Convex and Two-Convex Hypersurfaces Along Exterior Flows

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Statement of Originality

This thesis was supervised by Dr.Zhou Zhang. The first two chapters are a detailed review of work already done on mean curvature flow, mostly by Huisken and Huisken-Sinestrari, where appropriate I have clearly acknowledged the published work of others. Sections of the third chapter have been submitted to a journal where I am the sole author. The fourth chapter and sections of the appendix are intended as future work and I am the sole author.

I certify that the intellectual content of this thesis is the product of my own work and that all the assistance received in preparing this thesis and sources have been acknowledged.

Alexander Majchrowski

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Contents

Chapter 0

Introduction

In this thesis we are going to study the evolution of hypersurfaces embedded in Euclidean and Riemannian space by mean curvature flow and Brendle-Huisken G-Flow.

We begin by introducing mean curvature flow. We consider a compact, smooth immersion of an n-dimensional hypersurface in Euclidean space without boundary $F_0 : \mathcal{M} \to \mathbb{R}^{n+1}$ with $n \geq 2$. The evolution of $\mathcal{M}_0 = F_0(\mathcal{M})$ by mean curvature flow is the one parameter family of smooth immersions $F : \mathcal{M} \times [0, T) \to \mathbb{R}^{n+1}$ satisfying

$$\begin{aligned} \frac{\partial}{\partial t} F(p,t) &= -H(p,t)\nu(p,t), \quad p \in \mathcal{M},\ t \geq 0 \\ F(\cdot, 0) &= F_0(p) \end{aligned}$$

The motion of surfaces evolving by their mean curvature was initially studied by Brakke in [14], however the problem can be traced back to Mullins work in [83] to model the behaviour of grain boundaries in annealing pure metal. Motivation to study mean curvature flow arose from its geometric applications to obtain classification results for surfaces which satisfy certain initial curvature conditions. Another well studied flow with similar applications is the Ricci flow in which we study the evolution of the metric on manifolds. Ricci flow is defined by the following geometric evolution equation

$$\frac{\partial}{\partial t} g_{ij} = -2\mathrm{Ric}_{ij}.$$

There is a strong interplay between Ricci flow and mean curvature flow. Many results for mean curvature flow also hold true for Ricci flow. Moreover most techniques and ideas that can be applied to one flow can also be applied to the other. So when a breakthrough or new technique is discovered for one, it is exported and modified so that it can be applied to the other to yield a similar result. An example of this in recent times is when Colding and Minicozzi adapted results from Perelmans work on Ricci flow in his proof of the Poincaré conjecture [84] [86] [86] for mean curvature flow [21].

In the same way it appears that results from mean curvature flow can be exported over to Brendle-Huisken G-flow by using the same or modified techniques. Inspired by Andrews work on harmonic mean curvature flow [4], Brendle-Huisken introduced the following flow [16]. Fixing $n \geq 3$ consider a closed, embedded hypersurface $\mathcal{M}_0$ in an $(n+1)$-dimensional manifold $\mathcal{N}^{n+1}$, where $\mathcal{M}_0$ is κ-2-convex i.e. $\lambda_1 + \lambda_2 \geq 2\kappa$, where $\lambda_1 \leq \cdots \leq \lambda_n$ denote the principal curvatures. The evolution of $\mathcal{M}_0 = F_0(\mathcal{M})$ by G-flow flow is the one parameter family of smooth immersions $F : \mathcal{M} \times [0, T) \to \mathcal{N}^{n+1}$ satisfying

$$\begin{aligned} \frac{\partial}{\partial t} F(p,t) &= -G_\kappa(p,t)\nu(p,t), \quad p \in \mathcal{M},\ t \geq 0 \\ F(\cdot, 0) &= F_0(p) \end{aligned}$$

where

$$G_\kappa = \left(\sum_{i<j} \frac{1}{\lambda_i + \lambda_j - 2\kappa} \right)^{-1}.$$

This flow has the advantage of preserving 2-convexity in the Riemannian setting which unfortunately mean curvature does not. This greatly limits our ability to work with mean curvature flow in the Riemannian context.

Having given some context to the reader we continue by providing an outline of structure and contents of the thesis. The first two chapters focus on work done by Brendle, Evans, Head, Huisken, Lauer, Sinestrari and Spruck among others. In these chapters apart from extensively covering work already done for mean curvature flow, we also explore and develop our techniques, hoping to use and apply similar techniques later for Brendle-Huisken G-flow.

In the Chapter 1 we will restrict our attention to convex hypersurfaces. These are surfaces whose second fundamental form can be diagonalised such that all of our principal curvatures are strictly positive.

We begin by covering [58] in which Huisken showed that a uniformly convex, compact surface smoothly embedded in Euclidean space undergoing mean curvature flow shrinks to a round point in finite time. Moreover the normalised flow will converge to a sphere as $t \to \infty$. Huisken also showed the same is true for a surface embedded in a general Riemannian manifold with some restrictions [59]. We also discuss the two types of solutions to the mean curvature flow equation, type I and type II singularities. In [61] Huisken was able to show that singularities of the first type are asymptotically self-similar using his famous monotonicity formula. We finish dealing with the convex case for hypersurfaces embedded in Euclidean space by looking at Huisken and Sinestrari's paper [68] on ancient solutions for convex mean curvature flow. In it they give various conditions which are sufficient to ensure that a closed convex ancient solution is a shrinking sphere.

In the Chapter 2 we take the next natural step, which is to weaken our convexity assumption. In it we study two-convex hypersurfaces embedded in Euclidean space: these are surfaces where the sum of the smallest two principal curvatures must be strictly positive. In [67] Huisken-Sinestrari were able to classify surfaces undergoing mean curvature flow using the surgery algorithm Hamilton used for Ricci flow in [49]. Head [55] and Lauer [71] were then each able to independently show that as we take our surgery parameter to infinity, the curvature at which we do the surgery on the neck, that it will converge to the weak solution of level-set mean curvature flow as studied by Spruck-Evans in [36] and by Chen, Giga and Goto in [19] among others.

Ideally one would then like to extend the surgery algorithm to the Riemannian setting. However, as stated earlier in the introduction, two-convex mean curvature flow does not preserve two-convexity in the Riemannian setting. Instead we substitute it with the Brendle-Huisken G-flow, which does preserve two-convexity. In their paper [16] they were then able to use the Huisken-Sinestrari surgery algorithm for mean curvature flow for the Huisken-Brendle G-flow in both the Euclidean and Riemannian setting.

In Chapter 3 we explicitly develop the details implied in [16], and address some adaptations which must be made to the arguments from [67]. In order to do so adjustments have to be made to the gradient estimate from [16]. We are then required to make some more adjustments to prove the Neck Detection Lemma. The arguments of Section 7 and 8 from [67] then carry over unchanged with the exception of the proof of the Neck Continuation Theorem. To prove this theorem we would also need a lower bound for the time needed between two consecutive surgeries which we provide.

In the Chapter 4 we describe a level-set approach to G-flow. We are unable to derive an explicit level-set equation, however we show that for any extrinsic flow we can repeat this process and obtain a similar result. Suppose we have have a flow evolving by

$$\frac{\partial}{\partial t}F(p,t) = -\mathcal{A}\nu$$

then

$$|Du|\Delta_\infty u = -\left(\frac{1}{\mathcal{A}}\right)_t$$

where

$$\Delta_\infty u(x) = \sum_{i,j}\frac{\partial^2 u}{\partial x_i\,\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}$$

is the infinity Laplacian which has well known unique viscosity solutions which have recently been shown by Evans and Smart to be everywhere differentiable [32]. We leave the adaptation of Lauer's method [71] in Appendix 5.4 until a unique viscosity solution to G-flow is constructed.

Chapter 1

Mean Curvature Flow for Convex Hypersurfaces

In this chapter we cover some classic results from Huisken and Sinestrari's work on mean curvature flow in the convex hypersurface setting. The sources which provide the results and proofs are obtained from [58] [61] [65] and [68]. The proofs and results are not original work, they are included for completeness and we expand on the details of the calculations and proofs as an exercise to develop techniques required for studying geometric evolution equations.

1.1 Preliminaries

In this section we introduce notation and give a description of mean curvature flow. We also give some proofs and details about its main properties. Principal sources for this chapter are [58] [80].

Let $F_0 : \mathcal{M} \to \mathbb{R}^{n+1}$ be a compact, convex, smooth immersion of an n-dimensional hypersurface in Euclidean space without boundary, $n \geq 2$. The evolution of $\mathcal{M}_0 = F_0(\mathcal{M})$ by mean curvature flow is the one-parameter family of smooth immersions $F : \mathcal{M} \times [0, T) \to \mathbb{R}^{n+1}$ satisfying

$$\begin{aligned} \frac{\partial}{\partial t} F(p,t) &= -\Delta_t F(p,t), \quad p \in \mathcal{M},\ t \geq 0 \\ F(\cdot, 0) &= F_0(p) \end{aligned} \tag{1.1}$$

where Δ_t is the Laplace-Beltrami operator on the manifold $\mathcal{M}$ given by $F(\cdot, t)$.

We will use the following notation for the traces of the second fundamental form on $\mathcal{M}$:

$$H = g^{ij} h_{ij} \text{ and } |A|^2 = g^{ij} g^{kl} h_{ik} h_{jl}.$$

By $\langle \cdot, \cdot \rangle$ we denote the ordinary inner product on $\mathbb{R}^{n+1}$. If $\mathcal{M}$ is given locally by some F, the metric and second fundamental form can be computed as follows:

$$g_{ij}(\vec{x}) = \left\langle \frac{\partial F(\vec{x})}{\partial x_i}, \frac{\partial F(\vec{x})}{\partial x_j} \right\rangle, \quad h_{ij}(\vec{x}) = \left\langle \nu(\vec{x}), \frac{\partial^2 F(\vec{x})}{\partial x_i \partial x_j} \right\rangle, \ \vec{x} \in \mathbb{R}^{n+1}.$$

We also have the Gauss-Weingarten equations

$$\frac{\partial^2 F}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial F}{\partial x_k} = -h_{ij}\nu \tag{1.2}$$

$$\frac{\partial \nu}{\partial x_i} = h_{il} g^{lk} \frac{\partial F}{\partial x_k} \tag{1.3}$$

where $\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\frac{\partial}{\partial x_i} g_{jl} + \frac{\partial}{\partial x_j} g_{il} - \frac{\partial}{\partial x_l} g_{ij} \right)$.

Now working with (1.1) we can see that a short calculation gives

$$\begin{aligned} \Delta F &= g^{ij} \left(\frac{\partial^2 F}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial F}{\partial x_k} \right) \\ &= g^{ij} \left(\frac{\partial^2 F}{\partial x_i \partial x_j} - \nabla_i \frac{\partial F}{\partial x_j} \right) \\ &= g^{ij} \left(\frac{\partial^2 F}{\partial x_i \partial x_j} \right)^{\perp} \\ &= \vec{H} = -H\nu \end{aligned}$$

where $H(p,t)$ and $\nu(p,t)$ are the mean curvature and outer normal respectively at the point $F(p,t)$ on the surface $\mathcal{M}_t = F(\cdot,t)(\mathcal{M})$ and the signs are chosen such that $-H\nu = \vec{H}$ is the mean curvature vector and the mean curvature of a convex surface is positive.

It follows that the covariant derivation on $\mathcal{M}$ of a vector X is given by

$$\nabla_j X^i = \frac{\partial}{\partial x_j} X^i + \Gamma_{jk}^i X^k.$$

Moreover, the Riemann curvature tensor, the Ricci tensor and the scalar curvature are given by Gauss' equation

$$\begin{aligned} R_{ijkl} &= h_{ik} h_{jl} - h_{il} h_{jk} \\ R_{ik} &= H h_{ik} - h_{il} g^{lj} h_{jk} \\ R &= H^2 - |A|^2. \end{aligned}$$

With this notation we can obtain, for the interchange of two covariant derivatives

$$\begin{aligned} \nabla_i \nabla_j X^k - \nabla_j \nabla_i X^k &= R_i j k^l X^k = (h_{ik} h_{jl} - h_{il} h_{jk}) g^{kl} X^k \\ \nabla_i \nabla_j Y_k - \nabla_j \nabla_i Y_k &= R_{ijkl} g^{lm} Y_m = (h_{ik} h_{jl} - h_{il} h_{jk}) g^{lm} Y_m. \end{aligned}$$

And the Laplacian of a tensor given by

$$\Delta T_{jk}^i = g^{mn} \nabla_m \nabla_n T_{jk}^i$$

whereas the covariant derivative of T is denoted by $\nabla T = \{\nabla_l \nabla T_{jk}^i\}$.

The following lemma will be useful.

Lemma 1.4. *For any* $(0,2)$ *tensor* T_{ij} *we have*

$$\nabla_k \nabla_l T_{ij} = \nabla_l \nabla_k T_{ij} - R_{kli}^m T_{mj} - R_{klj}^m T_{im}.$$

Proof. We always work in geodesic coordinates i.e. $\nabla_{\nabla_k l} = 0$. Moreover $(T_{ij})_l = \partial l(T_{ij})$.

$$\begin{aligned}
\nabla_k \nabla_l T = \nabla_l \nabla_k T &= \nabla_k \nabla_l (T_{ij} dx^i \otimes dx^j) - \nabla_l \nabla_k (T_{ij} dx^i \otimes sx^j) \\
&= \nabla_k ((T_{ij})_l dx^i dx^j + T_{ij} \nabla_l dx^i \otimes dx^j + T_{ij} dx^i \otimes \nabla_l dx^j) \\
&\quad - \nabla_l ((T_{ij})_k dx^i dx^j + T_{ij} \nabla_k dx^i \otimes dx^j + T_{ij} dx^i \otimes \nabla_k dx^j) \\
&= T_{ij} \nabla_k \nabla_l dx^i \otimes dx^j + T_{ij} dx^i \otimes \nabla_k \nabla_l dx^j \\
&\quad - T_{ij} \nabla_l \nabla_k dx^i \otimes dx^j - T_{ij} dx^i \otimes \nabla_l \nabla_k dx^j \\
&= T_{ij} (\nabla_k \nabla_l - \nabla_l \nabla_k) dx^i \otimes dx^j + T_{ij} dx^i \otimes (\nabla_k \nabla_l - \nabla_l \nabla_k) dx^j \\
&= T R^i_{klm} dx^m \otimes dx^j + T R_k l m^j dx^i \otimes dx^m \\
&= -(R^m_{kli} T_{mj} + R^m_{klj} T_{im}).
\end{aligned}$$

□

We denote the Weingarten operator by $W = \{h^j_i\}$ and the principal curvatures by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. All of these quantities depend on $(p, t) \in \mathcal{M} \times [0, T)$.

We can now move on to prove our first theorem.

Theorem 1.5.

$$\int_{\mathcal{M}} div_{\mathcal{M}} X d\mu = \int_{\mathcal{M}} H \langle X, \nu \rangle d\mu.$$

Proof. Let $X = \frac{\partial F_t}{\partial t}|_{t=0}$. Choose a coordinate system s.t. at a point x it is orthonormal i.e., $g_{ij}(0) = \delta_{ij}$.

$$\begin{aligned}
\text{Then} \quad \frac{\partial}{\partial t} g_{ij}|_{t=0} &= \frac{\partial}{\partial t} \left\langle \frac{\partial F_t}{\partial x_i}, \frac{\partial F_t}{\partial x_j} \right\rangle \Bigg|_{t=0} \\
&= \left\langle \nabla_{\frac{\partial F_t}{\partial x_i}} X, \frac{\partial F_t}{\partial x_j} \right\rangle + \left\langle \nabla_{\frac{\partial F_t}{\partial x_j}} X, \frac{\partial F_t}{\partial x_i} \right\rangle
\end{aligned}$$

since the x_i and t derivatives commute,

$$\left[\frac{\partial F_t}{\partial t}, \frac{\partial F_t}{\partial x_i} \right] = 0.$$

This allows us to calculate the following

$$\begin{aligned}
\frac{\partial}{\partial t} \sqrt{\det g_{ij}} &= \frac{\sqrt{\det g_{ij}} g^{ij} \left(\left\langle \nabla_{\frac{\partial F_t}{\partial x_i}} X, \frac{\partial F_t}{\partial x_j} \right\rangle + \left\langle \nabla_{\frac{\partial F_t}{\partial x_j}} X, \frac{\partial F_t}{\partial x_i} \right\rangle \right)}{2} \\
&= (\mathrm{div}_{\mathcal{M}} X) \sqrt{\det g_{ij}}
\end{aligned}$$

Putting it all together gives, $\frac{\partial}{\partial t} \mathrm{Area}(F_t)|_{t=0} = \int_M \mathrm{div}_{\mathcal{M}} X d\mu$.

To obtain the other side, let X^M denote the tangential component of X,

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij}|_{t=0} &= \left\langle \frac{\partial X}{\partial x_i}, \frac{\partial F_t}{\partial x_j} \right\rangle + \left\langle \frac{\partial X}{\partial x_j}, \frac{\partial F_t}{\partial x_i} \right\rangle \\
&= \frac{\partial}{\partial x_i} \left\langle X, \frac{\partial F}{\partial x_j} \right\rangle + \frac{\partial}{\partial x_j} \left\langle X, \frac{\partial F}{\partial x_i} \right\rangle - 2 \left\langle X, \frac{\partial^2 F}{\partial x_i x_j} \right\rangle \\
&= \frac{\partial}{\partial x_i} \left\langle X^M, \frac{\partial F}{\partial x_j} \right\rangle + \frac{\partial}{\partial x_j} \left\langle X^M, \frac{\partial F}{\partial x_i} \right\rangle - 2\Gamma_{ij}^k \left\langle X^M, \frac{\partial F}{\partial x_k} \right\rangle + 2h_{ij} \langle X, \nu \rangle .
\end{aligned}$$

Where in the last step we used the Gauss-Weingarten equation, $\frac{\partial^2 F}{\partial x_i x_j} = \Gamma_{ij}^k \frac{\partial F}{\partial x_k} - h_{ij}\nu$.

Let ω be the 1-form defined by $\omega(Y) = g(F^*(X^M), Y)$. We can rewrite the above as

$$\begin{aligned}
g_{ij}|_{t=0} &= \frac{\partial \omega_j}{\partial x_i} + \frac{\partial \omega_i}{\partial x_j} - 2\Gamma_{ij}^k \omega_k + 2h_{ij} \langle X, \nu \rangle \\
&= \nabla_i \omega_j + \nabla_j \omega_i + 2h_{ij} \langle X, \nu \rangle
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{\partial}{\partial t} \sqrt{\det g_{ij}} &= \frac{\sqrt{\det g_{ij}} g^{ij} (\nabla_i \omega_j + \nabla_j \omega_i + 2h_{ij} \langle X, \nu \rangle)}{2} \\
&= \sqrt{\det g_{ij}} \left(\mathrm{div}_{\mathcal{M}} X^M + H \langle X, \nu \rangle \right)
\end{aligned}$$

Putting this all together we obtain

$$\begin{aligned}
\frac{\partial}{\partial t} Area(F_t)|_{t=0} &= \int_{\mathcal{M}} (div_{\mathcal{M}} X^M + H \langle X, \nu \rangle) d\mu \\
&= \int_{\mathcal{M}} H \langle X, \nu \rangle d\mu \quad \text{Stokes' Theorem}
\end{aligned}$$

□

Remark 1.6. *Alternatively one could use integration by parts on the tangential part to obtain a proof.*

Lemma 1.7. *We have the following identities.*

(i) $\Delta h_{ij} = \nabla_i \nabla_j H + H h_{il} g^{lm} h_{mj} - |A|^2 h_{ij}$ *(Simon's Identity)*

(ii) $\frac{1}{2} \Delta |A|^2 = \langle h_{ij}, \nabla_i \nabla_j H \rangle + |\nabla A|^2 + H(g^{ij} g^{kl} g^{mn} h_{ik} h_{lm} h_{nj}) - |A|^4$.

Proof. Using Lemma 1.4 we have

$$\nabla_k \nabla_l T_{ij} = \nabla_l \nabla_k T_{ij} - R_{kli}^m T_{mj} - R_{klj}^m T_{im} \tag{1.8}$$

We will also use the Gauss equations

$$R_{kij}^m = g^{ms}(h_{ij} h_{ks} - h_{kj} h_{is}) \tag{1.9}$$
$$R_{klj}^m = g^{ms}(h_{ks} h_{il} - h_{kl} h_{is}) \tag{1.10}$$

As well as the Codazzi equation

$$\nabla_i h_{kl} = \nabla_k h_{il} = \nabla_l h_{ik}. \tag{1.11}$$

Then

$$\begin{aligned}
\Delta h_{ik} =& g^{kl}\nabla_k\nabla_l h_{ij} \\
=& g^{kl}\nabla_k\nabla_i h_{jl} && \text{by (1.11)} \\
=& g^{kl}[\nabla_i\nabla_k h_{jl} - R^m_{kij}h_{ml} - R^m_{kil}h_{jm} && \text{by (1.8)} \\
=& g^{kl}\nabla_i\nabla_k h_{jl} - g^{kl}g^{ms}(h_{ij}h_{ks} - h_{kj}h_{is})h_{ml} \\
& - g^{kl}g^{ms}(h_{ks}h_{il} - h_{kl}h_{is})h_{jm} && \text{by (1.9)} \\
=& g^{kl}\nabla_i\nabla_j h_{kl} + g^{kl}h_{kl}h_{is}g^{sm}h_{mj} - g^{kl}h_{ks}g^{sm}h_{ml}h_{ij} \\
& + g^{kl}g^{ms}h_{kj}h_{is}h_{ml} - g^{kl}g^{ms}h_{ks}h_{il}h_{jm} \\
=& \nabla_i\nabla_j H + Hh_{il}g^{lm}h_{mj} - |A|^2 h_{ij}.
\end{aligned}$$

Make note that $h_i^j = g^{ik}h_{jk}$, $h^{ij} = g^{is}g^{jt}h_{st}$ and $|\nabla A|^2 = g^{ir}g^{js}g^{kt}\nabla_i h_{jk}\nabla_r h_{st}$. So we contract Simon's identity with h^{ij}.

$$\begin{aligned}
h^{ij}\nabla h_{ij} &= h^{ij}\nabla i\nabla_j h + h^{ij}H_{ik}g^{kl}h_{lj} - h^{ij}|A|^2 h_{ij} \\
&= h^{ij}\nabla_i\nabla_j H + Hg^{is}g^{jt}h_{st}h_{ik}g^{kl}h_{lj} - |A|^4 \\
&= h^{ij}\nabla_i\nabla_j H + Hh^i_t h^t_l h^l_i - |A|^4.
\end{aligned}$$

Now using the fact that $|A|^2 = h^i_j h^j_i$, we can obtain

$$\begin{aligned}
\Delta|A|^2 &= g^{ki}\nabla_k\nabla_i(h^m_j h^j_m) \\
&= 2h^j_m\nabla h^m_j + g^{ki}\nabla_k h^m_j\nabla_i h^j_m + g^{ki}\nabla_i h^m_j\nabla_k h^j_m.
\end{aligned}$$

By Ricci's Lemma $\nabla g \equiv 0$, so we obtain that $\Delta h^m_j = g^{mk}\Delta h_{kj}$ and $\nabla_i h^m_j = g^{mr}\nabla_i h_{rj}$. Whence,

$$\begin{aligned}
\Delta|A|^2 &= g^{jk}g^{mi}h_{km}\Delta h_{ij} + g^{ki}g^{ms}g^{jr}\nabla_k h_{sj}\nabla_i h_{rm} \\
&= g^{ki}g^{mr}g^{js}\nabla_i h_{rj}\nabla_k h_{sm} \\
&= 2h^{ij}\Delta h_{ij} + 2|\nabla A|^2.
\end{aligned}$$

So the result follows.

$\square$

We will also at times need to bound the gradient of the mean curvature by the gradient of the second fundamental form. To do this we will make use of the following Lemma.

Lemma 1.12.

$$|\nabla H|^2 \leq n|\nabla A|^2$$

Proof. Without loss of generality pick $g^{ij} = \delta^{ij}$, then

$$\begin{aligned}
|\nabla H|^2 &= |\nabla g^{ij} A_{ij}| \\
&= |g^{ij} (\nabla A)_{ij}| \\
&= |(\nabla A)_{ij}|^2 \\
&\leq n(|(\nabla A)_{11}|^2 + \cdots + |(\nabla A)_{nn})|^2 \\
&\leq n|\nabla A|^2
\end{aligned}$$

Where we have just applied Cauchy's inequality. $\square$

1.2 Mean Curvature Flow for Convex Hypersurfaces in Euclidean Space

In this section we summarise results and proofs from Huisken's paper [58]. In it he shows that a compact, uniformly convex, hypersurface embedded in $\mathbb{R}^{n+1}$ without boundary shrinks down to a point in finite time. By strictly convex we mean that the eigenvalues of the second fundamental form are strictly positive everywhere. In fact they satisfy a pinching condition which we will make explicit after the statement of the main theorem for this section.

While undergoing mean curvature flow the surface $\mathcal{M}_t$ will begin to look like a sphere quickly as the eigenvalues of the second fundamental form approach each other before shrinking to a point and no prior singularities will occur.

There is also an argument described in Section 9 [58] in which a normalisation procedure is carried out. In it, for any fixed time $t > 0$ such that a solution of $F(\cdot, t)$ exists, we let $\psi(t)$ be a positive factor chosen such that $\tilde{M}_t$ is given by

$$\tilde{F}(\cdot, t) = \psi(t)F(\cdot, t)$$

has total area $|\mathcal{M}_0|$. More precisely, for all t we have

$$\int_{\tilde{\mathcal{M}}_t} d\tilde{\mu} = |\mathcal{M}_0|.$$

After choosing a new time variable $\tilde{t}(t) = \int_0^t \psi^2(\tau)d\tau$ $\tilde{F}$ will satisfy

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{F}(\cdot, t) &= \tilde{\Delta}_{\tilde{t}}\tilde{F}(\cdot, t) + \frac{\int_{\tilde{\mathcal{M}}} \tilde{H}^2 d\tilde{\mu}}{n\int_{\tilde{\mathcal{M}}} d\tilde{\mu}} \\ \tilde{F}(\cdot, t) &= F_0\end{aligned}$$

These surfaces with fixed area will approach a round sphere as $t \to \infty$ and these surfaces are just homothetic expansions of our original mean curvature flow solution. This will not be covered here, but for more details refer to Section 9 [58].

Theorem 1.13 (Theorem 1.1 [58])**.** *Let $n \geq 2$ and assume that $\mathcal{M}_0$ is a uniformly convex, compact hypersurface without boundary. Then the evolution equation (1.1) has a smooth solution on a finite time interval $0 \leq t < T$, and the $\mathcal{M}_t$'s converge to a single point $\mathcal{O}$ as $t \to T$.*

We will go through some notation and background before outlining how to prove this Theorem.

If M_{ij} is a symmetric tensor, we call M_{ij} nonnegative, $M_{ij} \geq 0$, if all eigenvalues of M_{ij} are non-negative. Since we have assumed that all our eigenvalues of the second fundamental form of $\mathcal{M}_0$ are strictly positive, then there exists some $\epsilon > 0$ such that the inequality

$$h_{ij} \geq \epsilon H g_{ij} \tag{1.14}$$

holds everywhere on $\mathcal{M}_0$, this is our pinching condition.

Now we can go on to prove the following. These are essential later in proving that as the flow continues and $t \to T$ our eigenvalues will all approach the same value.

Lemma 1.15 (Lemma 2.3 [58])**.**

(i) $H(g^{ij}g^{kl}g^{mn}h_{ik}h_{lm}h_{nj}) - |A|^4 \geq n\epsilon^2 H^2(|A|^2 - \frac{H^2}{n})$

(ii) $|\nabla_i h_{kl} H - \nabla_i H h_{kl}|^2 \geq \frac{1}{2}\epsilon^2 H^2 |\nabla H|^2$.

Proof. (i) This is a point wise estimate, we can assume that $g_{ij} = \delta_{ij}$ and

$$\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & \text{0} & \\ & \text{0} & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

In this setting we have,

$$\begin{aligned} H(g^{ij}g^{kl}g^{mn}h_{ik}h_{lm}h_{nj}) - |A|^4 &= g[ij]h_{ij}g^{ij}g^{kl}g^{mn}h_{ik}h_{lm}h_{nj} - (g^{ij}g^{kl}h_{ik}h_{jl})^2 \\ &= \left(\sum_{i=1}^{n} \lambda_i\right)\left(\sum_{j=1}^{n} \lambda_j^3\right) - \left(\sum_{i=i}^{n} \lambda_i^2\right)^2 \\ &= \left(\sum_{i<j}^{n} \lambda_i\lambda_j^3 + \lambda_j\lambda_i^3\right) - \sum_{i<j}^{n} 2\lambda_i^2\lambda_j^2 \\ &= \sum_{i<j} \lambda_i\lambda_j(\lambda_i - \lambda_j)^2 \\ &\geq \epsilon^2 H^2 \sum_{I<j} (\lambda_i - \lambda_j)^2 \end{aligned}$$

and the result follows since,

$$|A|^2 - \frac{H^2}{n} = \frac{1}{n}\sum_{i<j}(\lambda_i - \lambda_j)^2.$$

(ii) We begin by looking at the quantity

$$\begin{aligned} |\nabla_i h_{kl} H - \nabla_i H h_{kl}|^2 &= |\nabla_i h_{kl} H - \frac{1}{2}(\nabla_i H h_{kl} + \nabla_k H h_{il}) - \frac{1}{2}(\nabla_i H h_{kl} - \nabla_k H h_{il})|^2 \\ &= |\nabla_i h_{kl} H - \frac{1}{2}(\nabla_i H h_{kl} + \nabla_k H h_{il})|^2 + \frac{1}{4}|\nabla_i H h_{kl} - \nabla_k H h_{il}|^2 \\ &\geq \frac{1}{4}|\nabla_i H h_{kl} - \nabla_k H h_{il}|^2. \end{aligned}$$

Since $\nabla_i h_{kl}$ is symmetric in (ik) by the Codazzi equations. Now we only have to consider points where the gradient of the mean curvature does not vanish. Around such a point pick an orthonormal frame $e_1, \dots, e_n$ such that $e_1 = \frac{\nabla H}{|\nabla H|}$. Then

$$\nabla_i H = \begin{cases} |\nabla H|, \ i = 1 \\ 0, \ i \geq 2 \end{cases} \tag{1.16}$$

in these coordinates. Therefore

$$\begin{aligned}\frac{1}{4}\sum_{i,k,l}(\nabla_i H h_{kl} - \nabla_k h_{il})^2 &\geq \frac{1}{4}(\nabla_1 H h_{22} - \nabla_2 H h_{12})^2 + \frac{1}{4}(\nabla_2 H h_{12} - \nabla_1 H h_{22})^2 \\ &= \frac{1}{2}h_{22}^2|\nabla H|^2 \\ &\geq \frac{1}{2}\epsilon^2 H^2 |\nabla H|^2\end{aligned}$$

since any eigenvalue, and hence any trace element h_{ij} is greater than ϵH.

□

We also have the following lemma which will be useful when trying to find a bound on $|\nabla H|$.

Lemma 1.17 (Lemma 2.2 [58]).

(i) $|\nabla A|^2 \geq \frac{3|\nabla H|^2}{(n+2)}$

(ii) $|\nabla A|^2 - \frac{|\nabla H|^2}{n} \geq \frac{2(n-1)|\nabla A|^2}{3n}$

Proof. For a proof refer to Lemma 2.2 of [58]. □

1.2.1 Evolution Equations

Here we continue to examine our mean curvature flow equation (1.1).

Since this equation is parabolic we know the the evolution equation has a solution $\mathcal{M}_t$ for a short time with any smooth compact initial surface $\mathcal{M} = \mathcal{M}_0$ at $t = 0$. For a proof of short time existence to (1.1) refer to Section 7 [40]. Therefore it makes sense to study how some of our important quantities also evolve under mean curvature flow, which we look at below.

Lemma 1.18. *If $\mathcal{M}_t$ evolves by mean curvature flow, the associated quantities above satisfy the following equations:*

(i) $\frac{\partial}{\partial t}g_{ij} = -2Hh_{ij}$

(ii) $\frac{\partial}{\partial t}d\mu = -H^2 d\mu$

(iii) $\frac{\partial}{\partial t}\nu = \nabla H$

(iv) $\frac{\partial}{\partial t}h_{ij} = \Delta h_{ij} - 2Hh_{il}g^{lm}h_{mj} + |A|^2 h_{ij}$

(v) $\frac{\partial}{\partial t}H = \Delta H + |A|^2 H$

(vi) $\frac{\partial}{\partial t}h^i_j = \Delta h^i_j + |A|^2 h^i_j$.

(vii) $\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4$.

Proof.

(i) The vectors $\frac{\partial F}{\partial x_i}$ are tangential to $\mathcal{M}$, and thus,

$$\left\langle \nu, \frac{\partial F}{\partial x_i} \right\rangle = 0 \quad h_{ij} = \left\langle \frac{\partial}{\partial x_i}\nu, \frac{\partial F}{\partial x_j} \right\rangle = \left\langle \frac{\partial}{\partial x_j}\nu, \frac{\partial F}{\partial x_i} \right\rangle.$$

From this we obtain

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= \frac{\partial}{\partial t}\left\langle \frac{\partial F}{\partial x_i}, \frac{\partial F}{\partial x_j} \right\rangle \\
&= \left\langle \frac{\partial}{\partial x_i}(-H\nu), \frac{\partial F}{\partial x_j} \right\rangle + \left\langle \frac{\partial}{\partial x_j}(-H\nu), \frac{\partial F}{\partial x_i} \right\rangle \\
&= -H\left\langle \frac{\partial}{\partial x_i}\nu, \frac{\partial F}{\partial x_j} \right\rangle - H\left\langle \frac{\partial F}{\partial x_i}, \frac{\partial}{\partial x_j}\nu \right\rangle \\
&= -2Hh_{ij}
\end{aligned}$$

(ii) If $d\mu_t = \mu_t(x)dx$ is the measure of $\mathcal{M}_t$, then $\mu_t = \sqrt{\det g_{ij}}$, so the result follows from a short calculation using the above.

$$\begin{aligned}
\frac{d}{dt}\mu &= \frac{\frac{1}{2}\det g_{ij}\,\mathrm{tr}(\frac{\partial}{\partial t}g_{ij})}{\sqrt{\det g_{ij}}} \\
&= \frac{1}{2}\sqrt{\det g_{ij}}\,\mathrm{tr}(-2Hh_{ij}) \\
&= -H^2\mu
\end{aligned}$$

(iii)
$$\begin{aligned}
\frac{\partial}{\partial t}\nu &= \left\langle \frac{\partial}{\partial t}\nu, \frac{\partial F}{\partial x_i} \right\rangle \frac{\partial F}{\partial x_j} g^{ij} \\
&= -\left\langle \nu, \frac{\partial}{\partial t}\frac{\partial F}{\partial x_i} \right\rangle \frac{\partial F}{\partial x_j} g^{ij} \\
&= \left\langle \nu, \frac{\partial}{\partial x_i}(H\nu) \right\rangle \frac{\partial F}{\partial x_j} g^{ij} \\
&= \frac{\partial}{\partial x_i} H \frac{\partial F}{\partial x_j} g^{ij} \\
&= \nabla H
\end{aligned}$$

(iv) We will make use of the Gauss Weingarten equations here.

$$\begin{aligned}
\frac{\partial}{\partial t}h_{ij} &= \frac{\partial}{\partial t}\left\langle \frac{\partial^2 F}{\partial x_i \partial x_j}, \nu \right\rangle \\
&= \left\langle \frac{\partial^2}{\partial x_i \partial x_j}(H\nu), \nu \right\rangle - \left\langle \frac{\partial F^2}{\partial x_i \partial x_j}, \frac{\partial}{\partial t}\nu \right\rangle \\
&= \left\langle \frac{\partial^2}{\partial x_i \partial x_j}, \nu \right\rangle - \left\langle \frac{\partial F^2}{\partial x_i \partial x_j}, \frac{\partial}{\partial x_l} H \frac{\partial F}{\partial x_m} g^{lm} \right\rangle \\
&= \frac{\partial^2}{\partial x_i \partial x_j} H + H \left\langle \frac{\partial}{\partial x_i}(\frac{\partial}{\partial x_j}\nu), \nu \right\rangle - \left\langle \frac{\partial^2 F}{\partial x_i \partial x_j}, \frac{\partial}{\partial x_l} H \frac{\partial}{\partial x_m} g^{lm} \right\rangle \\
&= \frac{\partial^2}{\partial x_i \partial x_j} H + H \left\langle \frac{\partial}{\partial x_i}(h_{jl} g^{lm} \frac{\partial F}{\partial x_m}), \nu \right\rangle - \left\langle \Gamma^k_{ij} \frac{\partial F}{\partial x_k} - h_{ij}\nu, \frac{\partial}{\partial x_l} H \frac{\partial F}{\partial x_m} g^{lm} \right\rangle \\
&= \frac{\partial^2}{\partial x_i \partial x_j} H - \Gamma^k_{ij} \frac{\partial}{\partial x_k} H + H h_{jm} g^{ml} \left(\Gamma^\sigma_{il} \frac{\partial F}{\partial x_\sigma} - h_{il}\nu, \nu \right) \\
&= \nabla_i \nabla_j H - H_{il} g^{lm} h_{mj} \quad \text{using } \left\langle \frac{\partial F}{\partial x_\sigma}, \nu \right\rangle = 0.
\end{aligned}$$

Now recall $\Delta h_{ij} = \nabla_i \nabla_j H + H h_{il} g^{lm} h_{mj} - |A|^2 h_{ij}$, and so the result follows.

(v) Using part (i) we have

$$\begin{aligned}
\frac{\partial}{\partial t} H &= \frac{\partial}{\partial t}(g^{ij} h_{ij}) \\
&= g^{ij} \frac{\partial}{\partial t}(h_{ij}) + \frac{\partial}{\partial t}(g^{ij}) h_{ij}
\end{aligned}$$

To find $\frac{\partial}{\partial t}(g^{ij})$ we use the fact that $g_{is} g^{sj} = \delta_i^j$ and differentiate both sides.

$$\begin{aligned}
&\frac{\partial}{\partial t}(g_{is} g^{sj}) = 0 \\
&\frac{\partial}{\partial t}(g_{is}) g^{sj} + g_{is} \frac{\partial}{\partial t}(g^{sj}) = 0
\end{aligned}$$

Which yields that

$$g_{is} \frac{\partial}{\partial t}(g^{sj}) = -\frac{\partial}{\partial t}(g_{is}) g^{sj}$$

and so

$$\frac{\partial}{\partial t}(g^{sj}) = -g^{is} \frac{\partial}{\partial t}(g_i s) g^{sj} \;\Rightarrow\; \frac{\partial}{\partial t} g^{ij} = -g^{is} \frac{\partial}{\partial t} g_{sl} g^{lj} = 2H g^{is} h_{sl} g^{lj}$$

So using the above and part (iv) we obtain the result.

(vi) Using parts (iv) and (v) above.

(vii)

$$\begin{aligned}\frac{\partial}{\partial t}|A|^2 &= \frac{\partial}{\partial t}(g^{ik}g^{jl}h_{ij}h_k l)\\ &= 4hG^{im}g^{kn}g^{jl}h_{mn}h_{ij}h_{kl} + 2g^{ik}g^{jl}h_{kl}(\Delta h_{ij} - 2Hh_{im}g^{mn}h_{nj} + |A|^2 h_{ij})\\ &= 2g^{ik}g^{jl}h_{kl}\Delta h_{ij} + 2|A|^4\end{aligned}$$

Now

$$\begin{aligned}\Delta|A|^2 &= g^{ki}\nabla_k\nabla_i(h_j^m h_m^j)\\ &= 2h_m^j\nabla h_j^m + g^{ki}\nabla_k h_j^m\nabla_i h_m^j + g^{ki}\nabla_i h_j^m\nabla_k h_m^j\\ &= 2g^{jk}h_{km}g^{mn}\nabla h_{jn} + g^{ki}g^{ms}g^{il}\nabla_k h_{sj}\nabla_i h_{ml} + g^{ki}g^{ms}g^{jl}\nabla_i h_{js}\nabla_k h_{lm}\\ &= 2h^{ij}\Delta h_{ij} + 2|\nabla A|^2\\ &= 2g^{ik}g^{js}h_{ks}\Delta h_{ij} + 2|\nabla A|^2.\end{aligned}$$

So $2g^{ik}g^{jl}h_{kl}\Delta h_{ij} = \Delta|A|^2 - 2|\nabla A|^2$. The result follows.

□

The evolution equations give us the following corollary which guarantees that the mean curvature will remain positive if initially positive and that the area of our hypersurface will continue to shrink as it undergoes mean curvature flow. However since $H_{\min}$ is not a differentiable function of time, but differentiable a.e. we will need the following theorem.

Theorem 1.19 (Theorem 2.1.1 [80])**.** *Assume that $g(t)$, for $t \in [0,T)$, is a family of smooth Riemannian metrics on some manifold $\mathcal{M}$. Let $u : \mathcal{M}\times[0,T) \to \mathbb{R}^{n+1}$ be a smooth function which satisfies the following*

$$\frac{\partial u}{\partial t} \le \Delta_{g(t)}u + \langle X(p,u,\nabla u,t), \nabla u\rangle_{g(t)} + b(u)$$

where X is a continuous vector field and b a locally Lipchitz function.

Then, suppose that for every $t \in [0,T)$ there exists a $\delta > 0$ and a compact subset $K \subset M \setminus \partial M$ such that at every time $t' \in (t-\delta, t+\delta)\cap[0,T)$ the maximum of $u(\cdot,t')$ is attained at aat least one point of K.

Setting $u_{\max}(t) = \max_{p\in\mathcal{M}} u(p,t)$ we have that the function $u_{\max}$ is locally Lipchitz, hence differentiable at almost every time $t \in [0,T]$ and at every differentiability time,

$$\frac{du_{\max}(t)}{dt} \le b(u_{\max}(t)).$$

Therefore if $h : [0,T') \to \mathbb{R}$ is a solution of the ODE

$$\begin{cases} h'(t) &= b(h(t))\\ h(0) &= u_{\max}(0),\end{cases}$$

for $T' \le T$, then $u \le h$ in $\mathcal{M}\times[0,T')$.

Corollary 1.20 (Corollary 3.6 [58])**.**

(i) If $d\mu_t = \mu_t(x)dx$ is the measure on $\mathcal{M}_t$ and the total area $|\mathcal{M}_t|$ of $\mathcal{M}_t$ is decreasing.

(ii) If the mean curvature of $\mathcal{M}_0$ is strictly positive everywhere, then it will be strictly positive as long as the solution exists.

Proof. (i) This follows straight from (i) and (ii) from Lemma 1.18.

(ii) Follows from Lemma 1.18 (v) and the maximum principle.

We argue by contradiction. Suppose there exists some interval $(t_0, t_1) \subset \mathbb{R}^+$ such that $H_{\min}(t) < 0$ and $H_{\min}(t_0) = 0$, where $H_{\min}(0) \geq 0$ and $H_{\min}$ is continuous in time.

In this interval let $|A|^2 \leq C$. Then

$$\frac{\partial H}{\partial t} = \Delta H + H|A|^2 \Rightarrow \frac{\partial H_{\min}}{\partial t} \geq CH_{\min}$$

for a.e. $t \in (t_0, t_1)$.

Integrating this in $[s, t] \in (t_0, t_1)$ we obtain that $H_{\min}(t) \geq e^{C(t-s)}H_{\min}(s)$. Letting $s \to t_0^+$ we conclude $H_{\min}(t) \geq 0$ for all $t \in (t_0, t_1)$.

Since $H \geq 0$ we get,

$$\begin{aligned}\frac{\partial H}{\partial t} &= \Delta H + H|A|^2 \\ &\geq \Delta H + \frac{H^3}{n}.\end{aligned}$$

Applying Theorem 1.19 with $u = -H$, $X = 0$ and $b(x) = \frac{x^3}{n}$ then, if $H_{\min}(0) = 0$ the solution is always zero, so if at some positive time $H_{\min}(\tau) = 0$, we have $H(\cdot, \tau)$ is constant zero on $\mathcal{M}_t$. However we know that there are no compact hypersurfaces with zero mean curvature. Therefore under mean curvature flow $H_{\min}$ is increasing and H is positive on all of $\mathcal{M}_t$ for every $t > 0$.

□

1.2.2 Preserving Convexity

Using Hamilton's maximum principle for tensors on manifolds [48], we will show that our pinching condition

$$h_{ij} \geq \epsilon H g_{ij}$$

holds as long as solutions to (1.1) exists.

Before we begin we say that a polynomial satisfies a null eigenvector condition if for any null eigenvector X of M_{ij} we have $N_{ij}X^iX^j \geq 0$.

Theorem 1.21 (Theorem 9.1 [48])**.** *Let u^k be a vector field and let g_{ij}, M_{ij} and N_{ij} be symmetric tensors on a compact manifold $\mathcal{M}$ which does not necessarily depend on t. Assume that $N_{ij} = p(M_{ij}, g_{ij})$ is a polynomial in M_{ij} formed by contracting products of M_{ij} with itself using the metric. Supposing that on $0 \leq t < T$ the evolution equation*

$$\frac{\partial}{\partial t}M_{ij} = \Delta M_{ij} + u^k\nabla_k M_{ij} + N_{ij}$$

holds, where $N_{ij} = p(M_{ij}, g_{ij})$ satisfies the null eigenvector condition. If $M_{ij} \geq 0$ at $t = 0$, then it remains so on $0 \leq t < T$.

Corollary 1.22 (Corollary 4.2 [58]). *If $h_{ij} \geq 0$ at $t = 0$ then it remains so for $0 \leq t < T$.*

Proof. Set $M_{ij} = h_{ij}$, $u^k \equiv 0$ and $N_{ij} = -2Hh_{il}g^{lm}h_{mj} + |A|^2 h_{ij}$ then it follows from the above Theorem and the evolution equation for h_{ij}. □

Theorem 1.23 (Theorem 4.2 [58]). *If $\epsilon H g_{ij} \leq h_{ij} \leq \beta H g_{ij}$, and $H > 0$ at the beginning for some constants $0 < \epsilon \leq \frac{1}{n} < \beta < 1$, then it remains so on $0 \leq t < T$.*

Remark 1.24. *Why the value of $\frac{1}{n}$? This value comes from contracting the pinching condition, we have*

$$\begin{aligned} h_{ij} &\geq \epsilon H g_{ij} \\ H &\geq \epsilon H n \\ \Rightarrow \epsilon &\leq \frac{1}{n}. \end{aligned}$$

Proof. To prove the first inequality, we wish to apply Theorem 1.21 with

$$\begin{aligned} M_i j = \frac{h_{ij}}{H} - \epsilon g_{ij}, \quad u^k = \frac{2}{H} g^{kl} \nabla_l H \\ N_{ij} = 2\epsilon H h_{ij} - 2h_{im} g^{ml} h_{lj}. \end{aligned}$$

With this choice the evolution equation in Hamilton's Theorem is satisfied since

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{h_{ij}}{H}\right) &= \frac{\frac{\partial}{\partial t}(h_{ij}H - \frac{\partial}{\partial t}H(h_{ij}))}{H^2} \\ &= \frac{\Delta h_{ij} H - \Delta H h_{ij}}{H^2} - 2h_{il}g^{lm}h_{mj}. \end{aligned}$$

We can also evaluate $\Delta\left(\frac{h_{ij}}{H}\right)$ using the following rule:

$$\Delta\left(\frac{f}{g}\right) = \frac{1}{g}\Delta f - \frac{f}{g^2}\Delta g - \frac{2}{g}\nabla\left(\frac{f}{g}\right)\nabla g. \tag{1.25}$$

Letting $f = h_{ij}$ and $g = H$, we obtain that

$$\Delta \frac{h_{ij}}{H} = \frac{H\Delta h_{ij} - h_{ij}H}{H^2} - \frac{2}{H} g^{kl} \nabla_k H \nabla_l \left(\frac{h_{ij}}{H}\right). \tag{1.26}$$

It just remains to check that N_{ij} is nonnegative on the null eigenvectors of M_{ij}. Assume that for some vector $X = \{X^i\}$, that $h_{ij}X^j = \epsilon H X_i$.

