TEACHING MUSICAL METER TO SCHOOL-AGE STUDENTS THROUGH THE SKI-HILL GRAPH

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This is to certify that to the best of my knowledge, the content of this thesis is my own work. This thesis has not been submitted for any degree or other purposes.

ABSTRACT

This thesis, *Teaching Musical Meter to School-Age Students Through the Ski-Hill Graph*, aims to demonstrate the "pedagogability" of modern meter theory, that is, that new scholarship on meter can translate into a coherent and practically implementable instructional curriculum, with various advantages for school-age students. The curriculum model developed in the thesis is derived from Richard Cohn's work on and approaches to meter theory, which focuses on "sound rather than notation" and graphic representations of meter through mathematical music theory. The materials set out in the thesis demonstrate ways students might be taught to articulate their experience of meter. A unified approach, it incorporates Cohn's ski-hill graph and other instruments of mathematical music theory such as the SkiHill app, numbering for counting meter, cyclic graphs, and beat-class theory. Among the outcomes it anticipates is a deeper engagement with music in classroom settings. Through this new foundation, where meter studies fits more compatibly with studies of tonality, students should logically be able to perform, compose, and analyse music both aurally and visually with increased confidence and understanding.

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Dedicated to Gerard, Catherine and Teresa

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CHAPTER 1

INTRODUCTION AND BACKGROUND

Introduction

The aim of this thesis is to explore the "pedagogability" (Cohn) of meter among school-age students using an approach that draws substantially on recent research in music theory, music psychology and neuroscience about what meter is and where meter is located.[1] The approach I propose with regard to teaching musical meter is ultimately derived from the theoretical and pedagogical writings of Dr Richard Cohn, who is Battell Professor of the Theory of Music at Yale University.[2] Although other music theorists have represented meter using mathematics, narrative, and visual representations no other music theorist has consistently used all three and developed an analytical model focused on sound rather than notation (Cohn, 2018).

Cohn's approach to teaching meter incorporates graphic representations of meter through mathematical music theory, the equal treatment of both meter and tonality, and the understanding that meter is a subjective and temporal experience.[3] A fundamental quality of Cohn's work on modern meter theory from the 1990's to the present day has been to promote the idea of the pedagogability of music theory. To do this, Cohn assimilates and distills meter theory contemporaneous to his own work and draws on music theory and mathematics from ancient Greece and the Medieval era to contribute to modern understandings of meter for the purposes of music education (Cohn, 2016c).

Although Cohn's ski-hill graph (Cohn, 2001, 2018a) is not the only way to represent meter it is arguably the most compact and efficient instrument of music theory for school-age students

[1] This term of Cohn's appears in the title of his 1998 article, Music Theory's New Pedagogability, *Music Theory Online*, *4*(2). Cohn's article resonated with my conviction that we are at a turning point where music theory and pedagogy is concerned in that important scholarly work, such as Cohn's on meter, can now be adapted to the needs of much younger audiences. This can occur through the development of teaching materials such as those proposed in Chapter 4 of this thesis. This would replenish an undernourished field of music theory (meter theory) in primary and high schools. See also Cohn (2015d).

[2] In 2015 I attended a semester of meter theory classes taught by Professor Richard Cohn at the Sydney Conservatorium of Music, following which I began teaching meter through ski-hill graphs with my private music students from age seven to young adults.

[3] By "mathematical music theory" I mean music theory as understood through the instruments and principles of mathematics.

to articulate their temporal experience of meter (Hilton, Calilhanna and Milne, 2018 p. 222). Unlike other representations of meter, students map all of the metric pulses they hear, to distinct duple and triple metric pathways. Through using nodes and edges, students are enabled to articulate their observations of the mathematical properties integral to the meter they experience. These properties include relations between sets of pulses and sets of meters, the depth of a meter, and the metric space or form of a piece of music. The unique ability of the ski-hill graph to function in these ways positions it as a potent instrument of mathematical music theory pedagogy through which students can represent their experience of meter.

In his seminal article on ski-hill graphs, dedicated to David Lewin, Cohn (2001) introduces the ski-hill graph as a new graphic technique, a two-dimensional matrix to represent hemiolas (see Chapter 2 "Origins of the Ski-Hill Graph"). Through mapping pulses to the ski-hill graph, duple divisions are represented by edges that slope downwards to the left and triple divisions are represented by edges that slope downwards to the right. The slowest pulse is at the top and the fastest at the bottom (see Figure 1).

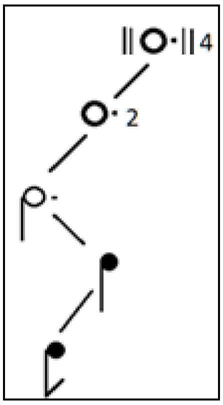

Figure 1 Ski-hill graph

The computerised version of the ski-hill graph, the SkiHill app, developed in 2016 by Dr Andrew Milne, MARCS Institute for Brain Behaviour and Development, Western Sydney University, provides instantaneous feedback in visualizations and sonifications (Hilton, Calilhanna and Milne, 2018 p. 228).[4] With the SkiHill app, students can choose traditional notation, fractions, polygons (cyclic graphs) and sounds to represent pulses and metric pathways (see also Chapter 2 of the thesis, "Recent Developments in the Use of the Ski-Hill

[4] See Footnote 5 for details about the availability of the SkiHill app (Milne, 2016).

Graph"). Figure 2 illustrates the computerized version of the ski-hill graph, the SkiHill app (Milne, 2016), using traditional notation, polygons, and sonifications to represent a simple hemiola (3:2).

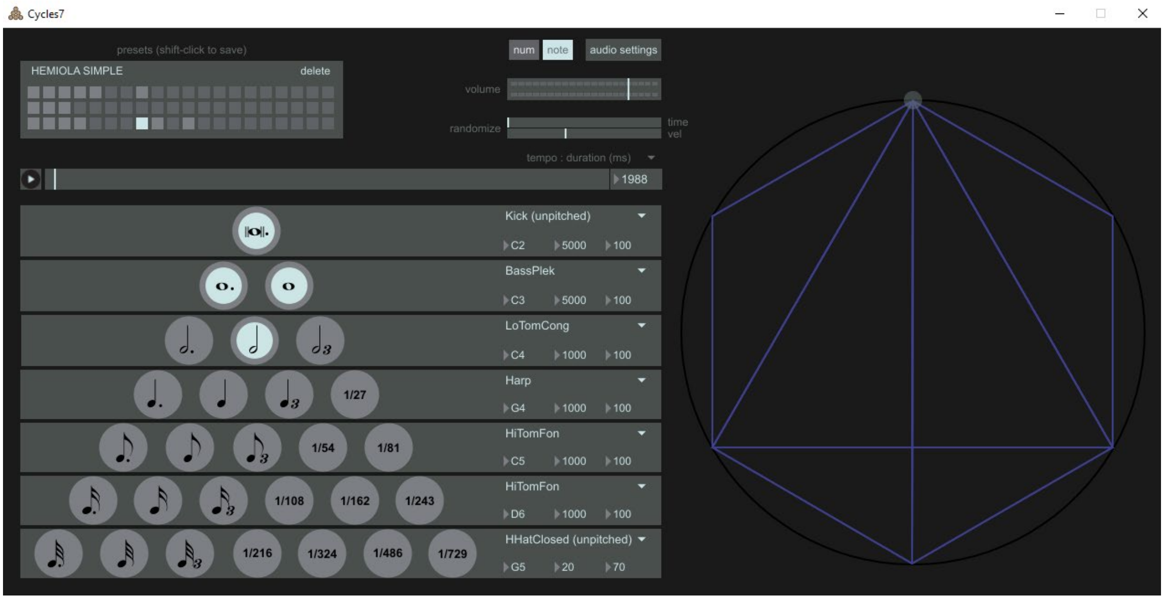

Figure 2 SkiHill app

In the thesis, I also demonstrate how Cohn's meter theory pedagogy draws on decades of research mainly from North American scholars, whose work I will also apply and adapt for school-age students. Among the outcomes of this approach to teaching meter is the likelihood that students will experience deeper engagement with their music classes – an outcome I have witnessed in my own teaching practice. With this new foundation of meter studies working compatibly with studies of tonality, students should logically be able to perform, compose, and analyse music, both aurally and visually, with additional confidence and understanding. This can occur through providing students with tools, language and experiences to help them articulate their understanding of meter so as to complement their tonality studies in music.

Methodology

This thesis synthesises and sequences concepts and principles related to understandings of musical meter as set out in scholarly publications, particularly those of Cohn. It aims to provide school-age students (K–12) and their teachers with access to Cohn's work and approaches to teaching meter theory. It begins by contextualizing the ski-hill graph through a survey and critique of classroom music-theory texts that are currently used to teach meter in tertiary institutions at undergraduate level, and those that are used in the training of pre-service primary and secondary music classroom teachers and studio music teachers. Also, I survey music textbooks accessed by classroom music teachers and students in secondary schools in Australia. In addition, to position Cohn's ski-hill graph in relation to recent research and thought on meter, I provide the reader with a summary of meter theory research that is contemporaneous to Cohn's work on meter.

Next I examine Cohn's writing about ski-hill graphs, as well as other scholars' writings on their use of ski-hill graphs to provide a contextual overview of the development of the ski-hill graph, its current use in scholarship, and future developments. I follow this with a study of a medieval precursor to Cohn's ski-hill graph – a triangular graph presented around 1330 by Johannes Torkesey (d.1340). I thus present these two triangular graphs in relationship so that, for the first time, they can be understood as representing pivotal moments in meter theory history.

After providing a context from which to view the ski-hill graph I ask the reader to explore the value and potential of the ski-hill graph by considering a new approach to teaching meter with school-age students. To do so, I present a conceptual framework from which to view the teaching of meter with new understandings. This provides the scaffolding for understanding and teaching the pedagogical materials I present. I also include analyses of music to demonstrate ways in which the ski-hill graph can serve as an indispensable instrument of music theory through which to teach school-age students meter.

Organisation of Chapters

The thesis is set out in four chapters. Chapter 1, "Introduction and Background," includes a "Review of Texts and Approaches for Teaching Meter" which is arranged into two sections, "Text Books for Tertiary Students" and "Text Books for Secondary School Use," and a further section "Recent Research and Thought on Teaching Meter."

Chapter 2 is organised into two parts. Part 1 provides an account of the ski-hill graph developed by Richard Cohn as an instrument to analyse and teach musical meter, including aspects of the ski-hill graph's recent and future development. Part 2 offers a review of elements of the history of graphic representations associated with meter, in particular a discussion of a medieval triangular graph presented by Johannes Torkesey (d. 1340) in his treatise *Trianguli et scuti declaratio de proportionibus musicae mensurabilis* (c. 1320-30) "Of the triangle and shield, an exposition of the proportions of mensural music."

In Chapter 3, "A Conceptual Framework for Teaching Musical Meter to School-age Students Through the Ski-Hill Graph," I outline the conceptual framework that I have developed to teach students prior to introducing the ski-hill graph and for teaching meter with new understandings. Chapter 4, "Teaching Meter to School-age Students," presents materials I have employed in my private music studio and with preservice music education students, that is, at the tertiary level. The majority of materials in Chapter 4 focus on teaching isochronous meter; however, I also provide examples of music and teaching strategies where metric dissonance occurs in music studied for exams by young learners such as through simple hemiolas and metric displacement (syncopation). In Chapter 4, I also demonstrate how the ski-hill graph can work as a unified approach to teach meter with the computerised version of the ski-hill graph (the SkiHill app) and other forms of metrical representation through visualizations and sonifications of mathematical music theory such as cyclic graphs, the XronoBeat app (Milne 2018), and beat-class theory (Cohn, 2018b).[5]

Review of Texts and Approaches for Teaching Meter

The review of literature examines a range of classroom music texts that are currently available to music educators and which include a section on teaching musical meter.[6] The literature included for survey comes from one of two categories: texts used in the training of pre-service

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[5] Through using beat-class theory, students learn to make observations about meter and rhythm by representing cyclic universes of different sizes, such as C6, C8 or C12, which represent the number of elements in a cycle, and through learning about the cardinality of each cycle, through mapping selected timepoint sets (see Cohn, 2018b and Chapters 3 and 4 of this thesis). Cohn (2016b, 2018a, 2018b) discusses his work with students at Yale University where he combines cyclic graphs with beat-class theory to teach meter and rhythm. The XronoBeat app (Milne, 2018) is the world's first computerized version of the cyclic graph which uses both visualisations and sonifications to represent meter as sets of timepoints using numbers for both small and large cyclic universes (see Chapter 4). The latest versions of the SkiHill and XronoBeat software applications will be available at the following URL's (below). At present, these are beta versions of the software, and have not yet been officially released. They are available for free. http://www.dynamictonality.com/dl/SkiHillApp_macOS.zip
http://www.dynamictonality.com/dl/SkiHillApp_win.zip
http://www.dynamictonality.com/dl/XronoBeat_macOS.zip
http://www.dynamictonality.com/dl/XronoBeat_win.zip

[6] My preference for spelling the word "meter," as used in the United States, is due to my understanding of meter theory as defined by Professor Richard Cohn, Yale University.

music teachers in tertiary institutions, which are also consulted by music teachers when teaching musical meter in both primary and secondary music classrooms and studios; and classroom music texts created for use in secondary schools, which are consulted by music teachers when teaching musical meter in both primary and secondary music classrooms and studios.[7]

The texts by Aldwell, Schachter and Gauldin were chosen by the author as they are considered to be fairly representative of the limited approach to meter in texts stemming from North America. These music theory texts aimed at tertiary students including those who will become music teachers in primary and secondary schools and music studios are also used in Australian universities. The text by Karpinski which is also aimed at tertiary students including those training to become music teachers of school-age students, demonstrates foresight in that it addresses some of the cognitive processes where meter is concerned. However, the music theory Karpinski provides to teach meter reveals some inconsistencies by referring to notational understandings of meter such as simple and compound meter. The texts by Kamien, Peterson, Dorricott, and Dorricott and Allan were also considered by the author as fairly representative of the limited approach to teaching meter in texts (largely notation-based understandings) sourced by classroom music teachers in secondary schools including those in Australian schools. Notably none of the authors addressed any problems encountered with the mathematics projected in the American system to teach meter, such as, the fractional terms.

The following review of literature begins with three well-known texts designed for tertiary music classrooms but which are sourced for materials in the secondary school music classroom and at times used as part of training pre-service music teachers: *Harmonic Practice in Tonal Music* (Second Edition, 2004) by Robert Gauldin; *Harmony and Voice Leading* (Fourth Edition, 2011) by Edward Aldwell and Carl Schachter with Allen Cadwallader; and *Manual for Ear Training and Sight Singing* (Second edition, 2017) by Gary Karpinski.[8] It should be noted that the primary purpose of the first two textbooks I review, by Gauldin (2004) and Aldwell and Schachter (2011), is to teach about harmony, not rhythm and meter.

[7] My goal here is not to criticize teachers rather I hope to have access to teaching materials where a theory of meter reflecting a modern understanding of meter in the fullest sense exists. For this reason, I am providing an objective review of available textbooks to see how pedagogical materials provide the study of meter for the purposes of understanding and teaching meter particularly for teaching school-age students.

[8] For a comparative survey of recent literature see Kari Jacinta McDonald (2001). *The role of harmony in expressed meter: a historical review and its placement in current pedagogy.*

Text Books for Tertiary Students

The author of *Harmonic Practice in Tonal Music* (Second Edition, 2004) Robert Gauldin describes his text as taking,

> a linear, functional approach to tonal music in the common practice era, not only showing students how individual chords function within the larger harmonic tendency, but also the interaction between melody and harmony. (p. xxi)

Gauldin's approach seeks to demonstrate how counterpoint is derived from harmonic function and through analysing voice-leading "more-insightful performances of the music" may form (p. xxi).

Chapter 2 "Rhythm and Meter: Beat, Meter, and Rhythmic Notation" begins by identifying rhythm as the collective term for the temporal elements of music and the "significant role" aural perception and memory play in how we hear or perform music (p. 20). For instance, to explore rhythm Gauldin focuses first on "how we hear different levels of metrical organisation, such as 'the beat,' its smaller divisions, and its larger groupings" (p. 20). Gauldin discusses the proportional divisions and multiples of the beat as represented in notation, to arm the reader with enough information to "interpret various meter signatures" (p. 20).

In the section "Metrical Grouping and Meter," Gauldin states,

> You will observe that your mind tends to group the beats in the music into larger units of equal duration, each of which begins with a stressed pulse. Each initial stronger downbeat is followed by several weaker beats, creating a series of regular groupings or units that contain the same number of beats. This pattern of stressed and unstressed beats results in a sense of metrical grouping or meter. (p. 22)

In "Division and Subdivision of the Beat," Gauldin describes division of the beat as simple or compound, duple, triple, or quadruple and provides guidelines for notating rhythms (p. 23).

Gauldin introduces an explanation of hypermeter in Chapter 12 as part of a section on "phrase periodicity" (p. 189). Chapter 18, "Rhythm and Meter II: Additional Meter Signatures and Rhythmic-Metrical Dissonance," begins with further instructions to interpret time signatures and notation for performances with very slow or very fast tempos (p. 307). For instance, where music is very rapid Gauldin explains each measure can be perceived "as a single beat" and grouped into "larger metrical groups called hypermeasures" (p. 309). In a following section, Gauldin introduces materials describing “Asymmetrical Meters” and some “Fast Complex Meters.”

In the section “Rhythmic-Metrical Consonance,” Gauldin states there are three levels within the metric hierarchy – “beat division, beat, and meter” – and provided they remain constant and regular those three levels form the basis of “rhythmical or metrical consonance” (p. 311). The author notes composers may introduce “deviant” elements into their music that disrupt the prevailing rhythmic and metrical regularity or consonance and Gauldin provides examples of “Rhythmic Dissonance” by describing “Substituted Beat Division,” “Superimposed Beat Division,” “Syncopation,” and “Displaced Accents” (pp. 311-15).[9] A further section, “Metrical Dissonance”, describes the use of “Hemiolas” as “simple-compound substitution,” which is followed by the sections “Substituted Meter,” “Polyrhythm,” “Metric Shift,” and “Changes of Meter” (pp. 315-321). Appendix 1 includes a brief section on “Duration and Length,” Appendix 3 includes the use of rhythm in “An Introduction to Species Counterpoint, and Appendix 5 discusses “Conducting Patterns.”

Harmony and Voice Leading (Fourth Edition, 2011) by Edward Aldwell and Carl Schachter with Allen Cadwallader presents a course of study in harmony and counterpoint in the music of the eighteenth and nineteenth centuries (pp. xi-xiii). The text contains an early unit entitled “Rhythm and Meter” covering beat, tempo and accent, meter and metrical accent, time signatures, rhythmic accent versus metrical accent, syncopation, hemiola, rhythmic groups, measure groups and phrases, hypermeter, dissonance, duration and accent, suspensions, and anticipations (pp. 38-45).

Beat, is described as “a span of time that recurs regularly; a succession of beats divides the flow of time into equal segments.” In a footnote the authors note that the segments are “approximately, rather than strictly equal” in performance (p. 35).

In the sections “Meter and Metrical Accent” and “Time Signatures” meter is defined as “A repetitive pattern that combines accented and unaccented beats” and a brief description of duple, triple and quadruple meter (“derived from duple meter”) is given. The authors state that composers indicate meter by means of time signatures placed at the beginning of a piece after the key signature and at any subsequent point where the meter changes. Also, “The inner

[9] For a further discussion of dissonance in Gauldin’s work see Kari Jacinta McDonald (2001) 17.

organisation of a divided beat mirrors in miniature the metrical organization of a measure" (p. 38).

The terms simple and compound are not mentioned other than to refer to, for example, the "so-called compound meters" as having accentual patterning on more than one level where "beats are grouped in multiples of three" (p. 39). Discussion of rhythm is included in Unit 5, "Introduction to Counterpoint," for second to fifth species counterpoint (pp. 69-91).

Manual for Ear Training and Sight Singing by Gary Karpinski (2017) is a classroom music textbook used in many universities to train undergraduates (including pre-service teachers) in aural skills. Karpinski states,

> *Manual for Ear Training and Sight Singing* is the first textbook that treats aural skills with the same breadth, depth, and attention to detail found in most harmony textbooks…The structure and content of this book have been shaped in large part by recent research in music cognition and perception. Knowledge about pulse perception informs discussions of meter…Discoveries about the nature of meter and hypermeter are incorporated into every discussion of temporal organization in music. These discussions treat meter primarily as an experiential phenomenon in which metric principles are derived from what listeners and performers hear and feel in music." (p. xiii)

Approaching the study of meter through practical and written materials Karpinski organises his chapters through a developmental approach using learning sequences. "Protonotation" is utilised to first develop students' aural skills without the constraints of learning music notation (pp. xviii – xix). Karpinski explains "protonotation" as,

> vertical lines to represent pulses and meter, horizontal lines to represent rhythms, and scale degree numbers and syllables to represent pitches. The purpose of this symbiology is to represent these features in a generic way without regard to how they might be notated in any specific meter sign, clef or key…to grasp the musical essence…and apply what they've learned in various specific contexts…Protonotation also allows students to represent what they hear, remember, and understand in dictation before they learn to notate it in any particular meter, clef, and key. (p. xix)

To establish a foundation of aural skills for students Karpinski restricts the first eleven chapters of the curriculum in terms of pitch, meter, and rhythm and in Chapter 12 introduces dotted notes and examples of "real notations" using three different meter signatures 3/8, 3/4, and 3/2, described by Karpinski as "various meters" (p. 55).

In Chapter 1, "The Fundamentals of Meter and Pitch," Karpinski describes "pulse" as the "regularly recurring feeling of stress in music." He states, "the pulse can occur at different levels in one piece of music" and "each pulse occurs at a point in time – it has no duration or length" (p. 1). He states, the primary pulse is "a pulse that regularly contains greater stress" and the secondary pulse is "a pulse that regularly contains less stress" (p. 1). The author defines "beat" as the [constant] duration between successive pulses" which can be used as "units of measurement" and he explains "A measure is the duration between successive primary pulses" (p. 2). Karpinski defines meter as "Meter is the organization of pulses into primary and secondary levels" (p. 1). He states, "In duple meter, there are two beats per measure. In triple meter, the pulses are grouped in three."

Chapter 5, "More About Meter and Rhythm," includes a discussion of the perception of music to introduce "the existence of more than two levels of pulse" where the "primary" and "secondary" pulses shift to a "broader level of pulse" (p. 20). The examples provided introduce simple meter – "In simple meters the beat is regularly divided in halves" (p. 20). The author states,

> Although musicians often feel several levels of pulse, most notated meter represents only two or three levels of pulse. It is important for you to be able to recognise and perform with an understanding of pulses beyond those made explicit in notation. (p. 21)

"Quadruple meter" is described as having "at least *three* different types of pulses." Furthermore,

> Quadruple meter is fundamentally more complex than duple or triple meter in that it explicitly recognizes three adjacent levels of pulse, whereas duple and triple recognise only two. (p. 21)

Karpinski discusses simple meter signs in Chapter 7, "Notating Rhythm and Meter," by explaining,

> In simple meters, the top number in the meter sign indicates the number of beats per measure. The bottom number indicates the note value (whole note, half notes etc.) equal to one beat (the beat value or beat unit). (p. 28)

Also,

> It's important to note that the top number in a meter sign is determined by the sound of a piece of music – the number of beats per measure is an audible result of the relationships between the primary and secondary pulses (although as we've already seen, there is often more than one level on which to view these relationships). The bottom number is not a product of sound – it is chosen by the composer, transcriber, arranger, or editor." (pp. 28-29)

In Chapter 16, "Compound Meters," the author defines compound meter as one in which "the beat is regularly divided into threes. In compound meters, the beat unit is equal to three of the note values represented by the bottom number of the meter sign" (p. 74). "Conducting Pulse Levels Other Than the Notated Beat" is a discussion of conducting patterns in relation to grouping in simple and compound meters (pp. 111-116). Syncopation is introduced in Chapter 32 and defined as follows: "Syncopation provides another means of altering regular metric divisions by shortening notes and skipping articulations on subsequent beats" (p. 154).

Karpinski discusses "harmonic rhythm" in Chapter 35 and "the collective rhythm of chord changes – the durations governed by successive chords" (p. 165). He uses "pulse-graphs" to represent the chord changes in relation to pulse, meter, and their relation to harmonic motion such as the tonic-dominant relationship, and cadences (pp. 165-169). Practical examples are provided in Chapter 36 to listen for rhythm in "Two-part music." "Diminution" and both "Suspensions and Anticipations" are briefly discussed in terms of rhythmic displacement in Chapter 46, "Voice-leading Techniques" (pp. 210-211).

"Advanced Triplets" are discussed in Chapter 52 in relation to simple meters where Karpinski notes, "It is also possible to divide a *two-beat* span in simple meters into three equal parts" (p. 255). In Chapter 66, "Hemiola," Karpinski discusses the "temporary shift between duple and triple groupings within a passage, a device known as a *hemiola."* He explains the implications for listening to hemiolas "At the very essence of hemiola is metric ambiguity" (p. 328). In Chapter 75, "Advanced Meters," Karpinski provides practical materials on "Quintuple and Septuple Meters," "Asymmetrical Compound Meters," and "Changing Meters." Karpinski states,

> In addition to duple, triple, and quadruple meters, you will also encounter meters with five or seven beats per measure and asymmetrical meters in which the beats in each measure are not all the same duration. (p. 384)

For "Asymmetrical Compound Meters" Karpinski explains,

> Compound meters with a multiple of three on the top of the meter sign (such as 6/8) are called **symmetrical** because their beats are consistently divisible into threes…[10] However, when a meter arises from groupings of *both* twos and threes, the parts are not equivalent, and the meter is **asymmetrical.** (p. 385)[11]

Chapter 76 investigates "divisions of the beat into smaller equal values as well as some irregular rhythms that span two or more beats" and "Other Irregular Rhythms" involving the division of more than one beat into various equal parts, such as quintuplets and other rhythms, are given as examples (pp. 391-393).

Hypermeter is introduced in Chapter 78 through protonotation, traditional notation, and practical exercises including conducting and singing. Karpinski states, "Many levels are not made obvious by the meter sign, but involve both smaller divisions of the beat and groupings broader than the measure" (p. 405). Karpinski notes,

> We have focused on groupings of either twos or threes because duple and triple are the two fundamental types of metric organization in Western music. Any other group is produced by some combination of twos and/or threes. For example, simple quadruple meter results from grouping at three different levels – beats group by twos to form half-measures, which group by twos to form measures. (p. 405)

He states, "Meter and hypermeter, no matter how complicated, can always be conceived of as some product of twos and/or threes." "Elision" is discussed in relation to hypermeter and primary and secondary pulse levels and patterns (p. 405).

In Chapter 79, "Form," the author instructs the reader on "analysing form entirely by ear" by mapping a visual representation of form on a grid. The grid "need not show the multiple levels that hypermetric grids do but they should show at least the downbeat of every measure you

[10] Karpinski notes that his use of the word *symmetrical* in this context refers to "equivalence among constituent parts" rather than "balance around a central point."

[11] The bold formatting in this passage is from the original text.

hear" (p. 410). Hearing the piece as being notated in different meter signatures is taken into account and sections are labelled according to the form (pp. 410-12).

Text Books for Secondary School Use

The following is a sampling of secondary music classroom texts sourced by classroom music teachers in Australian schools: *Music: An Appreciation* by Roger Kamien (2018); and *An Introduction to the Concepts of Music* by Nick Peterson (2011). *Listen to the Music Student Book* by Ian Dorricott, Sixth Edition (2015); *In Tune with Music. Books 1* and 2 Fourth Edition by Ian Dorricott and Bernice Allan (2011 and 2013).

In Chapter 3, "Rhythm," of Roger Kamien's *Music: An Appreciation* (2018)*,* meter is introduced as one of several interrelated aspects of rhythm including "beat, meter, accent and syncopation, and tempo" (p. 32). Kamien's section on meter provides a definition of meter as "The organisation of beats into regular groups" (p. 33). Measure or bar is defined as "a group containing a fixed number of beats" and he outlines the different types of meter based on the number of beats in a measure (he names and explains duple, triple, quadruple and quintuple, sextuple and septuple meter.

Kamien does not refer to simple, compound and complex meters to describe or categorise types of meter. In the section "Notating Meter," Kamien explains,

> A time signature (or meter signature) shows the meter of a piece. It appears at the beginning of the staff at the start of a piece (and again later if the meter changes). (p. 38)

In "Musical Styles since 1945" (p. 445), Kamien states,

> After 1945, some composers abandoned the concepts of beat and meter altogether. This is a natural outcome of electronic music, which needs no beat to keep performers together. In nonelectronic music, too, the composer may specify duration in absolute units such as seconds rather than in beats, which are relative units. In some recent music there may be several different speeds at the same time.

Kamien includes a paragraph on "Rhythm" in each of "Baroque Music (1600-1750) p. 125, "The Classical Style 1750-1820)" p. 190, and "Rock" p. 532, a paragraph on "Rhythm, Melody, and Harmony" in "Jazz Styles (1900-1950)" p. 486, and a paragraph on "Melody, Texture, and Rhythm" in "Nonwestern Music" p. 545.

An Introduction to the Concepts of Music by Nick Peterson (2011) is a secondary classroom music text generally aimed at the upper secondary school years commonly used throughout Australia. In the section "Meter: Prerequisite Information," Peterson explains,

> Once you have established that the music is based on a beat foundation, a comment can be made describing the metre of the beat. A comprehensive understanding of metre requires some prerequisite information. (p. 27)

Peterson follows this by providing a description of notation, beginning with, "In some metres, the beat is divided into 2 pulses; and in other metres, it is divided into 3 pulses" (p. 27). In "Metre" Peterson defines metre as: "the pattern of the beats' accents" and further explains that in music notation accents are indicated by the bar lines. "The bar lines indicate the end of each pattern" (pp. 28-29). Metre signatures "identify the metre of the music" and metres are categorised in the six categories of simple or compound, and either duple, triple or quadruple (p. 29-33).

Listen to the Music (Sixth Edition) by Ian Dorricott (2015) and *In Tune with Music. Books 1* and *2* (Fourth Edition) by Ian Dorricott and Bernice Allan (2011 and 2013) are designed for junior secondary school classroom music students. The text *Listen to the Music* defines "metre" as "the number of beats to the bar" (p. 258) and,

> The patterning of beats into repeated groups is called metre. The unit of time occupied by one group of beats (that is, from one accented beat to the next) is called a bar. Meter provides a framework of bars within which the sounds and silences of music occur. (p. 14)

Both texts categorise meter through the classifications simple or compound, and one of duple, triple or quadruple, complex metres, and mixed meters.

Recent Research and Thought on Teaching Meter

The following section provides a brief survey of recent research and thought on teaching meter including an introduction to Cohn's work. The literature presented below provides examples of work from research over the past 40 years or so by music theorists (mostly music educators) and those in the sciences who have studied meter (and rhythm). The handful of scholars mentioned below are among those who have contributed in some way towards a modern understanding of meter for the purposes of music education.

In *Theory of Suspensions: Study of Metrical and Pitch Relations in Tonal Music* (1971) Arthur J. Komar includes both metrical and pitch relations in his study of suspensions but without linking the analysis of rhythmic aspects to other sorts of musical groupings such as poetic meter.[12] He notes relations of "large-scale metrical structure" are not included in conventional notation and acknowledges the lack of a theory of meter to fall back on (p. v.). Komar states, "In the absence of a theory of meter, the general tendency is simply to accept the notated metrical relations in a piece as given; but of course, metrical notations are merely symptomatic, rather than determinant, of metrical structure" (p. v). Komar addresses what he identifies as "the central problem in tonal rhythm – providing a theory for deriving rhythmic values for *all* structural levels of a piece" (p. 5). Komar (p. 52) introduces the term "time-span" in defining "beat" as "the initial time-point of an equal subdivision of the background structural time-span."

Andrew Imbrie posits a new view on the experience of meter in his article "'Extra' Measures and Metrical Ambiguity in Beethoven" (1973) offering the argument that "two contradictory metrical interpretations of the same event can be simultaneously entertained in the mind of the performer and listener" (p. 51). Imbrie introduces the term "conservative listener" for those listeners who retain the same meter even if they hear a metric change and "radical listener" for those who swap or adjust their experienced meter to a new one when they hear the meter change.[13] Imbrie (1973, p. 53) also presents a spatial analogy where beats are described as durationless points in time.

In *The Stratification of Musical Rhythm* (1976) Maury Yeston presents a modern theory of "rhythm" which he hopes will,

> coexist with past theories that have seen rhythmic configurations as chains of strong- and weak-beat patterns, but…seeks to understand these patterns as they are formed by

[12] Komar (p. 5) notes that others had written prior about "large-scale rhythmic relationships…from a non-Schenkerian orientation" such as Alfred Lorenz (1924-33), Grosvenor W. Cooper and Leonard B. Meyer (1960), and Edward T. Cone (1968), who also wrote extensively about hypermeter. Berry (2016) shows evidence of Bricken's awareness of hypermeter but from a Schenkerian linear/motivic approach rather than a modern hierarchical approach.

[13] For further reading see also Fitch & Rosenfeld (2007).

> the interaction of two kinds of level of motion: (1) the music taken as unaccented, uninterrupted flow, and (2) middleground strata that are to be found beneath the level of motion of the musical surface. At these levels, the logical and structural bases by which different rhythmic shapes in a composition relate to one another can be discovered. (p. 34)

Yeston demonstrates the interaction between "attack" points and pitch to demonstrate how they form two broad structural categories: rhythmic "consonance" (where pulses are in a relation of inclusion) and rhythmic "dissonance" (such as hemiolas).

Carl Schachter's Schenkerian-inspired articles on rhythm and meter such as those in *The Music Forum* (1976, 1980, 1987) examine "the patterned movement…of musical rhythm." Schachter observes both tonal rhythm and durational rhythm, meter, accent, grouping and proportion which he illustrates through voice-leading reductions.

In 1981, David Lewin provided what might be perceived as an intuitive precursor to a theory of music which would later become known as Transformational theory in his article "On Harmony and Meter" in Brahms's op. 76, no. 8. in *Nineteenth-Century Music.* In the article Lewin treats meter and harmony equally for Brahms's piece in the sense that the different meters experienced by the listener can also be regarded as analogous to harmony, forming similar relations as tonic, dominant and subdominant depending on their function. As Cohn (2001b) explains, "Transformational theory stresses the dynamic nature of events and gestures as they engage time and articulate form." In *Generalized Musical Intervals and Transformations* (1987) (GMIT) Lewin introduced transformational theory where he applies mathematical group theory to music. "Based on a powerful metaphor of musical space, this theory can be applied to pitch, rhythm and metre, or even timbre. Moreover, it can be applied to both tonal and atonal repertories" (Rings 2011, 2).[14]

Jonathon D. Kramer (1981) wrote "New Temporalities in Music" which later contributed to *The Time of Music: New Meanings, New Temporalities, New Listening Strategies* (1988). In his abstract Kramer (1981) notes the contribution of "non-Western" music to the understanding of time:

> As this century has found new temporalities to replace linearity, discontinuities have become commonplace. Discontinuity, if carried to a pervasive extreme, destroys

[14] See also the discussion of Cohn' article from 2001.

> linearity… One factor contributing to the increase of discontinuity was the gradual absorption of music from totally different cultures, which had evolved over the centuries with virtually no contact with Western ideas…Cross-cultural exchange in music will, of course, never destroy aesthetic boundaries, but music of non-Western cultures continues to show Western composers new ways to use and experience time.

Fred Lerdahl and Ray Jackendoff in *A Generative Theory of Tonal Music* (1983) (GTTM) explain how music theory as psychology could offer a "formal description of the musical intuitions of a listener who is experienced in a musical idiom" (p 1). Restricting themselves to "the components of musical intuition that are hierarchical in nature" the authors proposed four such components:

> *[G]rouping structure* expresses a hierarchical segmentation of the piece into motives, phrases, and sections. *Metrical structure* expresses the intuition that the events of the piece are related to a regular alternation of strong and weak beats at a number of hierarchical levels. *Time-span reduction* assigns to the pitches of the piece a hierarchy of "structural importance" with respect to their position in grouping and metrical structure. *Prolongational reduction* assigns to the pitches a hierarchy that expresses harmonic and melodic tension and relation, continuity and progression. (pp. 8-9)[15]

GTTM presents a set of Metric Well-Formedness Rules (MWFR) and Metric Preference Rules (MPR) which are guidelines for making observations about the establishment of a metrical structure by the listener.

> These are not meant as prescriptions telling the reader how one should hear pieces of music or how music may be organized according to some abstract mathematical schema. Rather, it is evident that a listener perceives music as more than a mere sequence of notes with different pitches and durations; one hears music in organized patterns. (p. x)

Through applying the "rules" of grouping analysis, Lerdahl and Jackendoff study the hierarchical relations of meter and tonality through graphic representations from reductions using music tree structures and dot notation. Through analysing hierarchical elements of music rather than prominently motivic material, which is not hierarchical, as demonstrated in

[15] See also Lerdahl and Jackendoff (1983, 17). The italics are from the original text.

Schenker's approach to reductions, GTTM demonstrates important differences between syntaxes in language and music.

Music theory research increasingly sought to provide approaches to deal with all music rather than merely Western forms, for instance, ethnomusicologist Jay Rahn (1983) developed a theory to analyse both world music and Western art music. In response to the growing interest in these fields, in 1985, the Society for Music Theory published a special edition of *Music Theory Spectrum* on the topic "Time and Rhythm in Music."[16]

In the 1990's Richard Cohn advanced understandings of meter theory through developing mathematical music theory to represent meter with instruments and principles of mathematics such as set theory, demonstrated in his article, "The Dramatization of Hypermetric Conflicts in the Scherzo of Beethoven's Ninth Symphony" (1992b).[17] Concerned with the pedagogability of music theory (Cohn, 1998), Cohn (2001) introduced a new way of representing meter with "a two-dimensional matrix" which he referred to as a ski-hill graph in his article "Complex Hemiolas, Ski-hill Graphs and Metric Spaces." In his article from 2001, Cohn also applied the algebraic equation he introduced in 1991, as a method for calculating the number of duple and/or triple metric pathways connecting a span pulse (S: the longest unit pulse heard) to a unit pulse (L: the shortest pulse heard) in a piece of music.[18]

Jeanne Bamberger's (2000, 2003, 2013) pioneering research in music cognition and child development began in the 1970s later drawing on contemporary research such as that by Lerdahl and Jackendoff (1983) and many others to contribute new strategies for pedagogical approaches in music education with children.[19] Bamberger worked to find new understandings about learning and teaching which addressed deficits in traditional approaches to music education. Bamberger's work focused more generally on "thought and action" and prominent in her research were observations of the cognitive issues involved in children's

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[16] *Music Theory Spectrum* "Time and Rhythm in Music" Volume 7, Issue 1, 1 March, 1985, 1-215.

[17] Although Willi Apel (1948) had previously used numbers to represent meter Cohn's (1990) article advanced the theory through his discussion of hypermeter. See also Cohn (1992e) Transpositional combination of beat-class sets in Steve Reich's Phase-Shifting Music. *Perspectives of New Music*, 30 (2), 146 – 177.

[18] See Footnote 44 on Cohn's algebraic equation. Also see Chapter 2 for a more detailed description about the articles Cohn wrote which included his ski-hill graph and algebraic equation to represent meter and Chapter 4 for a detailed explanation of the practical application of the ski-hill graph and modern meter theory.

[19] Bruner (1956), Dewey, J. & Bentley, A.F. (1960) Schon (1984), Papavlasopoulou et al (2014), Seymour Papert (1980,1993) (a student of Piaget's), Wilensky (1999), Andrea diSessa et al (1991, 1998), Hasty (1997, 2000). See also Greher, G. R., & Ruthmann, S. A. (2012). On chunking, simples and paradoxes: Why Jeanne Bamberger's research matters. Visions of Research in Music Education, 20.

rhythmic understandings. This research involved the practice of children making their own notations from their own listening experience of rhythm and pitch. Howard Gardner notes,

> The observations that she has made and the distinctions which she has introduced (e.g. figural vs. formal, multiple representations study, the "mid-life crisis" in prodigies) is so widely known among music educators and cognitive psychologists that often they are no longer credited to Jeanne – they are simply assumed to be the basic knowledge of the field.[20]

With the advent of personal computers, Bamberger was quick to intuit their potential in music education and applied a cognitivist approach to software and design including through the software *Music Logo* and *Impromptu* (Bamberger, 2016). Students were enabled to learn about pitch and rhythm through manipulating the computer as a "mediator" between knowledge and information. In this way students would learn new knowledge rather than become "passive consumers of information" (Bamberger, 2018).

Bamberger (2016) notes at the same time music theorists and mathematicians were developing theories about the "functional connections between music and mathematics" her students were "reflecting on their simple composition materials which later developed into *Developing Musical Intuitions* and its computer music environment, *Impromptu*."[21] Thus, Bamberger's materials were intended to provide opportunities for students to be equipped with the skills they needed to understand the nature of music and then to be able to make their own music.[22]

Bamberger (2000) organised her textbook for middle school students, *Developing Musical Intuitions A Project-based Introduction to Making and Understanding Music,* and software, *Impromptu,* with projects for each section with an aim to foster creativity and curiosity in both student and teacher. Bamberger's text and software tools such as Tuneblocks, Drummer blocks, and Chord blocks were designed to help students comprehend musical structure in

[20] See also Gardner, H. (2012). Tribute to Jeanne Bamberger: Pre-eminent student of musical development and cognition in our time. Visions of Research in Music Education, 20.
[21] Bamberger was likely referring to scholars such as David Lewin, Clough, Douthett, and Cohn. For further reading see Douthett, J., Hyde, M. M., & Smith, C. J. (2008). *Music theory and mathematics: Chords, collections, and transformations*.
[22] Bamberger more recently added an iPad app Tuneblocks to her materials.

both tonality and meter. In "Building Meter," for example, the materials begin by asking students to participate in creative practical tasks with topics involving mathematics and graphic representations such as "Listen to and build metric hierarchies" and "Differentiate between and build duple and triple metric hierarchies."

Bamberger observed that where students learned both music and mathematics in a student-centred and interdisciplinary learning environment there resulted a relation of "mutually informing affinity" (Bamberger, 2003). In the Prelude to "Music as embodied mathematics: A Study of a mutually informing affinity" (2003), "Students' inquiry into the bases for their perceptions of musical coherence provides a path into the mathematics of ratio, proportion, fractions, and common multiples."

Also, Bamberger describes how students working with the graphical representations in *Impromptu* "make practical use of structures shared by music and mathematics" (p. 191).

> The first aspect is internal to the structure of music, particularly how music organizes time. The second aspect is the way these musical structures are represented in *Impromptu*. (p. 191)

Also,

> Overall, the children, working in an environment using multiple media and multiple representations (numbers, spatial representations, and sound-in-time) were learning about the reciprocal relationship between 'how much' (duration) and 'how many' (frequency); they were learning the connection between equivalent fractions and proportions embodied by pairs of iteratively sounding events that are different in absolute 'speed,' but the same in their internal relations. And they were also finding new meaning for 'That least common multiple stuff.' Finally, they were able to generate coherent structures using the principles of ratio and proportion as learned in school math' now expressed and experienced in novel situations. (p. 204)

Bamberger's quotation of Leibnitz's "Music is the arithmetic of the soul, which counts without being aware of it" becomes not only a reminder for readers about the connections between mathematics, music, and experience of music but perhaps also the quintessential theme of her extensive body of work.

Christopher Hasty's *Meter as Rhythm* (1997) re-evaluates the separation of meter and rhythm to provide a temporal theory of meter. He introduces the concept of "projection," which involves "the potential for a present event's duration to be reproduced for a successor" (p. 84).

New methods and approaches for teaching and learning meter and rhythm emerged during the twentieth and twenty-first centuries which included the addition of movement and practical musical activities through, for instance, Dalcroze, Kodaly, and Orff-Schulwerk. Audiation developed through Edwin Gordon's (1989, 2001) music learning theory where, for example, music pedagogy is taught as based in sound rather than notation, the location of the macrobeat is subjective, and quadruple meter is subsumed into duple meter (see also Dalby 2005, 2015).

Literature in the fields of meter and rhythm in music other than Western art music continues to grow: African Music (Agawu 1987, 1995, 2003), (Arom 2004), (Temperley 2000, 2001), (Tenzer M., & Roeder, J. Eds.); North Indian music (Clayton 2000); EDM (Butler 2001 2003); Stevie Wonder (Hughes 2003); metric ambiguity "Pyramid Song" by Radiohead (Hesselink 2013); metric dissonance in rock music (Biamonte 2014); Funky Rhythms (Cohn 2016b); Radiohead (Osborne 2017).

The fields of perception and cognition of meter and related studies are burgeoning (Pressing 1983), (Dirk-Jan & Essens 1985), (Palmer & Krumhansl 1990), (Parncutt 1994), (Keller 1997), (Repp 1998), (Krumhansl 2000), (Brochard et al 2003), (Huron 2006), (Fitch 2007), (Volk 2008), (Repp & Su 2013), (Patel and Iverson 2014); as are those in neuroscience on meter, rhythm, and pulse (Large & Snyder 2009); (Tierney & Kraus 2013), (Nozaradan 2018 et al). Numerous research papers are published every year in journals and handbooks with hundreds alone in *The Oxford Handbook of Music Psychology* (2016).

Justin London describes *Hearing in Time* (2004, 2nd edition 2012) as "a cross-cultural exploration of the perception and cognition of musical meter."[23] It incorporates psychological studies on metric perception, taking the view "that meter is a form of entrainment behaviour" that is "subject to a number of fundamental perceptual and cognitive constraints." London's point of departure with early modern theories on "division and hierarchy" is demonstrated in

[23] Justin London. https://people.carleton.edu/~jlondon/ Retrieved 29/8/18.

his theory of "well-formedness constraints" and diagrams where he considers non-isochronous meters are also meter in the strictest sense.

A growing number of scholars are researching the history of meter and related topics, notably Roger Matthew Grant (2014), and Carmel Raz (2018) "An Eighteenth-Century Theory of Musical Cognition*? John Holden's* Essay towards a Rational System of Music (1770)"; others present meter theory history to demonstrate their own theories and to examine historical practices, such as Harald Krebs, who expanded the treatment of metric dissonance through analysis of Schumann's music in *Fantasy Pieces* (1999), (Mirka 2009), (Karen Cook 2012), (DeFord 2015), and (Yust 2018). Recently, on the blog of the History of Music Theory SMT Interest Group & AMS Study Group, David E. Cohen (2018) wrote a post titled "Rhythm, Number, and Heraclitus's River," where he discussed a short passage recorded by "an anonymous student of Aristotle or his school in the section on music in the pseudo-Aristotelian work known as the *Problems*."[24] In this post rhythm is discussed:

> We enjoy rhythm because it possesses number both familiar and ordered, and moves us in an orderly way. For ordered movement is by nature more akin [to us] than disordered, as indeed [it is itself] more natural. [1]

Cohen says,

> All this implies that, for the writer and readers of our quoted passage, rhythm might have had a quality akin to that which we now call meter, characterized by periodicity, regularity, symmetry, the repetitive and predictable recurrence of identically "spaced" modules of identical "length." [4]

Cohen's post sheds light on one of the current problems in scholarship regarding meter – the use of the two words "meter" and "rhythm" to mean the same thing. Arguably this practice has played into the general fuzziness about understandings of meter and rhythm generally, in textbooks and pedagogy.

Evidence of the recent surge in interdisciplinary research on meter is demonstrated by the increase in meter and related research events such as Meter Symposium 1, 2 and 3 at the Sydney Conservatorium of Music between 2016-2018, Society for Music Theory "Rhythm and Meter Pedagogy" (2017), Symposium 3: "Microtiming and Musical Motion" at The

[24] See David E. Cohen *Rhythm, Number, and Heraclitus's River* (August 21, 2018) Retrieved August 21, 2018 from https://historyofmusictheory.wordpress.com/blog/.

University of British Columbia (2018), and the 15th International Conference on Music Perception and Cognition/10th triennial conference of the European Society for the Cognitive Sciences of Music, as well as through interdisciplinary societies such as the Australian Music Psychology Society (AMPS) and Society for Education and Music Psychology Research (SEMPRE).

Whereas music textbooks traditionally provided pedagogical information about music theory for students and teachers, social media is now providing "public music education" through, for example, YouTube, Facebook, Twitter, and blogs where current research about meter and rhythm is presented, such as that by Social Media Music Theory, 12tone, Adam Neely, David Kulma, John Harvey, and David Bruce Composer.[25]

Mathematical music theory and meter-related research in pedagogy continues to increase through the work of scholars (Cohn 2018b), (Douthett, Clampitt and Carey 2018), (Chew 2008), (Milne 2018), (Montiel 2018), (Gómez 2018), (Wilhelmi 2018), (Hall 2018), (Amiot 2018), (Johnson 2018), (Kochavi 2018), (Clampitt 2008), (Mannone 2018), (Hilton 2018), (Calilhanna 2018);[26] and through international conferences such as the Society for Mathematics and Computation in Music (SMCM). Teaching mathematical music theory with school-age students has also been piloted in secondary schools in Sydney, Australia through Richard Cohn's music theory and Andrew Milne's software applications (MARCS Institute for Brain Behaviour and Development, Western Sydney University) in the project *Teaching Mathematics with Music and Music with Mathematics* (Sydney 2017- ongoing).

In his article, "Why We Don't Teach Meter, and Why We Should" (2015d), Cohn focuses on the neglect of musical meter pedagogy in music curricula in comparison to the study of tonality. He reminds the reader of the two regulative systems through which we mentally filter sound when we listen to music: "tonality and meter" (p. 1). As Cohn points out the time spent on the study of meter is inadequate compared to the study of tonality in classroom music textbooks. Additionally, Cohn discusses how the traditional theories of meter in music

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[25] For John Harvey see "A different way to visualize rhythm." https://www.youtube.com/watch?v=2UphAzryVpY&t=216s See also William O'Hara (2018). Music Theory and the Epistemology of the Internet; or, Analyzing Music Under the New Thinkpiece Regime. *Analitica-Rivista online di studi musicali*, *10*.

[26] Montiel, M. & Gómez, F. (eds). (2018). *Theoretical and Practical Pedagogy of Mathematical Music Theory. Music for Mathematics and Mathematics for Musicians, From School to Postgraduate Levels.* World Scientific Press.

curriculum not only pertain to the focus on the classical music of European tonality but are wanting in terms of their contribution to a student's interpretation of a score in, for example, the preparation of a score for performance. Cohn (2015d) makes the salient point that the plethora of recent research on what musical meter is and where it is located has not yet found its way into music textbooks.

He reviews a number of prominent music textbooks and makes the following observations regarding the way they define meter:

> Four elements recur in these definitions: beats; patterns; grouping (arrangement, combination); and strong/weak (accented/unaccented, stressed/unstressed). The same four elements appear in Johann Phillip Kirnberger's definition of meter from 1776, but substituting "regularity" for pattern and "segment" for group. (p. 5)

Noted by Cohn (2015d), these four elements common to the classroom music textbooks reviewed in his article portray meter using a system to classify meter devised by Étienne Loulié (1696). This six-fold system classifies meter as either duple, triple or quadruple and simple or compound. Cohn states,

> What is actually being classified here is not the set of pulses and pulse relations that the listener is hearing; rather, it is the *meter signature,* representations that the performer is seeing, using the notational conventions that were developed for 18^{th-} century music. (p. 6)

Cohn proposes a "pseudo-curriculum" to hypothetically address this imbalance, "where the relative percentage of attention to meter vis-á-vis tonality is inverted" thus creating a metaphor for the current weakness in music education where meter pedagogy is concerned (p. 7).

