R^{\rho\delta} + R^2)\right) = 0\,.$$

Then, if a simple compactification[1] of the form $\mathcal{M}_4 \times S^1$ is considered, with $\mathcal{M}_4$ representing a four-dimensional asymptotically flat solution with spherical symmetry; then, one is unavoidably led to the conclusion that the only way for obtaining a non-trivial solution is that of setting one of the coefficients α or β to zero. Namely, the only possibilities turn out to be $\Lambda = \alpha = 0$ and $\Lambda = \beta = 0$, i. e., only for the pure Einstein-Hilbert or Gauss-Bonnet Lagrangians, but not for an arbitrary linear combination of them. In Section 2, this is discussed in detail and, furthermore, it is also shown that the same effect occurs for solutions of the form $\mathcal{M}_{D-p} \times T^p$, for arbitrary D and p. Depending on p and the spacetime dimension, the metric on $\mathcal{M}_{D-p}$ may describe a black hole or a spacetime with a conical singularity, so that the whole spacetime describes a black or a cosmic p-brane, respectively. For the purely Gauss-Bonnet theory it is shown that, if $\mathcal{M}_{D-p}$ is four-dimensional, a new exotic class of black hole solutions exists, for which spherical symmetry can be relaxed. It is also shown that simple compactifications of the form $\mathcal{M}_d \times T^p$ in $D = d + p$ acquire a similar structure for the case of a theory described by a Lagrangian given by an arbitrary single term $\mathcal{L}_k$ in eq. (1.1). The seven-dimensional case is instructive because it captures the whole structure; and it is discussed in section 3. Section 4 is devoted to the discussion of the general case, i.e., for a Lagrangian given by $\mathcal{L}_k$ in D dimensions and with arbitrary p. The thermodynamics of black p-branes is discussed in section 5, where it is shown that a thermodynamical analogue of the Gregory-Laflamme transition always occurs regardless the spacetime dimension or the theory considered. Section 6 is devoted to the summary and to the discussion of the exotic case as well as to show that the asymptotically flat behavior can be relaxed, and exact black brane solutions with a warp factor exist within the class of theories discussed in ref. [8].

Note added: The same day this paper was sent to arXiv:hep-th, the paper [9] appeared in the same database, which contains some overlap with our results.

2. Einstein-Gauss-Bonnet Lagrangian: selecting the theories

In this section, we look for asymptotically flat simple compactifications of the form $\mathcal{M}_{D-p} \times T^p$ in D-dimensional Einstein-Gauss-Bonnet theory. By demanding the manifold $\mathcal{M}_{D-p}$ to be spherically symmetric, it is shown that the Einstein-Gauss-Bonnet theory admits this class of solutions only for the pure Einstein-Hilbert or Gauss-Bonnet Lagrangians, but not for an arbitrary linear combination of them. Once these special cases have been selected, the requirement of spherical symmetry is no longer relevant since actually any solution

[1] By simple compactification we mean a space which is a solution of the vacuum field equations in the absence of Kaluza-Klein gauge fields and with a constant dilaton.

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of the pure Einstein or pure Gauss-Bonnet theories can then be cylindrically extended to higher dimensions. Depending on p and the spacetime dimension D, the metric on $\mathcal{M}_{D-p}$ may describe a black hole solution or a spacetime with a conical singularity, so that the whole spacetime describes a black or a cosmic p-brane, respectively. For the purely Gauss-Bonnet theory, it is shown that, if $\mathcal{M}_{D-p}$ is four-dimensional ($D-p=4$), a new exotic class of black hole solutions does exist, for which spherical symmetry can be relaxed.

Then, we are interested in solutions of the form

$$ds^2 = d\tilde{s}^2_{D-p} + \sum_{n=1}^{p} R_0^{(n)} d\phi_n^2 \ , \tag{2.1}$$

where $\mathcal{M}_{D-p}$ with metric $d\tilde{s}^2_{D-p}$ is assumed to have spherical symmetry and the element $\sum_{n=1}^{p} R_0^{(n)} d\phi_n^2$ denotes the flat metric on the T^p and $R_0^{(n)}$ is the radius of the n-th S^1 factor. The spacetime indices of the manifold $\mathcal{M}_{D-p} \times T^p$ will be split according to $\alpha_1, \ldots, \alpha_{D-p}$ for $\mathcal{M}_{D-p}$ and $\phi_1, \ldots, \phi_p$ for T^p. Analogously, the tangent space indices of the manifold $\mathcal{M}_{D-p} \times T^p$ will be denoted by capital letters in the begin of the alphabet i.e. $A, B, \ldots$ and will be split as $\mu_1, \ldots, \mu_{D-p}$ for $\mathcal{M}_{D-p}$ and $i_n = i_1, \ldots, i_p$ for T^p.

In the next subsection, in order to present the idea and sketch the procedure, we first discuss the particular case $D = 7$ and $p = 1$. In the following subsection, we analyze the case for any dimension D and arbitrary p. Besides, notice that for $p = D$, the entire manifold is T^D, while for $p = D-1$, the entire manifold is $\mathbb{R} \times T^{D-1}$. Thus in what follows, we consider $p < D-1$.

The details of our results can be enormously simplified, and thus explicitly done, by the use of differential forms. The Lovelock action then reads

$$I = \int \sum_{n=0}^{[\frac{D-1}{2}]} \alpha_n \mathcal{L}^n \ , \tag{2.2}$$

with

$$\mathcal{L}^n = \epsilon_{a_1 \ldots a_{2n} a_{2n+1} \ldots a_D} R^{a_1 a_2} \ldots R^{a_{2n-1} a_{2n}} e^{a_{2n+1}} \ldots e^{a_D} \ , \tag{2.3}$$

and where e^a and R^{ab} stand for the vielbein and the curvature two-form. Here, $[x]$ stands for the integer part of x. The field equations obtained through the variation with respect to the vielbein and the spin connection then read (see e.g. [10])

$$\sum_{n=0}^{[\frac{D-1}{2}]} (D-2n)\, \alpha_n \mathcal{E}_a^n = 0 \, , \tag{2.4}$$

$$\sum_{n=1}^{[\frac{D-1}{2}]} n\, (D-2n)\, \alpha_n \mathcal{E}_{ab}^n = 0 \, , \tag{2.5}$$

respectively, and where

$$\mathcal{E}_a^n = \epsilon_{ab_1 \ldots b_{D-1}} R^{b_1 b_2} \ldots R^{b_{2n-1} b_{2n}} e^{b_{2n+1}} \ldots e^{b_{D-1}} \ , \tag{2.6}$$

$$\mathcal{E}_{ab}^n = \epsilon_{abc_1 \ldots c_{D-2}} R^{c_1 c_2} \ldots R^{c_{2n-1} c_{2n}} T^{c_{2n+1}} e^{c_{2n+2}} \ldots e^{c_{D-1}} \ . \tag{2.7}$$

In what follows, we assume that the torsion two-form T^a vanishes, so that eq. (2.5) is trivially satisfied.