Then we derive,

$$\begin{aligned} N_{ij}X^iX^j &= 2\epsilon H h_{ij} X^i X^j - 2h_{im}g^{ml}h_{lj}X^iX^j \\ &= 2\epsilon^2 H^2 |X|^2 - 2\epsilon^2 H^2 |X|^2 \\ &= 0 \end{aligned}$$

The second inequality of the theorem follows from the same method after changing signs. □

1.2.3 The eigenvalues of A

In this section we want to show that along the flow all the eigenvalues of the second fundamental form, our principal curvatures, will approach the same value at the points where the mean curvature tends to infinity.

We look at the following quantity:

$$|A|^2 - \frac{H^2}{n} = \frac{1}{n}\sum_{i<j}(\lambda_i - \lambda_j)^2$$

which measures how far the principal curvatures λ_i diverge from each other. The idea is to show that the difference between $|A|^2 - \frac{H^2}{n}$ becomes small compared to the square of the mean curvature H^2.

Theorem 1.27 (Theorem 5.1 [58])**.** *There exist constants $\delta > 0$ and $C_0 < \infty$ depending only on $\mathcal{M}_0$, such that*

$$|A|^2 - \frac{H^2}{n} \leq C_0 H^{2-\delta}$$

for all times $0 \leq t < T$.

Our goal is to bound the function

$$g_\sigma = \frac{|A|^2 - \frac{H^2}{n}}{H^{2-\sigma}}$$

for sufficiently small σ.

Proof. For a proof refer to Section 5 of [58]. We omit the proof in this section as a very similar proof is shown in Section 1.4 for mean convex surfaces. □

Theorem 1.28. *A surface $\mathcal{M}_t$ undergoing mean curvature flow with initial conditions as in Theorem 1.13 has solutions on a finite time interval, $T < \infty$..*

Proof. Using the evolution equation for H Corollary 1.20(i), we are able to show that $T < \infty$.

$$\frac{\partial H}{\partial t} = \Delta H + H|A|^2 \geq \Delta H + \frac{H^3}{n}.$$

We introduce φ to be the solution to the ODE

$$\frac{\partial \varphi}{\partial t} = \frac{\varphi^3}{n}, \quad \varphi(0) = H_{\min}(0) > 0.$$

If we consider φ as a function on $\mathcal{M} \times [0, T)$, we get

$$\frac{\partial}{\partial t}(H - \varphi) \geq \Delta(H - \varphi) + \frac{1}{n}(H^3 - \varphi^3)$$

such that by the maximum principle $H \geq \varphi$ on $0 \leq t < T$.

Solving explicitly for φ we have

$$\varphi(t) = \frac{H_{\min}(0)}{\sqrt{1 - (\frac{2}{n})H^2_{\min}(0)t}},$$

and since $\varphi \to \infty$ as $t \to \frac{n}{2}H^{-2}_{\min}(0)$, the result follows.

□

1.2.4 A bound on $|\nabla H|$

If we want to compare the mean curvature at different points of the surface $\mathcal{M}_t$, we will need to compute a bound on $|\nabla H|$.

Theorem 1.29 (Theorem 6.1 [58]). *For any $\eta > 0$ there exists a constant C depending on $\eta, \mathcal{M}_0$ and n such that*

$$|\nabla H|^2 \leq \eta H^4 + C.$$

We will do this by bounding the function

$$f = \frac{|\nabla H|^2}{H} + N\left(|A|^2 - \frac{1}{n}H^2\right)H + NC_3|A|^2 - \eta H^3$$

for some large N depending only on n η. To do this we will need to find bound for the evolution equations which appear on the right hand side.

First we need the evolution equation for the gradient of the mean curvature.

Lemma 1.30 (Lemma 6.2 [58]). *We have the following evolution equation,*

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla H|^2 =& \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 + 2|A|^2|\nabla H|^2 + \\ &+ 2\langle \nabla_i H h_{mj}, \nabla_j H \nabla h_{im}\rangle + 2H\langle \nabla_i H, \nabla_i |A|^2\rangle.\end{aligned}$$

Proof.

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla H|^2 =& \frac{\partial}{\partial t}(g^{ij}\nabla_i H \nabla_j H)\\ =& 2H\langle h_{ij}, \nabla_i H \nabla_j H\rangle + 2g^{ij}\nabla_i(\Delta H)\nabla_j H\\ &+ 2g^{ij}\nabla_i(H|A|^2)\nabla_j H\\ =& 2H\langle h_{ij}, \nabla_i H \nabla_j H\rangle + 2g^{ij}\nabla_i(\Delta H)\nabla_j H\\ &+ 2g^{ij}\nabla_i H\nabla_j H|A|^2 + 2g^{ij}H\nabla_i|A|^2\nabla_j H.\end{aligned}$$

Now we also have that

$$\begin{aligned}\Delta|\nabla H|^2 &= g^{mn}\nabla_m\nabla_n g^{ij}\nabla_i H\nabla_j H\\ &= g^{mn}(2g^{ij}\nabla_m\nabla_n\nabla_i H\nabla_j H + 2g^{ij}\nabla_m\nabla_i\nabla_n\nabla_j H).\end{aligned}$$

With

$$\begin{aligned}g^{mn}g^{ij}\nabla_m\nabla_n\nabla_i H &= g^{mn}g^{ij}\nabla_m\nabla_i\nabla_n H\\ &= g^{mn}\nabla_i\nabla_m\nabla_n H - R_{imln}\nabla_l H\\ &= g^{mn}\nabla_i\nabla_m\nabla_n - g^{mn}(h_{il}h_{mn} + h_{in}h_{ml})\nabla_l H\\ &= \nabla_i(\Delta H) - H(h_{ij} + h_{in}g^{mn}h_{mj})\nabla_j H.\end{aligned}$$

This completes the proof. □

Corollary 1.31 (Corollary 6.3 [58]).

$$\frac{\partial}{\partial t}|\nabla H|^2 \leq \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 + 4|A|^2|\nabla H|^2 + 2H\langle \nabla_i H, \nabla_i |A|^2\rangle.$$

Proof. Using Lemma 1.30 together with the following relations yields the result,

$$\Delta|\nabla H|^2 = 2g^{kl}\Delta(\nabla_k H)\nabla_l H + 2|\nabla^2 H|^2$$
$$\Delta(\nabla_k H) = \nabla_k(\Delta H) + g^{ij}\nabla_i H(H h_{kj} - h_{km}g^{mn}h_{nj}).$$

□

Lemma 1.32 (Lemma 6.4 [58]). *We have the inequality*

$$\frac{\partial}{\partial t}\left(\frac{|\nabla H|^2}{H}\right) \leq \Delta\left(\frac{|\nabla H|^2}{H}\right) + 3|A|^2\left(\frac{|\nabla H|^2}{H}\right) + 2\langle\nabla_i H, \nabla_i|A|^2\rangle.$$

Proof. Using the result from before we have,

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{|\nabla H|^2}{H}\right) =& \frac{\frac{\partial}{\partial t}|\nabla H|^2 - H^2 - |\nabla H|^2\frac{\partial}{\partial t}H}{H^2}\\
\leq& \frac{\left(\Delta|\nabla H|^2 - 2|\nabla^2 H|^2 + 4|A|^2|\nabla H|^2 + 2H\langle\nabla_i H, \nabla_i|A|^2\rangle\right)H}{H^2}\\
&- \frac{|\nabla H|^2(\Delta H + |A|^2 H)}{H^2}\\
=& \frac{H\Delta|\nabla H|^2 - |\nabla H|^2\Delta H}{H^2} + \frac{3|A|^2|\nabla H|^2}{H} - \frac{2|\nabla^2 H|^2}{H}\\
&+ 2\langle\nabla_i H, \nabla_i|A|^2\rangle.
\end{aligned}$$

So, it remains to show

$$\frac{H\Delta|\nabla H|^2 - |\nabla H|^2\Delta H}{H^2} - \frac{2|\nabla^2 H|^2}{H} \leq \Delta\left(\frac{|\nabla H|^2}{H}\right) \tag{1.33}$$

Using (1.25) we obtain

$$\begin{aligned}
\Delta\left(\frac{f}{g}\right) &= \frac{1}{g}\Delta f - \frac{f}{g^2}\Delta g - \frac{2}{g}\nabla\left(\frac{f}{g}\right)\nabla g.\\
\Rightarrow \Delta\left(\frac{|\nabla H|^2}{H}\right) &= \frac{1}{H}\Delta|\nabla H|^2 - \frac{|\nabla H|^2}{H^2}\Delta H - \frac{2}{H}\nabla\left(\frac{|\nabla H|^2}{H}\right)\nabla H\\
\Rightarrow \frac{\Delta|\nabla H|^2 H - |\nabla H|^2\Delta H}{H^2} &= \Delta\left(\frac{|\nabla H|^2}{H}\right) + \frac{2}{H}\nabla\left(\frac{|\nabla H|^2}{H}\right)\nabla H.
\end{aligned}$$

So that the LHS of (1.33) becomes

$$\Delta\left(\frac{|\nabla H|^2}{H}\right) + \frac{2}{H}\nabla\left(\frac{|\nabla H|^2}{H}\right)\nabla H - \frac{2|\nabla^2 H|^2}{H} \tag{1.34}$$

Looking at the second last term of (1.34) we can obtain

$$\begin{aligned}
\frac{2}{H}\nabla\left(\frac{|\nabla H|^2}{H}\right)\nabla H =& \frac{2}{H}\left(-\frac{|\nabla H|^2}{H^2}\right)(\nabla H)^2 + \frac{4\langle\nabla_i\nabla_j H, \nabla_i H\nabla_j H\rangle}{H^2}\\
\leq& -\frac{2|\nabla H|^4}{H^3} + \frac{4|\nabla^2 H|\nabla H|^2}{H^2}
\end{aligned}$$

Where in the last line we have just applied Cauchy-Schwartz. Therefore

$$\begin{aligned}(1.34) &\leq \Delta\left(\frac{|\nabla H|^2}{H}\right) - \frac{2|\nabla H|^4}{H^3} + \frac{4|\nabla^2 H|\nabla H|^2}{H^2} - \frac{2|\nabla^2 H|^2}{H} \\ &\leq \Delta\left(\frac{|\nabla H|^2}{H}\right)\end{aligned}$$

where we have applied $ab \leq \frac{a^2+b^2}{2}$ with

$$a = \frac{|\nabla^2 H|H}{H^{\frac{3}{2}}} \quad \text{and} \quad b = \frac{|\nabla H|^2}{H^{\frac{3}{2}}}$$

and the result follows. □

To prove the main theorem of this section we still require two more evolution equations.

Lemma 1.35 (Lemma 6.5 [58])**.**

(i) $\frac{\partial}{\partial t} H^3 = \Delta H^3 - 6H|\nabla H|^2 + 3|A|^2 H^3$.

(ii) $\frac{\partial}{\partial t}\Big((|A|^2 - \frac{H^2}{n})H\Big) \leq \Delta\Big((|A|^2 - \frac{H^2}{n})H\Big) - \frac{2(n-1)}{3n} H|\nabla A|^2 + C_3|\nabla A|^2$
$+ 3|A|^2 H(|A|^2 - \frac{H^2}{n})$.

Proof. (i)

$$\frac{\partial}{\partial t} H^3 = 3H^2 \Delta H + 3|A|^2 H^3 \tag{1.36}$$

Now,

$$\begin{aligned}\Delta H^3 &= H^2\Delta H + H\Delta H^2 + 2\nabla H \nabla H^2 \\ &= H^2\Delta H + H(2H\Delta H + 2(\nabla H)^2) + 2\nabla H\,(2\nabla H \cdot H) \\ &= H^2 \Delta H + 2H^2\Delta H + 2H(\nabla H)^2 + 4H(\nabla H)^2 \\ &= 3H^2\Delta H + 6H(\nabla H)^2 \\ \Rightarrow 3H^2\Delta H &= \Delta H^3 - 6H(\nabla H)^2.\end{aligned}$$

Plugging into (1.36) the result follows.

(ii)

$$\begin{aligned}\frac{\partial}{\partial t}\left(\left(|A|^2 - \frac{1}{n}H^2\right)H\right) &= \Delta\left(\left(|A|^2 - \frac{1}{n}\right)H^2\right) - 2H\left(|\nabla A|^2 - \frac{1}{n}|\nabla H|^2\right) \\ &\quad - 2\left\langle \nabla_i H, \nabla_i\left(|A|^2 - \frac{1}{n}H^2\right)\right\rangle + 3|A|^2 H\left(|A|^2 - \frac{1}{n}H^2\right).\end{aligned}$$

Now using Theorem 1.27, $g_\sigma \leq H^\sigma$ and the following relations,

$$ab \leq \frac{\eta}{2}a^2 + \frac{1}{2\eta}b^2 \text{ and } |h^0_{ij}|^2 = \left(|A|^2 - \frac{1}{n}H^2\right) = g_\sigma H^{2-\sigma}.$$

using the Leibniz rule we can estimate

$$\begin{aligned} 2|\langle \nabla_i H, \nabla_i(|A|^2 - \frac{1}{n}H^2)\rangle| &= 4|\langle \nabla_i H h^0_{kl}, \nabla_i h^0_{kl}\rangle| \\ &\leq 4|\nabla H||h^0_{kl}||\nabla A|. \end{aligned} \tag{1.37}$$

Now

$$\begin{aligned} |\nabla H| = \nabla|g^{ij}h_{ij}| &= g^{ij}\nabla h_{ij} \leq |g^{ij}||\nabla h_{ij}| \leq n|\nabla A| \\ \text{and} \quad |h^0_{kl}|^2 &\leq C_0 H^{2-\delta} \Rightarrow |h^0_{kl}| \leq C_0^{\frac{1}{2}} H^{1-\frac{\delta}{2}}. \\ \Rightarrow \quad (1.37) &\leq 4nC_0^{\frac{1}{2}} H^{1-\frac{\delta}{2}}|\nabla A|^2 \\ &\leq \frac{2(n-1)}{3n}H|\nabla A|^2 + C|A|^2 \end{aligned}$$

where C depends on n, C_0 and δ. The result then follows from Lemma 1.17(ii).

□

We are now ready to bound f and prove the Theorem 1.29. From Lemma 1.32 and Lemma 1.35 we obtain

$$\begin{aligned} \frac{\partial f}{\partial t} =& \Delta f + 3|A|^2\left(\frac{|\nabla H|^2}{H}\right) + 2\langle \nabla_i H, \nabla_i |A|^2\rangle \\ &+ 6\eta H|\nabla H|^2 - N\frac{2(n-1)}{3n}H|\nabla A|^2 + 2NC_3|A|^4 \\ &+ 3N|A|^2 H\left(|A|^2 - \frac{1}{n}H^2\right) - 3\eta|A|^2H^3. \end{aligned}$$

Since

$$\frac{H^2}{n} \leq |A|^2 \leq H^2,\ |\nabla H|^2 \leq n|A|^2 \text{ and } \eta \leq 1$$

we can choose N depending only on n large enough such that

$$\frac{\partial f}{\partial t} \leq \Delta f + 2NC_3H^4 + 3NH^3\left(|A|^2 - \frac{H^2}{n}\right) - \frac{3}{n}\eta H^5.$$

By Theorem 1.27 we obtain

$$\begin{aligned} 2NC_3H^4 + 3NH^3\left(|A|^2 - \frac{1}{n}H^2\right) &\leq 2NC_3H^4 + 3NC_0H^{5-\delta} \\ &\leq \frac{3}{n}\eta H^5 + C \end{aligned}$$

where the constant C depends on η, δ, n, C_0 and C_3. and hence $\frac{\partial f}{\partial t} \leq \Delta f + C$, where C depends on η and $\mathcal{M}_0$.

This implies that $\max f(t) \leq \max f(0) + Ct$ and since we already have a bound for T, f is bounded by some constant C depending on η and $\mathcal{M}_0$.

Therefore

$$|\nabla H|^2 \leq \eta H^4 + CH \leq 2\eta H^4 + \tilde{C}(\eta, \mathcal{M}_0)$$

which proves Theorem 1.29 since η is arbitrary.

1.2.5 Higher Derivatives of $|A|$

For this section, we only state the theorems and lemmas as stated in [58], for detailed proofs or more information please refer to Section 7 in [58].

We write $S * T$ for any linear combination of tensors formed by contraction on S and T by g.

Theorem 1.38 (Theorem 7.1 [58])**.** *For any m we have the following*

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla^m A|^2 =& \Delta|\nabla^m A|^2 - 2|\nabla^{m+1}A|^2 \\ &+ \sum_{i+j+k=m} \nabla_i A * \nabla_j A * \nabla_k A.\end{aligned}$$

where the m-th iterated covariant derivative of A is denoted by $\nabla^m A$.

Proof. Refer to Section 13 of [48]. □

Lemma 1.39 (Lemma 7.2 [58])**.** *If T is any tensor and if $1 \leq i \leq m-1$, then given a constant C depending onn and m which is independent of the metric g and the connection Γ we have the estimate*

$$\int |\nabla^i T|^{\frac{2m}{i}} d\mu \leq C \max_{\mathcal{M}} |T|^{2\left(\frac{m}{i-1}\right)} \int |\nabla^m T|^2 d\mu.$$

Proof. Refer to Section 12 of [48]. □

Theorem 1.40 (Theorem 7.3 [58])**.** *We have the estimate*

$$\frac{\partial}{\partial t}\int_{\mathcal{M}_t} |\nabla^m A|^2 d\mu + 2\int_{\mathcal{M}_t} |\nabla^{m+1}A|^2 d\mu \leq C \max_{\mathcal{M}_t} |A|^2 \int_{\mathcal{M}_t} |\nabla^m A|^2 d\mu.$$

Proof. Integrate the identity from Theorem 1.38 and apply the generalised Hölder's inequality. Then apply the Lemma 1.39 from above.

For more detail refer to [58]. □

1.2.6 The Maximal Time Interval

Theorem 1.41 (Theorem 8.1 [58])**.** *The solution of equation (1.1) exists on a maximal time interval $0 \leq t < T < \infty$ and $\max_{\mathcal{M}_t} |A|^2$ becomes unbounded as $t \to T$.*

For a detailed proof refer to section 8 of [58].

Now we wish to compare the values of $H_{\max}$ and $H_{\min}$ as $t \to T$. Since $|A|^2 \leq H^2$ Theorem 1.41 tells us that $H_{\max}$ becomes unbounded as $t \to T$.

Huisken inspired by Hamilton's work uses Myers Theorem to prove the following result.

Theorem 1.42 (Theorem 8.4 [58])**.** $\frac{H_{\max}}{H_{\min}} \to 1$ *as* $t \to T$.

Below is Myers theorem and a necessary lemma to prove the result, the details can be found in Section 8 [58].

Theorem 1.43 (Theorem 8.5 [58])**.** *[Myers] If* $R_{ij} \geq (n-1)Kg_{ij}$ *along a geodesic of length greater or equal to* $\pi K^{-\frac{1}{2}}$ *on* $\mathcal{M}$*, then the geodesic has conjugate points.*

Lemma 1.44 (Lemma 8.6 [58])**.** *If* $h_{ij} \geq \epsilon H g_{ij}$ *holds on* $\mathcal{M}$ *with some* $0 < \epsilon \leq \frac{1}{n}$*, then*

$$R_{ij} \geq (n-1)\epsilon^2 H g_{ij}.$$

Theorem 1.42 leads us to the following result.

Theorem 1.45 (Theorem 8.7 [58])**.**

$$\int_0^T H_{\max}^2(\tau) d\tau = \infty.$$

Proof. The following DE:

$$\frac{\partial g}{\partial t} = H_{\max}^2 g \text{ with } g(0) = H_{\max}(t) \tag{1.46}$$

has a solution since $H_{\max}^2$ is continuous in t.

Furthermore Lemma 1.18 (v) gives us

$$\begin{aligned} &\frac{\partial}{\partial t} H \leq \Delta H + H_{\max}^2 H \\ \Rightarrow &\frac{\partial}{\partial t}(H-g) \leq \Delta(H-g) + H_{\max}^2(H-g). \end{aligned}$$

Applying the maximum principle we obtain $H \leq g$ for $0 \leq t < T$.

Therefore returning to (1.46)

$$\int_0^t H_{\max}^2(\tau) d\tau = \log\left(\frac{g(t)}{g(0)}\right) \to \infty \text{ as } t \to T.$$

where we have used the fact that $H_{\max} \to \infty$ [58].

□

Corollary 1.47.

$$\int_0^T H_{\min}^2(\tau)\, d\tau = \infty.$$

Proof. Combine Theorem 1.42 and Theorem 1.45. □

Corollary 1.48 (Corollary 8.8 [58])**.** *Let* $h = \frac{\int_{\mathcal{M}_t} H^2 d\mu}{\int_{\mathcal{M}_t} d\mu}$.

Then $\int_0^T h(\tau) d\tau = \infty$.

Proof. Follows from the Corollary 1.48 and Corollary 1.48. □

Corollary 1.49 (Corollary 8.9 [58]).

$$\frac{|A|^2}{H^2} - \frac{1}{n} \to 0 \; as \; t \to T.$$

Proof. Consequence of Theorem 1.27 since $H_{\min} \to \infty$ by Theorem 1.45. $\square$

We now have enough to prove Theorem 1.13. Since the surfaces are shrinking under the flow it is clear that $\mathcal{M}_{t_1}$ stays in the region $\mathbb{R}^{n+1}$ and is bounded by $\mathcal{M}_{t_2}$ for $t_1 > t_2$. Moreover $\frac{H_{\max}}{H_{\min}} \to 1$ as $t \to T$ so we know the diameter will tend to 0 as t approaches the singular time T.

1.3 Type I and Type II Singularities

In this section we discuss the two types of solutions to (1.1). A key starting point for singularity analysis is Huisken's monotonicity formula which he derived in [61]. He then went on to show that the first type, type I singularities, are asymptotically self-similar.

As in [61] we begin by providing a basic lower bound for the blow up rate of the curvature.

Lemma 1.50 (Lemma 1.2 [61]). *The function $U(t) = \max_{\mathcal{M}_t} |A|^2$ is Lipchitz continuous and satisfies $U(t) \geq \frac{1}{2(T-t)}$.*

Proof. It is clear that U is Lipchitz continuous so long as $|A|^2$ is bounded. We recall the evolution equation from Lemma 1.18 (vii) and the maximum principle we obtain

$$\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4.$$

using this evolution equation and the maximum principle we obtain

$$\frac{\partial}{\partial t}U(t) \leq 2(U(t))^2$$
$$\Rightarrow \frac{\partial}{\partial t}(U^{-1}(t)) \geq -2.$$

We can apply Theorem 1.41 to see that $U^{-1}(t) \to 0$ as $t \to T$ and we obtain

$$U(t) \geq \frac{1}{2(T-t)}.$$

□

For this section as in [61] we will assume that the blow-up rate of the curvature satisfies an upper bound of the form

$$U(t) = \max_{\mathcal{M}_t} |A|^2 \leq \frac{C_0}{2(T-t)} \tag{1.51}$$

In order to keep the curvature of our surface $\mathcal{M}$ bounded as $t \to T$ it will be useful to perform a rescaling for our surface near these singular points.

Now 1.51 tells us that for two times τ, t with $0 < \tau \leq t < T$

$$|F(p,t) - F(p,\tau)| \leq \int_\tau^t |H(p,\tau)| d\tau \leq C_0 \left((T-\tau)^{\frac{1}{2}} - (T-t)^{\frac{1}{2}}\right)$$

for all points $p \in \mathcal{M}$. Therefore $F(\cdot,t)$ converges uniformly as $t \to T$, which motivates the following definitions.

Definition 1.52. *Define $x \in \mathbb{R}^{n+1}$ to be a blow-up point if there is a $p \in \mathcal{M}$ such that $F(p,t) \to x$ as $t \to T$ and $|A|(p,t)$ becomes unbounded as $t \to T$.*

Definition 1.53. *Given a blow-up point $x \in \mathbb{R}^{n+1}$, we define a rescaled immersion $\tilde{F}(p,s)$ by*

$$\tilde{F}(p,s) = \left(2(T-t)^{\frac{1}{2}}\right) F(p,t), \quad s(t) = -\frac{1}{2}\log(T-t).$$

These rescaled surfaces $\tilde{\mathcal{M}}_s = \tilde{F}(\cdot, s)\mathcal{M}$ are defined for $-\frac{1}{2}\log T \le s < \infty$ and satisfy

$$\frac{d}{ds}\tilde{F}(p,s) = \tilde{H}(p,s) + \tilde{F}(p,s)$$

where $\tilde{H}$ in the mean curvature vector of $\tilde{\mathcal{M}}_s$.

These definitions lead us to consider the following.

Definition 1.54 (Type I Singularity)**.** *If there exists a positive constant $C > 0$ such that*

$$\max_{\mathcal{M}_t} |A|^2 \le \frac{C}{2(T-t)}$$

then we say that the flow undergoes a type-I singularity*.*

Definition 1.55. *If for the above definition no such C exists, then we say that the flow undergoes a* type-II singularity*.*

We will not study type-II singularities but we would also like to describe an example of one first conjectured by Hamilton for Ricci flow, described in Section 3 of [45].

It is possible to obtain a surface in the shape of a dumbbell in the following way.

For any $\lambda > 0$,

$$\phi_\lambda(x) = \sqrt{(1-x^2)(x^2+\lambda)}, \quad -1 \le x \le 1$$

where λ describes the radius of the neck.

Then for $n \ge 2$, we define $\mathcal{M}^\lambda$ to be the n-dimensional hypersurface obtained by rotating the graph of ϕ_λ in $\mathbb{R}^2$. Then [1] showed it is possible for the following to occur

(i) If λ is sufficiently large then $\mathcal{M}_t^\lambda$ will become convex and shrink to a point in finite time.

(ii) If λ is sufficiently small the $\mathcal{M}_t^\lambda$ we will obtain a standard neck pinch singularity which will be described in the two-convex case in Chapter 3.

(iii) There exist some range of value for λ between case (i) and case (ii) such that $\mathcal{M}_t^\lambda$ will still shrink to a point in finite time and has positive mean curvature up to the singular time T but will never become convex. Moreover the maximum curvature occurs at the two points where the surface meets the axis of rotation. This is a singularity of type II.

Returning to type-I singularities, we stated in the introduction that the main tool for studying these is Huisken's monotonicity formula. He showed that the flow is asymptotically self-similar near a given singularity and thus, is modelled by self-shrinking solutions of the flow.

Theorem 1.56 (Theorem 3.1 [61])**.** *Let $\rho(x,t)$ be the backward heat kernel at $(0, t_0)$,*

$$\rho(x,t) = \frac{1}{(4\pi(t_0-t))^{\frac{n}{2}}} \exp\left(-\frac{|x|^2}{4(t_0-t)}\right)$$

for $t < t_0$. Then if $\mathcal{M}_t$ is a surface satisfying (1.1) for $t < t_0$, we have the formula

$$\frac{\partial}{\partial t}\int_{\mathcal{M}_t} \rho(x,t)d\mu_t = -\int_{\mathcal{M}_t} \rho(x,t)\left|H + \frac{1}{2\tau}F^\perp\right|^2 d\mu_t,$$

where $F^\perp$ is the normal component of F and $\tau = (t_0 - t)$.

Proof. Refer to Section 3 of [61]. □

Theorem 1.57 (Theorem 3.5 [61])**.** *Any type I blow-up limit of a mean convex flow about a type I singularity satisfies*

$$H = \langle x, \nu \rangle$$

where x is the position vector, H is the mean curvature and ν is the outward pointing unit normal.

Proof. Integrate the monotonicity formula over the blow-up sequence, refer to [61] for a full proof. □

It is interesting to look at surfaces of positive mean curvature as those surfaces will continue to have positive mean curvature on $\mathcal{M}_t$ as long as a solution to (1.1) exists. In [61] Huisken proved that for $n \geq 2$ the sphere is the only compact hypersurface of positive mean curvature moving under self similarities.

Theorem 1.58 (Theorem 4.1 [61])**.** *If $\mathcal{M}$, $n \geq 2$, is compact, with nonnegative mean curvature and satisfies $H = \langle x, \nu \rangle$, then $\mathcal{M}$ is a sphere of radius $\sqrt{n}$.*

Proof. We begin by taking an orthonormal frame $e_1, e_2, \dots e_n$. Differentiating

$$\langle e_i, \nu \rangle = 0 \;\Rightarrow\; h_{ij} = e_i \nu_j. \tag{1.59}$$

We differentiate the expression $\langle \nu, \nu \rangle = 1$ to obtain that $\langle \nu_i, \nu \rangle = 0$ to see that ν_i can be expressed as a linear combination of tangent vectors

$$\Rightarrow\; \nu = a^j e_j.$$

Scalar multiplication with e_k together with (1.59) gives us

$$\begin{aligned} \langle \nu_i, e_k \rangle &= a^j e_j e_k \\ &= a^j \delta_{jk} = a^k. \end{aligned}$$

Therefore

$$\begin{aligned} a^k &= h_{ij} \text{ and } h_{ij} e_k \\ &\Rightarrow\; \nabla_k \nu = h_{ij} e_j. \end{aligned}$$

So differentiating

$$H = \langle x, \nu \rangle$$

we obtain

$$\begin{aligned} \nabla_i H &= \langle \nabla_i x, \nu + \langle x, \nabla_i \nu \rangle \\ &= \langle x, e_l \rangle h_{lj} \end{aligned}$$

since $\nabla_i x$ is perpendicular to ν.

We will also use Gauss' equation here $\nabla_k e_i = -h_{ik}\nu$.

$$\begin{aligned}\nabla_i \nabla_j H &= \nabla_i \langle x, e_l \rangle h_{lj} \\ &= \langle \nabla_i x, e_l \rangle h_{lj} + \langle x, \nabla_i e_l \rangle h_{lj} + \langle x, e_l \rangle \nabla_l h_{ij} \\ &= \langle e_i, e_l \rangle h_{lj} + \langle x, -h_{il}\nu \rangle h_{lj} + \langle x, e_l \rangle \nabla_l h_{ij} \\ &= h_{ij} - H h_{il} h_{lj} + \langle x, e_l \rangle \nabla_l h_{ij}.\end{aligned}$$

Contracting with g_{ij} and h_{ij} respectively we obtain

$$\Delta H = \Delta H - H|A|^2 + \langle x, e_l \rangle \nabla_l h_{ij} \tag{1.60}$$

$$h_{ij} \nabla_j \nabla_j H = |A|^2 - H \mathrm{tr}(A^3) + \langle x, e_l \rangle h_{ij} \nabla_l h_{ij}. \tag{1.61}$$

Applying Simon's identity from Lemma 1.7 to 1.61 we obtain

$$\Delta |A|^2 = 2|\nabla A|^2 + 2|A|^2 - 2|A|^4 + 2\langle x, e_l \rangle h_{ij} \nabla_l h_{ij}.$$

Applying the strong maximum principle to (1.60) we see since $\mathcal{M}_t$ is a closed, compact manifold we have the strict inequality $H > 0$. Since if there existed a point $p \in \mathcal{M}_0$ such that $H(p) = 0$ then the strong maximum principle would imply that $H = 0$ everywhere.

Simon's identity applied to (1.60) implies that

$$\begin{aligned}\Delta\left(\frac{|A|^2}{H^2}\right) =& \frac{2}{H^4}\Big(H^2|A|^2 + \frac{1}{2}\langle x, e_l \rangle \nabla_l |A|^2 H^2 - H|A|^2 \langle x, e_l \rangle \nabla_l H \\ & - 2H \nabla_i |A|^2 \nabla_i H + 3|A|^2 |\nabla H|^2\Big).\end{aligned}$$

Using

$$|h_{ij} \nabla_l H - \nabla_l h_{ij} H|^2 = |A|^2 |\nabla H|^2 + |\nabla A|^2 H^2 - H \nabla_l H \nabla_l |A|^2$$

we get that

$$\begin{aligned}\Delta\left(\frac{|A|^2}{H^2}\right) =& \frac{2}{H^4}|h_{ij}\nabla_l H - \nabla_l h_{ij} H|^2 + \frac{2}{H^4}\Big(2|A|^2|\nabla H|^2 - H\nabla_i |A|^2 \nabla_i H \\ & . + \frac{1}{2} H^2 \langle x, e_l \rangle \nabla_l |A|^2 - |A|^2 \langle x, e_l H \nabla_l H\Big).\end{aligned}$$

Moreover we have

$$\nabla_i\left(\frac{|A|^2}{H^2}\right) = \frac{\nabla_i |A|^2}{H^2} - 2\frac{|A|^2}{H^3}\nabla_i H$$

such that

$$\begin{aligned}\Delta\left(\frac{|A|^2}{H^2}\right) =& \frac{2}{H^4}|h_{ij}\nabla_l H - \nabla_l h_{ij} H|^2 - \frac{2}{H}\nabla_i H \nabla_i\left(\frac{|A|^2}{H^2}\right) \\ & + \langle x, e_i \rangle \nabla_i \left(\frac{|A|^2}{H^2}\right).\end{aligned} \tag{1.62}$$

Since $\mathcal{M}$ is compact, we apply the strong maximum principle to the equation above to obtain that there exists a fixed constant $\alpha > 0$ such that

$$|A|^2 = \alpha H^2.$$

together with (1.62) this tells us that

$$|h_{ij}\nabla_l H - \nabla_l h_{ij} H| \equiv 0 \tag{1.63}$$

on $\mathcal{M}$.

Now splitting (1.63) together with the Codazzi equation lets us obtain

$$|h_{ij}\nabla_l H - h_{il}\nabla_j H| = 0$$

And so if $e_1 = \frac{\nabla H}{|\nabla H|}$ (chosen such that it points in the direction of the gradient of the mean curvature) we have

$$|h_{ij}\nabla_l H - h_{il}\nabla_j H|^2 = 2|\nabla H|^2|A|^2 - 2h_{ij}h_{il}\nabla_l H\nabla_j H$$

where e_1 is the only direction H changes so $l = j = 1$ and $e_i H = 0$ for $i \geq 2$.

Thus at each part of $\mathcal{M}$ we either have $|A|^2 = \sum_{i=1}^n h_{li}^2$ or $|\nabla H|^2 \equiv 0$. If $|\nabla H|^2 \equiv 0$ it follows immediately that $\mathcal{M}$ is a sphere.

So suppose there is a point in $\mathcal{M}$ such that $|A|^2 = \sum_{i=1}^n h_{li}^2$, since

$$|A|^2 = h_{11}^2 + 2\sum_{i=2}^n h_{1i}^2 + \sum_{i,j\neq 1}^n h_{ij}^2$$

this would only be possible if $h_{ij} = 0$ unless $i = k = 1$. Then $|A|^2 = H^2$ at this point and therefore everywhere on $\mathcal{M}$. Integrating (1.60) we obtain

$$\begin{aligned}
0 = \int_{\mathcal{M}_t} \Delta H d\mu &= \int_{\mathcal{M}_t} H - H|A|^2 + \langle x, e_l\rangle \nabla_l H d\mu \\
&= \int_{\mathcal{M}_t} H - H^3 + \langle x, e_l\rangle \nabla_l H d\mu \\
\Rightarrow \int_{\mathcal{M}_t} H^3 d\mu &= \int_{\mathcal{M}_t} H d\mu + \int_{\mathcal{M}_t} \langle x, e_l\rangle \nabla_l H d\mu \\
&= \int_{\mathcal{M}_t} H d\mu - n\int_{\mathcal{M}_t} H d\mu + \int_{\mathcal{M}_t} \langle x, \nu\rangle H^2 d\mu.
\end{aligned}$$

Since $\langle x, \nu\rangle = H$, we derive

$$(n-1)\int_{\mathcal{M}_t} H d\mu = 0,$$

which is a contradiction for $n \geq 2$. This completes the proof. $\square$

Huisken then goes on to prove the following theorem, the proof of which we omit, but can be found in Section 5 of [61].

Theorem 1.64 (Theorem 5.5 [61]). *Let $\mathcal{M}_0^2 \subset \mathbb{R}^3$ be a two-dimensional rotationally symmetric hypersurface with positive mean curvature, defined by a graph along the whole x_1 axis. Then the solution of mean curvature flow develops a type I singularity as $t \to T$ for $0 \leq t < T < \infty$. Moreover at any blow-up point the rescaled surfaces $\tilde{\mathcal{M}}_s$ converge to a cylinder of radius* 1.

1.4 Convexity Estimates for Mean Curvature Flow

In the second chapter we will need to extend the flow past a singularity, in order to do this we will need to obtain results related to the singular behaviour as $t \to T$. Huisken and Sinestrari were able to achieve this by studying the elementary symmetric functions of the principal curvatures and deriving new a priori estimates for them using only the assumption of nonnegative mean curvature. They were then able to conclude that points where our mean curvature tends to infinity, have almost positive definite second fundamental form.

These surfaces are referred to as mean convex and do not necessarily satisfy a pinching condition, we require that the mean curvature is positive at all points, but not that each principal curvature is positive. The results and proofs in this section can be found in [66] [65] and [96].

Definition 1.65. *For any $k = 1, 2, \ldots, n$ we set*

$$S_k(\lambda) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_k}$$

for all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$.

It is clear from the definition that $S_1 = H$. Moreover we set $S_0 \equiv 1$ and $S_k \equiv 0$ for $k > n$.

The main Theorem from [65] is as follows.

Theorem 1.66 (Theorem 1.1 [66])**.** *Suppose $F_0 : \mathcal{M} \to \mathbb{R}^{n+1}$ is a smooth closed hypersurface immersion with nonnegative mean curvature. For each k, $2 \le k \le n$ and every $\eta > 0$ there is a constant $C_{\eta,k}$ depending only on n, k, η and the initial data such that everywhere on $\mathcal{M} \times [0, T)$ we have the estimate*

$$S_k \ge -\eta H^k - C_{\eta,k}.$$

Before we proceed we would like to describe the rescaling procedure for singularities in a bit more detail than done in Section 1.3, this is essential for a better understanding of Corollary 1.68.

In order to study the singular behaviour of hypersurfaces evolving geometric flows, techniques from PDE theory are applied. Solutions near a singularity are rescaled suitably so as to approach a non-trivial limit, we can then deduce the asymptotic profile of a surface near a singularity.

This process is described by Hamilton ([45], Section 16), and depends on the type of singularity.

We proceed as follows, with the only difference being in the choice of sequence chosen.

For a type-I singularity we can take any sequence of times $\{t_k\}$ with $t_k \to T$ and pick $p_k \in \mathcal{M}$ such that

$$|A|^2(p_k, t_k) = \max_{\mathcal{M}_{t_k}} |A|^2.$$

For a type-II singularity we pick a sequence (p_k, t_k) for any $k \ge 1$ such that for $t_k \in \left[0, T - \frac{1}{k}\right]$ and $p_k \in \mathcal{M}$ we have

$$|A|^2(p_k, t_k) \left(T - \frac{1}{k} - t_k \right) = \max_{\substack{t \le T - \frac{1}{k} \\ p \in \mathcal{M}}} |A|^2(p, t) \left(T - \frac{1}{k} - t \right).$$

Now for both types of singularities we set

$$L_k = |A|^2(p_k, t_k), \quad \alpha_k = -L_k t_k, \quad \omega_k = L_k\left(T - t_k - \frac{1}{t}\right),$$

then for any k we have the family of rescaled immersions,

$$F_k(\cdot,\tau) = \sqrt{L_k}\left(F\left(\cdot, \frac{\tau}{L_k} + t_k\right) - F\left(p_k, t_k\right)\right) \quad \tau \in [\alpha_k, \omega_k]$$

it is clear that $\mathcal{M}_{k,\tau} = F_k(\mathcal{M},\tau)$ evolve by mean curvature flow.

We then have the following properties.

Lemma 1.67. *As $k \to \infty$,*

$$t_k \to T, \quad L_k \to \infty, \quad \alpha_k \to -\infty, \quad \omega_k \to \Omega$$

where

$$\begin{cases} 0 < \Omega < \infty, & \text{for a type-I singularity} \\ \Omega = \infty, & \text{for a type-II singularity} \end{cases}$$

Moreover, for any T_0, T_1 such that $-\infty < T_0 < T_1 < \infty$ and k sufficient large, the surfaces $\mathcal{M}_{k,\tau}$ have uniformly bounded curvature for $\tau \in [T_0, T_1]$.

Proof. Refer to Lemma 4.4 [66]. □

It is then possible to show that a subsequence of the flows $\mathcal{M}_{k,\tau}$ converges smoothly to a limit evolving surface $\mathcal{M}_\tau$ defined for $\tau \in (-\infty, \Omega)$. This allows us to obtain the following corollary to Theorem 1.66.

Corollary 1.68 (Corollary 1.2 [66])**.** *Let $\tilde{\mathcal{M}}_t$ be the limit rescaling of a flow $\mathcal{M}_t$ of closed mean convex surfaces. Then the surface $\tilde{\mathcal{M}}_\tau$ are convex and the flow is defined for $\tau \in (-\infty, \infty)$.*

This follows from the main Theorem, as near a singularity $S_1 = H$ becomes unbounded and each S_k becomes nonnegative after rescaling. For more information on the rescaling procedure refer to [45][66][65][95] and [89].

Following from Section 1.23 we look for an upper bound for the function

$$\frac{|A|^2 - H^2}{H^{2-\sigma}}.$$

for some small positive σ. However, the argument of [58] does not carry over unchanged as it relies on some estimates which hold only for convex surfaces as opposed to mean convex surfaces. To overcome this we introduce a new parameter η and study the function $g_{\sigma,\eta}$. The proof which is described in detail below is an extremely powerful tool for mean curvature flow, similar proofs have been tried and tested by Huisken and Huisken-Sinestrari in many of their papers and obtained some very strong results.

Before we state the main theorem, we introduce the De Giorgi Iteration Lemma as in Lemma 4.1.1 [96]. This will be essential in proving the required result.

Lemma 1.69 (De Giorgi iteration Lemma).
Let $\varphi(t)$ be a non-negative and non-increasing function on $[k_0, \infty)$ satisfying $\varphi(h) \leq \left(\frac{M}{h-k}\right)^{\alpha} (\varphi(k))^{\beta}$ for all $h > k \geq k_0$ for some constants $M > 0, \alpha > 0, \beta > 1$. Then there exists $d > 0$ such that $\varphi(h) = 0$ for all $h \geq k_0 + d$.

Proof. Set $k_s = k_0 + d - \frac{d}{2s}$ for $s = 0, 1, 2, \cdots$, with constant d to be determined. Then our assumption implies the recursive formula

$$\varphi(k_{s+1}) \leq \frac{M^{\alpha} 2^{(s+1)\alpha}}{d^{\alpha}} (\varphi(k_s))^{\beta} \quad s = 0, 1, 2, \cdots \tag{1.70}$$

From this we can prove by induction

$$\varphi(k_s) \leq \frac{\varphi(k_0)}{r^s} \; s = 0, 1, 2 \cdots \tag{1.71}$$

with constant r to be chosen. Once this is proved, letting $s \to \infty$ we obtain $\varphi(k_0 + d) = 0$ and the conclusion of the Lemma holds since $\varphi(t)$ is non-increasing. So, for our proof by induction, suppose that (1.71) is valid for s, then using (1.70) we obtain

$$\begin{aligned}\varphi(k_{s+1}) &\leq \frac{M^{\alpha} 2^{(s+1)\alpha}}{d^{\alpha}} (\varphi(k_s))^{\beta} \\ &\leq \frac{\varphi(k_0) M^{\alpha} 2^{(s+1)\alpha}}{r^{s+1} d^{\alpha} r^{s(\beta-1)-1}} (\varphi(k_0))^{\beta-1}.\end{aligned}$$

Now picking $r = 2^{\frac{\alpha}{\beta-1}}$. Then

$$\varphi(k_{s+1}) \leq \frac{\varphi(k_0) M^{\alpha} 2^{\frac{\alpha\beta}{\beta-1}}}{r^{s+1} d^{\alpha}} (\varphi(k_0))^{\beta-1}.$$

From this, we see that if $d > 0$ satisfies

$$\frac{M^{\alpha} 2^{\frac{\alpha\beta}{\beta-1}}}{d^{\alpha}} (\varphi(k_0))^{\beta-1} \leq 1$$

i.e. $d \geq M 2^{\frac{\beta}{\beta-1}} (\varphi(k_0))^{\frac{\beta-1}{\alpha}}$, so that (1.71) is also valid for s replaced by $s+1$.

□

To prove the main theorem for general k refer to [65] where Huisken-Sinestrari prove the result using induction, the method will be similar to the one described in Section 2.13 of this thesis, however we do not go into great detail.

We follow Section 8 of [97] and Section 3 of [66] and prove the theorem for $k = 2$.

We do this by introducing the function

$$g_{\sigma,\eta} = \frac{|A|^2 - (1+\eta)H^2}{H^{2-\sigma}}. \tag{1.72}$$

The motivation for this function should be clear from the definition of S_k, for $k = 2$ as

$$\begin{aligned}H^2 &= \left(\lambda_1^2 + \lambda_2^2 + \cdots + \lambda_n^2\right) \\ &= |A|^2 - 2S_2.\end{aligned}$$

We need a uniform upper bound for (1.72) for some sufficiently small $\sigma > 0$. To do this we first compute the evolution equation of $g_{\sigma,\eta}$.

$$\begin{aligned}
\frac{\partial g_{\sigma,\eta}}{\partial t} =& \frac{\frac{\partial}{\partial t}|A|^2 - (1+\eta)2H\frac{\partial H}{\partial t}}{H^{2-\sigma}} - \frac{(2-\sigma)|A|^2 - (1+\eta)H^2}{H^{3-\sigma}}\frac{\partial H}{\partial t} \\
=& \frac{\Delta|A|^2 - 2|\nabla A|^2 + 2|A|^4 - 2(1+\eta)H(\Delta H + |A|^2 H)}{H^{2-\sigma}} \\
& - (2-\sigma)\frac{|A|^2 - (1+\eta)H^2}{H^{3-\sigma}}(\Delta H + |A|^2 H) \\
\nabla_i g_{\sigma,\eta} =& \frac{\nabla_i|A|^2 - 2(1+\eta)H\nabla_i H}{H^{2-\sigma}} - (2-\sigma)\left(\frac{|A|^2 - (1+\eta)H^2}{H^{3-\sigma}}\right)\nabla_i H \\
\Delta g_{\sigma,\eta} =& \frac{1}{H^{2-\sigma}}(\Delta|A|^2 - \sigma(1+\eta)H\Delta H + (1+\eta)\sigma(\sigma-3)|\nabla H|^2) \\
& - 2(2-\sigma)\frac{\nabla_i H}{H}\nabla_i g_{\sigma,\eta} + \frac{2-\sigma}{H^{4-\sigma}}(|A|^2|\nabla H|^2(\sigma-1) - H|A|^2\Delta H)
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\partial g_{\sigma,\eta}}{\partial t} =& \Delta g_{\sigma,\eta} + \frac{1}{H^{2-\sigma}}(\sigma(1+\eta)H\Delta H - (1+\eta)\sigma(\sigma-3)|\nabla H|^2 - 2|\nabla A|^2 + 2|A|^4 \\
& - 2(1+\eta)H(\Delta H + |A|^2 H)) + 2(2-\sigma)\frac{\nabla_i H}{H}\nabla_i g_{\sigma,\eta} \\
& - \frac{(2-\sigma)}{H^{4-\sigma}}((\sigma-1)|A|^2|\nabla H|^2 - H|A|^2\Delta H) \\
& - (2-\sigma)\frac{|A|^2 - (1+\eta)H^2}{H^{3-\sigma}}(\Delta H + H|A|^2) \\
=& g_{\sigma,\eta} + \frac{2(1-\sigma)}{H}\nabla_i H\nabla_i g_{\sigma,\eta} + \frac{1}{H^{4-\sigma}}[2H\nabla_i H\nabla_i|A|^2 - 4(1+\eta)H^2|\nabla H|^2 \\
& - 2(2-\sigma)|(|A|^2 - (1+\eta)H^2)|\nabla H|^2 - (1+\eta)\sigma(\sigma-3)H^2|\nabla H|^2 \\
& - 2H^2|\nabla A|^2 + 2H^2|A|^4 - 2(1+\eta)H^4|A|^2 - (2-\sigma)(\sigma-1)|A|^2|\nabla H|^2 \\
& - (2-\sigma)(|A|^2 - (1+\eta)H^2)H^2|A|^2] \\
=& \Delta g_{\sigma,\eta} + \frac{2(1-\sigma)}{H}\nabla_i H\nabla_i g_{\sigma,\eta} + \sigma(\sigma-1)g_{\sigma,\eta}\frac{|\nabla H|^2}{H^2} + \sigma g_{\sigma,\eta}|A|^2 \\
& - \frac{2}{H^{4-\sigma}}\left(|A|^2|\nabla H|^2 - H\nabla_i h\nabla_i|A|^2 + H^2|\nabla A|^2\right).
\end{aligned}$$

So that we get,

$$\begin{aligned}
\frac{\partial g_{\sigma,\eta}}{\partial t} =& \Delta g_{\sigma,\eta} + \frac{2(1-\sigma)}{H}\nabla_i H\nabla_i g_{\sigma,\eta} - \frac{\sigma(1-\sigma)}{H^2}g_{\sigma,\eta}|\nabla H|^2 \\
& - \frac{2}{H^{4-\sigma}}|h_{ij}\nabla_l H - \nabla_l h_{ij} H|^2 + \sigma|A|^2 g_{\sigma,\eta}.
\end{aligned} \tag{1.73}$$

In particular when $\sigma = 0$, $|A|^2 \leq C_0 H^2$ on $\mathcal{M} \times [0,t]$ for $C_0 = \max_{x\in\mathcal{M}}(|A|^2(x,0)/H^2(x,0))$ by the maximum principle.