In his article "Meter," Cohn (2018a) draws on recent research, mainly from North America (see above), and defines a meter as "a set of pulses" and classifies meter as "an ordered set of adjacent pairs of minimal meters." Cohn explains, "In traditional musics of the West minimal meters are classified as either *duple* or *triple."* A deep meter is "a set of at least three distinct pulses, each pair of which forms a minimal meter." Cohn refers to the *depth* of a meter to mean its' "number of constituent pulses. A minimal meter is 2-deep." Cohn explains, "The model encourages a view of meter as ever-changing and form-shaping."

To complement his definition of meter, Cohn (2018a) also provides data from recent research which indicates where meter is located, and he explains the two "critical terms" "projection" and "entrainment" involved in the process when humans create meter through listening to music. Recent research on the perception of meter and rhythm indicates that listeners typically structure meter by entrainment (understood here as temporal matching of consecutive pulses which engages the sensorimotor system) and grouping notionally isochronous pulses into 2s and 3s (isochronous here is "notional because human-generated pulses are elastic" (Cohn 2018a). This entrainment stimulates neural activity matching to future onset points by the listener who then projects (reproduces) duplicate pulses; furthermore, these groups are themselves grouped into sets of 2s or 3s; and so on, in a hierarchy. Also, key to Cohn's definition of meter is the understanding that listeners "subjectively metricize" multiplicative combinations and that listeners hear imagined pulses that contribute to meter. The assertion that pulse recognition is subjective indicates that one person's hearing of pulses that make up duple and/or triple meter(s) may not be the same as another's hearing of meter in the same music. (Brochard, Renaud, et al, 2003); (Povel and Essens, 1985); (Large and Snyder, 2009); (Trainor, 2009); (Repp, 1998; 2007); (Parncutt, 1994); (London, 2012); (Burger, B. et al, 2018); (Krumhansl, 1990); (Epstein, 1995); (Lerdahl and Jackendoff, 1983); (Hasty, 1997); (Nozaradan et al, 2011); (Cohn, 2018a).

In summary, the review of materials for teaching and learning meter in classroom music textbooks to educate university students studying to become music teachers in primary and secondary schools and also in textbooks for teaching school-age students indicates meter theory is normally presented with a largely notation-based understanding of meter. The literature reviewed also indicates that meter research has not yet dealt with how to teach with a modern understanding of meter to school-age students. Thus, not surprisingly, recent research findings regarding meter have not yet been incorporated into pedagogical materials suitable for school-age students. This thesis attempts to address this gap in the classroom music literature. Its curriculum model is derived from Richard Cohn's approach to meter theory, which focuses on "sound rather than notation" and graphic representations of meter through mathematical music theory.

CHAPTER 2

THE DEVELOPMENT OF THE SKI-HILL GRAPH AND A STUDY OF A MEDIEVAL PREDECESSOR

> *Ski-hill graphs, which appeared around 1994, allow you to see the distance between two paths, against a field of possibilities. In this way, they resemble more familiar structures, such as a circle of fifths which allows you to see how far apart two keys are, against a field of possible distances.* (Cohn, 2016)[27]

This chapter is organised into two parts. Part 1 provides an account of the ski-hill graph developed by Richard Cohn as an instrument to analyse and teach musical meter, including aspects of the ski-hill graph's recent and future development. Part 2 offers a review of elements of the history of graphic representations associated with meter, in particular a discussion of a medieval triangular graph presented by Johannes Torkesey (d.1340) in his treatise *Trianguli et scuti declaratio de proportionibus musicae mensurabilis* (c. 1320-30).[28]

Cohn's contribution of the ski-hill graph to the lexicon of modern meter theory has arguably been the main reason for Torkesey's work returning to the music history spotlight. Firstly, Cohn's (2016c p. 240) research identified Torkesey's graph as being the first known triangular graph to use the powers of the prime numbers 2 and 3 for mapping and teaching about the duration of notes, thus expanding the metric hierarchy and metric options available in mensural music. And secondly, through adapting the work of Nicomachus to develop the ski-hill graph and meter theory, Cohn has also, in a sense, extended and developed Torkesey's inspired work. Cohn's ski-hill graph and Torkesey's fourteenth-century graph should be considered pivotal moments in the long history of graphic representations in music theory and, above all, in meter theory history.

[27] Richard Cohn. Private correspondence 4/3/16.
[28] "Of the triangle and shield, an exposition of the proportions of mensural music."

Torkesey's "triangle" is often described as a table of proportions intended to teach monks about mensurations, yet, as Cohn (2016c) suggests, it can be interpreted as projecting a far more dynamic meaning. Torkesey's triangular graph of proportions also "constitutes a map of the various metric options available to a mensural composer" thus visually displaying a dynamic and complex landscape of metric relations available to composers of mensural music. Cohn notes,

> [I]f we take each node to indicate not just a single durational value, but also the rate of a pulse generated from that value, [t]his interpretation facilitates exploration of the various ways of combining multiple pulses by successive duple and triple divisions or groupings. (p. 241)

Notably, Cohn's earliest articles on meter recognize hyperpulse(s), hypermeter(s) and faster pulses, as playing an important part of meter.[29] Cohn (1992f) provides evidence of mathematics embedded in music through a formula for examining the number of permutations between the longest and shortest pulses that form meter to the hearing from inclusion relation in sets of 2s and 3s.

In 2001, Cohn rediscovered the Nicomachus triangle, populated it with modern durational values and called it a ski-hill graph. Cohn, p. 240-241 notes, Nicomachus (c. 100 AD) transformed Crantor's (c. 335 -275 BC) graph in the shape of a Greek lambda Λ, not only into a multiplication table but also into a "triangle of frequencies" by adding in the powers of 2 and 3 to the interior of Crantor's graph. Crantor's lambda represented the Pythagorean problem of approximation in tuning, the problem which Plato (428/427 or 424/423 BC) refers to by way of analogy in the *Republic* to political theory.[30] Cohn p. 241 explains, Nicomachus's extension of Crantor's lambda "indicates how transformation by some number of 2-generated octaves compresses the 3-generated tones into a single octave."

Cohn (2001) presents the two-dimensional ski-hill graph as an instrument for the representation of meter and also to show how his algebraic equation can be interpreted through a more intuitive graphic visualisation. Cohn (2001) demonstrates his algebraic equation to calculate the basic mathematical components involved in the experience of meter for the listener (see the next section "Origins of the Ski-Hill Graph). Thus Cohn's ski-hill

[29] Pulses which are not recognised by the notated meter signature in the metric system in current use devised by Étienne Loulié in 1696.

[30] See Footnote 52 and Cohn, 2016c p. 240.

graph and equation both demonstrate evidence of mathematics isomorphic to meter – a relationship normally omitted from the discussions of meter in classroom music theory texts.

Nicomachus's arrangement of the prime numbers 2 and 3 into a multiplication table was initially applied by musicians to Pythagorean tuning. Like Nicomachus's graph, Torkesey's "triangle" also displays the powers of 2 and 3 but Torkesey applied these numbers to durational rather than pitch proportions (Cohn, 2016c p. 240).

Torkesey developed his "trianguli" in the learning environment of a medieval Augustinian religious formation which included a Christian Neo-Platonic approach to the study of mathematics in the Quadrivium (arithmetic, geometry, astronomy and music) and his graph of duple and triple proportions can be understood as an instrument to teach Christian monks about God, through music, arithmetic, and geometry.[31] In its design, Torkesey may have been influenced by Boethius's writings on Nicomachus's mathematics, and also the writings of Saint Augustine and other scholars.[32] Boethius's writing, influenced by Pythagorean mathematics and the study of geometry, is evident in Torkesey's choice of visual representations such as a triangle, circles and the square of opposition.[33]

Torkesey's graph anticipates the ski-hill graph which also maps the interaction of pulses to form meter(s), but Cohn's meter theory expands this understanding. As an instrument of music theory, Cohn's ski-hill graph enables the listener to make observations regarding metric structure in music that are not possible through traditional notational means. Through graphic representations and mathematical music theory the listener is empowered to report evidence of their own hearing of meter and the syntaxes and metric spaces they perceive. Cohn's meter theory integrates recent research in music theory, music psychology, and neuroscience, and so enables the listener to gain a truer understanding of what meter is and where it is located.

[31] Morris Kline classified the four elements of the quadrivium as pure (arithmetic), stationary (geometry), moving (astronomy) and applied (music) number. See Morris Kline (1953). "The Sine of G Major." In *Mathematics in Western Culture*. Oxford University Press.

[32] Anicius Manlius Severinus Boethius (c. 477-c. 524 AD) also known as Saint Severinus Boethius, or Boethius.

[33] The square of opposition is included in the visual representation of Torkesey's treatise in MS Reg. lat. 1146 (see a further section in this thesis 'The Triangle, Shield, Square of Opposition, and Circle as Symbols').

Part 1: The Ski-Hill Graph

Origins of the Ski-Hill Graph

Cohn's development of the ski-hill graph began around 1990 in connection with his research on multiple-level hemiolas in the Scherzo of Beethoven's Ninth Symphony. In the article that followed, "The Dramatization of Hypermetric Conflicts in the Scherzo of Beethoven's Ninth Symphony" (1992b), Cohn modelled complex hemiolas with number representations, for example (2332). Cohn notes, "The ski-hill graph is a direct translation of those [(2332)] numbers and the numerical representations allow you to calculate the distance between two states."[34]

Cohn's examination of meter in Beethoven's Scherzo is arguably the first example of the analysis of an orchestral piece which embodies a thoroughly modern understanding of meter.[35] Cohn mentioned in 2017 that his article from 1992b "influenced all his other work about musical meter" written since then.[36]

In his article on Beethoven's Ninth Symphony, Cohn argues that meter and rhythm are principal players that contribute equally alongside tonality. Rather than playing supporting roles, Cohn demonstrates that meter and rhythm, like tonality and pitch, also form musical structure and act to "narrate, dramatize, polarize, and resolve" (Cohn, 1992b, p. 190). To do this, Cohn examines the multilevel hypermetric shifts by documenting the duple and triple hypermeters and their hypermetric conflicts. In this way, he illustrates how the duple and triple meters are the main protagonists (p. 191). The drama is played out even though none of the hyperpulses or hypermetric conflicts are represented in the notation in Beethoven's score or by the notated meter signatures. Rather, all is witnessed in the listener's imagination and Cohn only looks to the score for clues when necessary. Elsewhere, Cohn describes the Beethoven analysis as representing a large-scale design of indirect dissonance and resolution, which bears familiarity with tonal analysis where conflict and resolution are represented. He adds,

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[34] Cohn. Private correspondence 4/3/16.
[35] In Cohn's article "Meter" (2018a) he observes that modern metric theory "can be traced to the moment in the 1970's when music theorists segregated meter from other sorts of musical grouping (Komar, 1971, 5; Lerdahl & Jackendoff 1983, 17)."
[36] Richard Cohn. From a conversation in 2017.

> Direct hypermetric dissonances – essentially large scale hemiolas – serve as pivots through which indirect hypermetric dissonances are mediated. (Cohn, 1992f, p. 4)

In essence, Cohn's (1992b) article is an account of how hypermetric pulses and conflicts are rendered audible. He demonstrates how evidence of a listener's subjective hearing of hypermeter can be articulated through the prototype of the ski-hill graph as integers representing each meter in a hierarchy.

Additionally, Cohn (1992b, pp. 188-189) points to the rapid change since the mid-1970s where "the most significant work in tonal theory has focused on fundamental questions concerning rhythm and meter, especially beyond the boundaries of the bar line." This is significant in the discussion about ski-hill graphs because the scholars to whom he refers, such as Maury Yeston, Carl Schachter, Lerdahl and Jackendoff, Edward T. Cone, Arthur Komar, Andrew Imbrie, Harold Krebs, Jonathon Kramer, William Rothstein and many others, paved the way for the development of future instruments to analyse meter which incorporate and expand their research.

Another rich by-product connected to the early development of the ski-hill graph, is Cohn's brief but critical comment in Footnote 23 (Cohn, 1992b, p. 195),

> The terms 'simple' and 'compound' traditionally used to distinguish these interpretations, and assiduously learned by music students, have little theoretical value, since they over rely on the often arbitrary choice of the level at which downbeat and tactus are notated. A measure of 'compound' 6/8, two measures of 'simple' 3/4, and a measure of 'simple' tripleted 3/4 all manifest a [2 3] relation and, more importantly, are indistinguishable to perception.

Cohn makes the salient point that the same metrical structure could be expressed in multiple redundant notations. In other words, an astute reader of Cohn's paper may be prompted to adopt an understanding of notation which acknowledges the reality of metric equivalence (see Chapter 4 of this thesis).[37] In later work, Cohn describes notation as "arbitrary," meter as

[37] Although there are alternative terms, "metric equivalence" is a term I coined around 2015 when I started exploring the idea Cohn taught us that notation is arbitrary in the sense that music can be notated in a number of different ways to sound identical. I learned about the idea of metric equivalence and notation as arbitrary from Cohn's classes on meter in 2015 I attended in Sydney at the Sydney Conservatorium of Music where he used empty nodes on ski-hill graphs to represent meter as experienced. Thus, it was through Cohn's ski-hill graph that I first learned about the pedagogical potency of visual and graphic representations of meter and realised its potential for young students.

"independent of both the bar lines and meter signature(s)," and he gives reasons why Loulié's system from 1696 is more akin to a dead end where music theory is concerned:

> when incorporated into a general model of musical meter, Loulié's classification system is a poisonous pill. (Cohn, 2018a) [38]

In his article "Metric and Hypermetric Dissonance in the Menuetto of Mozart's Symphony in G minor, K. 550" (1992f), Cohn begins with a discussion of the recent hypothesized parallels between pitch and time in the work of Seeger (1930), Yeston (1976), Krebs (1987), and Pressing (1983).[39] Cohn (2018a) notes, that these parallels, or isomorphisms which "stretch back to antiquity," suggest an alternative orientation. This orientation encompasses meter and tonality as sharing modularity and perturbations, such as, neighbour tones or passing tones in the case of tonality, and syncopations or hemiolas in the case of meter. Cohn acknowledges that Krebs's (1987) work on direct and indirect metric dissonance shares parallels with tonal consonance and dissonance adding an important contribution to the discussion about pitch-time isomorphisms.

To further develop his discussion of meter at the level of hyperpulse and hypermeter to include duple and triple meter(s) at all metric levels, Cohn introduces an algebraic formula for observing and measuring varying degrees of metric consonance, and direct and indirect metric dissonance at each level in the metric hierarchy.[40] The formula (binary coefficients) Cohn introduces in his article about Mozart's Minuet K 550 (1992f), can be used to analyse a given unordered set of 2s and 3s – sets of adjacent pulses in a relation of inclusion that form duple and triple meters for the listener. The formula, integers, and the ski-hill graph as a two-dimensional matrix, would all later be synthesized, as instruments, to analyse meter in Cohn's seminal article about ski-hill graphs: "Complex Hemiolas, Ski-Hill Graphs and Metric Spaces," *Music Analysis* 20, no. 3 (2001).

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[38] Cohn used the expression "arbitrary" at the Melbourne Music Analysis Summer School lecture, 2016 when he explained notation could be written a number of different ways for the music to sound the same. Cohn explained how meter is "independent of both the bar lines and meter signature(s)," in a lecture for the Sydney Conservatorium of Music, Musicology Colloquium series on 26 April, 2017. "Poetic and empirical theories of musical meter: Cognitive dissonance in the historical archive, the laboratory, and the modern conservatory classroom."

[39] Cohn wrote his Mozart paper in 1991 and gave an oral presentation of it for the Society for Music Theory in the same year. Publisher *Integral* lists the issue as 1992 but the issue was produced in the later part of 1993. He dedicated the article to the memory of his colleague at Chicago University Howard Mayer Brown (d.1993).

[40] For further use of Cohn's equation see Cohn (2001).

Using the formula, Cohn identifies all of the permutations (the multiset) which results in the length (L), a single integer consisting of the metric pathways in time-span (XY) for example, L = 18 in Mozart's Minuet K 550. This is achieved by calculating, through factorization, all levels of pulse sets (time points) between the span pulse X, the slowest pulse, and Y, the fastest unit pulse, which form meter to the hearing. On observing the characteristics of spans, Cohn notes:

> The pure span arises, then, only when L is the power of a prime integer. Since {2} and {3} are the sole metric generators in practically all cases, the relevant underlying sets are {2}, pure duple, and {3}, pure triple … A mixed span arises if L is a multiple of two or more distinct primes.[41] (p. 9 and p. 11)

Cohn's formula, annotated scores, tables, dot notation, and narrative, provide detailed evidence for the presence of both duple and triple meter which form metric ambiguity, direct dissonance, and a double hemiola to the hearing in mm. 1-26 of Mozart's Minuet at the level of hypermeter. Notably, in this article Cohn, p. 13, introduces the new term "double hemiola" to describe 3:2 conflicts at two adjacent levels of the metric hierarchy.

Cohn provides two different readings (interpreted as "hearings") of hypermeter for mm. 21-28 – duple and triple, depending on the listener's awareness. Cohn (p. 20) refers to gestalt, the "rabbit/duck proposition," to discuss a percept of meter where both duple and triple hearings of meter are plausible. Measures 28-36 display a double hemiola at the hypermetric level engendering the previous measures as metrically dissonant.

In "Complex Hemiolas, Ski-Hill Graphs and Metric Spaces" (2001), Cohn's seminal article on ski-hill graphs, dedicated to David Lewin, Cohn introduces the ski-hill graph as a new graphic technique, a two-dimensional matrix to represent simple, double and complex hemiolas. Cohn begins his article by examining David Lewin's (1981) observations about harmony and meter in Brahms's *Capriccio* for piano, Op. 76 No. 8 mm. 1-10. Lewin describes the three meters as sharing a dualism with the three analogous tonal states – tonic, subdominant and dominant.

[41] Cohn introduces the new terms pure duple and pure triple for spans consisting of either duple meter or triple meter but not a combination of both duple and triple meter.

> Just as the acoustic tonic stands in a 3:2 ratio to its dominant and subdominant, so also the “rhythmic” tonic stands in a 3:2 ratio to its “dominant” and “subdominant” metres (Lewin, 1981)

To discuss the three meters observed by Lewin, Cohn notates the three symmetrical groupings of the twelve crotchet pulses reported by Lewin into three columns. (Cohn, 2001, p. 295) Cohn notes that each of the three meters is consonant because each pair of adjacent pulses in each column are related by duple or triple relations (ratios of 2:1 or 3:1). He then draws on the work of Harold Krebs (1987, 1999) to discuss the degree of dissonance between each meter, shown by the horizontal arrows across the columns. Each arrow represents a pair of pulses, on adjacent metric levels, which conflict in a ratio of 3:2 to form a simple hemiola. As Cohn indicates, the two hemiolas form a double hemiola, in this case, through indirect metric dissonance because the conflicting pulses are heard consecutively rather than simultaneously when the new pulse is introduced. Demonstrating other examples of three symmetrical partitions of a 12-unit span, Cohn provides examples from the nineteenth-century repertory, such as, the opening of Beethoven’s *Für Elise*, the Scherzo of Dvořák’s Symphony No. 7, the finales to the “Tempest” Sonata, and many others.

Cohn’s discussion of Lewin’s analytical observations, in a sense, provides a framework for his readers so as to introduce a new instrument of music theory, the ski-hill graph, to represent simple, double and complex hemiolas. In his article Cohn notes:

> Ski-hill graphs make finer distinctions than the graphic models introduced up to this point, whilst engaging the intuition of most readers more readily than the algebraic models found in my earlier articles. (p. 301)

Figure 3 illustrates a 3:2 conflict between two pulses on one metric level represented as a diamond graph:

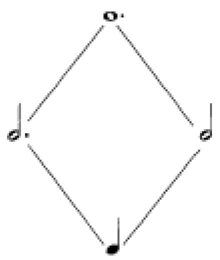

Figure 3: Simple hemiola

Duple divisions are represented by edges that slope downwards to the left and triple divisions are represented by edges that slope downwards to the right. The slowest pulse (the span pulse) is at the top and the fastest (unit pulse) at the bottom.[42]

Cohn describes a simple hemiola (see Figure 3) as occurring "when a span of time is trisected in place of an anticipated bisection."[43] The diamond form illustrates the switch of pulses in relations from either ratios 2:1 or 3:1 to a ratio of 3:2 thus the 'conflicting' pulses heard at the same time or consecutively, such as when approaching a cadential hemiola in an eighteenth-century minuet. Notably, the dotted half note and half note forming metric dissonance are the only two vertices in the graph whose pulses are not related by inclusion, that is, ratios which are integral (3:1 or 2:1). The span pulse (longest pulse) and unit pulse (fastest pulse) are connected by edges formed by duple and triple pathways thus all pathways of the graph share the span pulse and unit pulse. Cohn notes that,

> In most eighteenth-century cases, [the span pulse] is notationally represented as a hypermetric one, articulating every second downbeat. (p. 302)

Cohn uses the ski-hill graph to represent double hemiolas, that is, "the relationship between symmetrical divisions of a time-span that simultaneously bear 3:2 conflicts at two adjacent levels of the metric hierarchy" (p. 295 and Ex. 6 p. 303). Cohn then introduces the theory of metric space to analyse complex hemiolas where metric conflicts occur at three or more levels of the metric hierarchy in the Scherzo of Dvořák's Seventh Symphony No. 7 (see Figure 4). As illustrated in Figure 4 additional diamonds are needed to represent more complex hemiolas such as the complex hemiola reported by Cohn in Dvořák, Symphony No.7, Scherzo, mm. 155-160 (p. 303):

[42] Note that the orientation of the ski-hill graph places the span pulse (longest note) at the apex of the graph unlike Torkesey's "trianguli" where the shortest note (unit pulse) is at the apex.

[43] More recently Cohn also uses diamond graphs to represent simple hemiolas for 2:3 thus making 3:2 and 2:3 interchangeable for the ski-hill graph descriptions in Figure 3. Cohn. Ibid. 295.

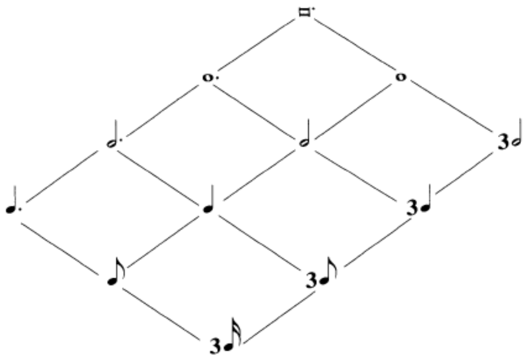

Figure 4: Complex hemiola

Influenced by the mathematician Euler's speculative acoustic work, Cohn also provides a useful algebraic equation (see Figure 5) as a method for calculating the number of duple and/or triple metric pathways connecting a span pulse (S: the longest unit pulse heard) to a unit pulse (L: the shortest pulse heard) in a piece of music (Cohn, p. 308):

$$S(L) = \frac{(m+n)!}{m!n!}$$

Figure 5: Cohn's algebraic equation

For example, Cohn translates the pulses that form meter to the hearing from the ski-hill graph for Dvořák, Symphony No.7, Scherzo, mm. 155-160 into his algebraic equation (see Figure 6) to calculate the number of pathways possible between the span pulse and unit pulse.[44] Cohn's equation provides an important compass for mapping a detailed study of the metric pathways in music. In this algebraic form Cohn's equation calculates the basic mathematical components involved in the experience of meter for the listener: span and unit pulse, powers of 2 and 3, depth of a meter, metric paths, and metric space.

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[44] The length of the span pulse is calculated through counting the number of duple pathways: $2^3 = 8$ and multiplying them by the triple pathways: $3^2 = 9$: 8 x 9 = 72. The total duration of the span pulse is 72 times its unit. To find the number of possible metric paths extract the exponents (the powers 3 and 2) and add them together 3+2 = 5; then calculate the factorials of the added powers: 5x4x3x2x1=120. Calculate the factorials of each power multiplied: (power of 3) 3x2x1 = 6 multiplied by (power of 2) 2x1 = 2: 6x2 = 12. Divide the factorials of the powers added together by the factorials of each power multiplied: 120÷12=10. Thus, there are 10 possible paths, or meters, between the span pulse and unit pulse in these measures.

Through audiation, mathematics embedded in music and embodied in the listener can now be explained in terms other than "the feel" or through fuzzy explanations and pedagogical approaches which ignore any connection whatsoever between music and mathematics.[45] The results of Cohn's short algebraic equation represent the isomorphic musical and mathematical structures experienced when examining musical meter through audiation. As Cohn demonstrates, this ironically compact form, can reveal a vastly complex and dynamic metric space as in Dvořák's Scherzo mm 155-160. Thus, Cohn's equation provides the listener with an instrument to enumerate the possibilities for a given span – detail never before introduced in the domain of musical meter pedagogy (Cohn, 2001. p. 311).[46]

$$S(72) = \frac{(3+2)!}{3!2!} = \frac{120}{12} = 10.$$

Figure 6: Cohn's algebraic equation for Dvořák, Symphony No.7, Scherzo, mm. 155-160

Also, in the 2001 article, Cohn examines musical meter in Brahms's *Von ewiger Liebe* expressing his findings in two different ways. Firstly, through narrative, Cohn provides broad observations about the song's plot through equal treatment of meter and tonality between the parts (voice/piano accompaniment) for each stanza. He refers sparingly to poetic meter and only does so if it contributes to the formation of pulses involved in the hearing of musical meter. In Cohn's examination of both meter and tonality in Brahms's *Von ewiger Liebe* he is particularly interested in describing metric dissonance wherever it occurs in the metric hierarchy, a feature overlooked in previous analyses by other scholars. Secondly, Cohn engages the use of the ski-hill graph to narrate a description of the pathways and to then provide a graph of the metric space which maps the six possible metric paths and their relations. In doing so, Cohn notes the similarities of his analysis "to a journey through

[45] I use the term "audiate" and "audiation" to mean active listening as in the following: "Audiation is not the same as aural perception, which occurs simultaneously with the reception of sound through the ears. It is a cognitive process by which the brain gives meaning to musical sounds." See The Gordon Institute for Music Learning. Retrieved 15/01/2018 from https://giml.org/mlt/audiation/ In my use of these terms I am also including the bodily response to sound and I have adopted the use of the terms without referring to Gordon's types and stages of audiation. I first learned about "audiation" when it was introduced into my post-graduate course as a secondary education classroom music specialist in 1990 but also without reference to Edwin Gordon's Learning Theory. For definitions of audiation see also Gordon (1987) and (2001).

[46] Cohn's equation also provides mathematical evidence for the property of metric equivalence where every piece of music can be notated in a number of different ways to be made to sound the same through maintaining the ratio value between pulse and tempo so that the notation looks different but sounds the same.

harmonic functional spaces" (p. 319). Prior to concluding his article with some important speculative statements about "the idea of a deep affinity between pitch and time," Cohn also suggests:

> Lewin's linear duality (tonic, dominant and subdominant with meter) is more properly included as one dimension of a two-dimensional array whose axes are generated by duple and triple metric proportions respectively. (p. 322)

Cohn's next published article which deals directly with ski-hill graphs was written in 2015 and published in 2016, "Graph-Theoretic and Geometric Models of Music." Cohn provides background to the prominence that graph theory, geometry and topology have maintained in music theory.[47] Before introducing various examples of music visualizations through graphic representations in current use including the ski-hill graph, Cohn outlines the reasoning behind visualizing music in such forms:

> Translating musical events into symbols, musical images place those events and their relations before our eyes and, freeze them in place. These representations facilitate fundamental analytic operations such as pattern recognition and projection, and also afford the capacity to map psychological intuitions, such as judgments of proximity and distance, conjunction and disjunction (Cohn, 2016c, p. 238).

Cohn explains that the crucial property of the graphic translation of duration and pitch is strict total ordering and, given two distinct time points, precedence is determinable (either $x > y$ or $y >$ x), while for any three distinct time points, precedence is transitive (if $x < y$ and $y < z$ then $x < z$). Cohn notes that pitch can be measured as uniform distances of scale steps or semitones and time can be measured as beats or pulses when punctuating the linear continua and both can be modelled by integers alone. In relation to meter theory, Cohn outlines the history of the ski-hill graph through a discussion of the work of Pythagoras,

[47] Cohn (2016c) p. 237 provides a summary of the different mathematical modes of "exploring, conceiving, and representing musical phenomena" which have contributed towards mathematical music theory regaining its central position in music theory today such as the symbolic languages of abstract algebra, especially sets and groups. However, Cohn states that since about 1995 there has been a shift towards "mathematical sub-disciplines that characteristically communicate through visual images: graph theory, geometry, and topology." He notes the current prevalence of musical graphs and geometries "not just polygons, and Cartesian graphs, but also multi-dimensional cylinders, including helices, torii, and complex orbifolds."

Plato, Crantor, Nicomachus, and Johannes Torkesey, whose work I will discuss later in this chapter.

Recent Developments in the Use of the Ski-Hill Graph

Cohn's recent article "Meter" (2018a) for *The Oxford Handbook of Critical Concepts in Music Theory*, is a distillation of the theory of meter he developed through the works discussed above and which is intended as an analytical model to understand musical meter. The article, which "focuses on sound rather than notation," includes a broad coverage of historical materials examining how meter has been understood, taught, classified and defined. "Meter" provides evidence and insights into how meter can now be understood in a modern sense based on recent research findings in music psychology, neuroscience and music theory, mainly from North American scholars. Cohn provides new classifications and descriptions for the different types of meter and their relations that depart from theories of meter written to describe the notation.

In the section "Locating Meter," Cohn acknowledges research which indicates that musical meter is located in the listener as a "mind and body" response to the stimulus of sound.[48] To complement this understanding, Cohn provides a definition of meter: "A meter is an inclusionally related set of distinct, notionally isochronous time-point sets."

This understanding about the location of meter and definition of meter runs contrary to the definitions and understandings of meter found in classroom music textbooks. Classroom music texts default to the meter signature to define meter and refer to some version of the systems to classify meter by and Étienne Loulié (1696) and Johann Phillip Kirnberger (1774) (see Chapter 1 of this thesis for a review of meter in classroom music textbooks).

Cohn includes a section on "Representing Meter" which discusses various ways deep meter can be represented, including an ordered multiset Apel (1949); dot notation; and Cohn's ski-hill graph (c. 2001) which are all "useful in different contexts." Cohn's ski-hill graph compresses the information and graphic style of dot notation which can be cumbersome and

[48] Cohn's earlier article, "Why We Don't Teach Meter, and Why We Should" (2015d) is a broad discussion of the "two systems (meter and tonality) through which we process when listening to music." Cohn notes the lack of pedagogical materials on meter in classroom music textbooks, even though there is a plethora of recent research indicating what meter is and where meter is located. See also Chapter 1 of this thesis for a further discussion on Cohn's article from 2015.

difficult to notate neatly by hand for classroom use with young students. The ski-hill graph removes the need for every repeated pulse to be depicted; instead, graphing in this way provides a summary of pulses in a relation of inclusion “distinct meters” which then becomes a network of pulse relations or sets of pulses in relation that make up meter or more usually meters. Cohn states,

> A third way to represent a meter is as a *ski-hill graph* such as Figure 4a (Gilles and Reaney, 1966; Cohn 2001). Each node represents a pulse, and each edge represents an adjacent minimal meter. Duple meters skew down to the left, and triple ones down to the right. Each graphic shape is associated with an abstract metric class. Filling the nodes with specific duration values or interonset periods realizes the abstract graph as a network (Lewin 1987). Figures 4 (b) and (c) realize the graph in two different ways: as Beethoven’s 3/4 meter (Figure 2b), and as a series of 12/8 measures (Figure 2e).

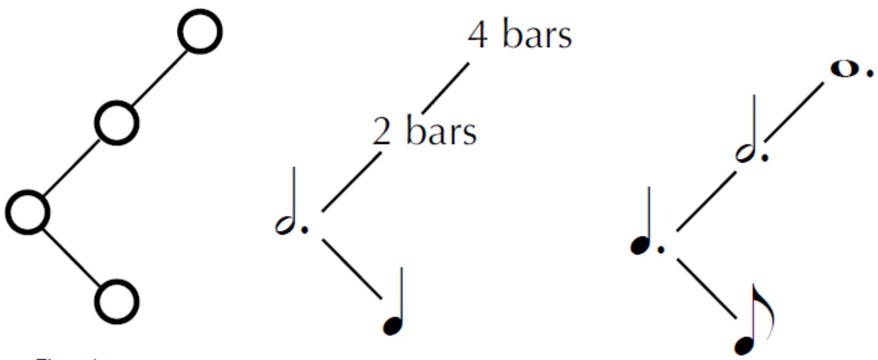

Figure 4a Figure 4b Figure 4c

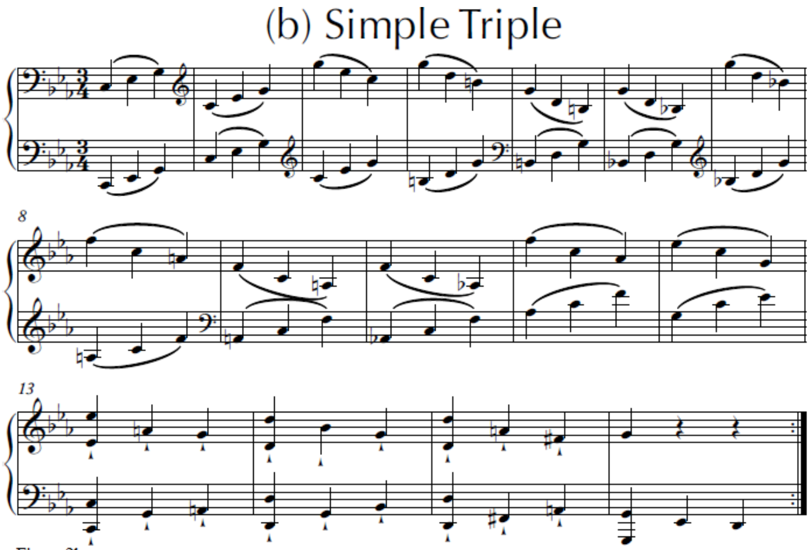

Figure 2b

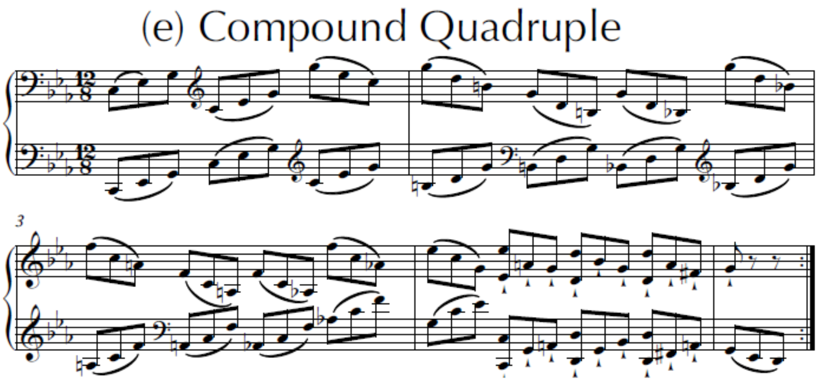

Figure 2e

In "Metric Form" Cohn (2018a) explains,

> As soon as we release meter from metric notation, and link it to music as heard, we license it to enter the arena of musical form, and we can quickly see that it has a vigorous contribution to make.

In "Relating Meters," Cohn includes ski-hill graphs and dot notation to explain the interaction of pulses for syncopations and hemiolas to occur. Cohn explains how "the progressive series of adjacency swaps" suggest a "scripted journey through a series of metric states, executing a departure/return scheme that has analogues in tonal theory" and he provides ski-hill graphs representing meter with complex hemiolas in Robert Schumann's Fantasy in C major, Op. 17 (1837).

Cohn's final section, "Expanding Meter," provides an explanation of meters (or "meters") that have irregular *quasi-pulses* and slow *hyper-pulses*. Cohn states,

> My open definition considers them to be meters, albeit not prototypical ones. A more closed definition that imposes context-free thresholds excludes them, so that they become “meters” whose status as meters is metaphorical (Hasty 1997).

During the last twelve years a number of scholars have incorporated Cohn’s work with ski-hill graphs to represent hemiolas, including Scott Murphy, "On Metre in the Rondo of Brahms’s Op. 25” *Music Analysis* 26, no. 3 (2007); Daphne Leong, "Humperdinck and Wagner Metric States, Symmetries, and Systems” *Journal of Music Theory* 51, no. 2 (2007); Yonatan Malin, *Songs in Motion* (Oxford University Press, 2010). Scholars such as Jason Yust (2018), along with a growing number of postgraduate students, have incorporated Cohn’s ski-hill graph into their research, such as Matthew Chiu’s (2018) lecture: "Form as Meter: Metric Forms through Fourier Space.”[49] Cohn lectures on ski-hill graphs with his own students at Yale University and elsewhere and he has written a handbook on ski-hill graphs (unpublished).

Since 2015, I incorporated Cohn’s ski-hill graphs into my music studio teaching of both school-age and adult learners. To my knowledge, no one had previously adapted either the ski-hill graph or Cohn’s understandings of and approaches to teaching meter for school-age students. I have witnessed the benefits it has brought in this context, and I elaborate on these in Chapters 3 and 4.[50] As I continued to teach meter in this new way, I began to wonder if a ski-hill graph app existed but found no one had yet produced one. After correspondence with Richard Cohn and Dr Andrew Milne from the MARCS Institute for Brain Behaviour and Development, Western Sydney University, Milne (2016) developed a ski-hill graph app suitable for use with school-age students and adult learners (see Footnote 5 for the availability of Milne’s SkiHill app).

The software version of the ski-hill graph gives students a choice of how to represent pulses, such as traditional durational symbols or fractions and a variety of sonifications through

[49] Matthew Chiu (Boston University) “Form as Meter: Metric Forms through Fourier Space.” GSMC: Panel 3 – Meter, Rhythm, and Form https://www.youtube.com/watch?v=_jBvgaSrHkM&t=1237s see also Chiu, M. (2018). *Form as meter: metric forms through Fourier space*. *Open.bu.edu*. Retrieved 18 August 2018, from https://open.bu.edu/handle/2144/30655

[50] Part of this section draws on an account I have written of my adaption of the ski-hill graph for school-age students and use of the SkiHill app prototype published in C. Hilton, A. Calilhanna, and A.J. Milne. Eds. M. Montiel and F. Gómez. “Visualizing and sonifying mathematical music theory with software applications: Implications of computer-based models for practice and education.” *Theoretical and Practical Pedagogy of Mathematical Music Theory. Music for Mathematics and Mathematics for Musicians, From School to Postgraduate Levels.* (World Scientific Press: 2018).

different sounds to choose from for each pulse. For example, a quarter note could be visually represented by its symbol, or by the fraction 1/4, and sonified with, for example, a drum or cymbal sound. As a computer-based instrument to analyse meter, the SkiHill app plays a unique role as a "mediator" between student and computer (Hilton, Calilhanna, Milne, 2018 p. 224). From listening to music and through movement such as tapping or clapping, or other forms of movement such as walking or conducting, students then map the adjacent pulses in inclusion relation they hear in a hierarchy.

In other words, through mapping the pulses they hear forming meter they manipulate the computer to "make" new knowledge by visualising the meter and metric space they hear. Thus, through articulating their own temporal experience of meter they are no longer passive consumers of information from a computer or teacher who tells them what pulses to hear based on Loulié's system. Meter is no longer a static notation at the head of a score, but a dynamic relationship of many pulses paired by the mind and body through entrainment and projection to form minimal meters and deep meters, hemiolas, metric consonance and metric dissonance, both indirect and direct.

The SkiHill app provides awareness of geometry through polygons which form according to the pulse relations and meters (metric pathways) which are represented both visually and through sonifications: duple 2:1 and/or triple meter(s) 3:1 and hemiolas such as 2:3; 4:3; and 9:8. A newer version of the SkiHill app is now in development, which includes expansions to other ratios of pulses which form hemiolas such as 5:2, complementing Cohn's research about what meter is and where meter is located (Cohn, 2015d, 2018a).

Looking to develop further materials to teach meter, in 2016, I realised that a ski-hill graph video, which was freely available for public music theory on the web, would be beneficial for a wider audience of students to learn about meter. After correspondence with music educator David Kulma and Richard Cohn, Kulma produced the YouTube video *"March V Waltz - A Short Intro to Meter and Ski-Hill Graphs"* on Music Corner Breve (December 2016): https://www.youtube.com/watch?v=0yxba7yoMSk.

At present I am part of an interdisciplinary research project, *Teaching Mathematics with Music and Music with Mathematics*, where the ski-hill graph forms part of the suite of materials designed for trialling in schools. Project manager Dr Andrew Milne (2018) states,

> The ongoing project – Teaching Mathematics with Music and Music with Mathematics – explores how commonalities between mathematics and music can be used to facilitate understanding and improve educational outcomes in both subjects.[51]

Part 2: A Medieval Triangular Graph Developed by Johannes Torkesey (d.1340)

Brief History

About 650 years before Cohn's ski-hill graph, English Augustinian Johannes Torkesey (d. 1340) wrote his short but influential music treatise *Trianguli et scuti declaratio de proportionibus musicae mensurabilis* in around 1320–1330 (Gilles and Reaney, 1966). The following material contextualises Torkesey's work with that of his predecessors and contemporaries, and also for a modern context within current meter theory. The material mainly focuses on music history, music treatises, meter theory research, relevant history of mathematics, and pertinent ancient Greek philosophy. Most of the works cited will refer to the measurable hierarchical properties of pulses in relation to each other as being innately duple and/or triple. Other relevant mathematical principles and approaches to understanding musical meter or mensurations are at times referred to which also place Torkesey's graph in a context of meter theory history.

Until recently, almost all writers on medieval and Renaissance music overlooked Torkesey's work or had given it only the briefest mention. Yet scholar Ronald Woodley (*Grove Music Online*) asserts that Torkesey's short work *"*was influential*"* in England in the 14th and 15th centuries and André Gilles and Gilbert Reaney note the "tremendous popularity" that Torkesey's triangle enjoyed "from the late fourteenth into the sixteenth century" (Gilles and Reaney, 1966). Fortunately, recent work by scholars such as Richard Cohn (2016c), Karen

[51] Quoted from the program notes of Meter Symposium 3 (2018) Retrieved 8/8/2018 from https://sydney.edu.au/music/our-research/research-events/meter-symposium-3.html. Also, Tara Hamilton et al "Teaching Mathematics with Music: a pilot study", Accepted for presentation at the *IEEE International Conference on Teaching, Assessment, and Learning for Engineering (TALE): Engineering Next-Generation Learning*, Australia, 4 - 7 December 2018.

Cook (2012), Peter Hauge (2010), Renata Pieragostini (2013), and Karen Desmond (2018) have begun restoring awareness of the importance of Torkesey's work. As mentioned, Torkesey's "triangle" and Cohn's ski-hill graph share many similarities, with Cohn's ski-hill graph bringing a new "version" of ancient ideas into a modern setting.

Important to the understanding of Torkesey's graph is music history associated with the recognition of the property of the prime numbers 2 and 3 in relation first to pitch then later to pulse. The numerical relationship to pitch is thought to have been first pronounced by Pythagoras (c. 570–c. 495 BC) who based his thinking about music on a fundamental observation that intervals can be expressed as measurable numerical ratios. As noted by Cohn,

> Pythagoras (c. 550) allegedly discovered the correspondence between musical pitch and number: the association of the number 2 with the octave, and the number 3 with the interval a perfect fifth greater than an octave. (Cohn, 2016c, p. 240)

Yet, octave relationships are determined by a factor of 2; fifth relationships by a factor of 3/2 thus both pitches can never meet (2 being even and 3 being odd) to form a useable or closed system without involving "some strategy of approximation, in which a small interval, or comma, is interpreted as an identity" (Cohn, 2016c, p. 240).

Cohn observes Pythagoras's problem of approximation became a preoccupation for Plato (428/427 or 424/423 BC) in his *Republic,* concerning political philosophy written around 380 BC.[52] Significant to this study is Torkesey's utilization centuries later of this same Pythagorean number theory and soon after for Willelmus, who before 1372 expanded Torkesey's "triangle" by adding another level of proportions (Gilles and Reaney, 1966, 6).

Also, Greek philosopher and disciple of Plato, Crantor (c. 335 -275 BC) presented what Cohn suggests is an effective illustration of the "problem" of the Pythagorean theory of approximation in the shape of a Greek lambda (Cohn, 2016c, 240). Crantor's lambda provides "a co-ordinated system where pitch is represented on both axes, powers of 2 to the left axis and powers of 3 on the right axis" (Cohn, 2015). Crantor's work, via both

[52] In his *Republic*, Plato refers to Pythagorean number theory (See Plato, *The Republic*. Translated by Desmond Lee. Penguin Books, 2007 Google Play Edition: 448, Footnote 25) by referring to "irrational" lines when explaining state affairs. The "irrational" lines could conceivably refer to the irreconcilable lines formed on Pythagorean triangles. See Cohn's (2016c) discussion of the Pythagorean problem of approximation solutions for tuning theory which for Plato "served by analogy as a model for the earliest European treatise on political theory."

Nicomachus and Boethius, it will be seen later, likely influenced Torkesey's approach to representing relationships between pulses on his triangular graph.

Johannes Torkesey (-d.1340) and *Trianguli et Scuti Declaratio De Proportionibus Musicae Mensurabilis* (1320-1330)

Contemporary of Johannes de Muris (c. 1290 - c. 1351) and Phillipe de Vitry (1291-1361), English Augustinian friar and music theorist, Johannes Torkesey (d. 1340), elected as Canon of the Augustinians at Elsham, North Lincolnshire, on 9 November 1339, adapted "the Nicomachus triangle to a new musical application in the fourteenth-century" (Cohn, 2016c, 240). Cohn notes, Torkesey "appended a durational value above each of the triangle's numbers" (Cohn, 2016c, 241). In his treatise *Trianguli et scuti declaratio de proportionibus musicae mensurabilis* (c. 1320 - 1330) (Gilles and Reaney, 1966) Torkesey does not attempt to expand Boethius's table of proportions in *De institutione arithmetica* beyond 243 as the longest note value. The expanded set of values in the *Declaratio* proceeded from the shortest 'simpla' to the longest (which was called "larga"), equivalent to 243 simplae. Karen Cook (2012, p. 94) includes Torkesey as one of a number of writers during the early fourteenth-century whose work reflects the development of the semiminim note to describe and name notes shorter than a minim.

Torkesey's addition of the simpla or semiminim note to his triangular graph was likely a demonstration of his deep understanding of Pythagorean number theory, in providing a graphic representation of those proportions and their duple and triple relationships. As Cook writes, instead of describing the mensural note values:

> Johannes Torkesey instructs his readers on how to determine the lengths of note values in relationship to others, all of which are hierarchically arranged on a shield (the *scuti* in the title) … the indivisible simpla is the unit by which all other units are measured. The next largest note value, the minim, is worth either two or three simplas. Torkesey does not mention the system of tempus and prolation that was in use in the areas following Murisian practice, but instead refers to note values as either

perfect or imperfect; the minim worth three simplas is perfect, that worth two simplas imperfect (Cook, 2012, p. 111).[53]

French Ars Nova theorist Johannes de Muris (born c. 1290 - after 1344) wrote five treatises on music: *Notitia artis musicae* (1319–21), *Compendium musicae practicae* (c. 1322), *Musica speculativa secundum Boetium* (1323), *Libellus cantus mensurabilis* (c. 1340), and *Ars contrapuncti* (post 1340) and, as Berger states, "his many writings constitute a *summa* of medieval speculative and practical traditions" (p. 636). In terms of the "new art" of the fourteenth-century, Berger describes Muris's *Libellus* as the following:

> It is the clearest and most influential presentation of the new mensural system. The *Libellus* was copied, translated, and quoted from extensively for the next hundred and fifty years, and used as a textbook in most schools and universities throughout the Middle Ages and Renaissance. (p. 636)

Berger describes how Muris presents the five basic note values beginning with the longest note value: "the *maxima*, the *longa*, the *brevis*, the *semibrevis*, and the *minima*… With the exception of the minim, each value can be divided into either two or three parts" (p. 636).

Muris provides descriptions of the various divisions and how they can be "distinguished and combined with one another to form mensurations" through modus, tempus, and prolatio. Berger states,

> Division of the long is termed the *modus*. If the division is triple (three breves), the mode is said to be perfect; if it is duple (two breves), the mode is imperfect. Division of the breve is termed the *tempus.* Again, if the division is triple (resulting in three semibreves) the *tempus* is said to be perfect, and imperfect if the division is in two. Finally, the division of the semibreve is termed *prolatio*, and it is distinguished by either *prolatio maior* (three minims) or *prolatio minor* (two minims). (p. 636)

Muris also provides four signs to represent the various mensurations of tempus and prolatio including, for instance, a complete circle to indicate perfection and incomplete circle for

[53] See also Anna Maria Busse Berger, The evolution of rhythmic notation (pp. 628-56), *The Cambridge History of Western Music Theory*; for a discussion John of Garland's treatise *De mensurabili musica* (c.1250) which contains "the first kind of rhythmic notation developed in the West." (p. 628); and Franco of Cologne's treatise *Ars cantus mensurabilis* (c.1280) ("The Art of Mensurable Music"). Berger states, "Franco is often celebrated, along with Boethius and Guido, as one of the most important music theorists – the "inventor" of mensural music" (p. 631).

imperfection (Berger p. 637).[54] However, Muris's discussions of diminution caused a great deal of confusion for the next two hundred years and it was during this period and climate that Johannes Torkesey's treatise appeared.

The opening statement of Torkesey's treatise begins:

> In order to have a complete understanding of the art of measured music, it should be known that for the praise of the One and Triune God there are three species of square note-shapes, from which six species of simple notes are formed.[55]

To help the reader obtain "a complete understanding of measured music" and praise God as Triune, Torkesey first provides the three primary species of square note shapes "Tres primae species quae prima figura vocatur" (see Figure 7):

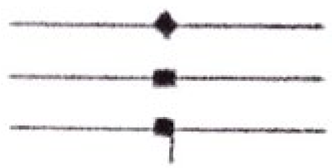

Figure 7: "Tres primae species quae prima figura vocatur"[56]

Figure 8 illustrates a shield (*scuti* of the title) representation of the six principal note-shapes, listed in a triangular shape, from the simpla (semiminim note) through to the larga (maxima). "Sex species simpliciter quae secunda figura vocatur." The six simple species, which is called the second figure:

[54] The dogma of The Holy Trinity played a part in this concept and the designation of mensuration terminology with the origin of the full circle symbolising God's Trinitarian perfection and the half circle symbolic of imperfection and hence duple mensurations see Riemann, *History of music theory,* 186-187.