The vielbeins and the curvature corresponding to (2.1) are given by

$$e^A = \begin{cases} e^{\mu_n} = \tilde{e}^{\mu_n} \\ e^{j_n} = R_0^{(j_n)} d\phi^{j_n} \end{cases} , \tag{2.8}$$

and

$$R^{AB} = \begin{pmatrix} R^{\mu_m \nu_n} & R^{\mu_m j_n} \\ R^{j_m \mu_n} & R^{\phi_m \phi_n} \end{pmatrix} = \begin{pmatrix} \tilde{R}^{\mu\nu} & 0 \\ 0 & 0 \end{pmatrix} . \tag{2.9}$$

Now, let us move to the seven dimensional case.

2.1 A working example: Einstein-Gauss-Bonnet theory in $D = 7$ and $p = 1$

Let us consider the Einstein-Gauss-Bonnet theory in $D = 7$ dimensions, whose field equations read

$$\epsilon_{ABCDEFG} \left(7\alpha_0 e^B e^C e^D e^E e^F e^G + 5\alpha_1 R^{BC} e^D e^E e^F e^G + 3\alpha_2 R^{BC} R^{DE} e^F e^G\right) = 0 . \tag{2.10}$$

In order to have an asymptotically flat solution, it is necessary to require that the coefficient α_0 that multiplies volume term vanishes. This can be easily seen as follows. Since eq. (2.10) can be factorized as

$$\gamma_2 \, \epsilon_{ABCDEFG} \left(R^{BC} + \gamma_1 e^B e^C\right) \left(R^{DE} + \gamma_0 e^D e^E\right) e^F e^G = 0 , \tag{2.11}$$

with $7\alpha_0 = \gamma_0\gamma_1\gamma_2$, it is apparent then that asymptotically flat solutions can only be found either if $\gamma_0 = 0$ or $\gamma_1 = 0$, which in turn means that α_0 must vanish. Thus, the suitable theory possessing asymptotically flat solutions has the following field equations

$$\mathcal{E}_A = \epsilon_{ABCDEFG} \left(3\alpha_2 R^{BC} R^{DE} e^F e^G + 5\alpha_1 R^{BC} e^D e^E e^F e^G\right) = 0 . \tag{2.12}$$

Considering a spacetime with geometry given by $\mathcal{M}_6 \times S^1$, and splitting the indices as $A = \{\mu, 1\}$, the field equations, $\mathcal{E}_A = 0$ in (2.12) become

$$\mathcal{E}_\mu = \epsilon_{\mu\nu\lambda\rho\sigma\tau 1} \left(3\alpha_2 \tilde{R}^{\nu\lambda} \tilde{R}^{\rho\sigma} \tilde{e}^\tau + 10\alpha_1 \tilde{R}^{\nu\lambda} \tilde{e}^\rho \tilde{e}^\sigma \tilde{e}^\tau\right) e^1 = 0 , \tag{2.13}$$

$$\mathcal{E}_1 = \epsilon_{1\nu\lambda\rho\sigma\tau\mu} \left(3\alpha_2 \tilde{R}^{\nu\lambda} \tilde{R}^{\rho\sigma} \tilde{e}^\tau \tilde{e}^\mu + 5\alpha_1 \tilde{R}^{\nu\lambda} \tilde{e}^\rho \tilde{e}^\sigma \tilde{e}^\tau \tilde{e}^\mu\right) = 0 . \tag{2.14}$$

Requiring spherical symmetry on $\mathcal{M}_6$, by virtue of the generalization of Birkhoff's theorem [11] and [12], eq. (2.13) implies that the metric $d\tilde{s}_6$ on $\mathcal{M}_6$ is the one found by Boulware and Deser [13]. On the other hand, considering the combinations $e^\mu \mathcal{E}_\mu - e^1 \mathcal{E}_1 = 0$, and $e^\mu \mathcal{E}_\mu - 2e^1 \mathcal{E}_1 = 0$ one obtains additional constraints on the geometry of $\mathcal{M}_6$

$$\alpha_1 \left(\epsilon_{1\nu\lambda\rho\sigma\tau\mu} \tilde{R}^{\nu\lambda} \tilde{e}^\rho \tilde{e}^\sigma \tilde{e}^\tau \tilde{e}^\mu\right) e^1 = 0 , \tag{2.15}$$

$$\alpha_2 \left(\epsilon_{1\nu\lambda\rho\sigma\tau\mu} \tilde{R}^{\nu\lambda} \tilde{R}^{\rho\sigma} \tilde{e}^\tau \tilde{e}^\mu\right) e^1 = 0 , \tag{2.16}$$

which is equivalent to say that each term in eq. (2.14) must vanish separately.

For the generic case where α_1 and α_2 are different from zero, the constraints (2.15) and (2.16) turn out to be too strong, since they imply that spacetime must be flat. In other words, in this case the constraints are satisfied by the Boulware-Deser solution only if the mass vanishes. Therefore, in order to circumvent this obstruction for the existence of nontrivial solutions, one has to require that either α_1 or α_2 vanish.

In the case $\alpha_2 = 0$, one recovers Einstein's theory and the remaining equation (2.15) (which just means the vanishing of the Ricci scalar) is no longer a constraint and generates no incompatibility since it just corresponds to the trace of the field equation (2.13).

The remaining possibility is to consider $\alpha_1 = 0$, i.e., the gravity theory described by the purely Gauss-Bonnet term. Analogously, in this case the remaining equation (2.16) is not a constraint since it is just the trace of the field equation (2.13) and the incompatibility is then removed.