Unfortunately when $\sigma > 0$, the last term in evolution equation (1.73) is positive and we cannot achieve our goal using just the maximum principle. Instead we will first establish the $\mathcal{L}^p$-estimate and use the De Giorgi type iteration argument to derive the upper bound for $g_{\sigma,\eta}$ with $\sigma > 0$.

In the following, we always assume $\sigma, \eta \in (0,1)$. We observe that

$$g_{\sigma,\eta} = \frac{|A|^2 - (1+\eta)H^2}{H^{2-\sigma}} \leq C_0 H^{\sigma}.$$

For simplicity denote $g_{\sigma,\eta} = g$ and $g_+(x,t) = \max\{g(x,t), 0\}$.

Lemma 1.74 (Lemma 3.4 [65]). *There exist constants C_1, C_2 depending only on η, C_0 such that*

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathcal{M}_t} g_+^p d\mu \leq & -\frac{p(p-1)}{2}\int_{\mathcal{M}_t} g_+^{p-2}|\nabla g|^2 d\mu - \frac{p}{C_2}\int_{\mathcal{M}_t}\frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu \\ & - p\int_{\mathcal{M}_t}\frac{g_+^{p-1}}{H^{4-\sigma}}|H\nabla_i h_{kl} - \nabla_i H h_{kl}|^2 d\mu + p\sigma\int_{\mathcal{M}_t}|A|^2 g_+^p d\mu\end{aligned}$$

for $p \geq C_1$.

Proof.

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathcal{M}_t} g_+^p d\mu =& \int_{\mathcal{M}_t}\left(\frac{\partial g_+^p}{\partial t} - H^2 g_+^p\right) d\mu \\ =& -p(p-1)\int_{\mathcal{M}_t} g_+^{p-2}|\nabla g|^2 d\mu + 2(1-\sigma)p\int_{\mathcal{M}_t}\frac{g_+^{p-1}}{H}\langle\nabla H, \nabla\rangle d\mu \\ & -\sigma(1-\sigma)\int_{\mathcal{M}_t}\frac{|\nabla H|^2}{H^2} p g_+^p d\mu - 2p\int_{\mathcal{M}_t}\frac{g_+^{p-1}}{H^{4-\sigma}}|h_{ij}\nabla_l H - \nabla_l h_{ij} H|^2 d\mu \\ & + p\sigma\int_{\mathcal{M}_t}|A|^2 g_+^p d\mu.\end{aligned}$$

Note that the $-\int_{\mathcal{M}_t} H^2 g_+^p d\mu$ is not important.

Moreover

$$\int_{\mathcal{M}_t} p g_+^{p-1}\Delta g = -p(p-1)\int_{\mathcal{M}_t} g_+^{p-2}|\nabla g|^2,$$

where we have used integration by parts and Stokes Theorem (since we are on a closed manifold $\int g\nabla g = -\int \nabla f \nabla g$).

Also note that

$$\begin{aligned}|h_{ij}\nabla_l H - \nabla_l h_{ij} H^2| &\geq \frac{1}{4}|h_{ij}\nabla_l H - h_{lj}\nabla_i H|^2 \\ &= \frac{1}{2}(|A|^2|\nabla H|^2 - |\nabla_i H h_{ij}|^2).\end{aligned}$$

Now in suitable coordinates, we have

$$|\nabla_i H h_{ij}|^2 = |\nabla_i \lambda_i|^2 \leq \lambda_n^2 |\nabla H|^2.$$

Thus,

$$\begin{aligned}
|h_{ij}\nabla_l h - \nabla_l h_{ij} H|^2 &\geq \frac{1}{2}\sum_{i=1}^{n-1}\lambda_i^2|\nabla H|^2 \\
&= \sum_{i=1}^{n-1}\lambda_i^2\lambda_n^2\frac{|\nabla H|^2}{2\lambda_n^2} \\
&\geq \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\lambda_i^2\lambda_j^2\frac{|\nabla H|^2}{2(n-1)|A|^2} \\
&\geq \frac{1}{\frac{n(n-1)}{2}}\left(\sum_{i<j}\lambda_i\lambda_j\right)^2\frac{|\nabla H|^2}{2(n-1)|A|^2} \quad \text{using Cauchy-Schwartz} \\
&\geq \frac{(|A|^2-H^2)}{4n(n-1)^2|A|^2}|\nabla H|^2 \\
&\geq \frac{\eta^2 H^2}{4n(n-1)^2 C_0}|\nabla H|^2
\end{aligned}$$

That is, $|h_{ij}\nabla_l H - \nabla_l h_{ij} H|^2 \geq \frac{\eta^2 H^2}{4n(n-1)^2 C_0}|\nabla H|^2$.

Now, defining a constant C such that $C \geq \frac{\eta^2}{4n(n-1)^2}$ allows us to obtain

$$\begin{aligned}
\frac{g_+^{p-1}}{H^{4-\sigma}}|h_{ij}\nabla_l H - \nabla_l h_{ij} H|^2 &\geq \frac{g_+^{p-1}}{CH^{2-\sigma}}|\nabla H|^2 \\
&\geq \frac{g_+^{p-1}}{2C}|\nabla H|^2\left(\frac{1}{H^{2-\sigma}} + \frac{g_+}{H^2 C_0}\right)
\end{aligned}$$

using the fact that $g_+ \leq C_0 H^\sigma$.

Applying Cauchy-Schwartz again we have,

$$\begin{aligned}
2(1-\sigma)p\frac{g_+^{p-1}}{H}\langle\nabla H, \nabla g\rangle \leq& 2p\frac{g_+^{p-1}}{H}|\nabla H||\nabla g| \\
\leq& \frac{p}{2CC_0}\frac{g_+^p}{H^2}|\nabla H|^2 + 2C_0 Cpg_+^{p-2}|\nabla g|^2 \\
\leq& p\frac{g_+^{p-1}}{H^{4-\sigma}}|h_{ij}\nabla_l H - \nabla_l h_{ij} H|^2 + \frac{p(p-1)}{2}g_+^{p-2}|\nabla g|^2 \\
&- p\frac{g_+^{-1}}{2CH^{2-\sigma}}|\nabla H|^2 \quad \text{provided } p \geq \max\{2, 1+4C_0\}.
\end{aligned}$$

where for the second inequality we have applied $ab \leq \frac{a^2+b^2}{2}$ where $a = |\nabla H|g^{p/2}$ and $b = |\nabla g|g_+^{\frac{p-2}{2}}$.

Therefore we obtain

$$\frac{\partial}{\partial t}\int_{\mathcal{M}_t} g_+^P d\mu \leq g_+^{p-2}|\nabla g|^2 d\mu - \frac{p}{2C}\int_{\mathcal{M}_t} \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu$$
$$- p\int_{\mathcal{M}_t} \frac{g_+^{p-1}}{H^{4-\sigma}}|h_{ij}\nabla_l H - \nabla_l h_{ij} H|^2 d\mu + p\sigma\int_{\mathcal{M}_t} |A|^2 g_+^p d\mu.$$

□

To absorb the positive integral on the right hand side of (1.73) we derive the following Poincare inequality.

Lemma 1.75 (Lemma 3.5 [65]). *There exists a constant C_3 depending only on C_0 such that*

$$\frac{1}{C_3}\int_{\mathcal{M}_t} |A|^2 g_+^p d\mu \leq (p+\frac{p}{\beta})\int_{\mathcal{M}_t} g_+^{p-2}|\nabla g|^2 d\mu + (1+\beta p)\int_{\mathcal{M}_t} \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu$$
$$+ \int_{\mathcal{M}_t} \frac{g_+^{p-1}}{H^{4-\sigma}}|h_{ij}\nabla_l H - \nabla_l h_{ij} H|^2 d\mu$$

for any $\beta > 0$ and $p > 2$.

Proof. Recall Lemma 1.7 (ii),

$$\frac{1}{2}\Delta|A|^2 = \langle h_{ij}, \nabla_i\nabla_j H\rangle + |\nabla A|^2 + H(g^{ij}g^{kl}g^{mn}h_{ik}h_{lm}h_{nj}) - |A|^4$$
$$\text{where} \quad H(g^{ij}g^{kl}g^{mn}h_{ik}h_{lm}h_{nj}) - |A|^4 = H\mathrm{tr}(A^3) - |A|^4 \quad \text{on}\{g(x,t)\geq 0\}.$$
$$\begin{aligned}\Rightarrow\ -2\left(H(g^{ij}g^{kl}g^{mn}h_{ik}h_{lm}h_{nj}) - |A|^4\right) &= 2(|A|^4 - H\mathrm{tr}(A^3))\\ &\geq 2(|A|^4 - |A|^3 H)\\ &\geq 2|A|^3(\sqrt{1+\eta}-1)H\\ &\geq 2\sqrt{1+\eta}(\sqrt{1+\eta}-1)|A|^2H^2\\ &\geq \eta|A|^2H^2\end{aligned}$$

Now we compute the Laplacian of the function g as,

$$\begin{aligned}
\Delta g &= \frac{1}{H^{2-\sigma}}(\Delta|A|^2 - \sigma(1+\eta)H\Delta H + (1+\eta)\sigma(\sigma-3)|\nabla H|^2) \\
&\quad - 2(2-\sigma)\frac{\nabla_i H}{H}\nabla_i g + \frac{2-\sigma}{H^{4-\sigma}}(|A|^2|\nabla H|^2(\sigma-1) - H|A|^2\Delta H) \\
&= \frac{2}{H^{2-\sigma}}\langle h_{ij}, \nabla_i\nabla_j H\rangle + \frac{2}{H^{2-\sigma}}(H(g^{ij}g^{kl}g^{mn}h_{ik}h_{lm}h_{nj}) - |A|^4) \\
&\quad - \frac{2(1-\sigma)}{H}\langle\nabla_i H, \nabla_i g\rangle + \frac{1}{H^{4-\sigma}}\Big(- 2H\nabla_i H\nabla_i|A|^2 + 4(1+\eta)H^2|\nabla H|^2 \\
&\quad + 2(2-\sigma)(|A|^2 - (1+\eta)H^2)|\nabla H|^2 + 2H^2|\nabla A|^2 - \sigma(1+\eta)H^3\Delta H \\
&\quad + (1+\eta)\sigma(\sigma-3)H^2|\nabla H|^2 + (2-\sigma)((\sigma-1)|A|^2|\nabla H|^2 - H|A|^2\Delta H)\Big) \\
&= \frac{2}{H^{4-\sigma}}\langle h_{ij}, \nabla_i\nabla_j H\rangle + \frac{2}{H^{2-\sigma}}(H(g^{ij}g^{kl}g^{mn}h_{ik}h_{lm}h_{nj}) - |A|^4) \\
&\quad - \frac{2(1-\sigma)}{H}\langle\nabla_i H, \nabla_i g\rangle + \sigma(1-\sigma)g\frac{|\nabla H|^2}{H^2} + \frac{1}{H^{4-\sigma}}\Big(- 2H\nabla_i H\nabla_i|A|^2 \\
&\quad + 2|A|^2|\nabla H|^2 + 2H^2|\nabla A|^2\Big) + \frac{1}{H^{4-\sigma}}\Big(- 2(1+\eta)H^3\Delta H - (2-\sigma)H^{3-\sigma}g\Delta H\Big).
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\Delta g =& \frac{2}{H^{2-\sigma}}\langle h_{ij}, \nabla_i\nabla)jH\rangle + \frac{2}{H^{2-\sigma}}(Hg^{ij}g^{kl}g^{mn}h_{ik}h_{lm}h_{nj}) - |A|^4) \\
&+ \frac{2}{H^{4-\sigma}}|h_{ij}\nabla_l H - \nabla_l h_{ij}H^2| + \sigma(1-\sigma)\frac{|\nabla H|^2}{H^2} \\
&- \frac{2(1-\sigma)}{H}\langle\nabla_i H, \nabla_i g\rangle - \left(\frac{(2-\sigma)}{H}g + 2(1+\eta)\frac{1}{H^{1-\sigma}}\right)\Delta H.
\end{aligned}$$

We multiply this equation by $g_+^p H^{-\sigma}$ and integrate by parts to obtain

$$\begin{aligned}
-p&\int_{\mathcal{M}_t}\frac{1}{H^\sigma}g_+^{p-1}|\nabla g|^2 d\mu + \sigma\int_{\mathcal{M}_t}\frac{g_+^p}{H^{1+\sigma}}\langle\nabla_i H, \nabla_i g\rangle d\mu \\
&= - 2\int_{\mathcal{M}_t} p\frac{g_+^{p-1}}{H^2}\langle h_{ij}, \nabla_i g\nabla_j H\rangle d\mu - 2\int_{\mathcal{M}_t}\frac{g_+^p}{H^2}|\nabla H|^2 d\mu \\
&\quad + 4\int_{\mathcal{M}_t}\frac{1}{H^3}\langle h_{ij}, \nabla_i\nabla_j H\rangle g_+^p d\mu + \int_{\mathcal{M}_t}\frac{2}{H^2}\Big(H(g^{ij}g^{kl}g^{mn}h_{ik}h_{lm}h_{nj}) - |A|^4\Big)g_+^p d\mu \\
&\quad + \int_{\mathcal{M}_t}\frac{2}{H^4}|H_{ij}\nabla_l H - \nabla_l h_{ij}H|^2 g_+^p d\mu + \sigma(1-\sigma)\int_{\mathcal{M}_t}\frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu \\
&\quad - 2(1+\eta)\int_{\mathcal{M}_t}\frac{g_+^p}{H^2}|\nabla H|^2 d\mu + 2(1+\eta)\int_{\mathcal{M}_t} p\frac{g_+^{p-1}}{H}\langle\nabla_i H, \nabla_i g\rangle d\mu.
\end{aligned}$$

Thus we have

$$
\begin{aligned}
&\int_{\mathcal{M}_t} \frac{-2g_+^p}{H^2} \left(H(g^{ij}g^{kl}g^{mn}h_{ik}h_{lm}h_{nj}) - |A|^4 \right) d\mu \\
&\quad = p\int_{\mathcal{M}_t} \frac{g_+^{p-1}}{H^\sigma}|\nabla g|^2 d\mu - 2p\int_{\mathcal{M}_t} \frac{g_+^{p-1}}{H^2}\langle h_{ij}, \nabla_i\nabla_j H\rangle \\
&\quad + \int_{\mathcal{M}_t} \frac{g_+^p}{H^3}\langle h_{ij}, \nabla_i\nabla_j H\rangle d\mu + 2\int_{\mathcal{M}_t} \frac{g_+^p}{H^4}|h_{ij}\nabla_l H - \nabla_l h_{ij} H|^2 d]mu \\
&\quad + \int_{\mathcal{M}_t} \left((2-\sigma)p\frac{g_+^p}{H^{1+\sigma}} + 2p(1+\eta)\frac{g_+^{p-1}}{H} \right) \langle \nabla_i H, \nabla_i g\rangle d\mu \\
&\quad - 2\left(\int_{\mathcal{M}_t} \frac{g_+^{p+1}}{H^{2-\sigma}} + (2+\eta)\frac{g_+^p}{H^{2-\sigma}} + (2+\eta)\frac{g_+^p}{H^2} \right) |\nabla H|^2 d\mu
\end{aligned}
$$

Using Cauchy-Schwartz and the fact that $g \leq C_0 H^\sigma$ we obtain that for any $\beta > 0$,

$$
\begin{aligned}
&\int_{\mathcal{M}_t} \frac{-2g_+^p}{H^2}(H(g^{ij}g^{kl}g^{mn}h_{ik}h_{lm}h_{nj}) - |A|^4) d\mu \\
&\quad \leq C_0\left(p + \frac{2(C_0+1)}{C_0\beta} \right) \int_{\mathcal{M}_t} g_+^{p-2}|\nabla g|^2 d\mu \\
&\quad + \left(4C_0^2 + 2pC_0(c_1+1)\beta \right) \int_{\mathcal{M}_t} \frac{g_+^{p-1}}{H^{2-\sigma}}|\nabla H|^2 d\mu \\
&\quad + 2C_0\int_{\mathcal{M}_t} \frac{g_+^{p-1}}{H^{4-\sigma}}\Big| h_{ij}\nabla_l H - \nabla_l h_{ij} H \Big|^2 d\mu.
\end{aligned}
$$

Where we have also looked at the term $\langle \nabla_i H, \nabla_i g\rangle$, and applied $ab \leq \frac{a^2+b^2}{2}$ with

$$
a = \Big|\nabla H\Big| \frac{g^{\frac{p-1}{2}}}{H^{1-\sigma/2}} \text{ and } b = \Big|\nabla g\Big| g^{\frac{p-2}{2}}.
$$

Then by Lemma 1.7 we obtain the Poincaré type inequality. $\square$

The combination of Lemma 1.74 and Lemma 1.75 gives the following L^p-estimate for the function g.

Proposition 1.76 (Proposition 3.6 [65])**.** *Given any $\eta \in (0,1)$ there are constants C_4, C_5 depending only on η, C_0 such that for $p \geq C_4$, $0 \leq \sigma \leq (C_5 p)^{-\frac{1}{2}}$*

$$
\frac{\partial}{\partial t}\int_{\mathcal{M}_t} (g_{\sigma,\eta})_+^p d\mu \leq \ 0 \quad \text{for } 0 \leq t < T.
$$

Proof. Choose $\beta \sim p^{-\frac{1}{2}}$ and $\sigma \sim Cp^{-\frac{1}{2}}$. $\square$

Now we can use a De Giorgi type iteration argument to derive an upper bound for $g_{\sigma,\eta}$ and prove Theorem 1.27.

Proof of Theorem 1.27. Denote $k_0 = \sup_{\sigma\in[0,1]} \sup_{\mathcal{M}_t}(g_{\sigma,\eta})_+$ for any $k \geq k_0$ we set

$$v = (g_{\sigma,\eta} - k)_+^{\frac{p}{2}} \quad \text{and} \quad A(k,t) = \{x \in \mathcal{M}_t | v(x, t \geq) 0\}.$$

By the same proof as in Lemma 1.74 we obtain for p large enough

$$\begin{aligned}
&\frac{\partial}{\partial t}\int_{\mathcal{M}_t} v^2 d\mu + \int_{\mathcal{M}_t} |\nabla v|^2 \\
&\leq \sigma p \int_{\mathcal{M}_t} |A|^2 v^2 d\mu \\
&\leq C_0 \sigma p \int_{\mathcal{M}_t} H^2 g_{\sigma,\eta}^p d\mu,
\end{aligned}$$

We also have the following Sobolev inequality due to Michael and Simon [81],

$$\begin{aligned}
\left(\int_{\mathcal{M}_t} v^{2q} d\mu\right)^{\frac{1}{q}} &\leq C(n)\int_{\mathcal{M}_t} |\nabla v|^2 d\mu \\
&\quad + C(n)\left(\int_{A(k,t)} H^n d\mu\right)^{\frac{2}{n}} \left(\int_{\mathcal{M}_t} v^{2q} d\mu\right)^{\frac{1}{q}}.
\end{aligned}$$

where $q = \frac{n}{n-2}$ if $n > 2$ and an arbitrary number greater than 1 if $n = 2$.

By Proposition 1.76 and denoting $\sigma' = \sigma + \frac{n}{p}$ we have

$$\begin{aligned}
\left(\int_{A(k,t)} H^n d\mu\right)^{\frac{2}{n}} &\leq k^{\frac{-2p}{n}} \left(\int_{A(k,t)} H^n g_{\sigma',\eta}^p d\mu\right)^{\frac{2}{n}} \\
&= k^{\frac{-2p}{n}} \left(\int_{A(k,t)} g_{\sigma',\eta}^p d\mu\right)^{\frac{2}{n}} \\
&\leq k^{\frac{-2p}{n}} \left(\int_{\mathcal{M}_t} (g_{\sigma',\eta})_+^p d\mu\right)^{\frac{2}{n}} \\
&\leq \left(\frac{k_0(1 + |\mathcal{M}_0|)}{k}\right)^{\frac{2p}{n}}.
\end{aligned}$$

Thus we fix $k_1 > k_0$ large enough such that for any $k > k_1$,

$$\frac{\partial}{\partial t}\int_{\mathcal{M}_t} v^2 d\mu + \frac{1}{C(n)}\left(\int_{\mathcal{M}_t} v^{2q} d\mu\right)^{\frac{1}{q}} \leq 2C_0\sigma p \int_{A(k,t)} H^2 g_{\sigma,\eta}^p d\mu$$

.

This is

$$\sup_{[0,\infty]} \int_{\mathcal{M}_t} v^2 d\mu + \frac{1}{C(n)}\int_0^T \left(\int_{\mathcal{M}_t} v^{2q} d\mu\right)^{\frac{1}{q}} dt \leq 2C_0\sigma p \int_0^T \int_{\mathcal{M}_t} H^2 g_{\sigma,\eta} d\mu\, dt.$$

Now we use the interpolation inequalities for L^p-spaces to obtain that,

$$\left(\int_{\mathcal{M}_t} v^{2q_0} d\mu\right)^{\frac{1}{q_0}} \leq \left(\int_{\mathcal{M}_t} v^{2q} d\mu\right)^{\frac{2}{q}} \left(\int_{\mathcal{M}_t} v^{2q} d\mu\right)^{1-\alpha},$$

where $1 < q_0 < q$, $\frac{1}{q_0} = \frac{\alpha}{q} + \frac{1-\alpha}{1}$ for $\alpha = \frac{1}{q_0} = \frac{1}{2-\frac{1}{q}}$ and $q > 2$.

Thus

$$\begin{aligned}
\left(\int_0^T \int_{\mathcal{M}_t} v^{2q_0} d\mu\right)^{\frac{1}{q_0}} &\leq \left(\int_0^T \left(\int_{\mathcal{M}_t} v^{2q} d\mu\right)^{\alpha \frac{q_0}{q}} \left(\int_{\mathcal{M}_t} v^{2} d\mu\right)^{(1-\alpha)q_0} dt\right)^{\frac{1}{q_0}} \\
&\leq \left(\sup_{[0,T]} \int_{\mathcal{M}_t} v^2 d\mu\right)^{1-\frac{1}{q_0}} \left(\int_0^T \left(\int_{\mathcal{M}_t} v^{2q} d\mu\right)^{\frac{1}{q}}\right)^{\frac{1}{q_0}} \\
&\leq 2C_0\sigma p \int_0^T \int_{\mathcal{M}_t} H^2 g_{\sigma,\eta}^p \, d\mu \, dt \\
&\leq 2C_0\sigma p \Big|\Big|A(k,t)\Big|\Big|^{1-\frac{1}{r}} \left(\int_0^T \int_{\mathcal{M}_t} H^{2r} g_{\sigma,\eta}^{pr} \, d\mu \, dt\right)^{\frac{1}{r}}
\end{aligned}$$

where for the last line we applied Hölder's inequality with r a positive number to be chosen and $||A(k,t)|| = \int_0^T \int_{\mathcal{M}_t} d\mu \, dt$.

This implies that

$$\begin{aligned}
\int_0^T \int_{\mathcal{M}_t} v^2 d\mu \, dt &\leq \Big|\Big|A(k,t)\Big|\Big|^{1-\frac{1}{q_0}} \left(\int_0^T \int_{A(k,t)} v^{2q_0} d\mu \, dt\right)^{\frac{1}{q_0}} \\
&\leq 2C_0\sigma p \Big|\Big|A(k,t)\Big|\Big|^{2-\frac{1}{q_0}-\frac{1}{r}} \left(\int_0^T \int_{A(k,t)} H^{2r} g_{\sigma,\eta}^{pr} d\mu \, dt\right)^{\frac{1}{r}}.
\end{aligned}$$

Let $1 < \gamma = 2 - \frac{1}{q_0} - \frac{1}{r}$ with r large enough and p, σ are suitably large.

For all $h > k \geq k_1$ we have

$$\begin{aligned}
|h-k|^p \cdot \Big|\Big|A(h,t)\Big|\Big| &\leq \int_0^T \int_{\mathcal{M}_t} v^2 d\mu \, dt \\
&\leq 2C_0\sigma p \left(\int_0^T \int_{A(k,t)} H^{2r} g_{\sigma,\eta}^{pr} d\mu \, dt\right)^{\frac{1}{r}} \Big|\Big|A(k,t)\Big|\Big|^{\gamma} \\
&= 2C_0\sigma p \left(\int_0^T \int_{A(k,t)} g_{\sigma''\eta}^{p} r d\mu \, dt\right)^{\frac{1}{r}} \Big|\Big|A(k,t)\Big|\Big|^{\gamma} \quad \left(\sigma'' = \sigma + \frac{2}{p}\right) \\
&\leq C(\sigma, p, T, |\mathcal{M}_0|, k_0) \Big|\Big|A(k,t)\Big|\Big|^{\gamma}.
\end{aligned}$$

Thus by the De Giorgi Iteration Lemma we conclude

$$\Big|\Big|A(k,t)\Big|\Big| = 0, \quad \forall\, k > k_1 + d,$$

where $d = C(\sigma, p, t, |\mathcal{M}_0|, k_0)^p 2^{\frac{\gamma}{\gamma-1}} \left|\left|A(k,t)\right|\right|^{\frac{\gamma-1}{p}}$. Therefore $g_{\sigma,\eta} \leq k_1 + d$ on $\mathcal{M}_t$ for $0 \leq t < T$. This proves the $k = 2$ case. $\square$

1.5 Ancient Solutions for Convex Hypersurfaces Undergoing Mean Curvature Flow

In this section we summarise the results and proofs from Huisken and Sinestrari's paper on ancient solutions [68]. In it they give various conditions ensuring that a closed convex ancient solution is a shrinking sphere. Ancient solutions arise in the study of singularities and of high curvature regions; an ancient solution to mean curvature flow is a solution defined for $t \in (-\infty, 0)$. Eternal solutions are solutions defined for all $t \in (-\infty, +\infty)$.

The simplest possible convex ancient solution for mean curvature flow is the shrinking sphere. Take $\mathcal{M}_t$, $t < 0$ to be the sphere of radius $\sqrt{-2nt}$, then $\mathcal{M}_t < 0$ is the only compact, convex self-similar solution to the mean curvature flow.

We also mention Haslhofer and Hershkovits paper, [53] in which they show the existence of ancient solutions that for $t \to 0$ converge to a round point but for $t \to -\infty$ have the following structure

(i) Near the centre they have asymptotic shrinkers modelled on the round cylinder $S^j \times \mathbb{R}^{n-j}$ and;

(ii) near the tips have asymptotic translators modelled on $\mathrm{Bowl}^{j+1} \times \mathbb{R}^{n-j-1}$.

this result applies for hypersurfaces $\mathcal{M}$ that are uniformly $n - j + 1$- convex, $\lambda_1 + \cdots + \lambda_{n-j+1} \geq \alpha_0 H$ for some $\alpha_0 > 0$ and not strictly convex hypersurfaces which we consider in this section.

Another interesting result is that of Angenent, Daskalopoulos and Sesum who show that compact convex ancient solution to (1.1) as has unique asymptotic as $t \to -\infty$. It is hoped that this result will result in a uniqueness result for ancient ovals with $O(k) \times O(l)$ symmetry, where $k + l = n + 1$, unique up to time translation and parabolic rescaling of spacetime.

An ancient oval is any ancient, compact, non-collapsed solution to mean curvature flow that is not self similar. For example, the Haslhofer and Hershkovits paper mentioned above prove the existence of an ancient solution that has $O(k) \times O(l)$ symmetry [53].

1.5.1 Main Result and Preliminaries

We begin by stating some preliminaries and definitions which will be required to prove the main theorem.

Definition 1.77. *If $\Omega \subset \mathbb{R}^{n+1}$ is a compact set with non-empty interior, the inner radius and out radius of Ω are defined as*

$$\rho_-(\Omega) = \max\{r > 0 |\ \forall B_r \ s.t.\ B_r \subset \Omega\}$$
$$\rho_+(\Omega) = \min\{R > 0 |\ \forall B_R \ s.t.\ \Omega \subset \bar{B}_R\}.$$

Definition 1.78. *The* width *of Ω in the direction of a unit vector $\nu \in \mathbb{R}^{n+1}$ is given by*

$$w(\nu, \Omega) = \max\{\langle y - x, \nu\rangle | x, y \in \Omega\}.$$

We can then define

$$w_-(\Omega) = \min_{|\nu|=1} w(\nu, \Omega); \quad w_+(\Omega) = \max_{|\nu|=1} w(\nu, \Omega).$$

It is easy to see that

$$w_+(\Omega) = \mathrm{diam}(\Omega). \tag{1.79}$$

We also have the following inequalities, where $\Omega \subset \mathbb{R}^{n+1}$,

$$\rho_+(\Omega) \leq \frac{w_+(\Omega)}{\sqrt{2}}, \quad \rho_-(\Omega) \geq \frac{w_-(\Omega)}{n+2} \tag{1.80}$$

We also denote by $\mathrm{diam}_I(\mathcal{M})$ the intrinsic diameter of $\mathcal{M}$, computed using the Riemannian distance on $\mathcal{M}$ induced by the immersion, in contrast to the extrinsic diameter $\mathrm{diam}(\mathcal{M})$, which is defined in terms of the distance in $\mathbb{R}^{n+1}$. These definitions lead us to the following if $\mathcal{M}$ is convex

$$\sqrt{2}\rho_+(\mathcal{M}) \leq \mathrm{diam}(\mathcal{M}) \leq \mathrm{diam}_I(\mathcal{M}) \leq \pi\rho_+(\mathcal{M}). \tag{1.81}$$

Throughout this section we will assume that the surface $\mathcal{M}$ has $n \geq 2$ and defined for $t \in (-\infty, 0)$. Where 0 is assumed to be the singular time of the flow, and the surfaces $\mathcal{M}$ shrink to a point as $t \to 0$ as seen in Section 1.1 .

The strong maximum principle for tensors applied to Lemma 1.18(vi) implies that all principal curvatures are strictly positive everywhere. Moreover if we consider Lemma 1.18(v) together with the inequalities

$$\frac{H^2}{n} \leq |A|^2 \leq H^2, \tag{1.82}$$

we obtain,

$$H_{\min} \leq \frac{\sqrt{n}}{\sqrt{-2t}}, \quad H_{\max} \geq \frac{1}{\sqrt{-2t}} \; \forall t \in (-\infty, 0). \tag{1.83}$$

Comparison with evolving spheres, along with the property that $\rho_-(\mathcal{M}_t) \to 0$ as $t \to 0$, yields the following bounds for $\mathcal{M}_t$:

$$\rho_-(\mathcal{M}_t) \leq \sqrt{-2nt} \leq \rho_+(\mathcal{M}_t), \; \forall\, t \in (-\infty, 0). \tag{1.84}$$

Lastly we will require Hamilton's Harnack estimate in the appropriate form for ancient solutions. We obtain this estimate by replacing t with $t - t_0$ in the original estimate found in [50] and taking the limit as $t \to -\infty$,

$$\frac{\partial H}{\partial t} - \frac{|\nabla H|^2}{H} \geq 0. \tag{1.85}$$

The above tells us that H is pointwise non-decreasing. This tells us that solutions have a uniformly bounded curvature on any time interval of the form $(-\infty, T_1]$, with $T_1 < 0$. Since H is the speed of our evolving solutions, we deduce for each solution the existence of a constant $K > 0$, s.t.

$$\rho_+(\mathcal{M}_t) \leq K(1 + |t|), \; \forall t < 0. \tag{1.86}$$

We can now state the main result for this section.

Theorem 1.87 (Theorem 1.1 [68]). *Let $\mathcal{M} \subset \mathbb{R}^{n+1}$ be a smooth, closed embedded n-dimensional hypersurface, we consider the compact set Ω such that $\mathcal{M} = \partial\Omega$ $\mathcal{M}_t$. Moreover $\mathcal{M}_t$ is a closed convex ancient solution of mean curvature flow. Then the following properties are equivalent:*

(i) $\mathcal{M}_t$ *is a family of shrinking spheres.*

(ii) The second fundamental form of $\mathcal{M}_t$ *satisfies the pinching condition* $h_{ij} \geq \epsilon H g_{ij}$ *for some* $\epsilon > 0$.

(iii) The diameter of $\mathcal{M}_t$ *satisfies* $diam(\mathcal{M}_t) \leq C_1(1+\sqrt{-t})$ *for some* $C_1 > 0$.

(iv) The outer and inner radius of $\mathcal{M}_t$ *satisfy* $\rho_+(t) \leq C_2\rho_-(t)$ *for some* $C_2 > 0$.

(v) $\mathcal{M}_t$ *satisfies* $H_{\max} \leq C_3 H_{\min}$ *for some* $C_3 > 0$.

(vi) $\mathcal{M}_t$ *satisfies the reverse isoperimetric inequality* $|\mathcal{M}_t|^{n+1} \leq C_4|\Omega_t|^n$ *for some* $C_4 > 0$.

(vii) $\mathcal{M}_t$ *is of type I, that is,* $\limsup_{t\to-\infty} \sqrt{-t}H_{\max} < \infty$.

1.5.2 Pinched Solutions

When looking at ancient solutions to convex mean curvature flow we will be considering ancient solutions that satisfy a pinching condition

$$h_{ij} \geq \epsilon H g_{ij} \tag{1.88}$$

for some $\epsilon > 0$ independent of t. In this section we will show that a solution that satisfies this property must necessarily be a family of shrinking spheres.

We consider the same function as in Section 2 of this chapter, which is only zero at umbilical points.

$$g_{\sigma,\eta} = \frac{|A|^2 - (1+\eta)H^2}{H^{2-\sigma}}, \tag{1.89}$$

with $\sigma, \eta > 0$. The result we desire will follow as a simple corollary from the following integral estimate which depends only on the pinching estimate and lifespan of the solution.

Theorem 1.90 (Theorem 3.1 [68]). *Let* $\mathcal{M}_t$*, with* $t \in [T_0, 0)$ *be a solution to the mean curvature flow such that* $\mathcal{M}_{T_0}$ *is closed, convex and satisfies 1.88 for some* $\epsilon > 0$*, and which becomes singular as* $t \to 0^-$*. Then there exist* $c_1, c_2, c_3 > 0$ *depending only on* n, ϵ *such that, for every* $p, \sigma > 0$ *satisfying*

$$p \geq c_1, \quad \sigma \leq \frac{c_2}{\sqrt{p}}, \quad p\sigma > n,$$

we have

$$\left(\int_{\mathcal{M}_t} g_\sigma^p dt\right)^{\frac{2}{\sigma p}} \leq \frac{c_3}{|T_0|^{1-\frac{n}{\sigma p}} - |t|^{1-\frac{n}{\sigma p}}},$$

for all $t \in [T_0, 0)$.

Proof. We know from Chapter 2 that $\mathcal{M}_t$ is convex and satisfies the pinching condition for all times t, $0 \leq t < T$ that a solution exists. □

The following Corollary to the above will give the equivalence between (i) and (ii) in Theorem 1.87.

Corollary 1.91 (Theorem 3.2 [68]). *Let $\mathcal{M}_t$, with $t \in (-\infty, 0)$, be an ancient solutions to the mean curvature flow such that every $\mathcal{M}_t$ is closed, convex and satisfies equation 1.88 for some $\epsilon > 0$ independent of t. Then $\mathcal{M}_t$ is a family of shrinking spheres.*

Proof. Arguing as in Lemma 5.5 of [58], which for the readers reference is similar Lemma 1.76 of this thesis we can obtain

$$\frac{\partial}{\partial t}\int_{\mathcal{M}_t} g_\sigma^p d\mu \leq -p\sigma \int_{\mathcal{M}_t} H^2 g_\sigma^p d\mu$$

Using (1.82) we see that $0 \leq g < H^\sigma$ and therefore

$$\frac{\partial}{\partial t}\int_{\mathcal{M}_t} g_\sigma^p d\mu \leq -p\sigma \left(\int_{\mathcal{M}_t} g_\sigma^p d\mu\right)^{1+\frac{2}{\sigma p}} |\mathcal{M}_t|^{-\frac{2}{p\sigma}} \tag{1.92}$$

using Hölder's inequality. Gauss-Bonnet tells us that,

$$\int_{\mathcal{M}_t} H^n d\mu \leq \epsilon^{-n} \int_{\mathcal{M}_t} K d\mu = C$$

where C depends on ϵ and n. This in conjunction with the evolution equation for $d\mu$, Lemma 1.18 (ii), allows us to bound the $|\mathcal{M}_t|$ term in equation (1.92).

$$\frac{\partial}{\partial t}|\mathcal{M}_t| = -\int_{\mathcal{M}_t} H^2 d\mu \geq -\left(\int_{\mathcal{M}_t} H^n d\mu\right)^{\frac{2}{n}} |\mathcal{M}_t|^{1-\frac{2}{n}} \geq -C|\mathcal{M}_t|^{1-\frac{2}{n}} \tag{1.93}$$

where C depends on ϵ and n.

Integrating the inequality over $[t, s]$ with $T_0 \leq t < s < 0$ we obtain,

$$|\mathcal{M}_s|^{\frac{2}{n}} - |\mathcal{M}_t|^{\frac{2}{n}} \geq -c(s-t).$$

Recall Theorem 1.13 from Section 1.1 tells us that $|\mathcal{M}_s| \to 0$ as $s \to 0$, this yields,

$$|\mathcal{M}_t| \leq C(-t)^{\frac{n}{2}} \tag{1.94}$$

as $s \to 0$.

Therefore (1.93) and (1.94) tell us that

$$\frac{\partial}{\partial t}\left(\int_{\mathcal{M}_t} g_\sigma^p d\mu\right)^{-\frac{2}{\sigma p}} \geq C(-t)^{-\frac{n}{\sigma p}}. \tag{1.95}$$

for any t so that $\int_{\mathcal{M}_t} g_\sigma^p d\mu > 0$. If $\int_{\mathcal{M}_t} g_\sigma^p d\mu > 0$ then $\int_{\mathcal{M}_s} g_\sigma^p d\mu > 0$ for all $s < t$. Therefore if we take $t \in [T_0, 0)$ such that $\int_{\mathcal{M}_t} g_\sigma^p d\mu > 0$ then we are able to integrate (1.95) over the interval $[T_0, t]$ to obtain,

$$\begin{aligned}\left(\int_{\mathcal{M}_t} g_\sigma^p d\mu\right)^{\frac{2}{\sigma p}} &\geq \left(\int_{\mathcal{M}_{T_0}} g_\sigma^p d\mu > 0\right) + C\int_{|t|}^{|T_0|} \tau^{-\frac{n}{\sigma p}} d\tau \\ &> C\int_{|t|}^{|T_0|} \tau^{-\frac{-n}{\sigma p}} \\ &> C(|T_0|^{1-\frac{n}{\sigma p}} - |t|^{1-\frac{n}{\sigma p}})\end{aligned}$$

where $\sigma p > n$. This proves the case when $\int_{\mathcal{M}_t} g_\sigma^p d\mu > 0$. The case $\int_{\mathcal{M}_t} g_\sigma^p d\mu = 0$ is trivial, as if $\int_{\mathcal{M}_t} g_\sigma^p d\mu = 0$ then $\mathcal{M}_t$ must be a sphere. □

Theorem 1.96. *Let $\mathcal{M}_t$ with $t \in (-\infty, 0)$ be an ancient solution to mean curvature flow such that every $\mathcal{M}_t$ is closed, convex and satisfies (1.88) for some $\epsilon > 0$. Then $\mathcal{M}_t$ is a family of shrinking spheres.*

Proof. Let $T_0 \to \infty$ in the previous theorem. Then $\int_{\mathcal{M}_t} g_\sigma^p d\mu$ is zero for every $t < 0$ having chosen the appropriate σ and p. This implies that every $\mathcal{M}_t$ is a sphere, as the only closed convex surfaces which contain only umbilical points are spheres. □

This proves the equivalence between (i) and (ii) in Theorem 1.87.

1.5.3 Solutions with a Diameter Bound

Huisken and Sinestrari first show that a growth bound of the order $O(\sqrt{t})$ on the diameter of the solution gives us control over the variation of the curvature at any fixed time.

Lemma 1.97 (Lemma 4.1 [68])**.** *Let $\mathcal{M}_t$ with $t \in (-\infty, 0)$ be a closed, convex ancient solution of the mean curvature flow. Then the following are equivalent:*

(i) There exists a constant $C_1 > 0$ such that

$$diam(\mathcal{M}_t) \leq C_1(1 + \sqrt{-t}) \tag{1.98}$$

or all $t < 0$.

(ii) There exist constants $C', C'' > 0$ such that

$$\frac{C'}{\sqrt{-t}} \leq H \leq \frac{C''}{\sqrt{-t}} \tag{1.99}$$

on $\mathcal{M}_t$ for all $t < 0$.

Lemma 1.100. *Integrating 1.85 we obtain the following classical type Harnack inequality:*

$$H(p_1, t_1) \leq H(p_2, t_2) \exp\left(\frac{diam_I^2(\mathcal{M}_{t_1})}{4(t_2 - t_1)}\right) \tag{1.101}$$

for any p_1, p_2 and $t_1 < t_2 < 0$.

A derivation of this result can be found in the appendix 5.15.

First we show that (i) implies (ii). Observe that property (1.81) tells us that the intrinsic diameter of our surfaces satisfies

$$\begin{aligned}\operatorname{diam}_I(\mathcal{M}_t) &\leq \pi\rho_+(\mathcal{M}_t) \\ &\leq \frac{\pi\omega_+(\Omega)}{\sqrt{2}} \\ &\leq \frac{\pi \operatorname{diam}(\mathcal{M}_t)}{\sqrt{2}} \\ &= C_1\frac{\pi}{\sqrt{2}}(1+\sqrt{-t}) \end{aligned} \tag{1.102}$$

for all $t < 0$. Moreover we also have

$$\operatorname{diam}_I(\mathcal{M}_t) \leq c\sqrt{-t} \tag{1.103}$$

for all $t < 0$ and a suitable constant $c > 0$. In fact, for t close to zero this follows from the convergence of $\mathcal{M}_t$ to a round point, whilst away from zero it follows from (1.102).

Now for any $t < 0$ we apply Lemma 1.101 together with equation (1.103), with $t_1 = t$ and $t_2 = \frac{t}{2}$ to obtain

$$H_{\max}(t) \leq e^{\frac{c^2}{2}} H_{\min}\left(\frac{t}{2}\right).$$

Using $H_{\min} \leq \frac{\sqrt{n}}{\sqrt{-2t}}$ we obtain

$$H_{\max} \leq e^{\frac{c^2}{2}}\sqrt{\frac{n}{-t}}.$$

Now we can apply $H_{\max} \geq \frac{1}{\sqrt{-2t}}$ and replacing t by $2t$ we obtain

$$\begin{aligned}H_{\min}(t) &\geq e^{\frac{-c^2}{2}} H_{\max}(2t) \\ &\geq e^{\frac{-c^2}{2}}\frac{1}{2\sqrt{-t}}\end{aligned}$$

for all $t < 0$.

The above inequalities imply (1.99).

Suppose now instead that (1.99) holds. Since $\mathcal{M}_t$ shrinks to a point as $t \to 0$, we find a pair of points $p, q \in \mathcal{M}_t$ such that

$$\begin{aligned}|F(p,t) - F(q,t)| &\leq \int_t^0 H(p,\tau)d\tau + \int_t^0 H(q,\tau)d\tau \\ &\leq 2C'' \int_t^0 \frac{d\tau}{\sqrt{-\tau}} \\ &= 4C''\sqrt{-t}\end{aligned}$$

which implies (1.98).

Before we move onto the next part we state the Cheeger-Gromov convergence theorem and definition [51]. This is a notion of convergence for Riemannian manifolds stronger than Gromov-Hausdorff convergence. It will be used to prove Theorem 1.106.

Definition 1.104. *A sequence $(\mathcal{M}_i, g_i)$ of closed Riemannian manifolds* Cheeger-Gromov converges *to a closed Riemannian manifold $(\mathcal{M}, g)$ with regularity $C^{k,\alpha}$ if there exists a sequence of diffeomorphisms $\phi_i : \mathcal{M} \to \mathcal{M}_i$ such that $\phi_i^* g_i \to g$ in $C^{k,\alpha}$.*

Theorem 1.105 (Cheeger-Gromov Compactness Theorem)**.** *If $(\mathcal{M}_i, g_i)$ is a sequence of closed Riemannian manifolds with uniform bounds*

$$|sec(\mathcal{M}_i)| \leq K \ Vol(\mathcal{M}, g_i) \leq V \text{ and } diam(\mathcal{M}, g_i) \leq D$$

for constants K, V and $D > 0$. Then there exists a subsequence $(\mathcal{M}_{i_k}, g_{i_k})$ that Cheeger-Gromov converges to a closed Riemannian manifold $(\mathcal{M}, g)$.

Theorem 1.106 (Theorem 4.2 [68])**.** *Let $\mathcal{M}_t$, with $t \in (-\infty, 0)$ be a closed convex ancient solution of the mean curvature flow satisfying (1.98) or (1.99). Then $\mathcal{M}_t$ is a family of shrinking spheres.*

Proof. Argue by contradiction that (1.88) does not hold.. In general for a contradiction, we take a sequence, rescale the singularity and look for what goes wrong.

Since (1.88) does not hold, then there exists a sequence of points and times $\{(p_k, t_k)\}$, with $t_k \to -\infty$ and

$$\left(\frac{\lambda_1(p_k, t_k)}{H(p_k, t_k)} \right) \to 0 \ \text{ as } k \to \infty.$$

Considering the flow $\mathcal{M}_t$ for $t \in [2t_k, t_k]$ and rescaling it by a factor of $\frac{1}{\sqrt{|t_k|}}$ in space and $\frac{1}{|t_k|}$ in time, we obtain a sequence of flows defined for $t \in [-2, -1]$. The previous Lemma guarantees curvature and diameter bounds from above and below, whilst results from [29] guarantee bounds on all derivatives of the curvature for $t \in [-\frac{3}{2}, -1]$. This limit solution is convex and compact but contains a point $\lambda_1 = 0$ at $t = -1$. Using Lemma 5.8, we see that the limit solution must split containing a flat factor (i.e. an infinite cylinder), contradicting the diameter bound. Therefore the original solution $\mathcal{M}_t$ must satisfy (1.88) and we obtain the result from Theorem 1.96. □

Corollary 1.107 (Corollary 4.3 [68])**.** *If our ancient solution $\mathcal{M}_t$ satisfies either of the two properties*

$$\rho_+(\mathcal{M}_t) \leq C \rho_-(\mathcal{M}_t) \ \forall \ t < 0$$
$$H_{\max} \leq C H_{\min} \ \forall \ t < 0$$

for a constant $C > 0$, then $\mathcal{M}_t$ is a family of shrinking spheres.