[55] "Ad habendam notitiam perfectam artis musicae mensurabilis, sciendum est quod ad laudandum trinum Deum et unicum tres sunt species figurarum quadratarum, ex quibus sex species notarum simplicium formantur." (Gilles and Reaney, 1966. p. 58). I acknowledge assistance by Dr Fiona Walsh with the English rendering of this passage and others. See the section *The Triangle, Shield, Square of Opposition, and Circle as Symbols* for an explanation of the significance of the number 3 in Torkesey's work. From their study of geometry Pythagoreans regarded 6 as the first perfect number (=1+2+3) which is also the area of the great 3,4,5 right triangle, see Hopper (2012, 36).

[56]

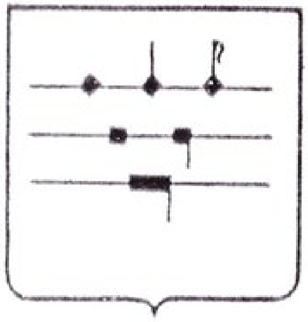

Figure 8: “Sex species simpliciter quae secunda figura vocatur”[57]

Torkesey includes a table of the corresponding rests Table of rests (see Figure 9):

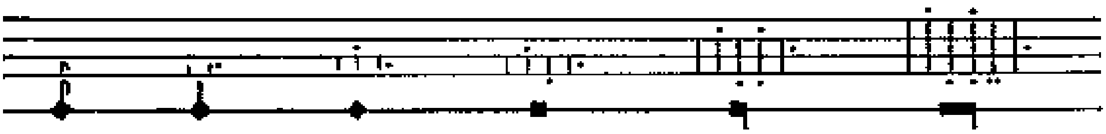

Figure 9: Table of rests[58]

To illustrate the mensural hierarchy the author of MS Reg. Lat 1146 provides an isosceles right triangle (see Figures 10 and 11) representing imperfection (duple divisions), beginning on the left side, and perfection, (triple divisions) from the right side, in the mensural hierarchy. He uses dots above and below the notes as well as to the right of the note to “indicate the perfection of values contained within as well as of the note itself” (Woodley 2016).

Figure 10 illustrates a modern transcription of Torkesey’s “trianguli” (triangle) from “Trianguli et scuti declaratio de proportionibus musicae mensurabilis” c. 1320 -1330 (c) 1966 American Institute of Musicology. Originally published in Corpus Scriptorum de Musica 12, p. 61. Used with permission:

[57] (c) 1966 American Institute of Musicology. Originally published in Corpus Scriptorum de Musica 12, p. 60. Used with permission. See also the later section in this thesis “The Triangle, Shield, Square of Opposition, and Circle as Symbols.”

[58] (c) 1966 American Institute of Musicology. Originally published in Corpus Scriptorum de Musica 12, p. 60. Used with permission.

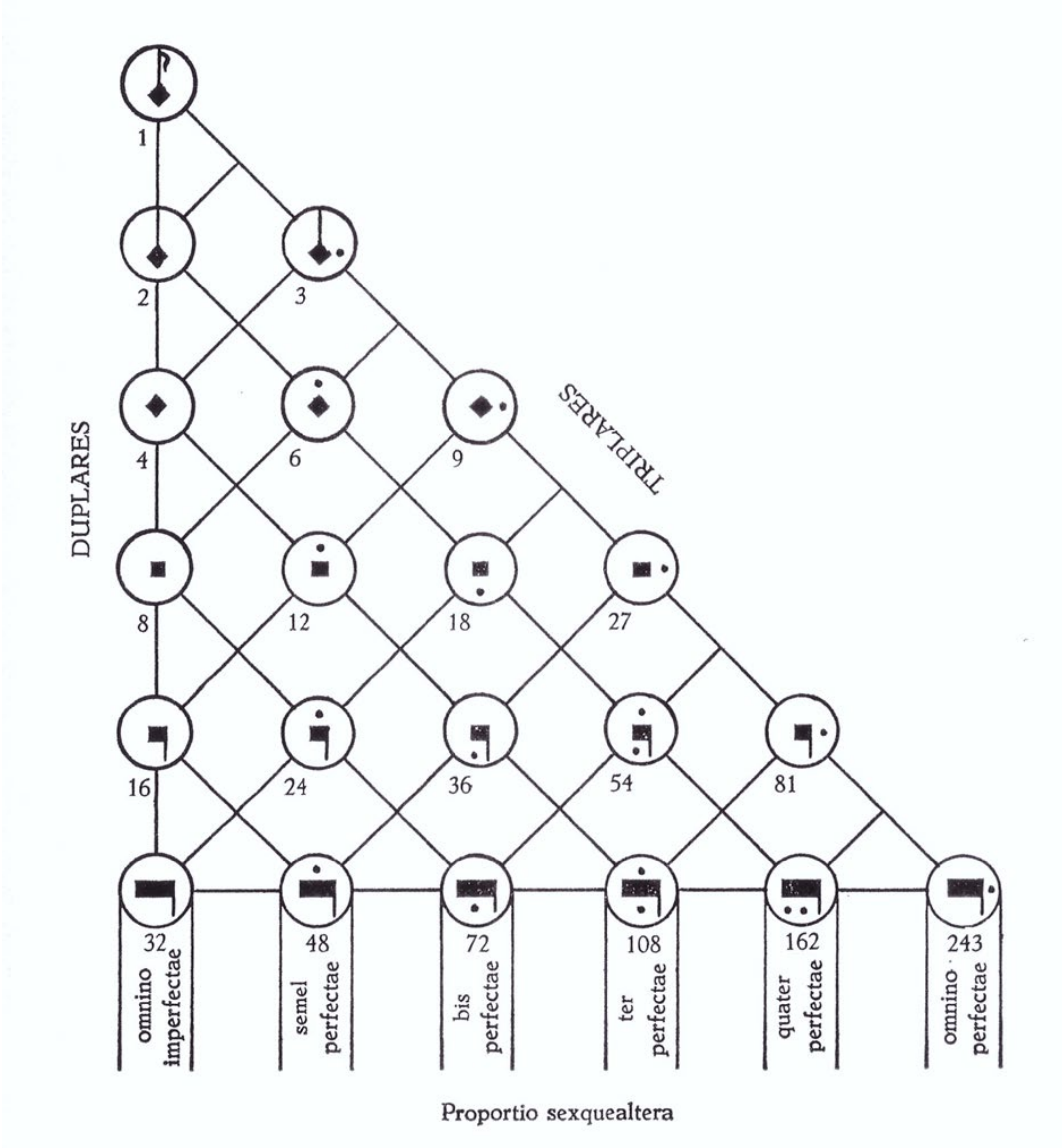

Figure 10: Modern transcription of Torkesey's "trianguli" (triangle)

Figure 11 illustrates Johannes Torkesey's "trianguli" from "Trianguli et scuti declaratio de proportionibus musicae mensurabilis" c. 1320 -1330, Vatican City, Biblioteca Apostolica Vaticana, MS Reginensi latini. 1146, f. 55v.[59] Used with permission.

[59] See also Biblioteca Apostolica Vaticana https://digi.vatlib.it/view/MSS_Reg.lat.1146 and "Boethius triangles" http://sound-colour-space.zhdk.ch/sets/10002.

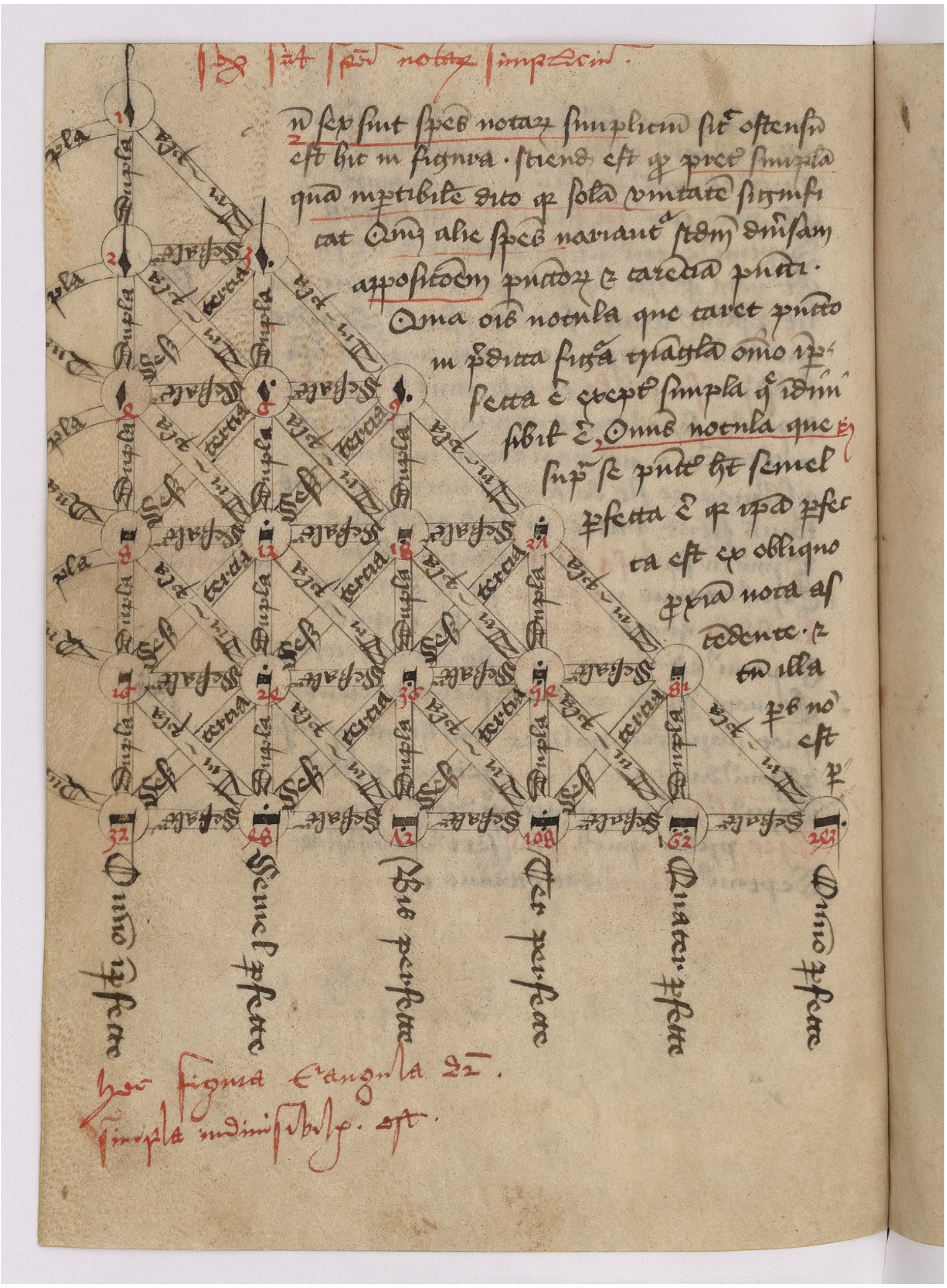

Figure 11: Johannes Torkesey, MS Reg. lat. 1146, f. 55.

Boethius's writings about Nicomachus's mathematics in *De institutione arithmetica* included a table of proportions in the shape of an isosceles right triangle using the same proportions presented in transcriptions of Torkesey's treatise c.1320-30.[60] Unlike earlier writings by Crantor and Nicomachus, Boethius presented his diagram in the shape of an isosceles right triangle (roughly to scale) to represent the Pythagorean tone system. The following annotation describes an arithmetic diagram (number triangle 2:3) from a copy of Boethius's *De institutione arithmetica*:

> Arithmetic diagram. The horizontal dimension "Latitudo" displays geometric progressions of factor 2, the diagonal dimension "Angularis" displays geometric progressions of factor 3. Therefore, the columns contain geometric progressions of factor 3/2. In other words, they form the continuous proportions of sequences of Pythagorean fifths. The diagram shows the integer numbers of the format $2^j \cdot 3^k$.[61]

The isosceles right triangle shape also appears in the manuscript Cambridge, Corpus Christi College, MS 352: Boethius, *De institutione arithmetica* f. 36v (ca. 800 A.D. - 999 A.D.).[62]

Torkesey's Augustinian Education

Torkesey's fourteenth-century quadrivial education as an Augustinian would likely have involved studying the works of his order's Patron Saint Augustine 354 AD-430 AD who in his writing drew on the number theory of the ancient Greeks including Pythagoras, Plato and

[60] The inclusion of the triangle and shield shape in London, British Library, Additional 21455, ff. 7v-8r (listed as a fifteenth century manuscript with an anonymous author) likely indicates that MS Add. 21455 is an earlier copy of Torkesey's treatise than MS Reginensi latini 1146 (currently listed as a fourteenth-century manuscript by Torkesey). In MS Add. 21455 f. 8r the six notes and rests and their successive duple and triple divisions are represented hierarchically as an equilateral triangle (roughly to scale) in a triangular shield, the 'trianguli et scuti' of the title, with each note surrounded by a red circle all except one: the first perfect mensuration of the indivisible simpla – the dotted minima "3". Although the shape of the figure in MS 21455 f. 8r more closely resembles that of Nicomachus's triangle to represent the Pythagorean tuning system as a triangle with equal sides, the proportions of the figure in MS 21455 f. 8r are those matching Boethius's triangle in *De institutione arithmetica.* At present there are no works written about MS Add. 21455 however a digitised version of Add MS 21455 has recently become available see British Library http://www.bl.uk/manuscripts/Viewer.aspx?ref=add_ms_21455_fs001r accessed on 1/10/19.

[61] Medeltidshandskrift 1 (Mh 1), Lund University Library, early 10th c., 4r Retrieved 25/7/18 from Sound Colour Space – A Virtual Museum", Zurich University of the Arts, 2017, "Boethius triangles" http://sound-colour-space.zhdk.ch/sets/10002. "Sound Colour Space – A Virtual Museum" provide an illustration and description of the number triangle (2 : 3) from Boethius's De institutione arithmetica.

[62] Chadwick 1990, pp. 37-39 discusses Neoplatonic themes about the "cartography of mathematics, and especially of geometry, in the map of human knowledge."

Nicomachus. Saint Augustine, in his tract on Christian instruction is certain that "music and number" are keys to unlock the exegesis of scripture. As Chadwick (1990) explains,

> Moreover, numbers are governed by immutable rules and are a signpost to the unchanging Creator. (17)[63] So the Augustinian tradition of Christian Platonism domesticated within the Church formed a substantial part of the old Platonic language about numbers and harmony as roads to the truth of the God who is. (p. 13)

Also, Torkesey likely studied the work of Saint Thomas Aquinas (1225-1274) whose influences included the writings of St Augustine and Boethius, all of whom wrote works concerning the Holy Trinity the "other" subject of Torkesey's treatise about proportions. Davies (2009, p. 192) notes, the fifth-century treatise by St Augustine, "was the first really major, systematic, and powerfully influential theological statement of how we can think of God as three in one."

The Triangle, Shield, Square of Opposition, and Circle as Symbols

Torkesey's choice of the symbolic forms – triangle and shield – in one sense is not surprising as the triangle and shield were already commonly known in his day for their religious symbolism and embedded mathematical meaning. The triangle was seen to reflect the mysterious and unfathomable equations 3=1 and 1=3. The geometric shield shape was also commonly in use around Torkesey's time and generally viewed in Christian symbolism as a defence against evil, representing the "heraldic arms of God and of the Trinity" (e.g. Shield of the Trinity).[64] Thus, the three-sided symbolic figure of the shield – the *scuti* in the title – is well-suited for the hierarchical display of the six pulses represented from the shortest to longest note.[65]

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[63] For (17) see Chadwick (1990) Notes: (17) [Saint Augustine] *De Doctrina Christiana* ii, 16, 26; ii, 38, 56.

[64] For examples of the medieval Shield of the Trinity see "Sound Colour Space – A Virtual Museum" http://sound-colour-space.zhdk.ch/archive?limit=15&order_by=date&match=OR&image_size=x-small&q=fulltext::trinity.

[65] Torkesey's choice of triangle and shield as symbols in his treatise (c.1340) indicate that the "shield" shape of Figure 8 can plausibly be illustrated as a medieval triangular shield, (rather than as a square-sided figure) see MS Add. 21455 f. 7v. http://www.bl.uk/manuscripts/Viewer.aspx?ref=add_ms_21455_fs001r See also Torkesey's six note shapes illustrated in the fifteenth-century manuscript copied by John Wylde (c.1460), precentor of the Augustinian priory of Holy Cross, Waltham, Essex: Lansdowne MS 763 f. 89v http://www.bl.uk/manuscripts/Viewer.aspx?ref=lansdowne_ms_763_f001r, and Cambridge 1441, fol. 53v.

As Hopper states, “It was Augustine who gave the final stamp of approval to number symbolism” (Hopper, 2000, 78) and who from the fifth century related the number 3 to perfection in the Godhead (Hopper, 83). Torkesey’s purpose in choosing these two symbols was certainly to help draw his students back to the subject of their enquiries: praise of The Holy Trinity, through a deeper understanding of mensural notation and number theory – all things to the glory of God.[66]

On closer inspection of Figures 10 and 11, the reader observes that the author also includes a circle, symbol of divine perfection, around each single note at the nodes or intersections. In earlier centuries, particularly during the twelfth and thirteenth centuries, circles had been used in a similar way to denote the placement of Father, Son and Holy Spirit at the three nodes of shields and triangles one at each edge. The medieval symbol for the material world, the square, (Trezise, 205, 294) is formed each time those four nodes appear on Torkesey’s triangle. This pattern forms The Square of Opposition (See Figure 12) which occurred in the writing of Aristotle and was passed on through Boethius. Late classical and medieval authors used diagrams such as these (squares) for a variety of purposes.[67]

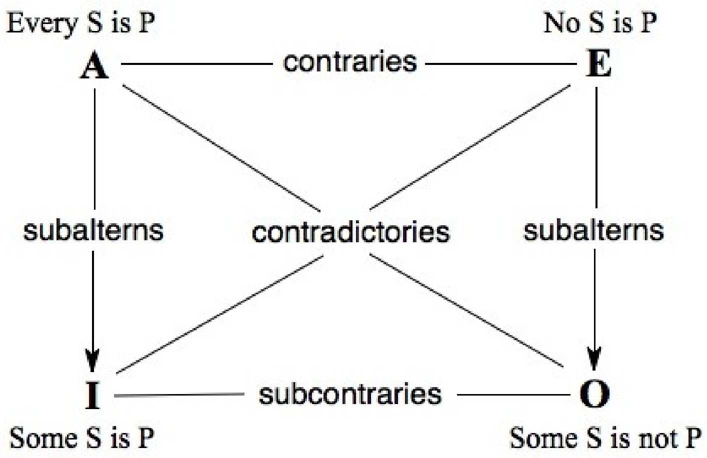

Figure 12: Square of Opposition

[66] A much later explanation of the symbolism of the triangle by Hauge regarding Fludd’s use of Torkesey’s “triangle” may reflect this earlier understanding: “The triangle…symbolised Trinity and hence God, the ultimate source of inspiration and knowledge. The triangle embodied the knowledge of proportions a knowledge which not only had a musical meaning but was also the ultimate expression of the relationship between man and God, microcosm and macrocosm” (Hauge 2008, 21).
[67] “The Traditional Square of Opposition” Retrieved June 16, 2016 from http://plato.stanford.edu/entries/square/ First published Friday, August 8, 1997; substantive revision Tuesday, August 21, 2012.

By placing the minimas (dotted and undotted) directly below the simpla (see Figures 10 and 11) Torkesey's treatise clearly demonstrates how the minim can be divided by 2 or 3 simplas. His compact graph demonstrates how these pulses continue to divide according to the mensuration in ratios of 2:1 (duple divisions on the left vertical axis N-S) or 3:1 (triple divisions to the right NW-SE on the oblique axis).

In MS Reg. lat. 1146 the author's detailed visual representation of Torkesey's text also labels the axes where 2:3 and 4:3 hemiolas occur (see Figure 11). On the horizontal axis the two pulses alongside each other in a ratio of 2:3 are labelled as sesquialtera (2 pulses "against" 3 such as in sections of music with a 2 "against" 3 feel). Pulses on the oblique NE-SW axis in a ratio of 4:3 he labels as sesquitertia (4 pulses "against" 3, for example, on Figures 10 and 11 the "first" sesquitertia displayed is equivalent to a modern quarter note against a dotted quaver note or four semiquaver notes against three semiquaver notes). By labelling his graph in this way attention is drawn to the directional and dynamic relation of pulses which form mensurations or meter. These sesquitertia and sesquialtera pulse relationships are in a mathematical sense in "opposition," being 4:3 and 2:3, hence making possible the inclusion of the Square of Opposition in the graphic representation of pulse relationships or mensurations for visualizing Torkesey's graph.

Torkesey, Classical Greek Mathematics and Christian Neo-Platonism

Torkesey probably bases the mathematical framework for his triangle and shield on the text *De institutione arithmetica* by Boethius who, as mentioned, drew heavily on the work of Nicomachus, preceding Boethius by about 450 years (Cohn, 2016c, pp. 240-41). As noted earlier, Crantor's lambda, developed by Nicomachus into a multiplication table and studied by Boethius as a table of pitch proportions also likely became Torkesey's model for his graphic representation of note proportions and their relationships.

Boethius discusses duplex and triplex proportions in *De institutione arithmetica* and provides a table of proportions (Masi, 1983, p. 114) which Torkesey likely imitated in his "triangle" and shield. The proportions listed by Boethius (see Figure 13) are those Torkesey includes in his treatise and they are listed in the same triangular shape (an isosceles right triangle) in transcriptions of Torkesey's work such as MS Reg. lat. 1146, (Figure 11). Torkesey is content to view the proportions from Boethius's writings on Nicomachus as suitable for his purpose to teach about music and a Trinitarian God.[68] Torkesey's pedagogy thus reflects both Neo-Platonist thinking and Augustinian influences.

1	2	4	6	16	32
	3	6	12	24	48
		9	18	36	72
			27	54	108
				81	162
					243

Figure 13: Boethius's table of duplex and triplex proportions in *De institutione arithmetica*

In the mid-fourteenth century, Willelmus expanded Torkesey's "triangle" through adding another multiple of numbers, the same numbers used by Nicomachus which were later used by musicians as a table of pitch proportions (see Cohn, 2016c p. 240-241). The largissima note (and largissima rest – in the corresponding table of rests) added by Willelmus are twice or three times the duration of the larga or maxima. As noted by Gilles and Reaney (1966, p. 9), Willelmus' addition to Torkesey's "triangle" meant there were now three species of breve (breve, semibreve and minim) and three species of long (largissima, larga and longa).

With 64 and 729 flanking the duple and triple outer limits of Torkesey's triangle Willelmus's symmetrical graph reveals a symbolic "closure" or perhaps more accurately "perfection" in a system of pulses (and rests) in proportion to each other. The vertices of the triangle in a sense "meet" at 216 through horizontal and vertical axes. The number 216, significant to Plato in his *The Republic*, would sit centrally at the conjunction of 1, 64 and 729 displaying a satisfying symmetry.[69]

The extent to which Pythagorean and Platonic number theory was involved in the design of these triangles is merely speculative and requires further investigation. However, I propose

[68] Although not explicitly stated by Torkesey, his choice of number symbolism, triune for God and quadruple for humanity (3+4=7) is likely influenced by his Augustinian formation. Augustine wrote of the dual nature of Christ as both divine and man and, as Hopper (2012, 84) notes in reference to Augustine*, City of God*, Book 20, 5, the number 7 is "the first number which implies totality" or in Augustine's words "the completeness of anything."

[69] See Footnote 52.

that although Torkesey likely understood that pulses could keep generating in either direction – faster or slower – he was content to display a "finite" number of pulses, mirrored in the work of Boethius, as a symbol of order in keeping with Christian Neoplatonic thinking where Classical Greek mathematics provided the foundation and framework for order and design in all things.[70]

Torkesey's astonishing clarity in his thinking and visual graphic representation of the nature of duple and triple meter as present in pulse relationships makes his graph, and even more compact shield, immediately accessible among the detailed and at times confusing literature that abounded in the medieval era on the topic of mensuration and proportion. The purpose of Torkesey's "Triangle," although not explicitly stated, like so many music treatises written by Catholic monks in the medieval era, was most likely intended as an instrument for teaching religious. Torkesey developed his treatise through an Augustinian formation, to teach students about mensuration and proportions through pulse relationships, especially in relation to an understanding of God as Triune.

Popularity of Torkesey's Work and the Chicago Manuscript 54.1

The basilica of San Pietro in Ciel d'Oro attached to the Augustinian's house in Pavia was renowned as Boethius's place of burial and it contained the relics of Augustinian patronal saints. Its scriptorium "produced several books for influential patrons" and many illuminated copies of other famous works such as *De Consolatione Philosophiae* by Boethius (Pieragostini, 2013, p. 70-1). Perhaps due to its popularity, Torkesey's "triangle" (in the equilateral version of the triangle included in *Breviarium regulare musicae* by Willelmus) was selected for inclusion in the now famous Chicago 54.1 manuscript (see Figure 14 Triangle of Willelmus in Chicago 54.1, fol. 9r. Courtesy

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[70] Cohn's discovery of Nicomachus's table of pitch proportions in 2001 (Cohn, 2016c) and Cohn's development of the ski-hill graph from around 1994 have recently led others to investigate the work of scholars whose use of mathematics and music are important to understanding more about visual representations of music and music theory more generally, such as, the ancient Greek and medieval scholars mentioned in this chapter. See Sound Colour Space http://sound-colour-space.zhdk.ch/ and the American Mathematical Association: https://www.maa.org/press/periodicals/convergence/mathematical-treasures-boethiuss-arithmetic accessed 11/1/2019.

Newberry Library, Chicago) copied by an English Friar at the Augustinian scriptorium in Pavia of San Pietro in Ciel d'Oro in 1391 during the Great Schism (1378-1415) (Pieragostini, 2013).

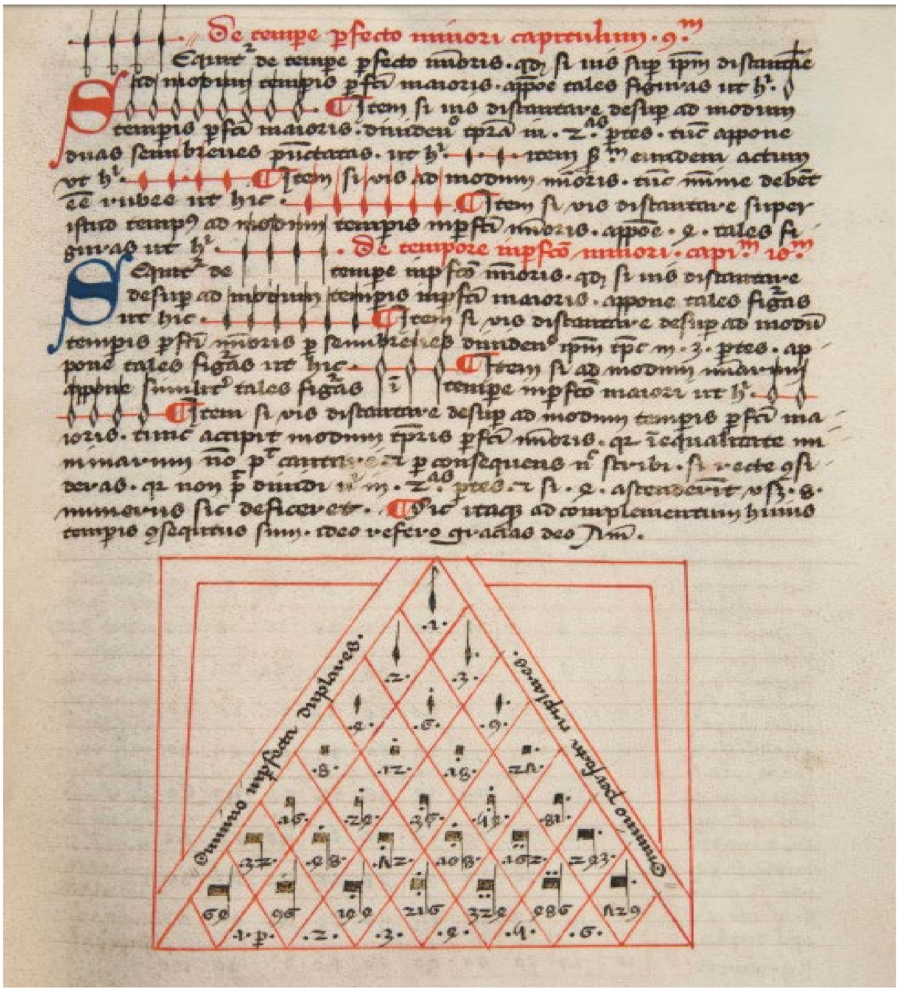


Figure 14: Triangle of Willelmus in Chicago 54.1, fol. 9r

This era saw the rise of the *studium* in monasteries to offer the highest level of education possible outside of the universities. England maintained obedience to the Roman Pope and opportunities arose at this time for an intensification of mobility, travel and appointments with Italian scholars. Included in Chicago 54.1 is the Ars subilitor song *La harpe de melodie* by Jacob Senleches in a roughly triangular harp-shaped notation; work by Marchetto da Padova; a late Ars Nova treatise *Tractatus Figurarum*, which discusses developments in notation; music treatises associated with Johannes de Muris and

Philippe de Vitry discussing mensuration; and materials on proportions and intervals (Pieragostini, p. 66).

Pieragostini (p. 73) notes the triangle from Torkesey's treatise had been considered an "obvious choice" for this influential treatise with a summation of the Western world's best-known music treatises of the time.[71] Torkesey's "Triangle" and "shield" from around 1330 lasted the test of time in this scholarly environment and was still popular around sixty years after his death within the confines of his order. The inclusion of the "trianguli" may indicate it was in use in the quadrivial studies of Augustinians in the studium in Pavia around this time.

Torkesey's two geometric graphs continued to be copied more often than the treatise itself for the next two hundred years, thus providing more evidence of its popularity, which appears to have been due to its accessible visual-symbolic qualities. Certainly, contributing to the triangular graph's importance is Torkesey's inclusion of the smaller note value the "simpla", the visual representation of hemiolas, and its ancestral mathematical roots to ancient Greece. Cohn's modern meter theory, which has brought Torkesey's "Triangle" back into the music history spotlight, allows both the ski-hill graph and Torkesey's work to be viewed, for the first time, as being pivotal achievements in the history of music.

[71] MS 54.1 Retrieved 31/7/2018 from http://digcoll.newberry.org/#/item/nby_music-4015

CHAPTER 3

A CONCEPTUAL FRAMEWORK FOR TEACHING MUSICAL METER TO SCHOOL-AGE STUDENTS THROUGH THE SKI-HILL GRAPH

In this chapter I provide a conceptual framework to support the proposal that the ski-hill graph, an instrument for analysing musical meter developed by Richard Cohn, enables school-age students to accurately report their perceptions of meter in graphic representations through mathematical music theory.[72] Materials for teaching meter through the ski-hill graph are introduced, and I briefly explain how beat-class theory (Cohn, 2018b), cyclic graphs (Cohn, 2018b) and the XronoBeat app (Milne, 2018) combined with the ski-hill graph and SkiHill app form an effective set of pedagogical tools to teach school students both meter and rhythm through mathematical music theory. Although the ski-hill graph is not the only way to describe meter, it is the most compact and efficient instrument of music theory for school-age students (K-12) to articulate – through visualisations and sonifications – graphic representations of their temporal experience of meter.

The pedagogical approach presented in this thesis for the analysis of meter, rejects the notion that the simple and compound classifications of Loulié are relevant to music other than that written in the common practice tradition of the seventeenth and eighteenth centuries, and even in that case, relevant only to a highly limited degree. In the approach advanced here, notation is only examined for clues to describe the meter formed by the listener.

[72] I noted in the Report of the *Understanding and Teaching Meter Survey* Project number: 2017/055 (see Appendices A and B) *Understanding and Teaching Meter Survey Report* (Calilhanna, Unpublished 2017), that mathematics was implicit in the language used by teachers to explain and teach meter but rarely taught as being explicitly connected to mathematics. The data suggested that there was a need for the inclusion in classroom music textbooks and pedagogical materials for meter theory and pedagogy which assisted teachers to express this music-mathematics connection with clarity and accuracy. A total of forty-seven music education practitioners representative of a range of educational bodies, programs and roles responded to the survey including tertiary music education lecturers, primary and secondary classroom music teachers, and qualified music studio teachers. The aim of the project was to develop a snapshot of music educators' thinking and practice in relation to meter. This exercise was conducted specifically as part of the research project presented in this thesis. See also Calilhanna and Webb (2018, unpublished).

The definition of meter used throughout this chapter is "an inclusionally related set of distinct, notionally isochronous time-point sets" (Cohn, 2018a). In conjunction with this definition Cohn acknowledges that musical meter is located in the listener who responds to sound through the entrainment of pulses in the body and projection of pulses in the mind.[73]

Part 1: Introduction

The following introductory section includes an overview of the organisation of materials throughout this chapter and brief contextual information for the conceptual framework provided to teach meter with school-age students.

Organisation of Materials

This chapter is arranged in three sections: Part One, Introduction; Part Two, Establishing a Conceptual Framework for Teaching Meter; and Part Three, Foundational Exercises to Introduce New Understandings of Meter with School Students.

In this chapter, I describe the concepts I find contribute to students' learning of meter with new understandings. This is presented, in a sense, as scaffolding for teachers so that they can incorporate and develop them for their own teaching. It is generally best to establish some groundwork and general guidelines about how meter is understood with new understandings before students use the new instruments of music theory discussed in this chapter.

The concepts presented in this chapter can best be understood as providing a framework for an "approach" rather than a "method" for teaching meter so as to avoid unnecessary rigidity of procedure. In Chapter 4 I provide a step-by-step approach for teaching meter according to the concepts outlined in this chapter but again they are only one of many ways to work within this conceptual framework.

[73] The materials provided in this chapter and Chapter 4 are those I have taught with school-age students since 2015 and with undergraduate students studying to become classroom music teachers.

Context for Teaching Meter With New Understandings in Schools

As reflected in the Review of Literature Chapter 1 of this thesis, classifications of meter in prominent classroom music textbooks return at some point to the meter signature and/or notation to identify meter. Thus, as Cohn (2015d) also points out, we learn almost nothing about meter in our music education. Learning meter using notational categorizations can be intimidating for students who intuitively hear legitimate pulses which form meter, but which do not "fit" notation-based categorizations. The underlying message projected in these instances can infer that the student is wrong, does not know music and is "unmusical."

Part 2: Establishing a Conceptual Framework for Teaching Meter

The following section provides music teachers of school-age students with the concepts necessary for teaching meter with truer understandings, through the ski-hill graph. It sets out a conceptual framework for teaching meter through the ski-hill graph with school-age students. I have chosen to provide a conceptual framework in order to avoid the situation where a ski-hill graph is presented by an educator as representing the only hearing or the "correct" way to hear meter in a particular piece, without the students first mapping their own experience of meter on a ski-hill graph.

Students need to be steered at times towards articulating accurate accounts of their experience of meter, but to tell the students what pulses to hear prior to their own listening would be to negate all pedagogical value of this approach to understanding meter. The motivation for providing such a conceptual framework is to assist teachers in developing adequate knowledge and understanding of meter to enable them to establish the ski-hill graph in their teaching of meter, steer their students when necessary, and develop their own materials from those presented in this chapter.

From Ski-Hill Graph to Cyclic Graphs and Beat-Class Theory

Since experiencing duple and triple meter begins with an intuitive process of "counting" the two small prime numbers 2 and 3, mapping pulses on the ski-hill graph requires objective graphing of subjective embodied mathematical relations and knowledge. It follows that to teach meter without acknowledging the important role mathematics plays in the experience and understanding of meter and music in general is to deny students access to a true understanding of the mathematical foundations of what they are hearing. The approach to teaching and learning meter presented here moves meter pedagogy and theory away from assumptions about how music "feels" to the listener on the basis of meter signature and notation, towards accurate mathematical depictions of certain aspects of sound relations relating to time organisation.

Another appealing aspect of learning meter through Cohn's ski-hill graph is the ease with which pulses can then be mapped onto other forms of metrical representation such as a cyclic graph as in the XronoBeat app (Milne 2018). Unlike traditional approaches to learning meter and recent computerized visualizations of rhythm, such as, the popular Groove Pizza app (Hein and November, 2016), Cohn's beat-class theory, on the other hand, uses modular arithmetic to enable students to articulate their observations about the patterns that form on cyclic graphs from their mapping of rhythms as points and polygons from listening to music.

Unlike other computer-based models of a cyclic graph, the XronoBeat app (Milne, 2018), allows students to experience in a perceptually immediate way, the theoretical information about music and mathematics they gather from their observations of cyclic universes through applying beat-class theory to a musical experience (Hilton, Calilhanna, Milne 2018 p.220). In other words, through beat class theory and cyclic graphs, students learn more about the music and mathematics they embody through their experience of cycles when listening to, performing or composing music.

To do this, students map rhythms to a circle which has numbered points distributed evenly around the circumference. For instance, a cyclic graph with 12 points is known as C12, and is much like a clockface, although the point at the top is labelled with a "zero" rather than a 12 thus displaying the numbers 0-11 (Cohn, 2017).

In this way, students learn about the beat-class of each cycle, for instance, the beat-class of C6 consists of 64 sets: 6 singles (6,1), 15 pairs (6, 2), 20 triplets (6, 20) and so on, which can then be explored through other practical musical activities. To learn about C6 a lesson might involve studying music notated in a 6/8 meter signature. Students can explore two meters in one measure where six quavers [012345] can be divided into two [03] for triple meter, and where the six quavers can be divided into pairs [024] for duple meter (Cohn, 2018b). Cohn

(2018b p. 132) also provides a useful table of the beat-class for each cycle (the sets of elements and counts (c, d) or cardinalities).

It is also important early in one's music training to introduce hemiolas in ratios of at least 3:2, 6:4, and 4:3 and other categories of meter such as quasi meters (mixed or additive meters also known as non-isochronous meters) so as to avoid problems with counting, multiplication, and division later. As an example, Cohn (p. 137) notes that sets for C6 (see above) can be superimposed as $\{03\} \cup \{024\} = \{0234\}$ to form a simple hemiola. The materials in Chapter 4 demonstrate how hemiolas can be taught effectively with school-age students through the ski-hill graph, cyclic graphs and beat-class theory.

Thus, through learning about the beat-class of each cycle, using modular arithmetic, students learn about isochronous and non-isochronous rhythms and polymeters to understand more deeply about why the music they listen to has a certain 'feel'. Through embedding an accurate understanding of meter with the aid of emphasising mathematical relationships, this approach to teaching and learning encourages a personal, subjective, intuitive and intellectual-mathematical encounter with meter, and music more generally.

Cohn (2018b p.128, 132) also notes, that beat-class theory, beginning with small cyclic universes, offers the student a means to explore not only the rhythmic/metric qualities of the smaller cycles but the chordal/scalar domain of a chromatic universe before exploring the 4096 distinct combinations of C12! (p. 128). Thus teaching mathematics with music and music with mathematics (Cohn, 2018b; Hamilton et al, 2018; Hilton, Calilhanna, Milne, 2018; Milne and Calilhanna, 2019. Manuscript submitted for publication) is now possible using Cohn's modern meter theory with school-age students.

Cyclicity and Hierarchy

Students explore cycles intuitively each time they perform or listen to music but through using ski-hill graphs, cyclic graphs and beat-class theory students become more aware of their musical intuitions and their "deeply embodied knowledge" (Cohn, 2018b). Through using basic mathematics to map pairs of pulses in a relation of inclusion in either 2s or 3s (divisions, ratios, proportions) on the ski-hill graph, students are measuring and calculating intrinsic musical qualities. These cyclical and hierarchical characteristics contribute to the formation of meter by the listener when experiencing meter's most important mathematical property: evenness.

For a set of pulses forming meter to possess evenness they need to be in a relation of inclusion and the distance or duration between the pulses identical – the mathematical qualities found in all meter (Cohn, 2017). Thus, each meter consists of evenly spaced sets of pairs of pulses in a relation of inclusion (minimal meters) which repeat in a cyclical process through entrainment in the body and projection by the mind. Each metric level is made up of pairs of pulses in 2:1 or 3:1 ratios which also describes their periodicity in relation to other pulses. These sets of pulses in relation to other sets of pulses in inclusion relation (deep meter) together form a cyclically-oriented hierarchy of pulses which form meter(s) in the listener.

Pedagogical Potency of Using Graphic Representations to Study Meter

Without a means to examine its dynamic operations in music, details about meter easily go unnoticed. A key strength of graphically representing meter through the ski-hill graph is its ability to "freeze [moments] in time" allowing us to then make observations. In his discussion, "Why visualize music?" (Cohn, 2016c) Cohn states,

> Translating musical events into symbols, musical images place those events and their relations before our eyes, and freeze them in place. (p. 238)

Thus, unlike the traditional understanding of meter presented in the classroom, through the ski-hill graph students are empowered to discuss their own representations of the metric structure (metric space) formed in the listener through the dynamic relations between meter(s) – or metric pathways – equally alongside tonality – pitch and key relations – in their music studies.

Analysing meter through the ski-hill graph and SkiHill app requires students to articulate their observations about meter through graphic representations using choices of abstract symbols, thus, sound to symbol. These symbols are presented in visualizations (and sonifications with the SkiHill app) such as traditional notation, fractions, numbers, polygons (cyclic graphs), and sonifications for each metric pulse and quasi pulse the students hears.

Through reporting pulses from listening, students map metric pathways in order to see and hear distances between meters that represent the metric structure (metric space). Graphic representations such as those formed on the ski-hill graph are comparable to the graphic representations possible with the analysis of tonal structures such as through a circle of fifths or the Tonnetz. Thus, analysing meter through the ski-hill graph and SkiHill app empowers students to give meter the prominence it deserves in order to increase one's understanding of the fundamental properties of music.

Traditional staff notation lacks the necessary graphic qualities to allow students to see structures preserved when meter forms to the hearing. On the other hand, mapping pulses on the ski-hill graph enables students to examine structures such as metric pathways, distances between meters, relations between pulses in inclusion relation, and those not in inclusion such as when hemiolas occur. In this way students map the metric space of a piece when they report their hearing of meter on the ski-hill graph giving recognition to music's deepest metric structures that are otherwise ignored or unrecognizable in traditional linear notation.

Pulses thus mapped can then be recorded in animations on cyclic graphs to make either a single rhythm with equal distances, or more than one rhythm to make polymeters. In this way students see spatial representation(s) of their temporal experience through polygon(s) which visualise and sonify the periodicity of each pulse in relation to each other (see Hilton, Calilhanna, Milne 2018 p. 211-213 and 219-220).

On the SkiHill app geometric polygons are sonified for each pulse, demonstrating the mathematical relation of hierarchy and inclusion as they cycle to form meter in evenness. Thus, through the ski-hill graph and app, meter can also be explained through graphic representations using abstract symbols to engage the listener's cognitive awareness of spatial intuitions (see also the section on Metric Equivalence in this chapter and More on Metric Equivalence in Chapter 4).

Meter, Subjectivity, and Pedagogical Implications

The subjective experience of meter involves engagement of the imagination, as not all pulses experienced are actually performed (Brochard et al, 2003). The pedagogical implications of understanding meter as a subjective and embodied experience which occurs as a result of listening to music, implies that students no longer need to make observations which propagate the obsolete notion that meter is located in the notation (in the measures and meter signature). Rather than understanding the meter signature as defining what pulses they are to concentrate on hearing students can articulate evidence of their personal hearing of meter instead of meter as heard by other listeners such as the teacher, students, or an author of a textbook or method.

Pedagogical Context of the Ski-Hill Graph and SkiHill App as Instruments of Music Theory

In a pedagogical setting with school-age students, the process of analysing meter through the ski-hill graph and SkiHill app can be applied to their studies of performance, composition, analysis, conducting, movement, and language. In other words, the ski-hill graph and SkiHill app were not developed to be standalone music theory tools to gather data without application to wider music-making, neither were they designed to "tell students" what pulses to hear. Rather they were created as instruments of music theory to analyse meter and to inform a wide variety of practice including, but not limited to, the discipline of music.

The Computer as "Mediator"

Through teaching meter with the SkiHill app the computer becomes, in a sense, a "mediator" between information and the student's subjective and objective knowledge. Instead of students rapidly absorbing information from the computer, through being told either directly or indirectly what to hear in a passive process that does not require learning, students use the computer's information to create new and objective knowledge through graphing meter. This engaging and empowering learning process occurs because of the unique ability of the ski-hill graph to connect a student's temporal experience with the computer via the SkiHill app. Thus, students learn more deeply about meter and mathematics as subjectively experienced.

Metric Equivalence

One of the ski-hill graph's unique strengths as an instrument to analyse meter is its ability to graphically represent metric equivalence, a mathematical property of meter which listeners embody when hearing meter. Metric equivalence can be said to be in play when the meter occurring in a piece can be notated in at least two different meter signatures. For instance, during early exercises to reinforce the notion of metric equivalence, ask students what meter signature(s) they think the pieces could be notated in. In exercises where the music is heard as duple meter students might answer, for example, 2/4, 4/4, or 2/2 which would all be correct, and if the music they hear is pure triple they might say meter signatures such as 9/8 or 3/4 depending on the piece. For music where duple and triple meter is present to the hearing students might answer any of the meter signatures mentioned above and sometimes many more.

Teaching metric equivalence through the ski-hill graph helps students to comprehend why notation is arbitrary in the sense that there are often multiple ways a piece of music could be notated to achieve the same aural outcome. Another possible exercise for students would be to listen to and perform well-known pieces, such as the Christmas carol *Silent Night,* through using different notations such as 3/4 and 6/8, and ask them if they can hear any difference in the meter. Most people hear the meter in these arrangements as identical yet when they look at the notations they realise that the 3/4 arrangement would traditionally be labelled as simple triple meter and the 6/8 arrangement would be labelled compound duple meter on, say, a written music exam.

Using a traditional notation-based approach to teach meter, students would normally be required to answer "simple duple" if asked to analyse "meter" for any notations of 2/4 even though the quarter note may be divided into three. And students would still be required to answer "simple triple" for pieces notated in 3/4 even if the quarter notes were all divided into three and there were levels of duple hypermeter no matter how they are audiated.

Confusion arises for the student and teacher about how to categorize "meter" in these "tricky" situations and much excellent music for teaching and/or in-depth discussion of meter is usually avoided in teaching materials. Thus, instead of rejecting an outmoded theory of meter for music other than that composed in the tradition of common practice era music, Loulié's system is usually perpetuated in music classrooms undermining valid teaching and learning of meter as audiated.

Because meter is a temporal experience, the mathematical property of metric equivalence proves – on perceptual, cognitive, and theoretical grounds – that theories of meter which are based on understanding meter as the notation, cannot enable students to learn deeply about meter. Chapter 4 sets out a step by step approach to teaching about metric equivalence and demonstrates how the ski-hill graph is ideal for graphically representing the same meter from different notations of the same piece, how notation is arbitrary, and how, as Cohn explains, meter is independent of time signatures and measures.

Cohn's definition of meter and classifications of meter used throughout this thesis solve the problem of classifying meter. This is possible because Cohn's classifications define and classify what the listener hears and the ski-hill graphs, cyclic graphs, and beat-class theory represent meter as experienced rather than classifying according to what the meter signature and notation dictate.

Part 3: Foundational Exercises to Introduce New Understandings of Meter with School Students

This section continues building a conceptual framework from which to teach meter through the ski-hill graph but also includes foundational exercises so as to introduce students to new understandings of meter. For the benefit of other teachers of school-age students I have presented the concepts in a sequence through which to learn the theoretical underpinnings of new understandings of meter. The following section provides exercises that initiate the process of understanding that meter is not located in the notation, meter signature, or measures.

"Notionally Constant Points in Time"

School-age music listeners generally intuit that music exists in a dimension of time – music comes and goes in a continuous flow through time – a "continuous spectrum of time with notionally constant points <u>in</u> time" (Cohn, 2015e). "Time" can be represented by a horizontal line with arrows at either end to indicate its temporal quality and vertical lines can represent, divisions into spans of time (Cohn, 2015). See Figure 15:

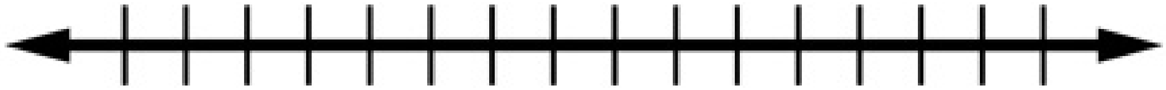


Figure 15: "Time" represented graphically as a horizontal line

To explain what is meant by "notionally constant points in time," ask students to listen to music which is characterised by rubato and which will likely form one of the following two meters for the listener: pure duple meter where all adjacent pulses pairs are in inclusion relations of 2:1, or pure triple meter for the listener where all of the pulses are in inclusion relations of 3:1. In this way students have an opportunity to experience both types of meter separately before being asked to identify them together in the same piece.

Repertoire chosen would be suitable for the class and include both familiar and unfamiliar pieces from recordings and through performing the music for them. Another way is to improvise on a piano or other instrument, music that is close to that with which students are familiar so that they can practice their new skills and understandings of meter. Improvising can help to solve the matter of finding appropriate materials; for instance, pieces in pure duple meter are numerous but pieces in pure triple are more difficult to find.

The "Beat"

The following points are a summary of how the concept of "beat" can be approached when teaching meter with new understandings:

1. Teach meter without referring to "the beat."
2. Emphasise that there will be many pulses to hear.
3. Explain that because all metric pulses contribute to meter, they are equal in importance and at different times serve different functions. If necessary, explain how the general textbook approach often gives preference to one or two "beats" as primary and secondary (such as a downbeat pulse and one or two faster) with the time signature representing only those two beats that contribute to a notation-based understanding of meter.

4. The word "pulse" in relation to meter is preferable to beat because in this approach what is being described is an onset (or timepoint) experienced in the listener as a response to sound. "Pulse" is more closely related to an organic understanding of an onset and more closely linked to a physical response to music: pulsations that are so fleeting that mostly go undetected and unnoticed without the listener stopping to observe what just happened.
5. Ask students to synchronize with one of the pulses they hear for the music they hear (see previous section) through some kind of movement they are comfortable with. Movement might include clapping, tapping, or clicking in a horizontal line in front of them, moving their arm across the body with the hand indicating the pulse onsets, walking to the pulse they hear, and/or playing a percussion instrument to the pulse.

Inclusion Relation

Next, introduce the term "inclusion" following from Cohn's (2018a) definition of meter "An inclusionally related set of distinct, notionally isochronous time-point sets."

1. Students should pair the pulse they chose with another pulse which "feels" most natural and intuitive. In most cases students choose a pulse twice as fast or twice as slow in a ratio of 2:1, if the music is audiated as pure duple meter, or, if the music is audiated as pure triple meter, they usually choose a pulse in a ratio of 3:1 which is three times as fast or three times as slow. These two pulses are in a "relation of inclusion" and a simple practical exercise such as described can be achieved in a matter of minutes depending on the age group and class.
2. Depending upon the age and ability of the students, some will entrain to and project three or more pulses at the same time through movement and/or sound, which is to be encouraged.
3. Ask the students to think about and articulate verbally what material in the music it was that motivated them to group the pulses in the way they chose such, as repetition or parallelism (Lerdahl and Jackendoff, 1983), melodic contour, change of harmony, and so forth. Mid to upper primary age students are capable of comprehending the words "entrain" and "project."