In sum, we have shown that requiring the existence of an asymptotically flat solution of the form $\mathcal{M}_6 \times S^1$ we obtain that the volume term must be absent ($\alpha_0 = 0$), and assuming the manifold $\mathcal{M}_6$ to be spherically symmetric, it is shown that the Einstein-Gauss-Bonnet theory admits this class of solutions only for the pure Einstein-Hilbert or Gauss-Bonnet Lagrangians, but not for an arbitrary linear combination of them. Furthermore, once these special cases have been selected, one may notice that the requirement of spherical symmetry is no longer relevant since actually any solution of the six-dimensional pure Einstein or pure Gauss-Bonnet theories can then be cylindrically extended to seven dimensions.

2.2 Einstein-Gauss-Bonnet theory for arbitrary D and p

The case of arbitrary p in D dimensions is performed by the straightforward generalization of the results found in the previous subsection.

The field equations for Einstein-Gauss-Bonnet theory now read

$$\epsilon_{ab_1\dots b_{D-1}}\Big(D\alpha_0 e^{b_1}\dots e^{b_{D-1}} + (D-2)\,\alpha_1 R^{b_1b_2}e^{b_3}\dots e^{B_{D-1}} \\ + (D-4)\,\alpha_2 R^{b_1b_2}e^{b_3}\dots e^{b_{D-1}}\Big) = 0\,,$$

and again, requiring the existence of asymptotically flat solutions forces one to choose $\alpha_0 = 0$. Then, the suitable theory possessing asymptotically flat solutions reads

$$(D-4)\alpha_2\epsilon_{AB_1\dots B_{D-1}}R^{B_1B_2}R^{B_3B_4}e^{B_5}\dots e^{B_{D-1}} \\ +(D-2)\alpha_1\epsilon_{AB_1\dots B_{D-1}}R^{B_1B_2}e^{B_3}e^{B_4}e^{B_5}\dots e^{B_{D-1}} = 0\,. \tag{2.17}$$

We consider spacetimes of the form $\mathcal{M}_{D-p}\times T^p$, where T^p stands for the p-dimensional flat torus. Indices now split so that greek ones μ,ν,λ run along $\mathcal{M}_{D-p}$, and latin indices i,j,k along T^p. Then, the field equations now split as

$$\mathcal{E}_{\mu_1} = \epsilon_{\mu_1\mu_2\dots\mu_{D-p}i_1\dots i_p}\begin{bmatrix}\binom{D-5}{p}(D-4)\alpha_2\tilde{R}^{\mu_2\mu_3}\tilde{R}^{\mu_4\mu_5}\tilde{e}^{\mu_6}\dots\tilde{e}^{\mu_{D-p}}e^{i_1}\dots e^{i_p} \\ +\binom{D-3}{p}(D-2)\alpha_1\tilde{R}^{\mu_2\mu_3}\tilde{e}^{\mu_4}\dots\tilde{e}^{\mu_{D-p}}e^{i_1}\dots e^{i_p}\end{bmatrix} = 0\;, \tag{2.18}$$

$$\mathcal{E}_{i_1} = \epsilon_{i_1\mu_1\ldots\mu_{D-p}i_2\ldots i_p}\begin{array}{l}\Big[\binom{D-5}{p-1}(D-4)\alpha_2\tilde{R}^{\mu_1\mu_2}\tilde{R}^{\mu_3\mu_4}\tilde{e}^{\mu_5}\ldots\tilde{e}^{\mu_{D-p}}e^{i_2}\ldots e^{i_p} \\ \quad +\binom{D-3}{p-1}(D-2)\alpha_1\tilde{R}^{\mu_1\mu_2}\tilde{e}^{\mu_3}\ldots\tilde{e}^{\mu_{D-p}}e^{i_2}\ldots e^{i_p}\Big]\end{array} = 0\,, \tag{2.19}$$

where the decomposition of the vielbein and the curvature as in eqs. (2.8) and (2.9) has been used.

Requiring spherical symmetry on $\mathcal{M}_{D-p}$, and using the generalization of Birkhoff's theorem [11] and [12], the field equation along $\mathcal{M}_{D-p}$, (2.18) implies that the metric of $\mathcal{M}_{D-p}$ corresponds to the Boulware-Deser solution [13]. Then, suitable linear combinations of the trace field equations allow give rise to the following constraints

$$\alpha_1\epsilon_{i_1\mu_1\ldots\mu_{D-p}i_2\ldots i_p}\tilde{R}^{\mu_1\mu_2}\tilde{e}^{\mu_3}\ldots\tilde{e}^{\mu_{D-p}}e^{i_2}\ldots e^{i_p} = 0\,, \tag{2.20}$$

$$\alpha_2\epsilon_{i_1\mu_1\ldots\mu_{D-p}i_2\ldots i_p}\tilde{R}^{\mu_1\mu_2}\tilde{R}^{\mu_3\mu_4}\tilde{e}^{\mu_5}\ldots\tilde{e}^{\mu_{D-p}}e^{i_2}\ldots e^{i_p} = 0\,, \tag{2.21}$$

implying that each term in eq. (2.19) must vanish separately.

Again, for the generic case where α_1 and α_2 are different from zero, the constraints (2.20) and (2.21) annihilate the mass of the Boulware-Deser solution implying that spacetime must be flat. Hence, in order to obtain nontrivial solutions it is necessary to require that either α_1 or α_2 vanish. In the case $\alpha_2 = 0$, the remaining equation (2.20) is no longer a constraint since it corresponds to the trace of the field equation (2.18). For the remaining possibility, $\alpha_1 = 0$, eq. (2.21) is not a constraint since it becomes the trace of the field equation (2.18) and the generic incompatibility is thus removed.