Proof. Starting with (1.81) we see that

$$\begin{aligned} \operatorname{diam}(\mathcal{M}_t) &\leq \pi \rho_+(\mathcal{M}_t) \\ &\leq C\pi \rho_-(\mathcal{M}_t) \quad \text{(using our current hypothesis)} \\ &\leq C\pi(\sqrt{-2nt}) \quad \text{by (1.84)} \end{aligned}$$

this gives us our diameter bound as in (1.98).

Whilst our second assumption on the mean curvature, implies that (1.99) holds. Therefore for both cases we obtain our conclusion using the previous theorem. □

Recall that $\mathcal{M}_t$ satisfies a uniform reverse isoperimetric estimate if there exists a constant $C > 0$ such that

$$|\mathcal{M}_t|^{n+1} \leq C|\Omega_t|^n \tag{1.108}$$

for all $t < 0$. Where $|\mathcal{M}_t|$ and $|\Omega_t|$ denote the n and $n+1$-dimensional measure of $\mathcal{M}_t$ and Ω_t respectively. This constant C will be greater than the optimal constant in the isoperimetric inequality achieved by the sphere. The following lemma will show that such an assumption implies a uniform bound on the ratio between the outer and inner radius.

Lemma 1.109 (Lemma 4.4 [68])**.** *Let $\mathcal{M}_t \subset \mathbb{R}^{n+1}$ be a closed convex n-dimensional hypersurface. Then for any $n \geq 1$ and $C_1 > 0$ there exists a constant $C_2 > 1$ depending on C_1 and n such that*

$$|\mathcal{M}|^n \leq C_1|\Omega|^n$$

where Ω is the region enclosed by $\mathcal{M}$. Then the outer and inner radius ρ_+ and ρ_- satisfy the following inequality

$$\frac{\rho_+}{\rho_-} \leq C_2.$$

Proof. Without loss of generality we can assume that the direction achieving minimal width $\omega_-(\mathcal{M})$ is in the x_{n+1} axis, and we denote by Σ the orthogonal projection of $\mathcal{M}$ onto the $\{x_{n+1} = 0\}$ hyperplane and $|\Sigma|$ its n-dimensional measure. Then we can estimate

$$\begin{aligned} |\Sigma| \leq \omega_-(\mathcal{M}) \text{ and } |\mathcal{M}| > |\Sigma| \\ \Rightarrow |\Sigma| \leq C_1\omega_-(\mathcal{M})^n. \end{aligned} \tag{1.110}$$

Moreover if p and q are any two points in $\mathcal{M}$ and p', q' are their projections onto the $\{x_{n+1} = 0\}$-hyperplane, then by Pythagoras' Theorem we know that

$$|p - q| \leq |p' - q'| + \omega_-(\mathcal{M}).$$

Moreover recalling that $\omega_+(\Omega) = \operatorname{diam}(\Omega)$ tells us that

$$\operatorname{diam}(\Sigma) \geq \operatorname{diam}(\mathcal{M}) - \omega_-(\mathcal{M}) = \omega_+ - \omega_-. \tag{1.111}$$

In the case $n = 1$, $\operatorname{diam}(\Sigma) = |\Sigma|$ and hence (1.110) and (1.111) give,

$$\omega_+(\mathcal{M}) \leq (C_1 + 1)\omega_-(\mathcal{M})$$

which by (1.80) obtains the result.

In the case that $n > 1$, we take an $(n + 1)$-dimensional ball B_0 of radius $\rho_-(\mathcal{M}) \subset \Omega$ and let $B_1 \subset \Sigma$ be its projection onto the $\{x_{n+1} = 0\}$-hyperplane with $p_1 \in \Sigma$ its centre. Then there exists some point $p_2 \in \Sigma$ such that $|p_2 - p_1| \geq \operatorname{diam}(\Sigma)/2$.

Intersecting B_1 through p_1 orthogonal to the direction $p_2 - p_1$ we obtain the $(n - 1)$-dimensional ball of radius $\rho_-(\mathcal{M})$

$$B_2 = \{p \in B_1 \mid \langle p - p_1, p_2 - p_1\rangle = 0\}.$$

Now $p_2 \in \Sigma$ and $B_2 \subset \Sigma$, moreover it is convex and contains the cone K with basis B_2, vertex p_2 and height $|p_2 - p_1| \geq \operatorname{diam}(\Sigma)/2$.

Therefore

$$|\Sigma| \geq |K| \geq \frac{\operatorname{diam}(\Sigma)\omega_{n-1}}{2n}\rho_-(\mathcal{M})^{n-1}$$

where ω_{n-1} is the volume of the unit $(n-1)$-dim ball. Applying (1.80) and (1.111) we can deduce

$$|\Sigma| \geq \frac{\omega_{n-1}\omega_-(\mathcal{M})^{n-1}(\omega_+(\mathcal{M}) - \omega_-(\mathcal{M})}{2n(n+2)^{n-1}} = \kappa_n\omega_-(\mathcal{M})^{n-1}\left(\omega_+(\mathcal{M}) - \omega_-(\mathcal{M})\right)$$

where κ_n is a constant which depends only on n. Applying (1.110) we obtain

$$C_1\omega_-(\mathcal{M}) \geq \kappa_n\Big(\omega_+(\mathcal{M}) - \omega_-(\mathcal{M})\Big)$$

which yields

$$\omega_+(\mathcal{M}) \leq \left(1 + \frac{C_1}{\kappa_n}\right)\omega_-(\mathcal{M}).$$

Applying (1.80) we obtain the desired result. □

Corollary 1.112 (Corollary 4.5 [68])**.** *Suppose that there exists a constant $C > 0$ such that the uniform reverse isoperimetric estimate (1.108) holds. Then $\mathcal{M}_t$ is a family of shrinking spheres.*

Proof. Combine Corollary (1.107) with Lemma (1.109). □

To conclude this Chapter we define a type I singularity for ancient solutions to mean curvature flow and show that closed convex solutions must be a family of shrinking spheres.

Definition 1.113. *An ancient solution to the mean curvature flow is of type I, if there exist constants $C > 0$ and $T_0 < 0$ such that*

$$H_{\max}H(\cdot,t) \leq \frac{C}{\sqrt{-t}} \tag{1.114}$$

for all $t \leq T_0$.

Proposition 1.115 (Proposition 4.6 [68])**.** *A closed convex ancient solution of the mean curvature flow of type I is a family of shrinking spheres.*

Proof. The results from Section 1.1 imply that (1.114) also holds for $t \in [T_0, 0)$. Arguing as in Lemma 1.100 we can obtain that a type I solution satisfies the estimates of Lemma 1.100 and so we can conclude using Theorem 1.106. □

1.6 Mean Curvature Flow for Convex Hypersurfaces in Riemannian Manifolds

1.6.1 Preliminaries

We will look at at summarise the results from Huisken's paper [59] in which he takes the next natural step and studies compact, convex, hypersurfaces $\mathcal{M}^n$, $n \geq 2$ without boundary, which are smoothly immersed in a Riemannian manifold $\mathcal{N}^{n+1}$. Let $\mathcal{M}^n = \mathcal{M}_0$ be given locally by some diffeomorphism

$$F_0 : U \subset \mathbb{R}^{n+1} \to F_0(U) \subset \mathcal{M}_0 \subset \mathcal{N}^{n+1}.$$

As before we move $\mathcal{M}_0$ by its mean curvature vector so that it satisfies equation 1.1.

However we will need to impose certain conditions on our ambient manifold $\mathcal{N}^{n+1}$, as it may interfere with the motion of our surfaces $\mathcal{M}_t$. By imposing these conditions we are able to prove a Theorem similar to Theorem 1.13.

Let $\overline{Rm} = \{\bar{R}_{\alpha\beta\gamma\delta}\}$ and $\overline{\nabla Rm} = \{\overline{\nabla}_\sigma \bar{R}_{\alpha\beta\gamma\delta}\}$ denote the curvature tensor of $\mathcal{N}$ and its covariant derivative. Moreover $\sigma_x(P)$ denotes the sectional curvature of a 2-plane P at $x \in \mathcal{N}$ and $i_x(\mathcal{N})$ denotes the injectivity radius of N at x.

Theorem 1.116 (Theorem 1.1 [59])**.** *Let $n \geq 2$ and $\mathcal{N}^{n+1}$ be a smooth complete Riemannian manifold without boundary which satisfies uniform bounds*

$$-K_1 \leq \sigma_x(P) \leq K_2 \quad K_1, K_2 \geq 0 \tag{1.117}$$

$$|\overline{\nabla}\, \overline{Rm}|^2 \leq L^2 \quad L \geq 0 \tag{1.118}$$

$$i_x(\mathcal{N}) \geq i(\mathcal{N}) > 0. \tag{1.119}$$

Let $\mathcal{M}_0$ be a compact connected hypersurface without boundary which is smoothly immersed in $\mathcal{N}$, and suppose that it satisfies the following pinching condition

$$Hh_{ij} > nK_1 g_{ij} + \frac{n^2}{H} L g_{ij}. \tag{1.120}$$

Then 1.1 has a smooth solution $\mathcal{M}_t$ on a finite time interval $0 \leq t < T$ and the $\mathcal{M}_t$'s converge uniformly to a single point $\mathcal{O}$ as $t \to T$.

Remark 1.121.

(i) (1.120) does not depend on K_2, so positive sectional curvature in the ambient Riemannian manifold works to contract under the flow, whereas negative sectional curvature will slow down the contraction. If $\mathcal{N}$ is locally symmetric ($\overline{\nabla}\, \overline{Rm} = 0$) then $L = 0$ and condition (1.120) is satisfied if the principal curvatures are bigger than $K_1^{\frac{1}{2}}$. If in addition the sectional curvature in the ambient manifold is nonnegative, then Theorem 1.116 is identical to Theorem 1.13.

(ii) Condition (1.120) implies that

$$H > nK_1^{\frac{1}{2}}. \tag{1.122}$$

We state without proof the following lemmas.

Lemma 1.123 (Lemma 2.1 [59]). *The following identities hold true.*

(i) $\Delta h_{ij} = \nabla_i \nabla_j H + H h_{il} H_j^l - |A|^2 h_{ij} + H \bar{R}_{0i0j} - h_{ij} \bar{R}_{0l0}{}^l + h_{jl} \bar{R}^l{}_{mi}{}^m$
$+ h_{il} \bar{R}^l{}_{mj}{}^m - 2 h_{lm} \bar{R}^l{}_i{}^m{}_j + \overline{\nabla}_J \bar{R}_{0li}{}^l + \overline{\nabla}_l \bar{R}_{0ij}{}^l.$

(ii) $\frac{1}{2}\Delta|A|^2 = \langle h_{ij}, \nabla_i \nabla_j H \rangle + |\nabla A|^2 + H(h_{ik} h_l^k h^{li}) - |A|^2 + H h^{ij} \bar{R}_{0i0j}$
$-|A|^2 \bar{R}_{0l0}{}^l + 2 h^{ij} h_{jl} \bar{R}^l{}_{mi}{}^m - 2 h^{ij} h^{lm} \bar{R}_{limj} + h^{ij} (\overline{\nabla}_j \bar{R}_{0li}{}^l + \overline{\nabla}_l \bar{R}_{0ij}{}^l).$

Proof. This is just an extension of Simon's identity as seen in Lemma 1.7. □

We will also need an extension of Lemma 1.17. We introduce the quantity $\{\omega_i\}$, the vector with components $\omega_i = \bar{R}_{0li}{}^l$, ω is the projection of $\overline{\mathrm{Ric}}(\nu, \cdot)$ on $\mathcal{M}$.

Lemma 1.124 (Lemma 2.2 [59]). *For any $\eta > 0$ we have the following inequalities*

(i) $|\nabla A|^2 \geq \left(\frac{3}{n+2} - \eta\right) |\nabla H|^2 - \frac{2}{n+2}\left(\frac{2}{n+2}\eta^{-1} - \frac{n}{n-1}\right)|\omega|^2.$

(ii) $|\nabla A|^2 - \frac{|\nabla H|^2}{n} \geq \frac{n-1}{2n+1}|\nabla A|^2 - \frac{2n}{(n-1)(2n+1)}|\omega|2$
$\geq \frac{n-1}{2n+1}|\nabla A|^2 - C(n, K_1, K_2).$

Proof. To prove (i) we decompose the tensor $\nabla A = \{\nabla_i h_{jk} = E_{ijk} + F_{ijk}\}$ where

$$E_{ijk} = \frac{1}{n+2}(\nabla_i H g_{jk} + \nabla_j H g_{ik} + \nabla_k H g_{ij}) - \frac{2}{(n+2)(n-1)}\omega_i g_{jk} + \frac{n}{(n+2)(n-1)}\Big(\omega_j g_{ik} + \omega_k g_{ij}\Big).$$

Then E_{ijk} has the same trace as $\nabla_i h_{jk}$ due to the Codazzi equations and $\langle E_{ijk}, F_{ijk}\rangle = 0$. Moreover the definition of E_{ijk} implies that

$$\begin{aligned} |E|^2 &= \frac{3}{n+2}|\nabla H|^2 + \frac{2n}{(n+2)(n-1)}|\omega|^2 - \frac{4}{n+2}\langle \omega_i, \nabla_i H\rangle \\ &\geq \left(\frac{3}{n+2} - \eta\right)|\nabla H|^2 + \frac{2}{n+2}\left(\frac{n}{n-1} - \frac{2}{n+2}\eta^{-1}\right)|\omega|^2 \end{aligned}$$

this proves part (i). Part (ii) follows from the first inequality with $\eta = \frac{2(n-1)}{n(n+2)}$. □

In a general Riemannian manifold $\mathcal{N}^{n+1}$ we take the indices α, ρ, σ to refer to a local coordinate system y^α. Then we can express the Gauss-Weingarten equations as follows

$$\begin{aligned} \frac{\partial^2 F^\alpha}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial F^\alpha}{\partial x_k} + \bar{\Gamma}^\alpha_{\rho\sigma} \frac{\partial F^\rho}{\partial x_i}\frac{\partial F^\sigma}{\partial x_j} &= -h_{ij} v^\alpha \\ \frac{\partial v^\alpha}{\partial x_j} + \bar{\Gamma}^\alpha_{\rho\sigma}\frac{\partial F^\rho}{\partial x_j} v^\sigma &= h_{jl} g^{lm} \frac{\partial F^\alpha}{\partial x_m}. \end{aligned} \tag{1.125}$$

and the evolution equation (1.1) becomes

$$\begin{aligned} \frac{\partial}{\partial t} F^\alpha(x,t) &= -H(x,t)\nu(x,t) \\ &= \Delta_t F^\alpha(x,t) + \left(\Gamma^\alpha_{\rho\sigma}\frac{\partial F^\rho}{\partial x_i}\frac{\partial F^\sigma}{\partial x_j} g^{ij}\right)(x,t). \end{aligned} \tag{1.126}$$

Lemma 1.127 (Lemma 3.1 [59]). *If the initial surface $\mathcal{M}_0$ is smooth, then (1.126) has a smooth solution on some maximal open time interval $0 \leq t < T \leq \infty$.*

Proof. Since this is a quasi linear parabolic system we can obtain a smooth solution on at least some short time interval. □

Since (1.126) is parabolic, we are able to obtain an avoidance principle for mean curvature flow which describes how two surfaces moving by their mean curvature will not overtake each other.

Lemma 1.128 (Lemma 3.2 [59]).

(i) Let $\mathcal{M}_{1,t}$ and $\mathcal{M}_{2,t}$ be two smooth closed surfaces moving by their mean curvature for $0 \leq t \leq t_1$. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are disjoint for $t = 0$, they remain disjoint on the whole interval $0 \leq t \leq t_1$.

(ii) If $\mathcal{M}_{1,t}$ is embedded for $t = 0$, then it remains so for $0 \leq t \leq t_1$.

Proof. We argue by contradiction and assume that the surfaces $\mathcal{M}_1$ and $\mathcal{M}_2$ are intersecting at some time t, $0 \leq t \leq t_1$. This implies that there exists a time t_0 at which the surfaces first touched at some point $p \in \mathcal{N}$. Let S be a fixed reference surface with the property that it is tangential to both $\mathcal{M}_{1,t_0}$ and $\mathcal{M}_{2,t_0}$ at p and assume that we have Gaussian coordinates in a neighbourhood of S, i.e. $y^0(q)$ is the length of the geodesic arc perpendicular to S through q and $y^i(q) = x_i(q)$ are the coordinates of the base point of the geodesic in S.

Then locally around p we can write $\mathcal{M}_{1,t}$ and $\mathcal{M}_{2,t}$ for $t \in (t_0 - \epsilon, t_0 + \epsilon)$ as graphs of functions $u_1(t)$ and $u_2(t)$ on S.

The unit normal to $\mathcal{M}_i$, $i = 1, 2$ is then given by

$$\nu_i = (1 + |\nabla_i u_i|^2)^{-\frac{1}{2}} \left(1, -\frac{\partial}{\partial x_1} u_i, \ldots, \frac{-\partial}{\partial x_n} u_i\right)$$

with u_i satisfying the evolution equation

$$\frac{\partial}{\partial t} u_i = -\left(1 + |\nabla u_i|^2\right)^{-\frac{1}{2}} H_i. \tag{1.129}$$

At the point (p, t_0) we have that $\nabla u_1 = \nabla u_2 = 0$, which makes (1.129) a uniformly parabolic equation in some small neighbourhood (p, t_0). Without loss of generality we assume that $u_1(t) > u_2(t)$ for $t < t_0$, however by applying the strong parabolic principle this leads us to a contradiction. Since u_1 is more convex and should be moving faster, this would contradict t_0 being the first time that they touched.

The proof for the second part of the lemma is similar. □

1.6.2 Evolution Equations

Now as in Section 1.1 we'll need to obtain evolution equations for mean curvature flow in this more complex setting. We assume that at (x_0, t_0) we have $g_{ij} = \delta_{ij}$ and that the normal coordinates y^α, $0 \leq \alpha \leq n$ for $\mathcal{N}$ are normal coordinates at $F(x_0, t_0)$. Moreover they are chosen such that $\nu^\alpha = -\delta_0^\alpha$ and $\frac{\partial F^\alpha}{\partial x_i} = \delta_i^\alpha$. They have been chosen in such a way so that all Christoffel symbols of the connection $\bar{\Gamma}$ will vanish at $F(x_0, t_0)$ and leave only the derivatives of the Christoffel symbols which will lead to curvature terms appearing along the way.

Lemma 1.130. *We have the following evolution equations,*

(i) $\frac{\partial}{\partial t} g_{ij} = -2Hh_{ij}$.

(ii) $\frac{\partial}{\partial t} \mu_t = -H^2 \mu_t$.

(iii) $\frac{\partial}{\partial t} \nu = \nabla H$.

(iv) $$\frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} - 2Hh_{il}h^l_j + |A|^2 h_{ij} + h_{ij}\bar{R}_{0l0}{}^l - h_{jl}\bar{R}^l{}_{mi}{}^m \\ - h_{il}\bar{R}^l{}_{mj}{}^m + 2h_{lm}\bar{R}^l{}_i{}^m{}_j - \overline{\nabla}_j \bar{R}_{0li}{}^l - \overline{\nabla}_l \bar{R}_{0ij}{}^l.$$

(v) $\frac{\partial}{\partial t} H = \Delta H + H(|A|^2 + \overline{Ric}(\nu, nu))$.

(vi) $$\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 + 2|A|^2\left(|\nabla A|^2 - \overline{Ric}(\nu,\nu)\right) \\ - 4(h^{ij}h^m_j \bar{R}_{mli}{}^l - h^{ij}h^{lm}\bar{R}_{milj}) \\ = 2h^{ij}(\overline{\nabla}_j \bar{R}_{0li}{}^l + \overline{\nabla}_l \bar{R}_{0ij}{}^l)$$

(vii) $$\frac{\partial}{\partial t}\left(|A|^2 - \frac{H^2}{n}\right) = \Delta\left(|A|^2 - \frac{1}{n}H^2\right) - 2\left(|\nabla A|^2 - \frac{|\nabla H|^2}{n}\right) \\ + 2\left(|A|^2 - \frac{H^2}{n}\right)\left(|A|^2 + \overline{Ric}(\nu, nu)\right) \\ - 2h^{ij}(\overline{\nabla}_j \bar{R}_{0li}{}^l + \overline{\nabla}_l \bar{R}_{0ij}{}^l) - 4(h^{ij}h^m_j \bar{R}_{mli}{}^l - h^{ij}h^{lm}\bar{R}_{milj}).$$

where $\overline{Ric}(\nu,\nu) = \bar{R}_{0l0}{}^l$.

Proof. (i) Similar to the Euclidean Case.

(ii) Follows from part (i).

(iii) Similar to the Euclidean Case.

(iv) From (1.126) and (1.125) we derive

$$\frac{\partial}{\partial t} h_{ij} = -\bar{g}\left\langle \frac{\partial}{\partial t}\left(\overline{\nabla}_{\frac{\partial F}{\partial x_i}} \frac{\partial F}{\partial x_i},\right), \nu\right\rangle - \bar{g}\left\langle \overline{\nabla}_{\frac{\partial F}{\partial x_i}} \frac{\partial F}{\partial x_i}, \frac{\partial \nu}{\partial t}\right\rangle.$$

Since we are using normal coordinates the last term vanishes, since $\frac{\partial \nu}{\partial t}$ is tangential and the spatial derivative is normal. So

$$\frac{\partial}{\partial t} h_{ij} = -\bar{g}\left\langle \frac{\partial}{\partial t}\left(\frac{\partial^2 F}{\partial x_i \partial x_j} + \bar{\Gamma}^\alpha_{\beta\gamma} \frac{\partial F^\beta}{\partial x_i}\frac{\partial F^\gamma}{\partial x_j}\frac{\partial}{\partial y^\alpha}\right), \nu\right\rangle.$$

Since $\bar{\Gamma}^\alpha_{\beta\gamma}$ terms vanish, we only need to look at the derivatives.

$$\frac{\partial}{\partial t} h_{ij} = -\bar{g}\left\langle \frac{\partial^2(-H\nu)}{\partial x_i \partial x_j} + \frac{\partial}{\partial t}\bar{\Gamma}^\alpha_{\beta\gamma} \frac{\partial F^\beta}{\partial x_i}\frac{\partial F^\gamma}{\partial x_j}\frac{\partial}{\partial y^\alpha}, \nu\right\rangle \\ = \bar{g}\left\langle \frac{\partial^2(-H\nu)}{\partial x_i \partial x_j} - H\nabla_\nu \bar{\Gamma}^\alpha_{\beta\gamma} \frac{\partial F^\beta}{\partial x_i}\frac{\partial F^\gamma}{\partial x_j}\frac{\partial}{\partial y^\alpha}, \nu\right\rangle.$$

At the point (x_0, t_0) the Weingarten equation gives us

$$\frac{\partial^2 \nu^\alpha}{\partial x_i} = h^j_i \frac{\partial F^\alpha}{\partial x_j} \quad \text{and}$$
$$\frac{\partial^2 \nu^\alpha}{\partial x_i \partial x_j} = \frac{\partial h^k_j}{\partial x_i}\frac{\partial F^\alpha}{\partial x_k} + h^k_j \frac{\partial^2 F^\alpha}{\partial x_i \partial x_k} - \frac{\partial \bar{\Gamma}^\alpha_{\beta\gamma}}{\partial x_i}\frac{\partial F^\beta}{\partial x_i}\nu^\alpha.$$

With $\langle \frac{\partial F}{\partial x_k}, \nu \rangle = 0$ and $\nu = \nu^\alpha \frac{\partial}{\partial y^\alpha}$. Therefore,

$$\bar{g}\left\langle \frac{\partial^2 \nu}{\partial x_i \partial x_j}, \nu \right\rangle = -h_j^k h_{ki} - \bar{g}\left\langle \overline{\nabla}_{\frac{\partial}{\partial x_i}} \bar{\Gamma}^\alpha_{\beta\gamma} \frac{\partial F^\beta}{\partial x_i} \nu^\alpha \frac{\partial}{\partial y^\alpha}, \nu \right\rangle.$$

Expanding yields,

$$\begin{aligned}\frac{\partial}{\partial t} h_{ij} &= \frac{\partial^2 H}{\partial x_i \partial x_j} - H h_{ik} h_j^k - H\bar{g}\left\langle \left(\overline{\nabla}_\nu \bar{\Gamma}^\alpha_{\beta\gamma} \frac{\partial F^\beta}{\partial x_i} \frac{\partial F^\gamma}{\partial x_j} - \overline{\nabla}_{\frac{\partial F}{\partial x_i}} \bar{\Gamma}^\alpha_{\beta\gamma} \frac{\partial F^\beta}{\partial x_j} v^\gamma \right) \frac{\partial}{\partial y^\alpha}, \nu \right\rangle \\ &= \bar{g}\left\langle \left(\frac{\bar{\partial}}{\partial y^\alpha} \bar{\Gamma}^\alpha_{\beta\gamma} - \overline{\nabla}_{\frac{\partial}{\partial y^\beta}} \bar{\Gamma}^\alpha_{\gamma\delta} \right) \frac{\partial}{\partial y^\alpha}, \nu \right\rangle \cdot \frac{\partial F^\beta}{\partial x_i} \frac{\partial F^\gamma}{\partial x_j} \nu^\delta.\end{aligned}$$

This matches the definition of the Riemann curvature tensor, $\bar{R}_{\alpha\beta\delta\gamma} \nu^\alpha \frac{\partial F^\beta}{\partial x_i} \frac{\partial F^\gamma}{\partial x_j} \nu^\delta = \bar{R}_{injn}$. Therefore

$$\frac{\partial}{\partial t} h_{ij} = \nabla_i \nabla_j H + H h_{il} h_j^l + H \bar{R}_{0i0j}.$$

and the result follow from Lemma 1.123.

(v)

$$\begin{aligned}\frac{\partial}{\partial t} H &= \left(\frac{\partial}{\partial t} g^{ij} \right) h_{ij} + g^{ij} \left(\frac{\partial}{\partial t} h_{ij} \right) \\ &= 2H g^{is} h_{sl} g^{lj} h_{ij} + g^{ij} \left(\nabla_i \nabla_j H - H h_{il} h_j^l + H \bar{R}_{0i0j} \right) \\ &= 2|A|^2 H + \Delta H - |A|^2 H + g^{ij} \left(H \bar{R}_{0i0j} \right) \\ &= 2|A|^2 H + \Delta H - |A|^2 H + \left(H \bar{R}_{0l0l} \right) \\ &= 2|A|^2 H + \Delta H - |A|^2 H + H \mathrm{Ric}(\nu, \nu).\end{aligned}$$

(vi)

$$\begin{aligned}\frac{\partial}{\partial t} |A|^2 =& \frac{\partial}{\partial t} \left(g^{ik} g^{jl} h_{ij} h_{kl} \right) \\ =& 4H g^{im} g^{kn} g^{jl} h_{mn} h_{ij} h_{kl} + 2 g^{ik} g^{jl} h_{kl} \left(\Delta h_{ij} - 2H h_{il} h_j^l \right) \\ &+ |A|^2 h_{ij} + h_{ij} \bar{R}_{0l0}{}^l - h_{jl} \bar{R}^m{}_{mi}{}^m - h_{il} \bar{R}^l{}_{mj}{}^m \\ &+ 2h_{lm} \bar{R}^l{}_i{}^m{}_j - \overline{\nabla}_j \bar{R}_{0li}{}^l - \overline{\nabla}_l \bar{R}_{0ij}{}^l \\ =& \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^2 \left(|A|^2 + \overline{\mathrm{Ric}}(\nu, \nu) \right) \\ &+ 2 g^{ik} g^{jl} h_{kl} \left(-h_{jl} \bar{R}^l{}_{mi}{}^m - h_{il} \bar{R}^l{}_{mj}{}^m + 2h_{lm} \bar{R}^l{}_i{}^m{}_j - \overline{\nabla}_j \bar{R}_{0li}{}^l - \overline{\nabla}_l \bar{R}_{0ijl} \right) \\ =& \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^2 \left(|A|^2 + \overline{\mathrm{Ric}}(\nu, \nu) \right) \\ &- 4(h^{ij} h_j^m \bar{R}_{mli}{}^l - h^{ij} h^{lm} \bar{R}_{milj}) \\ &- 2h^{ij} \left(\overline{\nabla}_j \bar{R}_{0li}{}^l + \overline{\nabla}_l \bar{R}_{0ij}{}^l \right).\end{aligned}$$

(vii) Combination of the above identities.

□

1.6.3 A Lower Bound for the Eigenvalues of A

Here we show that (1.120) and (1.122) are preserved under the flow. In view of (1.120) there exists some $\epsilon_1, \epsilon_2 > 0$ such that

$$H^2 \geq n^2 K_1 + n\epsilon_2 H^2 \tag{1.131}$$

$$Hh_{ij} \geq nK_1 g_{ij} + \frac{n^2}{H} L g_{ij} + \epsilon_1 (H^2 - n^2 L) g_{ij} \tag{1.132}$$

hold on $\mathcal{M}_0$. Since $|A|^2 \geq \frac{H^2}{n}$ and $\overline{\mathrm{Ric}}(\nu, nu) = \bar{R}_{0l0}{}^{l} \geq -nK_1$ we can obtain that

$$\begin{aligned}
\frac{\partial}{\partial t} H &= \Delta H + H|A|^2 + \overline{\mathrm{Ric}}(\nu, nu) \\
&\geq \Delta H + H\left(\frac{H^2}{n} - nK_1\right) \\
&\geq \Delta H + H\left(\frac{1}{n}(n^2 K_1 + n\epsilon_2 H^2) - nK_1\right) \\
&\geq \Delta H + \epsilon_2 H^3
\end{aligned}$$

Therefore by the maximum principle we know that $H_{\min}(0)$ is increasing. So

$$H^2 - n^2 K_1 \geq \epsilon_2 H^2$$

on $\mathcal{M}_0$. Therefore by the above we know that

$$(1 - \epsilon_2) H^2 \geq n^2 K_1$$

for $0 \leq t < T$ since H is increasing. So it remains true under the flow.

In view of this we obtain the follow lemma.

Lemma 1.133 (Lemma 4.1 [59]). *If (1.131) holds on $\mathcal{M}_0$, then it remains true on $\mathcal{M}_t$ for $0 \leq t < T$ and we have $T \leq \frac{1}{2}\epsilon_2^{-1} H_{\min}^{-2}(0)$.*

Proof.

$$\begin{aligned}
\frac{\partial H_{\min}}{H_{\min}^3} &\geq \partial t \epsilon_2 \\
-\frac{1}{H^2} &\geq 2\epsilon_2 t - H_{\min}^{-2}(0) \\
H &\leq \frac{1}{\sqrt{2\epsilon_2 T - H_{\min}^{-2}(0)}}.
\end{aligned}$$

□

Now we derive a lower bound for the eigenvalues of $|A|$.

Theorem 1.134 (Theorem 4.2 [59]). *If for some $0 < \epsilon_1 < \frac{1}{n}$ the inequality*

$$Hh_{ij} > \frac{h_{ij}}{H} - \epsilon_1 g_{ij} - \frac{n(1 - n\epsilon_1)}{H^2} K_1 g_{ij} - \frac{n^2}{H^3} L g_{ij}$$

is valid of $\mathcal{M}_0$, then it remains true for $\mathcal{M}_t$, $0 \leq t < T$.

Proof. We show that all eigenvalues of

$$M_{ij} = \frac{h_{ij}}{H} - \epsilon_1 g_{ij} - \frac{n(1-n\epsilon_1)}{H^2} K_1 g_{ij} - \frac{n^2}{H^3} L g_{ij} \tag{1.135}$$

remain nonnegative. First of all we need an evolution equation for M_{ij}. Using Lemma 1.130 (iiii) and (v), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\frac{1}{H^\alpha} &= -\frac{\alpha}{H^{\alpha+1}}\left(\Delta H + H(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu))\right)\\ &= \frac{-\alpha\Delta H}{H^{\alpha+1}} - \frac{\alpha}{H^\alpha}(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu))).\end{aligned}$$

Moreover we have the following identities

$$\nabla\left(\frac{1}{H^\alpha}\right) = -\frac{-\alpha}{H^{\alpha+1}}\nabla_i H \tag{1.136}$$

$$\Delta\left(\frac{1}{H^\alpha}\right) = -\frac{\alpha}{H^{\alpha+1}}\Delta H + \frac{\alpha(\alpha+1)}{H^{\alpha+2}}|\nabla H|^2. \tag{1.137}$$

We will derive the second of these estimates later in this Section. Rearranging and substituting into the above gives,

$$\begin{aligned}\frac{\partial}{\partial t}\frac{1}{H^\alpha} =& \Delta\left(\frac{1}{H^\alpha}\right) - \frac{\alpha(\alpha+1)}{H^{\alpha+2}}|\nabla H|^2 - \frac{\alpha}{H^\alpha}(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu))\\ =& \Delta\left(\frac{1}{H^\alpha}\right) + \frac{2}{H}\left\langle\nabla_l H, \nabla_l\left(\frac{1}{H^\alpha}\right)\right\rangle \text{ by (1.136)}\\ &- \alpha(\alpha-1)\frac{1}{H^{\alpha+2}}|\nabla H|^2 - \frac{\alpha}{H^\alpha}\left(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\right) \text{ by (1.137)}.\end{aligned}$$

We can derive as in Section 1.2.2. of this chapter that

$$\frac{\partial}{\partial t}M_{ij} = \Delta M_{ij} + \frac{2}{H}\langle\nabla_l H, \nabla_l M_{ij}\rangle + N_{ij}.$$

with

$$\begin{aligned}N_{ij} =& -2h_{il}h_j^l + 2\epsilon_1 H h_{ij} + \frac{2n(1-n\epsilon_1)}{H}K_1 h_{ij} + \frac{2n^2}{H^2}L h_{ij}\\ &+ \frac{2n(1-n\epsilon_1)}{H^4}K_1|\nabla H|^2 g_{ij} + \frac{6n^2}{H^2}L|\nabla H|^2 g_{ij}\\ &+ \frac{1}{H}\left(2h_{lm}\bar{R}^{l\ m}_{\ i\ j} - h_{jl}\bar{R}^{l\ \ m}_{\ mi} - h_{il}\bar{R}^{l\ \ m}_{\ mj}\right) - \frac{1}{H}\left(\overline{\nabla}_j\bar{R}_{0li}{}^i + \overline{\nabla}_l\bar{R}_{0ij}{}^l\right)\\ &+ \left(\frac{2n(1-n\epsilon_1)}{H^2}K_1 + \frac{3n^2}{H^3}L\right)\left(|A|^2 + \overline{\mathrm{Ric}}(\nu,nu)\right)g_{ij}.\end{aligned}$$

And we can continue the argument as in Section 1.2.2. of this Chapter or as stated in [48] Theorem 9.1. Since $\overline{Rm}$ is smooth then the argument still holds. We are only required to check that $N_{ij}v^iv^j$ is nonnegative the first time t_0, where at some point $p \in \mathcal{M}_0$ a zero eigenvector $v = \{v^i\}$ occurs. Choosing an orthonormal basis $(e_1,\ldots,e_n)$ for $T_p\mathcal{M}_0$ such that h_{ij} becomes diagonal. Let us assume that $v = e_1$. Then from $M_{11} = 0$ it follows that

$$\lambda_1 = \epsilon_1 H + \frac{n(1-n\epsilon_1)}{H}K_1 + \frac{n^2}{H^2}L. \tag{1.138}$$

Since $M_{11} = 0$ we can use (1.135) to obtain

$$\frac{\lambda_1}{H} = \frac{h_{11}}{H} = \epsilon_1 g_{11} + \frac{n(1-n\epsilon_1)}{H^2} K_1 g_{11} - \frac{n^2}{H^3} L g_{11}$$
$$\Rightarrow (1.138).$$

We wish to obtain a better expression for N_{ij}. We will do this using the following identities,

$$\begin{aligned}
&\frac{1}{H}(2h_{lm}\bar{R}^{l\ \ m}_{\ 1\ \ 1} - h_{1l}\bar{R}^{l\ \ \ m}_{\ m1} - h_{1l}\bar{R}^{l\ \ \ m}_{\ m1}) \\
=&\frac{1}{H}(2h_{11}\bar{R}^{1\ \ 1}_{\ 1\ \ 1} + 2h_{22}\bar{R}^{2\ \ 2}_{\ 1\ \ 1} + \cdots + 2h_{11}\bar{R}^{1\ \ \ m}_{\ m1}) \\
=&\sum_{l=2}^{n} \bar{R}_{1l1l}(\lambda_l - \lambda_1)\frac{2}{H} \\
\geq& - K_1 \frac{2}{H}\sum_{l=2}^{n}(\lambda_l - \lambda_1) \text{ since } \lambda_1 \text{ is the smallest eigenvalue} \\
=& - K_1\frac{2}{H}\lambda_1(H - n\lambda_1) \\
=& - 2K_1\lambda_1 + K_1\frac{2n}{H}\lambda_1\left(\epsilon_1 H + \frac{n(1-n\epsilon_1)}{H}K_1 + \frac{n^2}{H^2}L\right)
\end{aligned}$$

Using $|A|^2 \geq \frac{H^2}{n}$ and $\overline{\text{Ric}}(\nu, nu) \geq -nK_1$ we got

$$\left(\frac{2n(1-n\epsilon_1)}{H^2}K_1 + \frac{3n^2}{H^3}L\right)\left(\frac{H^2}{n} - nK_1\right)$$

lastly using $|\nabla_\alpha \bar{R}_{\beta\gamma\delta\sigma}| \leq L$,

$$-\frac{1}{H}\left(\nabla_1 \bar{R}_{0l1} + \bar{R}_{011}{}^{l}\right) \geq -\frac{2}{H}L.$$

Also applying the identity from Lemma 1.133 we obtain our final expression for N_{ij},

$$N_{ij}v^i v^j = N_{11} \geq \frac{n}{H}L - \frac{n^3}{H^3}LK_1 \geq 0.$$

This completes the proof. □

1.6.4 The Pinching Estimate

Just as in Section 3 of the previous Chapter, we want to show that the eigenvalues of the second fundamental form approach each other as the mean curvature gets very large.

Theorem 1.139 (Theorem 5.1 [59])**.** *There exist constants $\delta > 0$ and $C_0 < \infty$ depending only on $\mathcal{M}_0$ and our curvature bounds K_1, K_2, L and $i(\mathcal{N})$ such that*

$$|A|^2 - \frac{H^2}{n} \leq C_0 H^{2-\delta}$$

holds on $0 \leq t < T$.

The proof for this is very similar to the calculation in Section 1.4 and Section 5 of [58]. We outline some extra remarks needed for this computation that were not present in the previous section. For full details please refer to Section 5 of [59].

Lemma 1.140. *We have the following identity,*

$$h^{ij}(\overline{\nabla}_j \bar{R}_{0li}{}^l + \overline{\nabla}_l \bar{R}_{0ij}{}^l) = h^{0ij}\left(\overline{\nabla}_j \bar{R}_{0li}{}^l + \overline{\nabla}_l \bar{R}_{0ij}{}^l\right),$$

where $h^0_{ij} = h_{ij} - \frac{H}{n} g_{ij}$ is the traceless second fundamental form.

Proof. We take the trace of both sides and check that it is zero. This is clear for the right hand side. For the left hand side we have

$$\begin{aligned}\overline{\nabla}_j \bar{R}_{0ljl} + \overline{\nabla}_l \bar{R}_{0jjl} &= \overline{\nabla}_j \bar{R}_{0ljl} + \overline{\nabla}_j \bar{R}_{0llj} \\ &= \overline{\nabla}_j \bar{R}_{0ljl} - \overline{\nabla}_j \bar{R}_{0ljl}.\end{aligned}$$

□

Lemma 1.141.

$$\frac{1}{2}\Delta|A|^2 \geq \langle h_{ij}, \nabla_i\nabla_j H\rangle + H\left(h_{ik}h^k_l h^{li}\right) - |A|^4 + |\nabla A|^2 - CH^2 - C$$

where C is a constant depending on n, K_1, K_2 and L.

Proof. Recall the following identity from (ii) Lemma 1.123

$$\begin{aligned}\frac{1}{2}\Delta|A|^2 =& \langle h_{ij}, \nabla_i\nabla_j H\rangle + |\nabla A|^2 + H(h_{ik}h^k_l h^{li}) - |A|^2 + Hh^{ij}\bar{R}_{0i0j} \\ & - |A|^2 \bar{R}_{0l0}{}^l + 2h^{ij}h_{jl}\bar{R}^l{}_{mi}{}^m - 2h^{ij}h^{lm}\bar{R}_{limj} + h^{ij}(\overline{\nabla}_j \bar{R}_{0li}{}^l + \overline{\nabla}_l \bar{R}_{0ij}{}^l).\end{aligned}$$

Then we just need to look at the negative terms. Also remember that our curvature bound implies a sectional bound which gives us a tensor bound.

Then we just use the following algebraic manipulation

$$\begin{aligned}(\lambda_1 + \cdots + \lambda_n) &= H \\ (\lambda_1 + C) + \cdots + (\lambda_n + C) &= H + nC \\ \text{using} \quad \lambda_1^2 + \cdots + \lambda_n^2 &= |A|^2 \leq C + CH^2.\end{aligned}$$

This completes the proof. □

Lastly we will require a Sobolev inequality derived by Hoffman and Spruck from [57], for submanifolds of Riemannian manifolds. This is where our injectivity radius condition is required.

Lemma 1.142 (Lemma 5.7 [59]). *Let v be a Lipchitz function of $\mathcal{M}$. Moreover take α to be a free parameter, $0 < \alpha < 1$, ω_n to be the volume of the unit ball and*

$$\rho_0 = K_2^{-1} \arcsin(K_2(1-\alpha)^{-\frac{1}{n}}(\omega_n^{-1}|suppv|)^{\frac{1}{n}}).$$

Then

$$\left(\int_{\mathcal{M}_t} |v|^{\frac{n}{n-1}} d\mu\right)^{\frac{n-1}{n}} \leq C_n \left(\int_{\mathcal{M}_t} |\nabla v| d\mu + \int_{\mathcal{M}_t} H|v| d\mu\right),$$

if $K_2^2(1-\alpha)^{-\frac{2}{n}}(\omega_n^{-1}|supp\ v|)^{\frac{2}{n}} \leq 1$. and $2\rho_0 \leq i(\mathcal{N})$. With

$$C_n = \pi 2^{n-1}\alpha^{-1}(1-\alpha)^{-\frac{1}{n}}\frac{n}{n-1}\omega_n^{-\frac{1}{n}}.$$

1.6.5 The Gradient Bound

The same gradient estimate for the mean curvature as in the Section 1.1.4 is also valid in this context.

Theorem 1.143 (Theorem 6.1 [59])**.** *For any $\eta > 0$ there exists a constant $C_\eta < \infty$ depending on $\eta, C_0, \delta, \mathcal{M}_0, K_1, K_2, n$ and L such that*

$$|\nabla H|^2 \leq \eta H^4 + C_\eta.$$

We will require the following lemma.

Lemma 1.144. *We have the following identities*

(i) $\Delta(\nabla_k H) = \nabla_k(\Delta H) + g^{ij}\nabla_i H(Hh_{kj} - h_{mj}g^{mn}h_{kn} + \bar{R}_{kn})$.

(ii) $\nabla_i(\overline{Ric}(\nu,\nu)) = \overline{\nabla}_i \bar{R}_{0l0}{}^l + 2\bar{R}_{ml0}{}^l h_i^m$.

Proof. (i) We will make use of the following identities $\nabla_i\nabla_j T_k - \nabla_j\nabla_i T_k = R^l{}_{ijk}T_k$ and $R_{ikjl} = \bar{R}_{ikjl} + h_{ij}h_{kl} - h_{il}h_{kj}$. Then we have

$$\begin{aligned}
\Delta(\nabla_k H) &= g^{ij}\nabla_i\nabla_j\nabla_k H = g^{ij}\nabla_j\nabla_i\nabla_k H \\
&= g^{ij}(\nabla_k\nabla_i\nabla_j H) + g^{ij}R^l{}_{ikj}\nabla_l H) \\
&= \nabla_k(\Delta H) + g^{ij}g^{ln}\nabla_l H(h_{ij}h_{kn} - h_{in}h_{kj} + \bar{R}_{ikjn}) \\
&= \nabla_k(\Delta H) + g^{ln}\nabla_l H(Hh_{kn} - h_{in}g^{ij}h_{kj} + \bar{R}_{kn}).
\end{aligned}$$

(ii) We know that

$$\nabla_i(\overline{\mathrm{Ric}}(\nu,\nu)) = \overline{\nabla}_i\bar{R}_{0l0}{}^l + 2\overline{\mathrm{Ric}}(\nabla_i\nu,\nu).$$

Now $(\nabla_i\nu, \frac{\partial F}{\partial x_j}) = h_{ij}$ let

$$\begin{aligned}
\nabla_i\nu = a_i^l\frac{\partial F}{\partial x_l} &\Rightarrow a_i^l g_{jl} = h_{ij} \\
&\Rightarrow a_i^l = h_{ij}g^{jl}.
\end{aligned}$$

Therefore $\nabla_i\nu = h_{ij}g^{jm}\frac{\partial F}{\partial x_m}$ and the result follows. □

Lemma 1.145 (Lemma 6.2 [59])**.** *We have the evolution equation*

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla H|^2 =& \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 + 2|A|^2|\nabla H|^2 + 2\langle\nabla_i H h_{mj}, \nabla_j H h_{im}\rangle \\
&+ 2H\langle\nabla_i H, \nabla_i|A|^2\rangle + 2\overline{Ric}(\nu,\nu)|\nabla H|^2 - 2\bar{R}_{ij}\nabla^i H\nabla^j H \\
&+ 2H\langle\overline{\nabla}_i\bar{R}_{0l0}{}^l, \nabla_i H\rangle + 4H\langle\bar{R}_{ml0l}h_i^m, \nabla_i H\rangle.
\end{aligned}$$

Proof. Since

$$\frac{\partial}{\partial t}|\nabla H|^2 = \frac{\partial}{\partial t}(g^{ij}\nabla_i H\nabla_j H).$$

we can use the evolution equations for g^{ij} and H to obtain the result. □

Corollary 1.146 (Corollary 6.3 [59]). *We have the estimate*

$$\begin{aligned}\frac{\partial}{\partial t}|\nabla H|^2 \leq & \Delta|\nabla H|^2 - 2|\nabla^2 H|^2 + 6|A|^2|\nabla H|^2 + 2H\langle \nabla_i H, \nabla_i |A|^2\rangle \\ & + C|\nabla H|^2 + CH^2\end{aligned}$$

where C depends on K_1, K_2 and L.

Proof. We apply the following bounds to the previous lemma,

$$\begin{aligned}2\langle \nabla_i H h_{mj}, \nabla_j H h_{im}\rangle &\leq 2|A|^2|\nabla H|^2 \\ 2\overline{\mathrm{Ric}}(\nu,\nu)|\nabla H|^2 &\leq C|\nabla H|^2 \\ 2\bar{R}_{ij}\nabla^i H \nabla^j H &\leq C|\nabla H|^2 \\ 2H\langle \nabla_i \bar{R}_{0l0}{}^l, \nabla_i H\rangle &\leq C|\nabla H| H \\ 4H\langle \bar{R}_{ml0}{}^l h_i^m, \nabla_i H\rangle &\leq 4Hh\nabla H \\ &\leq 4H^2 + 4|A|^2|\nabla H|^2.\end{aligned}$$

□

Lemma 1.147 (Lemma 6.4 [59]). *We have*

(i) $\frac{\partial}{\partial t}H^3 \geq \Delta H^3 - 6H|\nabla H|^2 + 3\epsilon_2 H^5$

(ii) $\frac{\partial}{\partial t}\left(H\left(|A|^2 - \frac{H^2}{n}\right)\right) \leq \Delta\left(H\left(|A|^2 - \frac{H^2}{n}\right)\right) - \frac{n-1}{2n+1}H|\nabla A|^2$
$+C_2|\nabla A|^2 + C_3 H^3 + 3|A|^2 H\left(|A|^2 - \frac{H^2}{n}\right)$

Proof. (i) Using

$$\frac{\partial}{\partial t}H^3 = \Delta H^3 - 6H|\nabla H|^2 + 3H^3\left(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\right)$$

and the fact that $|A|^2 \geq \frac{H^2}{n}$ the first inequality holds due to Lemma 1.133.