4. Students may be very inexperienced or have no musical training at all and so answers such as “I feel this in 2s” or “I can count in 3s” are also excellent answers which are to be encouraged at this early stage of learning meter.
5. Students are then asked to think about why they chose their original pulse to help them realise that when listening to metric music we often choose one pulse to tap our foot to or bob our heads to (entrain to and project) but that we do this because we “feel” “2s” or “3s” (duple meter or triple meter).
6. Normally students are not aware that when we bob our head or tap our foot we are already intuitively grouping in sets that one pulse with at least one other pulse in a ratio of either 2:1 or 3:1 to form a minimal meter.
7. Explain that, although we do this grouping, these two pulses, do not take on any special status where meter is concerned other than to signify one level of meter and that in most music there are many pairs of pulses and levels of meter.
8. If students are already familiar with traditional notation, explain that we often learn only about the downbeat pulse and one faster because meter is usually taught as understood through the time signature and notation which only represents these two or three “pulses” in the notation. In reality, however, all sets of pulses in inclusion relation form a hierarchy as part of our temporal experience of meter and that is why we learn about all of them in this approach.
9. Through this exercise, students learn that if you stop to examine what is happening when we tap our foot to music, there is “counting” going on (embodied mathematics), otherwise we would not experience “2s” and “3s” to form duple or triple meter.
10. In later exercises, students will experience “2s” and “3s” at the same time to form duple meter, triple meter, and polymeters with hemiolas (where inclusion with all pulses does not occur).

In Chapter 4, I will explain why Cohn’s ski-hill, cyclic graphs, and beat-class theory are excellent instruments of music theory to “freeze in time” these observations about our often fleeting temporal experiences in order to examine them and how teaching meter with school students can incorporate these instruments.

Movement

A simple exercise of listening to music while sitting or standing as still as possible helps students observe that meter can be experienced without any perceivable outward movement of the body. This knowledge helps students understand that meter is not dependent on voluntary body movement, rather meter is initially an activity of the mind (Cohn, 2018a). While movement is to be encouraged as a part of experiencing meter it should be put into perspective, especially as there may be students who are uncomfortable about movement in school music classes.

These students should be encouraged to use small gestures and/or instruments such as drums; over time they may warm to the idea of more movement. Generally, school students enjoy using percussion instruments, such as stackable conga drums, which make a satisfying sound and can be easily stored away after class. In this way, students express themselves with the focus shifted to the sound the drum makes and their internal subjective experience of sound rather than dealing with a potentially distracting focus on body movement.

"Notionally Isochronous Time Point Sets"

When listening to music with rubato students observe – as they entrain to their pulse(s) – that the distances between the timespans "lengthen" or "shorten" on their imagined timeline according to the onset(s) of their chosen pulse(s). Students often notice that where they anticipated and expected to place their next pulse slightly changes. Students remark that they notice the effect rubato has on onsets occurs both in their imagination and in the gesture they make with their body movement from entraining to their pulse(s) (embodiment of meter). Through discussing this process students describe how the distances between timespans widen as music becomes slower and how the timespans become closer as the music becomes faster, see Figures 16 and 17:

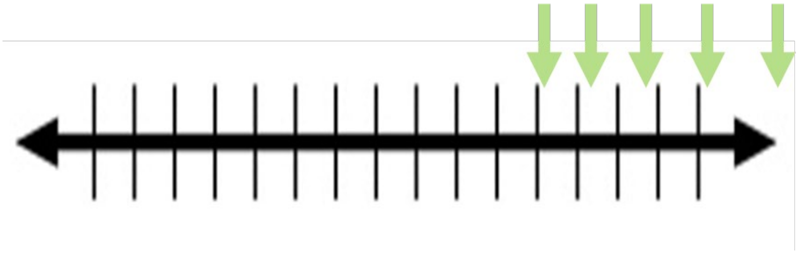


Figure 16: Timespans widen (green arrows) as music becomes slower

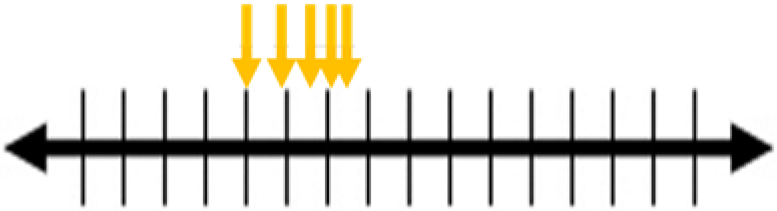


Figure 17: Timespans narrow (yellow arrows) as music becomes faster

Through this exercise students observe that as tempo fluctuates it is not pulse(s) that change rather it is the elastic quality of the timespans which are "notionally constant" (Cohn, 2018a). Everything shifts and stays in place hierarchically even as the gaps between time-points widen or shorten. The performer's onsets of pulses, in a sense, "stretch" the timespans during performance of music yet, in another sense, these timespans remain "constant" or grid-like because of memory and act as "reference points" to the continued formation of sets by the listener.

Students soon learn that meter is formed in the listener through entraining and projecting sets of pulses (time points) which themselves are sets of sets, hence, a hierarchy of 2s and 3s in inclusion relation which are measurable by spans of time. Onsets (in this case pulses in inclusion relation) are placed around the expected or anticipated timespans generated in the listener but at times these sets "stretch" or "shrink" timespans according to the onset of the pulses by the performer. Sometimes this exercise takes place during part of a music lesson and lasts only 5 or 10 minutes because most students grasp intuitively that we process meter internally.

Slides for Teaching Notionally Isochronous Time Point Sets

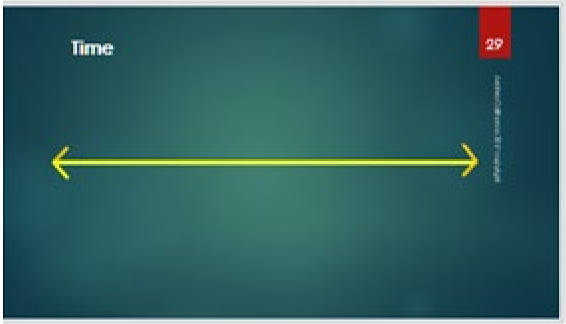
The following teaching materials are part of a set of slides I use in a music education module "Music Theory – Focus on Meter" at the Sydney Conservatorium of Music, The University of Sydney. The materials prepare undergraduate students studying to become classroom music teachers in schools to teach meter with ski-hill graphs, cyclic graphs, and beat-class set theory.

The slides in Figure 18 introduce students to visualizations of the operations of notionally isochronous time point sets as described above. The slides represent time, spans of time, time points, inclusion, and the concept of notionally isochronous time-point sets demonstrated through showing the "stretchy" nature of timespans when music becomes slower and faster.

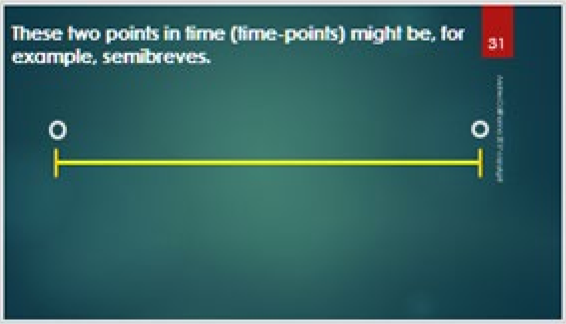
Further to the presentation of the materials each slide can then represent metric equivalence through replacing Xs or dots for each note illustrated. In this way, students are taught that many different notations are possible but, in order to communicate music, choices are made about notation(s) according to, for example, the purpose of the music: playability, saleability, publisher expectations etc.

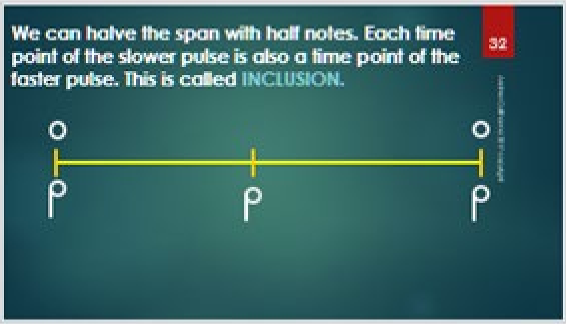
The slides marked 31, 32 and 34 normally include an animation to show how timespans shift according to onsets by the performer. Other than to slow down or speed up the tempo of music performers often use onsets which result in very uneven spacing between timespans for expressive purposes, such as pausing on an onset marked with a fermata represented in slide 39. Yet, as illustrated in slide 39, the performer maintains the metric structure from pulses remaining in inclusion relation but "stretched out" thus illustrating notionally isochronous time-point sets. Slide 88 illustrates where a timespan is both halved, then tripled, and halved again representing a 3/4 notation with all pulses in a hierarchical relation of inclusion. (Read the following slides from the left to the right.)[74] See Figure 18: Instructional slides to introduce students to visualizations of the operations of notionally isochronous time point sets:[75]

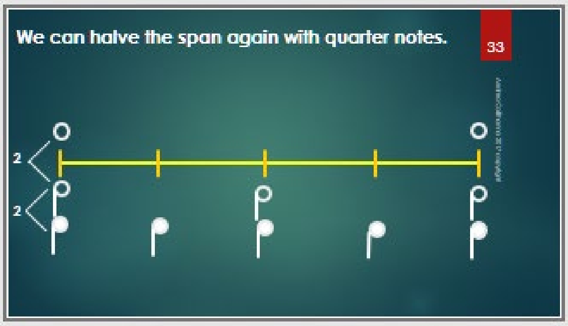
[74] Throughout this thesis I consistently use the terms half note, quarter note, etc., which are not in common usage in Australia. Use of these terms make more sense in this context but see pp.105-107 and pp.139-140 where significant inadequacies in this system are problematized.
[75] Note that the red tabs on slides are only decorative and should be ignored throughout this chapter.

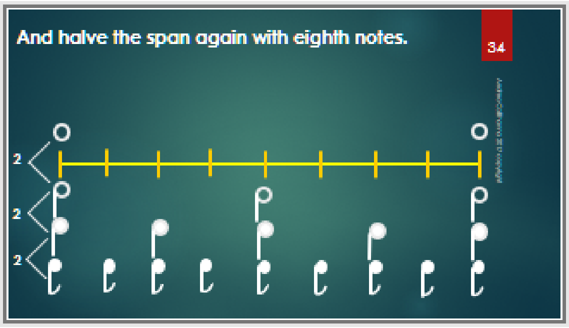

Time
29

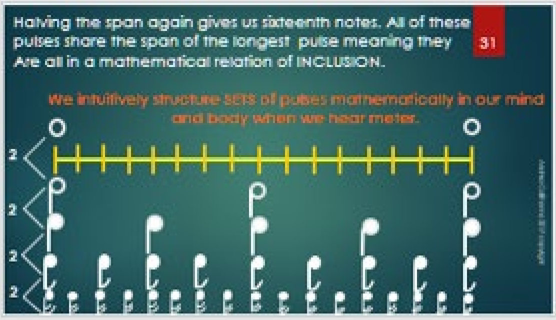

Span of time
30

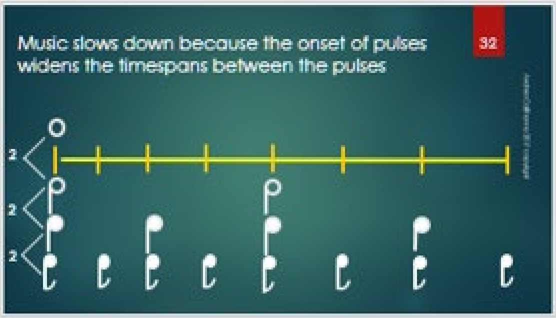

These two points in time (time-points) might be, for example, semibreves.
31

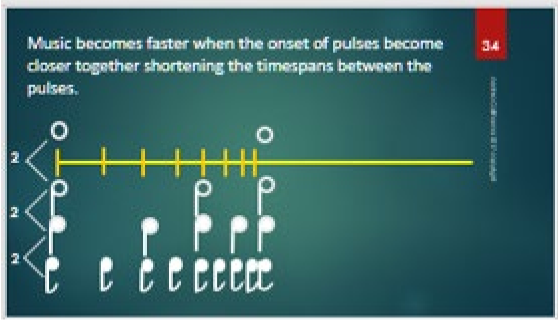

We can halve the span with half notes. Each time point of the slower pulse is also a time point of the faster pulse. This is called INCLUSION.
32

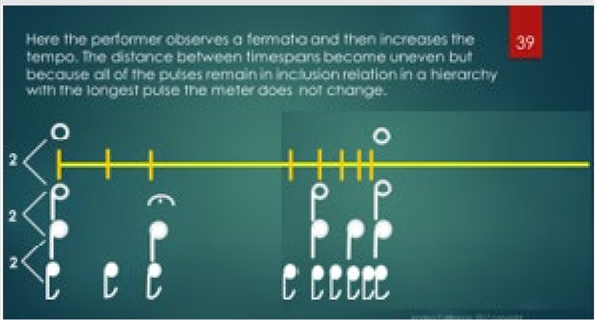

We can halve the span again with quarter notes.
33


And halve the span again with eighth notes.
34


Halving the span again gives us sixteenth notes. All of these pulses share the span of the longest pulse meaning they Are all in a mathematical relation of INCLUSION.
31
We intuitively structure SETS of pulses mathematically in our mind and body when we hear meter.


Music slows down because the onset of pulses widens the timespans between the pulses
32


Music becomes faster when the onset of pulses become closer together shortening the timespans between the pulses.
34


Here the performer observes a fermata and then increases the tempo. The distance between timespans become uneven but because all of the pulses remain in inclusion relation in a hierarchy with the longest pulse the meter does not change.
39

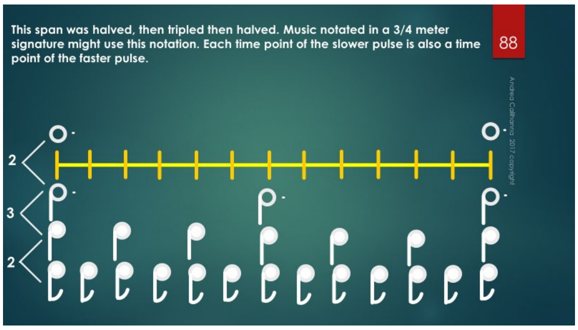


Figure 18: Instructional slides

Presenting students with a visualisation of a recording of a live performance on software such as Audacity also helps them observe that even where there is no deliberate change in tempo by a human performer, graphing indicates humans rarely place onsets on isochronous timepoints to perform sets of perfectly even pulses. Cohn's term "notionally isochronous" acknowledges the reality that humans onset "stretchy" or approximately even time-point sets when performing music and/or listening to music where we usually "imagine" sets as (more or less, i.e. notionally) isochronous.

Meter, Notation, Approximation, and Interpretation in Performance

When relating the experience of meter and notionally constant time points to traditional notation, primary school-age students observe that when notating music composers place pulses which are "heard" only approximately at the places we expect them to occur in relation to timespans according to the changes in tempo. Students learn that traditional notation does not reflect or explain the idea of notionally constant points in time or the "stretchy" nature of timespans in any great detail. Instead, horizontal divisions on the staff are not given equal space according to pulse, for example, a minim will not, in most cases, occupy the exact same amount of lateral space as two crotchets. Students soon realise that they, as the performer, decide where to place the onsets approximated from reading the notation. They use the kinesthetic gesture of onset for expressive purposes during the process of experiencing meter in performance.

“Swung” Eighth Notes

Many young musicians play music with “swung” eighth notes in their early examination repertoire. Teaching them meter theory through practical exercises and explanations as just described offers solid preparation for the style. Students learn that jazz musicians often exploit the idea of approximation in their performances to achieve a “swung” feel. The eighth notes are performed in triple meter with the quarter note yet, when notated, the eighth notes are usually written in “straight” eighth notes not tripleted eighth notes in duple meter with the quarter note. In other words, the notation of “swung” eighth notes in this way is not only approximate nor does it reflect the audiated experience by the listener.

These types of exercises help students solve problems when interpreting music for performance, especially at home between lessons. They soon realise that subtle changes in performance are made possible through their own intuition, counting, hearing, and interpretation, and not through being “told” what to do by the notation. For many students this will be the first time they understand that notation is an abstract form of representation and that notation oftentimes has very little relation to what is heard in performance.

From these practical exercises in music theory students learn that notation is not where meter is located. Rather, as audiating listeners, we ourselves generate meter when we listen to, imagine, analyse, perform, compose, conduct, and move to music.

Hyperpulse and Hypermeter

Hyperpulses, or pulses which are in inclusion relation with two or more downbeat pulses in duple meter or triple meter, are not represented by any meter signatures using traditional notation, yet they are often reported by students. Most students will be unfamiliar with the terms, concept and practice of hyperpulses and hypermeter, and for this reason it is important to teach school-age students about hyperpulses and hypermeter from the beginning of their training in meter, rhythm and note values.

As performers become aware of the importance of “playing to” the hyperpulse at times, they learn how composers often exploit the expressive value of hypermeter in their work, and conductors will shape their interpretations of performances around the hypermetric pulse where appropriate. For school-age students, awareness of hyperpulses and hypermeter often results in increased confidence in performance through attaining a more musically satisfying performance result. This occurs because composers often write music which, if performed, for example, to the downbeat and not to the hyperpulse will likely sound "choppy" and less satisfying musically. In Chapter 4, I explain how teaching students about the hyperpulse and hypermeter through learning meter with new understandings can also help students to improve their performances.

This chapter provided a conceptual framework and foundational materials through which to view teaching meter with new and increased understanding. The instructional curriculum model in Chapter 4 demonstrates how meter can be taught through the ski-hill graph with school-age students. The curriculum model shows how meter can be understood as "experienced," and taught with a focus on "sound rather than notation" using graphic representations of meter through mathematical music theory.

Derived from Richard Cohn's work on and approaches to meter theory, the curriculum model in Chapter 4 presents new understandings of meter firmly grounded in recent meter research. In addition, the curriculum model addresses findings and recommendations from the *Understanding and Teaching Meter Survey Report* (Calilhanna, 2017 unpublished) which identified a number of areas in need of attention for teachers.[76] Calilhanna (2017 unpublished, p. 18) offered the following recommendations:

- Development and implementation of music curriculum materials and instruments of music theory to analyse meter as a temporal experience rather than as understood as notation in K – Tertiary music education to teach meter in schools, universities and studios.
- Training in the approach to learning about meter by Professor Richard Cohn (Yale University) which involves graphic representations of meter through mathematical music theory.
- Musical meter curriculum materials based on recent research to promote whole-student education through valuing their individual temporal experience and the promotion of student-centred learning.

[76] *Understanding and Teaching Meter Survey Report* (Calilhanna, Unpublished 2017): Project number: 2017/055. See Appendices A and B. See also Calilhanna and Webb (2018, unpublished).

- Training for teachers and students to encompass a modern understanding of meter which includes awareness and skills to teach about the mathematics embedded in music in both notated and non-notated forms and embodied in the listener.
- Meter to be taught equally alongside tonality to address music education in the two systems through which we process music.
- Implementation of tools to analyse meter such as the ski-hill graph, SkiHill app, cyclic graphs, XronoBeat app, and beat-class theory in all music genres.
- Development of resources such as YouTube videos e.g. “March V Waltz - A Short Intro to Meter and Ski-Hill Graphs” Music Corner Breve by David Kulma: https://www.youtube.com/watch?v=0yxba7yoMSk
- Use of software to include the SkiHill app and XronoBeat new apps designed by Dr Andrew Milne, which empower students and teachers to articulate evidence of meter as a temporal experience.

CHAPTER 4

TEACHING METER TO SCHOOL-AGE STUDENTS

All of the materials presented in this chapter have been developed and successfully trialled with school-age students as well as preservice music teachers. Many of the figures in the chapter are derived from slides I have used in my teaching. The majority of the materials focus on isochronous meter: duple, triple, and duple and triple meter where there is metric consonance and no conflicting pulses because all pulses are in a relation of inclusion. However, students may encounter metric dissonance in music they are required to study for exams, such as through simple hemiolas and metric displacement (syncopation) and I also provide examples of music and teaching strategies where this occurs.

As outlined in Chapter 1, the absence of the understanding of meter *as experienced* rather than as notation from both music textbooks and university courses explains why school teachers avoid treating meter to the same extent as tonality. The materials in this chapter demonstrate how a music curriculum can accord equal weighting to the three key areas of music learning: Performance, Theory and Composition. Thus, it is intended that the materials in this chapter based on Cohn's approach to teaching meter will contribute to enriching what is arguably the most undernourished field in primary and secondary school music curricula: music theory.

This chapter is divided into two parts: part one focuses on instances of metric consonance while part two considers ways of dealing with direct and indirect metric dissonance (hemiolas etc).

Part 1: Classifying Meter

Introducing the Ski-Hill Graph

Understanding the different types of meter and their classifications is critical to teaching and learning meter with new understandings. When students have demonstrated their comprehension of meter's quintessential property of pulses experienced in a relation of inclusion in ratios of 2:1 or 3:1 in a hierarchy through simple practical activities I instruct them on the function and use of the ski-hill graph as an instrument of music theory to map pulses that form meter as experienced.

Mapping Pulses to Ski-Hill Graphs

When students (ages 7 to adults) first map pulses they hear to the ski-hill graph, ask them to use "empty" nodes to demonstrate the fundamental quality of musical meter as either duple or triple and to introduce the concept of metric equivalence. Teachers can have students map traditional notation in the nodes first, instead of leaving it blank if they need to work with the symbols they are familiar with before applying their new knowledge to another abstract form, but this is usually the exception rather than the norm.

As the following materials demonstrate, when graphically representing meter on the ski-hill graph students map to nodes all of the adjacent pulses in inclusion relation they hear forming duple and/or triple meter in pieces of music.[77] In this way students become familiar with graphing duple meter to the left pathways and adjacent pulses in inclusion relation they hear forming triple meter to the right pathways (see Figure 19).

Ideally, the following preliminary exercises should be taught when tonality is also being discussed and can take place in as little as five minutes or over a lesson depending on the age group and experience of students. Teachers should encourage students to use some kind of movement such as clapping or tapping as they listen to help identify and map the pulses they hear.

[77] For pulses to be adjacent they are connected by an edge on a ski-hill graph.

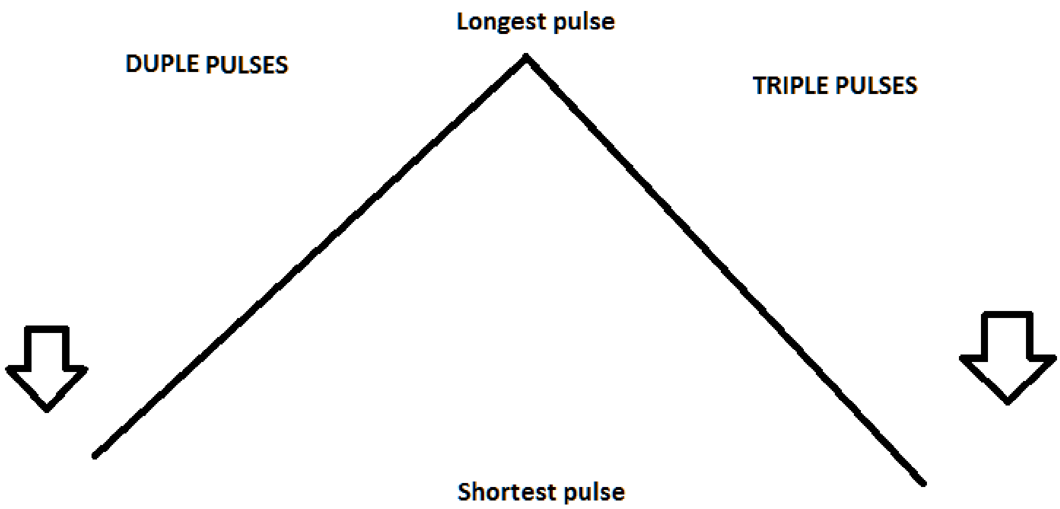


Figure 19: Ski-hill graph instructions for direction of mapping for duple and triple meters

Minimal Meter

When first graphically representing meter it is not essential for young students to use the terms “minimal” or “deep” to describe meter, but it is essential for those teaching and learning the materials to have a solid understanding of what constitutes a meter and how most of the music we hear is made up of many minimal meters.

Through the following exercises students can be taught Cohn’s (2015e) definition of minimal meter, where (there) is “a relation between two pulses of different speeds, the slower of which is included in the faster.”

Students listen to two pulses repeated, one twice as long as the other, they identify the ratio 2:1 and demonstrate this themselves through clicking, playing a drum, tapping, or walking and clapping, and so on. They then notate this relationship of the pulses with a hand drawn ski-hill graph in their exercise books or device.

Discussion about inclusion takes place not just where pulses are concerned but in other related subjects such as mathematics (how many ones are there in 2? 1+1 = 2; 2÷1 = 2); two bowls fitting into each other; a nest of tables; or a large and small box fitting into each other. To clearly make this point takes on average one minute, even with lower primary students.

The process can be repeated with the teacher visibly clicking two pulses in a ratio of 3:1 with the steps above repeated to learn about the mathematical property of inclusion in music.

Figure 20 illustrates the two different sets of ski-hill graph nodes students have sketched, one representing minimal duple meter (mapped to the left direction) and the other representing minimal triple meter (mapped to the right direction).

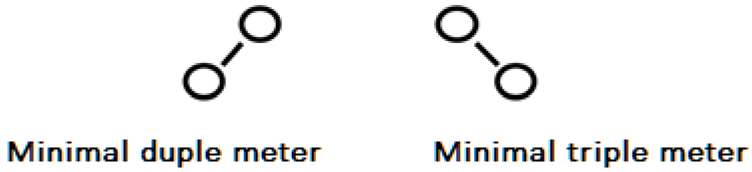


Figure 20: Minimal duple and triple meter represented by "empty" nodes[78]

In graphic representations the nodes of Figure 20 form "pathways" in opposite directions that result from mapping duple meter's 2:1 ratio of two adjacent pulses in a relation of inclusion, and triple meter's 3:1 ratio of two adjacent pulses in a relation of inclusion.

If students are familiar with traditional notation I guide them to the understanding that each set of two nodes could potentially be filled with different sets of pairs of pulses using traditional notation. Each set of two pulses illustrated in Figure 21 represent a minimal duple meter using traditional notation to encourage students to learn more about their experience of meter through understanding more deeply what notation(s) represent, the mathematical property of inclusion, and metric equivalence.

All of the sets in Figure 21 are related by metric equivalence because they share the same mathematical ratio of 2:1, thus they sound identical to the hearing. Students observe and tap selected pairs from this figure:[79]

---

[78] Nodes of the ski-hill graph can be described as being "empty" when there is nothing physically notated inside them such as traditional notation or a fraction.

[79] Hyperpulses are represented using traditional notation with the number of downbeat pulses to which they are in inclusion relation written next to the pulse. The first hyperpulse listed below would span four measures to the hearing and the second pulse listed would span two measures to the hearing: 

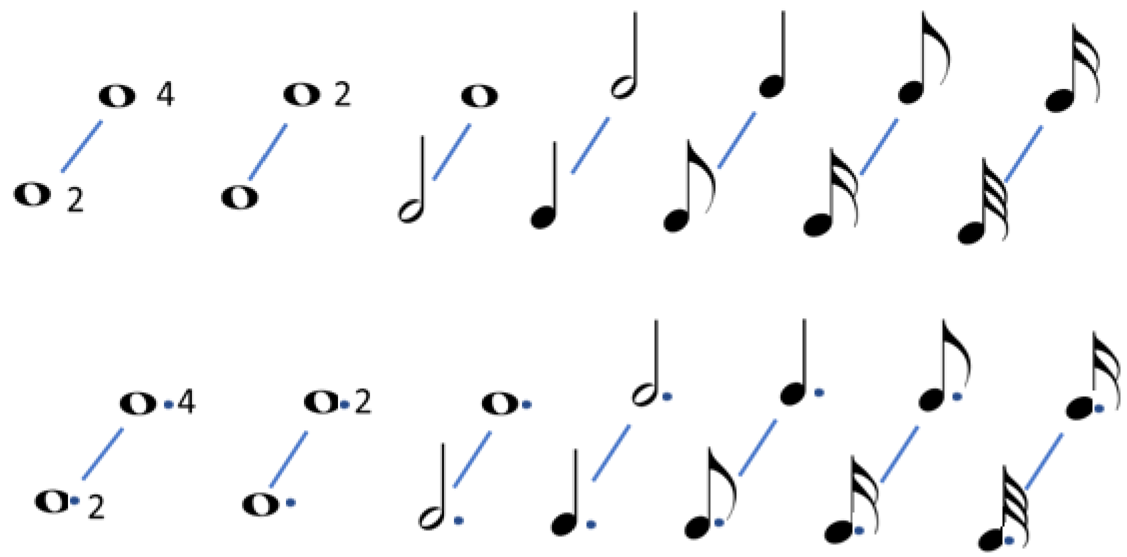


Figure 21: Pairs of pulses related by metric equivalence representing minimal duple meter

The ski-hill graphs of Figure 22 illustrate sets of pulses related by metric equivalence forming minimal triple meter to the hearing in ratios of 3:1. Any of the following pairs of pulses illustrated in Figure 22 represent the inclusion relation of 3:1 through using traditional notation.

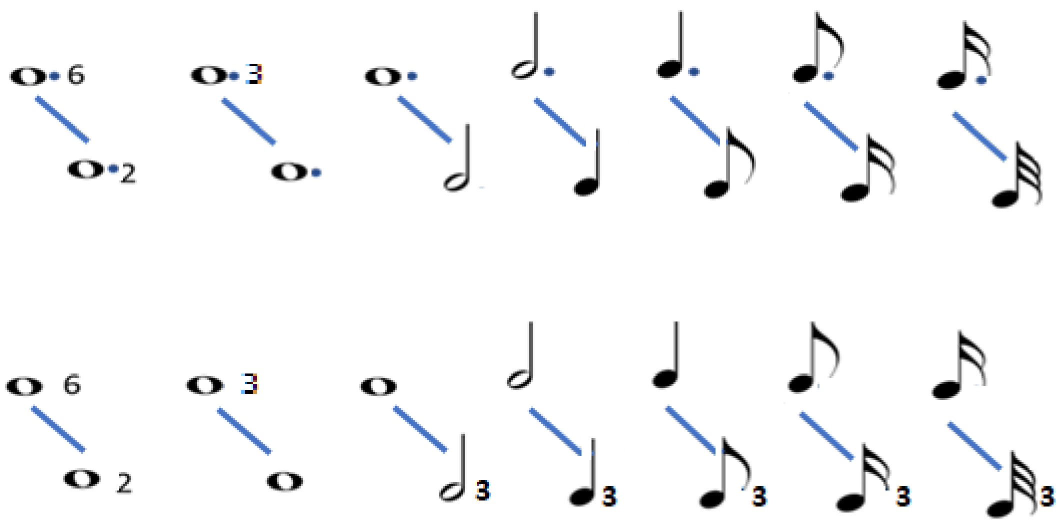


Figure 22: Pairs of pulses related by metric equivalence representing minimal triple meter

In learning about inclusion experientially and through graphically representing their hearing, students soon learn that meter is cyclical. By engaging in these exercises students also learn that they experience meter because they entrain to and project sets of two pulses. From here students understand that inclusion is a mathematical property of meter they experience and at this point students map the two pulses in a minimal duple meter from their ski-hill graph to a cyclic graph using introductory set theory (Cohn, 2017).

As the size of the cyclical universe increases, students learn more about the mathematical properties of cyclicity, inclusion, ratio, complementation, rotation, and periodicity when experiencing meter. Alternatively, a cyclic graph (see Figure 23) can be introduced prior to or alongside the ski-hill graph to teach meter, as both graphs have different pedagogical strengths as instruments for teaching students about meter as explained in this chapter. Additionally, these instruments of music theory are particularly helpful when students are learning to read music as they represent the length of notes experienced and in relation to each other. In this way learning note values is a more logical process based on sound mathematical principles. See the further section in this chapter "Teaching Note Values Through Ski-Hill and Cyclic Graphs."

By mapping pulses from the ski-hill graph to a cyclic graph either by hand drawing or manipulating an app such as XronoBeat (Milne 2018) (see Figures 23 and 24), students can discuss and observe meter, beginning with arguably the "purest" form of meter possible in what Cohn (2015e) describes as minimal duple meter – the first 2:1. Thus, students can study the mathematical properties of the metric relationship of the pulses, which is otherwise not possible with traditional notation.

XronoBeat is the only app so far developed which uses both visualisations and sonifications (see Figure 24) to represent meter as sets of timepoints using numbers.[80] In the terminology of Cohn's beat-class theory (2018b), the two pulses can be represented as sets {0} and {1}.[81]

On the XronoBeat app two different sounds can be chosen to distinguish sets {0} and {1} and students can recite {0} as "O" ("oh") while a differentiating sound such as a chime

[80] See Footnote 5 for the availability of XronoBeat.
[81] The curly brackets are used by Cohn (2018b p. 133) to model sets of integers which can be represented in three different ways: cyclic graphs, repeating patterns and integers. Applied to modular arithmetic through cyclic graphs the sets can represent rhythmic sets. In this way students learn about similarities and differences between rhythmic sets and through applying their new knowledge to music learn how "graphic, arithmetic, and musical relations reciprocally model each other" (Cohn, 2018b p. 134).

distinguishes {1} from {0}.[82] The word "O" for {0} should be used as it contains only one syllable as opposed to "zero" which has two syllables, and as Cohn points out, may "offset" the next fastest pulse. As this cycle continues we begin to entrain to {01} thus forming a second pulse twice as fast as {0} to form duple meter: a set of two timepoints (onsets or points selected on a timeline are the selected points heard in a performance of a piece studied). Through Cohn's beat-class theory students can articulate their identification of the mathematical relation of the two timepoints {0} and {1} where ∪={0,1} (Cohn 2017):

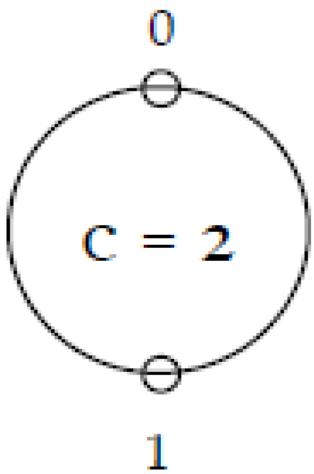


Figure 23: Cohn's cyclic graph representing minimal duple meter

[82] Cohn's (2017) word for {0} is "zee" for "zero" but here I have chosen to use the word 'O' (pronounced "oh") for use in Australian schools.

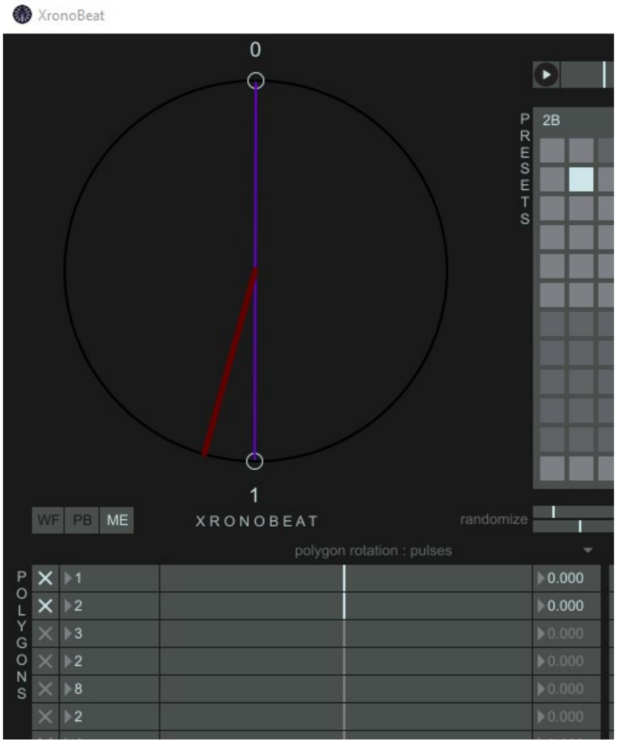


Figure 24: XronoBeat app (Milne, 2018) representing minimal duple meter

As Cohn (2017) explains, two identical pulses are formed by the listener which can be labelled as {0} and {1} but as we listen, one {1} in a sense, "becomes" or is perceived as being twice as fast as the pulse {0} forming the pulse {01}. Thus, when we bob our head or tap our foot to what seems like one pulse we reproduce an "on beat" and an "off-beat" or what Cohn (2018a) categorizes as a minimal duple meter. This occurs because there are two pulses on the timepoint labelled {0}, the "on-beat" (one longer and one "twice" as fast), and only one pulse on the timepoint {1} the "off-beat." Through these practical exercises students recognise that every timepoint of the slower {0} pulse is also a timepoint of the faster {01} pulse which forms the basis of Cohn's definition of both minimal and deep meter as explained in the next section.

When mapping the pulses representing minimal triple meter of Figure 22 from the ski-hill graph to the XronoBeat app, two different sounds can be used to distinguish {0} from both {1} and {2}. As students recite {0} as "O" ("oh") a click or chime distinguishes {1} and {2} from {0}and most people begin to entrain to {012} thus forming a second pulse three times as fast as {0} to form minimal triple meter: a set of timepoints {0}, {1}, and {2} where ∪ ={0,1,2}:

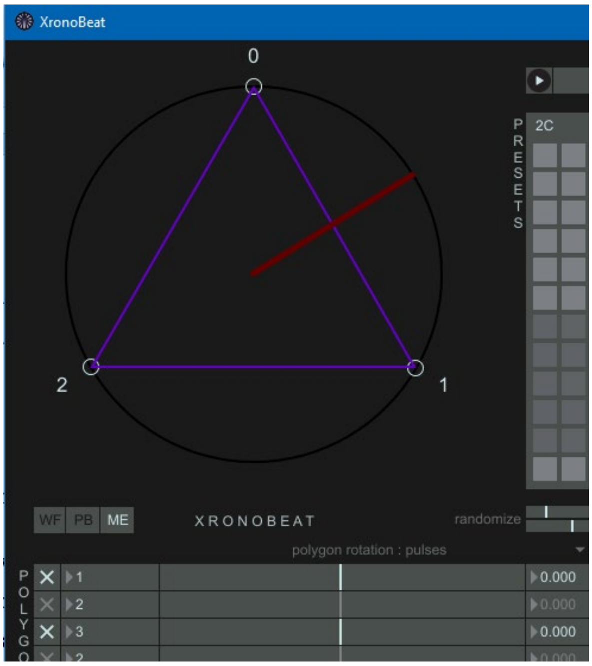


Figure 25: XronoBeat app (Milne, 2018) minimal triple meter

Hence, three identical pulses are formed by the listener which can be labelled as {0}, {1} and {2} but as we listen two of the pulses at timepoints {1} and {2} in a sense, "become" three times as fast as the pulse {0} forming the pulse {012}. Thus, when we bob our head or tap our foot to what seems like one pulse we reproduce an "on beat" and two "off beats" or what Cohn (2017) categorizes as minimal triple meter. This occurs because there are two pulses on the timepoint labelled {0} the "on-beat" (one longer and one three times as fast), and only one pulse on both timepoints {1} and {2}, the "off-beats."

Deep Meter

This section provides a definition of deep meter, followed by materials for teaching and learning deep meter with music examples. This section is organized into three parts: pure duple meter, pure triple meter, and duple and triple meter.

Metric music is formed through hearing sets of minimal meters in a hierarchy, thus forming deep meter:

A deep meter is a relation between three or more pulses of different speeds, each pair of which forms a minimal meter (Cohn, 2018a).

Deep Meter: Pure Duple Meter

The object of this section is to help students understand meter and the mathematical properties that contribute to the metric hierarchy more deeply through experiencing meter from listening to music made up of more than one set of minimal meters. To achieve this, I set out an approach for teaching junior secondary school students about meter in Bach's Prelude in C major BWV 846. Similar outcomes can be achieved for younger students by simplifying the steps described and selecting relevant repertoire in pure duple meter.

Suggested year levels: Years 9/10 Music.

This material is also suitable as an introduction to ski-hill graphs with senior secondary music students and undergraduate pre-service teachers who are exposed to new meter theory for the first time. While also suitable for the music studio, what follows is designed for classroom music and organised to modules or topics: Introduction to Music Theory, Baroque Music, Keyboard Music, A Study of Music in Pure Duple Meter Through the Ages from Around the World.

Music textbooks generally describe the meter in Bach's Prelude as simple quadruple time with four quarter note beats per measure. Nevertheless, the dynamic role meter plays in pieces containing pure duple deep meter can be articulated in much greater detail and deserves further analysis.

<u>A suggested sequence of tasks.</u>

If this is the first time students are notating a ski-hill graph, it can be more effective if they undertake each step then discuss the work as a whole class before progressing to the next step. The materials are presented with this "first-time" approach in mind, but as students gain more experience in representing meter graphically, and articulating evidence of hearing meter, they will be able to work independently. For instance, with more experience of learning meter through this approach, these tasks could be undertaken individually in class, discussed and workshopped as a class, or they could be set as homework or as an assignment or project, included as part of a listening skills examination, or as a combination of these approaches.

Students first listen to Bach's Prelude without the score in order to detect pulses that make minimal meters. During their listening they draw two ski-hill graphs, one with "empty" nodes and another using staff notation note values. This process is then discussed by the class, ideas are workshopped, and ski-hill graphs are notated on the board. They then fill out a listening log to report evidence of their hearing of the pulses that in their experience forms meter (see the section "Listening Log").[83] They consult the score for clues, annotate it with numbers for counting meter, count the meter out loud with the class, and using the SkiHill app they map the pulses from their hand-drawn ski-hill graph (they can map directly to the SkiHill app if available).

Next, students map to a cyclic graph, either by hand or through the XronoBeat app, the pulses from their ski-hill graphs as timepoints in order to observe the mathematical properties of evenness, cyclicity, periodicity, inclusion, complementation, and rotation (Cohn, 2018b). If using the SkiHill and XronoBeat apps students have the advantage of seeing and hearing the pulses they mapped on their graphs interacting visually, in sonifications as animations. If the app is not available students should listen to the piece once again with the graphs after all the evidence has been gathered.

Ideally, students should listen to Bach's Prelude as they observe the pulses they mapped, then where possible practice and perform the pieces individually or in groups to experience the pulses they mapped on their graphs in performance. I have noticed that students demonstrate increased rhythmic security and produce more satisfying performances of isochronous music as a result of learning through direct experience about the metric hierarchy.

Most students draw ski-hill graphs resembling those in Figure 26 to represent hearings of pulses in pairs of minimal meters making a deep meter – pure duple meter. Illustrated in

[83] The listening log is an introductory means to help students organise their reporting of meter as understood as sound rather than notation through answering three short questions (see the section "Listening Log"). This process of articulating their experience of meter, which is new to school-age students, requires the student to represent a hierarchy of pulses on a ski-hill graph. Thus, the listening log assists students who are learning new understandings of meter, to organise their findings, particularly, for the discussion of meter either verbally or in written forms such as narrative (see Chapter 4 "Writing a Meter Narrative") to complement their analysis of meter through the ski hill graph.

Figure 26 are two ski-hill graphs representing pure duple meter from a hearing of Prelude in C major BWV 846 by J S Bach: (left) a ski-hill graph with “empty” nodes to represent metric equivalence and (right) a ski-hill graph with traditional notation.

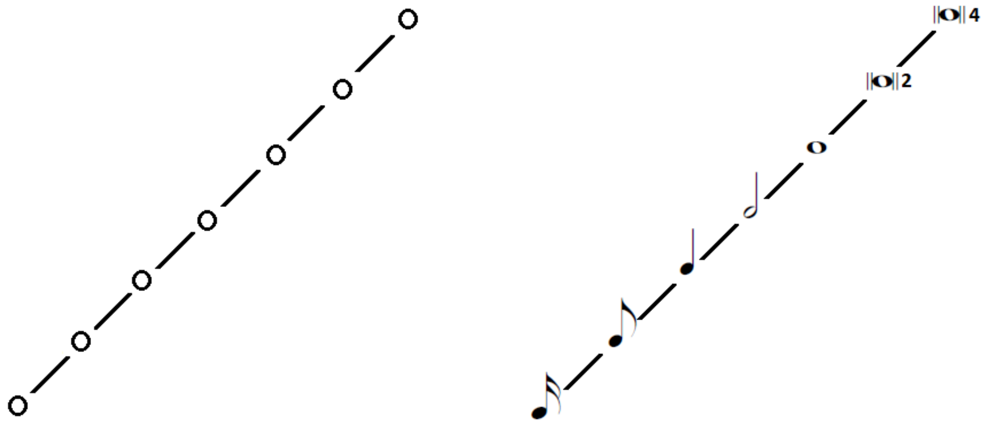


Figure 26: Illustrates two ski-hill graphs representing pure duple meter <2>

Cohn (2015e) describes pure duple meter as the set <2> because all of the minimal meters are in sets of 2:1 (minimal duple meters), thus no other adjacent meter is present to the hearing.[84] Figure 26 is a typical response of a students’ reported hearings of Bach’s Prelude where at least seven pulses are graphed.

Understandably, as students first learn to draw ski-hill graphs, they will sometimes “leave out” an adjacent pulse, or only map those pulses they have seen notated on the score if they are already familiar with it. This occurs largely because the experience of listening for all of the pulses that make up meter, let alone graphing pulses, are foreign concepts and practices for most students. Students are generally used to listening mainly for the downbeat pulse and one faster as required in music textbook explanations of time signature and meter theory.

Also, pulses “omitted” from ski-hill graphs by students are usually those heard in the imagination and not physically present on the notated score. Students who have a notation-based understanding of meter will not normally consider these pulses as having significance

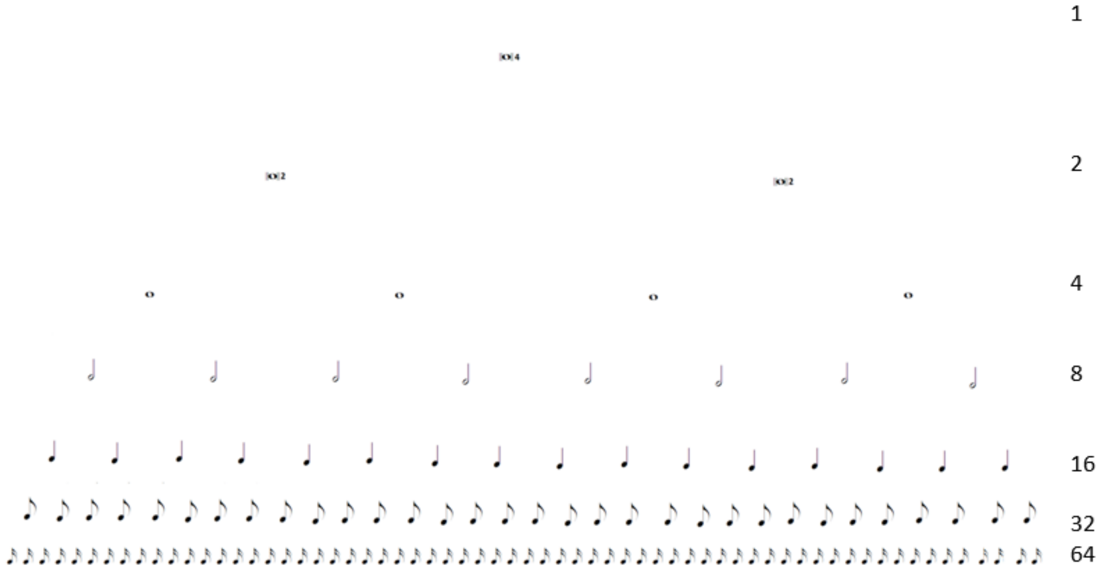
[84] Meters are parsimoniously represented with Cohn’s *ordered set notation* (Cohn, 1992b), which show the inclusion relations of all of the metric levels for example, pure duple meter is represented as <2>; and pure triple meter as <3>. Meter represented as <322> indicates that a) duple and triple meters are heard together; b) there four metrical levels; and c) each pulse of the slowest metrical level contains 3 pulses of the next faster metrical level, which contains 2 pulses.

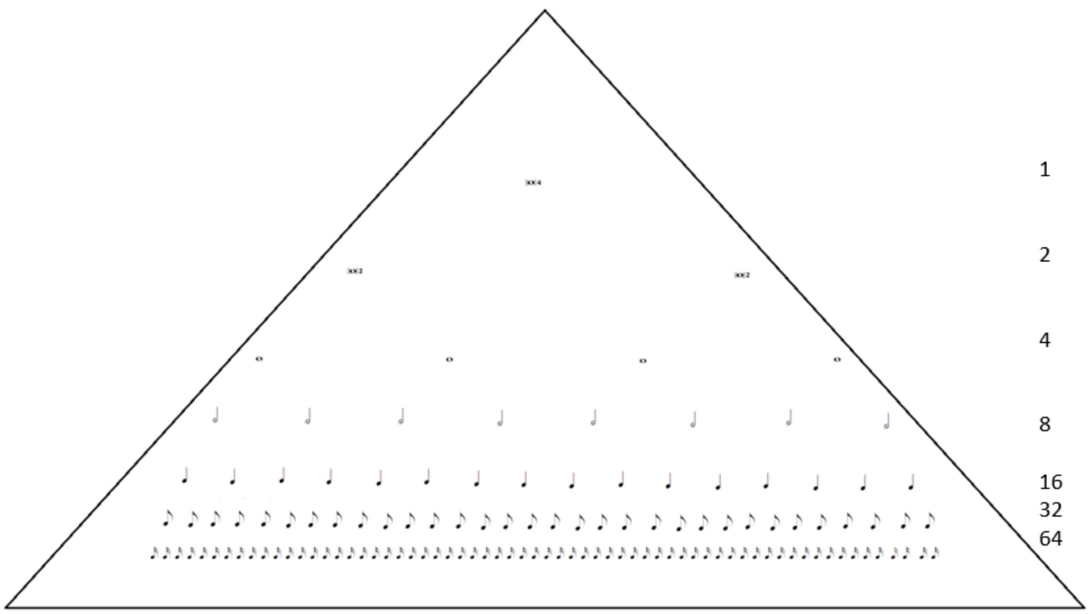
in the description of the meter of the piece. Thus, it is more effective pedagogically where possible for students to listen for meter first without viewing the notated score prior to introducing the ski-hill graph.

When first mapping pulses to ski-hill graphs, students rarely jot down the full list of adjacent pulses they hear in a hierarchical order, graphing first every pulse from the slowest to fastest, nor is it important that they do so. In other words, the order in which students place the pulses in the hierarchy is completely arbitrary and no one order of listing pulses in the hierarchy is "correct." For instance, choosing to begin mapping pulses to their ski-hill graph by graphing a downbeat pulse and one faster does not make these two pulses more "significant"; rather, they may play a prominent role in the listener's formation of meter, either throughout the piece or in a particular section. Also, it is understandable that the listener with a notation-based understanding of meter often notates the minimal meter they hear from the relation between the downbeat pulse and one faster pulse, because of the weight accorded in a notation-based understanding of meter as conveyed by the notated time signature and measures in music text books.

Through workshopping ideas, students can be steered and taught about the metric hierarchy, and how to more accurately articulate evidence of their hearing. With a little training, very young students are able to consistently acknowledge and report their awareness of two- and four-measure hypermetric pulses, as in Figure 26.