Therefore, it has been shown that requiring the existence of an asymptotically flat solution of the form $\mathcal{M}_{D-p} \times T^p$, the volume term must be absent ($\alpha_0 = 0$), and assuming the manifold $\mathcal{M}_{D-p}$ to be spherically symmetric, the Einstein-Gauss-Bonnet theory was shown to admit this class of solutions only for the pure Einstein-Hilbert or pure Gauss-Bonnet cases, but not for an arbitrary linear combination of them. Having selected these special cases, the requirement of spherical symmetry can be dropped out since actually any solution of the $(D-p)$-dimensional pure Einstein or pure Gauss-Bonnet theories can then be toroidally extended to D dimensions.[2]

2.3 Summary and extension to theories with higher powers in the curvature

We have shown that the existence of non trivial toroidal extensions of asymptotically flat and spherically symmetric solutions for the Einstein-Gauss-Bonnet theory is only achieved for the cases where the Lagrangian is selected in the form

$$\mathcal{L} = \mathcal{L}^{(k)}\ \ ,\ \text{for } k \geq 1\,,$$

which corresponds to the pure Gauss-Bonnet Lagrangian ($\mathcal{L}^{(2)} = \epsilon RRe\ldots e$), or for the pure Einstein-Hilbert case ($\mathcal{L}^{(1)} = \epsilon Re\ldots e$), and that in these cases the requirement of spherical symmetry can be dropped out because for this special class of theories any solution of the $(D-p)$-dimensional pure Einstein or pure Gauss-Bonnet theories can then be toroidally extended to D dimensions.

[2] Let us mention here that cylindrical extensions of four-dimensional solutions in Einstein-Gauss-Bonnet theory were also discussed in references [14, 15] and [16]

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One may then naturally wonder whether this results extend to theories with higher powers in the curvature.

Based on the previous results, we consider the theory described by a Lagrangian given by the dimensional continuation of the Euler density of dimension $2k$, i.e.,

$$I^{(k)} = \frac{\kappa_{D,k}}{(D-2k)} \int \epsilon_{B_1 \ldots B_{2k} B_{2k+1} \ldots B_D} R^{B_1 B_2} \ldots R^{B_{2k-1} B_{2k}} e^{B_{2k+1}} \ldots e^{B_D} \,. \tag{2.22}$$

This action possesses well behaved spherically symmetric black hole solutions [8] for $k < \left[\frac{D}{2}\right]$ with metric

$$ds_d^2 = -\left(1 - \left(\frac{2G_k m}{r^{D-2k-1}}\right)^{\frac{1}{k}}\right) dt^2 + \frac{dr^2}{\left(1 - \left(\frac{2G_k m}{r^{D-2k-1}}\right)^{\frac{1}{k}}\right)} + r^2 d\Omega_{D-2}^2 \,, \tag{2.23}$$

where m is the mass and the gravitational constant[3] G_k is related to the coupling constant in the action $\kappa_{D,k}$ by

$$\kappa_{D,k} = \frac{1}{2\,(D-2)!\Omega_{D-2} G_k} \,. \tag{2.24}$$

For $D = 2k+1$, (2.23) describes a conical singularity.

One can then see that these theories are special in the same sense described above, since any of their solutions for a given dimension can be toroidally extended to higher dimensions within the same theory. Therefore the class of black hole solutions (2.23) can be extended to black p-branes. It is then natural to see whether there exists a Gregory-Laflamme-like transition for these kind of objects within this class of theories which are quite different from General Relativity.

It is also shown here that for some particular cases, the toroidally extended solutions may describe cosmic strings as well as black strings with exotic topology in the transverse section. This last kind of exotic black p-branes objects belong to a completely different class of solutions, since they do not correspond to the toroidal extension of the black holes found in [8].

In the next section we discuss the seven-dimensional case since it is a simple and good representative that captures the whole structure present in the general case.

3. Scanning the seven-dimensional case with extended objects

The field equations for the seven-dimensional class of theories described by the action $I^{(k)}$ in eq. (2.22) are given by

$$\epsilon_{AB_1 \ldots B_{2k} B_{2k+1} \ldots B_6} R^{B_1 B_2} \ldots R^{B_{2k-1} B_{2k}} \overbrace{e^{B_{2k+1}} \ldots e^{B_6}}^{6-2k} = 0 \,. \tag{3.1}$$

Let us consider a spacetime of the form $\mathcal{M}_{7-p} \times T^p$, whose metric is of the form (2.1). Before entering into the details we summarize how the structure depends on the values of k and p:

[3] The usual Newton constant is related with this one through $8\pi G_{Newton} = (d-2)\Omega_d G_{k=1}$, where Ω_d is the volume of unit sphere in d dimensions, being $\Omega_d = 2\pi^{\frac{d+1}{2}}/\Gamma(\frac{d+1}{2})$.

- $p < 6 - 2k$: Black p-brane, where $\mathcal{M}_{7-p}$ is a black hole of the form (2.23).
- $p = 6-2k$: Cosmic p-brane, since in this case $\mathcal{M}_{7-p}$ is given by (2.23) which describes a conical singularity.
- $p = 7 - 2k$: Exotic black p-brane, where $\mathcal{M}_{7-p}$ is given by a new kind of black hole geometry for which the Euler density vanishes.
- $6 > p \geq 8 - 2k$: The manifold $\mathcal{M}_{7-p}$ is arbitrary.

In the following subsections we describe in detail these results.

3.1 Case $k = 1$: The Einstein theory

The toroidal extensions of the Einstein-Hilbert theory are well-known and we discuss them here just for completeness. For $p \geq 6$ the whole spacetime is locally flat, while for $p = 5$ the manifold $\mathcal{M}_2$ must be locally flat, which in two dimensions is trivially equivalent to say that its Euler density vanishes, i.e., $\varepsilon_2(\mathcal{M}) = 0$. For $p = 4$, one recovers the cosmic brane, since in this case the manifold $\mathcal{M}_3$ solves the three-dimensional Einstein equations without cosmological constant whose spherically symmetric solutions have a conical singularity. The remaining cases correspond to $p < 4$, where the solutions are Black p-branes where $\mathcal{M}_{7-p}$ is endowed with the Schwarzschild metric.

3.2 Case $k = 2$: The pure Gauss-Bonnet theory

For $k = 2$, the equations of motion are given by

$$\epsilon_{ABCDEFG} R^{BC} R^{DE} e^F e^G = 0 \,. \tag{3.2}$$

3.2.1 $p < 2$: Black string and black hole

The black hole (2.23) is obviously recovered for $p = 0$.