(ii)

$$\begin{aligned}\frac{\partial}{\partial t}\left(H\left(|A|^2 - \frac{H^2}{n}\right)\right) =& \Big(\Delta H + H\left(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\right)\Big)\left(|A|^2 - \frac{H^2}{n}\right) \\ & + H\Bigg(\Delta\left(|A|^2 - \frac{H^2}{n}\right) - 2\left(|\nabla A|^2 - \frac{|\nabla H|^2}{n}\right) \\ & + 2\left(|A|^2 - \frac{H^2}{n}\right)\left(|A|^2 + \overline{\mathrm{Ric}}(\nu,\nu)\right) \\ & - 2h^{ij}\left(\overline{\nabla}_j \bar{R}_{0li}{}^l + \overline{\nabla}_l \bar{R}_{0ij}{}^l\right) - 4\left(h^{ij}h_j^m \bar{R}_{mli}{}^l - h^{ij}h^{lm}\bar{R}_{milj}\right)\Bigg).\end{aligned}$$

It is clear that the first part

$$\begin{aligned}\leq & \Delta\left(H\left(|A|^2 - \frac{H^2}{n}\right)\right) - 2\left(|\nabla|A|^2 - \frac{|\nabla H|^2}{n}\right)H \\ & - 2\left\langle \nabla_i H, \nabla_i\left(|A|^2 - \frac{H^2}{n}\right)\right\rangle\end{aligned}$$

where the last term is a consequence of the Laplacian.

The rest is then bounded by CH^3, since we have a bound on $\overline{\mathrm{Ric}}$, $\bar{R}$ and $\overline{\nabla}\bar{R}$. All we need is a bound on the second fundamental form, the h_{ij}'s. That we have since

$$\begin{aligned}|A|^2 &\leq \frac{H^2}{n} + CH^{2-\sigma}\\ &\leq \frac{H^2}{n} + C + H^2\\ &\leq \frac{H^2}{n} + CH^2\end{aligned}$$

since H has a lower bound.

So we can obtain that

$$\begin{aligned}\frac{\partial}{\partial t}\left(H\left(|A|^2 - \frac{H^2}{n}\right)\right) \leq \Delta\left(H\left(|A|^2 - \frac{H^2}{n}\right)\right) &- 2\left(|\nabla A|^2 - \frac{|\nabla H|^2}{n}\right)H\\ &- 2\langle\nabla_i H, \nabla_i\left(|A|^2 - \frac{H^2}{n}\right)\rangle + 3|A|^2 H\left(|A|^2 - \frac{H^2}{n}\right)\\ &+ CH^3.\end{aligned}$$

where C depends on K_1, K_2, L and $H_{\min}^{-1}(0)$. Using Theorem 1.139 we can estimate

$$\begin{aligned}\left|\left\langle\nabla_i H, \nabla_i\left(|A|^2 - \frac{H^2}{n}\right)\right\rangle\right| &= 2|\langle\nabla_i H h_{kl}^0, \nabla_i h_{kl}^0\rangle|\\ &\leq 2|\nabla H||h_{kl}^0||\nabla A|\\ &\leq 2nC_0^{\frac{1}{2}}H^{1-\frac{\delta}{2}}|\nabla A|^2\\ &\leq \frac{n-1}{2n+1}H|\nabla A|^2 + C|\nabla A|^2\end{aligned}$$

where C depends on n, C_0 and δ and we have applied the following

$$\begin{aligned}&|h_{kl}^0|^2 = |A|^2 - \frac{H^2}{n}\\ \Rightarrow\quad & h_{kl}^0 \leq nC_0^{\frac{1}{2}}H^{1-\frac{\delta}{2}}\\ \text{and}\quad & |\nabla H| = \nabla|g^{ij}h_{ij}| = g^{ij}\nabla h_{ij} \leq |g^{ij}||\nabla h_{ij}| \leq n|\nabla A|^2.\end{aligned}$$

Now the result follows as a direct consequence of Lemma 1.124 (ii). □

Now we study the function

$$f = \frac{|\nabla H|^2}{H} + P\left(|A|^2 - \frac{H^2}{n}\right) + PC_4|A|^2 - \eta H^3$$

where P depending only on $\mathcal{N}$ is large and $C_4 > 0$ depends on K_1, K_2, l and C_2. Using Corollary 1.146, Lemma 1.147 and Lemma 1.130(vi) we obtain that

$$\frac{\partial f}{\partial t} \leq \Delta f + C$$

where C depends on $\eta, \mathcal{M}_0, C_0, \delta, K_1, K_2, l$ and ϵ_2. This proves Theorem 1.143.

1.6.6 Contraction to a Point

Here we follow as in Section 7 of [59]. Let $0 \leq t < T$ be the maximal time interval where the smooth solution of (1.126) exists.

Then we have the following Theorem.

Theorem 1.148 (Theorem 7.1 [59]). *The quantity* $\max_{\mathcal{M}_t} |A|^2$ *becomes unbounded as* $t \to T$.

Proof. Argue by contradiction assuming that there exists some constant C_5 such that

$$\max_{\mathcal{M}_t} |A|^2 \leq C_5.$$

For more details refer to Theorem 7.1 in [59]. □

We will also require a lower bound for the intrinsic Ricci curvature R_{ij} of the surfaces $\mathcal{M}_t$.

Lemma 1.149 (Lemma 7.3 [59]). *The intrinsic Ricci curvature* R_{ij} *of* $\mathcal{M}_t$ *satisfies*

$$R_{ij} \geq (n-1)\epsilon_1\epsilon_2 H^2 g_{ij}.$$

Proof. The Ricci curvature on $\mathcal{M}$ is given by the Gauss equation

$$R_{ij} = \bar{R}_{ilj}{}^{l} + Hh_{ij} + h_{il}h_j^l.$$

Suppose that R_{ij} is diagonal at the point of consideration, then $\bar{R}_{ili}{}^{l}$ is the sum of $(n-1)$ sectional curvatures and therefore larger than $-(n-1)K_1$.

Any eigenvalue of $Hh_{ij} - h_{il}h_j^l$ is larger than $\frac{(n-1)}{n}\lambda_1 H$, but we know from (1.120), (1.131) and (1.132) that

$$H\lambda_1 \geq \epsilon_1(n^2K_1 + n\epsilon_2 H^2) + nK_1 - n^2\epsilon_1 K_1$$

and so the result follows. □

Theorem 1.150 (Theorem 7.4 [59]). $\frac{H_{\max}}{H_{\min}} \to 1$ *as* $t \to T$.

Proof. Arguing as in the previous Chapter, combining Theorem 1.143, Theorem 1.148 and Lemma 1.149. □

Using Theorem 1.148 it follows that $H_{\max}$ and $H_{\min}$ tend to ∞ as $t \to T$ and so the diameter of $\mathcal{M}_t$ tends to zero. Since the injectivity radius is bounded from below we know that there exists some $\theta < T$ such that $\mathcal{M}_\theta \subset B_\rho(p) = \{q \in \mathcal{N} | \operatorname{dist}(p,q) < p\}$ where ρ is small compared to the injectivity radius and $(K_1 + K_2)^{-1}$.

The elliptic maximum principle then ensures that the $\mathcal{M}_t$'s stay in $B_\rho(p)$ for all $\theta \leq t < T$. As $H_{\min} \to \infty$ as $t \to T$, Theorem 1.139 ensures that the principal curvatures approach the same value. Therefore for t close to T, $\mathcal{M}_t$ is an embedded sphere bounding a convex region. For $t_2 > t_1 \geq 0$ the region $\mathcal{M}_{t_2}$ is enclosed by $\mathcal{M}_{t_1}$ since the surfaces are shrinking under the flow and so the $\mathcal{M}_t$'s shrink to a single point as $t \to T$.

Chapter 2

Mean Curvature Flow for Two-Convex Hypersurfaces

In this chapter we take the next step and consider what happens if we loosen our convexity assumption. Here we will require that our initial surface is 2-convex rather than strictly convex. This means that $\lambda_1 + \lambda_2 \geq \alpha_0 H$ for some $\alpha_0 > 0$.

2.1 Mean Curvature Flow with Surgeries of Two-Convex Hypersurfaces

The results and proofs of this section originate from [67], when we use results from elsewhere this will be explicitly stated.

The two-convexity assumption presents a new challenge as Huisken and Sinestrari had to develop a surgery procedure for mean curvature flow [67], similar to that which Hamilton developed for Ricci flow in [49] and [45]. The focus was to be able to continue the flow past the first singular time T in a way that would allow us to keep track of the topological changes that occur and allow us to classify all possible geometries for the initial manifold. This is in contrast to weak solutions which succeeded in continuing the flow but did not yield a classification result, refer to [14], [19],[1] and [36]. To do this a surgery procedure is constructed and the flow is restarted after our first singular time.The surgery is controlled in terms of a few parameters which depend only on our initial manifold $\mathcal{M}_0$. Huisken and Sinestrari then went on to show that this procedure will terminate after finitely many steps after all components are recognised as being diffeomorphic to copies of S^n or $S^{n-1} \times S^1$.

The main result of Huisken and Sinestrari's paper is as follows.

Theorem 2.1 (Theorem 1.1 and Corollary 1.2 [67]). *Let $n \geq 3$ and $F_0 : \mathcal{M} \to \mathbb{R}^{n+1}$ a smooth immersion of a closed, 2-convex n-dimensional hypersurface. Then there exists a mean curvature flow with surgeries starting from $\mathcal{M}_0$ which terminates after a finite number of steps. Moreover any such initial surface $\mathcal{M}_0$ is diffeomorphic to S^n or a finite connected sum of $\mathcal{S}^{n-1} \times S^1$.*

2.1.1 Preliminaries

We begin this section by stating some results which are unique to the 2-convex setting.

Lemma 2.2 (Lemma 2.3 [67])**.** *Let $\mathcal{M}$ be a smooth n-dimensional hypersurface such that $S_1, S_2, ..., S_N > 0$, where S_k is defined in Section 1.1.4. Then $\lambda_1 + \lambda_2 > 0$.*

Proof. Argue by induction. □

Although the following definition may not be immediately intuitive it is required in order to study the flow in the 2-convex case. These classes of surfaces are controlled by a few parameters which remain invariant under the flow and surgery construction.

Definition 2.3. *For a positive set of constants $R, \alpha_0, \alpha_1, \alpha_2$ we denote by $\mathcal{C}(R, \alpha)$ with $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ the class of all smooth and closed hypersurface immersions $F : \mathcal{M} \to \mathbb{R}^{n+1}$ satisfying the estimates*

(i) $\lambda_1 + \lambda_2 \geq \alpha_0 H$

(ii) $H \geq \alpha_1 R^{-1}$

(iii) $|\mathcal{M}| \leq \alpha_2 R^n$

R is our scaling parameter and makes α scaling invariant. R is chosen such that $|A|^2 \leq R^{-2}$ on the initial surface $\mathcal{M}_0$. This is only required on the initial surface as this property will not be preserved by the flow.

Proposition 2.4 (Proposition 2.6 [67])**.**

(i) Given any smooth, closed, weakly 2-convex hypersurface immersion $\mathcal{M}_0$, the solutions $\mathcal{M}_t$ of mean curvature flow is strictly convex for each $t > 0$.

(ii) For every strictly 2-convex, smooth closed hypersurface $\mathcal{M}$ we can choose R and α such that $\mathcal{M} \in \mathcal{C}(R, \alpha)$ and $|A|^2 \leq R^{-2}$ everywhere on $\mathcal{M}$.

(iii) Each class $\mathcal{M} \in \mathcal{C}(R, \alpha)$ is invariant under smooth mean curvature flow.

Proof. For a full proof refer to Proposition 2.6 of [67]. To prove (i) we use the maximum principle for tensors found in Section 5.1.3. looking at the evolution equation for the components h_i^j of the Weingarten operator. To prove (ii) define $\sup |A|^2 \leq R^{-2}$ then the existence of α's follows from compactness of $\mathcal{M}$. The last part follows from evolution equations for $d\mu$ and H found in Lemma 1.18. □

To be able to extend the flow past a singular time for two-convex hypersurfaces, we introduce the mean curvature flow with surgeries algorithm. The idea is to combine mean curvature flow with surgeries like Hamilton did for Ricci flow in [49]. Huisken and Sinestrari describe the process as follows:

[Section 2, [67]] **Mean curvature flow with surgeries algorithm**
Mean curvature flow with surgeries is determined by an algorithm that assigns to each initial smooth closed two-convex hypersurface immersion $F_0 : \mathcal{M} \to \mathbb{R}^{n+1}$ in some class $\mathcal{C}(R, \alpha)$ a sequence of intervals $[T_0, T_1], [T_1, T_2], ..., T_{N-1}, T_N$, a sequence of manifolds $\mathcal{M}_i$, $1 \leq i \leq N$ and a sequence of smooth mean curvature flows $F_t^i : \mathcal{M}_i \to \mathbb{R}^{n+1}$, $t \in [T_{i-1}, T_i]$ such that the following is true:

(i) The initial hypersurface for the family F^1 is given by $F_0 : \mathcal{M}_1 \to \mathbb{R}^{n+1}$.

(ii) The initial hypersurface for the flow $F^i_t : \mathcal{M}_i \to \mathbb{R}^{n+1}$ n $[T_{i-1}, T_i]$ for $2 \leq i \leq N$ is obtained from $F^{i-1}_{T_{i-1}}$ by the following 2-step procedure:

(1) A hypersurface $\hat{F}^{i-1}_{Ti-1}$ is obtained from $F^{i-1}_{T_{i-1}} : \mathcal{M}_{i-1} \to \mathbb{R}^{n+1}$ by standard surgery, replacing finitely many necks with disjoint spherical caps.

(2) Finitely many disconnected components are removed from the surface $\hat{F}^{i-1}_{Ti-1}$ that are recognised as being diffeomorphic to $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{S}^1$, resulting in $F^i_{T_{i-1}}$ on $[T_{i-1}, T_i]$.

2.1.2 Necks and Surgery

In order to do surgery on a neck it will be essential for us to be able to detect necks in the first place. To do this we introduce the notion of a curvature neck and a geometric neck. A curvature neck is a region with intrinsic curvature resembling that of a cylinder. It relates to the pointwise nature of curvature. By contrast a geometric neck has an actual cylindrical parametrisation with metric close to standard cylinder. In [49] Hamilton showed these two are essentially the same. A large enough curvature neck possesses a suitable subset with can be parametrised as a geometric neck. This is discussed in more detail in the appendix 5.3.

This is important, in order to perform surgery on necks for two-convex surfaces undergoing mean curvature flow we need both notions of a neck. In order to detect necks we will require a priori estimates on curvature quantities satisfied by solutions to the flow, so curvature necks are required. However, in order to perform surgery we will need regions which are diffeomorphic to a cylinder and so geometric necks are required.

Definition 2.5 (Extrinsic curvature necks)**.** *Let $\mathcal{M}^n \to \mathbb{R}^{n+1}$ be a smooth hypersurface and $p \in \mathcal{M}^n$.*

(i) We say the extrinsic curvature is ϵ-cylindrical *at p is there exists an orthonormal frame at p such that*

$$|W(p) - \bar{W}(p)| \leq \epsilon \tag{2.6}$$

where $\bar{W}(p)$ is the Weingarten map on the tangent space to $\mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n+1}$ in a standard frame.

(ii) We say the extrinsic curvature is (ϵ, k)-parallel *at p if*

$$|\nabla^l W(p)| \leq \epsilon \quad \textit{for } 1 \leq l \leq k. \tag{2.7}$$

(iii) We say the extrinsic curvature is (ϵ, k, L)-cylindrical *around p if it is ϵ-cylindrical and (ϵ, k)-parallel at every point in the intrinsic ball of radius L around p.*

(iv) We say that p lies at the centre of a (ϵ, k, L) extrinsic curvature neck if it is (ϵ, k)-parallel $\in B_L(p)$ and the extrinsic curvature is (ϵ, k, L)-hypothetically cylindrical around p.

Proposition 2.8 (Proposition 3.5 [67]). *Let $k \geq 1$. For all $L \geq 10$ there exists $\epsilon(n, L) > 0$ and $c(n, L)$ such that any point $p \in \mathcal{M}$ which lies at the centre of a (ϵ, k, L)-extrinsic curvature neck with $0 < \epsilon \leq \epsilon(n, L)$ has a neighbourhood which after appropriate rescaling can be written as (cylindrical) graph of a function $u : \mathbb{S}^{n-1} \times [-(L-1), (L-1)] \to \mathbb{R}$ over some standard cylinder in $\mathbb{R}^{n+1}$ satisfying*

$$||u||_{C^{k+2}} \leq c(n, L)\epsilon.$$

Proof. Refer to Section 3 of [67]. □

Definition 2.9 (Geometric Neck). *The local diffeomorphism $N : \mathbb{S}^{n-1} \times [a, b] \to (\mathcal{M}, g)$ is called an* intrinsic (ϵ, k)-cylindrical geometric neck *if it satisfies the following conditions:*

(i) The conformal metric $\hat{g} = r^{-2}(z)g$ satisfies the estimates

$$|\hat{g} - \bar{g}|_{\bar{g}} \leq \epsilon, \quad |\bar{D}^j \hat{g}|_{\bar{g}} \leq \epsilon \quad \text{for } 1 \leq j \leq k \tag{2.10}$$

uniformly on $\mathbb{S}^{n-1} \times [a, b]$.

(ii) The mean radius function $r : [a, b] \to \mathbb{R}$ satisfies the estimate

$$\left| \left(\frac{d}{dz} \right)^j \log r(z) \right| \leq \epsilon \tag{2.11}$$

for all $1 \leq j \leq k$ everywhere on $[a, b]$.

Moreover we can say that N is an (ϵ, k)-cylindrical hypersurface neck *if in addition to the above assumptions we also have:*

$$|W(q) - r(z)^{-1}\bar{W}| \leq \epsilon r(z)^{-1} \quad \text{and} \tag{2.12}$$

$$|\nabla^l W(q)| \leq \epsilon r(z)^{-l-1}, \quad 1 \leq l \leq k, \tag{2.13}$$

for all $q \in \mathbb{S}^{n-1} \times z$ and all $z \in [a, b]$.

Hamilton was then able to show that so long as we were sufficiently far from the boundary and if we were to choose ϵ and k appropriately that every geometrical (ϵ, k) neck is diffeomorphic to a normal neck which is unique up to isometries of the standard cylinder.

Definition 2.14. *We call an (ϵ, k)-cylindrical hypersurface neck N a* maximal normal (ϵ, k)-cylindrical hypersurface neck *if N is normal and if whenever N^* is another such neck with $N = N^* \circ G$ for some diffeomorphism G then the map G is onto.*

We are now able to obtain uniqueness and existence among other properties on (ϵ, k)-cylindrical hypersurface necks.

Theorem 2.15 (Theorem 3.12 [67]). *Let $F : \mathcal{M} \to \mathbb{R}^{n+1}$ be a smooth closed hypersurface with $n \geq 3$.*

(i) For any $\delta > 0$ we can choose $\epsilon > 0$, k and $N : \mathbb{S}^{n-1} \times [a, b] \to \mathcal{M}$ to be an (ϵ, k)-cylindrical hypersurface neck with $b - a \geq 3\delta$. Then we can find a normal neck N^* and a diffeomorphism G of the domain cylinder of N^* onto a region in the domain cylinder of N containing all points at least δ from the ends, such that $N^* = N \circ G$.

(ii) For any $\delta > 0$ and any (ϵ', k') we can choose (ϵ, k) so that the normal neck N^* in (i) is an (ϵ', k')-cylindrical hypersurface neck.

(iii) For $0 \le \epsilon \le \epsilon(n)$ sufficiently small and $k \ge 1$, take N_1 and N_2 to be normal necks which are (ϵ, k)-hypersurface necks. If there is a diffeomorphism G of the corresponding cylinders such that $N_2 = G \circ N_1$, then G is an isometry in the standard metrics on the cylinders.

(iv) For $k \ge 1$ and any $\Lambda > 0$ there is $\tilde{\epsilon}(\Lambda, n) > 0$ such that any two normal (ϵ, k)-hypersurface necks N_1, N_2 with $0 < \epsilon \le \tilde{\epsilon}(\Lambda, n)$ that overlap on some collar $\mathbb{S}^{n-1} \times [z_0, z_0 + \Lambda]$ agree on that collar up to isometries of the standard cylinder and can be combined into a common (ϵ, k)-hypersurface neck.

(v) The normal neck N^* constructed in (i) is contained in a maximal normal (ϵ, k)-hypersurface neck unless the tangent hypersurface $\mathcal{M}$ is diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$.

Theorem 2.16 (Theorem 3.14 [67])**.** *For every (ϵ, k, L) with $L \ge 10$ there exist (ϵ', k') such that if the extrinsic curvature is (ϵ', k', L)-cylindrical about $p \in \mathcal{M}$ then p lies at the centre of a normal (ϵ, k)-cylindrical hypersurface neck $N : \mathbb{S}^{n-1} \times [-(L-1), (L-1)] \to \mathcal{M}$, which is contained in a maximal normal (ϵ, k)-hypersurface neck unless the target hypersurface $\mathcal{M}$ is diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$.*

Proof. By Proposition 2.8, we see that p has a neighbourhood which after rescaling can be written as graph over the standard cylinder $\mathbb{S}^{n-1} \times [-(L-1), (L-1)]$ which is C^{k+2}-close to the standard cylinder. Then by the Theorem 2.15 (i) yields a normal parametrisation and (v) gives the extension to a maximal normal hypersurface neck. □

We now move on to describe the standard surgery for a maximal normal (ϵ, k) hypersurface neck $N : S^{n-1} \times [a, b] \to \mathcal{M}$. Let $z_0 \in [a, b]$ be at a sufficient distance from either end of the neck i.e $z \in [a - 4\Lambda, b - 4\Lambda]$ for some $\Lambda > 0$.

[Section 3 [67]] Given a pair (N, z_0) we have surgery parameters $0 < \tau < 1$ and $B > 10\Lambda$ with $\Lambda \ge 10$. Denote by $\bar{C}_{z_0} : S^{n-1} \times \mathbb{R} \to \mathbb{R}^{n+1}$ the straight line cylinder best approximating $\mathcal{M}$ at Σ_{z_0} with radius $r(z_0) = r_0$. A point on its axis is given by the centre of mass of Σ_{z_0} with its induced metric, and its axis is parallel to the average of the unit normal field to Σ_{z_0}. The standard surgery with parameters τ and B is performed as follows:

(a) The surgery takes place in the middle of the neck and leaves the ends $S^{n-1} \times [a, z_0 - 3\Lambda]$ and $S^{n-1} \times [z_0 + 3\Lambda, b]$ unchanged.

(b) It replaces the two cylinders $N(S^{n-1} \times [z_0 - 3\Lambda, z_0]$ and $N(S^{n-1} \times [z_0, z_0 + 3\Lambda])$ with two n-balls attached smoothly to the cross sections $\Sigma_{z_0 - 3\Lambda}$ and $\Sigma_{z_0 + 3\Lambda}$. Without loss of generality we describe the procedure for the left side of the neck $[z_0 - 4\Lambda, z_0]$. From now on we let $z_0 - 4\Lambda = 0$ and consider a normal parametrisation $N : S^{n-1} \times [0, 4\Lambda] \to \mathcal{M}$.

(c) Define the function $u(z) \equiv r_0 \exp\left(-\frac{B}{z-\Lambda}\right)$ on $[\Lambda, 3\Lambda]$ for $B > 10\Lambda$ to bend the surface inwards so that it is strictly convex on $S^{n-1} \times [2\Lambda, 3\Lambda]$ for $0 < \tau < 1$:

$$\tilde{N}(\omega, z) := N(\omega, z) - \tau u(z) \nu(\omega, z).$$

(d) Now we need to blend this surface into an axially symmetric one. Pick a smooth transition function $\phi : [0, 4\Lambda] \to \mathbb{R}^+$ with $\varphi = 1$ on $[0, 2\Lambda]$ and $\varphi = 0$ on $[3, \Lambda, 4\lambda]$ with

$\varphi' \leq 0$. Taking $\tilde{C}_{z_0} = \bar{C}_{z_0} - \tau u(z)\nu(\omega, z)$ defined on $S^{n-1} \times [0, 4\Lambda]$ we interpolate to obtain

$$\hat{N}(\omega, z) := \varphi(z)\tilde{N}(\omega, z) + (1 - \varphi(z))\tilde{C}_{z_0}.$$

(e) Now we extend the function u to a suitable function $\hat{u}$ defined on $[3\Lambda, 4\Lambda]$ to guarantee that $\tau\hat{u}(z) \to r(z_0) = r_0$ as z approaches some $z_1 \in (3\Lambda, 4\Lambda]$, such that $\bar{C}_{z_0}[3\Lambda, 4\Lambda]$ is a smoothly attached axially symmetric and uniformly convex cap. Since the last deformation occurs only on $[3\Lambda, 4\Lambda]$ only concerns the axially symmetric case, it can be made for each pair τ, B of parameters in such a way that on the attached convex cap there is an upper bound for the curvature and each of its derivatives, independent of $\Lambda \geq 10$ and the surgery parameters τ, B.

We now want to look at how the bending of a neck affects its curvature. For a neck $N : \mathbb{S}^{n-1} \times [0, 4\Lambda] \to \mathcal{M} \subset \mathbb{R}^{n+1}$ in normal parametrisation we take

$$u(z) = r_0 f(z) = r_0 \exp\left(-\frac{B}{z - \Lambda}\right), \quad z \in [\Lambda, 4\Lambda], \tag{2.17}$$

where $r_0 = r(z_0) = r(4\Lambda)$ and $B > 1$ large enough.

Theorem 2.18 (Theorem 3.19)**.** *For any $\theta > 0$ and $\Lambda \geq 10$ we may choose $k \geq 1$ and $0 \leq \epsilon < \epsilon_0$, and then fix $0 < \tau_0 < 1$ small enough such that the second fundamental form of the deformed surface $\tilde{N}^{\tau_0}(p) = N(p) - \tau_0 r_0 f(z)\nu(p)$ satisfies*

(i) $|h_{ij}^{\tau_0} - (h_{ij} + \tau_0\delta_i^1\delta_j^1 f'' - \tau_0 r_0 f h_{il}h_j^l)| \leq \theta\tau_0 r_0 f''$,

(ii) $|\tilde{h}_j^{\tau_0 i} - (h_j^i + \tau_0 r_0 g^{li}\delta_j^1 f'' + \tau_0 r_0 f h_l^i h_j^l)| \leq \theta\tau_0 r_0^{-1} f''$

on $[\Lambda, 4\Lambda]$ for any (ϵ, k)-cylindrical neck $N : \mathbb{S}^{n-1} \times [0, 4\Lambda] \to \mathcal{M} \subset \mathbb{R}^{n+1}$ with normal parametrisation.

Proof. Refer to Theorem 3.19 of [67]. □

Taking an orthonormal frame $e_1, \ldots, e_n$ which diagonalises the second fundamental form at some point of the neck with e_1 the smallest eigenvalue yields the following result.

Corollary 2.19 (Corollary 3.21 [67])**.** *For any $\Lambda \geq 10$ we may choose $k_0 \geq 1, 0 < \epsilon_0, 0 < \tau < 1$ and B large enough such that for all $0 < \epsilon \leq \epsilon_0, k \geq k_0$ large enough the deformed surface $\tilde{N}^{\tau_0}$ satisfies:*

(i) $\tilde{H} \geq H,\ \tilde{\lambda}_1 + \tilde{\lambda}_2 \geq \lambda_1 + \lambda_2,\ \sqrt{det\tilde{g}} \leq \sqrt{detg}$ *on* $[\Lambda, 4\Lambda]$,

(ii) $\tilde{\lambda}_1 \geq \frac{1}{2}\tau_0 D_1 D_1(r_0 u),\ \tilde{\lambda}_1 + \tilde{\lambda}_2 \geq \lambda_1 + \lambda_2 + \frac{1}{2}\tau_0 D_1 D_1(r_0 u)$ *on* $[2\Lambda, 4\Lambda]$,

(iii) $\tilde{\lambda}_1 + \tilde{\lambda}_2/\tilde{H} \geq \lambda_1 + \lambda + 2/H$ *on* $[\Lambda, 4\Lambda]$,

(iv) $\tilde{H} \geq H + \frac{1}{2}\tau_0 D_1 D_1(r_0 u),\ \sqrt{det\tilde{g}} \leq \sqrt{detg}(1 - \frac{1}{2}\tau_0 u H)$ *on* $[2\Lambda, 3\Lambda]$.

Proof. Refer to Remark 3.20 in [67]. □

The following theorem show that the class $\mathcal{C}(R, \alpha)$ remains invariant under the surgery construction.

Theorem 2.20 (Theorem 3.22 [67]). *For any $\Lambda \geq 10$ we may choose $k_0 \geq 1, 0 < \epsilon_0, 0 < \tau_0$ small enough and B large enough such that for all α and $R > 0$ the class $C(R,\alpha)$ is invariant under standard surgery with parameters τ_0, B on a normal (ϵ, k)-hypersurface neck $N : \mathbb{S}^{n-1} \times [-4\Lambda, 4\Lambda] \to \mathcal{M}$, for all $0 < \epsilon \leq \epsilon_0$, and $k \geq k_0$.*

Proof. In the region $[\Lambda, 2\Lambda]$ the claim follows directly from Corollary 2.19 (i) and the fact that on a cylinder $\lambda_2 > \frac{1}{n}H$.

In the region $[2\Lambda, 3\Lambda]$ we can ensure the interpolated surface

$$\hat{N}(\omega, z) := \varphi(z)\tilde{N}(\omega, z) + (1 - \varphi(z))\tilde{C}(\omega, z)$$

is approximately close to $\tilde{N}$ in any norm if $k \geq 1$ and ϵ are chosen appropriately. Therefore $\hat{N}$ will also satisfy the estimates of $H, \lambda_1 + \lambda_2$ and $\sqrt{\det g}$.

Finally by making an appropriate choice of $\hat{u}$ we can smoothly attach the strictly convex cap in $[3\Lambda, 4\Lambda]$. Increasing the curvature and decreasing the area. □

We conclude this section we look at how Huisken and Sinestrari showed that the topological properties of $\mathcal{M}$ before the surgery can be recovered from the properties of the surface $\tilde{\mathcal{M}}$ after the surgery.

The following proposition will not be proved as it is a direct consequence of our surgery construction.

Proposition 2.21 (Proposition 3.23 [67]). *There exist parameters $\Lambda \geq 10$, $0 < \epsilon \leq \epsilon_0$ and $k \geq 0$ depending on n, such that the following is true. Suppose we perform a standard surgery procedure on a normal (ϵ, k)-hypersurface neck $N : S^{n-1} \times [-4\Lambda, 4\Lambda] \to \mathcal{M}$ in some connected smooth closed immersed hypersurface $F : \mathcal{M} \to \mathbb{R}^{n+1}$ which results in a new smooth hypersurface $\tilde{\mathcal{M}}$.*

Then there exist three possibilities.

(a) $\tilde{\mathcal{M}}$ is connected and $\mathcal{M}$ is diffeomorphic to the manifold obtained from $\tilde{\mathcal{M}}$ by a standard connected sum with itself.

(b) $\tilde{\mathcal{M}}$ is disconnected with two components $\tilde{\mathcal{M}}_1$ and $\tilde{\mathcal{M}}_2$. Then $\mathcal{M}$ is diffeomorphic to the connected sum of $\tilde{\mathcal{M}}_1$ and $\tilde{\mathcal{M}}_2$.

(c) $\tilde{\mathcal{M}}$ is disconnected and $\tilde{\mathcal{M}}_1$ is diffeomorphic to S^n, then $\tilde{\mathcal{M}}_2$ is diffeomorphic to $\mathcal{M}$.

Huisken and Sinestrari then prove the following lemma which in fact holds for k-convex surfaces and not just the 2-convex case.

It shows that embedded 2-convex surfaces in $\mathbb{R}^{n+1}$ are still embedded after surgery.

Lemma 2.22 (Lemma 3.24 [67]). *Let $\mathcal{M}^n = \mathcal{M} \subset \mathbb{R}^{n+1}$ with $n \geq 3$ be a smoothly embedded, closed connected hypersurface. Now suppose in addition that $\mathcal{M}$ is strictly k-convex. Let $E^n = E \subset \mathbb{R}^{n+1}$ be the hyperplane transverse to $\mathcal{M}$ such that $\emptyset \neq \Sigma^{n-1} = \Sigma = E \cap \mathcal{M}$ is a smooth closed hypersurface of E. Then each component of Σ is strictly k-convex and bounds a region in E that does not contain another component of Σ.*

Proof. Refer to Lemma 3.24 of [67]. □

We now introduce the notion of a solid tube enclosed by a normal hypersurface neck along with some of its characteristics. This will be essential throughout this chapter, including when we show that the flow with surgeries converges to the weak set flow.

Proposition 2.23 (Proposition 3.25 [67]). *Given a normal (ϵ, k)-hypersurface neck $N : S^{n-1} \times [0, L] \to M^n \subset \mathbb{R}^{n+1}$ with parameters $L \geq 20 + 8\Lambda \geq 100$, $0 < \epsilon \leq \epsilon_0$ and $k \geq k_0$ depending on n, there exists a unique local diffeomorphism*

$$G : \bar{B}_1^n \times [0, L] \to \mathbb{R}^{n+1}$$

such that

(i) G (restricted to the cylinder) agrees with N;

(ii) Each cross-section $G(\bar{B}_1^n \times \{z_0\}) \subset \mathbb{R}^{n+1}$ is an embedded area minimising hypersurface;

(iii) G restricted to each slice $B_1^n \times \{z_0\}$ is a harmonic diffeomorphism; and

(iv) G is ϵ-close in C^{k+1}-norm to the standard isometric embedding of a solid cylinder in $\mathbb{R}^{n+1}$.

Proof. Refer to Proposition 3.25 [67]. □

Theorem 2.24 (Theorem 3.26 [67]). *There is a range of parameters $\Lambda \geq 10, 0 < \epsilon \leq \epsilon_0$ and $k \geq k_0$ depending only on n such that the following is true. Suppose $\mathcal{M} \subset \mathbb{R}^{n+1}, n \geq 3$ is a connected, smooth, closed and embedded hypersurface which is strictly 2-convex. Let U be the closed bounded region enclosed by $\mathcal{M}$.*

(i) If standard surgery is performed on a normal (ϵ, k)-hypersurface neck $N : \mathbb{S}^{n-1} \times [-4\Lambda, 4\Lambda] \to \mathcal{M}$ then the resulting hypersurface $\tilde{\mathcal{M}}$ is again embedded.

(ii) If $\tilde{\mathcal{M}}$ is connected with the resulting bounded region $\tilde{U}$, then the region U is diffeomorphic to a connected sum of $\tilde{U}$ with itself. If $\tilde{U}$ is disconnected consisting of two disjoint bounded region $\tilde{U}^1$ and $\tilde{U}^2$, then U is diffeomorphic to the connected sum of $\tilde{U}^1$ and $\tilde{U}^2$. In particular, if $\tilde{U}^2$ is diffeomorphic to a standard closed disc $\bar{B}_1^n \subset \mathbb{R}^{n+1}$, then U is diffeomorphic to $\tilde{U}^1$.

Proof. Refer to Theorem 3.26 [67]. □

2.1.3 Convexity Estimates in the Presence of Surgery.

Continuing on from Section 1.4.4., we want to show $S_m \geq -\delta H^m - C_\delta$ on $\mathcal{M}_t$ still holds for mean curvature flow with surgeries in class $C(R, \alpha)$, provided surgery is done on (ϵ, k)-necks with $k \geq 2$, $0 < \epsilon \leq \epsilon_0$ small depending only on n. We have to split it up into two cases, $m = 2$ and $m > 2$.

Theorem 2.25 (Theorem 4.1 [67]). *Let $\mathcal{M}_t, t \in [0, T)$ be a family of smooth closed n-dimensional surfaces immersed in $\mathbb{R}^{n+1}$ evolving the mean curvature flow. Suppose that $\mathcal{M}_0$ has positive mean curvature. Then for any $\delta > 0$, there exists $C_\delta > 0$ where C_δ depends only on $\mathcal{M}_0$ such that for all $m = 2, ..., n$ we have*

$$S_m \geq -\delta H^m - C_\delta \quad \text{on } \mathcal{M}_t \text{ for all } t \in [0, T) \tag{2.26}$$

We want to show that this estimate still holds for mean curvature flow with surgeries in a class $\mathcal{C}(R, \alpha)$.

For the $m = 2$ case we follow the proof as in [66]. We begin as in [66] by introducing the function,

$$g_{\sigma,\eta} = \frac{|A|^2 - (1 + \eta)H^2}{H^{2-\sigma}}. \tag{2.27}$$

Theorem 2.28 (Theorem 4.3 [67]). *$\mathcal{M}_0$ a closed n-dimensional surface such that*

$$H \geq \beta_1 |A| > 0 \quad \text{on } \mathcal{M}_0 \tag{2.29}$$

for some $1 \geq \beta > 0, \mathcal{M}_t$ smooth evolution, then there exists $c_1, c_2 > 0$ depending on n, η, β such that for any $\sigma \leq \frac{1}{c_1}$ and $p \geq \frac{c_2}{\sigma^2}$, the integral

$$\int_{\mathcal{M}_t} (g_{\sigma,\eta})_+^p d\mu$$

is a decreasing function of t.

Proof. Refer to Remark 4.4 of [67]. □

We also want to show that the integral defined above cannot increase under standard surgery with surgery parameters defined as in the previous section.

Proposition 2.30 (Proposition 4.5 [67]).

(i) We can choose $\epsilon_0 > 0, \eta_0 > 0$ and $\sigma_0 > 0$ small enough such that $(g_{\sigma,\eta})_+$, is non-increasing under standard surgery with surgery parameters as in the previous section on a normal (ϵ, k)-hypersurface neck for any $0 < \sigma < \sigma_0$,$0 < \eta < \eta_0$ and any $0 < \epsilon < \epsilon_0, k \geq 2$. By this we mean that $(g_{\sigma,\eta})_+$ is non-increasing in the region $[0, 3\Lambda]$ of the surface modified by surgery and it is zero on the regions such as $[3\Lambda, 4\Lambda]$ which is added by the surgery.

(ii) The statement of the above Theorem holds for mean curvature flow in a class $\mathcal{C}(R, \alpha)$ with surgeries as determined in (i).

Proof. On an approximate cylinder $|A|^2 \approxeq \frac{1}{n-1} H^2$. Since $n \geq 3$ it follows that for sufficiently small ϵ_0 the function $(g_{\sigma,\eta})_+$ vanishes everywhere on the region affected by surgery proving (i).

(ii) Follows from part (i) as the inequality $H \geq \beta_1 |A|$ for $1 \geq \beta_1 > 0$ is not affected by surgery and therefore the constants c_1, c_2 of Theorem 2.26 do not change. □

The rest of the proof follows much like [66], if $\mathcal{M}_t$ is a flow with surgeries we only require (2.29) and the monotonicity of the L^p norm, and these properties are preserved by the surgeries.

In the case when $m > 2$ an induction procedure is used. As in [66] we define the quotient $Q_{m+1} := S_{m+1}/S_m$ and consider a perturbation of the second fundamental form

$$b_{ij;\rho,D} := h_{ij} + (\rho H + D) g_{ij} \quad \text{for a given } 0 < \rho < 1/n, D > 0$$

as the quotient may not be well-defined given that S_m is not guaranteed to be nonzero by our assumption (2.29). Using a similar procedure to the $m = 2$ replacing $g_{\sigma,\eta}$ with the perturbed

$$g_{\sigma,\eta,\rho,D_\rho} = -\frac{-Q^b_{m+1;\rho,D} - \eta H^b_{\rho,D}}{(H^b_{\rho,D})^{1-\sigma}}$$

will lead us to the following result for our initial surface in the class $\mathcal{C}(R, \alpha)$:

Theorem 2.31 (Theorem 4.6 [67]). *Let $\mathcal{M}_0 \in \mathcal{C}(R,\alpha)$ be a surface satisfying $|A|^2 \leq R^{-2}$. Then for any $\delta > 0$ there is a constant β_m depending on n, δ and α such that a solution $\mathcal{M}_t, t \in [0,T)$, of mean curvature flow with initial data $\mathcal{M}_0$ and with surgeries satisfies the estimates*

$$S_m \geq -\delta H^m - \beta_m R^{-m}.$$

2.1.4 Cylindrical Estimates

We want to show that any rescaling near a singularity which is not strictly convex must be cylindrical, [61]. In order to do so we make use of both 2-convexity and estimates from Theorem 2.31.

Want to show that points where λ_1 is small have curvature close to the curvature of a cylinder.

Theorem 2.32 (Theorem 5.3 [67]).

(i) Let $\mathcal{M}_t, t \in [0,T)$ be a smooth solution of mean curvature flow in $\mathcal{C}(R,\alpha)$ with $n \geq 3$ and initial data satisfying $|A|^2 \leq R^{-2}$. Then for any $\eta > 0$ there exists a constant $C_\eta = C_\eta(n,\alpha) > 0$ such that

$$|A|^2 - \frac{1}{n-1}H^2 \leq \eta H^2 + C_\eta R^{-2}$$

on $\mathcal{M}_t$ for any $t \in [0,T)$.

(ii) We define

$$g_{\sigma,\eta} = \frac{|A|^2 - (\frac{1}{n-1} + \eta)H^2}{H^{2-\sigma}}.$$

Then for all $\Lambda \geq 10$ we can choose $k_0 \geq 2, \epsilon_0 > 0$, surgery parameters B, τ_0, as well as parameters $\eta_0 > 0, \sigma_0 > 0$ such that $(g_\sigma, \eta)_+$ id non-increasing under standard surgery on a normal (ϵ, k)-hypersurface neck for any $0 < \sigma \leq \sigma_0, 0 < \eta \leq \eta_0$ and any $0 < \epsilon \leq \epsilon_0, k \geq k_0$. For mean curvature flow with surgeries and parameters $o < \eta, \eta_0$ we then have the same estimate as in (i).

Notice that $g_{\sigma,\eta}$ is slightly different to before. The factor in front of the H^2 is chosen such that if $\eta = 0$, the function vanishes on a cylinder. Moreover if $\eta = 0$ and $\lambda = 0$, then the numerator is nonnegative and vanishes if and only if $\lambda_2 = \cdots = \lambda_n$.

To prove this we argue as usual for $g_{\sigma,\eta}$.

The above, together with

$$|A|^2 - \frac{1}{n-1}H^2 = \frac{1}{n-1}\left(\sum_{1<i<j}(\lambda_i\lambda_j)^2 + \lambda_1(n\lambda_1 - 2H)\right)$$

yields the cylindrical estimate.

Theorem 2.33 (Theorem 1.5 [67]). *[Cylindrical Estimate] For a given smooth closed two-convex initial hypersurface $\mathcal{M}_0$ in $\mathbb{R}^{n+1}$, $n \geq 3$, the parameters of standard surgery can be chosen in such a way that the solution $\mathcal{M}_t, t \in [0,T)$ of mean curvature flow with surgery satisfies the following estimate: for any $\eta > 0$, there exists $C_\eta = C_\eta(\mathcal{M}_0) > 0$ such that at every point we have the property*

$$|\lambda_1| \leq \eta H \Rightarrow |\lambda_i - \lambda_j|^2 \leq c(n)\eta H^2 + C_\eta, \quad \forall i,j \geq 2.$$

2.1.5 Derivative Estimates for the Curvature

Another key tool in the study of geometric evolution equations are bounds on the derivatives of our curvature terms. In this section we will derive a pointwise derivative estimate for the curvature for 2-convex surfaces along mean curvature flow, it will depend on the mean curvature at a point instead of some maximum value of curvature.

Theorem 2.34 (Theorem 6.1 [67]). *[Gradient Estimate] Let $\mathcal{M}_t$ in $\mathcal{C}(R,\alpha)$ be a solution to mean curvature flow with surgery and normalised initial data. Then there is a constant γ_2 depending on n and a constant γ_3 depending on n and α such that for suitable surgery parameters as in Sections 2.1.3. and 2.1.4. the flow satisfies the uniform estimate*

$$|\nabla A|^2 \leq \gamma_2 |A|^4 + \gamma_3 R^{-4}$$

for every $t \geq \frac{1}{4}R^2$.

Proof. Define $g_i = a_i H^2 - |A|^2 + C_i R^{-2}$. Where $g_1 = (\frac{1}{n-1} + \eta),\ g_2 = \frac{3}{n+2}$ come from Theorem 2.32. And C_1 and C_2 are chosen such that g_1 and g_2 are strictly positive.

We then use Theorem 2.32 to find a C_η depending on n, η and α such that $g_1 \geq C_\eta R^{-2}$.

Using the relevant evolution equations we want to estimate the following function $\frac{|A|^2}{g_1 g_2}$ using the maximum principle.

Taking combinations of derivatives we are able to arrive at

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{|\nabla A|^2}{g_1 g_2}\right) - \Delta\left(\frac{|\nabla A|^2}{g_1 g_2}\right) - \frac{2}{g_2}\left\langle \nabla g_2, \nabla\left(\frac{|\nabla A|^2}{g_1 g_2}\right)\right\rangle \\ \leq \frac{|\nabla A|^2 |A|^2}{g_1 g_2}\left((c_n + 4) - 2\kappa_n^2)\frac{n+2}{3n}\frac{|\nabla A|^2}{g_1 g_2}\right)\end{aligned}$$

where $\kappa_n = \frac{1}{2}(\frac{3}{n+2} - \frac{1}{n-1})$ and c_n is as defined by Theorem 1.38.

First consider the case without surgeries. Then in view of Proposition 2.7(v) in [67] we obtain that at time $t_0 = (1/4)R^{-2}$ we have an upper bound $|\nabla A|^2 \leq m_0 R^{-4}$ where m_0 depends on n and α. Applying the maximum principle and recalling that $g_1 g_2 \geq R^{-4}$, we obtain

$$\frac{|\nabla A^2|}{g_1 g_2} \leq \max\left\{m_0, \frac{3n(c_n + 4)}{2\kappa_n 2(n+2)}\right\}$$

completing the proof.

For the case with surgeries we pick $\eta = \kappa_n$ in the definition fo g_1. On an exact cylinder we have $|A|^2 - \frac{1}{n-1}H^2$, by our surgery construction, and taking suitable surgery parameters we have the estimate $\frac{1}{n-1}H^2 - |A|^2 \geq -\frac{\kappa_n H^2}{2}$ in the region of the surface affected by surgery. Therefore, in such a region we have $g_1 \geq \frac{\kappa_n H^2}{2}$ and $g_2 = \frac{3\kappa_n H^2}{2}$.

Moreover given any (ϵ, k)-neck with $k \geq 1$, the fact that $|\nabla A|^2 = 0$ on a standard cylinder implies $|\nabla A|^2 \ll H^4$.

For a given choice of transition function φ and of convex cap in steps (d) and (e) of the surgery construction there is a fixed constant μ_0 depending only on n such that for all surgery parameters considered we have the uniform estimate $|\nabla A|^2 \leq \mu_0 H^4$ on the region altered by surgery and hence

$$\frac{|\nabla A|^2}{g_1 g_2} \leq \frac{4\mu_0}{3\kappa_n^2}.$$

Iterating the argument in every time interval between two surgeries we find

$$\frac{|\nabla A^2|}{g_1 g_2} \leq \max\left\{m_0, \frac{3n(c_n+4)}{2\kappa_n^2(n+2)}, \frac{4\mu_0}{3\kappa_n^2}\right\}.$$

Since we only chose $\eta = \kappa_n$, the corresponding constant C_η depends only on n, α. Thus,

$$g_1 g_2 \leq H^4 + C_\eta R^{-4}$$

and so the above estimate implies

$$|\nabla A|^2 \leq c(n)|A|^4 + C_\eta R^{-4}.$$

□

It is also interesting to note Remark 6.2 from [67] which states that on a neck we have, up to lower order terms, $g_1 \approx \eta H^2$ and so on a neck gradient estimate $|\nabla A|^2 \leq c g_1 g_2$ implies that $|\nabla A|^2 \leq c\eta H^4 + C$.