When meter is taught through the ski-hill graph, school students and adult learners, including undergraduates, often learn for the first time that meter is structured hierarchically. Thus, it is critical for them to understand what a hierarchy is and how meter is structured in this way. Figure 27 illustrates how the pulses represented on the ski-hill graph are a hypothetical summary of the extensive metric hierarchy of pulses they potentially experience when hearing (structuring) pure duple meter in Bach's Prelude*,* and in all the music they hear in pure duple meter in music of all genres and styles, Western and otherwise:


1
2
4
8
16
32
64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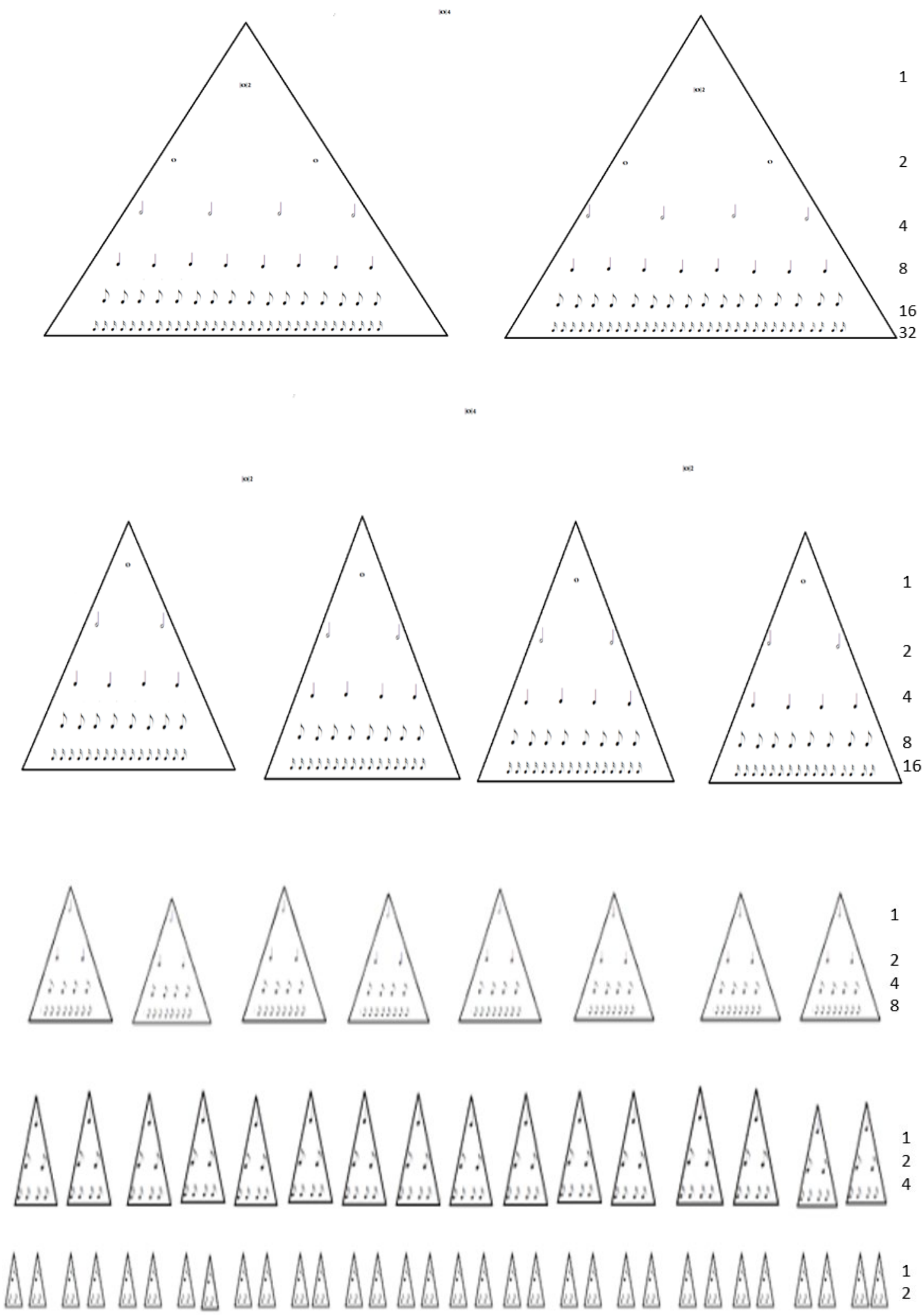


Figure 27: Metric hierarchy and fractal-like formations

The metric hierarchy notated in this way illustrates the fractal-like formations meter creates when structured by the listener. The numbers listed to the right of the divisions should be explained to students in simple terms, such as division and ratio, and explained as being representative of metric equivalence, where the note looks different but sounds the same. Students soon realise that the same numbers can be applied to different pulses but they will still sound the same. Visualising metric hierarchies in the ways described above helps students to recognise the efficacy of writing just one set of 2:1 pulses (minimal meters) on a ski-hill graph, otherwise the full gamut of pulses spirals out of practical application when notated as a table.

Students' attention can be drawn to the section of the table where the divisions match the labelling of note values used in the current system where the whole note is (1), half note (2), quarter note (4), eighth note (8), and sixteenth note (16) and so on. At some stage, however, students should be taught that in reality any note can become the "whole" note, including a dotted note.[85] For instance, where a dotted whole note is the "whole" note, the half note becomes a dotted half note in a notation in 6/4, and a third note in a notation in 3/2; the quarter note becomes a sixth note in 3/2 and 6/4; the eighth note becomes a twelfth note in 12/8, 3/2, and 6/4 and so on. The numbers to the right-hand side of the tables above also reflect those of the fractions displayed on the SkiHill app, an option students have when wishing to observe the mathematical relations between pulses notated in traditional notation.

In a recently conducted survey (Calilhanna, Unpublished 2017), teachers reported that counting and division were a significant problem for their students (of all ages) when learning about meter.[86] Teachers in the survey noted the emphasis on duple meter in the initial stages of teaching about meter, which, the survey Report summary noted, often cements problems with counting in later years. Arguably, it is not the students who have difficulty dividing and counting with small numbers; it is a system where the mathematics does not add up. Teaching note values through Cohn's meter theory can be seen as more logical. For instance, traditional

[85] The current understanding of the semibreve as the whole note reflects that only minor prolation or *imperfect prolation* remained in Loulié's time signature system. Ironically, using names such as crotchet and quaver etc. obviates the mathematical problem in cases where, such as in dotted values and 3:1 ratios, it becomes confusing. Only until Cohn's meter theory is adopted and Loulié's archaic system is no longer in use will the labelling of notation make sense mathematically based on the listening experience.

[86] *Understanding and Teaching Meter Survey*: Project number: 2017/055. See Appendices A and B, Calilhanna, (Unpublished 2017), also Calilhanna and Webb (2018, unpublished).

pedagogical approaches to the early learning of note values avoids an equal study of triple meter, and combinations of both duple and triple meter. Teaching mainly duple meter in the early years of a student's music education, where the quarter note is always taught as being of "one" beat in value, often leads to later difficulty in comprehending meter. For example, a common problem students encounter in reading notation is the case where a note such as a quarter note is followed by a dot and is worth three of the pulses or 1/3 of its own value (3:1), but is also called "one," as in a 6/8 notation, thus raising the question: "Wasn't a quarter note worth 'one' ?"

The preoccupation with traditionally labelling duple meter divisions for triple meter divisions such as those mentioned above, for example, labelling the triple note divisions of a dotted whole note as three half notes instead of three triple notes, and in 12/8 the triple division of a dotted quarter note as three eighth notes instead of three twelfth notes is one which needs addressing in meter theory and pedagogy. Arguably, presenting students with music theory tenets, without there being a logical pedagogical explanation as to why notes are labelled in this way, likely contributes to students' later trepidation of counting where meter and rhythm are concerned due to the mathematics that just do not "add up." Often, students who think through the processes and systems of their aural work learned traditionally, particularly those who employ simple mathematics to make sense of pattern in music, will be ill at ease in a system which does not make good logical mathematical and reasoned sense.

Unfortunately, students who are uncomfortable with traditional approaches to aural skills pedagogy in high school and tertiary programs often cannot put their finger on why they are hearing differently the same music and/or hearing more possibilities then their teacher or other students convey. This is a many faceted issue but the traditional notation-based understanding of meter includes: traditional labelling of note values, the lack of teaching about metric equivalence, and an over-emphasis on teaching duple meter. These are three misleading tenets of music theory, perpetuated in traditional music theory, which contribute to why students do not learn deeply about meter and rhythm.

From their observations of meter and hierarchy, students soon observe that pulses share equal importance in the metric hierarchy but at times pulses will serve different functions in the context of the listening experience. These functions give pulses not so much an increase in the degree of importance as more an opportunity to be categorised and explained as fulfilling a certain role; for instance, a span, unit, counting, downbeat, intermediary, or hypermetric pulse. In labelling notes with functional terms which can be explained logically through mathematical music theory, Cohn's theory ensures that note values can be used in a coherent and implementable music curriculum. Thus, through utilising the materials in this chapter the student is positioned to examine the dynamic operations of the many pulses that make up the meter they hear.

Listening Log

To assist students to organise their evidence of experiencing meter reported through the ski-hill graph, provide them with a listening log (see Figure 28) to fill out. When students are first learning how to use ski-hill graphs, workshop with them the data they collect on their listening log. The purpose of the listening log is to establish a framework for thinking and writing about meter through prompting students to answer three short questions as they listen for meter:

What pulses do I hear?
What meters am I structuring?
Give reasons for the meter(s) I structure.

Meter(s)	**Pulse(s)**	**Reasons for structuring meter(s):**

Figure 28: Listening Log

After using the listening log a few times students may find they remember the three questions as an intuitive framework for thinking about meter and write and graph evidence without using the listening log sheet.

Figure 29 is a typical example of a listening log workshopped in a 45 minute piano/theory lesson. In the process of preparation for a practical piano examination a student will:

1. Report the meter they hear while listening to Bach's Prelude in C major BWV 846.
2. Draw a ski-hill graph and fill out ideas about the questions on the listening log.
3. Look at the score for clues as to why they hear the pulses they mapped, and to discuss their listening log ideas adding any new ideas to their report.

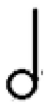
One reason students study meter so closely is to achieve a more sensitive performance. For instance, students frequently accent nearly every downbeat with equal volume. A student might perform a piece in this manner because they observe the meter signature as 4/4 and diligently learn the notes by practicing at home to a single metronome beat representing only a quarter note pulse; this can eventually flavor a performance as "choppy."

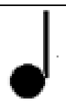
Instead of telling students how to perform a piece it is best to first let them "discover" for themselves the best solutions for interpretation and only steer them where necessary. In this way students are better equipped to apply their new knowledge and understanding of meter to other works without anyone present to assist them.

On the student's listening log "MPR 1" refers to Parallelism (see Figure 29). Here I draw on the work of Fred Lerdahl and Ray Jackendoff through using the Metric Preference Rules provided in *A Generative Theory of Tonal Music* (1983). MPR 1: "Where two or more groups or parts of groups can be construed as parallel, they preferably receive parallel metrical structure" (Lerdahl and Jackendoff 1983, p. 75). In other words, when we hear the same hierarchical pattern of pulses recurring (meter) we usually expect it to happen again (Huron, 2006), projecting these pulses with our mind (Hasty, 1997), and entraining to them with our body (London, 2012) thus forming metric structure until a new pattern occurs.

Parallelism is not mentioned in the discussions about meter in Bach's Prelude in classroom music textbooks for school students, although, in responses students report as evidence for their formation of the experience of meter, parallelism features prominently at all metric levels, as illustrated in Figure 29:

Meter	**Pulse**	**Reasons for structuring meter**
Pure duple meter <2>	||O|| 4	MPR 1 (Parallelism) I entrain to and project a four-measure hyperpulse – the span pulse – an imagined pulse not indicated by the meter signature and not physically present in the notation. This hyperpulse (the span or longest pulse) occurs on a downbeat which at regular timespans shares the most number of pulses of any set of timepoints in this piece: seven pulses as indicated on my ski-hill graph and annotated score. The four measure hyperpulse forms a minimal duple meter with a two measure hyperpulse. I group this pulse in this way because I hear changes of harmony on the downbeat each four measures which usually results in some kind of harmonic resolution. Although in measure 5 the arrival of the relative minor chord A minor contrasts with the C major chord of measure 4 to create tension rather than resolution. Melodic contour contributes to my formation of the four measure hyperpulse in measure 5 because combined with the chord change there is also the highest note in the section - the root note of the new chord – which draws my attention to this set of timepoints beginning with the four measure hyperpulse.
	||O|| 2	MPR 1 (Parallelism) I entrain to and project a two-measure hyperpulse which occurs on a downbeat at regular timespans sharing six pulses as indicated on my ski-hill

		graph. The two-measure hyperpulse is imagined, not indicated by the meter signature, and not present in the physical sound or notation. The two measure hyperpulse and downbeat form a minimal duple meter. The main reason I hear this pulse is because of a harmonic accent or grouping of timepoints into sets which occurs every two measures which in most cases forms a sense of relaxation or “release of tension” with only a few exceptions. All chords are arpeggiated except the final chord m. 37.
		MPR 1 (Parallelism) I entrain to and project a semibreve note: The downbeat, an imagined semibreve note pulse, brings with it a harmonic change and therefore a harmonic accent. This happens because I had been entraining to a projected pattern then I noticed something new happened with the patterns of pitch which began either on the onset of every downbeat or just after with the second or third semiquaver notes. The semibreve note is formed as part of the longer and short pulses such as the minim note in a duple relationship and the two measure hyperpulse.
		MPR 1 (Parallelism) Entrainment and projection of the minim pulse due to: The minim is the first pulse of each set of “eight” sounded semiquaver notes which I entrain and project throughout the whole piece.

		The minim note occurs on the downbeat bass note bringing with it harmonic change and therefore a harmonic accent. The repeated bass note at the same pitch twice in each measure once with the first onset on the downbeat and the second with the onset on the third crotchet note pulse caused me to project this regular pattern into the next section where I heard it again regularly throughout the piece.
		MPR 1 (Parallelism) Entrainment and projection of an "imagined" crotchet note pulse due to: Grouping into sets of four the faster semiquaver pulses. Pulse indicated by the meter signature and not present in the physical sound or notation. Melodic contour accents due to the first note in the first set of four semiquaver notes being the lowest pitch in the measure and likewise the first note of the second set of four semiquaver notes is the highest note in the measure. The pattern repeats in each measure.
		MPR 1 (Parallelism) Entrainment and projection of an imagined quaver pulse due to: Pairing of rising skips (melodic contour accent) in semiquaver notes with one interval descending in a skip and the same pattern repeats each measure.
		MPR 1 (Parallelism) Unit pulse (shortest pulse I heard).

		Entrainment and projection of the semiquaver – the unit pulse – due to: Eight semiquaver notes which repeats once per measure creating a pattern of short durations.

Figure 29: Listening Log description of pulses and meter

After students have mapped pulses to a ski-hill graph and filled out evidence of their hearing of meter on their listening log they:

1. Map the pulses they hear to the SkiHill app (See the Mapping Pulses to the SkiHill App and following sections).
2. Use the score to annotate the pulses they hear (See the next section).
3. Listen again to the piece more than once if necessary (Repeated listening is to be encouraged).
4. Observe the score to look for clues as to why they experienced meter and reported pulses.
5. Reflect on their listening log and mapping to their ski-hill graph or SkiHill app to adjust any of their reporting of pulses if needed.
6. Fill out numbers for counting meter notated from their mapping of pulses on their ski-hill graph and annotation on the score. (See the next section – Counting Meter.)
7. Consider whether counting meter hierarchically and seeing the pulse stacks above onsets leads them to adjust their reporting of pulses if needed.
8. Analyze the harmonic progression (See the next section).
9. Make observations about the structures of both the harmonic progression and meter (see the following sections).
10. Apply what they have learned about meter and harmonic progression to their performances (See the following sections).

Counting Meter

In this section I introduce a new approach for counting meter for Bach's Prelude based on meter as experienced and meter understood as a hierarchy.[87] In my *Understanding and Teaching Meter Survey Report* (Calilhanna, unpublished 2017), over half of the teachers responding to the question: "What challenges have you faced in helping your students gain a clearer understanding of meter?" reported that their students had difficulty counting when learning meter. This difficulty may be connected to students learning methods of rhythm training where syllables instead of numbers are used.[88] The syllables used can vary from method to method and students can be faced with learning a new rhythm training method each time they change teachers or schools.

Another contributing factor to fear of counting in students appears to arise from students' excess of exposure to only duple meter in the early years of music training.[89] This cements confusion for young learners when they encounter, for instance, quarter notes that are not worth "1," as in 3/8 and 6/8, even though all of their prior training likely drilled that a quarter note is worth "1" and a half note is worth "2," etc. Dots next to a note can confuse children and not so young children because of the lack of training in the mathematics involved in note length.

To solve the confusion about counting meter, Figure 30 illustrates an approach to counting meter annotated from evidence of the temporal experience of hearing meter mapped as pulses on the ski-hill graph. This approach to count meter also provides a mathematical explanation for students as to why there are stronger and weaker onsets (downbeats and upbeats) in metric music:[90]

---

[87] Richard Cohn (email correspondence, 2017) told me about an approach for counting meter based on the metric hierarchy of pulses which I adopted for practical use with students. Cohn hadn't taught students using this approach for counting meter.

[88] In the *Understanding and Teaching Meter Survey Report* (Calilhanna, Unpublished 2017) a significant 31.81% of teachers responded to the question, "What challenges have you faced in helping your students gain a clearer understanding of meter?" that their students have difficulty understanding pulse: division and subdivision of beats and have fear of counting. This data may indicate that current methods of counting (including use of syllables) available in textbooks and approaches may not be effective in helping students when learning about meter.

[89] In the *Understanding and Teaching Meter Survey Report* (Calilhanna, Unpublished 2017) a significant 20.18% of teachers responded to the question, "What challenges have you faced in helping your students gain a clearer understanding of meter?" "Student experience with mainly duple meter" and also "crotchet taught as always being worth '1.' "

[90] See also Brochard et al (2003) for an explanation of "subjective accent."

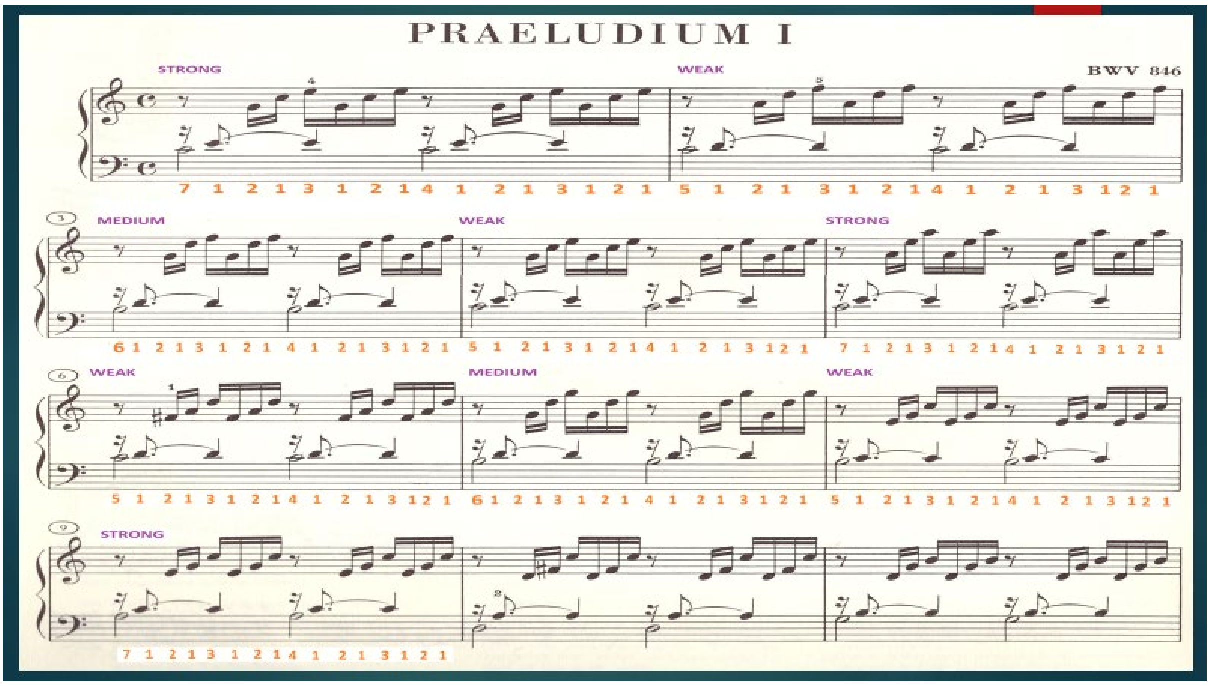


Figure 30: Numbers for counting meter - Prelude in C major BWV 846 by J. S. Bach[91]

The numbers represent the total number of simultaneous pulses experienced at any given timepoint in metric music; these can be recited by students. In learning meter through mapping pulses to their ski-hill graphs, then counting meter, students can make further observations about hearing meter in relation to the notation to form a deeper understanding of metric structure. Because meter occurs experientially as a hierarchy of pulses, it is therefore logical to base counting meter on the mathematical principles that underpin how humans generate meter to be able to represent it as a cyclical pattern of even pulses in notionally isochronous sets in a hierarchy.[92]

Through notating numbers directly to their scores to represent the number of pulses they hear at each onset, students observe meter as a hierarchy allowing them to relate the numbers to the function of each pulse. For instance, the most number of pulses occur on downbeats, but not every downbeat – only where they could hear a four measure hyperpulse when all of the pulses (seven) were experienced at the same time.

[91] Johannes Sebastian Bach. Prelude in C major BWV 846, *Das Wohltemperierte Klavier. Tiel 1.* Copyright 1974, G. Henle Verlag, Munich. Printed by permission.
[92] See Chapter 3 for a discussion of Cohn's term "notionally isochronous"

Using numbers to count meter instead of using only syllables taps into a student's intuitions about music and mathematics as it directly represents the metric structure formed in the listener. Rather than a system based solely on syllables, an approach to count meter which uses numbers gives witness and acknowledgement to the mathematics embedded in the music and both the mathematics and music embodied in the listener.

Figure 31 illustrates an annotated score using traditional notation for the Prelude in C major, mm. 1-5. Each number corresponds to the number of pulses for each onset reflected in the compact metric hierarchy represented on a student's ski-hill graph:

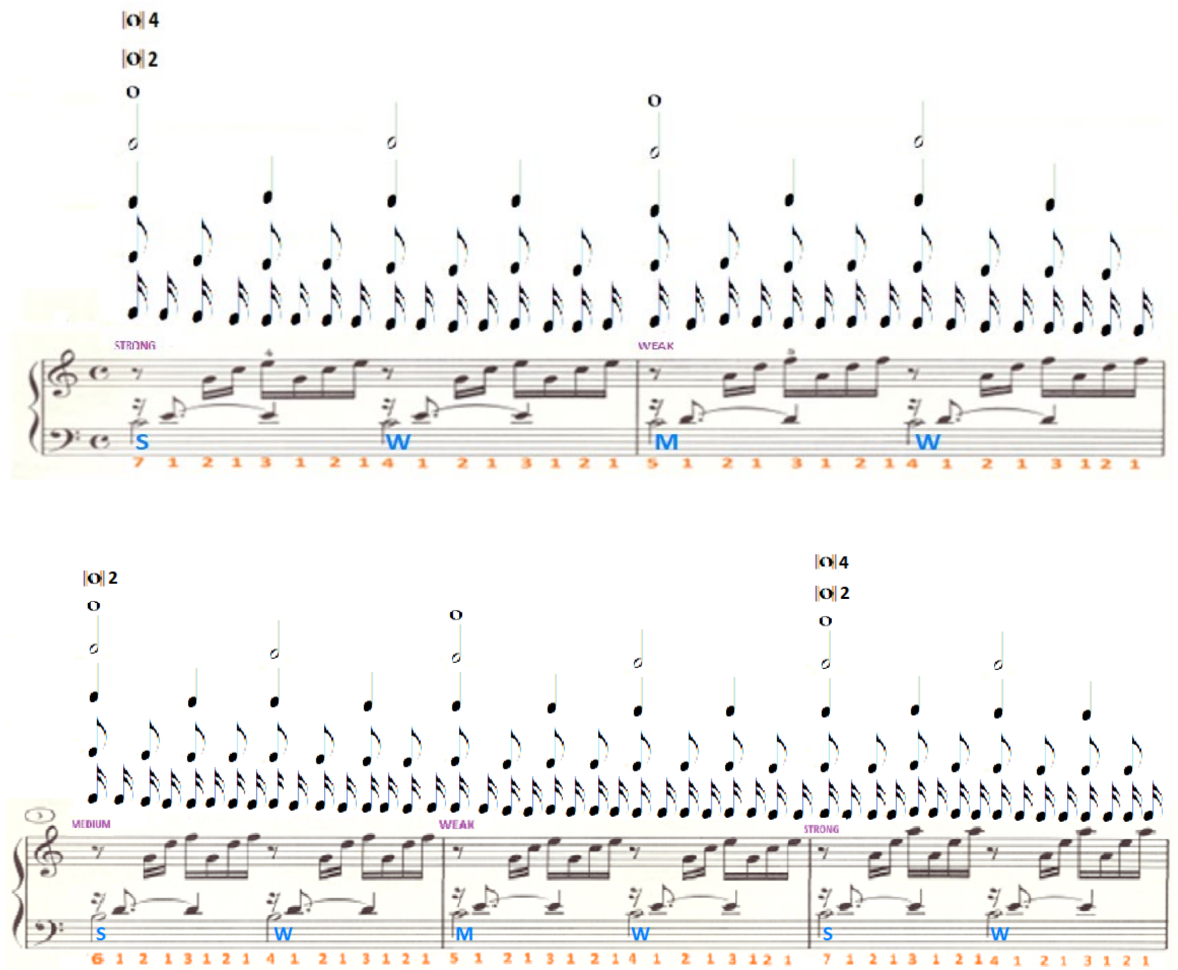


Figure 31: Annotated score with numbers corresponding to pulses heard

The letters in blue indicate a further indication of strong ("S"), medium ("M") and weak ("W") pulses at the level of the half note. It is entirely possible to add in SWMW at the level of the quarter note, eighth note, and sixteenth note.

Figure 32 illustrates a sample of the tonality and meter analysed (ski-hill graph and numbers for counting meter) for the Prelude in C major, mm. 1-11. Notably, the four-measure, functional cycle of harmonic/voice-leading through I–ii$^{4/2}$–V$^{6/5}$–I is a strong determinant of the four-measure hyperpulse. The next four-measure unit is marked by a sequential progression (two-measure model, mm. 5-6 stepping down in mm. 7-8) thus enabling listeners to entrain to and project two- and four-measure hyperpulses.

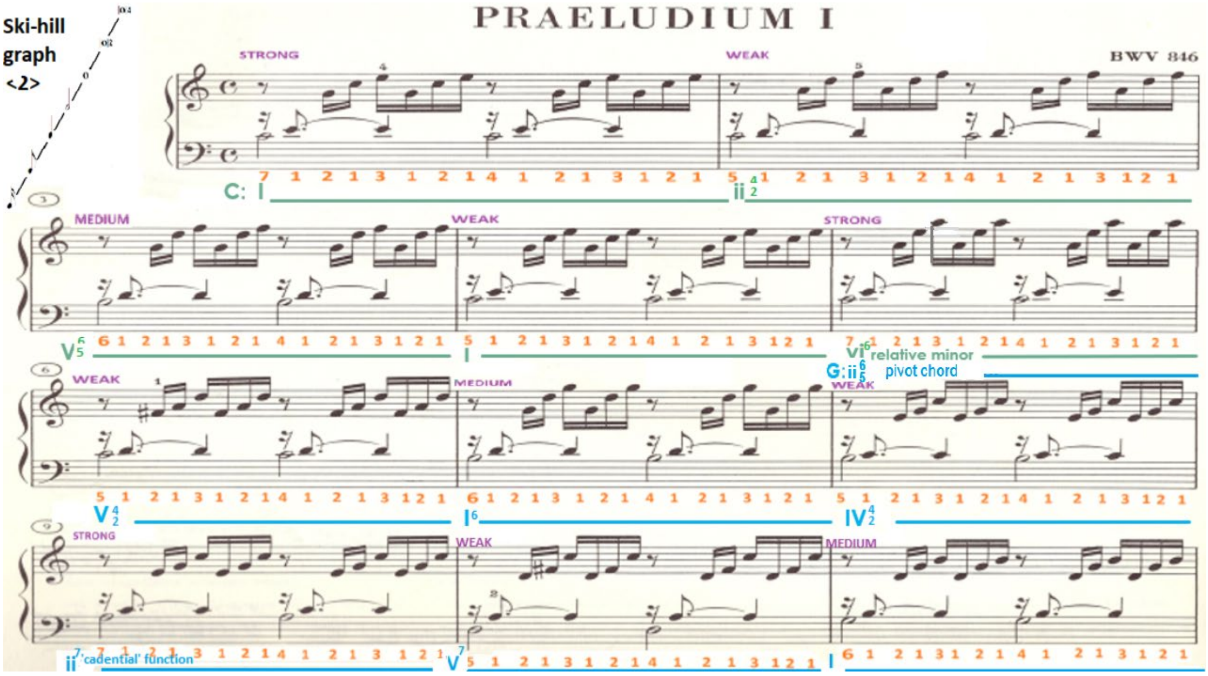


Figure 32: Tonality and meter analysed

Analysing meter and tonality, and then annotating a score enables high school students to make observations not only about the tonality but also how the tonality and meter work together isomorphically to form tonal and metric structures. This then helps students to understand why one performance may sound more "musical" or have a better "feel" than another performance.

Mapping Pulses to the SkiHill app

When mapping pulses to the SkiHill app, students can choose traditional notation, fractions, and polygons, which provide instantaneous feedback about the pulses they map so as to articulate observations and new knowledge about the meter they experience. Figure 33 illustrates pure duple meter mapped to the SkiHill app from listening to the Prelude in C major, mm. 1-11:

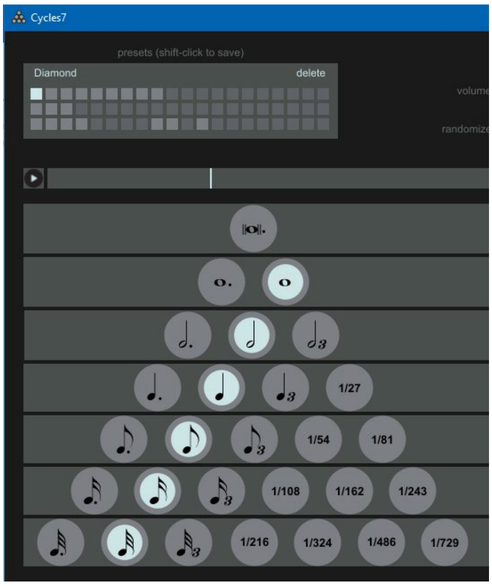


Figure 33: Pure duple meter represented by pulses mapped to the SkiHill app

Students have options for choosing different sounds (sonifications) and visualizations for each pulse as the nodes light up and pulsate during playback, thus enabling the students to make observations about the interactions of the pulses as sets of minimal meters that form deep meters through metric pathways and metric space. Experiencing these visualizations and sonifications means students become more deeply aware of their subjective experience of meter through seeing and hearing the pulses they graph in relation to each other. Thus, in graphically representing meter in this way students are then equipped to understand the mathematics they embody each time they experience meter. Figure 34 represents pure duple meter <2> or what would equate to a single measure of the Prelude in C major illustrated in fractions:

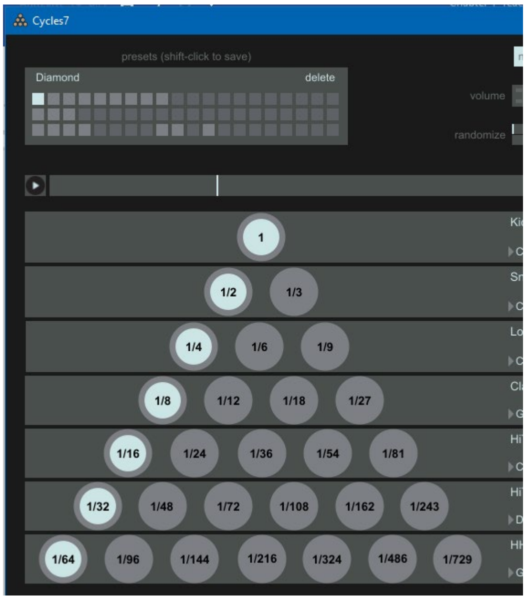


Figure 34: Pure duple meter <2> represented in fractions

Polygons, Geometry, and Cyclicity

On the right-hand side of the SkiHill app a polygon is represented for each pulse and its periodicity, so that students can make observations about its relation to other pulses and the mathematical properties we experience when forming meter. Unlike traditional notation, the geometric polygons graphically represent many mathematical properties of meter experienced by the listener. For instance, each polygon represents a pulse and the number of sides of each polygon represents its periodicity, allowing us to see and hear each polygon in relation to the other polygons or pulses.

When we experience meter the even patterns we entrain to are cyclical and we repeat the same pattern through entrainment and projection until there is a change in our perception. And because all of the slower pulses that make meter can be divided into the faster pulses in a hierarchy, the polygons also represent the mathematical property of inclusion because the pulses and their polygons divide into each other in sets. The SkiHill app in Figure 35 illustrates a visual representation of pure duple meter using fractions and polygons from a hearing of the Prelude in C major:

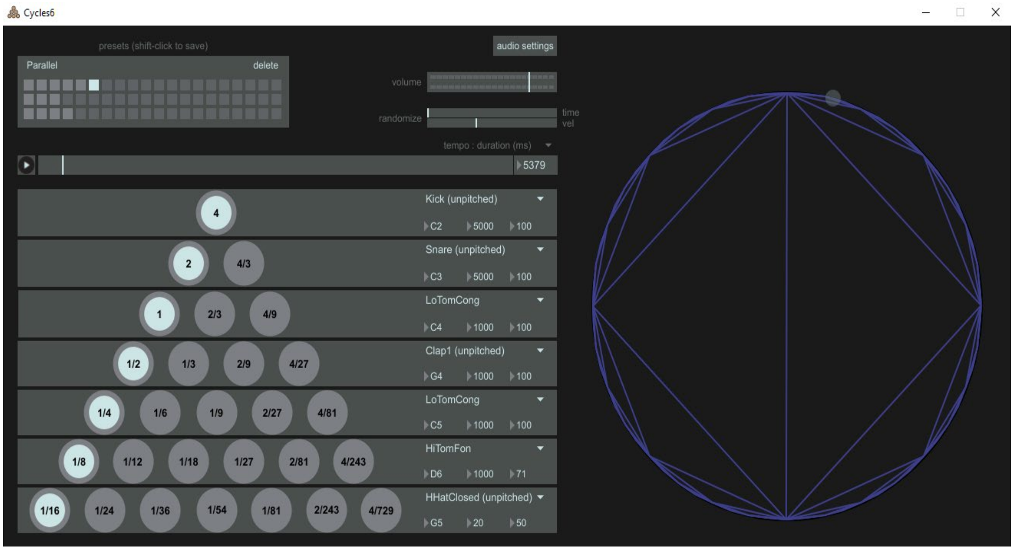


Figure 35: SkiHill app representing pure duple meter <2> using fractions and polygons

With the help of the SkiHill app I teach students how pure duple meter can be notated as fractions (see Figure 35) and also as a set of dotted pulses (see Figure 36). Students observe how dotted notes as well as the traditional set of undotted pulses normally pictured on typical "note family trees" to illustrate pulses that form duple meter sound identical to the hearing. In other words, both sets of pulses, dotted and undotted, are related by metric equivalence because they are both made up of pairs of minimal meters. Thus, any sets of pulses mapped exclusively to the left on the ski-hill graph will sound and be labelled as duple meter <2> and any pulse sets mapped exclusively to the right will sound and be labelled triple meter <3>. When heard this set of dotted notes sound as pure duple meter see Figure 36:

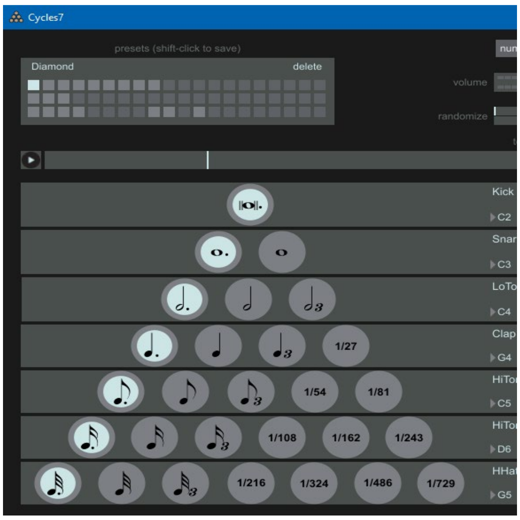


Figure 36: Pure duple meter <2>

In order to make observations about meter as cyclical, and to apply meter's hierarchical structure to the mathematical property of inclusion, students map the pulses they hear as timepoints to a cyclic graph such as XronoBeat (see Figure 37). The cycle in Figure 37 represents what would be notated as one measure of Bach's Prelude. Students benefit from working through each metric level before they perform this piece. The timepoints can be understood as integers and students can tap or clap to different pulses, either solo with the app or in groups.

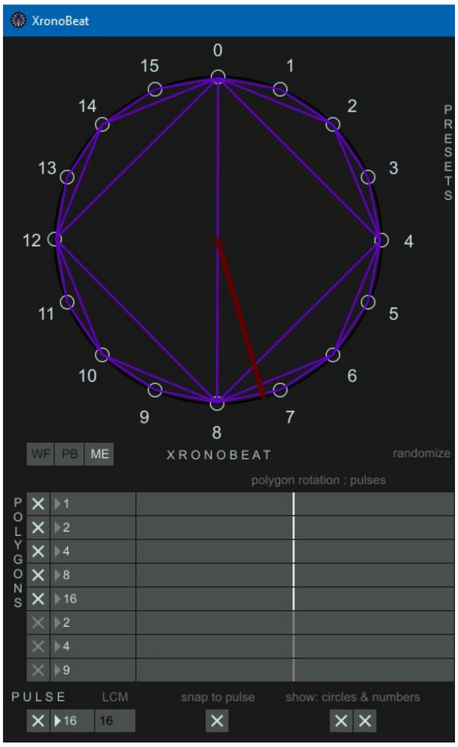


Figure 37: XronoBeat representing pure duple meter

Through mapping all of the pulses they hear making meter to a ski-hill graph, SkiHill app and then the XronoBeat app, students studying Bach's Prelude in C Major are well positioned to hear and see all of the pulses they mapped operating in a relation of inclusion in a hierarchy as they perform. In this way students no longer feel compelled to perform to a quarter note and become much more at ease experiencing a number of different pulses together at the same time as they perform their work.

The process of using the SkiHill app as a multi-pulse metronome in lessons and practice sessions helps students relate their theoretical understandings of meter to their practice of music. This is also a good example of how the computer becomes a “mediator” between information and knowledge for the students.

Whereas metronomes traditionally have one “beat” for students to gauge tempo according to the number of beats per minute, the ski-hill graph as an instrument of music theory also offers students sonification and visualization of the many pulses they hear when performing. In this way, students can then switch more easily to perform to particular pulses that help them stabilise the meter and rhythm of a section of music.

Many students preparing performances appreciate that the traditional metronome does not always represent the pulses that will help them in their performances. The process of taking a new look at the meaning of time signatures, their notation-based interpretations, and new understandings of meter, broadens students’ appreciation of the ski-hill graph, particularly as a means of representing all of the pulses they hear in music.

Learning meter with this approach brings many benefits for students, for instance, students learning Bach’s Prelude find they can see and hear a broader metric structure with a mathematical explanation not possible solely with traditional notation. With new awareness of the metric structure represented on a ski-hill graph, students are well-equipped with the knowledge to group pulses that form meter to their hearing in a hierarchy.

New knowledge about hypermetric pulses means students play the downbeat pulses as strong, weak, medium, weak to achieve more of a sense of movement and vitality in their performances. Mapping pulses to the ski-hill graph, app, and cyclic graph lead students to a deeper understanding of meter as being cyclical, experienced, entrained and projected, meter as made up of pulses in sets of pairs of pulses in inclusion relation in a hierarchy, pulses as having periodicity, options to study pulses in relation to each other as fractions, polygons, and in metric pathways in a metric space.

Critically, rather than first “telling” students how to hear or how to perform, this approach to learning meter equips students with experiences, information, and knowledge to lead them to more satisfying outcomes for their performances. Hallam (2016, p. 73) notes that there is “increasing recognition of the importance of executive meta-cognitive skills, concerned with the planning, monitoring, and evaluation of learning.” She then provides a summary of meta-cognitive skills and attributes, and musical and professional activities that,

> those wishing to engage in musical activities may require. Some are applicable to all musical activities, others apply selectively to particular tasks, while others also apply to a wide range of non-musical activities. (p. 73)[93]

Through making observations about their own listening experience with the assistance of the ski-hill graph, SkiHill app, numbers for counting meter, XronoBeat app, and basic mathematics, students are able to discover solutions to make informed decisions about performing more musically. In doing so, students learn skills to enable them to apply their new knowledge to other music where they hear meter.

Some ideas for follow-on classroom activities involving this piece may include performing Bach's Prelude; studying Gounod's *Ave Maria* to observe how he used the same tonal (chord pattern) and metric structures as Bach's Prelude; composing an 8 or 16 measure piece in the style of Bach's Prelude and provide a ski-hill graph, a listening log, numbers for counting meter, and an account of the compositional process discussing both tonality and meter.

Deep Meter: Pure Triple Meter

The SkiHill app in Figure 38 illustrates two representations of a hearing of a deep meter pure triple meter which is rarely taught in music classes with school-age students:

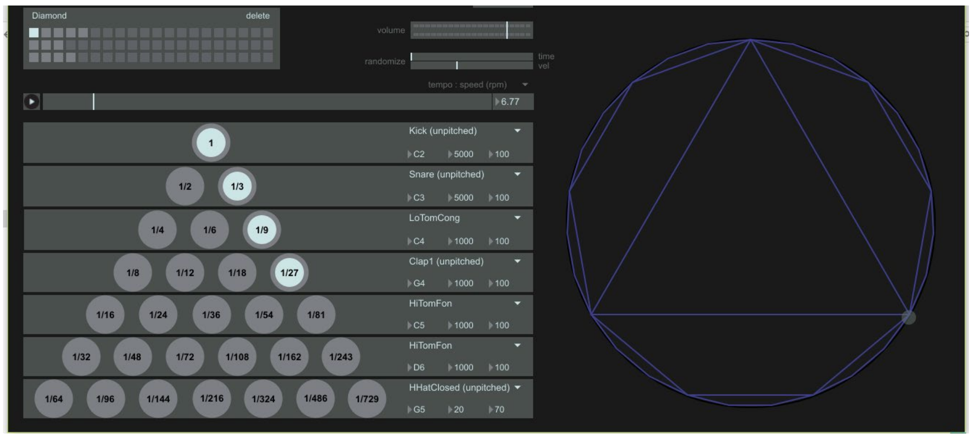


Figure 38: Pure triple meter <3>

The periodicity of each pulse is also graphically represented as a polygon through the composition of equal sides or distances illustrated for each pulse's polygon. Each pulse is

[93] See also Hallam (Table 4.1 p. 73), and McPherson (2016) *The Child as Musician A Handbook of Musical Development.*

also represented as a fraction and with the option of traditional notation (where possible) in a relation of inclusion in a hierarchy. Graphing the pulses as pathways in relation to each other in a hierarchy on the ski-hill graph, then as polygons, rather than traditional notation, positions students to more effectively observe visually and through sonifications how the periodicity of each pulse divides evenly. This division occurs in ratios of 3:1 to form the mathematical property of evenness individually as a set or pulse and as a set in relation to other sets or pulses (minimal meters – a set of two pulses in inclusion relation, and deep meters – sets of three or more minimal meters in a relation of inclusion).

To teach students about pure triple meter, I improvise music on the piano using the triple divisions represented on the SkiHill app as a guide (see Figure 38). While listening to the SkiHill app sound four pulses representing pure triple meter (a hierarchical set of three minimal triple meters) for three full iterations of the cycle, I first onset the timepoints for a span pulse (the three-measure hyperpulse) and the downbeat. During the next three iterations of the cycle, I add the next fastest pulse for a melody line, and for the next three cycles I add in the unit (fastest) pulse.

If teaching about a piece notated in 3/8, the denominator and fractions in Figure 38 represent: the span pulse – a three measure hyperpulse notated as the denominator "1" which equals a "whole" and is represented on the SkiHill app as a circle (or in staff notation as a dotted half note tied to a dotted quarter note). The span pulse forms a minimal triple meter with a dotted quarter note – the downbeat, represented as an equilateral triangle =1/3 division. The downbeat is in a 3:1 ratio with the eighth note (nonagon = 1/9), which is in a 3:1 relation with the unit pulse: tripleted sixteenth notes (27-gon = 1/27).

Some other suggestions for teaching minimal triple meter are to provide students with a recording of a composition, or, if teaching a class of students, the class can be divided into groups and perform pulses in sets of minimal meters spiralling the metric hierarchy. If teaching an individual without improvising on an instrument, the teacher and student/s can perform through tapping or drumming various pulses in inclusion relation in triple meter with or without the SkiHill app.

Through symbol, sound, movement, and narrative students are by now equipped to articulate their understanding of meter by using basic mathematics through graphic representations of their experience of both pure duple meter and pure triple meter.

Deep Meter: Duple and Triple Meter

Building on the previous section, the following task teaches students about hearing both duple and triple meter together in an orchestral piece notated in a 3/4 meter signature, the third movement of Beethoven's Symphony No. 6 in F, Op. 68, *"Pastoral,"* written in 1808. Through this task students learn about meter through mapping pulses and metric pathways with graphic representations, and through analysing meter with new understandings alongside tonality.

Traditionally the metric analysis for mm. 1-16 of this movement for school students would involve labelling the meter as being in 3/4, simple triple time with three quarter note beats per measure, and that this was to be performed in a lively dance-like manner. But the meter plays a much more dynamic role and by this stage students will now be equipped with the materials of graphic representation and language to analyze meter equally alongside tonality.

Suggested units: This piece may be included in a unit on Classical Music or Music of the Nineteenth Century (Early Romantic Music) to demonstrate how Beethoven's music sits at the junction between the Classical and Romantic eras by sharing ideas and practices from each era. Otherwise it could be included in a unit about the history of Rock Music to demonstrate how a rock song notated in a 3/4 meter signature, such as *I Dig a Pony* by The Beatles, can share certain similarities with Beethoven's movement.

Suggested level: Years 10-11, but all materials can be adapted for younger students or for music students more advanced in their music theory studies. For instance, students in Year 7 might only be expected to draw a ski-hill graph and study evenness through broad discussions about periodicity of pulses in a relation of inclusion and application of their learning through compositions and performances. For more advanced students the materials could be set as an aural skills assignment, classwork, and/or homework task. Additional tasks may include setting a major assignment or part of a unit around these and the following materials.

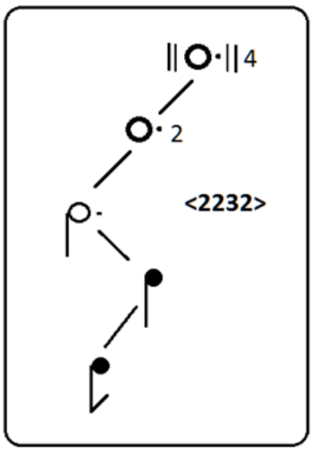
Students –

1. Draw a ski-hill graph from listening without the score to the third movement of Beethoven's Op. 68 up to m. 164 (the meter changes at measure 165).
2. Map the pulses to the SkiHill app.

3. Fill out a listening log for a shorter section, for example, mm. 1-16
4. Annotate numbers for counting meter in these measures on a score.

Figure 39 illustrates a ski-hill graph representing my hearing of duple and triple meter in mm. 1-16 of Beethoven's movement which is metrically consonant (all of the pulses are in a relation of inclusion) and can be represented as the meter <2232>. At the apex is the longest pulse, a four-measure dotted whole note hyperpulse, followed by a two-measure dotted whole note hyperpulse, a dotted half note downbeat, a quarter note pulse, and an eighth note pulse.

Notably the ski-hill graphs in Figure 39 illustrate a hearing of at least five pulses yet the score of measures 1-16 reveals mainly quarter note and dotted half note pulses (see Figure 40). This is an effective piece for discussing with young learners the many pulses we hear in our imagination – those not physically present in the notation and/or sound. Figure 39 represents Deep meter: duple and triple. This reproduction of a hand-drawn ski-hill graph (top) and SkiHill app (below) represent a hearing of the metric pathway or meter <2232> in the First and Second themes of mm. 1-164 of this movement.



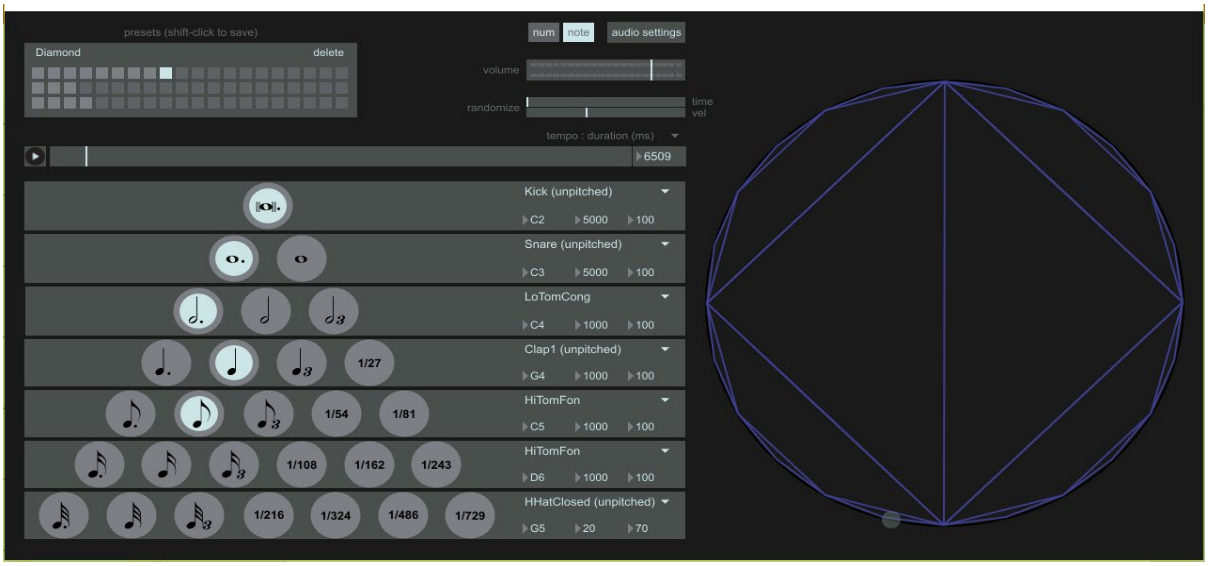


Figure 39: Deep meter: duple and triple meter <2232>

Figure 40: Beethoven, Symphony No. 6, Mvt 3, mm. 1–16 (piano reduction)[94]

As this approach to music pedagogy involves teaching students how to analyze meter and tonality equally, students learn how to provide annotations of the score illustrating both tonal analysis (tonal structure) and metric analysis (metric structure) separately, and both the tonal and metric structures annotated together on the one score. To do this, junior secondary school students should learn how to write a written analytical report of both the tonality and meter

[94] Beethoven, Symphony No. 6, Mvt 3, mm. 1–16 (piano reduction) Arranged by Franz Liszt (1811–1886). Leipzig: Breitkopf & Härtel, n.d. [1840]. Plate 6007. Public Domain. Retrieved from http://hz.imslp.info/files/imglnks/usimg/1/1c/IMSLP268150-PMLP01595- LvBeethoven_Symphony_No.6, Op.68_pianosolo_Liszt.pdf

they observe from their own experience and through referring to the scores they annotate. Ideally, they would begin with a brief history of the piece to contextualize the work with some details about the genre, style, date(s) and ideas among Beethoven's works.