For $p = 1$ the geometry describes a Black string. In this case, the field equations split as

$$\epsilon_{\mu\nu\lambda\rho\sigma\tau} \tilde{R}^{\nu\lambda} \tilde{R}^{\rho\sigma} \tilde{e}^{\tau} = 0 \,, \tag{3.3}$$

$$\epsilon_{\mu\nu\lambda\rho\sigma\tau} \tilde{R}^{\nu\lambda} \tilde{R}^{\rho\sigma} \tilde{e}^{\tau} \tilde{e}^{\mu} = 0 \,, \tag{3.4}$$

where the second eq. gives no extra restrictions since it is just the trace of (3.3). The problem has then been reduced to solve the same equations for $\mathcal{M}_6$. Therefore, as any solution of the theory with $k = 2$ can be cylindrically extended, a black string is obtained from to the cylindrical extension of the six-dimensional black hole (2.23) with $k = 2$.

3.2.2 $p = 2$: Cosmic membrane

For $p = 2$ the field equations split according to

$$\epsilon_{\mu\nu\lambda\rho\tau} \tilde{R}^{\nu\lambda} \tilde{R}^{\rho\tau} = 0 \,, \tag{3.5}$$

$$\epsilon_{\mu\nu\lambda\rho\tau} \tilde{R}^{\nu\lambda} \tilde{R}^{\rho\tau} \tilde{e}^{\mu} = 0 \,, \tag{3.6}$$

and again the second equation is the trace of the equation for $\mathcal{M}_5$. The problem has been reduced to the one of finding a solution for the purely Gauss-Bonnet theory in five dimensions. Since the spherically symmetric solution is a spacetime with a conical singularity, its toroidal extension corresponds to a Cosmic membrane.

3.2.3 $p = 3$: The exotic black three-brane

In the case of $p = 3$ a very interesting phenomenon occurs, since in this case the projection of the field equations along $\mathcal{M}_4$ is trivially satisfied ($\mathcal{E}_\mu \equiv 0$) because at least one of the indices along T^3 lies in a curvature. The remaining field equation then reads

$$\varepsilon_4 \left(\mathcal{M}_4\right) := \epsilon_{\mu\nu\lambda\rho} \tilde{R}^{\mu\nu} \tilde{R}^{\lambda\rho} = 0 \,. \tag{3.7}$$

This means that we have a single scalar equation for the four-manifold $\mathcal{M}_4$ which states that its Euler density vanishes. Note that this is a very weak condition on the geometry of $\mathcal{M}_4$, as compared with the equations for a standard gravity theory on a four-dimensional manifold. An explicit solution of eq. (3.7) that includes a black hole with exotic topology is presented in section 6. Therefore its toroidal extension originates the exotic black 3-brane.

3.2.4 $6 > p \geq 4$: The manifold $\mathcal{M}_{7-p}$ is arbitrary

In this case the field equations are trivially satisfied ($\mathcal{E}_\mu \equiv 0$, $\mathcal{E}_i \equiv 0$) since the torus is "big enough" to ensure that at least one of the indices along T^p lies always in the curvatures. Therefore, as the equations of motion are identically solved, one obtains no restriction on the geometry of $\mathcal{M}_{7-p}$.

3.3 Case $k = 3$: Beyond the Einstein and Gauss-Bonnet theories

The field equations for $k = 3$ read

$$\epsilon_{ABCDEFG} R^{BC} R^{DE} R^{FG} = 0 \,. \tag{3.8}$$

In the case of $p = 0$, the spherically symmetric solution with conical singularity is recovered from eq. (2.23), and no seven-dimensional toroidal extension of the black holes found in [8] exist for any value of p.

3.3.1 $p = 1$: The exotic string

In the case of $p = 1$ the projection of the field equations along $\mathcal{M}_6$ is again trivially satisfied ($\mathcal{E}_\mu \equiv 0$). The remaining field equation now reads

$$\varepsilon_6 \left(\mathcal{M}_6\right) := \epsilon_{\mu\nu\lambda\rho\sigma\tau} \tilde{R}^{\mu\nu} \tilde{R}^{\lambda\rho} \tilde{R}^{\sigma\tau} = 0 \;, \tag{3.9}$$

which means that the Euler density of $\mathcal{M}_6$ vanishes. A solution for eq. (3.9) that includes a black hole with exotic topology is presented in section 6, so that its cylindrical extension is the exotic black string.

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3.3.2 $6 > p \geq 2$: The manifold $\mathcal{M}_{7-p}$ is arbitrary

In this case the field equations are again trivially satisfied since the dimension of the torus is large enough. This means that there is no restriction on the geometry of $\mathcal{M}_{7-p}$.

In the next section the results in seven dimensions are generalized for arbitrary D, k and p.

4. Arbitrary D, k and p

The field equations read

$$\epsilon_{AB_1 \ldots B_{2k} B_{2k+1} \ldots B_{D-1}} R^{B_1 B_2} \ldots R^{B_{2k-1} B_{2k}} \overbrace{e^{B_{2k+1}} \ldots e^{B_{D-1}}}^{D-2k-1} = 0 \ . \tag{4.1}$$

Consider a spacetime of the form $\mathcal{M}_{D-p} \times T^p$, with a metric given by (2.1). The summary of how the structure depends on p for the D-dimensional theory with k curvatures given by (2.22) is:

- $p < D-2k-1$: Black p-brane, where $\mathcal{M}_{D-p}$ has a black hole metric of the form (2.23).
- $p = D-2k-1$: Cosmic p-brane, where $\mathcal{M}_{D-p}$ is given by (2.23) and describes a spacetime with a conical singularity.
- $p = D-2k$: Exotic black p-brane, where $\mathcal{M}_{D-p}$ has a new kind of black hole metrics for which the Euler density vanishes.
- $D-1 > p \geq D-2k+1$: The manifold $\mathcal{M}_{D-p}$ is arbitrary.