Theorem 2.35 (Corollary 6.4 [67]). *Let $\mathcal{M}_t$ in $\mathcal{C}(R,\alpha)$ be a solution to mean curvature flow with surgery and normalised initial data. Then there is a constant γ_4' depending only on n and γ'' depending on n and α such that for suitable surgery parameters as in the previous two sections we have the estimate*

$$|\partial_t^h \nabla^h A|^2 \leq \gamma' |A|^{4h+2m+2} + \gamma'' R^{-(4h+2m+2)}$$

for all $h, n \geq 0$ such that $2h + m \leq k_0$.

Here we abuse notation slightly by writing $\partial_t A$ with the convention that at a surgery time we are taking one sided time derivatives.

We are also able to obtain the following as a special case, which will be useful for the analysis of regions with large curvature.

Corollary 2.36 (Corollary 6.5 [67]). *Let $\mathcal{M}_t$ be a mean curvature flow with surgeries staring from a surface in $\mathcal{C}(R,\alpha)$. Then we can find $c^\# > 0, H^\# > 0$ such that, for all $p \in \mathcal{M}$ and $t > 0$,*

$$H(p,t) \geq H^\# \Rightarrow |\nabla H(p,t)| \leq c^\# H^2(p,t),\ |\partial_t H(p,t)| \leq c^3 H^3(p,t) \tag{2.37}$$

where $c^\#$ depends on n and $H^\# = h_0 R^{-1}$ with h_0 depending on n and α.

Lemma 2.38 (Lemma 6.6 [67]). *Let $F : \mathcal{M} \to \mathbb{R}^{n+1}$ be an n-dimensional immersed surface. Suppose that there are $c^\#, H^\# > 0$ such that $|\nabla H(p)| \leq c^\# H^2(p)$ for any $p \in \mathcal{M}$ such that $H(p) \geq H^\#$. Let $p_0 \in \mathcal{M}$ satisfy $H(p_0) \geq \gamma H^\#$ for some $\gamma > 1$. Then*

$$H(q) \geq \frac{H(p_0)}{1 + c^\# d(p_0,q) H(p_0)} \geq \frac{H(p_0)}{\gamma} \text{ for all } q$$

such that

$$d(p_0,q) \leq \frac{\gamma - 1}{c^\#} \frac{1}{H(p_0)}.$$

Proof. Refer to Lemma 6.6 [67] or the analogous proof for G-flow in Lemma 3.17. □

2.1.6 Neck Detection

We want to show that the surgery procedure can be used to alter mean curvature flow before a singular time whilst keeping the mean curvature bounded. Unless surface is convex or of type $\mathbb{S}^{n-1} \times \mathbb{S}^n$ in which case it will be discarded. We do this by showing if we are close enough to a singular time and the surface is not uniformly convex, then the regions with largest curvature are necks where we perform the surgery. In this section and the following section we will omit proofs and not go into extensive detail, the reasoning behind this is that we will prove similar results for the G-flow in Chapter 3, however for completeness of this thesis it is important to state the results here for the mean curvature flow case.

Definition 2.39. *Given t, θ such that $0 \leq t-\theta < t \leq T_0$, we define the* backward parabolic neighbourhood *of (p,t) by,*

$$\mathcal{P}(p,t,r,\theta) = \{(q,s) | q \in \mathcal{B}_{g(t)}(p,r), s \in [t-\theta, t]\}. \tag{2.40}$$

where $\mathcal{B}_{g(t)}(p,r) \subset \mathcal{M}$ is the closed ball of radius r with respect to the the metric $g(t)$.

We now extend the definition of a backward parabolic neighbourhood to the case of a flow with surgeries. We have a family of flows $F^i : \mathcal{M}_i \times [T_{i-1}, T_i] \to \mathbb{R}^{n+1}$, where $T_0 = 0$ is the initial time and $< T_1 < \cdots < T_n < \infty$ are the surgery times. The neighbourhood $B_{g(t)}(p,r) \in \mathcal{M}_i$ corresponding to the interval $[T_{i-1}, T_i]$ containing t. At a surgery time $t = T_i$ we write $g(t-)$ and $g(t+)$ to denote the manifold before and after the surgery. As per convention $g(t) = g(t-)$, at a surgery time our flow is continuous from the left. This motivates the following definition.

Definition 2.41. *Let $F^i : \mathcal{M}_i \times [T_{i-1}, T_i] \to \mathbb{R}^{n+1}$, $i = 1, 2, ..., n$ be a mean curvature flow with surgeries. Let $(p,t) \in \mathcal{M}_i \times [T_{i-1}, T_i]$ for some i and $\theta \in (0, T]$ and $r > 0$. We say that $B_{g(t)}(p,r)$ has not been changed by surgeries in the interval $[t-\theta, t]$ if there are no points of $B_{g(t)}(p,r)$ which belong to a region changed by a surgery occurred at a time $s \in (t\theta, t]$. In this case we define the backward parabolic neighbourhood $\mathcal{P}(p,t,r,\theta)$ as in the smooth case. We also describe this behaviour by saying that $\mathcal{P}(p,t,r,\theta)$ does not contain surgeries.*

Remark 2.42. *The above definition allows for the presence of surgeries in the time interval $(t-\theta, t]$ provided they are performed on parts of the surface disjoint from $B_{g(t)}(p,r)$.*

We define the following to simplify the analysis of necks.

$$\hat{r}(p,t) := \frac{n-1}{H(p,t)}, \quad \hat{\mathcal{P}}(p,t,l,\theta) := \mathcal{P}(p,t,\hat{r}(p,t), \hat{r}(p,t)^2\theta). \tag{2.43}$$

Then if (p,t) lies on a neck, then $\hat{r}(p,t)$ is approximately equal to the radius of the necks. Moreover if we rescale the flow in space and time such that $\hat{r}(p,t) = 1$ then $\hat{\mathcal{P}}(p,t,L,\theta) = \mathcal{P}(p,t,\hat{r}(p,t), \hat{r}(p,t)^2\theta)$.

The following lemma will prove useful. In particular (ii) tells us that if $H(p,t_1) \gg H(q,t_2)$ where (p,t_1) is any point and (q,t_2) is a point modified by previous surgeries, then a suitable backward parabolic neighbourhood of (p,t) will be surgery-free.

Lemma 2.44 (Lemma 7.2 [67])**.** *Let $c^\#, H^\#$ be as defined in the previous section. Define $d^\# := (8(n-1)^2 c^\#)^{-1}$. Then the following properties hold.*

(i) Let (p,t) satisfy $H(p,t) \geq 2H^{\#}$. Then, given any $r, \theta \in (0, d^{\#})$ such that $\mathcal{P}(p,t,r,\theta)$ does not contain surgeries, we have

$$\frac{H(p,t)}{2} \leq H(q,s) \leq 2H(p,t)$$

for all $(q,s) \in \mathcal{P}(p,t,r,\theta)$.

(ii) Suppose that, for any surgery performed at time less than t, the regions modified by surgery have mean curvature less than K, for some $K \geq H^{\#}$. Let (p,t) satisfy $H(p,t) \geq 2K$. Then the parabolic neighbourhood

$$\mathcal{P}\left(p, t, \frac{1}{8c^{\#}K}, \frac{1}{8c^{\#}K^2}\right)$$

does not contain surgeries. In particular, the neighbourhood $\hat{\mathcal{P}}(p,t,d^{\#},d^{\#})$ does not contain surgeries and all points (q,s) contained there satisfy (i).

Proof. Both estimates are obtained from integrating the estimates for Corollary 2.36. For more details refer to Lemma 7.2 [67] or the analogous proof for G-flow in Lemma 3.21. □

Definition 2.45. *We say that a point (p_0,t_0) lies at the centre of a (ϵ,k,L,θ)-shrinking curvature neck, if after setting $r_0 = \hat{r}_0(p_0,t_0)$ and $\mathcal{B}_0 = B_{g(t_0)}(p_0, r_0 L)$, the following properties hold:*

(i) The parabolic neighbourhood $\hat{\mathcal{P}}(p_0,t_0,L,\theta)$ does not contain surgeries;

(ii) For every $t \in [t_0 - r_0^2\theta+, t_0]$, the region $\mathcal{B}_0$, w.r.t. the immersion $F(\cdot,t)$ multiplied by the scaling factor $\rho(r_0, t-t_0)^{-1}$, is ϵ-cylindrical and (ϵ,k)-parallel at every point.

The notation $t_0 - r_0^2\theta+$ means the limit from the right where $t_0 - r_0^2\theta$ is a surgery time. Definition 2.45 says that at any point of $\mathcal{P}(p_0,t_0,r_0L,r_0^2\theta)$ the Weingarten operator of our surface and its spatial derivatives, up to order k, are ϵ close to that of a standard cylinder after a possible rescaling.

In order to define a flow beyond a singular time using our surgery procedure we want to show that the surface develops large curvature as the singular time is approached.

Lemma 2.46 (Lemma 7.4 [67])**.** *[Neck Detection Lemma] Let $\mathcal{M}_t, t \in [0,T)$ be a mean curvature flow with surgeries as in the previous sections, starting from an initial manifold $\mathcal{M}_0 \in \mathcal{C}(R,\alpha)$. Let $\epsilon, \theta, L > 0$, and $k \geq k_0$ be given (where $k_0 \geq 2$ is the parameter measuring the regularity of the necks where surgeries are performed). Then we can find η_0, H_0 with the following property:*

Suppose that $p_0 \in \mathcal{M}_0$ and $t_0 \in [0,T)$ are such that:

(ND1) $H(p_0,t_0) \geq H_0$, $\frac{\lambda_1(p_0,t_0)}{H(p_0,t_0)} \leq \eta_0$

(ND2) The neighbourhood $\hat{\mathcal{P}}(p_0,t_0,L,\theta)$ does not contain surgeries.

Then

(i) The neighbourhood $\hat{\mathcal{P}}(p_0,t_0,L,\theta)$ is an $(\epsilon,k_0-1,L,\theta)$-shrinking curvature neck;

(ii) The neighbourhood $\hat{\mathcal{P}}(p_0,t_0,L-1,\theta/2)$ is an $(\epsilon,k,L-1,\theta/2)$-shrinking curvature neck.

With constants $\eta_0(\alpha, \epsilon, k, L, \theta)$ and $H_0 = h_0 R^{-1}$ where h_0 depends on α, ϵ, k, L and θ.

Proof. We argue by contradiction based on a rescaling procedure. To prove (i), we assume that for some ϵ, L, θ the conclusion is not true, no matter how we choose η_0, H_0. Then we can find a sequence $\{\mathcal{M}_t^j\}$ of solutions to the flow, a sequence of times t_j, and a sequence of points $p_j \in \mathcal{M}^j$, such that by setting $\hat{r}_j = \frac{n-1}{H_j}$ we have

(a) Each flow starts from a manifold belonging to the same class $\mathcal{C}(R, \alpha)$, and therefore satisfies the estimates of the previous sections with the same constants;

(b) The parabolic neighbourhood $\mathcal{P}^j(p_j, t_j, \hat{r}_j L, \hat{r}_j^2 \theta)$ is not changed by surgeries;

(c) $H_j \to \infty$, $\lambda_{1,j} H_j \to 0$ as $j \to \infty$;

(d) (p_j, t_j) does not lie at the centre of an $(\epsilon, k_0 - 1, L, \theta)$-shrinking neck.

We then perform a parabolic rescaling of each flow such that the $H(p_j, t_j) = n - 1$ and translate the point to the origin and t_j becomes 0. We know by (b) that such a neighbourhood contains no surgeries, and the aim is to show that the restrictions of the rescaled flows converge, up to a subsequence, to a limit flow which is a portion of the shrinking cylinder. This will yield the contradiction. Part (ii) is proved similarly. For the whole proof refer to [67]. $\square$

Remark 2.47 (Remark 7.5 [67]). *Lemma 2.46(i) concerns the whole parabolic neighbourhood which is surgery free, but can be arbitrarily close to surgery, the points of the neighbourhood are even allowed to be modified by a surgery at the initial time $t_0 - \theta r_0^2$. Therefore the description goes up to $k_0 - 1$ derivatives. Part (ii) is concerned with a smaller parabolic neighbourhood, where we can apply interior parabolic regularity and as many derivatives as we wish.*

Corollary 2.48 (Corollary 7.7 [67]). *Given $\epsilon, \theta > 0, L \geq 10$ and $k > 0$ integer, we can find $\eta_0, H_0 > 0$ such that the following holds. Let p_0, t_0 satisfy (ND1) and (ND2). Then*

(i) The point (p_0, t_0) lies at the centre of a cylindrical graph of length $2(L-2)$ and C^{k+2}-norm less than ϵ;

(ii) The point (p_0, t_0) lies at the centre of a normal $(\epsilon, k, L-2)$-hypersurface neck.

The next lemma shows us that the shrinking curvature necks obtained by Lemma 2.46 are equivalent to hypersurface necks for any given time, even surgery times.

Lemma 2.49 (Lemma 7.9 [67]). *In the Neck Detection Lemma, Lemma 2.46 we can choose the constants η_0, H_0 so that the additional following property holds. Suppose that $L \geq 10$ and that $\theta \leq d^{\#}$. Denote as usual*

$$r_0 = \frac{n-1}{H(p,t)}, \ \mathcal{B}_0 = \mathcal{B}_{g(t_0)}(p_0, r_0 L).$$

Then for any $t \in [t_0 - \theta r_0^2 +, t_0]$, the point (p_0, t_0) lies at the centre of a $(\epsilon, k_0 - 1)$-hypersurface neck $\mathcal{N}_t \subset \mathcal{B}_0$, satisfying the following properties:

(i) The mean radius $r(z)$ of every cross section of $\mathcal{N}_t$ is equal to $\rho(r_0, t - t_0)(1 + O(\epsilon))$;

(ii) The length of $\mathcal{N}_t$ is at least $L - 2$;

(iii) There exists a unit vector $\omega \in \mathbb{R}^{n+1}$ such that $|\nu(p,t) \cdot \omega| \leq \epsilon$ for any $p \in \mathcal{N}_t$.

Proof. Refer to Lemma 7.9 [67] or the analogous proof for G-flow in Lemma 3.35. □

Assumption *(ND2)* is essential in the proof of the Neck Detection Lemma. However modifications to the lemma have to be made for some cases. The next result ensures that *(ND2)* will follow from our other assumptions in the Neck Detection Lemma so long as the curvature at the point (p,t) is large compared to the curvature of regions previously modified by surgeries.

Lemma 2.50 (Lemma 7.10 [67]). *Consider a flow with surgeries satisfying the same assumptions of Lemma 2.46. Let $d^{\#}$ be as before and let ϵ, k, L, θ be given with $\theta < d^{\#}$. Then we can find η_0, H_0 with the following property. Let (p_0, t_0) be any point satisfying*

$$H(p_0,t_0) \geq \max\{H_0, 5K\}, \ \frac{\lambda_1(p_0,t_0)}{H(p_0,t_0)} \leq \eta_0$$

where K is the maximum of the curvature at the points changed in the surgeries at times before t_0. Then (p_0,t_0) satisfies hypothesis (ND2) and the conclusions (i) and (ii) of Lemma 2.46. In addition, the neighbourhood

$$\mathcal{P}\left(p_0, t_0, \frac{n-1}{H(p_0,t_0)}L, \frac{(n-1)^2}{K^2}L\right),$$

which is larger in time than (ND2) does not contain surgeries.

Proof. Refer to Lemma 7.10 [67] or the analogous proof for G-flow in Lemma 3.36. □

Definition 2.51. *We say that the parabolic neighbourhood $\mathcal{P}(p_0,t_0,r,\tau)$ is* adjacent *to a surgery region if it has not been changed by surgeries, but there exists $p \in \mathcal{M}$ such that $d_{g(t_0)}(p,p_0) = r$, and which belongs to the boundary of a region changed by a surgery at a time $s \in [t_0-\tau, t_0]$. We say that a hypersurface neck $\mathcal{N} \subset \mathcal{M}$ is bordered on one side by a disc if one of the two components of $\partial\mathcal{N}$ is also the boundary of a closed domain $\mathcal{D} \subset \mathcal{M}$, which is diffeomorphic to a disc and has no interior points in common with $\mathcal{N}$.*

[Section 7, [67]] For the next result we assume that our flow with surgeries satisfies certain properties, which we will list below:

(s1) Pick a fixed value $K^* > 2H^{\#}$, all surgeries will take place at cross-sections Σ_{z_0} of normal necks with radius $r(z_0) = r^* = \frac{(n-1)}{K^*}$.

(s2) On normal necks where the surgery has taken place we will have two portions with the following properties. One of the portions will belong to a component which will be discarded after the surgery. On the other portion, the part of the neck which has been left unchanged by the surgery has the following structure: the cross section which coincides with the boundary of the region changed by surgery satisfies $r(z) \leq (11/10)r^*$, on the last section $r(z) \geq 2r^*$ and in the sections in between $r^* \leq r(z) \leq 2r^*$.

(s3) Surgery is responsible for removing regions with curvature larger than $10K^*$. For example, looking back at a previous surgery, we will find the components which were discarded to have curvature larger than $10K^*$, so if they surgery had not taken place it would have not been disconnected from the surface.

Combining (s1) and (s3) tells us that the regions with largest curvature are the discarded components with known topology and not ones removed by the surgery construction. Property (s2) guarantees that the surgery procedure takes place a certain distance away from the end of the neck such that there is a portion of the neck left which has a radius twice as large. This leftover part will be necessary for the next lemma.

Lemma 2.52 (Lemma 7.12 [67]). *Consider a flow with surgeries satisfying our usual assumptions, and in addition properties (s1)-(s3) above. Let $L, \theta > 0$ be such that $\theta \leq d^{\#}$, where $d^{\#}$ is as defined previously, and that $L \geq 20$. Then there exist η_0, H_0 such that the following property holds. Let (p_0, t_0) satisfy properties (ND1),(ND2). Suppose in addition that the parabolic neighbourhood $\hat{\mathcal{P}}(p_0, t_0, L, \theta)$ is adjacent to a surgery region. Then (p_0, t_0) lies at the centre of a hypersurface neck $\mathcal{N}$ of length at least $L - 3$, which is bordered on one side by a disc $\mathcal{D}$. The mean curvature on $\mathcal{N} \cup \mathcal{D}$ at time $t_0 < 5K^*$, where K^* is as defined in (s1).*

Proof. Refer to Lemma 7.12 [67] or the analogous proof for G-flow in Lemma 3.37. □

Just like we dealt with the special case that (*ND2*) does not hold we will also have to deal with the special case that (*ND1*) does not hold. In this case we require a result for when the point under consideration $\frac{\lambda_1}{G}$ may not be small. This is a general property of hypersurfaces and not related to geometric flows, so the proof is exactly as in [67].

Theorem 2.53 (Theorem 7.14 [67]). *Let $F : \mathcal{M} \to \mathbb{R}^{n+1}$, with $n > 1$, be a smooth connected immersed hypersurface (not necessarily closed). Suppose that there exist $c^{\#}, H^{\#} > 0$ such that $|\nabla H(p)| \leq c^{\#} H^2(p)$ for all $p \in \mathcal{M}$ such that $H(p) \geq H^{\#}$. Then for any $\eta_0 > 0$ we can find $\alpha_0 > 0$ and $\gamma_0 > 1$ depending on $c^{\#}$ and η_0 such that the following holds. Let $p \in \mathcal{M}$ satisfy $\lambda_1(p) > \eta_0 H(p)$ and $H(p) \geq \gamma_0 H^{\#}$. Then either $\mathcal{M}$ is closed with $\lambda_1 > \eta_0 H > 0$ everywhere, or there exists a point $q \in \mathcal{M}$ such that*

(i) $\lambda_1(q) \leq \eta_0 H(q)$,

(ii) $d(p, q) \leq \frac{\alpha_0}{H(p)}$,

(iii) $H(q') \geq H(p)\gamma_0$ *for all* $q' \in \mathcal{M}$ *such that* $d(p, q') \leq \frac{\alpha_0}{H(p)}$*; in particular,* $H(q) \geq \frac{H(p)}{\gamma_0}$.

Proof. Refer to Theorem 7.14 [67] or the analogous proof for G-flow in Theorem 3.38. □

To conclude this section we can state a result about the existence of necks before the first singular time is approached.

Corollary 2.54 (Corollary 7.15 [67]). *Let $\mathcal{M}_t$ be a smooth mean curvature flow of two-convex hypersurfaces. Given neck parameters ϵ, k, L, there exists H^* (depending on initial data) such that if $H_{\max}(t_0) \geq H^*$, then the hypersurface at time t_0 either contains an (ϵ, k, L)-hypersurface neck or it is convex.*

Proof. We combine Corollary 2.48 and Theorem 2.53. Since we assume the flow is smooth, the parabolic neighbourhood in hypothesis *(ND2)* trivially does not contain surgeries. □

There are more results from Section 7 of[67], which are essential in proving the Neck Continuation Theorem stated in the next part. However, the theorems and proofs are omitted as a similar proof will be covered for the Brendle-Huisken G-flow introduced in Chapter 3.

2.1.7 The Flow with Surgeries

Theorem 2.55 (Theorem 8.1)**.** *Let $\mathcal{M}_0 \in C(R,\alpha)$ be a smooth closed two-convex hypersurface immersed in $\mathbb{R}^{n+1}$, with $n \geq 3$, satisfying $|A|^2 \leq R^{-2}$. Then there exist constants $H_1 < H_2 < H_3$ and a mean curvature flow with surgeries starting from $\mathcal{M}_0$ with the following properties:*

- *Each surgery takes place at a time T_i such that $H_{\max}(T_i-) = H_3$.*
- *After the surgery, all the components of the manifold satisfy $H_{\max}(T_i+) \leq H_2$, except for those diffeomorphic to spheres or to $\mathbb{S}^{n+1} \times \mathbb{S}^1$, which are neglected afterwards.*
- *Each surgery starts from a cross section of a normal hypersurface neck with mean radius $r(z_o) = \frac{n-1}{H}$.*
- *The flow with surgeries terminates after finitely many steps.*

The constants H_i can be any values such that $H_1 \geq \omega_1 R^{-1}$, $H_2 = \omega_2 H_1$ and $H_3 = \omega_3 H2$, with $\omega_i > 1$ depending only on the parameter α.

Proof. Refer to Theorem 8.1 [67] or the analogous proof for G-flow in Theorem 3.45. □

[Section 8, [67]] In proving the Theorem 2.55, we will define the surgery algorithm such that the following properties are satisfied:

(S) Each surgery is performed on a normal (ϵ_0, k_0)-hypersurface neck. The surgery is performed at times T_i such that $H_{\max}(T_i) = H_3$. After the surgeries are performed, and we remove suitable components whose topology is known and we are left with $H_{\max}(T_i+) \leq H_2$. In addition, all surgeries satisfy properties (s1)-(s3) with $K^* = H_1$.

We state without proof the Neck Continuation Theorem which is required to prove the Theorem 2.55. The proof relies heavily on the results of Section 7 of [67]. For a detailed proof refer to [67].

Theorem 2.56 (Theorem 8.2 [67])**.** *[Neck Continuation Theorem] Suppose that $\mathcal{M}_t$ with $t \in [0, t_0]$, is mean curvature flow with surgeries satisfying (S), and let $\max_{\mathcal{M}_{t_0}} H \geq H_3$. Moreover let p_0 be such that*

$$H(p_0, t_0) \geq 10H_1, \quad \lambda_1(p_0, t_0) < \eta_1 H(p_0, t_0), \tag{2.57}$$

where η_1, H_1 are as defined in (P0)-(P7), [Section 8, [67]] (these conditions will also be stated in Chapter 3 for the g-flow case). Then (p_0, t_0) lies on some (ϵ_0, k_0)-hypersurface neck N_0 in normal form, which either covers the whole component of $\mathcal{M}_{t_0}$ including p_0 or has a boundary consisting of two cross-sections Σ_1, Σ_2, each of which satisfies either of the two following properties:

(i) Σ has mean radius $\frac{2(n-1)}{H}$

(ii) The cross-section of Σ is the boundary of a region D, diffeomorphic to a disc where the curvature is at least H/Θ. The region D lies after the cross-section Σ and is disjoint from N_0.

Proof. Refer to Theorem 8.2 [67] or the analogous proof for G-flow in Theorem 3.49. □

Huisken and Sinestrari are then able to prove Theorem 2.55. Again a proof of this is omitted as a similar proof will be presented in Chapter 3 for the Brendle-Huisken G-flow.

2.2 Reconciliation Between the Flow with Surgeries and the Weak Solution

2.2.1 Level-Set Mean Curvature Flow

Here we study the level-set formulation for mean curvature flow as described by Evans and Spruck in [36].

We begin by considering a smooth function $u = u(x,t)$ such that $Du = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right)$ does not vanish on some open subset of $\mathbb{R}^n \times [0,T)$.

Assume further that each level-set of u smoothly evolves according to mean curvature flow. We focus our attention on any one such level-set, and consider its zero sets given by

$$\Gamma_t = \{x \in \mathbb{R}^n \mid u(x,t) = 0\}$$

Then the mean curvature vector field is given by $\mathrm{div}(\nu)\nu$ and the point x evolves according to the ODE:

$$\begin{cases} \dot{x} = [-\mathrm{div}(\nu)\nu](x(s),s)\ (s > t) \\ x(t) = x \end{cases} \tag{2.58}$$

Fixing $s \in [0,T)$, we know that since $x(s) \in \Gamma_s$ for $s \geq t$, $\tilde{u}(x,s) = 0$ for all $s > t$ and so

$$\begin{aligned} 0 &= \frac{d}{ds}\tilde{u}(x(s),s) && (2.59)\\ &= -[(D(\tilde{u})\nu)\mathrm{div}(\nu)](x(s),s) + \tilde{u}_t(x(s),s). && (2.60) \end{aligned}$$

Setting $s = t$ we obtain

$$u_t = (D\tilde{u} \cdot \nu)\mathrm{div}(\nu)$$

at (x,t). Choosing $\nu \equiv \frac{D\tilde{u}}{|D\tilde{u}|}$ we get

$$\tilde{u}_t = |D\tilde{u}|\mathrm{div}\left(\frac{D\tilde{u}}{|D\tilde{u}|}\right)$$

.

Then we have $\tilde{u}(x,t) = 0$ is a level surface of dimension n.

$$(x_1, \ldots, x_{n+1}, t) \to (x_1, \ldots, x_{n+1}, \tilde{u}(x_1, \ldots, x_{n+1}, t))$$

. We wish to change $\tilde{u}(x_1, \ldots, x_{n+1}, t)$ to $u(x_1 \ldots, x_{n+1}) - t = 0$. Looking at the mapping $(x_1, \ldots, x_{n+1}, t) \to (x_1, \ldots, x_{n+1}, \tilde{u})$ the Jacobian is

$$\begin{bmatrix} \frac{d\tilde{u}}{dt} \neq 0 & A \\ 0 & I_{n+1} \end{bmatrix}$$

$\Rightarrow \quad t = \phi(x_1, \ldots, x_n, \tilde{u})$. But taking the level-set $\tilde{u} = 0 \Rightarrow \quad t = \phi(x_1, \ldots, x_{n+1})$. So $\Gamma = \{x \in \mathbb{R}^{n+1} | \tilde{u}(x,t) = u(x) - t = 0\}$.

It remains only to check that $u(x,t) - t$ still satisfies the mean curvature flow equation. This is the same calculation as before with $u_t = -1$.

Therefore rearranging we will obtain the mean curvature flow equation for u:

$$\begin{cases} \operatorname{div}\left(\frac{Du}{|Du|}\right) = -\frac{1}{|Du|} \\ u|_{\partial\Omega} = 0 \end{cases} \tag{2.61}$$

Expressing this in terms of our coordinates we can calculate

$$\begin{aligned} |Du|\operatorname{div}\left(\frac{Du}{|Du|}\right) &= |Du|\left(\left\langle \nabla, \frac{\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n}\right)}{\sqrt{\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \dots + \left(\frac{\partial u}{\partial x_n}\right)^2}} \right\rangle\right) \\ &= |Du|\left(\frac{\partial}{\partial x_1}\left(\frac{\frac{\partial u}{\partial x_1}}{|Du|}\right) + \frac{\partial}{\partial x_2}\left(\frac{\frac{\partial u}{\partial x_2}}{|Du|}\right) + \dots + \frac{\partial}{\partial x_n}\left(\frac{\frac{\partial u}{\partial x_n}}{|Du|}\right)\right) \end{aligned}$$

Now

$$\begin{aligned} &\frac{\partial}{\partial x_i}\left(\frac{\frac{\partial u}{\partial x_i}}{\sqrt{\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \dots + \left(\frac{\partial u}{\partial x_n}\right)^2}}\right) \\ &= \frac{\frac{\partial^2 u}{\partial x_i^2}}{\sqrt{\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \dots + \left(\frac{\partial u}{\partial x_n}\right)^2}} - \frac{\frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}}{\sqrt{\left(\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \dots + \left(\frac{\partial u}{\partial x_n}\right)^2\right)^3}} \end{aligned}$$

Therefore

$$-1 = |Du|\operatorname{div}\left(\frac{Du}{|Du|}\right) = \left(\delta_{ij} - \frac{\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}}{|Du|^2}\right)\frac{\partial^2 u}{\partial x_i \partial x_j} \tag{2.62}$$

Conversely assume that u is a solution of (2.61) or equivalently (2.62). Fixing $t > 0$, we look at the ODE (2.58). Differentiating u w.r.t s and using $\nu \equiv \frac{Du}{|Du|}$, we obtain (2.60), and since u solves (2.61), we deduce that $u(x(s), s) = 0$, the zero sets evolve by their mean curvature. Similarly, the level-sets of u will evolve according to their mean curvatures.

In Section 3 of [36], Spruck and Evans were able to prove uniqueness of a weak solution and in Section 4 they proved existence. Please refer to [36] for more details.

The following definition will be useful later when we go through Head's method on showing that the mean curvature flow with surgeries converges to the weak solution.

Definition 2.63. *Given $u \in C^{0,1}(\bar{\Omega})$ such that $|Du|^{-1} \in L^1(\Omega)$, $u > 0$ on Ω, and $\{u = 0\} = \partial\Omega$, we say that u is a weak solution of 2.61 on Ω if*

$$\int_\Omega \left(|Du| - \frac{u}{|Du|}\right) dx \le \int_\Omega \left(|Dv| - \frac{v}{|Du|}\right) dx = J_u(v).$$

for any Lipchitz continuous function v on Ω such that $\{u \neq v\} \subset\subset \Omega$.

To give some justification for this definition we take a small perturbation $v = u + \epsilon w$ for $\epsilon \ll 1$ and take the derivative $\frac{d}{d\epsilon} J_u(v)|_{\epsilon=0}$

$$\frac{d}{d\epsilon}\left(|Dv| - \frac{v}{|Du|}\right)\Big|_{\epsilon=0} = \left(\frac{Dv(\frac{d}{d\epsilon}(Dv))}{|Dv|} - \frac{w|Du|}{|Du|^2}\right)\Bigg|_{\epsilon=0}$$
$$= \left(\frac{(Du)(Dw)}{|Dv|} - \frac{w}{|Du|^2}\right)\Bigg|_{\epsilon=0}$$
$$= \frac{(Du)(Dw)}{|Du|} - \frac{w}{|Du|^2}$$

This gives us

$$\frac{d}{d\epsilon}J_u(v)|_{\epsilon=0} = \int_\Omega = \frac{(Du)(Dw)}{|Du|}dx - \int_\Omega \frac{w}{|Du|^2}dx$$
$$= -\int_\Omega wD(\frac{Du}{|Du|})dx - \int_\Omega \frac{w}{|Du|}dx \text{ using IBP}$$
$$\Rightarrow D(\frac{Du}{|Du|}) + \frac{1}{|Du|} = 0$$
$$\Rightarrow D(\frac{Du}{|Du|}) = -\frac{1}{|Du|} = 0$$

Remark 2.64. *Why is it true that for a vector field X we have*

$$\int_\Omega XDw\, dV_g = -\int_\Omega div(X)w\, dV_g$$

Since D is over a manifold we need to change from $\nabla_{\frac{\partial F}{\partial x^i}}X$ to $\nabla_{\frac{\partial}{\partial x^i}}X$.

Taking a compactly supported partition of unity ρ_j such that $\sum_{j=1}^m \rho_j = 1$ and using normal coordinates such that dV_g doesn't contribute curvature terms we just apply Stokes Theorem which says

$$\int_\Omega d\alpha = \int_{\partial\Omega} \alpha = 0$$

since $\partial\Omega$ is empty. So we have

$$\int_\Omega XDwdV_g$$
$$= \int_\Omega \sum_{j=1}^m \rho_j \sum_{i=1}^n X_i \frac{\partial W}{\partial x^i} dV_g$$
$$= -\int_\Omega \sum_{j=1}^m \frac{\partial \rho_j}{\partial x_i} \sum_{i=1}^n X_i w\, dV_g + \int_\Omega \sum_{j=1}^m \rho_j \sum_{i=1}^n \frac{\partial X_i}{\partial x^i} w\, dV_g$$
$$= -\int_\Omega div(X)w\, dV_g$$

since $\sum \frac{\partial \rho_j}{\partial x^i} = 0$.

From now on we denote the weak solution of the level-set flow by u_L and define

$$\Gamma_t = \begin{cases} \partial\{x \in \Omega | u_L > t\} & \text{for all } t \leq T \\ \emptyset & \text{for all } t > T \end{cases}$$

to be the t-slices of u_L.

For more information on how to construct a weak solution to the mean curvature equation, refer to Section 2 of [36].

Theorem 2.65 ([36])**.** *[Properties of Weak Solutions] Let $\Omega \subset \mathbb{R}^{n+1}$ be open and bounded such that $\partial\Omega$ has nonnegative mean curvature flow. Then there exists a unique weak solution u_L of 2.61 such that*

(i) Γ_t agrees with the smooth solution $\mathcal{M}_t$ of mean curvature flow starting from $\mathcal{M}_0 = \partial\Omega$ if and as long as the latter exists, and

(ii) if $\mathcal{M}_t$, $t_1 \leq t \leq t_2$, is any smooth, compact mean curvature flow with positive mean curvature then

$$\mathcal{M}_{t_1} \cap \Gamma_{t_1} = \emptyset \;\Rightarrow\; \mathcal{M}_t \cap \Gamma_t = \emptyset \text{ for all } t_1 \leq t \leq t_2.$$

It is easy to see that (ii) implies $\frac{d}{dt}\text{dist}(\mathcal{M}_t, \Gamma_t) = 0$.

Definition 2.66. *$U \subset \mathbb{R}^{n+1}$ is an open set. $E \subset \mathbb{R}^{n+1}$ is* outward minimising *in U if for the reduced boundary $\partial^* E$ and $\partial^* F$ (refer to Chapter 15 [76]) we have*

$$|\partial^* E \cap K| \leq |\partial^* F \cap K| \tag{2.67}$$

or any $F \supset E$ such that $F \setminus E$ is relatively compact in U and any compact set $K \supset (F \setminus E)$.

Remark 2.68. *In the future we will use $A \subset\subset B$ to denote a set A which is relatively compact in B.*

Proposition 2.69 ([94])**.** *[Outward Minimising] Let $\Omega \subset \mathbb{R}^{n+1}$ be open and bounded and suppose $\partial\Omega$ has non-negative mean curvature. Then the sets $\Omega_t = \{u_L > t\}$ enclosed by the level-sets of the weak solution of mean curvature flow generated by Ω are outward minimising in Ω.*

Proof. For a proof or more information refer to [94]. □

2.2.2 Head's Method

As described at the beginning of this chapter there are two well-known solutions to the mean curvature evolution of a smooth, closed, two-convex hypersurfaces in Euclidean space, the Huisken-Sinestrari surgery algorithm and the weak solution of the level-set flow. Head and Lauer were able to provide a reconciliation between these methods in [55] and [71]. In this section we go through Head's method which rely on geometric estimates for certain L^p-norms of the mean curvature. The results and proofs of this section are as in [55] and direct references are given for these.

The problem with the Huisken-Sinestrari surgery algorithm is that it relies on a non-canonical modification of the surface at each surgery time. As we saw in the previous section it is controlled by a set of parameters H_1, H_2, H_3, which determine when the surgery occurs.

We stop when $H_{\max} = H_3$ and perform the surgery so that the curvature drops by some fixed amount to H_2.

Since these surgery parameters are not unique, Head looked at an increasing sequence of parameters $\{H_1^i, H_2^i, H_3^i\}$ for which the surgery times grow and the necks being modified become increasingly thin. As these values of H increase more surgeries are required and we want to make sure that only finitely many are needed and we are not faced with an accumulation of infinitely many surgeries to perform.

Head was able to show that as we take the limit $H_1^i, H_2^i, H_3^i \to \infty$ that the surgery construction and weak solution agree in a precise quantitative sense. He did this by combining his geometric estimates for L^p-norms, a geometric barrier argument and Brakke's clearing out Lemma.

Integral Estimates for smooth Mean Curvature Flow

Now we may use the evolution equations found in Lemma 1.18(v),(iii) as well as the Cylindrical Estimate from Theorem 2.32 to compute the following,

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{M}_t} H^p d\mu =& p\int_{\mathcal{M}_t} H^{p-1}\frac{\partial H}{\partial t} + \int_{\mathcal{M}_t} H^p \frac{\partial d\mu}{\partial t} \\
=& \int_{\mathcal{M}_t} pH^{p-1}(\Delta H + |A|^2 H) - H^{p-2} d\mu \\
=& -p(p-1)\int_{\mathcal{M}_t} |\nabla H|^2 H^{p-2} d\mu + \int_{\mathcal{M}_t} H^p(p|A|^2 - H^2) d\mu \\
\leq& -p(p-1)\int_{\mathcal{M}_t} |\nabla H|^2 H^{p-2} d\mu + \left(\rho\eta - \frac{n-1-p}{n-1}\right)\int_{\mathcal{M}_t} H^{p+2} d\mu \\
&+ pC_\eta R^{-2}\int_{\mathcal{M}_t} H^p d\mu.
\end{aligned}$$

We restrict our attention to $p < n-1$. More formally, let $\epsilon > 0$ and fix $p = n-1-\epsilon$. We then choose an appropriate $\eta_\epsilon = \frac{\epsilon}{2(n-1)(n-1-\epsilon)}$. Henceforth we write C_η in place of C_{η_ϵ}. If our ϵ is not small enough to satisfy $\eta < \tilde{\eta}$ we instead take

$$\eta = \min\left\{\frac{\epsilon}{2(n-1)(n-1-\epsilon)}, \frac{\tilde{\eta}}{2}\right\}$$

in the cylindrical estimate from Theorem 2.32.

We obtain

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathcal{M}_t} H^p d\mu \leq & -p(p-1)\int_{\mathcal{M}_t} |\nabla H|^2 H^{p-2} d\mu - \frac{\epsilon}{2(n-1)}\int_{\mathcal{M}_t} H^{p+2} d\mu \\
& + pC_\epsilon R^{-2}\int_{\mathcal{M}_t} H^p d\mu \\
\leq & -p(p-1)\int_{\mathcal{M}_t} |\nabla H|^2 H^{p-2} d\mu - \frac{\epsilon}{2(n-1)}\left(\int_{\mathcal{M}_t} H^p d\mu\right)^{\frac{p-2}{p}} |\mathcal{M}_t|^{-\frac{2}{p}} \\
& + pC_\epsilon R^{-2}\int_{\mathcal{M}_t} H^p d\mu \qquad \text{(Hölder's inequality)} \\
\leq & -p(p-1)\int_{\mathcal{M}_t} |\nabla H|^2 H^{p-2} d\mu \\
& - \frac{\epsilon}{2(n-1)}\alpha_2^{\frac{2}{p}} R^{\frac{-2n}{p}}\left(\int_{\mathcal{M}_t} H^p d\mu\right)^{\frac{p-2}{p}} + pC_\epsilon R^{-2}\int_{\mathcal{M}_t} H^p d\mu
\end{aligned}
$$

where we have used the definition of class $|\mathcal{M}_0| \leq \alpha_2 R^{-2}$. Let

$$\varphi = \exp\left(-\frac{pC_\epsilon}{R^2}t\right)\int_{\mathcal{M}_t} H^p d\mu.$$

We have now proved φ is non-increasing under the smooth evolution in the two-convex setting for all $p < n-1$.

Lemma 2.70. *Let $\mathcal{M}_t$ be a smooth solution of mean curvature flow starting from $\mathcal{M}_0 \in C(R,\alpha)$ and fix $p = n-1-\epsilon$. Then there exists a constant $C_\epsilon = C_\epsilon$ depending on $(\mathcal{M}_0)$ such that*

$$
\begin{aligned}
\frac{d}{dt}\varphi \leq & -p(p-1)\exp\left(-\frac{pC_\epsilon}{R^2}t\right)\int_{\mathcal{M}_t} |\nabla H|^2 H^{p-2} d\mu \\
& - \frac{\epsilon}{2(n-1)}\alpha_2^{\frac{2}{p}} R^{-\frac{2n}{p}}\exp\left(-\frac{pC_\epsilon}{R^2}t\right)\exp\left(-\frac{pC_\epsilon}{R^2}t\right)\left(\int_{\mathcal{M}_t} H^p d\mu\right)^{\frac{p+2}{p}}.
\end{aligned}
$$

for all $\epsilon > 0$ as long as the solution remains smooth.

Hence and L^p-norm of the mean curvature is bounded under smooth mean curvature flow on any finite time interval for all $p < n-1$. In fact, solving the ODE

$$\frac{d}{dt}\phi \leq -\frac{\epsilon}{2(n-1)}\alpha_2^{-\frac{2}{p}} R^{\frac{-2n}{p}}\exp\left(\frac{2C_\epsilon}{R^2}t\right)\phi^{\frac{p+2}{p}}$$

we conclude that

$$\phi \leq \alpha_2 R^{n-p}\left(\frac{\epsilon}{2(n-1)pC_\epsilon}\right)\left(\exp\left(\frac{2C_\epsilon}{R^2}t\right) - 1\right)^{-\frac{p}{2}}$$

.

We have arrived at an L^p-estimate for the mean curvature flow which will behave like $t^{-\frac{1}{2}}$ for small values of t.

Theorem 2.71 (Proposition 2.4 [55]). *[Smooth L^p-estimate] Let $\mathcal{M}_t$ be a smooth solution of mean curvature flow starting from $\mathcal{M}_0 \in C(R, \alpha)$ and set $p = n-1-\epsilon$. Then there exists a constant $C_\epsilon = C_\epsilon$ depending on $(\mathcal{M}_0)$ such that*

$$||H||_{L^p(\mathcal{M}_t)} \leq \alpha^{\frac{1}{p}} R^{\frac{n-p}{p}} \exp\left(\frac{C_\epsilon}{R^2} t\right) \left(\frac{\epsilon}{2(n-1)pC_\epsilon} (\exp\left(\frac{2C_\epsilon}{R^2} t\right) - 1)\right)^{-\frac{1}{2}}$$

for all $\epsilon > 0$ and for al $t > 0$ as long as the solution remains smooth.

Integral Estimates for Mean Curvature Flow with Surgeries

We will now devote our attention into obtaining an L^p-estimate across surgery and combine it with the above Theorem to obtain one for mean curvature flow with surgeries.

Lemma 2.72 (Lemma 3.4 [55]). *[L^p-estimate Across Surgery] For each $p \geq 0$ the following property holds. We can choose L depending on (n) sufficiently large such that*

$$\int_{\mathcal{M}^-} H^p d\mu - \int_{\mathcal{M}^+} H^p d\mu \geq C(r_0)^{n-p}$$

where C depending on n and L, $\mathcal{M}^+$ denotes the hypersurface obtained from $\mathcal{M}^-$ after performing standard surgery, and r_0 is the mean radius.

Proof. Let $N^- : S^{n-1} \times [0, L] \to \mathcal{M}^- \to \mathbb{R}^{n+1}$ be an (ϵ, k)-hypersurface neck with mean radius r_0 in normal form. Choosing ϵ sufficiently small we can arrange that

$$H(p) \geq \tfrac{9}{10}(\tfrac{n-1}{r_0}) \text{ for all } p = (\omega, z) \in N^- \text{ such that } z \in [0, L].$$

Let $U^- \subset \mathcal{M}^-$ be the subset of $\mathcal{M}^-$ altered by the given surgery and $U^+ \subset \mathcal{M}^+$ replacing U^-. We then estimate $|U^-| \geq (9/10)\ L\ \omega_{n-1}\ (r_0)^n$, where ω_{n-1} is the area of the standard unit $(n-1)$-sphere.

Remark 2.73. *The $\frac{9}{10}$ estimate arises from Proposition 3.4 in [67]. As will the $\frac{11}{10}$ further on.*

Using the estimate for $|U^-|$ we have

$$\int_{U^-} H^p d\mu \geq C_1\ L\ r_0^{n-p}$$

for some C_1 depending only on n. Without loss of generality we focus on the left hand side of the neck. As described in the surgery procedure we pinch the neck on the interval $[\lambda, 3\lambda]$ and attach a convex cap on $[3\lambda, 4\lambda]$. We choose our parameter τ so that in the surgery construction our curvature remains close to that of the cylinder on $[\lambda, 3\lambda]$, such that

$$\tfrac{9}{10}(\tfrac{n-1}{r_0}) \leq H(p) \leq \tfrac{11}{10}(\tfrac{n-1}{r_0}) \text{ for all } p = (\omega, z) \in N^+ \text{ such that } z \in [\lambda, 3\lambda].$$

Similarly the curvature of the convex cap attached can be made as close as we like to that of the standard sphere:

$$\tfrac{9}{10}(\tfrac{n}{r_0}) \leq H(p) \leq \tfrac{11}{10}(\tfrac{n}{r_0}) \text{ for all } p = (\omega, z) \in N^+ \text{ such that } z \in [3\lambda, 4\lambda].$$

We apply the same analysis to the right hand side of the neck. We decompose U^+ such that U_1^+ denotes the bent cylinder and U_2^+ denotes the convex cap attached to $N_{3\lambda}$. We can rearrange to obtain $\frac{9}{5}\ \Lambda\ \omega_{n-1}\ (r_0)^n \leq |U_1^+| \leq \frac{11}{5}\ \Lambda\ \omega_{n-1}\ (r_0)^n$ and

$$C_2\Lambda(r_0)^{n-p} \leq \int_{U_1^+} H^p d\mu \leq C_3\Lambda(r_0)^{n-p}$$

for constants C_2 and C_3 depending on n. Finally we can modify the capping off such that $\left(\frac{9}{10}\right)\omega_n\ (r_0)^n \leq |U_2^+| \leq \left(\frac{11}{10}\right)\omega_n\ (r_0)^n$ and

$$C_4(r_0)^{n-p} \leq \int_{U_2^+} H^p d\mu \leq C_5 r_0^{n-p}.$$

for constants C_4 and C_5 depending only on n. Making an appropriate choice for Λ depending only on n we can obtain $L = C + 8\Lambda \geq 20 + 8\Lambda$ depending only on n. It is chosen sufficiently large such that for each $p \geq 0$ we have

$$\int_{U^-} H^p d\mu \geq 2\int_{U^+} H^p d\mu.$$

This completes the proof. □

Theorem 2.74 (Theorem 3.6 [55]). *[L^p-estimate for Flow with Surgeries] We can choose L depending only on n, sufficiently large such that the following property holds. Let $\mathcal{M}_0 \in C(R,\alpha)$ with $n \geq 3$ and fix $p = n - 1 - \epsilon$. Then the solution $\mathcal{M}_t$ of mean curvature flow with surgeries starting from $\mathcal{M}_0$ satisfies*

$$\begin{aligned} C\int_{\mathcal{M}_0} H^p d\mu \geq \int_{\mathcal{M}_T} H^p d\mu + p(p-1)\int_0^T \int_{\mathcal{M}_t} |\nabla H|^2 H^{p-2} d\mu\, dt \\ + \frac{\epsilon}{2(n-1)}\int_0^T \int_{\mathcal{M}_t} H^{p+2} d\mu\, dt \end{aligned}$$

for all $\epsilon > 0$ and for all $0 < T \leq T_N < \infty$, where T_N denotes the final surgery time. The constant C depends on ϵ, T and $\mathcal{M}_0$.