I would also recommend that junior secondary students begin learning in their first years of high school how to write separate analyses for tonality and meter until they are confident to then learn how to integrate the two interactive fields of tonality and meter. However, writing about and discussing meter and tonality together should be encouraged at all times through conversation and written text modelling for students in short sections to demonstrate how this is possible. Scaffolding should be provided for school students to assist them to organize their analytical thoughts into a concise and orderly format.

As students progress into the senior secondary school and their ability to analyze music advances, discussing both meter and tonality together in the same written reports about their annotated scores can become quite detailed. Each narrative should include some contextual details about the piece itself, as listed above, and by the senior school years and latter part of Year 10, students should be integrating how meter and tonality both contribute to the music they are experiencing through analysis and/or music making.

Through this approach, senior secondary school students would describe how meter and tonality each play their role in a given piece of music to project a certain style and genre, imagery, topics and metaphors where appropriate to the piece. Scaffolding should continue to be provided where necessary to assist students to present economical and well-organized writing.

Figure 41 is an example of an annotated score for an analysis of the harmonic structure of the third movement of Beethoven's *Pastoral*, mm. 1-16. Annotating scores with colour assists students to indicate various elements significant to the structures (harmonic and metric) discussed in the narrative. For students in the junior secondary school these kinds of materials can be used as part of their guided training as a class group for labelling sections of music and for workshopping annotations of scores.

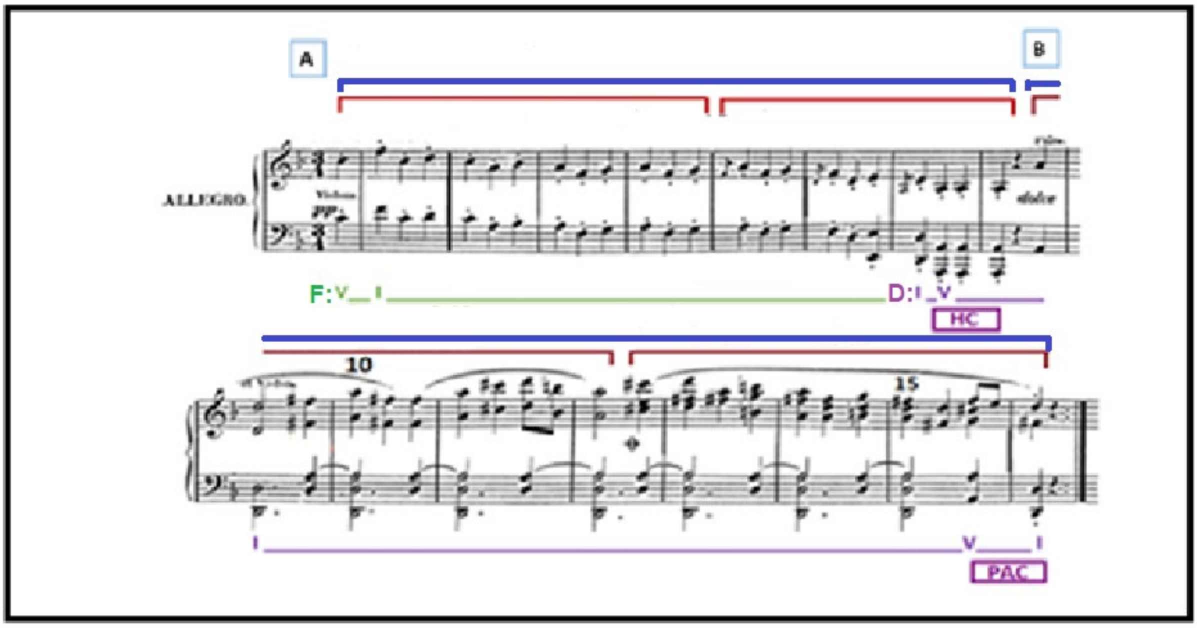


Figure 41: Harmonic structure

In Figure 41 annotations on the score indicate that the scherzo of Beethoven's *Pastoral* begins with two contrasting 8-bar phrases. The first phrase mm. 1-8 divides symmetrically into two parts 4 + 4 and can be labelled the A section. Measures 1-8 contain a three-note staccato motif which is treated sequentially and with repetition. In mm. 5-8 the motif is embellished and modulates from F major to the seemingly remote key of D major which occurs with a HC (half cadence).

The modulation to D major locates satisfactorily to the ear because of the parsimonious or smooth and minimal voice leading of the F major tonic triad tones transforming into those of the D major tonic triad.[95] The tone F moves only one semitone to F#, A remains constant to both, and C moves only two semitones thus both keys share very close tonal relations to the hearing. Also, tones of the tonic chord F major dominate every measure in mm. 1-8 until the modulation where tones of the D major tonic triad dominate every measure until measure 16.

Like the A section, mm. 9-16, the B section, also divides symmetrically into two parts 4+4 and presents new thematic material in mm. 9-12 featuring double octaves and a contrasting lyrical legato melody. In mm. 13-16 material developed from the motif in the A section ends with a PAC (perfect authentic cadence) V-1 in D major.

[95] Richard Cohn (1997) used the term "parsimonious" in the context of his article "Neo-Riemannian Operations, Parsimonious Trichords, and Their "Tonnetz" Representations" thus injecting it into current music theory.

Figure 42 illustrates an annotated score with the numbers for counting representing the number of pulses heard for each onset:

Figure 42: Numbers representing pulses for counting meter

Figure 43 illustrates the same annotation but with all of the pulses compactly represented by a ski-hill graph, pulse stacks, and numbers for counting meter notated on the score:

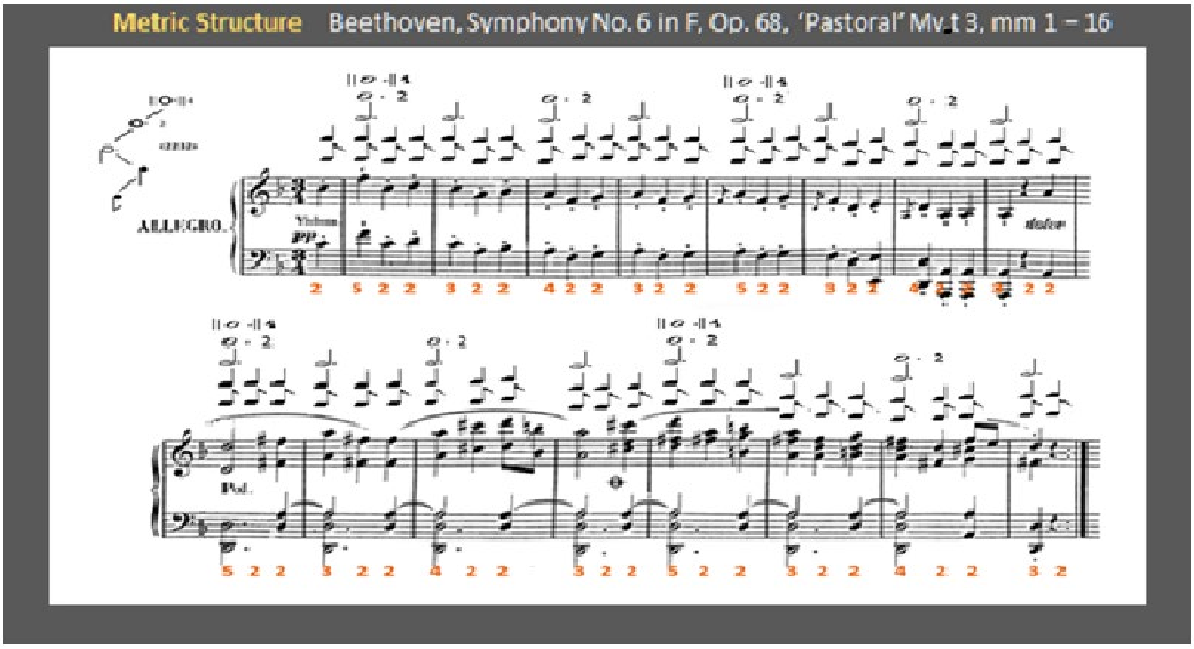


Figure 43: Ski-hill graph, pulse stacks, and numbers for counting meter

Figure 44 illustrates the meter I mapped on a ski-hill graph where each pulse is colour coded to match Figure 46:

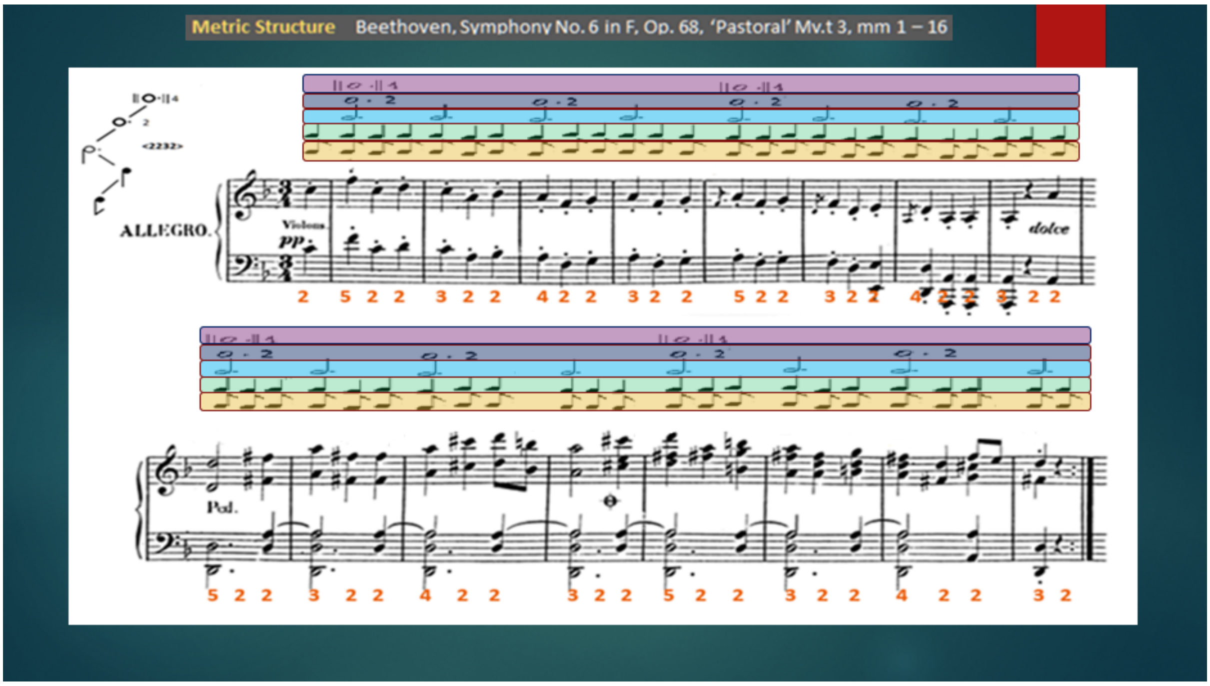


Figure 44: Colour-coding

Writing a Meter Narrative

An analytical report of meter written by a Year 10-11 student familiar with learning meter through the ski-hill graph might resemble the narrative below. In addition, the ideas for a narrative could be workshopped in-class using a listening log and written in more detail for homework or as an assignment.

In paragraph 2 of the following narrative, I begin to show how tonality can be discussed in relation to meter to help students begin the process of integrating both sets of data about tonality and meter. Figure 45, above the narrative, illustrates both harmony and meter analysed and annotated on the score to demonstrate how the pulses contribute equally towards the form by aligning with the harmonic structure of the excerpt to the hearing.

Figure 45: Harmony and meter analysed

Narrative:

"Listening to Beethoven's *Pastoral* mm. 1-16 I heard both duple meter and triple meter which I mapped onto a ski-hill graph as a deep meter: set <2232>. The pulses I heard included a four measure hyperpulse (imagined) in duple meter (see Figure 45 blue arrows) with a two measure hyperpulse (imagined) (see Figure 45 yellow stars) which formed duple meter with the downbeat pulse, a dotted half note. The dotted half note (physically notated for about half the section) formed triple meter with a quarter note pulse, and the quarter note pulse formed duple meter with an eighth note pulse (imagined but briefly "appearing" in measures 11 and 15). The music is metrically consonant because all of the pulses are in a relation of inclusion in a hierarchy as represented on my ski-hill graph (Figures 39 and 45).

I began entraining to sets of three quarter notes after the "anacrusis" where the dominant chord V led to the tonic chord F major on the downbeat bringing with it a harmonic accent. Listening retrospectively, I interpret the weak upbeat onset of the opening onset of this piece as occurring in what would be the last measure of the first of the four hypermetric timespan sets.

I continued entraining and projecting sets of three quarter notes with a dotted half note which formed a minimal triple meter because the downbeat took a melodic contour accent due to the onset of the highest pitch in the melody which was also a note of the tonic chord on each downbeat in measures 1-8 and 13-16. Although in measures 9-12 the material contrasts with mm. 1-8, I continued to project a set of three quarter notes and a dotted half note downbeat until measure 16 largely because of the onset of D major triad tones on the downbeats and parallelism of the new pattern in the bass.

In measure 3 I heard a two measure hyperpulse due to a melodic accent with the lowest tone of the tonic triad of F major placed on the downbeat and also measure 7 where the new tonic of D major is placed on the downbeat. In measures 5 and 7 the embellishments of acciaccaturas before the downbeats contribute to the formation of a four measure hyperpulse and two measure hyperpulse. At measure 13 the melodic contour contributes to my hearing of the four measure hyperpulse with a leading note rising to the tonic of the new key of D major (see Figure 45)."

The annotated score of Figure 46 demonstrates how both the structures of meter and harmony align in Beethoven's *Pastoral* mm. 1-16:

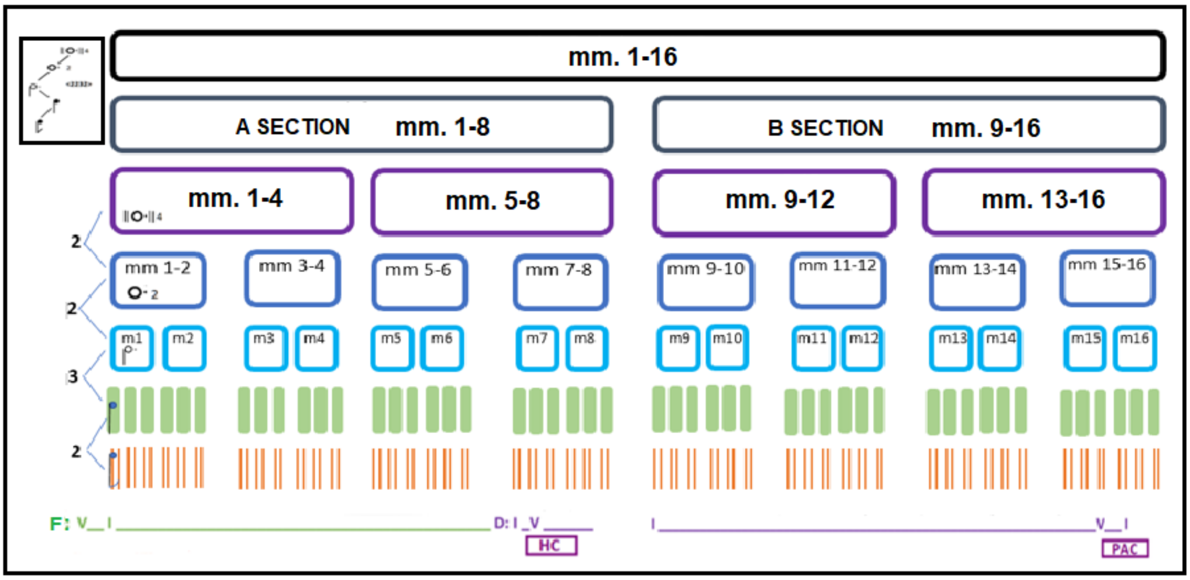


Figure 46: Hierarchical metric and harmonic structure (colour coded)

Figure 46 is a graphic representation of the metric and harmonic structures, in other words, the form, based on the pulses I mapped on the ski-hill graph. This graphic representation enables students to see similarities between both structures of meter and tonality.[96] In this

[96] In this graphic I included two additional structural levels not represented by the pulses I heard on my ski-hill graph which includes only 5 pulses. The first "extra" level included is the whole section mm. 1-16 and the next

piece, both meter and tonality share isomorphisms of structural symmetry, they align and form sets within sets forming a tonal and metric hierarchy. I have not included the opening onset of the “anacrusis” but another graphic of the piece might illustrate the first set of four measures which begins the formation of the four measure hyperpulse. Students should listen to Beethoven’s *Pastoral* while reading the annotated scores of Figures 45 and 46 as the teacher tracks the progression of the structure for the class.[97]

With so many similarities and connections between meter and tonality for listeners of music, it is important to study both fields equally so as not to overlook important structural material constituting the form of a piece of music such as illustrated in Figure 46. From spending more time observing music in this way, students are situated in a much stronger position to compose, perform and understand music through participating in a more deeply satisfying experience. Thus, as demonstrated through the materials above, I would conjecture that students would benefit from spending more time on less repetition of content, or content which is not integral to their progress, in the school music curriculum to assist them to understand music more deeply over a shorter amount of time.

Teaching Note Values Through Ski-Hill and Cyclic Graphs

In the following section I explain how traditional notation for illustrating and understanding note length can be taught through ski-hill and cyclic graphs, and I also explain how this method raises questions about how we traditionally understand and label note values. The materials used Bach’s Prelude and are intended to complement the materials in the previous sections by either preceding them or being used concurrently or independently of them. With more experience analysing meter, other music can be adapted for school-age students. This includes music which uses triple meter, and duple and triple meter together in the same music so that students can eventually notate more complex rhythms, including non-isochronous rhythms, polymeters, and music which is not metric (see later sections Teaching Simple

level is the piece halved into two symmetrical parts. My inclusion of the “extra” levels was motivated by a conviction that it is important to direct student attention to hierarchical structures in music which can be represented using mathematical principles. These other levels of meter which may be impossible to hear are still measurable with, for example, computers.

[97] Ideally an app could be developed for classroom use by students for the purposes of representing the isomorphisms in metric and tonal structures such as represented in Figure 46 from first analysing meter through the ski-hill graph (as demonstrated above).

Hemiolas Through the Ski-Hill Graph, SkiHill App, Cyclic Graphs, and Beat-Class Theory, and Sample Lesson Materials: Teaching Simple Hemiolas With Direct Metric Dissonance).

Not all students will be able to read traditional notation to map pulses to their ski-hill graphs in music classes, yet through identifying timepoints as integers using cyclic graphs, students who are unfamiliar with reading note values for traditional music can be taught how to achieve this skill. Most teenagers I teach about meter already know how to read, perform, and compose music using traditional notation, but until I started teaching them meter none of them had ever been taught how mathematics might help them understand their experience of music.

When showing teenage students how notation can be learnt through ski-hill and cyclic graphs, first analyse the periodicity of a pulse, find the polygon which represents the pulse, and observe its relation to other pulses through its sharing of timepoints. Then use a table pre-prepared to "match" the traditional notation to the pulse set (integers) and compose, read, and perform using traditional notation with new understandings. Students will need to be reminded that the expected answers on music tests will be their previous understanding of meter as notation-based until the system changes to adopt Cohn's modern meter theory. Regardless, students are interested to know more about this new development in music curriculum and they can see good sense in applying mathematics to their music studies.

If students have access to the app XronoBeat, the following exercise can be adapted and students can map directly onto the app; otherwise instructions for handwritten cyclic graphs are described below:

1. Draw a circle on the board and ask the students to copy the circle to their page about the size of a large jam jar lid (the XronoBeat app automatically displays a circle and if the classroom is equipped this can be projected onto the whiteboard and/or students can use a computer with XronoBeat).

2. Tell the students that they will use their circle to represent the rhythms they hear, and through discussion explain that because the rhythms repeat, it is good to use a circle because it also symbolizes repetition, pattern, and cyclicity; that

is, you can notate a rhythm once in a circle rather than a number of times in a straight line.

3. Explain to the students you would like them to first listen to a short section of Bach's Prelude.

4. After they listen to the section, briefly discuss J.S. Bach, the Baroque Era, stylistic characteristics, and similarities with music to which the students are familiar.

5. Ask the students to listen again, steering them to count the fastest pulse for what would equate to one measure: 16 unit pulses (sixteenth notes). Ideally students will know from previous materials that the pulses are also timepoints and that music "moves" through time and they can revise and/or be reminded of their previous lessons.

6. Write on the board c=16 and d=16 and explain that the number of timepoints equals sixteen and that these are also all performed as onsets in this particular piece.

7. Notate sixteen sixteenth notes, (low on the board so as to work up to the longest pulse to later correspond to the direction of pulses on the ski-hill graph) and explain/discuss the reason this pulse is called a sixteenth note. Demonstrate that the sixteenth notes can be grouped in eight sets of two – draw eight eighth notes above the sixteenth notes; draw four quarter notes above the eighth notes; two half notes above the quarter notes; and one whole note above the two half notes.

8. Model for the students sixteen equally-spaced (isochronous) unit pulses drawn on the circle using dots for sixteenth note pulses and ask them to do the same. On XronoBeat students can choose 16 as their unit pulse which displays as 16 timepoints numbered 0-15.

9. If hand-drawing, ask the students to label each unit pulse or timepoint as an integer from 0-15.

10. Repeat the process through listening to the Prelude guiding them through divisions of the circle into the next slower pulse one at a time. Use a different colour pencil or pen for each pulse: eight (eighth notes); four (quarter notes); two (half notes); one (whole note).

11. For each new division of the circle, ask the students to draw the polygon that represents each pulse. Discuss reasons for choosing each pulse.

12. The following Table represents the mathematical properties of meter for one measure of Bach's Prelude notated in a 4/4 meter signature: each division of the circle (metric cycle) (c), the periodicity of each pulse (d), the pulses notated using traditional notation, timepoint sets (pulses) notated as integers, inclusion, hierarchy, ratio, and polygons to represent each pulse.

Division	**Pulse**	**(c, d)** **c = size of the cycle** **d = periodicity**	**Sets**	**Polygon**
1	𝅝	(16,1)	{0}	Circle (red)
2	𝅗𝅥	(16, 2)	{0,8}	Digon (pink)
4	♩	(16, 4)	{0,4,8,12}	Square (green)
8	♪	(16, 8)	{0,2,4,6,8,10,12,14}	Octagon (yellow)

16		(16, 16)	{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15}	Hexa-decagon or 16-gon (purple)

Figure 47: Mathematical properties of meter

13. The students should now have a cyclic graph which looks similar to Figure 48. Figure 48 illustrates a cyclic graph representing rhythms constituting pure duple meter experienced by a listener equal to one measure of Bach's Prelude notated in a 4/4 meter signature. The red line on an angle to the right is the sweep which rotates clockwise as the pulses represented proceed in sonifications and/or animations through their hierarchical cycle, forming meter in the listener:

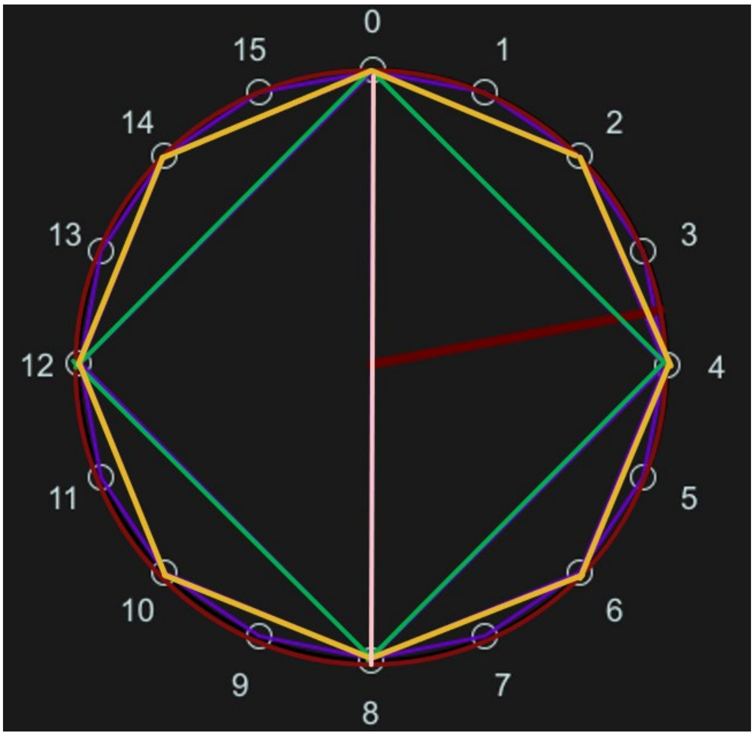


Figure 48: Cyclic graph: pure duple meter

14. Explain to the students that while they listen to the Prelude they will perform pure duple meter as a class in groups by reciting the rhythms each pulse makes. The syllable "O" ("oh") can be used instead of the word zero ("zero" has two syllables which may offset a faster pulse). Each group is assigned a pulse and all timepoints sets are performed in groups.

15. Looking to the score for clues, discuss the pulses, their periodicity, and the divisions of the circle which represent each pulse and ask the students if all of the pulses they heard are written on the score. Discuss that not all pulses we hear are physically present in the notation or sound, rather, they are in our imagination.

16. Discuss the mathematical properties of meter: inclusion, division, multiplication, ratio, hierarchy, and evenness. Compare the cyclic graph and traditional notation, and emphasize the strength of cyclic graphs to represent these mathematical properties.

17. Discuss how longer pulses than whole notes make longer cycles and that many listeners hear Bach's Prelude such as possessing two-measure hyperpulses in 32 cycles, and two- and four-measure hyperpulses in 64 cycles (map these with more advanced students or assign as homework assignments).[98]

18. Map to the ski-hill graph the pulses heard for the 16-cycle in a 4/4 notation, and add the two- and four-measure hyperpulses discussed.

19. If students are ready they can be shown other metrically equivalent notations such as notations in 2/4, and 2/2; or they might write these out for homework on ski-hill graphs.

20. Map duple divisions as "empty" nodes on a ski-hill graph.

21. Discuss applicable mathematical properties (see Ex.12.) such as periodicity, division, inclusion, cyclicity, hierarchy, ratio, metric pathways, and embodied mathematics (music theory in practice).

[98] An interesting exercise for more advanced students is to ask them to extend the table to fit the two-and four-measure hyperpulses – the longest pulse becomes the 'whole note' (1). Thus, to teach about extending the table, it will be necessary draw on the mathematical property of meter – metric equivalence, and that notation is arbitrary when experiencing meter.

It is important to teach the students that the table in Figure 48 works well for duple meters but that they may notice anomalies when triple meters are introduced. For instance, where a dotted whole note becomes the whole note in Figure 49 for a notation such as 3/2, the division at the level of a sixteenth note would be equal to and could logically be labelled as a "twelfth" note with twelve "sixteenth" notes equal to a dotted whole note (whole note plus a half note).

In Figure 49, the table and cyclic graph represents one measure for each notated meter signature. Pulses mapped onto ski-hill graphs in Figure 39 are notated in the table in Figure 49 using colour-coding for each pulse. The same pulses are represented on a cyclic graph (right) with each pulse represented by its corresponding colour. Tables such as this can be used for all metrically-equivalent notations.

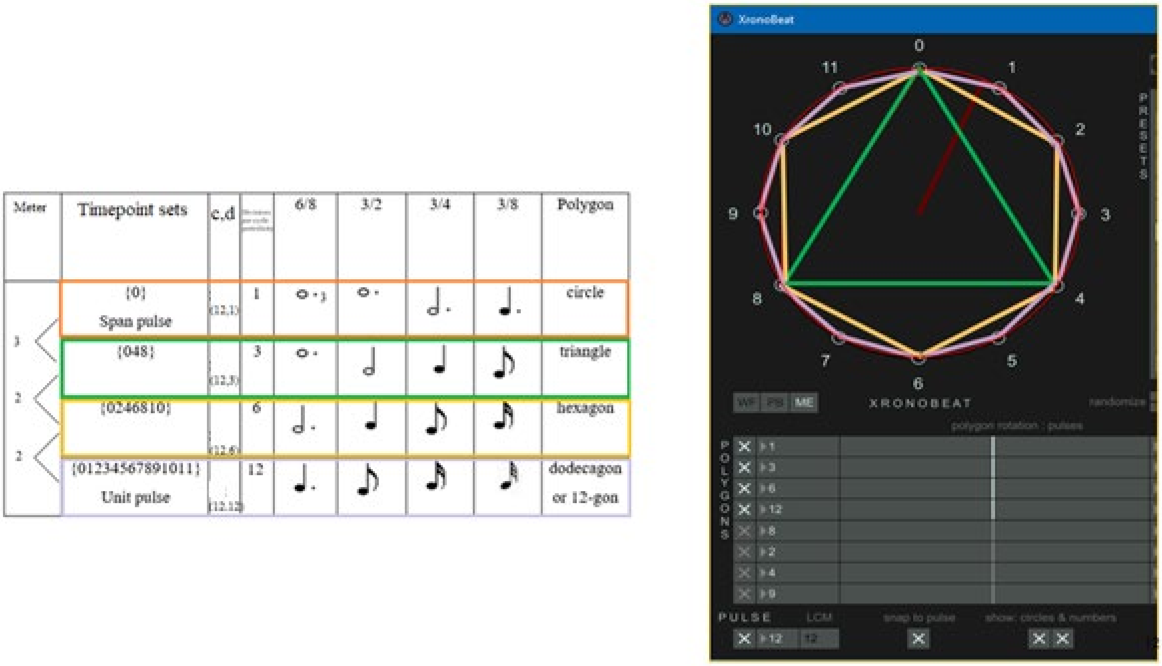


Figure 49: Note values, cyclic graphs, and colour coding

The system of notation most music students are familiar with is based on an understanding of notation where the undotted whole note, half note, and quarter notes, for instance, are usually labelled as notes with duple relations regardless of their divisions of a cycle. For instance, when a dotted whole note is designated the position of the whole, such as in a 3/2 notation, divisions at each metric level occur which should affect the naming of notes according to their new value mathematically. As an example, if a piece is notated using three half notes per measure, the label logical to the division into three is “third” notes, and quarter notes would become “sixth” notes, eighth notes become “twelfth” notes, but when tripleted the “eighth” notes become “eighteenth” notes!

When meter is understood as experienced rather than as notation, any note can be designated the position of the whole note; for example, a four measure hyperpulse will sound metrically equivalent to a number of different “whole notes” depending on the piece as heard. Obviously, the system currently in use delegates labels for notes which do not make sense but are also not thought about deeply by students and teachers; otherwise we would be teaching their meaning very differently. Although I am not proposing what the new names for notes should be, I do believe it is important that students understand how divisions of notes work in music making so that they can apply the meaning of a note (music theory) to practice (music theory in practice); and to do this they will need to be taught about the history of why notes are called different names for different purposes (historically informed theory).[99] Because notes can only be divided by either two or three, I conjecture we should be looking more closely at what Cohn proposes with his new mathematical music theory to unravel the mysteries of basic notation and how we teach music theory.

Through this approach to learning how to notate what they hear, students understand more deeply what it is they are representing when writing abstract representations such as traditional notation. Teaching notation in this way neutralizes potential marginalization of students who may not be able to read traditional notation when studying notated music, or students who intuit connections between mathematics and music who might raise rich questions about what they are hearing and/or the current system for labelling note values. Thus, through this approach to learning about notation the playing field is levelled to provide all students with the opportunity to learn how to notate music from the ground up.

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[99] Historically informed theory (HIT) is a term I coined in November 2016 in a paper I presented at the Melbourne Music Analysis Summer School: *Teaching Meter in Secondary School Music Programs.* In the paper I discussed the importance of developing secondary music education curricula to include a modern theory of meter, music theory history, and research in music psychology and neuroscience about meter to inform pedagogical practice and understanding. In Cohn’s 2015 classes on Meter Theory at the Sydney Conservatorium of Music, I wrote my first paper about introducing a modern understanding of meter through ski-hill graphs into the secondary music curriculum and classroom music textbooks.

More About Metric Equivalence

As explained in the previous section, when mapping pulses to the SkiHill app to represent meter, each polygon represents the periodicity of a pulse in a metric cycle, and students can be taught to identify and notate, compose and perform pulses and rhythms represented as timepoints that contribute to the formation of meter for the listener. In this way students can see and hear how the pulses interact with each other and how the pathways mapped on the SkiHill app are direct representations of their own experience of meter.

For instance, Figure 50 illustrates a deep meter possessing both duple and triple meter with pulses represented on nodes forming pathways from three minimal meters. These minimal meters can be represented as the deep meter the set <322>, which forms a hierarchy because of their relation of inclusion. Each pulse is in a relation of inclusion with the next fastest pulse thus each timepoint of the slower pulse is also a time point of the faster pulse.

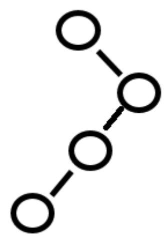

Figure 50: Ski-hill graph representing meter <322>

Figure 51 illustrates a hearing of the meter represented as the set <322> illustrated on ski-hill graphs in four different notations. Because they all share the same set of minimal meters, they will sound identical, thus they are all metrically equivalent:

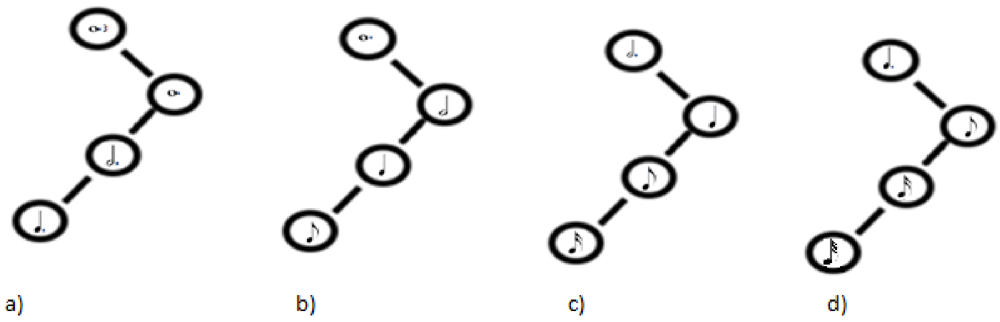

Figure 51: Deep meter <322> represented on ski-hill graphs

The nodes of each ski-hill graph are “filled” with notations which when sounded are identical to the hearing, thus they are all related by metric equivalence. Thus, Figure 51 demonstrates what Cohn refers to as notation being “arbitrary” because the ski-hill graphs represent notations in a) 6/8 meter signature; b) 3/2; c) 3/4; and d) 3/8. The pulse at the apex of the ski-hill graph is the Span pulse (the longest pulse) and the lowest pulse is the Unit pulse (the fastest pulse).

Figure 52 illustrates the same meter represented in Figures 50 and 51 mapped to a cyclic graph using XronoBeat:

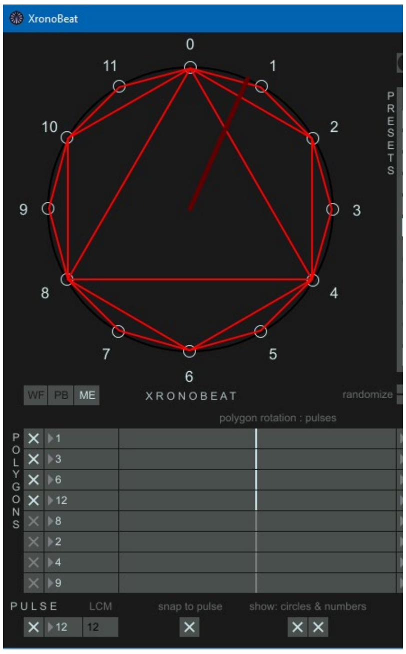

Figure 52: Deep meter <322> mapped to XronoBeat

Prior to using the ski-hill graph to analyse meter, it can be very important for students to listen to pieces which are familiar to them but notated in different meter signatures to help them develop their understandings of what meter is, the mathematical property of metric equivalence, “notation as arbitrary,” and to experience where meter is located – in the listener as a response to sound. An exercise I use with students from primary school-age to lower secondary students to explain metric equivalence is to listen to and perform well-known pieces, such as, the Christmas carol *Silent Night* by Franz Gruber, through using different notations such as 3/4 and 6/8 and to ask them if they can hear any difference in the meter (see Figure 53).

Most students hear the meter in the two arrangements of *Silent Night* in Figure 53 as identical, yet when they look at the notations they realise that the 3/4 arrangement would traditionally be labelled as simple triple meter, and the 6/8 version would be labelled compound duple meter on a written music exam.

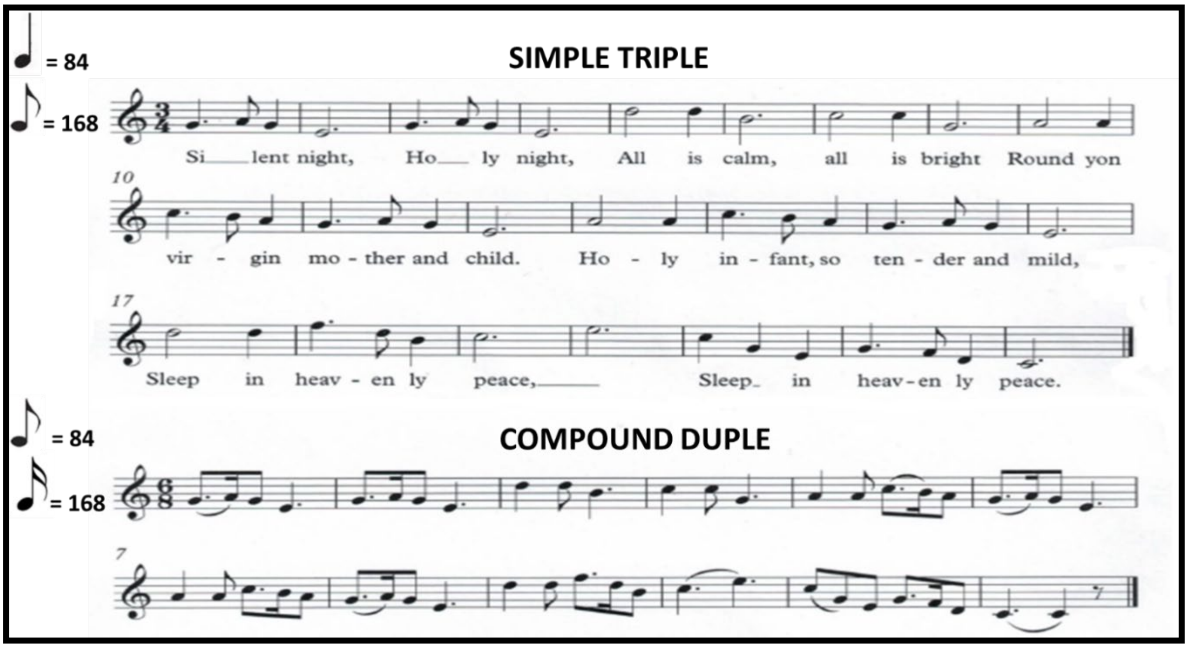

Figure 53: Christmas carol *Silent Night* by Franz Gruber notated in 3/4 and 6/8

Figure 54 illustrates a hearing of meter in *Silent Night* notated in vertical stacks reflecting two different notations, 6/8 and 3/4. On the left side of each pulse stack is a list of the minimal meters each pair of adjacent pulses formed to my hearing, ordered from the longest to shortest pulse. Thus, the set <2232> duple, duple, triple, and duple:

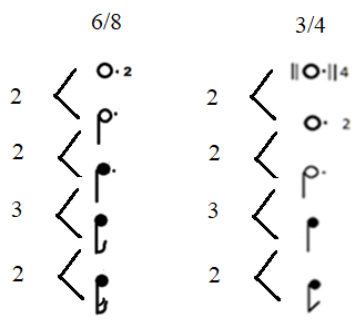

Figure 54: Metric equivalence represented by pulse stacks

These two different notations of *Silent Night* (3/4 and 6/8) would be considered two different meters if using Loulié's system to categorise meter as understood by the notation. Thus, metric equivalence provides the mathematical explanation for students as to why meter sounds identical in these two different notations of the same piece.

Through mapping the pulses heard by the listener onto ski-hill graphs, meter can then be observed more closely in order to make observations about differences or distances between pulses and meters, thus pathways of meters and their dynamic relation to each other.

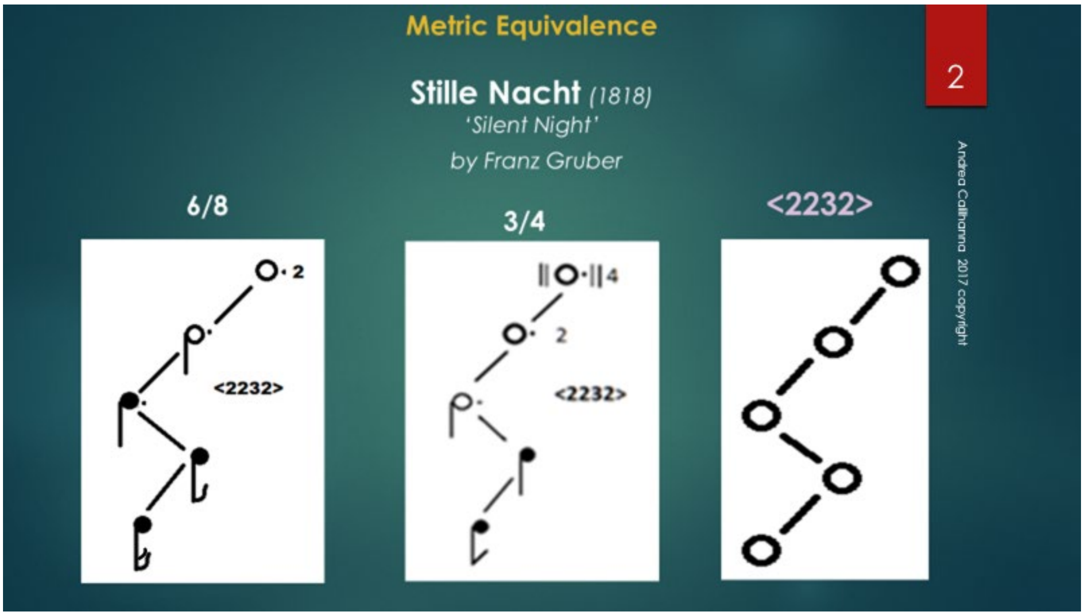

Figure 55: Ski-hill graphs representing meter <2232>

An example of using the ski-hill graph incorrectly is illustrated in Figure 56 a) and b), where the ski-hill graphs represent meter as the notation rather than meter as experienced by the listener for *Silent Night:* a) 6/8 and b) 3/4:[100]

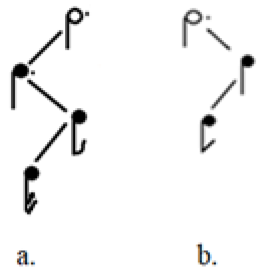

Figure 56: The ski-hill graph used incorrectly

Meter is not processed by the listener as measures when the meter of these two notations is heard as being metrically equivalent. Rather, where listeners hear these two arrangements as identical, meter is formed by the listener through the pairing of adjacent pulses in 2:1 and 3:1 relations – sets of duple and triple minimal meters such as <2232> duple, duple, triple, and duple together which form a deep meter. As mentioned earlier this approach recognises hypermetric pulses as contributing equally to the formation of meter for the listener. Thus, as Cohn explained in a lecture at Sydney University (2017), meter is independent of the meter signature and notated measures because there are many ways to notate music for the meter to sound identical.

Depending on the students, at generally around Year 10, students would proceed to listen to and perform pieces of a more challenging level such as Johann Sebastian Bach's *Jesu, Joy of Man's Desiring* in different arrangements, such as 3/4 and 9/8, to learn more about meter as dependent not on the notation but on what is heard.

When the mathematical property of metric equivalence is firmly grounded, and confusing labels of simple and compound are not involved in the discussion of meter, students comprehend more readily how it was possible Bach originally wrote *Jesu, Joy of Man's*

[100] I am not referring here to instances where students when first learning to map pulses to ski-hill graphs may leave out the hyperpulses due to unfamiliarity with these pulses.

Desiring in two different meter signatures. See Figure 57 for arrangements of *Jesu, Joy of Man's Desiring* in 9/8 and 3/4:

Figure 57: *Jesu, Joy of Man's Desiring* by J.S.Bach[101]

However, discussing *tempo giusto* in the context of historically informed theory is a pedagogically sound approach to teaching about Bach's choice of time signatures to suit the style and purpose of the music. See Figure 58: Johannes Sebastian Bach, *Jesu, Joy of Man's Desiring*, from the 10th movement of cantata "Herz und Mund und Tat und Leben," BWV 147 ("Heart and Mouth and Deed and Life"):

[101] (Top) *Jesu, Joy of Man's Desiring* by J.S. Bach for piano solo. Retrieved 8/10/17 from www.virtualsheetmusic.com. Used with permission; (Below) *Jesu, Joy of Man's Desiring* measures 1-9 arranged for Flute & Piano by Andrea Calilhanna (2018).

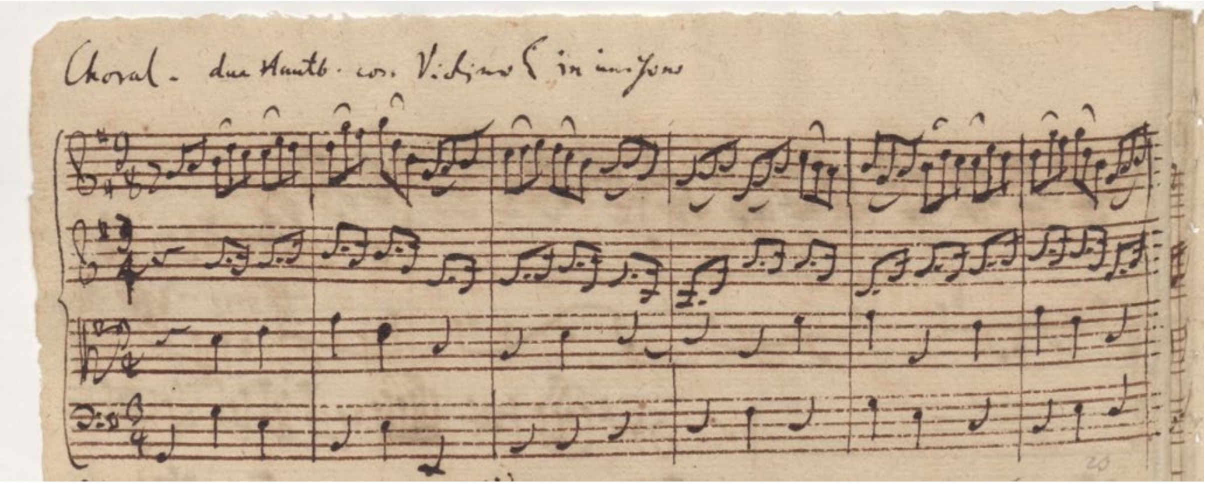

Figure 58: *Jesu, Joy of Man's Desiring* by J.S. Bach[102]

Students may well ask why Bach wrote *Jesu, Joy of Man's Desiring* in two different meter signatures – 3/4 with a quarter note pulse dividing into duple divisions and 9/8 with a dotted quarter note pulse dividing into triple divisions, and how this works in performance, because they do not hear this difference when listening. Students deserve to receive historically-informed answers about the music theory they encounter in their music lessons – historically informed theory.

For instance, the German music theorist Kirnberger, a student of Bach's who wrote about Bach's music, discussed the practice of using two meter signatures (Kirnberger, 1774, 396). Teachers can explain to students that a practice of notation from the eighteenth century was to choose one or two meter signatures to suit the style and purpose of the music in the tradition of *tempo giusto*. Because the function of *Jesu, Joy of Man's Desiring* is for liturgical purposes the 3/4 meter signature, like a minuet in 3/4, would indicate that the piece was not to be performed too fast, especially as it would be heard in an echoey church and required a certain restraint and prayerfulness.

Kirnberger noted that where music had predominantly quarter movement in sets of threes and 9/8 and 3/4 were notated simultaneously the parts in 3/4 were played lighter than the 9/8 parts and suggests the quarter note of the 3/4 meter signature was to be played as a dotted quarter note. In Bach's *Jesu, Joy of Man's Desiring* the melody is notated in 9/8, suggesting

[102] Autograph manuscript. Staatsbibliothek zu Berlin Mus. ms. Bach p. 102. Retrieved 8/10/17 from https://digital.staatsbibliothek-berlin.de/werkansicht?PPN=PPN84777578X&PHYSID=PHYS_0014&DMDID=DMDLOG_0001&view=overview-toc

the quarter note from the 3/4 parts are played as a dotted quarter note (the “tactus” in the tradition of *tempo giusto*).

A further comparison could be made to the stylistic practice for notating jazz pieces such as those used in many practical exams today where eighth notes are notated “straight” (somewhat like Bach’s eighth notes in the original score grouped “square” as four sixteenth notes) but played “swung” like the triplets we see in the 3/4 arrangement and 9/8 version of Bach’s piece in Figure 59.

In other words, students need to be taught to identify meter based on what they hear not on the notation and with Cohn’s contemporary meter theory they are now empowered to articulate and represent new classifications of meter. Students and teachers no longer need to grapple with using confusing labels that were used by composers and schools in various countries in different ways hundreds of years ago, except where necessary to have an understanding of theory as historically informed.