The splitting of the fields equations generically is

$$\mathcal{E}_\mu = \epsilon_{\mu\nu_1 \ldots \nu_{2k} \ldots \nu_{D-p-1} j_1 \ldots j_p} \tilde{R}^{\nu_1\nu_2} \ldots \tilde{R}^{\nu_{2k-1}\nu_{2k}} \overbrace{\underbrace{\tilde{e}^{\nu_{2k+1}} \ldots \tilde{e}^{\nu_{D-p-1}}}_{D-2k-1-p} e^{j_1} \ldots e^{j_p}}^{D-2k-1} = 0 \ , \tag{4.2}$$

$$\mathcal{E}_i = \epsilon_{i\nu_1 \ldots \nu_{2k}\nu_{2k+1} \ldots \nu_{D-p} j_1 \ldots j_{p-1}} \tilde{R}^{\nu_1\nu_2} \ldots \tilde{R}^{\nu_{2k-1}\nu_{2k}} \overbrace{\underbrace{\tilde{e}^{\nu_{2k+1}} \ldots \tilde{e}^{\nu_{D-p}}}_{D-2k-p} e^{j_1} \ldots e^{j_{p-1}}}^{D-2k-1} = 0 \ , \tag{4.3}$$

and the analysis goes as follows.

4.1 $p < D-2k-1$: The black p-branes (The problem reduces to find a black hole for the same theory in $(D-p)$-dimensions)

For $p = 0$ the black hole (2.23) is directly recovered.

For the rest of the allowed range of p the geometry correspond to a black p-brane. In this case, the field equations now split in the following way

$$\epsilon_{\mu\nu_1 \ldots \nu_{2k} \ldots \nu_{D-p-1}} \tilde{R}^{\nu_1\nu_2} \ldots \tilde{R}^{\nu_{2k-1}\nu_{2k}} \tilde{e}^{\nu_{2k+1}} \ldots \tilde{e}^{\nu_{D-p-1}} = 0 \ , \tag{4.4}$$

$$\epsilon_{\nu_1 \ldots \nu_{2k}\nu_{2k+1} \ldots \nu_{D-p-1}\nu_{D-p}} \tilde{R}^{\nu_1\nu_2} \ldots \tilde{R}^{\nu_{2k-1}\nu_{2k}} \tilde{e}^{\nu_{2k+1}} \ldots \tilde{e}^{\nu_{D-p-1}} \tilde{e}^{\nu_{D-p}} = 0 \ . \tag{4.5}$$

The field equations along T^p give no extra conditions since they reduce to the trace of the field equation along $\mathcal{M}_{D-p}$. Therefore the problem reduces to solve the field equations for the same theory in $(D-p)$-dimensions. Consequently the black p-branes are described by the toroidal extension of the black holes discussed in [8].

4.2 $p = D - 2k - 1$: The cosmic p-branes

In this case the field equations split as follows

$$\epsilon_{\mu\nu_1\ldots\nu_{D-p-1}}\tilde{R}^{\nu_1\nu_2}\ldots\tilde{R}^{\nu_{D-p-2}\nu_{D-p-1}} = 0 \ ,$$
$$\epsilon_{\nu_1\ldots\nu_{D-p-1}\nu_{D-p}}\tilde{R}^{\nu_1\nu_2}\ldots\tilde{R}^{\nu_{D-2p-2}\nu_{D-2p-1}}\tilde{e}^{\nu_{D-p}} = 0 \ ,$$

so that the equations along T^p give again the trace of the equation along $\mathcal{M}_{2k+1}$. As the problem is reduced the one of finding a solution for the same theory in $(2k+1)$-dimensions, one can use the spherically symmetric conical solutions for $\mathcal{M}_{2k+1}$ to obtain the cosmic p-branes from their toroidal extensions.

4.3 $p = D - 2k$: The exotic black p-branes with vanishing Euler density on $\mathcal{M}_{2k}$

For $p = D-2k$, the projection of the field equations along $\mathcal{M}_{2k}$ is trivially satisfied ($\mathcal{E}_\mu \equiv 0$) because the torus T^p is large enough so as at least one of associated indices always lies in a curvature. A very interesting phenomenon then occurs, since the remaining field equation implies the vanishing of the Euler density of $\mathcal{M}_{2k}$, i.e. :

$$\varepsilon_{2k}\left(\mathcal{M}_{2k}\right) := \epsilon_{\nu_1\ldots\nu_{D-p}}\tilde{R}^{\nu_1\nu_2}\ldots\tilde{R}^{\nu_{D-p-1}\nu_{D-p}} = 0 \ . \tag{4.6}$$

This is a very weak condition on the geometry of $\mathcal{M}_{2k}$, since it is just a scalar equation. An explicit solution for eq. (4.6) including a black hole with exotic topology is discussed in section 6. The toroidal extensions of these black objects generate the exotic black p-branes.

4.4 $D - 1 > p \geq D - 2k + 1$: The manifold $\mathcal{M}_{D-p}$ is arbitrary

In this case the dimension of the torus is large enough so as to ensure that the field equations are trivially satisfied, which means that the geometry of $\mathcal{M}_{D-p}$ is completely unrestricted.

5. Thermodynamics

In this section, we study the thermodynamics of the black p-brane solutions that can be obtained from the toroidal extension of the black holes in eq. (2.23). The computation of the mass and the entropy is explicitly performed, and their thermodynamical stability is briefly discussed for the microcanonical ensemble, which suggest the extension of the Gregory-Laflamme instability [17] for the class of theories considered here. We first discuss the particular case $p = 1$ and then describe the thermodynamics for $p < D - 2k - 1$.

We have shown that for the cases under consideration, the equations of motion for the whole space $\mathcal{M}_{D-p} \times T^p$ in the theory (2.22) with k powers in the curvature reduce to the field equations of the same theory for the $(D-p)$-dimensional manifold $\mathcal{M}_{D-p}$.

Analogously, it is simple to show that the Euclidean action evaluated on the black p-brane in D dimensions is proportional to the Euclidean action of the black hole in $\mathcal{M}_{D-p}$. This means that there is a mapping between the thermodynamics of the black p-branes and the thermodynamics of the black holes in $\mathcal{M}_{D-p}$. The thermodynamics of these black holes was studied in [8], and the temperature, the mass and the entropy for the black holes in eq. (2.23) where shown to be given by

$$T = \frac{1}{4\pi k_B} \frac{(D-2k-1)}{k} \frac{1}{r_+} , \tag{5.1}$$

$$m = \frac{1}{2G_k} r_+^{D-2k-1} , \tag{5.2}$$

$$S_k = \frac{2\pi k_B}{G_k} \frac{k}{(D-2k)} r_+^{D-2k} , \tag{5.3}$$

respectively. Here k_B stands for the Boltzmann constant.