Proof. The proof of Theorem 2.71 relies only on Theorem 2.32. Since the cylindrical estimate survives surgery without any modification the constants, we conclude Lemma 2.70 applies to each smooth time interval, $[0,T_1],[T_1,T_2],\ldots,[T_{N-1},T_N]$. Therefore we can integrate on each time interval and sum the contributions. Furthermore $\exp(-pC_\epsilon R^{-2}t)$ is continuous in t and from Lemma 2.72 we have

$$\int_{\mathcal{M}_{T_{j+1}^-}} H^p d\mu > \int_{\mathcal{M}_{T_{j+1}^+}} H^p d\mu$$

for each $p \geq 0$ and for all $j \geq 0$. Hence we simply disregard any contribution made by the components discarded at surgery time. This completes the proof. □

Number of Surgeries

There are two arguments that show the flow must terminate after a finite number of steps. The first combines the evolution equations for H and $d\mu$ with the two-convex inequality $|A|^2 \leq nH^2$ which yields $\frac{\partial}{\partial t} \leq \delta H + nH^3$.

Comparison with the associated ODE yields a uniform lower bound $\delta T \geq C(n,\alpha)(H_2)^{-2}$ on the time interval δT separating two consecutive surgery times. Since the mean curvature has to increase from H_2 to $H_3 = \omega_3 H_2 > H_2$ during this time, the number of surgery times satisfies the bound $N \leq C(n,\alpha)R^2(H_2)^2$. Refer to Remark 7.17 in [67] for more details.

The second argument is as follows. By the definition of a class of two-convex surfaces we know that there exist constants $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and R where $\max_{\mathcal{M}_0} |A|^2 \leq R^{-2}$, such that $|\mathcal{M}_t| \leq \alpha_2 R^n$. Therefore each time surgery is performed the area of the surface decreases by some fixed multiple of $(H_k)^{-n}$, $(k = 1, 2, 3)$. It follows that $N \leq C(H_k)^n$, for some constant C depending only on n. This is sufficient to show that mean curvature flow with surgeries must terminate after a finite number of surgery times for each finite choice of H_k.

However the estimate on the number of surgeries needed in the surgery procedure to obtain the reconciliation result between the solution to mean curvature flow with surgeries and the weak solution of level-set flow, after taking appropriate limits, needs to be bettered. Otherwise the size of his time-translation Nt_ω needed to prove the lower barrier result, seen later, blows up.

Now the second argument outlined is applied to the higher L^p-norms of the mean curvature. It follows from Theorem 2.74, the definition of class and the above remarks on the number of surgeries, that there exists a uniform constant $C > 0$ depending on n, ϵ and α such that

$$\int_{\mathcal{M}_t} H^{n-1-\epsilon} \leq CR^{1+\epsilon}$$

on $[0, T_N]$. In addition Lemma 2.72 guarantees that each surgery consumes

$$\int_{\mathcal{U}^-} H^{n-1-\epsilon} d\mu - \int_{\mathcal{U}^+} H^{n-1-\epsilon} \geq CL(H_1)^{-1-\epsilon}$$

Hence there exists a constant C depending on n, ϵ, L and $\mathcal{M}_0$ such that the number of surgeries is bounded,

$$N \leq C(H_1)^{1+\epsilon}. \tag{2.75}$$

Mean Curvature flow with surgeries and the weak solution

Let $\Omega \subset \mathbb{R}^{n+1}$ such that $\partial\Omega$ is 2-convex where $\partial\Omega = \mathcal{M}_0 \in C(R, \alpha)$. Then we have $\mathcal{M}_t, t \in [0, T_N]$, with surgery times $T_1, T_2, \ldots, T_N$ and surgery parameters $\{H_1, H_2, H_3\}$. $\mathcal{M}_t = \{x \in \Omega | u = t\}$ for $t \notin T_1, T_2, \ldots T_N$.

Consider any surgery time T_j. Let $E_{T_j^-}$ be the enclosed domain bounded by $\mathcal{M}_{T_j^-}$ and $F_{T_j^+}$ be the open set in $\mathbb{R}^{n+1}$ enclosed by $\mathcal{M}_{T_j^+}$.

$$u = \begin{cases} t & \text{for all } x \in \mathcal{M}_t \\ T_j & \text{for all } x \in E_{T_j^-} \setminus F_{T_j^+} \end{cases}$$

We define the following: $\mathcal{M}_{T_j^-} = \partial(int\{x \in \Omega | u \geq T_j\})$, $\mathcal{M}_{T_j^+} = \partial\{x \in \Omega | u > T_j\}$. It will also be convenient to define the following $\Sigma_t := \{u > t\}$ and $\tilde{\Sigma}_t := \text{int}\{u \geq t\}$. We will often have to consider the regions $\Omega_t := \{u_L > t\}$ enclosed by the level-sets $\Gamma_t = \partial\Omega_t$ of the weak solution.

Theorem 2.76 (Theorem 4.3 [55]). *[Convergence to a weak solution] Let $\mathcal{M}_0 \in C(R, \alpha)$ with $n \geq 3$ such that $\mathcal{M}_0 = \partial\Omega$ for some open, bounded $\Omega \subset \mathbb{R}^{n+1}$. Let u_L be the solution of weak level-set flow on Ω, and denote by u_i the level-set functions representing the solutions*

$\mathcal{M}_t^i$ *of mean curvature flow with surgeries starting from* $\mathcal{M}_0$ *with parameters* H_1^i, H_2^i, H_3^i. *Then for all sufficiently small* $\epsilon > 0$ *we have*

$$\sup_{\bar{\Omega}} |u_i - u_L| \leq C(H_1^i)^{-1+\epsilon}$$

where C *depends on* n, ϵ *and* $\mathcal{M}_0$.

Note: As i increases, the necks become thinner and our surgery parameter H_1 at which we do surgery increases.

Barrier Result

We want to compare the Huisken-Sinestrari mean curvature flow with surgeries procedure to the weak solution obtained using the level set method. It is clear that before the first surgery time $\mathcal{M}_t^i$ and Γ_t agree. The first step in doing so will be to show u_i is bounded from above by u_L for each i.

It is clear that at the first surgery time $\mathcal{M}^i_{T_1^+} \subset E_{T_1^-} = \bar{\Omega}_{T_1}$, but what happens for $t > T_1$.

We will use the Tearing Apart Lemma discussed in Section 4 of [34], which, as the name suggests, states that two surfaces which agree except for on some subset, must separate instantaneously under the smooth evolution.

Lemma 2.77. *Let* $W \subset \mathbb{R}^{n+1}$ *be open and bounded. Consider a subset* $\hat{W} \subset W$. *Suppose that* $\mathcal{M}_0 = \partial W$ *and* $\hat{M}_0 = \partial \hat{W}$ *are smooth and mean-convex with* $\hat{M}_0 \subset \hat{W}$ *and* $\hat{M}_0 \neq \mathcal{M}_0$. *Then the corresponding solutions* $\mathcal{M}_t$, $\hat{\mathcal{M}}_t$ *of mean curvature flow satisfy*

$$\hat{M}_t \cap \mathcal{M}_t = \emptyset$$

for $t > 0$ *as long as they remain smooth.*

Therefore $\mathcal{M}_t^i$ is trapped inside Ω_t for all $t > T_1$. This corresponds to the following global barrier result.

Lemma 2.78 (Lemma 4.4 [55]). *Let* Ω, u^i *and* u_L *be as in Theorem 2.76. Then for each* i *we have*

$$u_i(x) \leq u_L(x)$$

for all $x \in \bar{\Omega}$.

Proof. $\mathcal{M}_0^i = \Gamma_0 = \partial\Omega$ which tells us that $\mathcal{M}_\delta^i \subset\subset \Omega$ for all $\delta > 0$. Therefore $\frac{d}{dt} dist(\mathcal{M}^i_{t+\delta}, \Gamma_t) \geq 0$ as long as $\mathcal{M}^i_{t+\delta}$ remains smooth. However this is clearly preserved by the Huisken-Sinestrari surgery construction.

Now recall the definition of a solid tube as in Proposition 2.23. Each standard surgery is performed on an (ϵ, k)-hypersurface neck N_0 of length L which encloses a solid tube $G_0 : \bar{B}_1^n \times [0, L] \to \mathbb{R}^{n+1}$. Denote the two regions diffeomorphic to discs introduced by each standard surgery by U^+. Then by construction it follows that $U^+ \subset G_0(\bar{B}_1^n \times [0, L])$ so it follows:

$$\begin{aligned} &\Rightarrow \mathcal{M}^i_{T_j^+} \subset E_{T_j^-} \\ &\Rightarrow \mathcal{M}^i_{T_j^- + \delta} \subset\subset \Omega_{T_j} \\ &\Rightarrow \mathcal{M}^i_{T_j^+ + \delta} \subset\subset \Omega_{T_j} \end{aligned}$$

In fact $\operatorname{dist}(\mathcal{M}^i_{t+\delta}, \Gamma_t)$ is non-decreasing across each surgery time T_j.
And

$$\mathcal{M}^i_{t+\delta} \cap \Gamma_t = \emptyset \ \Rightarrow \ \mathcal{M}^i_{t+\delta} \subset\subset \Omega_t \ \forall \, t > 0.$$

Since there are only finitely many surgeries and u_i and u_L are both continuous, the result follows. ☐

Time shifting the weak solution

We have established that $u_i \leq u_L$ on $\bar{\Omega}$ for each i. Now we want to translate u_L vertically in time until it sits below u_i. We will then use the Clearing Out Lemma [26] to obtain the necessary result.

As usual let $\partial\Omega = \mathcal{M}_0 \in C(R, \alpha)$ and the flow with surgeries $\mathcal{M}^i_t$ for a given choice of surgery parameters.

At the first surgery time, T_1^-, $\mathcal{M}^i_t$ agrees with the classical and weak solutions. The idea is that we "freeze" $\mathcal{M}^i_{T_1^-}$ and run the weak solution a little longer until

$$\Gamma_{T_1+t_\omega} \subset\subset \Sigma_{T_1}$$

This allows the weak solution to vacate the regions modified by surgery. This must happen for some constant t_ω due to the 2-convexity assumption.

Then show that t_ω can be controlled explicitly in terms of the surgery parameters. Next we perform surgery on $\mathcal{M}^i_{T_1^-}$ after which

$$\mathcal{M}^i_{T_1^+} \cap \Gamma_{T_1+t_\omega} = \emptyset$$

and restart both evolutions. Now suppose that at any surgery time T_j we have

$$\mathcal{M}^i_{T_j^+} \cap \Gamma_{T_j+jt_\omega} \quad \text{and} \quad \Gamma_{T_j+jt_\omega} \subset\subset \Sigma_{T_j}$$

then the avoidance principle guarantees

$$\mathcal{M}^i_t \cap \Gamma_{t+jt_\omega} \emptyset \text{ on } [T_j^+, T_{j+1}^-] \text{ by avoidance.}$$

We will not need to keep track on the distance between the two solutions. At each subsequent surgery time T_{j+1}, we again freeze $\mathcal{M}_{T_{j+1}^-}$ and apply an additional translation.

Proposition 2.79 (Proposition 4.6 [55])**.** *Ω, u_i and u_L as before. We can choose $L = L(n)$ sufficiently large such that for each i we have*

$$u_L(x) - Nt_\omega \leq u_i(x) \ \forall \, x \in \bar{\Omega}, \tag{2.80}$$

where t_ω satisfies $t_\omega \leq CL^2(H_1^i)^{-2}$ for a constant C depending on n and N is the number of surgeries times associated with u_i.

We will also require the Clearing Out Lemma stated below. It guarantees that if the surface has a small area ratio with respect to a ball of given radius, then the solution of mean curvature flow must clear out a smaller concentric ball in a controlled way.

Lemma 2.81 (Theorem 4.7 [55]). *There exist constants θ and C depending on n such that for all $x_0 \in \mathbb{R}^{n+1}$ and $\rho > 0$, the estimate*

$$|\Gamma_{t_0} \cap B_\rho(x_0)| \leq \theta\rho^n$$
$$\Rightarrow \Gamma_t \cap B_{\frac{\rho}{2}}(x_0) = \emptyset$$

where $t - t_0 \leq C\rho^2$.

We first deal with regions that are directly affected by the surgery procedure. The point $x \in \mathbb{R}^{n+1}$ is modified by the surgery procedure if it belongs to the part of a solid tube G which is changed by surgery, i.e. $x \in G(\bar{B}_1^n \times [\Lambda, L - \Lambda])$.

Lemma 2.82 (Lemma 4.9 [55]). *[Regions modified by surgery procedure] Suppose $n \geq 3$ and let $\Omega \in \mathbb{R}^{n+1}$ be an open, bounded set such that $\partial\Omega \in C(R, \alpha)$ for some R, α. We can choose L depending on n sufficiently large such that the following holds. Let T_j, $j \in \{1, ..., N\}$, be a surgery time for u_i and assume that $t_0 > T_j \geq 0$ is such that $H^n(\Gamma_{t_0}) = H^n(\partial^*\Omega_{t_0})$ and $\Gamma_{t_0} \subset \tilde{\Sigma}^i_{T_j}$. Then there exist constants C_1, C_2 depending only on n such that $\Gamma_{t_0+\bar{t}} \cap B_{\frac{\rho_0}{2}}(x) = \emptyset$ for all $x \in \Sigma^i_{T_j}$ modified by the surgery procedure, where $\rho_0 = C_1 L (H_1^i)^{-1}$ and $\bar{t} \leq C_2 L^2 (H_1^i)^{-2}$.*

Proof. Consider any $x \in G_0(\bar{B}_1^n \times [\Lambda, L - \Lambda])$, where $G_0 : \bar{B}_1^n \times [0, L] \to \mathbb{R}^{n+1}$ is the solid tube enclosed by a neck N_0 with scale r_0^i and centre p_0. Since G_0 can be made as close as we wish to the standard isometric embedding of a piece of the solid cylinder in $\mathbb{R}^{n+1}$, therefore we can arrange that at each such x we have

$$|N_0 \cap B_{\Lambda r_0^i}(x)| \leq \frac{4\Lambda^n \omega_{n-1}(r_0^i)^n}{\Lambda^{n-1}} = 4\Lambda\omega_{n-1}(r_0^i)^n.$$

We then choose Λ, L sufficiently large such that

$$\frac{\omega_{n-1}}{\Lambda^{n-1}} \leq \frac{\theta}{4}$$

which implies that

$$|N_0 \cap B_{\Lambda r_0^i}(x)| \leq \theta(\Lambda r_0^i)^n.$$

Given this choice of Λ set,

$$\rho_p = \frac{(n-1)\Lambda}{H_1^i}.$$

Now we verify that a weak solution trapped inside N_0 satisfies

$$|\Gamma_{t_0} \cap B_{\rho_0}(x)| \leq \theta\rho_0^n.$$

The result then follows from the Clearing Out Lemma.

We can now use Proposition 2.66, the area minimisation property of the weak solution, direct comparison of the set $\Omega_{t_0} \cap G_0$ with the perturbation $\Omega_{t_0} \cup G_0$ yields the estimate

$$|(\Gamma_{t_0} \cap G_0) \cap B_{\rho_0}(x)| \leq |N_0 \cap B_{\rho_0}(x)|.$$

To complete the proof, we need to confirm that no other part of the surface can interfere with $B_{\rho_0}(x)$, i.e. $B_{\rho_0}(x) \cap (\bar{\Omega}_{t_0} \setminus G_0) = \emptyset$.

We let, $B_{g(t)}(p, r) = \{q \in M^n | d_{g(t)(p,q) \leq r} \subset M^n\}$. Consider a spacetime point (p_0, T_j) such that p_0 lies at the centre of the neck $N_0 \subset \mathcal{M}^i_{T_j}$ and set $R = \frac{n-1}{H(p_0,T_j)}$ and $B_- = B_{g(T_j)}(p_0, R_0 L)$.

From the Huisken-Sinestrari surgery construction we know that the solutions to the flow with surgeries is a family of smooth flows $F^j : \mathcal{M}_j \times [T_{j-1}, T_j] \to \mathbb{R}^{n+1}$. As long as we are not close to a surgery time , the ball $B_{g(t)}(p, r)$ belongs to the manifold $\mathcal{M}_j$ corresponding to the interval $[T_{j-1}, T_j]$ containing t. If t corresponds to a surgery time we will need to distinguish between the manifolds before and after the surgery procedure.

Assume that surgery has not taken place at any points in the ball $B_{g(T_j)}(p, r)$ for times between $T_j - r_0^2\omega$ and T_j. However surgery may have occurred elsewhere during this time interval, but must be disjoint from $B_{T_j}(P_0, R_0 L)$.

Huisken and Sinestrari showed that any point in $\mathcal{P}(p_0, T_j, R_0 L, R_0^2\omega)$ the Weingarten operator of the surface and its spatial derivatives are ϵ-close to the corresponding quantities associated with the standard shrinking cylinder. Furthermore, for any $t \in [T_j - \omega R_0^2, T_j]$, we know that the point (p_0, t) lies at the centre of an $(\epsilon, k_0 - 1)$-hypersurface neck $N_t \subset B_0$ of length at least $L - 2$. Let

$$\sigma(r, s) = (r^2 - 2(n-1)s)^{\frac{1}{2}}$$

for $s \leq 0$. We let $\sigma(r, s)$ denote the radius at time s of a standard n-dimensional cylinder along mean curvature flow. We then know that the mean radius $r(z)$ of every cross section of N_t is given by $\sigma(R_0, t - T_j)(1 + O(\epsilon))$ and that there exists a unit vector $\chi \in \mathbb{R}^{n+1}$ such that $| < \nu(, p, t), \chi > | \leq \epsilon$ for all $p \in N_t$.

We able to choose $\omega = CL^2$ where C depends only on n, sufficiently large to ensure that $B_{\rho_0}(x)$ is completely contained in the solid tube enclosed by the hypersurface neck at an earlier time $[T_j - \omega r_0^2]$. Since the surgery scale is less than H_1^i no surgery can interfere with N_t on the time interval $[T_j - \omega R_0^2, T_j]$. By the curvature assumption on the initial data, each point $x \in \mathbb{R}^{n+1}$ satisfies $x \in \Gamma_t$ for at most one t. This ensures that the ball does not touch any part of the weak solution outside the neck N_0 and therefore completes the proof. $\square$

The proof of Theorem 8.1 in [67] (as well as the proof of Theorem 3.45), establishes that, at each surgery time, the regions with mean curvature exceeding H_2^i are contained in one of finitely many disjoint regions $\mathcal{A}_i$. Let $r_\delta^i \equiv \frac{2(n-1)}{H_1^i}$. Each $\mathcal{A}_i$ must admit one of five possible structures:

(i) $\mathcal{A}_i$ is uniformly convex and diffeomorphic to S^n;

(ii) $\mathcal{A}_i$ is the union of a neck N_0 with two discs and forms a connected component diffeomorphic to S^n;

(iii) $\mathcal{A}_i$ is a maximal hypersurface neck N_0 which covers an entire connected component of $\mathcal{M}^i_{T_j^-}$ and is diffeomorphic to $S^{n-1} \times S^1$;

(iv) $\mathcal{A}_i$ is the union of a neck N_0 with a region diffeomorphic to a disc and has one boundary component with mean radius r_δ^i;

(v) $\mathcal{A}_i$ is a neck N_0 with two boundary components (each of which has mean radius r_δ^i) and is therefore diffeomorphic to $S^{n-1} \times [0, 1]$.

Components of known topology are discarded at the surgery time. In addition, one standard surgery is performed at the cross-section nearest to each boundary component with mean radius $r_0^i \equiv \frac{n-1}{H_1^i}$, forming a component diffeomorphic to S^n which is also discarded. It is necessary to deal with the points affected by step two of the surgery procedure. Let T_j be the surgery time. Consider any $\mathcal{A}_i \subset \mathcal{M}^i_{T_j}$ and the corresponding domain $\mathcal{G}_i \subset R^{n+1}$ enclosed by $\mathcal{A}_i$. Let $S \subset \mathcal{G}_i$ be the open set in $\mathbb{R}^{n+1}$ enveloped by a component removed at the surgery time T_j. The following lemma will provide us with an upper bound on the extinction time $T_S \equiv \sup\{t \geq 0 | \Gamma_t \neq \emptyset\}$ of the weak solution generated by S.

Lemma 2.83 (Lemma 4.10 [55])**.** *[Discarded Components] Suppose $n \geq 3$ and let $\Omega \subset \mathbb{R}^{n+1}$ be an open, bounded set such that $\partial\Omega \in C(R, \alpha)$ for some $R < \alpha$. Let $\mathcal{M}^i_t$ be the solution of the flow with surgeries starting from $\partial\Omega$ and with the parameters H_1^i, H_2^i, H_3^i. In addition, let T_j be any surgery time for $\mathcal{M}^i_t$ and consider any discarded component ∂S produced by the solution $\mathcal{M}^i_t$ at time T_j. Denote by $S \subset \mathbb{R}^{n+1}$ the open set enveloped by ∂S and let $u_S : \bar{S} \to \mathbb{R}$ be the weak solution generated by the domain S. Then there exists a constant $C > 0$ depending only on n, α such that $T_S \leq CL^2(H_1^i)^{-2}$, where T_S denotes the extinction time of u_S.*

Proof. Discarded components must be diffeomorphic to S^n or $S^{n-1} \times S^1$. The only case in which the latter occurs is in the form of a maximal normal (ϵ, k)-hypersurface neck without boundary. An argument similar to Lemma 2.82 together with the clearing out Lemma yields the estimate. This argument can be applied to any neck which arises as a subset of a discarded component.

In the remaining cases ∂S is diffeomorphic to S^n, therefore we have three different possibilities.

(1) As a uniformly convex component,

(2) As the union of a hypersurface neck with two regions diffeomorphic to discs or

(3) as a component which becomes disconnected from the rest of the surface as he result of surgery.

If two surgeries are performed on a region $\mathcal{A}_i$ with two boundary components then the resulted connected component satisfies the estimate by the argument above.

In case (1) we can use the curvature bound from Theorem 2.53 in combination with Myers Theorem and an appropriate spherical barrier to obtain the appropriate estimate.

We are now left to deal with the remaining convex regions diffeomorphic to discs. Huisken and Sinestrari showed that a neck can either close up with a convex cap or border a disc which was inserted by a previous surgery. In either situation we use a straight cylinder as a smooth barrier.

By Theorem 2.56 the curvature of this cylinder is bounded below by H_1^i up to a constant. We know that after a time bounded above by $C_1(n)(H_k^i)^{-2}$ the weak solution must clear out the bordering neck. By the curvature assumption, it cannot re-enter the collar of the neck. Then by comparison with the smooth evolution of a standard cylinder, the weak solution disappears completely after an additional time bounded above by $C_3(H_k^i)^{-2}$ where C_3 is a constant depending only on n. Choosing $C = \max\{C_1, C_2\} + C_3$ complete the proof. $\square$

Proof of Proposition 2.79. We have $\mathcal{M}_0^i = \Gamma_0 = \partial\Omega$ and therefore $\Gamma_\delta \subset\subset \Omega$ for $\delta > 0$. The avoidance principle guarantees that $\mathrm{dis}(\Gamma_{\delta+t}, \mathcal{M}^i_t)$ is non-decreasing in t for all $\delta > 0$ and for all $0 < t \leq T_1^-$ until the first surgery time for $\mathcal{M}_t^i$ - that is, as long as $\mathcal{M}_t^i$ remains smooth. Therefore

$$\Gamma_{\delta+t} \subset\subset \tilde{\Sigma}_t^i,\ 0 \leq t \leq T_1, \quad \text{and} \quad \Gamma_{\delta+t} \subset\subset \Sigma_t^i,\ 0 \leq t \leq T,$$

for all $\delta > 0$. Let $t_\delta \equiv \delta + T_1$.

We will now show that $\Gamma_{t_\delta+t_\omega} \subset\subset \Sigma_{T_1}^i$ for all $\delta > 0$ where $t_\omega \leq CL^2(H_1^i)^{-2}$ for a constant C depending on n and α. Let $\partial\mathcal{S}_i$ denote the finitely many components discarded by the solution $\mathcal{M}_t^i$ at time T_i. Applying Lemma 2.82, we obtain $\Gamma_{t_\delta+\bar{t}} \subset\subset (\Sigma_{T_1^i} \cup (\cup)_i\mathcal{S}_i)$ for all small $\delta > 0$, where $\bar{t} \leq CL^2(H_1^i)^{-2}$ for a constant C depending only on n. The avoidance principle for weak solutions yields $\Gamma_{t_\delta+t_\omega} \subset\subset \Sigma_{T_1}^i$ for all $\delta > 0$, where $t_\omega \equiv \bar{t} + \max_i T_{\mathcal{S}_i}$ and T_{S_l} denotes the extinction time of the weak solution generated by $\mathcal{S}_i$. Using Lemma 2.83, we conclude that $\max_l T_{S_l} \leq CL^2(H_1^I)^{-2}$ where the constant C depends on n and α.

We then invoke the avoidance principle on the next smooth time interval and iterate the argument finitely many times. This establishes that $\Gamma_{\tilde{t_\delta}+t} \subset\subset \Sigma_t^i$ for all $t \geq 0$ and for all small $\delta > 0$, where $\tilde{t_\delta} \equiv \delta + Nt_\omega$. The proposition then follows from the continuity of the level-set functions u_i, u_l. $\square$

Proof of Theorem 2.76. Combine Lemma 2.78, Proposition 2.79 and (2.75). $\square$

2.2.3 Lauer's Convergence Method

Lauer was able to also prove that the surgery process converges to level-set flow as we take the limit of our surgery parameter. Whilst Head obtained and used explicit estimates to get the result, Lauer was able to prove the same result using a maximum principle argument. The results and proofs are as in [71].

Definition 2.84 (Weak Set Flow)**.** *Let $K \subset \mathbb{R}^{n+1}$ be closed and $\{K_t\}_{t\geq 0}$ a one parameter family of closed sets such that the spacetime track $\cup(K_t \times \{t\}) \subset \mathbb{R}^{n+2}$ is closed. Then $\{K_t\}_{t\geq 0}$ is* a weak set flow *for K if every smooth mean curvature flow Σ_t on $[a,b]$ we have $K_a \cap \Sigma_a = \emptyset \Rightarrow K_t \cap \Sigma_t = \emptyset$ for all $t \in [a,b]$.*

Definition 2.85 (Level Set Flow)**.** *The level-set flow of a compact set $K \subset \mathbb{R}^{n+1}$, is the* maximal weak set flow. *$K \subset \mathbb{R}^{n+1}$ is level flow flow if for any weak set flow $\hat{K}$ we have $\hat{K} \subset K_t$ for all $t \geq 0$.*

The existence of a maximal weak set flow is verified by taking the closure of the union of all weak set flows with given initial data. If K_t is the weak set flow of K, we denote by $\hat{K}$ the spacetime track swept out by K_t. That is,

$$\hat{K} = \cup_{t\geq 0} K_t \times \{t\} \subset \mathbb{R}^{n+2}.$$

Let $\Sigma_H \subset \mathbb{R}^{n+2}$ denote the spacetime track swept out by the hypersurfaces. Here we use H_3 to denote our surgery parameter.

We work with regions bounded by the evolving hypersurface. Let $K \subset \mathbb{R}^{n+1}$ be a compact domain such that ∂K is a two-convex hypersurface. Then if ∂K_H is mean curvature flow with surgeries, we define $K_H \subset \mathbb{R}^{n+2}$ to be the region of spacetime such that $t = T$ time-slice of K_H is the compact domain bounded by $(\partial K_H)_T$.

Theorem 2.86 (Theorem A [71]). *Let $K \subset \mathbb{R}^{n+1}$ with $n \geq 3$ be compact with ∂K two-convex. Then for H sufficiently large, let K_H be the result of mean curvature flow with surgeries performed with parameter H_3, and initial condition $(K_H)_0 = K$. Then*

$$\lim_{H \to \infty} K_H = \hat{K}.$$

The key ingredient in proving this theorem is the following lemma.

Lemma 2.87 (Lemma 2.2 [71]). *Given $\epsilon > 0$ there exists $H_0 > 0$ such that if $H \geq H_0$, T a surgery time and $x \in \mathbb{R}^{n+1}$ such that*

$$B_\epsilon(x) \subset (K_H)_T^- \Rightarrow B_\epsilon(x) \subset (K_H)_T^+.$$

where we use $(\partial K_H)_T^-$ and $(\partial K_H)_T^+$ to refer to the pre- and post-surgery hypersurfaces at surgery time T and $(K_H)_T^-$ and $(K_H)_T^+$ to the regions they bound.

Proof. Refer to the proof in [71], it just relies on the surgery construction from [67]. □

Recall the definition of Hausdorff distance.

Definition 2.88. *X and Y two non-empty subsets of a metric space (M, d). Then*

$$dist_H(X,Y) = \max\{\sup_{x\in X} \inf_{y \in Y} dist(x,y), \sup_{y \in Y} \inf_{x \in X} dist(x,y)\}. \tag{2.89}$$

Equivalently

$$dist_H(X,Y) = \inf\{\epsilon > 0 | X \subset Y_\epsilon \; and Y \subset X_\epsilon\} \; where$$
$$X_\epsilon := \cup_{x \in X}\{z \in M | dist(z,x) \leq \epsilon\}.$$

The intuitive way to think of this is what is the largest ball we can attach to any point $x \in X$ such that the ball remains in Y.

We now are able to prove the Theorem 2.86.

Proof. Given an $\epsilon > 0$ sufficiently small let $t_\epsilon > 0$ be the time such that

$$\mathrm{dist}(\partial K, \partial K_{t_\epsilon}) = \epsilon.$$

Such a time exists since ∂K is two-convex. Let $\Omega_\epsilon \subset \mathbb{R}^{n+2}$ be the level-set flow K_{t_ϵ}. We now claim that $\Omega_\epsilon \subset K_H$ for all $H \geq H_0$.

We pick our ϵ large enough depending on H_0, such that at the first surgery time T for K_H, Ω_ϵ has vacated the region affected by surgery, we know such an ϵ exists as the region is two-convex. Now since the distance between the weak set flow and mean curvature flow with surgeries is non-decreasing on the interval $[0, T)$ we know that $d((\Omega_\epsilon)_T, (\partial K_H)_T^-) \geq \epsilon$. By applying Lemma 2.87 and the definition of Hausdorff distance we know that $d((\Omega_\epsilon)_T, (\partial K_H)_T^+) \geq \epsilon$. Since $(\partial K_H)_T^+$ is a smooth hypersurface we can repeat this argument for each subsequent surgery time. This proves our claim.

Since $\lim_{\epsilon \to 0} \Omega_\epsilon = \hat{K}$, the claim implies that $\hat{K} \subset \lim_{H \to \infty} K_H$ as the limit of closed sets is closed.

Lastly, since each mean curvature flow with surgeries is also a weak set flow for K, we have $\lim_{H \to \infty} K_H \subset \hat{K}$.

□

Chapter 3

The Surgery Procedure for Brendle-Huisken G-Flow

In the first chapter we began by outlining Huisken's results for mean curvature flow in the convex Euclidean setting. He was then able to extend his classification result to the Riemannian setting with some extra restrictions.

Taking the next natural step Huisken and Sinestrari weakened the convexity assumption and used their surgery algorithm to study two-convex surfaces in the Euclidean setting. Ideally the next step would then go on to prove a similar result for mean curvature flow of two-convex hypersurfaces embedded in a Riemannian manifolds. However in this setting 2-convexity is not preserved by the flow. Inspired by Andrews work on harmonic mean curvature flow, [4], Brendle-Huisken introduced the following flow which has the advantage of preserving 2-convexity in the Riemannian setting.

Fixing $n \geq 3$ consider a closed, embedded hypersurface $\mathcal{M}_0$ in $\mathbb{R}^{n+1}$. $\mathcal{M}_0$ is κ-two-convex if $\lambda_1 + \lambda_2 \geq 2\kappa$, where $\lambda_1 \leq \cdots \leq \lambda_n$ denote the principal curvatures. We evolve $\mathcal{M}_0$ with normal velocity

$$G_\kappa = \left(\sum_{i<j} \frac{1}{\lambda_i + \lambda_j - 2\kappa} \right)^{-1}.$$

This is called Brendle-Huisken G-flow, but we will sometimes abbreviate it to G-flow. Brendle and Huisken were able to extend the surgery algorithm of Huisken and Sinestrari to this G-flow in both the Euclidean setting and Riemannian setting. In order to do this they obtained a convexity estimate, cylindrical estimate and gradient estimate for the flow which is described in Section 3.1.

The main theorem of this chapter is as follows.

Theorem 3.1. *Let $\mathcal{M}_t$ be a smooth Brendle-Huisken G-flow of a closed, compact 2-convex hypersurface. Given our neck parameters, there exists a constant G^* depending on $\mathcal{M}_0$ such that if $G_{\max}(t_0) \geq G^*$, then the hypersurface at time t_0 either contains an (ϵ, k, L)-hypersurface neck or is convex.*

When studying this flow in the Euclidean setting it suffices to check the $\kappa = 0$ case. In order to argue as in Section 2.1.6 for this flow, adjustments have to be made for the gradient estimate from [16]. Unfortunately we are not able to integrate the gradient estimate in its current form to obtain results relating to our backward parabolic neighbourhoods being surgery free as in Lemma 2.44. This is crucial in our proof of the Neck Detection Lemma. Some small adjustments will also be required in our proof of the Neck Detection Lemma for this setting.

After making these changes we can follow Section 2.1.6. and Section 7 of [67] to obtain the other necessary results for when certain conditions in the Neck Detection Lemma are not met. Firstly we may not know that the backward parabolic neighbourhood about a point is surgery free, in this case we can obtain the required result as long as the curvature at our point is large enough compared to the curvature of the regions changed by previous surgeries. We must also deal with the case when $\frac{\lambda_1}{G}$ is not small, however the proof here does not rely on gradient estimates and is instead a general property of hypersurfaces as shown in Theorem 2.53.

We also wish to prove the following result analogous to Theorem 2.55, which relates to the existence and classification of surgically modified flows.

Theorem 3.2. *Let $\mathcal{M}_0$ be a smooth closed two-convex hypersurface immersed in $\mathbb{R}^{n+1}$, with $n \geq 3$. Then there exist constants $G_1 < G_2 < G_3$ and a G-flow with surgeries starting from $\mathcal{M}_0$ with the following properties:*

- *Each surgery takes place at a time T_i such that $G_{\max}(T_i-) = G_3$.*
- *After the surgery, all the components of the manifold satisfy $G_{\max}(T_i+) \leq G_2$, except for those diffeomorphic to spheres of to $S^{n+1} \times S^1$, which are neglected afterwards.*
- *Each surgery starts from a cross section of a normal hypersurface neck with mean radius $r(z_o) = \frac{(n-1)(n-2)}{2G}$.*
- *The flow with surgeries terminates after finitely many steps.*

The constants G_i can be any values such that $G_1 \geq \omega_1$,$G_2 = \omega G_1$ and $G_3 = \omega_3 G_2$, with $\omega_i > 1$.

In order to prove this theorem and cover the arguments of Section 8 from [67] we need to obtain a lower bound for the time between surgeries, which we do in Section 3.3.

3.1 Evolution Equations and Necessary Estimates

In this section we go over some preliminary results obtained from Brendle and Huiskens paper on G-flow [16]. We will need the evolution equation for G, convexity estimate, cylindrical estimate, as well as our new gradient estimate which allows us to control the size of the curvature in the neighbourhood of a given point.

Firstly we give some introductory results regarding G-flow, stated by Brendle and Huisken in [16].

Proposition 3.3. *Given G as above we have the following properties:*

(i) $G_\kappa \leq C_1 H$, *where* $C_1 > 0$ *depends only* n.

(ii) $0 \leq \frac{\partial G_\kappa}{\partial h_{ij}} \leq C_2 g_{ij}$, *where* $C_2 > 0$ *depends only on* n.

Proof. (i) Clear.

(ii) This is equivalent to observing $\frac{\partial G}{\partial \lambda_i}$ being bounded for each i, because

$$\frac{\partial G}{\partial h_{kl}} = \frac{\partial G}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial h_{kl}}.$$

Now

$$(\lambda_i) = O(h_{kl})O^T$$

for some orthogonal matrix O. Expanding

$$\begin{aligned} \lambda_i &= \sum_{k,l} o_{ik} h_{kl} o_{li} \\ \Rightarrow \quad \frac{\partial \lambda_i}{\partial h_{kl}} &= o_{ik} o_{li} \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial G}{\partial h_{kl}} &= \sum_i \frac{\partial G}{\partial \lambda_i} o_{ik} o_{li} \\ \Rightarrow \frac{\partial G}{\partial h_{kl}} &= O\left(\frac{\partial G}{\partial \lambda_i}\right)O^T. \end{aligned}$$

Now calculating $\frac{\partial G}{\partial \lambda_i}$ we obtain,

$$\frac{\partial G}{\partial \lambda_i} = -G^{-2}\left(\sum_{k=1,k\neq i}^{n} \frac{-1}{(\lambda_i + \lambda_k)^2}\right)$$

The above line together with two-convexity gives me the lower bound. To obtain the upper bound observe that this is less than

$$\begin{aligned}
&\leq G^{-2}\left(\sum_{k=1,k\neq i}^{n}\frac{1}{(\lambda_i+\lambda_k)}\right)^2\\
&\leq G^{-2}G^2 \quad \text{again using two-convexity}\\
&= 1
\end{aligned}$$

□

Now we are able to obtain the necessary evolution equations for G.

Lemma 3.4. *If $\mathcal{M}_t$ evolves by G-flow, the associated quantities above satisfy the following equations:*

(i) $\frac{\partial}{\partial t}g_{ij} = -2Gh_{ij}$

(ii) $\frac{\partial}{\partial t}g^{ij} = 2Gh^{ij}$

(iii) $\frac{\partial}{\partial t}\nu = \nabla G$

(iv) $\frac{\partial}{\partial t}h_{ij} = D_iD_jG - Gh_{il}g^{lm}h_{mj}$

(v) $\frac{\partial}{\partial t}G = \frac{\partial G}{\partial h_{ij}}(D_iD_jG + h_{ik}h_{jk}G)$

(vi) $\frac{\partial}{\partial t}H = \Delta|G| + |h|^2G.$

(vii) $\frac{\partial}{\partial t}d\mu \leq -\frac{G^2}{C}d\mu.$

Proof. (i)

$$\begin{aligned}
\frac{\partial}{\partial t}g_{ij} &= \frac{\partial}{\partial t}\left\langle\frac{\partial F}{\partial x_i},\frac{\partial F}{\partial x_j}\right\rangle\\
&= \left\langle\frac{\partial}{\partial x_i}(-G\nu),\frac{\partial F}{\partial x_j}\right\rangle + \left\langle\frac{\partial}{\partial x_j}(-G\nu),\frac{\partial F}{\partial x_i}\right\rangle\\
&= -G\left\langle\frac{\partial}{\partial x_i}\nu,\frac{\partial F}{\partial x_j}\right\rangle - G\left\langle\frac{\partial F}{\partial x_i},\frac{\partial}{\partial x_j}\nu\right\rangle\\
&= -2Gh_{ij}
\end{aligned}$$

(ii) Obtained by differentiating $g_{il}g^{lj} = \delta_i^j$.

(iii)

$$\begin{aligned}
\frac{\partial}{\partial t}\nu &= \left\langle \frac{\partial}{\partial t}\nu, \frac{\partial F}{\partial x_i}\right\rangle \frac{\partial F}{\partial x_j} g^{ij} \\
&= -\left\langle \nu, \frac{\partial}{\partial t}\frac{\partial F}{\partial x_i}\right\rangle \frac{\partial F}{\partial x_j} g^{ij} \\
&= \left\langle \nu, \frac{\partial}{\partial x_i}(G\nu)\right\rangle \frac{\partial F}{\partial x_j} g^{ij} \\
&= \frac{\partial}{\partial x_i} G \frac{\partial F}{\partial x_j} g^{ij} \\
&= \nabla G.
\end{aligned}$$

(iv) In this proof we will make use of the Gauss-Weingarten equations.

$$\begin{aligned}
\frac{\partial}{\partial t}h_{ij} =& -\frac{\partial}{\partial t}\left\langle \frac{\partial^2 F}{\partial x_i \partial x_j}, \nu\right\rangle \\
=& \left\langle \frac{\partial^2}{\partial x_i \partial x_j}(G\nu), \nu\right\rangle - \left\langle \frac{\partial^2 F}{\partial x_i \partial x_j}, \frac{\partial}{\partial t}\nu\right\rangle \\
=& \left\langle \frac{\partial^2}{\partial x_i \partial x_j}(G\nu), \nu\right\rangle - \left\langle \frac{\partial^2 F}{\partial x_i \partial x_j}, \frac{\partial}{\partial x_l} G \frac{\partial F}{\partial x_m} g^{lm}\right\rangle \\
=& \frac{\partial^2}{\partial x_i \partial x_j} G + G\left\langle \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}\nu\right), \nu\right\rangle - \left\langle \frac{\partial^2 F}{\partial x_i \partial x_j}, \frac{\partial}{\partial x_l} G \frac{\partial F}{\partial x_m} g^{lm}\right\rangle \\
=& \frac{\partial^2}{\partial x_i \partial x_j} G + G\left\langle \frac{\partial}{\partial x_i}\left(h_{jl} g^{lm} \frac{\partial F}{\partial x_m}\right), \nu\right\rangle \\
& - \left\langle \Gamma^k_{ij}\frac{\partial F}{\partial x_k} - h_{ij}\nu, \frac{\partial}{\partial x_l} G \frac{\partial F}{\partial x_m} g^{lm}\right\rangle \\
=& \frac{\partial^2}{\partial x_i \partial x_j} G - \Gamma^k_{ij}\frac{\partial}{\partial x_k} G + G h_{jm} g^{ml}\left\langle \Gamma^p_{il}\frac{\partial F}{\partial x_p} - h_{il}\nu, \nu\right\rangle \\
=& D_i D_j G - G h_{il} g^{lm} h_{mj}.
\end{aligned}$$

(v) Here we just use the fact that $\frac{\partial G}{\partial t} = \frac{\partial G}{\partial h^j_i}\frac{\partial}{\partial t}h^j_i$ as well as part (ii) and (iv).

(vi)

$$\begin{aligned}
\frac{\partial}{\partial t}H &= \frac{\partial}{\partial t}(h_{ij} g^{ij}) \\
&= g^{ij}\frac{\partial}{\partial t}h_{ij} + 2G h^{ij} h_{ij} \\
&= g^{ij}(D_i D_j G - G h_{il} g^{lm} h_m j) + 2G g^{ik} g^{jl} h_{kl} h_{ij} \\
&= \Delta G + |h|^2 G.
\end{aligned}$$

(vii) $\frac{\partial}{\partial t} d\mu = \frac{1}{2}\sqrt{\det g_{ij}}\,\mathrm{tr}(-2G h_{ij}) = -GH\mu$. Using Proposition 3.3(i) the result follows. $\square$

We will also often make use of the following proposition which states that on any given bounded time interval, the mean curvature is bounded from above by a constant multiple of G. We often use this assumption without statement.

Proposition 3.5 (Proposition 2.4 [16]). *We have $G \geq \beta H$ for all $t \in [0, T)$ where β is a positive constant that depends only on T and $\mathcal{M}_0$.*

Proof. Refer to Proposition 2.4 from [16]. $\square$

The convexity estimate, is necessary in order to know that the nearly singular regions of the surface become asymptotically convex as a singular time is approached.

Theorem 3.6 (Corollary 7.7 [16]). *(Convexity Estimate) Suppose that $\mathcal{M}_t$, $t \in [0, T)$ is a surgically modified G-flow starting from a closed, embedded, 2-convex hypersurface $\mathcal{M}_0$ then for any $\delta > 0$*

$$\lambda_1 \geq \delta G - C$$

where C is a positive constant that depends only on δ, n and T.

Next we need a cylindrical estimate which implies that at points where λ_1 is small, we have curvature close to the curvature of a cylinder.

Theorem 3.7 (Theorem 3.1 [16]). *(Cylindrical Estimate) Let $\mathcal{M}_t$ be a family of closed, two-convex hypersurfaces moving with speed G, then for all $\eta > 0$ there exists a constant $C > 0$ depending on δ, T and n such that*

$$H \leq \frac{(n-1)^2(n+2)}{4}(1+\delta)G + C_{\eta,T}.$$

The following is the gradient estimate.

Theorem 3.8 (Theorem 7.12 [16]). *(Gradient Estimate) For a closed, embedded, two-convex hypersurface $\mathcal{M}_0 = \delta\Omega_0$. We can find a constant $G^{\#}$, depending only on $\mathcal{M}_0$ such that the following holds: Suppose that Ω_t, $t \in [0, T)$, is a one-parameter family of smooth open domains with the property that the hypersurfaces $\mathcal{M}_t = \partial\Omega_t$ form a surgically modified flow starting from $\mathcal{M}_0$ with surgery scale $G_* \geq G^{\#}$. Then we have*

$$\alpha^2 G^{-2}|\nabla h| + \alpha^3 G^{-3}|\nabla^3 h| \leq \Lambda \tag{3.9}$$

for all points in spacetime satisfying $G \geq G^{\#}$. Here α is a constant depending on T and n is the constant in Proposition 7.8 ([16]), and Λ is a constant depending on T and n is the constant appearing in Corollary 7.11 [16].

Now we want to modify this gradient estimate using the following lemma, in order to allow us to integrate our gradient estimates and obtain necessary results related to the backward parabolic neighbourhood as done in Lemma 2.44.

Theorem 3.10. *Let $\mathcal{M}$ be a G-flow with surgeries. Then the inequalities $\alpha^2 G^{-2}|\nabla h| \leq C$ and $\alpha^3 G^{-3}|\nabla^2 h| \leq C$ from Theorem 3.8 allow us to find $c^{\#} > 0, G^{\#} > 0$ such that for all $p \in M$ and $t > 0$,*

$$G(p,t) > G^{\#} > 0 \;\Rightarrow\; |\nabla G(p,t)| \leq c^{\#} G^2(p,t),\; |\partial_t G(p,t)| \leq c^{\#} G^3(p,t), \tag{3.11}$$

where $c^{\#}$ only depends on the dimension of n, T.

Proof. For the n-dimensional case we look at the following. We know that

$$\det(\lambda I - h_{ij}) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$
$$\Rightarrow \det\begin{bmatrix} \lambda - h_{11} & \dots & -h_{1n} \\ \vdots & \ddots & \vdots \\ -h_{n1} & \dots & \lambda - h_{nn} \end{bmatrix} = \lambda^n - \lambda^{n-1}(\lambda_1 + \cdots + \lambda^n) + \cdots - \lambda_1 \cdots - \lambda_n.$$

Here we will need to introduce some notation. Let $Q_k(h_{ij})$ denote a k-degree polynomial in terms of of h_{ij}'s, such that no lower degree can appear. Using a degree argument and equating terms on either side we will obtain

$$\lambda^p Q_{n-p}(h_{ij}) = \lambda^p \lambda_j^{n-p}. \tag{3.12}$$

This guarantees that we can rewrite our principal curvature values in terms of the second fundamental form. Rewriting G as follows,

$$G = \frac{\prod_{i<j}(\lambda_i + \lambda_j)}{\sum_{i<j} \frac{1}{\lambda_i + \lambda_j} \prod_{i<j}(\lambda_i + \lambda_j)} \tag{3.13}$$

applying the result of (3.12) to our rewritten G we see that we can write out the λ^p using our h_{ij}^p terms,

$$G = \frac{Q_n(h_{ij})}{Q_{n-1}(h_{ij})}$$

Moreover from our definition of G, (3.13) and 2-convexity, we can see that

$$\frac{\lambda_1 + \lambda_2}{n} \leq G \leq \lambda_1 + \lambda_2. \tag{3.14}$$

Moreover from Proposition 3.5 we know that $H \leq \beta_0 G$ for some constant C. This tells us that

$$\begin{aligned} \lambda_1 + \cdots + \lambda_n = H &\leq \beta_0 G \\ \Rightarrow \lambda_i &\leq \beta_1 G \\ \Rightarrow |h_{ij}| &\leq |\beta_2 G| \end{aligned} \tag{3.15}$$

for some constants β_1, β_2 depending on n. The last step is clear as we know $(h_{ij}) = O(\lambda_i)O^T$, where O is an orthonormal matrix and (λ_i) is the diagonal matrix of principal curvatures. So,

$$\begin{aligned} |\nabla G| &= \left| \frac{Q_1(\nabla h_{ij})Q_{n-1}(h_{ij})Q_{n-1}(h_{ij}) - Q_1(\nabla h_{ij})Q_{n-2}(h_{ij})Q_n(h_{ij})}{Q_{2n-2}(h_{ij})} \right| \\ &= \left| \frac{Q_1(\nabla h_{ij})Q_{2n-2}(h_{ij})}{Q_{2n-2}(h_{ij})} \right| \\ &\leq \left| \frac{\beta_3 G^{n-2} Q_1(\nabla h_{ij})}{\beta_4 G^{2n-2}} \right| \text{ by (3.15) and (3.14) where } \beta_3, \beta_4 \text{ are constants.} \\ \Rightarrow \ |\nabla G| &\leq \left| \frac{\beta G^2}{\alpha^2} \right| \end{aligned}$$

were by equality we refer to equality in the degree and in the last line we have applied Theorem 3.8. This proves (i).