The meter of the two arrangements in Figure 59 sounds identical because they are metrically equivalent. This quality can be explained through mathematics and abstract representations using numbers and single notes to represent sets of pulses which form meters (see Figure 60):

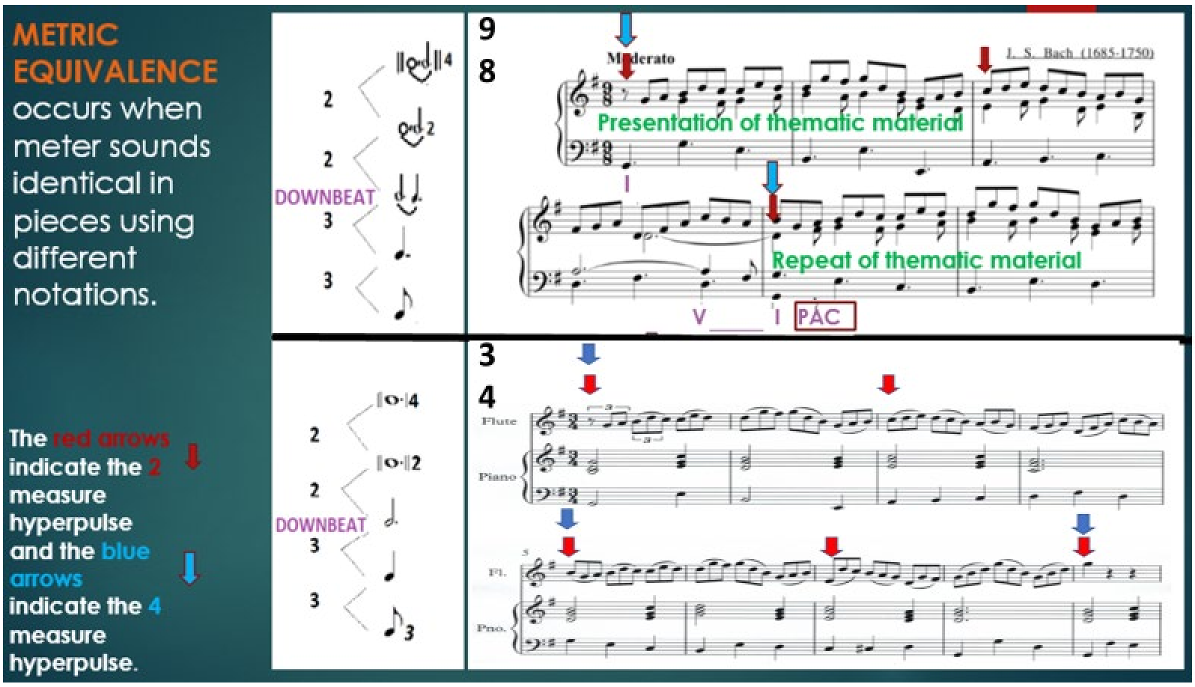

Figure 59: *Jesu, Joy of Man’s Desiring* by J.S.Bach

The numbers labelling the sets of pairs of vertical pulse stacks in Figure 59 represent the mathematical explanation for why both versions of these notations sound metrically equivalent. They both have an identical set of minimal meters formed by pairs of pulses in a relation of inclusion: two sets in ratios of 2:1 forming duple meter to the hearing and two sets in ratios of 3:1 forming triple meter to the hearing.

In each pulse stack there are two hyperpulses which formed duple meter to the hearing <2> and another two pulses which formed duple meter also labelled <2>. Next there are two levels of triple meter between the downbeat pulse and one faster and one faster again both labelled with <3>. The numbers within the brackets form sets made up of two pulses per set which make a minimal meter to the hearing. Thus, for *Jesu, Joy of Man's Desiring* the total set of minimal meters making meter can be summed up as <2233>.

Figure 60 illustrates how students can represent the metric equivalence discussed here for *Jesu, Joy of Man's Desiring*, through using the ski-hill graph where duple meters are graphed to the left and triple meters are graphed to the right. Meter is represented as five nodes with two edges forming duple meter to the hearing and two edges forming triple meter to the hearing. This can be summarized as a set called <2233> and this same representation can be applied for any music related by metric equivalence where this combination of minimal meters is heard as the set <2233>:

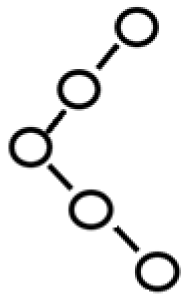

Figure 60: Ski-hill graph representing meter <2233>

Evidence of meter mapped from pairs of pulses in a relation of inclusion heard when listening to music can then be mapped onto the nodes of a ski-hill graph using traditional notation. Figures 61 and 62 illustrate ski-hill graphs representing the metric pathways mapped from listening to Bach's *Jesu, Joy of Man's Desiring* notated in two different meter signatures 9/8 and 3/4 (see Figure 61) thus demonstrating the mathematical property of metric equivalence:

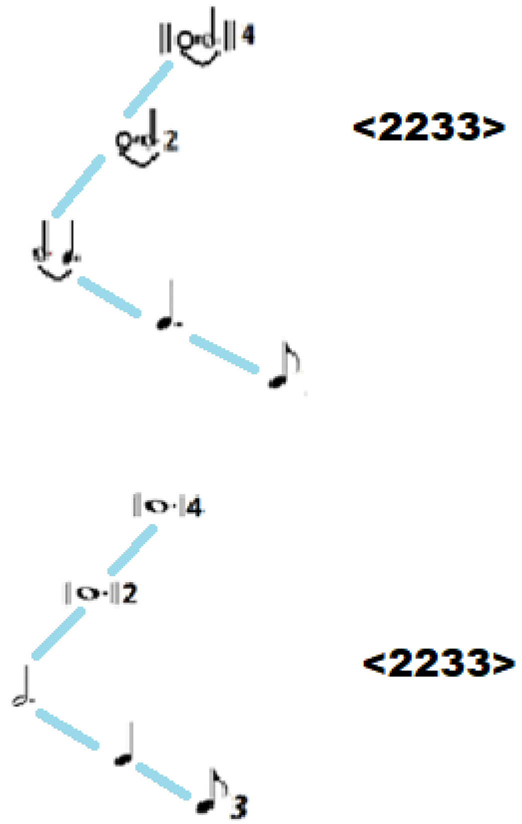

Figure 61: Ski-hill graphs representing meter <2233>

Figure 62 illustrates Meter <2233> represented on the SkiHill app as fractions and geometric polygons:

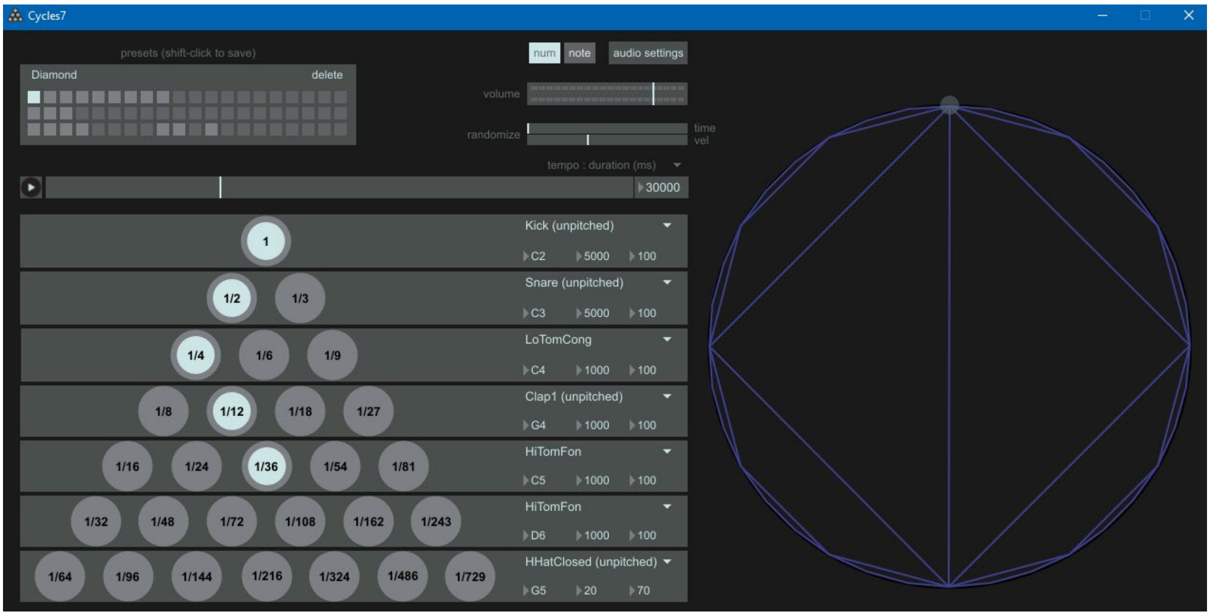

Figure 62: Meter <2233> represented on the SkiHill app

As mentioned in an earlier section, "Counting Meter," students are often bewildered as how to count when dotted notes appear on a score, for instance, they may never have been taught that a note with a dot is worth three of its next faster adjacent pulse in the metric hierarchy. Music theory in textbooks rarely addresses, in any depth, situations where students count a dotted half note or dotted quarter note as "three" then explain that those same notes can also be counted as one, one and a half, two, or three, and so on, depending on the mathematical relation of pulses in relation to each other in a hierarchy. Counting meter using numbers in this way offers a solution where confusion arises about counting note values.

In Figure 63 I have included all of the pulses notated as well as the numbers for counting meter to illustrate how the pulses correspond to those mapped on the ski-hill graph for Bach's *Jesu, Joy of Man's Desiring* in a notation in 9/8:

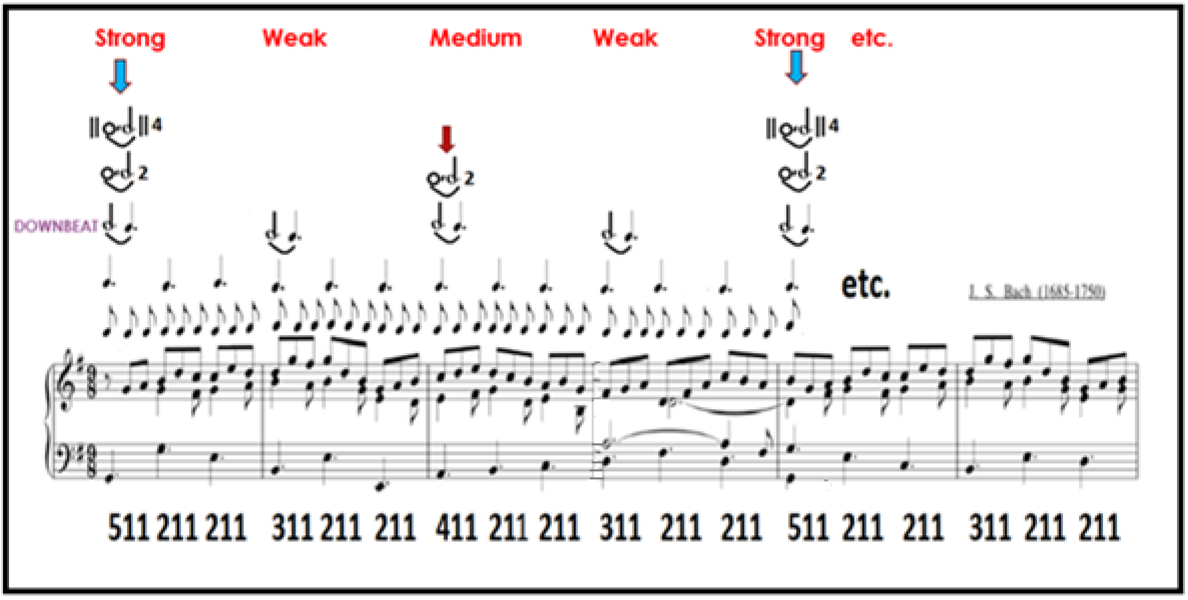

Figure 63: Annotated score *Jesu, Joy of Man's Desiring* by J. S. Bach

Figure 63 illustrates Bach's *Jesu, Joy of Man's Desiring* with numbers to represent the number of pulses I heard on each onset. Listening retrospectively, in measure one the downbeat had the most pulses with at least five pulses making it a strong onset at the hypermetric level (the blue arrows represent the four measure hyperpulse and the red arrows represent the two measure hyperpulse).

In measure one the following two eighth notes on upbeats have only one pulse on G and one pulse on A making them weak onsets. The next onset has two pulses but is comparatively weak. The music continues in this way until measure two where three pulses are heard on the downbeat making a slightly stronger accent than those previous but a weak accent at the hypermetric level. The downbeat of measure three has a medium accent because here the two measure hyperpulse can also be heard but the downbeat of measure four is weaker with only three pulses heard. This cycle repeats at measure five with the return of the four measure hyperpulse making a stronger accent.

Asking students to sing *Jesu, Joy of Man's Desiring* using numbers to represent the number of pulses stacked on each onset mapped from their ski-hill graphs can be very productive, especially if students are asked to see whether they intuitively place more stress on one onset than another. Naturally, no one is bound to perform a piece in any particular way, because meter is experienced subjectively. However, I have found counting meter in a hierarchy for metric music rather than using more traditional approaches can be very helpful for school-age students who are learning the basics about meter as well as more advanced students who want to learn how to interpret their pieces to perform more musically and to learn how downbeats and upbeats occur mathematically in their performance experience. This occurs because counting meter in this way takes into account all of the pulses which contribute to making meter. For instance, students involve hyperpulses in their musical performance, pulses which are not represented by the meter signature, which can shape longer sections of music for a more satisfying performance.

Annotating the score with numbers for counting through first mapping the pulses heard on a ski-hill graph allows students to see how the pulses and meters they form are directly related to the music they perform. This enables the student to observe more closely how the composer played their part in providing notation (if the music is notated) as an abstract communication for the performer to interpret. Critically, a young student may never learn about hyperpulses or how to make their performances more musical and rhythmically stable at all metric levels if their teacher is not knowledgeable of meter at every level.

Beat-class theory (Cohn, 2017) takes counting meter a step further by assigning a number to each onset so that the mathematical properties of meter, and rhythms can be described through set theory including: evenness, cyclicity, periodicity, timespans, timepoints, geometry, rotation, complementation, and symmetry. Unlike traditional notation, these observations are made possible because of the sonifications and visualisations using the app XronoBeat where students add integers to the timepoints of cyclic graphs and, through modular arithmetic, explain the distances between pulses and the meter that is formed in the listener as embodied mathematics.

The following cyclic graph in Figure 64 represents a cycle of the pulses making meter to the hearing in what could be described as one measure of Bach's *Jesu, Joy of Man's Desiring* notated in, for instance, 3/4 and/or 9/8:

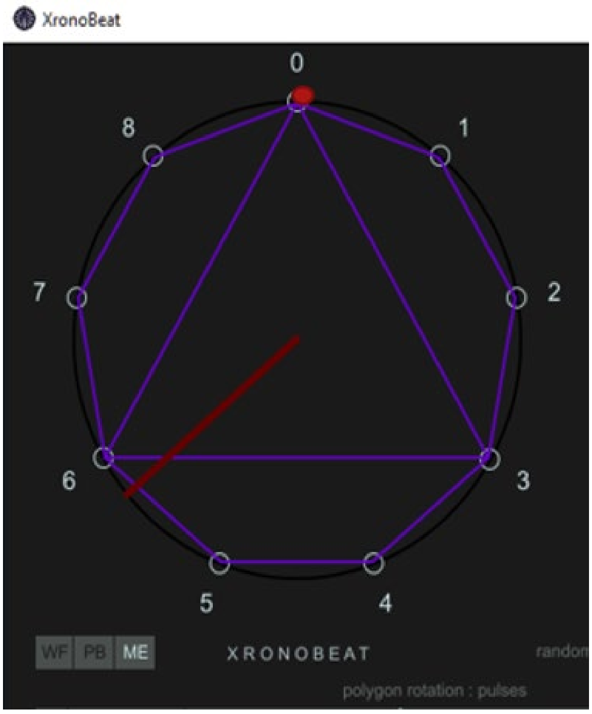

Figure 64: Cyclic graph

If students are already familiar with traditional notation they can make observations about cyclic graphs so that questions such as the following can be part of lessons and exam papers: “If this is one measure of a 3/4 notation and the red dot is the downbeat pulse what does the triangle represent?” Answer: Three quarter note pulses. “What pulses make up the nonagon (9-sided polygon), in other words, what are the fastest pulses?” Answer: Tripleted eighth notes. “Map the onsets mapped on the cyclic graph to a ski-hill graph using empty nodes.” “What duple, triple, or duple and triple meter do these nodes represent? Write your answer as a set <   >.” “Name at least three meter signatures a composer could use to notate music which has the same meter when heard as represented on this ski-hill graph.”

If students are not yet familiar with reading traditional notation, they can be taught the skill of understanding note values through using ski-hill and cyclic graphs to teach notation (see an earlier section Teaching Note Values Through Ski-Hill and Cyclic graphs). Through cyclic graphs and beat-class theory students can make observations about their experience of meter such as: the onsets are evenly spaced; the pulses divide evenly into each other in a hierarchy (inclusion); the pulses are in sets, and each polygon represents the periodicity of one pulse.

With beat-class theory the size of the cycle is labelled as c and here c = 9 because the fastest pulse – the unit pulse – is a quarter triplet for a 3/4 notation and there are 9 unit pulses in one cycle, d = the number of onsets for each pulse, that is, the periodicity of each pulse. Figure 65 illustrates three different divisions of the 9-cycle resulting in three pulses, their polygons, and graphic representation of inclusion in an evenly spaced hierarchy:

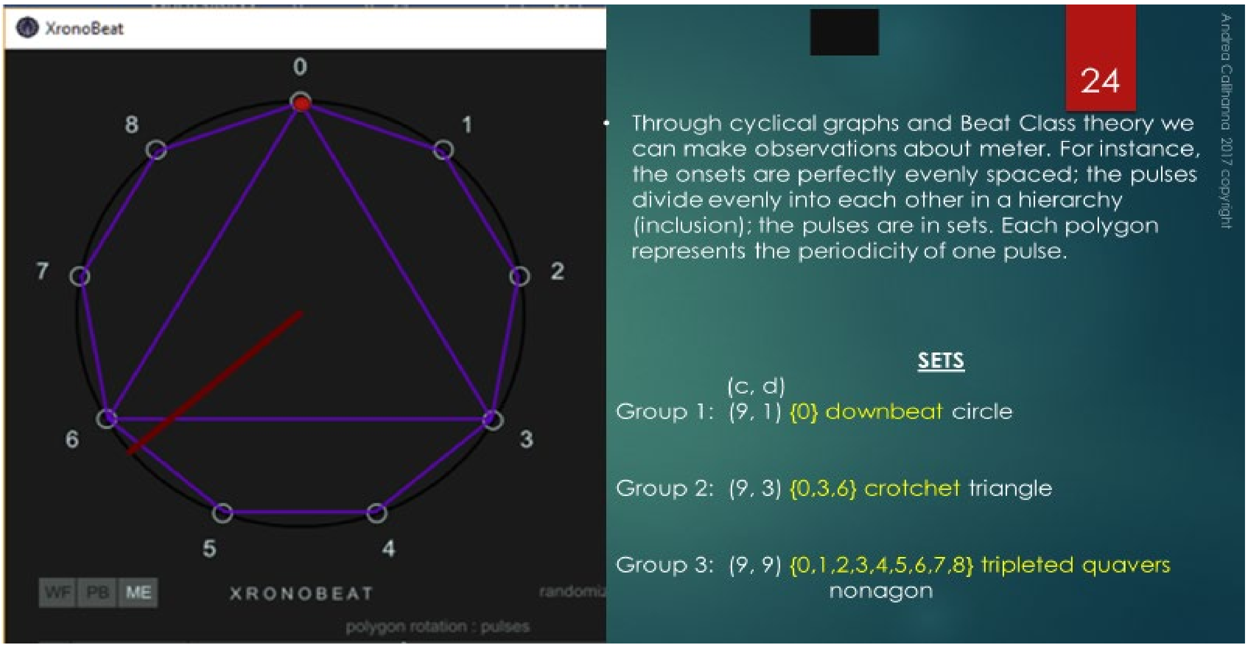

Figure 65: Metric hierarchy, cyclic graph and beat-class theory

With undergraduate students (pre-service classroom music teachers) I asked them to split into three groups to clap the meter for Bach's *Jesu, Joy of Man's Desiring*:

(9,1) Group 1 0

(9,3) Group 2 0 3 6

(9,9) Group 3 0 1 2 3 4 5 6 7 8

During the recitation I followed the cycle around the circle. The undergraduate students recited the numbers with zero based counting – the advantage of which is to show the generating integer – here it is 3.

I also provided larger cycles where students could then apply their new knowledge of hypermetric pulses to counting meter represented through cyclic graphs. Figure 66 (left to right) represents the pulses from hearing one, two and three measures of Bach's *Jesu, Joy of Man's Desiring*.

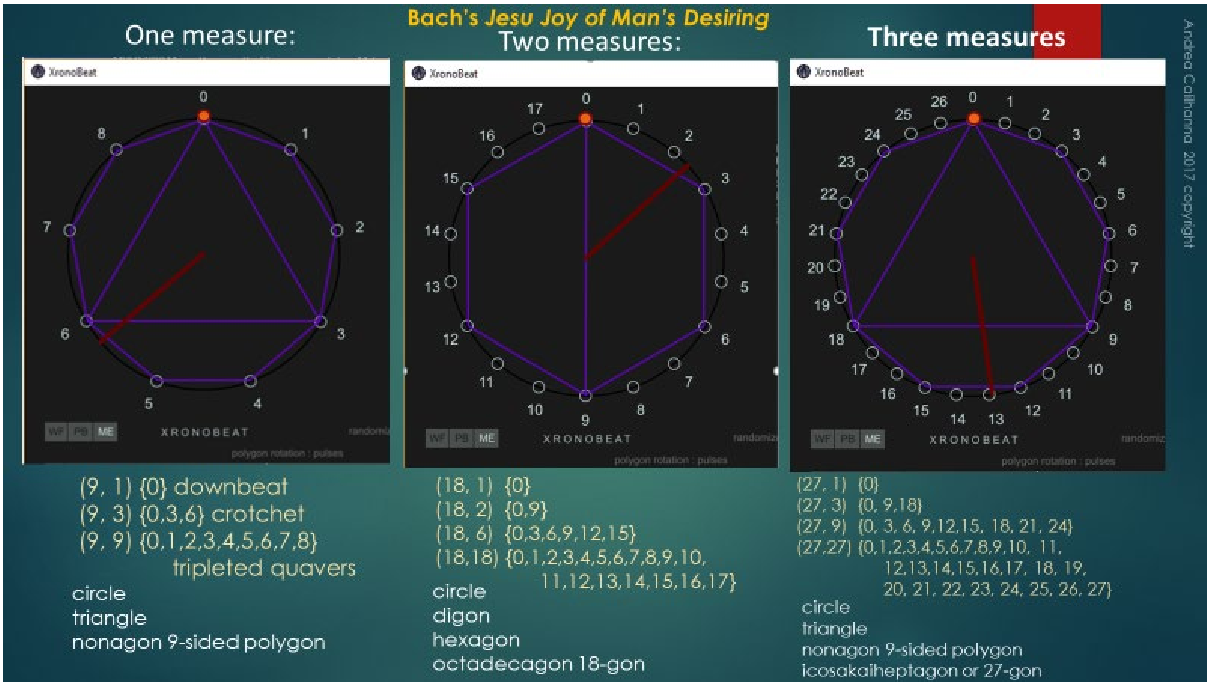

Figure 66: Hypermeter, cyclic graphs, and beat-class theory

In Figure 67, the cyclical graph XronoBeat, the SkiHill app (using a 3/4 notation) and beat-class theory, represent meter from a hearing of four measures of Bach's *Jesu, Joy of Man's Desiring* (notated in both 9/8 and 3/4) so as to include representations of the four measure hyperpulse in the metric hierarchy.

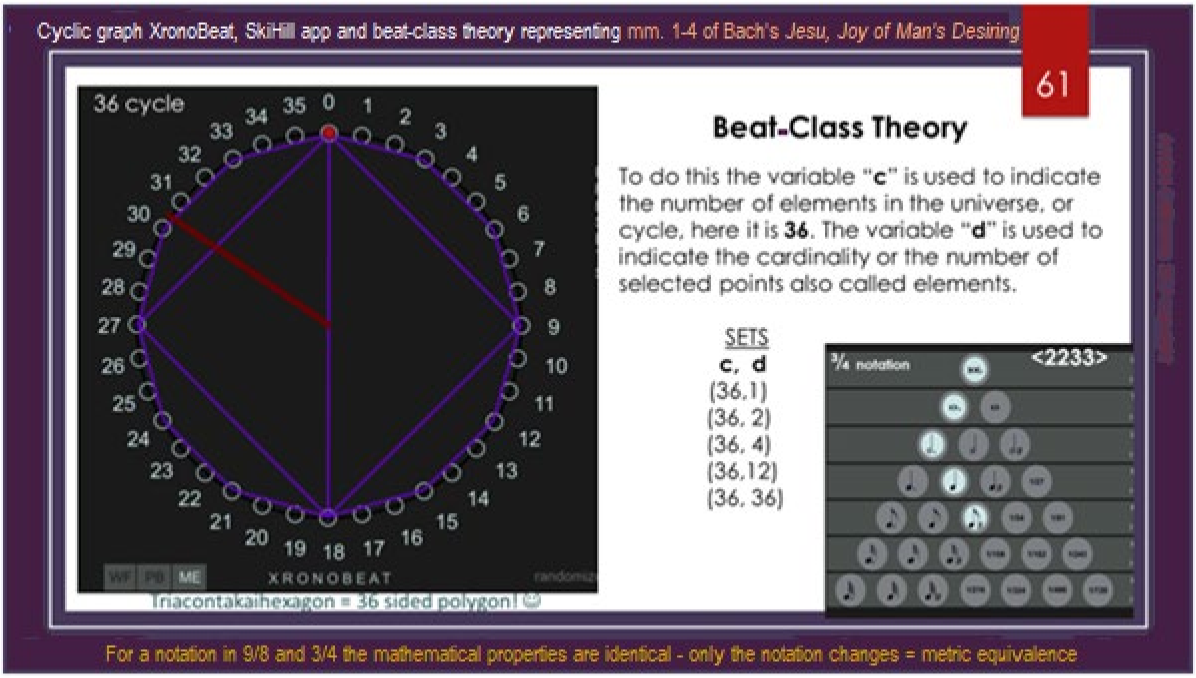

Figure 67: Hypermeter, ski-hill graph, cyclic graph, and beat-class theory

Through these theoretical and practical activities using instruments of mathematical music theory, students are thus provided with music pedagogy to enable them to observe mathematical properties of meter. For instance, as demonstrated in Figure 67, the mathematical properties of the sounded music are observed as being identical for both the 9/8 and 3/4 notations, that is, they are metrically equivalent.

Figure 68 is an instructional slide for illustrating the periodicity of each pulse:

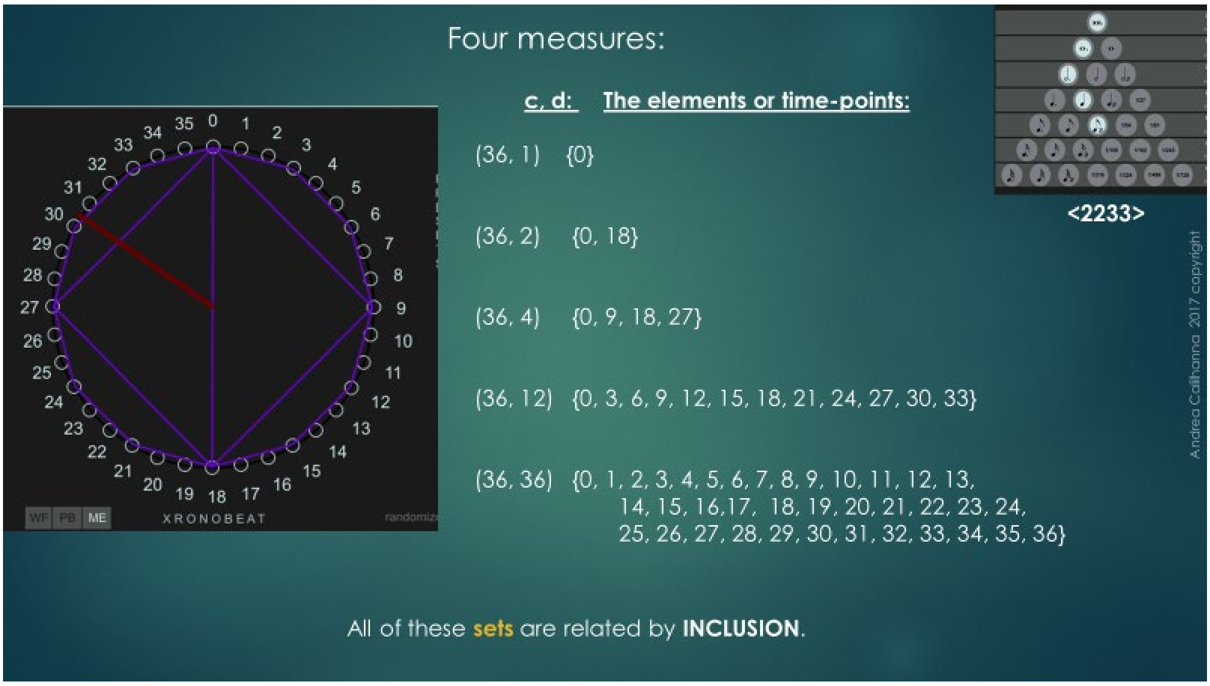

Figure 68: Instructional slide

Figure 69 is an instructional slide illustrating traditional notation and the periodicity of each pulse:

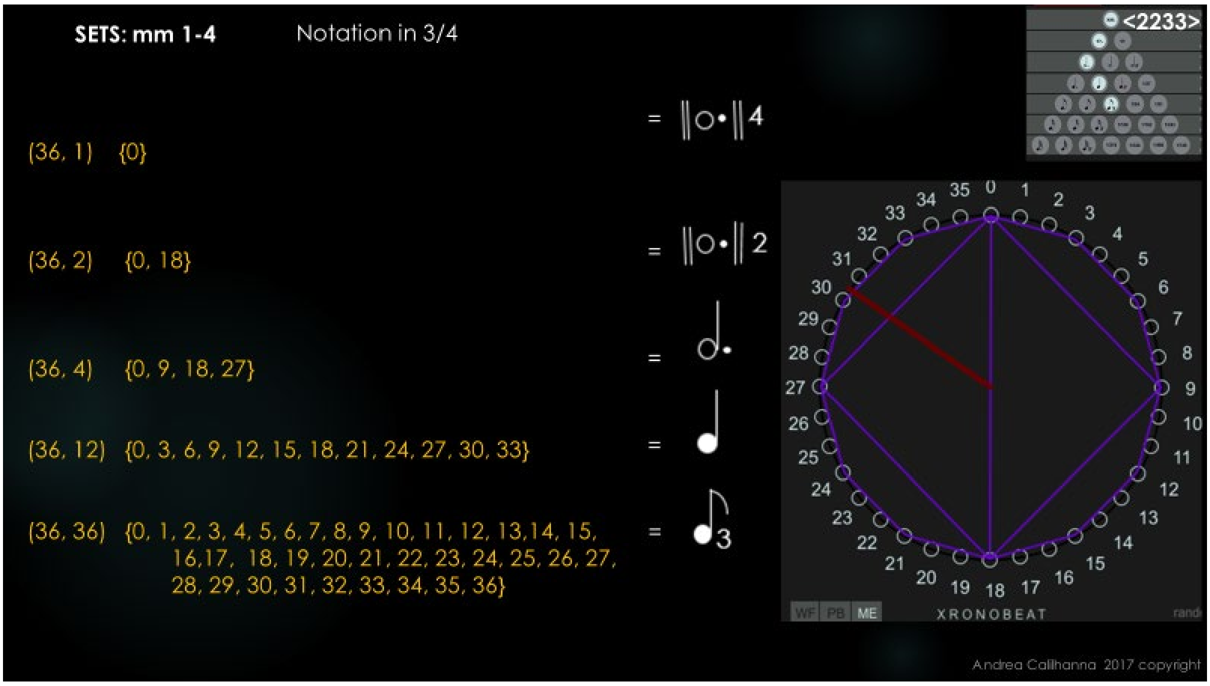

Figure 69: Instructional slide

For most students learning meter through pulse stacks, ski-hill, and cyclic graphs, beat-class theory and counting meter in a hierarchy is the first time they realize the importance of the hypermetric pulse when performing music, thus opening up new ways to view a phrase and musical line. Other benefits for students include development of listening skills through participating in aural training which utilizes audiation, improvement in sight reading skills from more practice in and application of logical hierarchical counting, and increased confidence in performance through developing deeper knowledge about meter to apply to performance preparation.

Through learning meter with the help of graphic representations and basic mathematics, students are equipped with the knowledge they need to convey expression based on an understanding of the metric and tonal structure which generates the form of a piece of music. This occurs because all of the pulses that make meter to the hearing contribute to the form including the hyperpulse which often aligns with the tonal structure of a piece. Knowledge of hyperpulse(s) can then be taught with its application to more musical performances in mind for young students, for instance, to reduce over-accenting downbeats in a performance. For instance, in Figure 63, measure 5, at the hypermetric level the metric and tonal structures align with a harmonic accent occurring on the downbeat of measure 5. The significant chord change and bass note movement V-I of the PAC in mm. 4-5 usher in the repeat of the opening thematic material forming a structural "boundary" of the four measure hyperpulse and hypermeter. Thus, teaching music in this way helps students to realise how studying both tonality and meter together contribute to achieving a more satisfying performance and a deeper understanding of their experience of metric and tonal structure in music. I find this personalised style of learning about music motivates curiosity and engagement in students because the results have direct currency for their analyses, compositions, and performances. The information and knowledge they assimilate is not the result of a teacher telling them how to hear or what to think: rather they learn new ways of understanding meter because Cohn's meter theory helps students describe what is taking place for them as the listener.

Part 2: Teaching Simple Hemiolas Through the Ski-Hill Graph, SkiHill App, Cyclic Graphs, and Beat-Class Theory

All of the materials presented to this point can be described as being metrically consonant because all of the pulses that make meter have been in a relation of inclusion. School-age students who learn musical instruments and music theory at some stage study pieces for their exams where they can hear pure duple meter, and duple and triple meter in the same piece. Many pieces students learn are metrically consonant but sometimes a hemiola occurs through the regrouping of sets of pulses where metric dissonance occurs, such as in a section before a cadence, and most often it is necessary to assist students with the performance of these pulses.

The following section describes how students can learn about indirect metric dissonance through the ski-hill graph where hemiolas occur. Hemiolas can occur either as indirect metric dissonance where the conflicting pulses in a ratio of 3:2 are heard consecutively or as direct metric dissonance where the "conflicting" pulses (pulses not in a relation of inclusion) are heard together at the same time.

Etude No. 13, *Petite Valse*, Op. 45 by Stephen Heller is a Sixth Grade piano piece where the majority of the music is notated in a 3/4 meter signature and is mostly metrically consonant to the hearing. School-age students preparing for a piano examination may complain that they have trouble performing the rhythm of measures 33-40 (see Figure 71). This can occur because they had been taught previously to understand the time signature as the meter and to perform the piece according to the time signature.

Sometimes students diligently attempt to perform mm. 33-40, a section notated in and sounded as 6/8, with an understanding of performing as if it were written in a 3/4 meter signature. In many ways the problem is quite straightforward, and a teacher could easily tell a student how to perform the piece by regrouping the eighth notes from "twos" to "threes." Where students have experience with ski-hill graphs for understanding metrically consonant music additional benefits occur from learning more about meter through mapping their hearing of the metric dissonance resulting from a simple hemiola in their pieces.

To help students solve the problems they experience with the rhythm in this piece a teacher might perform *Petite Valse* and workshop together a ski-hill graph to map the pulses that the student hears, as in Figure 70:

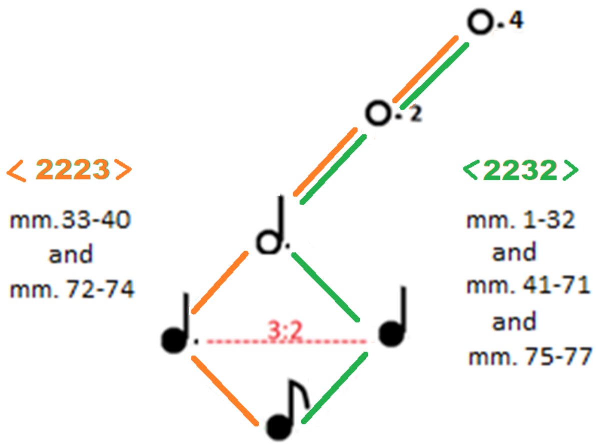

Figure 70: Ski-hill graph representing a simple hemiola 3:2

From seeing a graphic visualisation of their experience of meter students solve problems by realising they need to regroup the eighth notes from sets of two to grouping sets of three eighth notes.[103] Through discussing the hierarchical relation of the pulses in inclusion relation students are able to understand that even though the downbeat dotted half note and the number of eighth notes remained the same throughout the piece, there were two different meters occurring in the piece, which they had not realised before. Critically, students can now see and hear the two pulses that were not in a relation of inclusion and where the mathematical ratio of 2:3 was the cause of indirect metric dissonance.[104]

[103] Although using colour is not critical to graphing representations of meter through ski-hill graphs there are additional benefits for students visually and spatially through colour-matching the edges of the ski-kill graph to their annotation of a score, such as the coloured arrows shown in the examples. In this way, students are better positioned to identify metric pathways and metric structure to then explain the relations between pulses and meters in a piece of music.

[104] For a description of how I use the SkiHill app with students in their practical examination preparation to learn about simple hemiolas see the book chapter: Hilton, C., Calilhanna A., Milne, A.J. Visualizing and sonifying mathematical music theory with software applications: Implications of computer-based models for practice and education. *Theoretical and Practical Pedagogy of Mathematical Music Theory. Music for Mathematics and Mathematics for Musicians, From School to Postgraduate Levels. Eds. Montiel, M. & Gomez, F.* (World Scientific Press, December 2018). Note the ski-hill graph and score annotations were originally in colour but in the book they are given in black and white.

As part of the lesson the student and teacher can also look to the score for further clues as to how the composer achieved indirect metric dissonance with a simple hemiola. See Figure 71, an annotated score of mm. 28-39 of *Petite Valse*:[105]

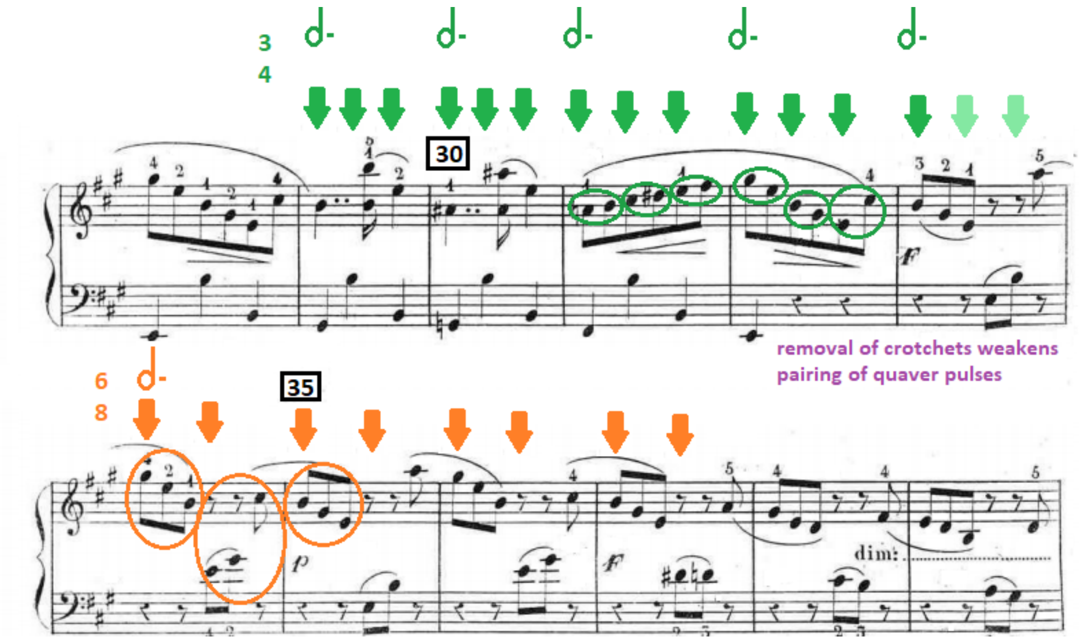

Figure 71: Annotated score

Through applying new knowledge about meter by mapping pulses to a ski-hill graph, students are now equipped to annotate their score to observe how the composer shifted to the new meter. Students realise they had been entraining to the quarter note pulse (see the green arrows in Figure 71) and projecting sets of pairs of eighth notes in a minimal duple meter with the quarter note pulse (see green circles in Figure 71). Through listening carefully, mapping pulses, and studying the score students also notice at measure 32 that the composer began to "weaken" the meter <2232> first by removing the quarter note pulse in the left hand (see Figure 71, mm. 32-33).

As the meter of *Petite Valse* morphs from <2232> to <2223>, beginning with the removal of the quarter notes at measure 32, the piece also modulates to the dominant key of E major through an imperfect authentic cadence (IAC).[106] Thus, through listening, mapping

[105] Stephen Heller. 25 Etudes mélodiques. Etude No. 13 *Petite Valse* Opus 45 mm. 28-39 Retrieved from http://ks.imslp.net/files/imglnks/usimg/4/43/IMSLP253721-PMLP25002-SHeller_25_Etudes,_Op.45_BNE.pdf

[106] An imperfect authentic cadence (IAC) is defined by Caplin (2013 p.708) as an authentic cadence but where the soprano voice ends on the third scale degree.

representations of meter, and observing the score, students can make observations about how the composer changes the character of the piece through both meter and tonality.

Students may still entrain and project minimal duple meter with the quarter note and quaver pulses through measure 32 and into 33 although they realise the quarter note pulse was in their imagination during this section. Through this approach students notice the physical removal of the quarter note pulse contributed to a short section where the meter seemed ambiguous at measure 33 (see Figure 71, the pale green arrows), and where it was possible to group in either 2s or 3s starting with the first quaver. Students learn that it was here the meter was morphing into a new meter <2223>.

The student and teacher can discuss how, through the experience of parallelism due to the "sequential" nature of the material beginning from the last quaver in measure 32, the student was then able to confidently entrain to and project eighth notes in sets of threes in a minimal triple meter, with the new pulse the dotted quarter note (see the orange arrows Figure 71) to establish the new meter <2223>. The addition of an eighth note (+1 ♪) to the sets of "two" eighth notes to make sets of "three" eighth notes is called metric displacement because the quarter note has been "displaced" by one eighth note to introduce a dotted quarter note.

By learning and understanding meter as experienced and through reporting evidence of the temporal experience through a ski-hill graph, students are able to solve performance problems. This occurs because teaching meter through the ski-hill graph enables students to have a better understanding about how indirect metric dissonance, experienced as a simple hemiola, can enhance a performance by understanding and applying simple mathematical principles. Through using basic mathematics, students are then equipped with the information and knowledge they need to articulate their understanding of meter through mapping the pulses, annotating the score, discussing their understanding of timing problem(s), and performing with more confidence.

Introducing hemiolas as early as primary school helps students to map, count, and perform duple and triple meters separately, and together develops their rhythmic awareness and performance skills. Experience counting "two against three," "three against two," and other combinations such as "four against three," and "five against four", and "six against four" familiarizes students with aspects of meter that exist in music from many different cultures and traditions from all around the world enabling students to articulate their understanding of these similarities.

In the following section I provide a detailed study for teaching simple hemiolas 3:2 with school-age students through cyclic graphs and beat-class theory. Depending on their prior exposure to ski-hill graphs, cyclic graphs, and beat-class theory the activity may only take 15 minutes or so. The materials are mainly intended for the teacher to gain a better understanding of the mathematical components of a 3:2 hemiola so as to be able to teach with an in-depth understanding of the subject.

Although it is entirely possible for students to learn how to perform hemiolas prior to any of the preliminary materials introduced in this chapter, the benefits from understanding more about the performing experience beyond how a hemiola "feels" to perform or a basic "two against three" explanation are immense. Students should be familiar with the representations of both duple and triple minimal and deep meters on ski-hill graphs and cyclic graphs prior to teaching hemiolas in this way.

Cohn defines simple hemiolas as possessing three conditions:

> 1) there are two distinct meters, 2) each of which has three pulses, 3) of which two, the slowest and fastest, are shared. Because the meters are distinct, this insures that the remaining intermediate pulses are distinct. (Cohn, 2016)[107]

Complex hemiolas are defined by Cohn (2016) as having the three conditions listed above for simple hemiolas except that they relate three or more distinct meters as in 4:3 and 6:4 hemiolas rather than only two meters as in a 3:2 hemiola.

In this practical approach to teaching music theory, students begin counting with the single syllable "O" (pronounced "oh") to represent zero so that students can group sets of pulses according to a generating integer. Cohn (2017) outlines the advantages of using zero-based counting rather than 1-based counting:

Beats (at a particular level) are integral multiples of their generating integer (=duration). Thus, 9/8 meter as traditionally constructed in the West is counted as **1** 2 3 **4** 5 6 **7** 8 9; here we count it as **0** 1 2 **3** 4 5 **6** 7 8. The generating integer is 3 and the generating duration is ♩. at this level of beat. The generating integer here is 2: **0**1 **2**3 **4**5 **6**7 and the generating duration is ♩ at this level of beat.[108]

[107] Richard Cohn. (2016 Unpublished) *Double and Complex Hemiolas.* Yale University, USA.
[108] See also Cohn (2017) and Milne and Calilhanna (2019 Manuscript submitted for publication).

Figure 72 illustrates an example of a simple hemiola mapped on the SkiHill app using single notes using traditional notation and polygons.

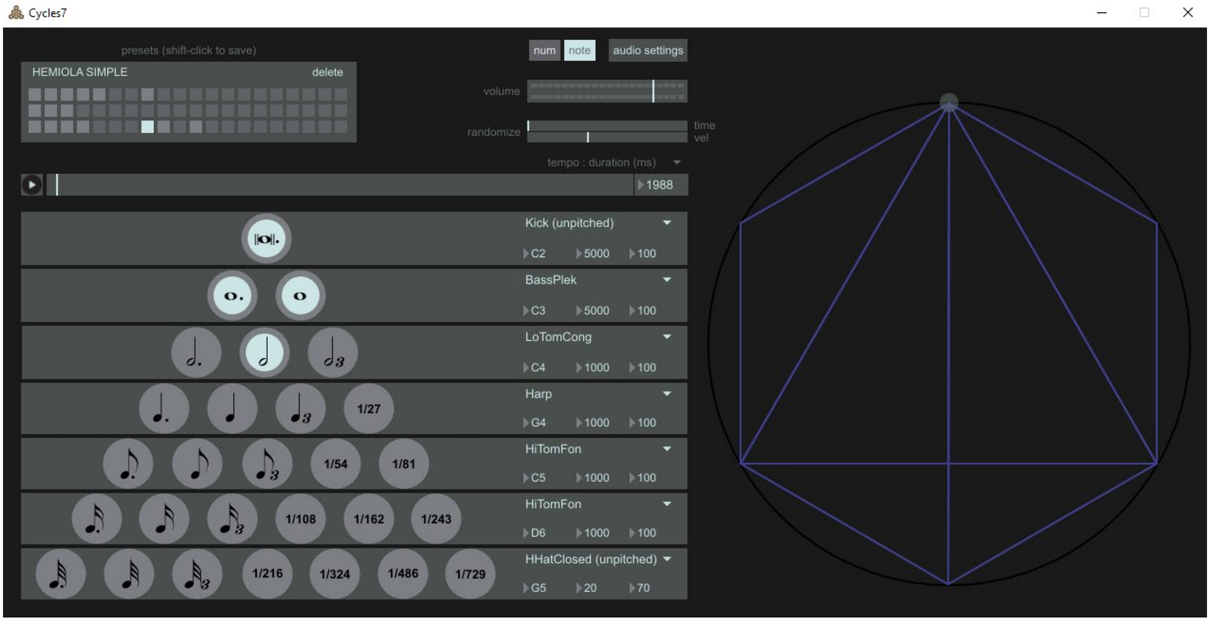

Figure 72: Simple hemiola using traditional notation, polygons, and sonifications

The ski-hill graph in Figure 73 represents the same simple hemiola as Figure 72 but with pulses mapped as fractions 1/2 and 1/3:

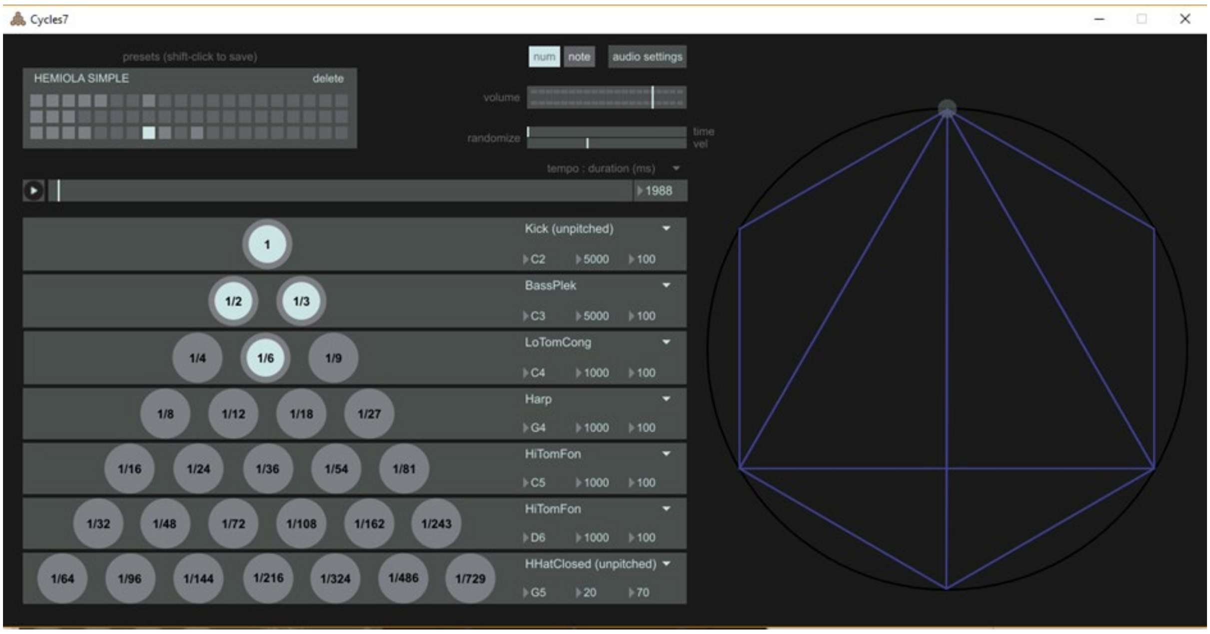

Figure 73: Simple hemiola using fractions, polygons, and sonifications

As students work through the materials to study a 3:2 hemiola, they will make observations about the metric pathways they mapped on a ski-hill graph or SkiHill app, apply those pulses to cyclic graphs, and label all of the elements of a simple hemiola using beat-class theory and cyclic graphs. Through using cyclic graphs and beat-class theory, students examine the components of a hemiola in order to discover how the hemiola achieves its "feel" in music. To do this the variable "c" is used to indicate the number of elements in the universe or cycle, and the variable "d" is used to indicate the depth or the number of selected points, also called elements. For a simple hemiola 3:2 the cyclical universe has six elements and the number of selected points is four which can be summarized as c=6 and d=4 and the set for a simple hemiola can be labelled as {0234} to represent the selected onsets in a universe of 6 elements. It is important for students to understand that {1} and {5} are complements of {0234} because they are the two elements not included in {0234} but those which complete the cycle. Cohn (2018b p. 137) states,

> Juxtaposed, these meters model the Baroque pre-cadential hemiola. Superposed as {03} ∪ {024} = {0234}, they convey the polymetric tug of a waltz, as well as the 3-against-2 cross-rhythms of West African and Afro-Caribbean repertoires.