5.1 The action for the black p-brane

Let us begin with the simplest case $p = 1$, though for arbitrary dimension D and with arbitrary k. Evaluating the Euclidean action (2.22) for any simple compactification of the form $\mathcal{M}_{D-1} \times S^1$ gives

$$\begin{aligned} I_{D,k} &= 2\pi R_0 \kappa_D \int \epsilon_{\mu_1 \ldots \mu_{2k}\mu_{2k+1} \cdots \mu_{D-1}\phi} \tilde{R}^{\mu_1\mu_2} \ldots \tilde{R}^{\mu_{2k-1}\mu_{2k}} \tilde{e}^{\mu_{2k+1}} \ldots \tilde{e}^{\mu_{D-1}} && (5.4) \\ &= \frac{\kappa'_{D-1}}{(D-1-2k)} \int \epsilon_{\mu_1 \ldots \mu_{2k}\mu_{2k+1} \cdots \mu_{D-1}\phi} \tilde{R}^{\mu_1\mu_2} \ldots \tilde{R}^{\mu_{2k-1}\mu_{2k}} \tilde{e}^{\mu_{2k+1}} \ldots \tilde{e}^{\mu_{D-1}} && (5.5) \\ &= I'_{D-1,k} , && (5.6) \end{aligned}$$

where we have defined

$$\kappa'_{D-1} = 2\pi R_0 \kappa_D \left(D - 1 - 2k\right) . \tag{5.7}$$

This means that the Euclidean action for the simple compactification $\mathcal{M}_{D-1} \times S^1$ reduces to the Euclidean action for the same theory on $\mathcal{M}_{D-1}$ with a modified gravitational constant, which by means of eq. (2.24), is given by[4]

$$G'_{d,k} = \frac{(d-1)}{(d-2k)} \frac{\Omega_{d-1}}{\Omega_{d-2}} \frac{1}{2\pi R_0} G_{d+1,k} . \tag{5.8}$$

Therefore we have shown that the thermodynamics for the black p-branes can be reproduced following the lines of [8], considering the metric of $\mathcal{M}_{D-1}$ to be given by (2.23) with the modified gravitational constant.[5]

[4]The subindex d in $G_{d,k}$ stands to explicitly refer to the dimension, which turns out to be very important in the discussion here; cf. eq. (2.23).

[5]It would also be interesting to analyze the thermodynamics following alternative approaches, see e.g. [18], [19], [20] and references therein, as well as to see the mass loss rates and lifetimes as in [21]

The entropy for the black string $\mathcal{M}_{D-1} \times S^1$ is then given by

$$S_k^{Bs} = 2\pi k_B \frac{1}{G'_{d,k}} \frac{k r_+^{(d-2k)}}{(d-2k)} \tag{5.9}$$

$$= \frac{4\pi^2 k_B k}{(d-1)} \frac{R_0 \Omega_{d-2}}{\Omega_{d-1}} \frac{r_+^{(d-2k)}}{G_{d+1,k}}, \tag{5.10}$$

where the horizon radius r_+ can be expressed in term of the black hole mass m in $\mathcal{M}_{D-1}$, namely

$$r_+ = (2G_{k,d} m)^{\frac{1}{d-2k-1}} \ . \tag{5.11}$$

Hence, the horizon radius can be written also in terms of the modified gravitational constant and the string mass since

$$r_+ = \left(2G'_{k,d} m_{string}\right)^{\frac{1}{d-2k-1}} \ . \tag{5.12}$$

This implies that the entropy of the black string expressed in terms of the string mass reads

$$S_k^{Bs} = A_k \ (m_{string})^{\frac{D-2k-1}{D-2k-2}}, \tag{5.13}$$

where the coefficient A_k is given by

$$A_k = k \frac{4\pi k_B}{(D-2k-1)} \left(\frac{2\,(D-2)}{(D-2k-1)} \frac{\Omega_{D-2}}{\Omega_{D-3}} \frac{G_{D,k}}{2\pi R_0} \right)^{\frac{1}{D-2k-2}} . \tag{5.14}$$

Proceeding in the same way, in the generic case for arbitrary p and k, the entropy of the black p-brane of the form $\mathcal{M}_d \times T^p$ in $D = d + p$ dimensions for the theory (2.22) with k powers in the curvature is found to be given by

$$S_k^{Bp\text{-}b} = 4\pi \kappa_B \frac{k}{(D-p-2k)} \ (m_{p\text{-}b})^{\frac{D-2k-p}{D-2k-p-1}} (2A_{p,k} G_{D,k})^{\frac{D-2k-p}{D-2k-p-1} - 1} \ , \tag{5.15}$$

where $m_{p\text{-}b}$ is the mass of the black p-brane, where the factor $A_{p,k}$ is

$$A_{p,k} = \frac{(D-2k-p-1)!(D-2)!}{(D-2k-1)!(D-2-p)!} \times \frac{\Omega_{D-2}}{\Omega_{D-p-2} Vol(T^p)} \ .$$

Here $Vol(T^p)$ corresponds to the volume of the p-torus.

It is easy to check that the expression for the entropy above reproduces the right factors as well as the area law for the Einstein theory which is recovered in the case $k = 1$.

5.2 Thermodynamical instability: Black holes v/s Black p-branes

Now, we have the suitable tools to study the thermodynamical stability of the black p-branes solutions. A naive, but interesting result can be obtained comparing the entropy of a D-dimensional black hole with the one for a black p-brane in the microcanonical ensemble, i.e., for a fixed mass m. In this case the quotient between both entropies is

$$\frac{S_k^{Bp\text{-}b}}{S_k^{Bh}} = \frac{D-2k}{D-2k-p} (2G_{k,D})^a \ (A_{p,k})^{\frac{1}{D-2k-p-1}} \ m^b \ . \tag{5.16}$$

with a and b given by

$$a := \frac{1}{D-2k-p-1} - \frac{1}{D-2k-1},$$
$$b := \frac{D-2k-p}{D-2k-p-1} - \frac{D-2k}{D-2k-1}.$$

This equation implies the existence of a critical mass m_c for which both entropies agree, $S_k^{B\text{-}pb} = S_k^{Bh}$, that explicitly depends on D, k and p. The critical mass defined the point where the thermodynamic transition occurs, since for $m > m_c$ the black p-brane has an entropy greater than the one of the black hole, meaning that in this case the black p-brane is thermodynamically favoured, and in this sense stable unlike the black hole. The converse is obtained for $m < m_c$. This transition suggests that a thermodynamic analogue of the Gregory-Laflamme instability should also exist beyond General Relativity, at least for the class of theories considered here.[6] It is worth pointing out that when the ratio $S_k^{Bp\text{-}b}/S_k^{Bh}$ is expressed in terms of the horizon radius, the scaling of such quantity agrees with the one for the Einstein theory, even though the entropies do not follow the area law.

6. Discussion and summary

Asymptotically flat simple compactifications of the form $\mathcal{M}_{D-p} \times T^p$ were shown to exist for the class of theories described by the action (2.22). If the manifold $\mathcal{M}_{D-p}$ is assumed to be spherically symmetric, it was shown that the Einstein-Gauss-Bonnet theory admits this class of solutions only for the pure Einstein-Hilbert or Gauss-Bonnet Lagrangians, but not for an arbitrary linear combination of them. Once these special cases have been selected, the requirement of spherical symmetry is no longer relevant since actually any solution of the pure Einstein or pure Gauss-Bonnet theories can then be toroidally extended to higher dimensions. Depending on p and the spacetime dimension, the metric on $\mathcal{M}_{D-p}$ may describe a black hole or a spacetime with a conical singularity, so that the whole spacetime describes a black or a cosmic p-brane, respectively. Under the same assumptions, it was also shown that simple compactifications acquire a similar structure for the whole class of theories defined by the action (2.22).

The thermodynamics of black p-branes was also discussed, and it was shown that a thermodynamical analogue of the Gregory-Laflamme transition should be expected to occur regardless the value of k and the spacetime dimension, and hence not only for General Relativity.

A new class of exotic black p-branes exist for $p = D-2k$, for which the manifold $\mathcal{M}_{2k}$ possesses a metric that makes the Euler density to vanish. As we have seen, the case when $p = D-2k$ tell us that the Euler density of the $2k$-dimensional M manifold must vanish. A black hole metric of this kind for $\mathcal{M}_{2k}$ can be found as follows:

Consider an ansatz of the form

$$d\tilde{s}^2 = -f^2(r)\,dt^2 + \frac{dr^2}{f^2(r)} + r^2 d\Sigma_\gamma^2 \,, \tag{6.1}$$

[6] It is worth pointing out that, even for black holes in Gauss-Bonnet theories the stability possesses a fairly different behavior as compared with the Schwarzschild solution [22 – 25].

where $d\Sigma_\gamma^2$ stands for the line element of a manifold with constant curvature given by γ. In the four dimensional case the equation

$$\varepsilon_4\left(\mathcal{M}_4\right) := \epsilon_{\mu\nu\lambda\rho}\tilde{R}^{\mu\nu}\tilde{R}^{\lambda\rho} = 0 \ ,$$

has the following solution:

$$ds^2 = -\left(\pm\sqrt{2Cr+B}+\gamma\right)dt^2 + \frac{dr^2}{\left(\pm\sqrt{2Cr+B}+\gamma\right)} + r^2 d\Sigma_\gamma^2 \ , \tag{6.2}$$

for certain integration constants B and C, with Σ a 2-dimensional manifold of constant curvature γ. This solution describes a black hole in the case for negative γ and for the branch with the “plus” sign.

In the even dimensional case the equation

$$\varepsilon_{2k}\left(\mathcal{M}_{2k}\right) := \epsilon_{\nu_1\ldots\nu_{D-p}}\tilde{R}^{\nu_1\nu_2}\ldots\tilde{R}^{\nu_{D-p-1}\nu_{D-p}} = 0 \ , \tag{6.3}$$

admits the following solution

$$d\tilde{s}^2 = -\left(\gamma-\sigma\left(nCr+B\right)^{\frac{1}{n}}\right)dt^2 + \frac{dr^2}{\left(\gamma-\sigma\left(nCr+B\right)^{\frac{1}{n}}\right)} + d\Sigma_\gamma^2 \ , \tag{6.4}$$

where $\sigma = (\pm 1)^{n+1}$. Therefore in general the metric distinguishes between the cases where the dimension is $d = 4m$ or $d = 4m+2$, and a black hole is obtained when Σ_γ is a manifold of negative constant curvature. A detailed analysis of this kind of objects as well as their thermodynamics is an interesting open problem.

We have considered so far the class of theories given by the action (2.22) which possesses the special property of having a single maximally symmetric vacuum which is flat [8]. These theories were obtained from the vanishing cosmological constant limit of the class of theories with an action given by

$$I_k = \kappa\int\sum_{n=0}^{k}c_n^k\mathcal{L}^n \ , \tag{6.5}$$

with the following choice of coefficients

$$c_n^k = \begin{cases} \frac{l^{2(n-k)}}{(D-2n)}\begin{pmatrix} k \\ n \end{pmatrix} & , \ n \leq k \\ 0 & , \ n > k \end{cases} \tag{6.6}$$

and $1 \leq k \leq \left[\frac{D-1}{2}\right]$. This class of theories possesses a unique maximally symmetric AdS vacuum with radius l. The de Sitter case is obtained making $l \rightarrow il$. Exact spherically symmetric black hole solutions for these class of theories exist, and their generalization to the case with topologically nontrivial AdS asymptotics was done in [26].

Extended black objects within these theories can be obtained from the following class of metrics

$$ds_D^2 = f^2\left(z\right)d\tilde{s}_{D-1}^2 + dz^2 \ , \tag{6.7}$$

where $d\tilde{s}^2_{D-1}$ can be any solution of the same theory (as the ones found in [8], [26]) in one dimension below provided that the warp factor has the form

$$f(z) = A \sinh\left(\sqrt{\lambda}\,(z - z_0)\right) \,, \tag{6.8}$$

where λ is determined by the D-dimensional cosmological constant, and A is related to the cosmological constant of the same theory in $(D-1)$ dimensions. When the cosmological constant of the theory in $(D-1)$-dimensions tends to zero the warp factor tends to $\exp(\sqrt{\lambda}\,(z - z_0))$.

The thermodynamics of this class of extended objects is also an open problem.

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