Unlike traditional linear notation, graphic visualizations of the ski-hill graph and polygons enable students to provide evidence for a mathematical explanation of the cause of the "conflict" between the two pulses – the dotted whole note and whole note. Rather than both being mapped down to the left to form a relation of inclusion as a minimal duple meter or to the right to form a minimal triple meter these two pulses are placed horizontally on the ski-hill graph thus forming a mathematical relation of a 2:3 ratio hence the "conflict" we hear in music with a hemiola.

Simple hemiolas possess a third pulse, the unit or fastest pulse; see Figure 74, where the unit pulse of a half note is in a relation of inclusion with all of the pulses. The unit pulse is isochronous, and being the fastest pulse, it is 1/6 of the denominator. And a fourth pulse, the span pulse, the slowest pulse which is also in a relation of inclusion with all of the other pulses – here it is a dotted breve, represented also as the denominator (labelled as the integer "1" to indicate a "whole note").

The hexagon represents a single cycle of the fastest pulse – the unit pulse – each edge represents the length of a pulse; thus, there are 6 unit pulses for each 3:2 hemiola. Each pulse can then be explored in relation to other longer or shorter pulses in relations of 3:1 or triple minimal meter or 2:1 duple minimal meter as sets. The two edges of the digon, represent two pulses, and the three edges of the triangle represent three pulses, creating the "conflicting" two against three "feel."

Studying polygons in this way reveals for students interesting mathematical properties, such as evenness: sets of even distances (timespans) which form minimal duple meter (hexagon and triangle 2:1) and another set forming minimal triple meter (hexagon and digon 3:1). A second level of minimal meters can be explored through studying the relation of the longest pulse, the span pulse to pulses sounding twice as fast represented by a digon to form duple meter 2:1, and span pulse to pulses sounding three times as fast represented by a triangle to form minimal triple meter 3:1.

When the four minimal meters (or two deep meters <23> and <32> as in Figures 74 and 75) are observed as a set performed together at the same time and superimposed in a visual and/or sonified graphic representation, students can see and hear how the hemiola occurs (see Figure 76). Performed together the two deep meters form a "two against three" feel through a pulse ratio of 2:3. This occurs because the two pulses not related by inclusion: the dotted whole note and whole note cannot divide evenly into each other to form a minimal meter. Note that in the following Figures I use colours to represent the meters and polygons corresponding to the periodicity of pulses making meter to the hearing. The color used is additional to the XronoBeat app and I have superimposed color as a means of demonstrating for the reader the pedagogical process more clearly. The color represents the metric pathways and polygons I point to, direct student attention to, and focus on for each step in the lesson.

Figures 74-76 illustrate the two distinct meters that together form a simple hemiola 3:2. Figure 74 Illustrates a ski-hill graph and cyclic graph representing minimal meters <23> a distinct meter (blue) forming a hexagon and digon in a relation of inclusion:

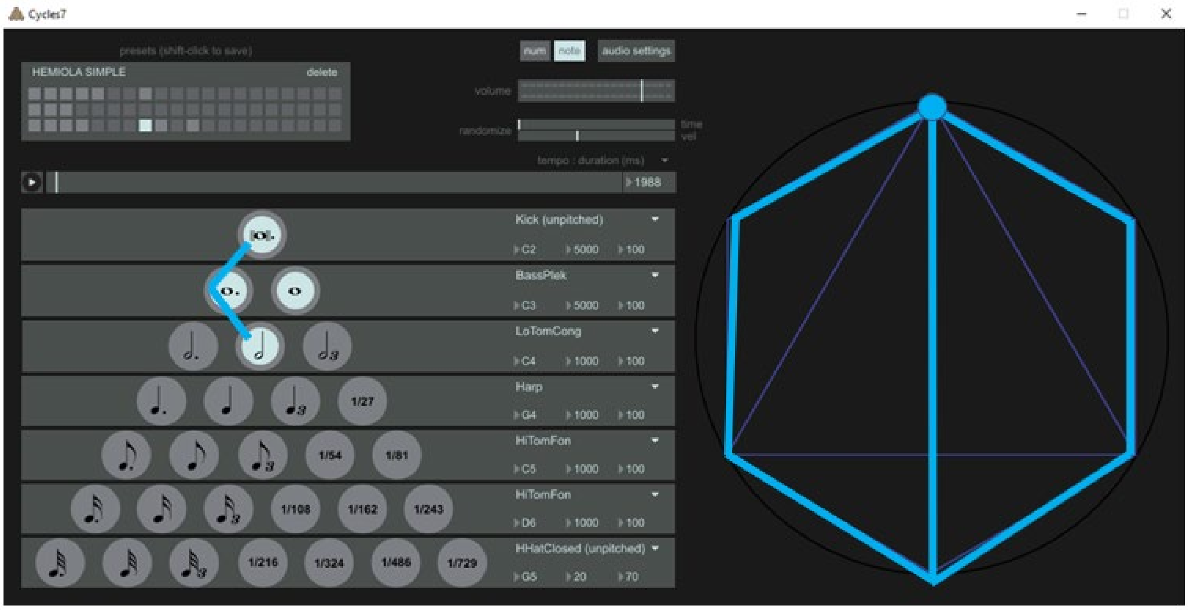

Figure 74: Minimal meters <23>

Figure 75 Illustrates a ski-hill graph and cyclic graph representing minimal meters <32> a distinct meter (red) forming a hexagon and triangle in a relation of inclusion:

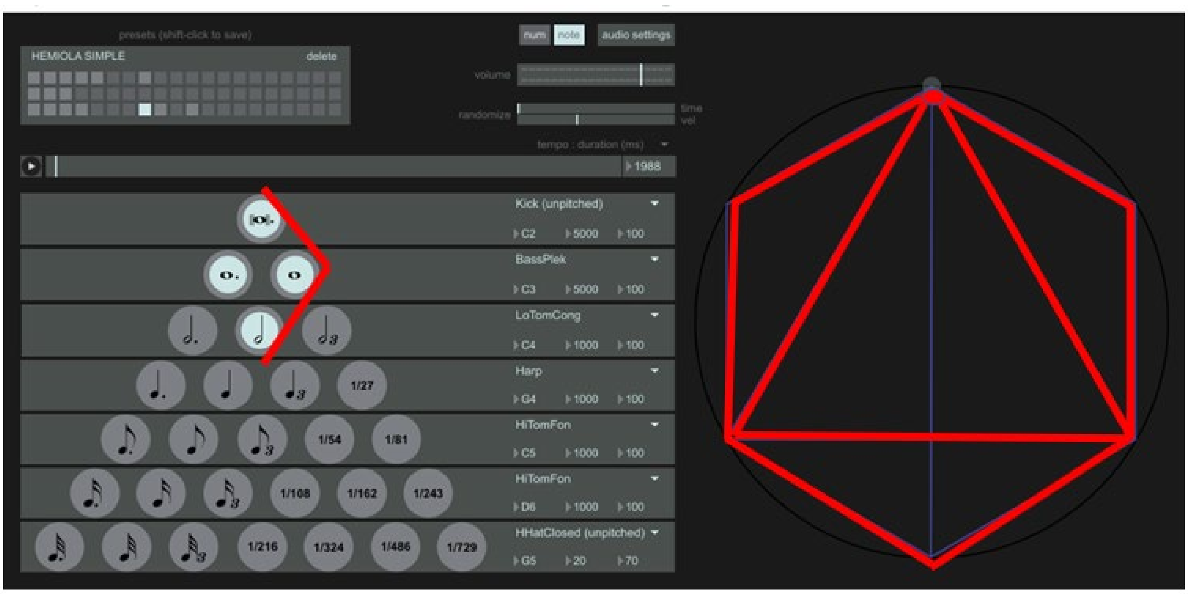

Figure 75: Minimal meters <32>

Figure 76 Illustrates a ski-hill graph with a simple hemiola mapped by a listener. Both sets of distinct minimal meters <23> (blue) and <32> (red) are mapped and a green dotted line is placed horizontally to indicate the two pulses "conflicting" in a 3:2 ratio.

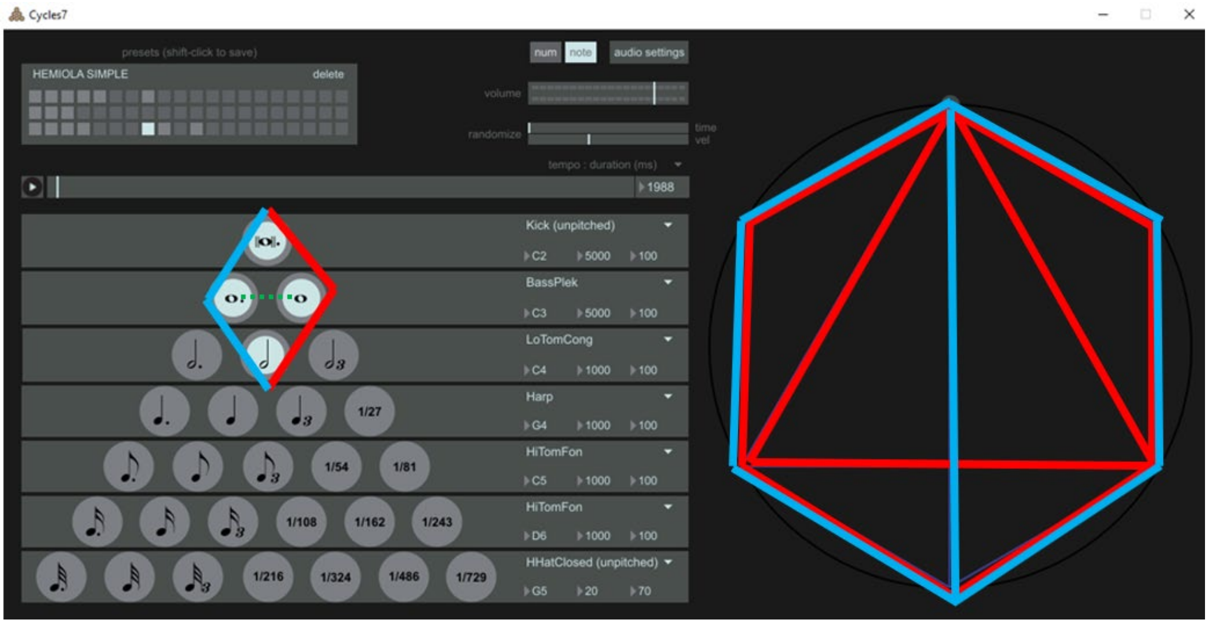

Figure 76: Ski-hill graph and cyclic graph representing a simple hemiola

Unlike traditional notation, graphic representation of pulses as polygons allows the student to visualize and sonify how each distinct minimal meter possesses the property of isochrony. This is possible because students hear and see polygons with perfectly even sets of equally spaced pulses. Here however, the two distinct meters form a simple hemiola because two of the pulses are not in a periodic relation of inclusion and therefore the two meters combined form direct metric dissonance. These graphic representations for simple hemiolas can be used for all genres of music where both direct and indirect metric dissonance occurs.

The following are a set of hemiolas mapped onto a ski-hill graph which are related by metric equivalence (see Figure 77). These graphs can be incorporated into lessons after the previous exercises are worked through to help students understand that notation is arbitrary. It would be helpful if the students could replicate the images themselves on the SkiHill app and/or if they were also part of their music text or handouts to study further. Composition and performance tasks can be developed from these music theory activities to teach students about composers' use of hemiolas throughout the history of music.

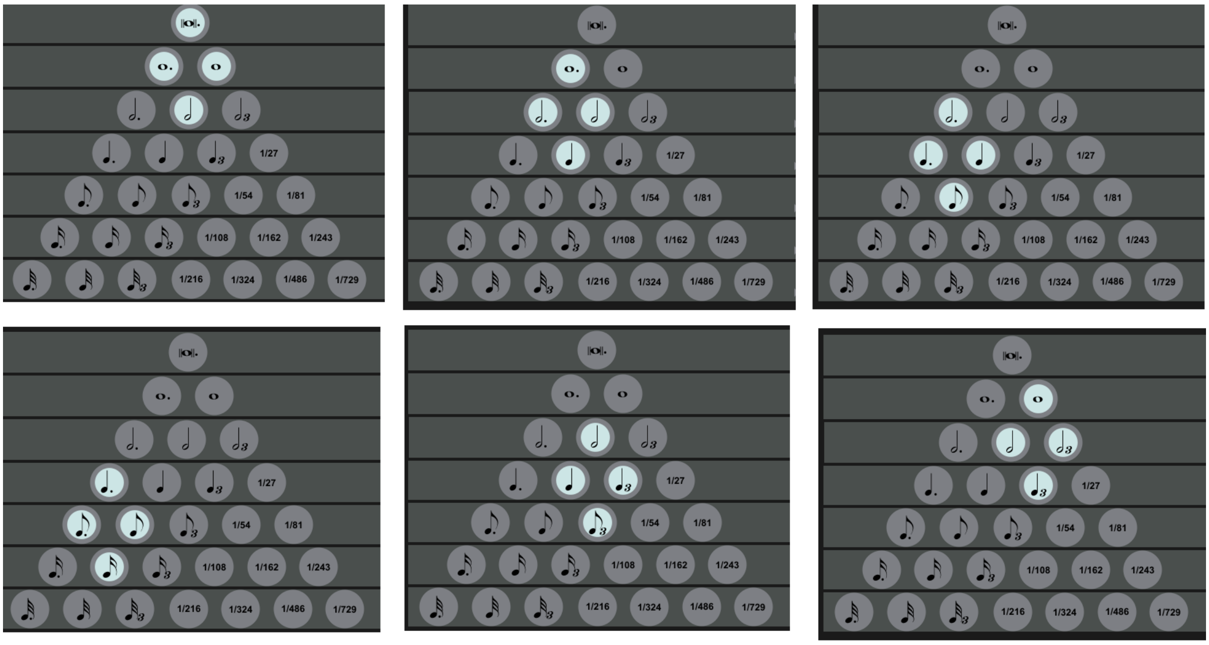

Figure 77: Hemiolas related by metric equivalence

Working with the ski-hill graph, cyclic graphs, polygons, and beat-class theory in the classroom and studio is an efficacious pedagogical approach to introduce primary and secondary school students to the experience of performing, composing, and analysing a simple hemiola through, for instance, clapping, tapping, walking, counting and/or drumming.

I have chosen to sequence the materials in this order so that students can first acknowledge their intuitive awareness of the mathematical components of a hemiola through mapping pulses in inclusion relation on the ski-hill graph and through physical movement. It is, however, possible for students to learn about the mathematical components of a hemiola entirely through the visualizations and sonifications of the SkiHill app and beat-class theory without any outward movement. But ideally, movement should also be encouraged so as to maximize the pedagogical outcomes for students through broadening students' experience of learning meter.

Students perform the materials in Figures 74-77 to "build" a 3:2 hemiola from the ground up. As soon as students have mapped a hemiola on the ski-hill graph and become aware of the movement and counting involved in performing minimal meters (embodied mathematics) I introduce beat-class theory and cyclic graphs. Ordering the materials in this way helps students to first familiarize themselves with new theory learnt through a process of listening, composing, performing, and analysing a hemiola. Each component adds a new level of information and understanding before adding the further dimension by articulating their experience of hemiolas with the language of mathematics through beat-class theory. Including the mathematical component of beat-class theory in lessons on meter moves the level of teaching and learning from the sometimes fuzzy level of "feel" to arrive at a mathematical explanation of how a hemiola forms to the hearing (how the "feel" happens). Thus, students are equipped with new knowledge and skills to empower them to articulate their ideas through symbol, sound, movement, and narrative.

When mapping hemiolas, if students are not familiar with traditional notation they can use fractions (see Figure 78) or, if they are not confident with fractions, they can use "empty" nodes with or without colour (see Figures 79 and 80). Most students intuit the connection with meter and mathematics and many will work out how many of each different pulse is repeated in each cycle (the periodicity) especially if drawn to their attention. Thus, students who are not confident with traditional notation may at first prefer to write a number for each node on the ski-hill graph (see Figure 81), which should be encouraged as should their application of this understanding to traditional notation and the concept of metric equivalence.

Prior to this stage of learning about meter, ideally all students should have a good understanding of metric equivalence from working through the previous materials, so that they can understand how different notations can be used in the "empty" nodes to achieve an identical sound, and how this can be explained using mathematics. All students, including those not proficient in using traditional notation, would benefit from seeing the ski-hill graph of Figure 80 where a number of hemiolas are represented, all related by metric equivalence mapped through colour and position, which students can then compare to notations in Figure 77.

Figure 78 illustrates a simple hemiola mapped to the SkiHill app using visualisations to articulate the experience of meter through fractions, polygons, and also through sonifications for each pulse:

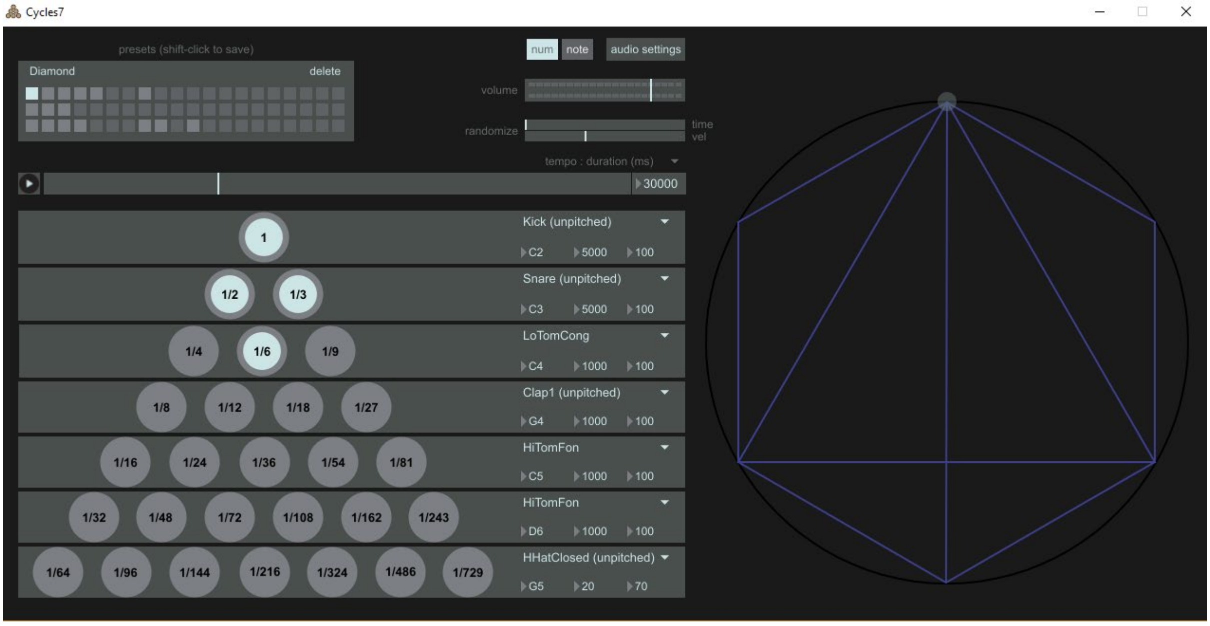

Figure 78: Simple hemiola

Figure 79 illustrates a hemiola represented in visualisations as coloured nodes, polygons, and sonifications mapped onto the SkiHill app.

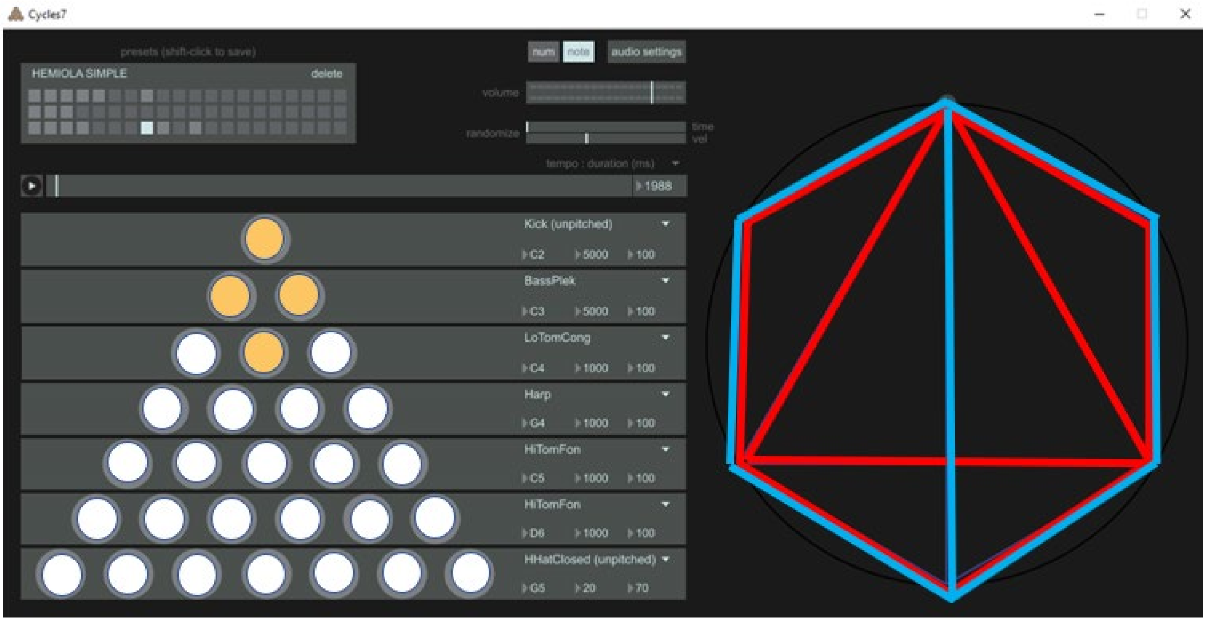

Figure 79: Simple hemiola

In Figure 80, a simple hemiola is represented in visualisations as coloured nodes, polygons, and sonifications mapped onto the SkiHill app to demonstrate the mathematical property of metric equivalence.

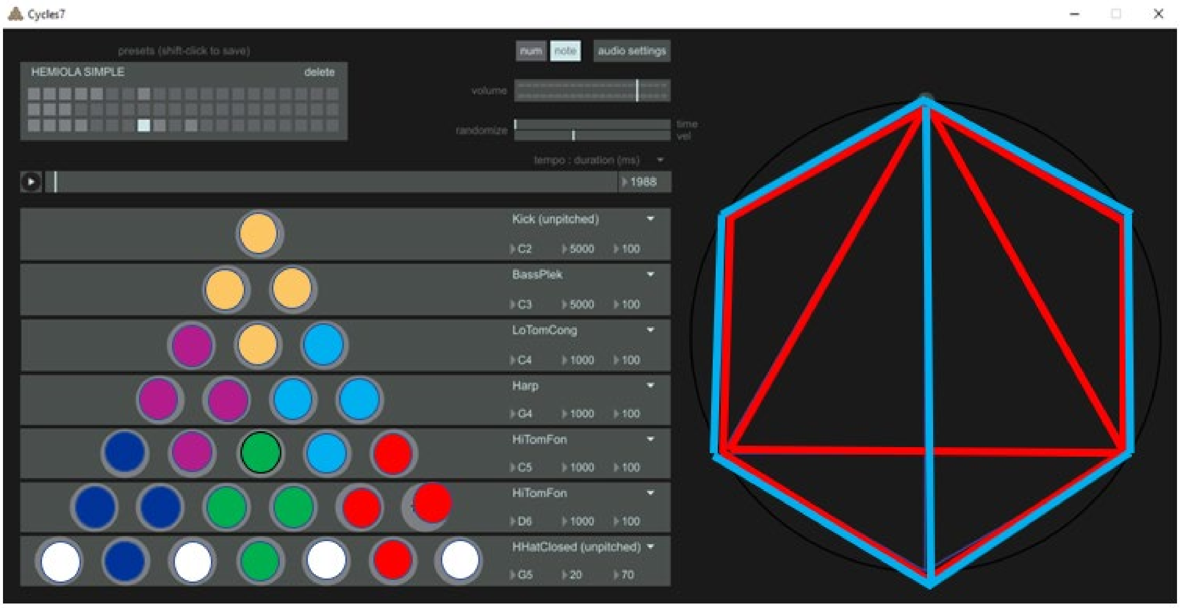

Figure 80: Simple hemiola

Figure 81 Illustrates a simple hemiola with numbers to graphically represent the 2:3 ratio between pulses, coloured nodes, polygons, and sonifications mapped onto the SkiHill app.

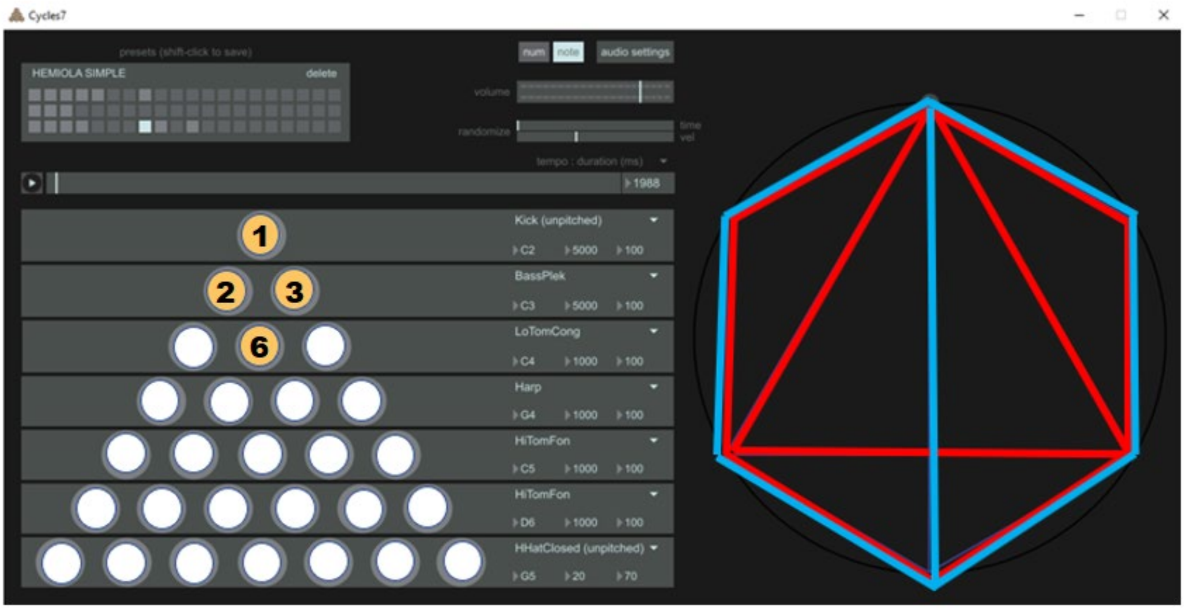

Figure 81: Simple hemiola

Sample Lesson Materials: Teaching Simple Hemiolas With Direct Metric Dissonance

Years 7-9

The following materials to teach 3:2 hemiolas and direct metric dissonance are best approached after students are well-acquainted with mapping on the ski-hill graph and/or SkiHill app both duple and triple meters for music which is metrically consonant. This lesson on 3:2 hemiolas can take place ideally over one double or two short lessons.

The sequence of materials presented below follows a basic pattern but, depending on the age group and experience of the students, teachers should reorder some materials to suit the group as the lesson proceeds, if necessary. Working through the content of each of the following steps enables students to learn more deeply about meter as temporal and to articulate their understanding of the intrinsic mathematical properties of meter. This occurs because students are equipped to both articulate through symbol, sound, movement and narrative why a hemiola occurs to the hearing, and apply their new music theory skills to other materials.

The following lesson on 3:2 hemiolas includes the following tasks:

- **Listening** for meter to experience the "feel" of a 3:2 hemiola where direct metric dissonance occurs (embodied mathematics).
- **Graphing** a hemiola in visualizations and/or sonifications through the ski-hill graph and SkiHill app using traditional notation, fractions, "empty" nodes, colour, and/or numbers.
- **Performing** the components of a simple hemiola: four cycling adjacent pulses two of which are common to both meters (unit and span pulse) and the two pulses which are not in a relation of inclusion 3:2 thus creating a "conflict," four minimal meters or two distinct meters, then all components performed together (including counting with sets).
- **Analysing** a simple hemiola through beat-class theory to apply labels of set theory to each pulse on a cyclic graph, and to discuss the periodicity of each pulse and the relation of the pulses to each other to form meters.

- **Discussion** of the "feel" of a hemiola and the mathematical explanation of how that feel happens through knowledge of the components of hemiolas, meter, and mathematics.
- **Composing** metrically equivalent notations of 3:2 hemiolas.

1. Students first listen without notation to various pieces of music which possess 3:2 hemiolas, both direct and indirect, from different eras, styles, and genres to become acquainted with the "feel" of hemiolas and direct and indirect metric dissonance. Students might express that a hemiola invokes characteristics such as, "clashing," "conflicting," "crunching," "different timing," three "against" two or two "against" three "feel" and others will use the terms direct metric dissonance and/or indirect metric dissonance, depending on the example.

2. Students learn how to perform the actions in Figures 82 and 83 along with the 3:2 hemiolas in the music they are listening so as to experience the "feel" of a 3:2 hemiola with direct dissonance. If students are having trouble with coordinating "two against three" they can repeat the words and actions "together, left, right, left" to direct their hand movements to achieve familiarity with the gesture involved in counting "two against three." Students use numbers to count the two different meters of the hemiola using the sets {0}, {03}, {024}, {012345}. The differentiation between direct metric dissonance and indirect metric dissonance is discussed and explored in practical ways.

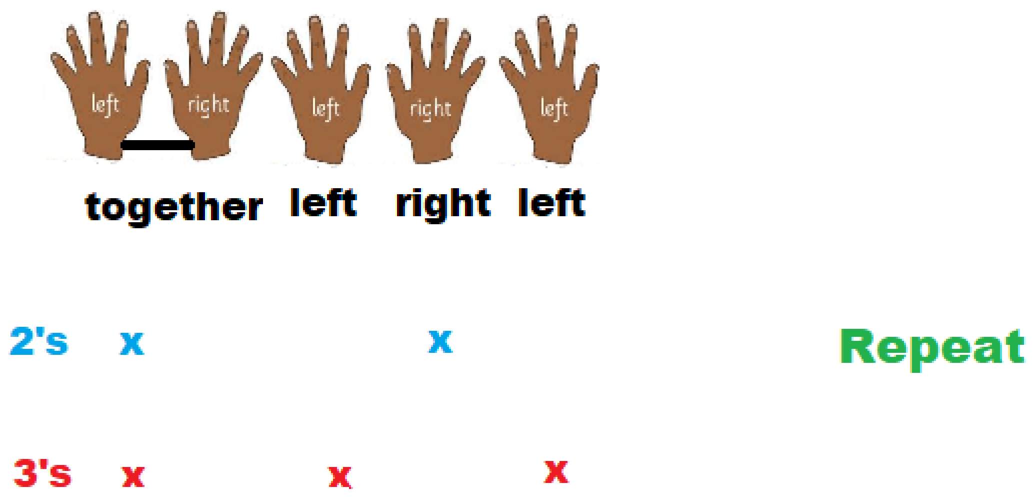

Figure 82: Instructional graphic for performing a simple hemiola 3:2

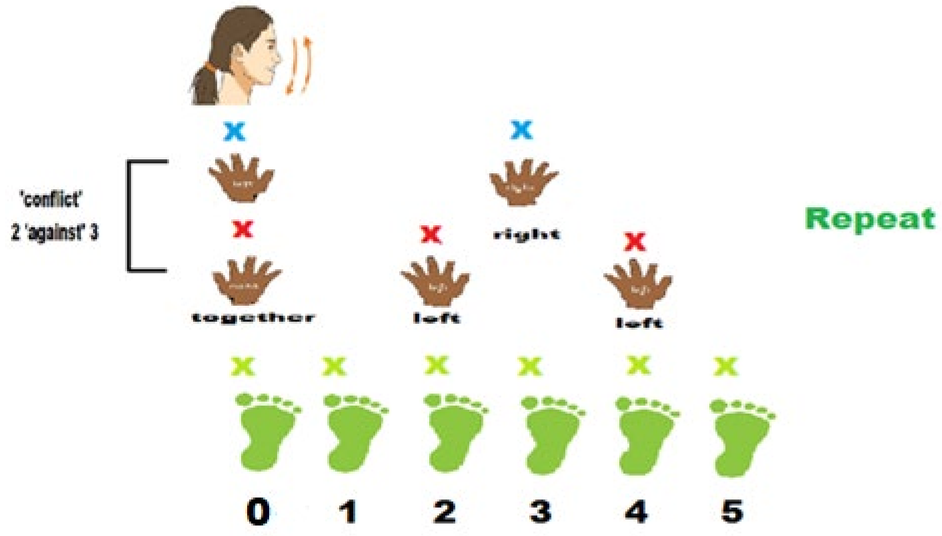

Figure 83: Instructional graphic for performing a simple hemiola 3:2

Ideally, students should perform different parts of the hemiola with other students in groups or pairs. In this way students develop ensemble skills and confidence to perform and practice the hemiola by themselves and/or as a solo.

Pieces heard might include: “America” by Bernstein (indirect metric dissonance) or “This is Prophetic” from Nixon in China by Adams (direct metric dissonance), hemiola patterns also used for example, in African bell pattern, sub-Saharan African music, Cuban Palo music, music by Bach and Handel prior to a cadence, the 3:2 hemiola on the YouTube video **“***A different way to visualize rhythm”* by John Varney (begin at 2.17 secs) (direct metric dissonance) https://www.youtube.com/watch?v=2UphAzryVpY&list=PLboLXq4Fo7iue-2TN-T2jO5dDFynviDzp&index=63 , or the song “The Farmer Refuted” from the musical *Hamilton* (direct metric dissonance).

3. Students map a hemiola in visualizations and/or sonifications through the ski-hill graph and SkiHill app using traditional notation, fractions, “empty” nodes, colour, and/or numbers.

4. Students analyse their simple hemiola through beat-class theory to apply labels of set theory to each pulse on a cyclic graph to discuss the periodicity of each pulse and the relation of the pulses to each other to form meters.

5. The class discusses the “feel” of a hemiola and the mathematical explanation of how that feel happens through knowledge of the components of hemiolas, meter, and mathematics.

6. Students compose and perform metrically equivalent notations of 3:2 hemiolas where there is direct metric dissonance to the hearing such as those in Figure 84.

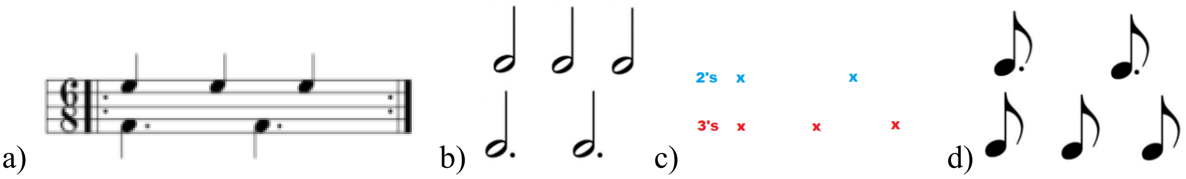

Figure 84: Simple hemiola and direct metric dissonance

Students also compose and perform 3:2 hemiolas possessing indirect metric dissonance to the hearing such as the example in Figure 85. Here two pulses are heard consecutively that are not in a relation of inclusion forming a “conflict” of “3” against “2” such as that used in the musical “America” by Leonard Bernstein:

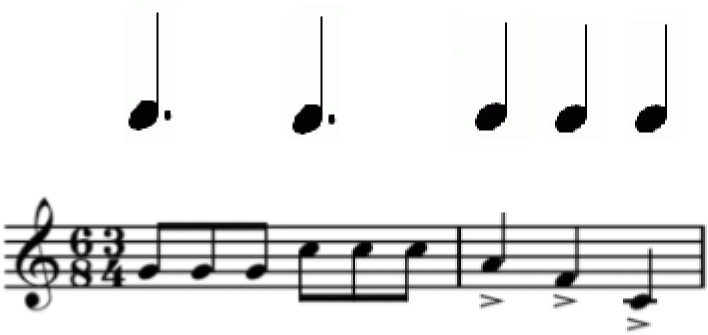

Figure 85: Simple hemiola and indirect metric dissonance

CONCLUSION

In this thesis I have demonstrated how adapting Cohn's approach to teaching meter through mathematical music theory equips school-age students with instruments and skills necessary to articulate their findings relating to meter through symbol, sound, movement and narrative, for all of the metric music they encounter. The thesis also shows how the traditional and largely notation-based understanding and approach to teaching meter with school-age students is obsolete and needs replacing with an approach which is firmly grounded in recent research about what meter is and where meter is located. To demonstrate for the reader how modern meter theory can translate into teaching materials and practices with school-age students, it has provided a model for the teaching of meter that is aimed largely at secondary music students who are in their middle years. In choosing this age group the reader is able to imagine how the materials might be adapted for either younger or older students.

In the thesis I refer to the *Understanding and Teaching Meter Survey* Project number: 2017/055 and the *Understanding and Teaching Meter Survey Report* (Calilhanna, 2017 unpublished) the aim of which was to collect data that indicated what understandings of meter are actually being taught. In addition, the survey data revealed how teachers actually understand meter and the challenges they face in teaching the concept.

As mentioned in the Report (p. 3), through the collection of this data

> We are able to pursue valid directions in the future in relation to addressing the issue of meter in music education. It is our hope that this data will assist us to make a contribution towards the improvement of practice in this area of music education in New South Wales.

The thesis has proposed solutions to the problems that need to be addressed in order for teachers to teach meter to school-age students. It has accomplished this by:

- Presenting a pedagogical approach to meter based on Cohn's meter theory which defines meter as *an inclusionally related set of distinct, notionally isochronous time-point sets* (Cohn 2018a), which distils decades of research mainly from North America, and focuses on "sound rather than notation" and the understanding that meter is a subjective and temporal experience.

- Offering a coherent and practically implementable curriculum to teach new meter theory to school-age students.
- Instructing how to graphically represent meter through Cohn's ski-hill graph in a unified approach with other instruments of mathematical music theory such as the SkiHill app, numbering for counting meter, cyclic graphs, and beat-class theory.
- Advocating for the equal treatment of both meter and tonality in music analysed.
- Setting out materials that contextualise the ski-hill graph in relation to the history of meter theory.

In summary, it offers a conceptual and practical framework from and through which to view teaching meter with new and increased understanding.

Teaching meter with the proposals presented impacts upon student thinking and current approaches to understanding and learning meter. Rather than continuing to a pursue a traditional notation-based approach to meter, students are "permitted" to listen to music and experience meter in order to discover the dynamic and transformational role it plays in isochronous music.

In this approach to learning meter students acknowledge the roles all pulses play in the formation of meter, not just the two pulses recognised by the meter signature. For instance, they also learn about the slower pulses (hypermetric or span pulses), faster pulses, or unit pulses, and those pulses in the imagination and which are not identified by a meter signature.

Students benefit from applying their new knowledge of meter, such as the hypermetric pulse and hypermeter, to improve their musical performances due to achieving a more secure metrical and rhythmic grounding. Also beneficial to students is their awareness that notation is arbitrary, and the knowledge that meter is independent of the bar-lines and meter signatures.

Through studying meter with the proposals in the thesis, students learn that musical meter is structured hierarchically, that all pulses contribute to meter, and so learn that mathematics is intrinsic to a deeper understanding of meter. Students learn that graphic representations in visualisations and sonifications using mathematical music theory are an engaging, effective and efficient means to articulate their experience of meter. Students learn how to graphically represent meter through the ski-hill graph, SkiHill app, beat-class theory, cyclic graphs and through counting meter using a hierarchy of numbers.

Studying the mathematical properties of music to understand meter also involves studying tonality equally alongside meter. Through their graphic representations of music students discover that meter and tonality are of equal importance in the analysis of music.

Finally, through the processes of listening to music, graphically representing meter using Cohn's ski-hill graph and other instruments of mathematical music theory, annotating scores, and counting hierarchically to comprehend meter, students benefit from learning about the mathematical properties of meter and related mathematical operations such as its structure, pulse or beat, duple meter, triple meter, beat-class, note values, metric equivalence, hypermeter, hemiola, polymeter, metric displacement, metric consonance, metric dissonance, sets, cardinality, elements, time points, timespans, timelines, cycles, groupings, hierarchy, pattern, counting, duration, isochrony, fractions, division, subtraction, addition, structure, cyclicity, periodicity, ratio, proportion, evenness, geometry, polygons, symmetry, integers, "rotation, complementation, and inclusion" (Cohn, 2018a, 2018b), metric pathways, graphing and mapping pulses.

Further research with school-age students will consider best practice and approaches to teaching quasi pulses and quasi meters through Cohn's beat-class theory and cyclic graphs. Such research would involve students' understanding of "rhythms" that use quasi (non-isochronous) pulses to form quasi meters. Complex polymeters represented through cyclic graphs and beat-class theory and complex hemiolas through ski-hill graphs and the SkiHill app can be researched with school-age students. Metrically ambiguous music and non-metric music can be explored in terms of mathematical representation with school-age students.

Meter and tonality in music from diverse cultures can be explored through research with school-age students in a new curriculum where Performance, Theory, and Composition all have equal weighting, a topic that awaits further research. A K-12 music curriculum can be piloted in schools and universities to research how, through Cohn's mathematical music theory, meter and tonality inform both Performance and Composition. Further research on the isomorphic relations between meter, tonality, and mathematics can be explored in pedagogical materials with school-age students in order to learn more about best practices for teaching both tonality and meter equally.

In implementing new understandings of meter such as through the ski-hill graph and with the benefit of further research that trials these, the undernourished field of music theory, in particular meter theory, may be seen to deliver even greater benefits to teachers and students.

My hope is that this thesis might contribute in some way to the wider adoption of Cohn's ski-hill graph and meter theory, and the incorporation of Milne's SkiHill app, XronoBeat and other software in classroom music programs, textbooks and on-line materials. This will be made possible by addressing meter in the light of recent research findings about what meter is and where meter is located. As Cohn (2015d) reminds us, learning about music involves:

> "mentally filtering sound through two regulative systems both tonality and meter."

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Appendix A

Ethics Approval Letter

Research Integrity & Ethics Administration
Human Research Ethics Committee

Wednesday, 8 February 2017

Dr Michael Webb
Music Education Unit; Sydney Conservatorium of Music
Email: michael.webb@sydney.edu.au

Dear Michael

The University of Sydney Human Research Ethics Committee (HREC) has considered your application.

I am pleased to inform you that your project has been approved.

Approval is granted for a period of four years from **08 February 2017** to **08 February 2021**

Project title:	**Understanding and Teaching Meter**
Project no.:	**2017/055**
First Annual Report due:	**08 February 2018**
Authorised Personnel:	**Webb Michael; Calilhanna Andrea;**

Documents Approved:

Date Uploaded	Version number	Document Name
12/01/2017	Version 1	PIS - Understanding and Teaching Meter Survey

Special Condition/s of Approval

- The PIS appears to be slated to a specific conclusion and should be made neutral. Please reword the PIS at point 1 by removing the words "why meter is not a prominent part of music teaching and learning, and why this might be the case" and replace with "…the importance given to meter in the teaching of music" or words to that effect.

Condition/s of Approval

- Research must be conducted according to the approved proposal.
- An annual progress report must be submitted to the Ethics Office on or before the anniversary of approval and on completion of the project.
- You must report as soon as practicable anything that might warrant review of ethical approval of the project including:
  - Serious or unexpected adverse events (which should be reported within 72 hours).
  - Unforeseen events that might affect continued ethical acceptability of the project.
- Any changes to the proposal must be approved prior to their implementation (except where an amendment is undertaken to eliminate *immediate* risk to participants).

THE UNIVERSITY OF SYDNEY

- Personnel working on this project must be sufficiently qualified by education, training and experience for their role, or adequately supervised. Changes to personnel must be reported and approved.

- Personnel must disclose any actual or potential conflicts of interest, including any financial or other interest or affiliation, as relevant to this project.

- Data and primary materials must be retained and stored in accordance with the relevant legislation and University guidelines.

- Ethics approval is dependent upon ongoing compliance of the research with the *National Statement on Ethical Conduct in Human Research*, the *Australian Code for the Responsible Conduct of Research*, applicable legal requirements, and with University policies, procedures and governance requirements.

- The Ethics Office may conduct audits on approved projects.

- The Chief Investigator has ultimate responsibility for the conduct of the research and is responsible for ensuring all others involved will conduct the research in accordance with the above.

This letter constitutes ethical approval only.

Please contact the Ethics Office should you require further information or clarification.

Sincerely

Glen Davis
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Professor Glen Davis
Chair
Human Research Ethics Committee

The University of Sydney HRECs are constituted and operate in accordance with the National Health and Medical Research Council's (NHMRC) National Statement on Ethical Conduct in Human Research (2007) and the NHMRC's Australian Code for the Responsible Conduct of Research (2007).

Appendix B

Participant Information Statement

THE UNIVERSITY OF SYDNEY

Sydney Conservatorium of Music
Faculty of Music Education

ABN 15 211 513 464

CHIEF INVESTIGATOR (SUPERVISOR)
Dr Michael Webb

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Understanding and Teaching Meter Survey

PARTICIPANT INFORMATION STATEMENT

(1) What is this study about?

You are invited to take part in a research study about the concept of meter in music education. It seeks to know what you understand by the term, ways you currently approach the teaching of meter, whether and why you think teaching meter is important, how much time you devote to imparting an understanding of meter, resources and repertoires you use to teach meter (if any), and what challenges you face in helping your students to gain a clear understanding of meter.

You have been invited to participate in this study because as a music educator we believe you have something worthwhile to tell us about the importance given to meter in the teaching of music and why this might be the case.

This Participant Information Statement tells you about the research study. Knowing what is involved will help you decide if you want to take part in the research. Please read this sheet carefully and ask questions about anything that you don't understand or want to know more about.

Participation in this research study is voluntary.

By giving your consent to take part in this study you are telling us that you:

- ✓ Understand what you have read.
- ✓ Agree to take part in the research study as outlined below.
- ✓ Agree to the use of your personal information as described.

You will be given a copy of this Participant Information Statement to keep.

(2) Who is running the study?

The study is being carried out by the following researchers:

- Chief Investigator: Dr Michael Webb, Senior Lecturer and Chair Music Education Unit.
- Assistant Investigator: Andrea Calilhanna, Graduate research student, Master of Music, (Musicology).

In the role of research assistant Andrea Calilhanna will prepare this survey on behalf of Dr Michael Webb. Andrea Calilhanna will also prepare a written report from the survey data which may be referred to during Meter Symposium 2 at the Sydney Conservatorium of Music 24-25 February, 2017. Also, Dr Michael Webb and co-author Andrea Calilhanna will write an article from the survey report. Andrea Calilhanna will refer to the same article in her thesis.

Andrea Calilhanna is receiving financial assistance in her role as research assistant from funding approved by the Research Committee at the Sydney Conservatorium of Music.

(3) What will the study involve for me?

You will be asked to answer ten short questions in an anonymous on-line survey titled Understanding and Teaching Meter Survey. The survey questions have been prepared by using the software program Survey Monkey. This data will then be collated by Andrea Calilhanna for the purposes of writing a report which will be referred to at Meter Symposium 2 a special research event held at the Sydney Conservatorium of Music on February 24-25, 2017. Dr Michael Webb and co-author Andrea Calilhanna will also write an article from the collated data for publication in a peer-reviewed education journal.

The following is a list of the survey questions:

1. As a teacher how would you explain meter?
2. Why is it important to teach students to understand meter?
3. How do you approach the teaching of meter?
4. What repertoire do you concentrate on in the teaching of meter? Give genres and examples.
5. Please provide an example of a resource you use in the teaching of meter, if any.
6. What challenges have you faced in helping your students gain a clearer understanding of meter?
7. Approximately how much of your teaching is devoted to meter? Very little/Occasional concentrated periods or lessons/Some or part of almost every lesson. Elaborate if possible.
8. In what role(s) do you teach meter e.g. classroom teacher, studio teacher, university lecturer, university student, educational administrator, other (please state which).
9. Number of years of teaching experience.
10. Please state your highest musical qualification.

(4) How much of my time will the study take?

The amount of time required to fill out the Understanding and Teaching Meter Survey will vary according to participant but on average should take approximately 10 -15 minutes.

(5) Who can take part in the study?

The Understanding and Teaching Meter survey has been designed to gather data from practicing music teachers so as to ascertain current practices of teaching meter. This data will then be used to compare and contrast current teaching methods to potentially develop music curriculum based on best practice. No one will be excluded from answering the questions on this survey.

(6) Do I have to be in the study? Can I withdraw from the study once I've started?

Being in this study is completely voluntary and you do not have to take part. Your decision whether to participate will not affect your current or future relationship with the researchers or anyone else at the University of Sydney.

Submitting your completed questionnaire is an indication of your consent to participate in the study. You can withdraw your responses any time before you have submitted the questionnaire. Once you have submitted it, your responses cannot be withdrawn because they are anonymous and therefore we will not be able to tell which one is yours.

(7) Are there any risks or costs associated with being in the study?

Aside from giving up your time, we do not expect that there will be any risks or costs associated with taking part in this study.

(8) Are there any benefits associated with being in the study?

We cannot guarantee that you will receive any direct benefits from being in the study.

(9) What will happen to information about me that is collected during the study?

By providing your consent, you are agreeing to us collecting personal information about you for the purposes of this research study. Your information will only be used for the purposes outlined in this Participant Information Statement, unless you consent otherwise.

Your information will be stored securely and your identity/information will be kept strictly confidential, except as required by law. Study findings may be published, but you will not be individually identifiable in these publications.

We will keep the information we collect for this study, and we may use it in future projects. By providing your consent you are allowing us to use your information in future projects. We intend to give the information from this project to other researchers in the form of a report and through a research article in a peer-reviewed education journal so that they can use it in their projects. They won't know that you participated in the project and they won't be able to link you to any of the information you provided.

(10) Can I tell other people about the study?

Yes, you are welcome to tell other people about the study.

(11) What if I would like further information about the study?

When you have read this information, Dr Michael Webb or Andrea Calilhanna will be available to discuss it with you further and answer any questions you may have. If you would like to know more at any stage during the study, please feel free to contact Dr Michael Webb Senior Lecturer Music Education Unit michael.webb@sydney.edu.au or Andrea Calilhanna acal8111@uni.sydney.edu.au Master of Music (Musicology) graduate student Sydney Conservatorium of Music.

(12) Will I be told the results of the study?

You have a right to receive feedback about the overall results of this study. You can tell us that you wish to receive feedback by sending an email to request a copy of the survey report to Dr Michael Webb michael.webb@sydney.edu.au or Andrea Calilhanna acal8111@uni.sydney.edu.au.

(13) What if I have a complaint or any concerns about the study?

Research involving humans in Australia is reviewed by an independent group of people called a Human Research Ethics Committee (HREC). The ethical aspects of this study have been approved by the HREC of the University of Sydney *[INSERT* protocol number once approval is obtained*]*. As part of this process, we have agreed to carry out the study according to the *National Statement on Ethical Conduct in Human Research (2007)*. This statement has been developed to protect people who agree to take part in research studies.

If you are concerned about the way this study is being conducted or you wish to make a complaint to someone independent from the study, please contact the university using the details outlined below. Please quote the study title and protocol number.

The Manager, Ethics Administration, University of Sydney:

- **Telephone:** +61 2 8627 8176
- **Email:** human.ethics@sydney.edu.au
- **Fax:** +61 2 8627 8177 (Facsimile)

This information sheet is for you to keep