Now we prove part (ii). From 3.4 (v) we know that

$$\partial_t G = \frac{\partial G}{\partial h_{ij}} (\nabla_i \nabla_j G - h_{ij} h_{jk} G). \tag{3.16}$$

We can control $\frac{\partial G}{\partial h_{ij}}$ using Proposition 3.3(ii).

Next by applying (3.15) we can bound the $h_{ij}h_{jk}G$ term by βG^3 for some constant β.

Lastly $\nabla_i \nabla_j G$ will give terms of the form $\nabla^2 h_{ij}$ and $\nabla h_{ij} G$. Using (i) as well Proposition 3.8 we see that $|\partial_t G| \leq |KG^3|$ for some constant K. This completes the proof of (ii). $\square$

These estimates allow us to control the size of the curvature in a neighbourhood of a given point.

3.2 The Neck Detection Lemma

Using our new gradient estimate we will now obtain an result analogous to Lemma 2.38 relating to the size of the curvature in a neighbourhood of a given point. For the ease of understanding the proofs of these results will follow the template set out by Huisken and Sinestrari in [67] Section 7, with only the necessary modifications, we are able to follow their template due to our new gradient estimate (Theorem 3.10) .

Lemma 3.17 (Analogous to Lemma 2.38 and Lemma 6.6 [67])**.** *Let $F : \mathcal{M} \to \mathbb{R}^{n+1}$ be an n-dimensional immersed surface. Suppose that there are $c^{\#}, G^{\#} > 0$ such that $|\nabla G(p)| \leq c^{\#} G^2(p)$ for any $p \in \mathcal{M}$ such that $G(p) \geq G^{\#}$. Let $p_0 \in \mathcal{M}$ satisfy $G(p_0) \geq \gamma G^{\#}$ for some $\gamma > 1$. Then*

$$G(q) \geq \frac{G(p_0)}{1 + c^{\#} d(p_0, q) G(p_0)} \geq \frac{G(p_0)}{\gamma} \quad \textit{for all } q$$

such that

$$d(p_0, q) \leq \frac{\gamma - 1}{c^{\#}} \frac{1}{G(p_0)}.$$

Proof. Consider points $q \in \mathcal{M}$ such that $G(q) < \frac{G(p_0)}{\gamma}$. Take q_0 to be a point with this property with minimal distance from p, and set $d_0 = d(p_0, q_0)G(p_0)$ and $\theta_0 = \min\{d_0, \frac{\gamma-1}{c^{\#}}\}$. Now for any point $q \in \mathcal{M}$ with $d(p_0, q) \leq \frac{\theta_0}{G(p_0)}$, let $\xi : [0, d(p_0, q)] \to \mathcal{M}$ be a geodesic from p_0 to q.

Then from our definition of θ_0 it follows that $G(\xi(s)) \geq \frac{G(p_0)}{\gamma} \geq G^{\#}$ for any $s \in [0, d(p_0, q)]$. Then we can apply Lemma 3.8 to obtain $|\nabla G(\xi(s))| \leq c^{\#} G^2(\xi(s))$ and

$$\frac{d}{ds} G(\xi(s)) \geq -c^{\#} G^2(\xi(s))$$

for all $s \in [0, d(p_0, q)]$ since it is a geodesic. Integrating this inequality we obtain

$$G(\xi(s)) \geq \frac{G(p_0)}{1 + c^{\#} s G(p_0)}, \; s \in [0, d(p_0, q)],$$

which implies

$$G(q) \geq \frac{G(p_0)}{1 + c^{\#} d(p_0, q) G(p_0)} \geq \frac{G(p_0)}{1 + c^{\#} \theta_0}. \tag{3.18}$$

This holds for all q such that $d(p_0, q) \leq \frac{\theta_0}{G(p_0)}$. Now suppose $d_0 < \frac{\gamma-1}{c^{\#}}$, then $d_0 = \theta_0$ and (3.18) holds for $q = q_0$. But that implies $G(q_0) > \frac{G(p_0)}{\gamma}$ which is a contradiction. Therefore $d_0 \geq \frac{\gamma-1}{c^{\#}}$, which implies $\theta_0 = \frac{\gamma-1}{c^{\#}}$, which proves (3.18).

In the case where $G(q) \geq \frac{G(p_0)}{\gamma}$ for all $q \in \mathcal{M}$, then we have $|\nabla G| \leq c^{\#} G^2$ everywhere, and our result follows more directly from the same argument. □

Next we introduce a backward parabolic neighbourhood. This will be essential in dealing with necks.

Definition 3.19. *Given t, θ such that $0 \leq t-\theta < t \leq T_0$, we define the* backward parabolic neighbourhood *of (p,t) by,*

$$\mathcal{P}(p,t,r,\theta) = \{(q,s) | q \in \mathcal{B}_{g(t)}(p,r), s \in [t-\theta, t]\}. \tag{3.20}$$

where $\mathcal{B}_{g(t)}(p,r) \subset \mathcal{M}$ is the closed ball of radius r with respect to the metric $g(t)$.

Before we go on to prove the next lemma, analogous to Lemma 2.44, we need to define $\hat{r}_G = \frac{(n-1)(n-2)}{2G}$ and $\hat{\mathcal{P}}_G = \mathcal{P}\left(p,t,\hat{r}_G(p,t)L, \hat{r}^2(p,t\theta)\right)$. If (p,t) lies on a neck then $\hat{r}_G(p,t)$ is approximately equal to the radius of the neck.

Lemma 3.21 (Analogous to Lemma 2.44 and Lemma 7.2 [67])**.** *Let $c^\#$ and $G^\#$ be the constant from Theorem 3.10. Define*
$d^\# = (2(n-1)^2(n-2)^2c^\#)^{-1}$. Then the following properties hold.

(i) Let (p,t) satisfy $G(p,t) \geq 2G^\#$. Then, given any $r, \theta \in (0, d^\#]$ such that $\hat{\mathcal{P}}_G(p,t,r,\theta)$ does not contain surgeries, we have

$$\frac{G(p,t)}{2} \leq G(q,s) \leq G(p,t) \tag{3.22}$$

for all $(q,s) \in \hat{\mathcal{P}}_G(p,t,r,\theta)$.

(ii) Suppose that for any surgery performed at time less than t, the regions modified by surgery have G-curvature less than K, for some $K \geq G^\#$. Let (p,t) satisfy $G(p,t) \geq 2K$. Then, the parabolic neighbourhood

$$\mathcal{P}\left(p,t,\frac{1}{8c^\# K}, \frac{1}{8c^\# K^2}\right) \tag{3.23}$$

does not contain surgeries. In particular, the neighbourhood $\hat{\mathcal{P}}_G\left(p,t,d^\#,d^\#\right)$ does not contain surgeries and all points (q,s) contained there satisfy (i).

Proof. First we prove (ii).

Suppose the neighbourhood in (3.23) is modified by surgeries. Take a point (q,s) which is modified by surgery, with s the maximal time at which we can find such a point. Then by assumption we have $G(q,s+) \leq K$. Integrating the estimate on $\partial_t G$ from Theorem 3.10,

$$\begin{aligned}
\int_s^t \frac{\partial G}{G^3} &\leq \int_s^t c^\# \partial t \\
-\frac{1}{G^2(q,t)} + \frac{1}{G^2(q,s)} &\leq c^\#(t-s) \\
\frac{1}{G^2(q,t)} \geq \frac{1}{G^2(q,s)} &- 2c^\#(t-s) \geq \frac{3}{4K^2}
\end{aligned}$$

where in the last line we used our assumption on $H(q,s)$ and that $t-s \leq \frac{1}{8c^\# K^2}$. Then we integrate along a geodesic from q to p at time t and use the estimate on ∇G

$$\frac{1}{G(p,t)} \geq \frac{1}{G(q,t)} - c^\# d_{g(t)}(p,q) \geq \frac{4\sqrt{3}-1}{8K} > \frac{1}{2K},$$

where in the last line we used our estimate on $G^2(q,t)$ and that $d_{g(t)} \le \frac{1}{8c^{\#}K}$. This contradicts our assumption that (3.23) contains surgeries since $G(p,t) \ge 2K$.

In this argument in order to apply the results of Theorem 3.10 we have had to assume that $G \ge G^{\#}$ along the integration paths. If this were not the case, we could choose the last point along the path with $G \le G^{\#}$ and integrate from that point onwards, obtaining a contradiction using the same argument.

Now we use the definition of $d^{\#}$ to see that $\hat{\mathcal{P}}_G(p,t,d^{\#},d^{\#})$ is contained in the neighbourhood (3.23),

$$\begin{aligned} &\frac{(n-1)(n-2)d^{\#}}{2G(p,t)} \le \frac{1}{8K(n-1)(n-2)c^{\#}} \le \frac{1}{8c^{\#}K} \\ \text{and} \quad &\frac{(n-1)^2(n-2)^2 d^{\#}}{2G(p,t)^2} \le \frac{1}{16K^2c^{\#}} \le \frac{1}{8c^{\#}K^2}. \end{aligned}$$

Therefore $\hat{\mathcal{P}}_G(p,t,d^{\#},d^{\#})$ does not contain surgeries and part (i) can be applied to this neighbourhood.

To prove (i), we integrate the same inequalities and use the assumption that $\hat{\mathcal{P}}_G$ is surgery free. $\square$

Lemma 3.24 (Analogous to Lemma 2.46 and Lemma 7.4 [67])**.** *[Neck Detection Lemma] Let $\mathcal{M}_t, t \in [0,T)$ be G-flow with surgeries, starting from an initial manifold $\mathcal{M}_0$. Let $\epsilon, \theta, L > 0$, and $k \ge k_0$ be given (where $k_0 \ge 2$ is the parameter measuring the regularity of the necks where surgeries are performed). Then we can find η_0, G_0 with the following property:*

Suppose that $p_0 \in \mathcal{M}_0$ and $t_0 \in [0,T)$ are such that:

(ND1) $G(p_0,t_0) \ge G_0$, $\frac{\lambda_1(p_0,t_0)}{G(p_0,t_0)} \le \eta_0$

(ND2) The neighbourhood $\hat{\mathcal{P}}(p_0,t_0,L,\theta)$ does not contain surgeries.

Then,

(i) The neighbourhood $\hat{\mathcal{P}}(p_0,t_0,L,\theta)$ is an $(\epsilon, k_0-1, L, \theta)$-shrinking curvature neck;

(ii) The neighbourhood $\hat{\mathcal{P}}(p_0,t_0,L-1,\theta/2)$ is an $(\epsilon, k, L-1, \theta/2)$-shrinking curvature neck.

With constants η_0 and G_0 depending on α, ϵ, k, L and θ.

Proof. Here we argue by contradiction. Suppose that for some values of ϵ, L, θ the conclusion does not hold. No matter how we pick η_0 or G_0. Take a sequence $\{\mathcal{M}_t^j\}_{j\ge1}$ of solutions to the flow. Then a sequence of times t_j and points p_j such that

(a) $\lambda_1 + \lambda_2 \ge \alpha_0 G$, this is clear from (3.14).

(b) The parabolic neighbourhood $\mathcal{P}^j(p_j, t_j, \hat{r}_j L, \hat{r}_j^2\theta)$ is not changed by surgeries.

(c) $G_j \to \infty$ since curvature of the flows uniformly bounded at $t=0$ and $\frac{\lambda_{1,j}}{G_j} \to 0$ as $j \to \infty$. Equivalently $\frac{\lambda_{1,j}}{H_j} \to 0$ as $j \to \infty$.

(d) (p_j, t_j) does not lie at the centre of an $(\epsilon, k_0-1, L, \theta)$-shrinking neck.

where , $H_j, G_j, \lambda_{1,j}$ denote the values of H, G and λ_1 at $F_j(p_j, t_j) \in \mathcal{M}^j_{t_j}$. We now continue with a parabolic rescaling such that $H(p_j, t_j) = 1$ and p_j is translated to the origin $0 \in \mathbb{R}^{n+1}$ and t_j becomes 0. We define

$$\tilde{F}(p,\tau) = \frac{1}{H_j}[F_j(p, \hat{r}^2\tau + t_j) - F(p_j, t_j)]$$

where $\bar{\mathcal{M}}^j_\tau$ denotes our rescaled surface.

Moreover by Theorem 3.8 we know that the first and second derivatives of the second fundamental form are bounded. Therefore the rescaled flows converge to a smooth limit flow $\tilde{M}^\infty_\tau$. Moreover by (c) since $\frac{\lambda_{1,j}}{G_j} \to 0$ we know there exists a point on the limit flow where $\lambda_1 = 0$.

Now passing to the limit in the cylindrical estimate yields $H - \frac{(n-1)^2(n+2)}{4}G \leq 0$.

Scaling by H we know that the principal curvatures on a cylinder are $\lambda_1 = 0$ and $\lambda_j = \frac{1}{n-1}$ for all $j \geq 2$

$$\begin{aligned}
&\Rightarrow H = 1\\
&\Rightarrow G = \left((n-1)(n-1) + \frac{(n-1)(n-2)}{2}\frac{(n-1)}{2}\right)^{-1}\\
&\quad\;\; = \left(\frac{(n-1)^2(n+2)}{4}\right)^{-1}.
\end{aligned}$$

On a cylinder $H - \frac{(n-1)^2(n+2)}{4}G = 0$.

We want to see, $G(0, a_1, ..., a_{n-1}) \leq G(0, \frac{1}{n-1}, ..., \frac{1}{n-1})$ with equality when the a_i's are equal.

Picking from $(0, a_1, ..., a_{n-1})$ we know,

$$G^{-1} = \frac{1}{a_1} + \cdots + \frac{1}{a_{n-1}} + \sum \frac{1}{a_i + a_j} + \lambda(1 - a_1 + \cdots + a_{n-1})$$

where λ is the Lagrange multiplier with the constraint that $\sum_i a_i's = 1$. Taking partial derivatives to find a local maximum we see that for any two i, k,

$$\begin{aligned}
\frac{\partial G^{-1}}{\partial a_i} &= -\frac{1}{a_i^2} - \sum_{j\neq i}\frac{1}{(a_i + a_j)^2} - \lambda = 0\\
\frac{\partial G^{-1}}{\partial a_k} &= -\frac{1}{a_k^2} - \sum_{j\neq k}\frac{1}{(a_k + a_j)^2} - \lambda = 0\\
\Rightarrow -\frac{1}{a_i^2} + \frac{1}{a_k^2} &- \sum_{j\neq i,k}\frac{1}{(a_i + a_j)^2} + \sum_{j\neq i,k}\frac{1}{(a_k + a_j)^2} = 0
\end{aligned}$$

$\Rightarrow$ $a_i = a_k$ and all $a_i's$ are equal. Now on the boundary when $a_l \to 0$ for some $l > 1$ tells us that $G \to 0$ and we have a minimum, therefore we have a maximum when they are equal on the interior. This tells us that $G = CH$ for some constant C depending on n, so on the limit we see that the fully nonlinear G-flow is the same as mean curvature flow. This allows us to argue as in Lemma 7.4 [67] and Theorem 5.1[62] to obtain that $\tilde{\mathcal{M}}^\infty_t$ on $\tilde{\mathcal{P}}^\infty(0, 0, d, d)$ is a portion of a shrinking cylinder.

Now we can continue as in the proof of Lemma 7.4 [67]. We iterate the procedure to that the neighbourhoods $\bar{\mathcal{P}}_j^\infty(0,0,L,\theta)$ of the rescaled flow converge to a cylinder. From the first step we know that, for j large enough, the curvature on $\bar{\mathcal{P}}_j^\infty(0,0,d,d)$ is close to the curvature of a unit cylinder. Then, using the gradient estimates we have uniform bounds on $\bar{G}_j$ also on some larger neighbourhoods, i.e. $\bar{\mathcal{P}}_j^\infty(0,0,2d,2d)$, we can repeat the previous argument to prove convergence to a cylinder there. After a finite number of iterations we can obtain convergence of the neighbourhoods $\bar{\mathcal{P}}_j^\infty(0,0,L,\theta)$. The immersions converge in the C^{k_0-1}-norm and this ensures that for j large enough, the neighbourhoods are $(\epsilon, k_0-1, L, \theta)$-shrinking necks. This contradicts assumption (d) and proves part (i) of the Lemma.

To prove part (ii) of the Lemma we argue in a similar fashion. Again we argue by contradiction and take a sequence of rescaled flows. Consider the smaller parabolic neighbourhoods $\bar{\mathcal{P}}(0,0,L-1,\frac{\theta}{2})$ and apply interior regularity results from [29] to find bounds in the C^{k+1} norm as well. This yields compactness in the C^k-norm, which gives the desired result. □

Remark 3.25. *Once we know that $H = CH$ for some constant C depending only n we could argue as in Proposition 3.8 [16]. Looking at the evolution equation for G Lemma 3.4(v) and the upper bound of the evolution equation for H found in Lemma 3.2 of [16] we have:*

$$\frac{\partial}{\partial t}G = \frac{\partial G}{\partial h_{ij}}(D_iD_jG + h_{ik}h_{jk}G)$$

and

$$\frac{\partial}{\partial t}H \leq \frac{\partial G}{\partial h_{ij}}(D_iD_jH + h_{ik}h_{jk}H) - \frac{1}{C}\frac{|\nabla h|^2}{G}.$$

Since $H = CH$, we can conclude that the second fundamental form is parallel, i.e. $|\nabla h|^2 = 0$ at each point in spacetime, therefore $\mathcal{M}_t$ is contained in a cylinder.

Remark 3.26. *Alternatively one could also argue as in in the proof of the Neck Detection Lemma in [16]. In which they look at the sequence of manifolds $\mathcal{M}_{t_k}^{(k)}$ and replace condition (b) with*

$$G(p_k,t_k) \geq k \quad \textit{and} \quad \frac{\lambda_1(p_0,t_0)}{G(p_0,t_0)} \leq \frac{1}{k}.$$

Since this is true for all values of k there exists a point such that $\lambda_1(p_k,t_k) \leq 0$. Then we can apply Proposition 3.8 from [16] analogous to the above remark to obtain the result.

Remark 3.27. *Here we wish to briefly show why we cannot prove the neck detection lemma in the same way as in Lemma 7.4 [67]. Arguing as in Section 3 [65] we take $P(W)$ a symmetric polynomial of degree α where W is the Weingarten map, then*

$$\begin{aligned}
\nabla_p\nabla_q P &= \nabla_p\left(\frac{\partial P}{\partial h_{lm}}\nabla_q h_{lm}\right)\\
&= \left(\frac{\partial^2 P}{\partial h_{lm}\partial h_{ab}}\nabla_p h_{ab}\nabla_q h_{lm} + \frac{\partial P}{\partial h_{lm}}\nabla_p\nabla_q h_{lm}\right)\\
\Rightarrow \frac{\partial P}{\partial h_{lm}}\nabla_p\nabla_q h_{lm} &= \left(\nabla_p\nabla_q P - \frac{\partial^2 P}{\partial h_{lm}\partial h_{ab}}\nabla_p h_{ab}\nabla_q h_{lm}\right) \qquad (3.28)
\end{aligned}$$

Now

$$\frac{\partial P}{\partial t} = \frac{\partial P}{\partial h_{lm}}\frac{\partial h_{lm}}{\partial t}$$
$$= \frac{\partial P}{\partial h_{lm}}\left(\nabla_l\nabla_m G + h_{lk}h_{mk}G\right) \tag{3.29}$$

Since $G = G(h_{ij})$ *we have*

$$\begin{aligned}\nabla_l\nabla_m G &= \nabla_l\left(\frac{\partial G}{\partial h_{pq}}\nabla_m h_{pq}\right)\\ &= \frac{\partial^2 G}{\partial h_{pq}\partial h_{ab}}\nabla_l h_{ab}\nabla_m h_{pq} + \frac{\partial G}{\partial h_{pq}}\nabla_l\nabla_m h_{pq}\\ &= \frac{\partial^2 G}{\partial h_{pq}\partial h_{ab}}\nabla_l h_{ab}\nabla_m h_{pq} + \frac{\partial G}{\partial h_{pq}}\nabla_p\nabla_q h_{lm} \quad \textit{using geodesic and Codazzi.}\end{aligned} \tag{3.30}$$

Using (3.28) and (3.29) we obtain

$$\begin{aligned}\frac{\partial P}{\partial t} =& \frac{\partial G}{\partial h_{pq}}\left(\nabla_p\nabla_q P - \frac{\partial^2 P}{\partial h_{lm}\partial h_{ab}}\nabla_p h_{ab}\nabla_q h_{lm}\right)\\ &+ \frac{\partial P}{\partial h_{lm}}\left(\frac{\partial^2 G}{\partial h_{pq}\partial h_{ab}}\nabla_l h_{ab}\nabla_m h_{pq} + h_{lk}h_{mk}G\right).\end{aligned} \tag{3.31}$$

Substituting $Q_n = \frac{S_n}{S_{n-1}}$ *for* P *into (3.31) we obtain*

$$\begin{aligned}\frac{\partial Q}{\partial t} =& \frac{\partial G}{\partial h_{lm}}\left(\nabla_l\nabla_m Q - \frac{\partial^2 Q}{\partial h_{pq}\partial h_{ab}}\nabla_l h_{ab}\nabla_m h_{pq}\right)\\ &+ \frac{\partial Q}{\partial h_{lm}}\left(\frac{\partial^2 G}{\partial h_{pq}\partial h_{ab}}\nabla_l h_{ab}\nabla_m h_{pq} + h_{lk}h_{mk}G\right).\end{aligned}$$

It follows that to apply the maximum principle for the convex case it remains to show that

$$-\frac{\partial G}{\partial h_{lm}}\frac{\partial^2 Q}{\partial h_{pq}\partial h_{ab}} + \frac{\partial Q}{\partial h_{lm}}\frac{\partial^2 G}{\partial h_{pq}\partial h_{ab}} \geq 0.$$

This is equivalent to showing that

$$F(\lambda) = -\frac{\partial G}{\partial \lambda_k}\frac{\partial^2 Q}{\partial \lambda_i\partial \lambda_j} + \frac{\partial Q}{\partial \lambda_k}\frac{\partial^2 G}{\partial \lambda_i\partial \lambda_j} \geq 0.$$

However we have currently not been able to show that this is true.

Without this we are unable to argue as in the proof of Lemma 7.4 [67] to show that $\tilde{Q}_n \equiv 0$ *and hence* $\tilde{\lambda}_1 \equiv 0$.

Corollary 3.32 (Analogous to Corollarly 7.7 [67])**.** *Given* $\epsilon, \theta > 0, L \geq 10$ *and* $k > 0$ *an integer, we can find* $\eta_0, G_0 > 0$ *such that the following holds Let* p_0, t_0 *satisfy* $(ND1)$ *and* $(ND2)$ *of Lemma 3.24. Then*

(i) The point (p_0, t_0) *lies at the centre of a cylindrical graph of length* $2(L-2)$ *and* C^{k+2}*-norm less than* ϵ*;*

(ii) The point (p_0, t_0) lies at the centre of a normal $(\epsilon, k, L-2)$-hypersurface neck.

Proof. Using Proposition 5.17 and Theorem 2.16 both properties are true is (p_0, t_0) lies at the centre of a $(\epsilon', k', L-1)$-curvature neck for suitable ϵ', k'. Therefore it is clear that the properties hold if (p_0, t_0) lies at the centre of a $(\epsilon', k', L-1, \frac{\theta}{2})$-shrinking curvature neck. It remains to apply Neck Detection Lemma(ii) with parameters $(\epsilon', k', L, \theta)$ and use the corresponding values of η_0, G_0. ☐

The next lemma shows us that the shrinking curvature necks obtained by Lemma 2.46 are equivalent to hypersurface necks for any given time, even surgery times.

We will require the following concept and notation. A point (p, t) lies at the centre of a neck if $p \in \mathcal{M}$ lies at the centre of a neck with respect to the immersion $F(\cdot, t)$. We want a formula for the radius at time s of a standard n-dimensional cylinder evolving by G-flow, to do this we introduce the following for $s \leq 0$,

$$\rho(r, s) = \sqrt{r^2 - 2(n-1)s} \tag{3.33}$$

where the radius is equal to r at time $s = 0$. Moreover we have

$$r \leq \rho(r, s) \leq 2r \text{ for all } s \in [d^{\#} r^2, 0] \tag{3.34}$$

otherwise the cylinder would violate Lemma 3.21.

Lemma 3.35 (Analogous to Lemma 2.49 and Lemma 7.9 [67])**.** *In Lemma 3.24, we can choose the constants η_0, G_0 so that the additional following property holds. Suppose that $L \geq 10$ and that $\theta \leq d^{\#}$. Denote*

$$r_0 = \frac{(n-1)(n-2)}{2G(p,t)}, \ B_0 = B_{g(t_0)}(p_0, r_0 L).$$

Then for any $t \in [t_0 - \theta r_0^2, t_0]$, the point (p_0, t) lies at the centre of a $(\epsilon, k_0 - 1)$-hypersurface neck $N_t \subset B_0$, satisfying the following properties:

(i) The mean radius $r(z)$ of every cross section of N_t is equal to $\rho(r_0, t - t_0)(1 + O(\epsilon))$;

(ii) The length of N_t is at least $L - 2$;

(iii) There exists a unit vector $\omega \in \mathbb{R}^{n+1}$ such that $|\nu(p, t) \cdot \omega| \leq \epsilon$ for any $p \in N_t$.

Proof. Proved in the same way as part Lemma(i) 3.24. By contradiction, we take a suitable η_0, G_0, then our parabolic neighbourhood is as close as we wish to an exact cylinder evolving by G-flow over the same time interval. The cylinder has radius r_0 at the final time, hence it has radius $\rho(r_0, t - t_0)$ at previous times.

At the final time, C_{t_0} is a neighbourhood of radius $r_0 L$ of p_0. Let $B_L \subset C$ be the set of points of C having intrinsic distance less than L from p_0. Clearly, B_L cannot be written in the form $S^{n-1} \times [a, b]$ for any $a, b,$. However, it is easy to see that for $L \geq (\pi^2 + 1)/2$, then $S^{n-1} \times [-(L-1), (L-1)] \subset B_L \subset S^{n-1} \times [-L, L]$. Using this logic, we can see C_{t_0} contains a sub-cylinder of length $2(L-1)$. The same sub-cylinder is contained in C_t for $t < t_0$; however since the scaling factor is given by $\rho(r_0, t - t_0)$ rather than r_0, then length of the sub-cylinder becomes $\frac{2r_0(L-1)}{\rho(r_0, t-t_0)}$. Recalling (3.34), we see the sub-cylinder has length at least $2(L-1)$ for the times under consideration. Since we can make our parabolic neighbourhood as close as we wish in the $(k_0 - 1)$-norm to the cylinder C_t we can find a geometric neck parametrizing the part of the neighbourhood corresponding to the sub-cylinder found above, and this neck will satisfy properties (i) and (ii). Property (iii) follows from choosing ω to be the axis of our cylinder C_t. $\square$

Just as before in the mean curvature flow case, we rely on *(ND2)* and seen it is an essential assumption in the proof of the Neck Detection Lemma. The next result ensures that *ND2* will follow from our other assumptions in the Neck Detection Lemma so long as the curvature at the point (p,t) is large compared the curvature of regions previously modified by surgeries.

Lemma 3.36 (Analogous to Lemma 2.50 and Lemma 7.10 [67])**.** *Consider a flow with surgeries satisfying the same assumptions of Lemma 3.24 excluding (ND2). Let* $d^{\#}$ *be as before and let* ϵ, k, L, θ *be given with* $\theta < d^{\#}$*. Then we can find* η_0, G_0 *with the following property. Let* (p_0, t_0) *be any point satisfying*

$$G(p_0,t_0) \geq \max\{G_0, 5K\}, \ \frac{\lambda_1(p_0,t_0)}{G(p_0,t_0)} \leq \eta_0$$

where K *is the maximum of the curvature at the points changed in the surgeries at times before* t_0*. Then* (p_0, t_0) *satisfies hypothesis* $(ND2)$ *and the conclusions Lemma 3.24 hold. In addition, the neighbourhood*

$$\mathcal{P}\left(p_0, t_0, \frac{(n-1)(n-2)}{2G(p_0,t_0)}L, \frac{(n-1)^2(n-2)^2}{4K^2}d^{\#}\right),$$

which is larger in time than $(ND2)$ *does not contain surgeries.*

Proof. Let ϵ, k, L, θ be given, with $\theta \leq d^{\#}$. The constants η_0, G_0 depend continuously on the parameters L, θ measuring the size of the parabolic neighbourhood. Thus, if $L_2 > L_1$ and $\theta_2 > \theta_1 > 0$ it is possible to find η_0, H_0 which apply to any $L \in [L_1, L_2]$ and $\theta \in [\theta_1, \theta_2]$. Thus we can find values η_0, G_0 such that the conclusions of the Neck Detection Lemma hold for our choice of (ϵ, k, L, θ), and also if we replace L with any $L' \in [d^{\#}, L]$. In addition, we can also assume that $G_0 \geq 2G^{\#}$. We claim that such values of η_0, G_0 satisfy the conclusions of the present lemma.

By our choice of η_0, G_0, the conclusions fail only if *(ND2)* is not satisfied, i.e.$\hat{\mathcal{P}}(p_0, t_0, L, \theta)$ contains surgeries.

By Lemma 3.21, at least the neighbourhood $\hat{\mathcal{P}}(p_0, t_0, d^{\#}, 0)$ does not contain surgeries. Therefore, if *(ND2)* s violated, there exists a maximal $L' \in [d^{\#}, L]$ such that $\hat{\mathcal{P}}(p_0, t_0, L, \theta)$ does not contain surgeries. We apply the Neck Detection Lemma to this neighbourhood and deduce that it is an $(\epsilon, k_0 - 1, L', \theta)$-shrinking neck. In particular $G(p_0, t_0)(1 + O(\epsilon)) \geq 4K$ for all p such that $d_{g(t_0)}(p_0, p) \leq \hat{r}(p_0, t_0)d^{\#}$. But then Lemma 3.21 shows that the larger neighbourhood $\hat{\mathcal{P}}(p_0, t_0, L' + d^{\#}, \theta)$ does not contain surgeries as well, contradicting the maximality of L'. This proves *(ND2)* holds and that the Neck Detection Lemma can be applied to the whole neighbourhood $\hat{\mathcal{P}}(p_0, t_0, L, \theta)$.

To obtain the last claim, take any q such that

$$d_{g(t_0)}(q, p_0) \leq \frac{(n-1)(n-2)}{2G(p_0,t_0)}L.$$

By the previous part of the statement, $G(q, t_0) = G(p_0, t_0)(1 + O(\epsilon)) > 2K$. Then, Lemma 3.21(ii) implies that q has not been affected by any surgery between time $t_0 -$

$(n-1)^2 d^{\#}/K^2$ and t_0. Since this holds for any q in the neighbourhood, the statement is proved. $\square$

[Section 7, [67]] For the next result we assume that our flow with surgeries satisfies certain properties, which we will list below:

(g1) Pick a fixed value $K^* > 2H^{\#}$, all surgeries will take place at cross-sections Σ_{z_0} of normal necks with radius $r(z_0) = r^* = \frac{(n-1)}{K^*}$.

(g2) On normal necks where the surgery has taken place we will have two portions with the following properties. One of the portions will belong to a component which will be discarded after the surgery. On the other portion, the part of the neck which has been left unchanged by the surgery has the following structure: the cross section which coincides with the boundary of the region changed by surgery satisfies $r(z) \leq (11/10)r^*$, on the last section $r(z) \geq 2r^*$ and in the sections in between $r^* \leq r(z) \leq 2r^*$.

(g3) Surgery is responsible for removing regions with G-curvature larger than $10K^*$. For example, looking back at a previous surgery, we will find the components which were discarded to have G-curvature larger than $10K^*$, so if they surgery had not taken place it would have not been disconnected from the surface.

If the neck parameter ϵ_0 is chosen small enough, then (g1) tells us that the areas modified by surgery will have G between $K^*/2$ and $2K^*$ after the surgery. (g1) also implies that $r(z) \leq (11/10)r^*$ on the first cross section. Property (g3) is a natural assumption as we wish the reduce the curvature by a certain amount each time we perform surgery. Whilst (g1) and (g3) together imply that the regions with largest curvature are not the ones affected by surgery but the ones that become disconnected from the surface and removed as they have known topology. Lastly, property (g3) tells us that surgeries are actually performed at a certain distance away from the ends of the neck, this will be useful in the next lemma.

Lemma 3.37 (Analogous to Lemma 2.52 and Lemma 7.12 [67])**.** *Consider a flow with surgeries with our usual assumptions, and (g1)-(g3). Let $L, \theta > 0$ be such that $\theta \leq d^{\#}$ and $L \geq 20$. Then there exist η_0, G_0 such that the following property holds. Let (p_0, t_0) satisfy $(ND1)$ and $(ND2)$ of the Neck Detection Lemma, and suppose in addition that the parabolic neighbourhood $\hat{\mathcal{P}}_G(p_0, t_0, L, \theta)$ is adjacent to a surgery region. Then (p_0, t_0) lies at the centre of a hypersurface neck N of length at least $L-3$, which is bordered on one side by a disc D. The value of G on $N \cup D$ at time t_0 is less than $5K^*$, where K^* is defined above in property (g1).*

Proof. Begin by applying Neck Detection Lemma(i) to find η_0,G_0 such that any point (p_0, t_0) satisfying *(ND1)* and *(ND2)* lies at the centre of a $(\epsilon, k_0 - 1, L, \theta)$ shrinking curvature neck. By refining the choice of η_0, G_0 we can also obtain that for all times under consideration the neck can be parametrised as a geometric neck. Let us also set

$$r_0 = \frac{(n-1)(n-2)}{2G},\ B_0 = \{p \in \mathcal{M} \mid d_{g(t_0)}(p, p_0) \leq r_0 L\}$$

Our assumptions are that B_0 is not modified by any surgery for $t \in [t_0 - \theta r_0^2, t_0]$, but that there is a point $q_0 \in \partial B_0$ and a time $s_0 \in [t_0 - \theta r_0^2, t_0]$ such that q_0 lies in the closure of a region modified by surgery at time s_0. Our aim is to now show that the structure is not affected by the other surgeries which may occur between time s_0 and t_0.

Let us denote by D^* the region modified by the surgery which includes q_0 in its closure, and let N^* be the part of the neck left unchanged with the properties described in (g2). Let us denote by Σ_1^* and Σ_2^* the two components of ∂N^* having mean radius less than $(11/10)r^*$ and greater than $2r^*$ respectively. By (g2), $\Sigma_1^* = \partial D^*$, and so $q_0 \in \Sigma_1^*$. It follows that the mean radius of Σ_1^* is equal to $\frac{(n-1)(n-2)}{2G(q_0,s_0)}$ up to an error of order $O(\epsilon)$. Then we know that $G(p, s_0) \geq \frac{(n-1)(n-2)}{2r^*}(10/11 + O(\epsilon)) = K^*(10/11 + O(\epsilon))$ for all $p \in B_0$, because the fully non-linear curvature G is constant up to $O(\epsilon)$ on B_0 at any fixed time.

We claim that B_0 must be contained in N^*. In fact, we know that B_0 has not been changed by the surgery at time s_0, and so it has no common points with D^*. If B_0 were not contained in N^*, then it would intersect the other component Σ_2^* of ∂N^*. But this is impossible, since at time s_0 the points in B_0 and in Σ_2^* have a value of G respectively greater than $(10/11)K^*$. and less than $K^*/2$ up to $O(\epsilon)$.

Let $z \in [z_1, z_2]$ be the parameter describing the cross-sections of N^*, where $z = z_i$ corresponds to Σ_i^*. Then we can find a maximal interval $[a, b] \subset [z_1, z_2]$ such that the neck corresponding to $z \in [a, b]$ is centred at p_0 and is contained in B_0. Let us denote by N_0 this neck. Arguing as in Lemma 3.35, we can see that N_0 has a length at least $L - 2$.

Let us now denote with N' the part of N^* corresponding to $z \in [z_1, a]$. Then we have that p_0 belongs to N_0, which is a normal k_0-hypersurface neck of length at least $L-2$, and which is bordered on one side by the region $N' \cup D^*$, which is diffeomorphic to a disc. This is the statement of our theorem, except it holds at time s_0 rather than the final t_0.

It remains to show that, if there are any surgeries between time s_0 and t_0, they do not affected the region $N_0 \cup N' \cup D^*$. Observe that in this region $G(p, s_0) \leq 2K^*$ for any p in this region. By our choice of $D^\#, G^\#$ we have $G(p, t) \leq 4K^*$ for any $p \in N_0 \cup N' \cup D^*$ and t between s_0 and either t_0 or the first surgery time, if it exists, that affect this region. But this shows that there cannot be any such surgery. Since N_0 is contained in B_0, which by assumption is not changed by surgeries in $[s_0, t_0]$, the neck N_0 disconnects the region $D^* \cup N'$ from the rest of the manifold. By (g3), if a surgery changes this part, it must disconnect a region contained in $N_0 \cup N' \cup D^*$ here the maximum of the curvature is at least $10K^*$. This contradicts the bound on the curvature we just found, which proves that the topology of the region does not change up to time t_0, and that the curvature remains below the value $5K^*$ in this region.

To conclude the proof, it suffices to parametrise the geometric neck N_0 in normal form at the final time t_0, using the property that $N_0 \subset B_0$ which is an $(\epsilon, k_0 - 1)$ curvature neck at any fixed time. □

Just like we dealt with the special case that (*ND2*) does not hold we will also have to deal with the special case that (*ND1*) does not hold. In this case we require a result for when the point under consideration $\frac{\lambda_1}{G}$ may not be small. This is a general property of hypersurfaces and not related to geometric flows, so the proof is exactly as in [67].

Theorem 3.38 (Analogous to Theorem 2.53 and Theorem 7.14 [67]). *Let $F : \mathcal{M} \to \mathbb{R}^{n+1}$, with $n > 1$ be a smooth connected immersed hypersurface (not necessarily closed). Suppose that there exist $c^\#, G^\# > 0$ such that $|\nabla G(p)| \leq c^\# G^2(p)$ for all $p \in \mathcal{M}$ such that $G(p) \geq G^\#$. Then, for any $\eta_0 > 0$ we can find $\alpha_0 > 0$ and $\gamma_0 > 1$ depending only on $c^\#$ and η_0, such that the following holds. Let $p \in \mathcal{M}$ satisfy $\lambda_1(p) > \eta_0 G(p)$ and $G(p) \geq \gamma_0 G^\#$. Then either $\mathcal{M}$ is closed with $\lambda_1 > \eta_0 G > 0$ everywhere, or there exists a point $q \in \mathcal{M}$ such that*

(i) $\lambda_1(q) \leq \eta_0 G(q)$

(ii) $d(p,q) \leq \frac{\alpha_0}{G(p)}$

(iii) $G(q') \geq \frac{G(p)}{\gamma_0}$ *for all* $q' \in \mathcal{M}$ *such that* $dist(p,q') \leq \frac{\alpha_0}{G(p)}$*; in particular,* $G(q) \geq \frac{G(p)}{\gamma_0}$.

Proof. Given $\alpha_0 > 0$, set $\gamma_0 = 1 + c^{\#}\alpha_0$. For a given $p \in \mathcal{M}$, let us set $\mathcal{M}_{p,\alpha_0} = \{q \in \mathcal{M} | d(p,q) \leq \alpha_0/G(p)\}$. By 3.17, we obtain that, if $G(p) \geq \gamma_0 G^{\#}$, then

$$G(q) \geq \frac{G(p)}{1 + c^{\#}d(p,q)G(p)} \geq \frac{G(p)}{\gamma_0}$$

for all $q \in \mathcal{M}_{p,\alpha_0}$.

To prove the theorem suppose now that $p \in \mathcal{M}$ is such that $G(p) \geq \gamma_0 G^{\#}$ and that $\lambda_1(q) > \eta_0 G(q)$ for all $q \in \mathcal{M}_{p,\alpha_0}$. We claim that is α_0 is suitably large, these properties imply that $\mathcal{M}$ coincides with $\mathcal{M}_{p,\alpha_0}$ and is therefore compact with $\lambda_1 > \eta_0 G$ everywhere.

To prove this, we show that the Gauss map $\nu : \mathcal{M}_{p,\alpha_0} \to S^n$ is surjective. Take any $\omega \in S^n$, such that $\omega \neq \pm\nu(p)$. We consider the curve γ as the solution of the ODE, $\dot{\gamma} = \frac{\omega^T(\gamma)}{|\omega^T(\gamma)|}$ with $\gamma(0) = p$, where for any $q \in \mathcal{M}, \omega^T(q) = \omega - \langle\omega, \nu(p)\rangle\nu(q)$ is the component of ω tangential to $\mathcal{M}$ at q. Since $|\dot{\gamma}| = 1$, the curve γ will be parametrized by arc length. The curve can be continued until $|\omega^T(\gamma)| \neq 0$, i.e. $\nu(\gamma) \pm \omega$. As long as $\gamma(s)$ is contained in $\mathcal{M}_{p,\alpha_0}$, which is the case if $s \in [0, \alpha_0/G(p)]$, we can use the property $\lambda_1 > \eta_0 G$ to derive some estimate.

Namely, if we take an orthonormal basis $e_1, \ldots, e_n$ of the tangent space to $\mathcal{M}$ at point $\gamma(s)$, we have

$$\begin{aligned}
\frac{d}{ds}\langle\nu, \omega\rangle &= \sum_{i=1}^{n} \langle\dot{\gamma}, e_i\rangle\langle\nabla_{e_i}\nu, \omega\rangle = \frac{1}{\omega^T} h_{ij}\langle\omega, e_i\rangle\langle\omega, e_j\rangle \\
&\geq \frac{1}{\omega^T}\eta_0 H|\omega^T|^2 \\
&= \eta_0 G\sqrt{1 - \langle\nu, \omega\rangle^2},
\end{aligned}$$

which implies $\frac{d}{ds}\arcsin\langle\nu, \omega\rangle \geq \eta_0 G$.

Now suppose that $\gamma(s)$ exists for $s \in [0, \alpha_0/G(p)]$. Then we have

$$\begin{aligned}
\pi &> \arcsin\left\langle \nu\left(\gamma\left(\frac{\alpha_0}{G(p)}\right)\right), \omega\right\rangle - \arcsin\langle\nu(p), \omega\rangle \\
&\geq \eta_0 \int_0^{\alpha_o/G(p)} G(\gamma(s))ds \\
&\geq \eta_0 \int_0^{\alpha_0/G(p)} \frac{ds}{G(p)^{-1} + c^{\#}s} \\
&= \frac{\eta_0}{c^{\#}} \ln(1 + c^{\#}\alpha_0).
\end{aligned}$$

Thus, if α_0 is large enough to have

$$\alpha_0 > \tfrac{1}{c^{\#}}(\exp(\tfrac{c^{\#}\pi}{\eta_0}) - 1)$$

we obtain a contradiction. Therefore there exists $s^* \in (0, \alpha_0/G(p)]$ such that either $\langle\nu(\gamma(s)), \omega\rangle \to 1$ or $\langle\nu(\gamma(s)), \omega\rangle \to -1$ as $s \to s^*$. Since $\arcsin\langle\nu, \omega\rangle$ is increasing, only the first possibility can occur. This shows that $\gamma(s)$ converges, as $s \to s^*$, to some point $q^* \in \mathcal{M}_{p,\alpha_0}$ such that $\nu(q^*) = \omega$, as desired.

It remains to consider the case when $\omega = \pm\nu(p)$, when ω trivially belongs to the image of the Gauss map. If instead we have $\omega = -\nu(p)$, it suffices to replace p with another point p' sufficiently close to p: by convexity, we have $\nu(p') \neq \nu(p) = -\omega$ and the previous argument can be applied.

Thus, we have proved that the Gauss map is surjective from $\mathcal{M}_{p,\alpha_0}$ to S^n. Since $\lambda_1 > 0$ on $\mathcal{M}_{p,\alpha_0}$, the Gauss map is also a local diffeomorphism. Then, since S^n is simply connected for $n > 1$, it follows that the map is a global diffeomorphism. $\square$

Putting all this together, we are able to provide a result about the existence of necks before a first singular time is approached.

Corollary 3.39 (Analogous to Corollary 2.54 and Corollary 7.15 [67])**.** *Let $\mathcal{M}_t$ be a smooth G-flow of closed 2-convex hypersurfaces. Given neck parameters ϵ, k, L there exists G^* (depending on $\mathcal{M}_0$) such that, if $G_{max}(t_0) \geq G^*$, then the hypersurface at time t_0 either contains an (ϵ, k, L)-hypersurface neck or it is convex.*

Proof. Combine Lemma (3.37) with Theorem (3.38). Since we assume the flow is smooth, the parabolic neighbourhood in hypothesis *(ND2)* trivially does not contain surgeries. $\square$

Before we move onto the next section, we will need to prove two more results which are required to prove the main theorem in the Section 3.2. The results here are analogous and as described in the latter part of Section 7 from [67].

Let N be an (ϵ, k)-hypersurface neck contained in a closed hypersurface $\mathcal{M}$, with $k \geq 1$ where z is the parameter along the neck. We know that locally N can be represented as a cylindrical graph, we pick a point $p_0 \in N$ such that p_0 is at the centre of a cylindrical graph $N_1 \subset N$ on a C^1-norm less than $\epsilon_1 > 0$. Next we choose a unit vector ω such that ω is parallel to the axis of N_1 which we will denote by x_{n+1}, moreover we orient ω such that it points in the direction of increasing x_{n+1}. We then set $y = x_{n+1}$ and assume p_0 lies on the $y = 0$ plane, we call the vertical direction the direction of the y-axis and any direction which is orthogonal to the y-axis horizontal.

There are two different parametrisations for N_1; the cylindrical graph and the one induced by the normal parametrisation of N. The two are very similar, with the exception that where z is constant, y can vary by as much as $O(\epsilon)$ and vice versa. Note that z is scale invariant, whilst y is not, so as we increase Δy in the y-coordinate corresponds to an approximate increase $r(z)\Delta y$ in the z-coordinate. We assume an orientation in such a way that the directions of increasing y and z agree.

The key object to look at in this setting is $\omega \cdot \nu$ and how it will vary over N. If $\langle \omega, \nu \rangle$ is small at some point on N we know that the axis of N is almost parallel ω. Moreover if $\langle \omega, \nu \rangle > 0$ then we can deduce that the radius of our neck is getting smaller.

To study this quantity we will introduce an ODE. Let Σ_0 be the intersection of the cylindrical graph N_1 with the $y = 0$ plane. Then by construction, we have $|\omega \cdot \nu(p)| \leq \epsilon_1$ for all $p \in \Sigma_0$. Let us consider, for any $p \in \Sigma_0$, the curve $\gamma(p, \tau)$ satisfying the equation

$$\begin{cases} \dot{\gamma} = \frac{\omega^T(\gamma)}{|\omega^T(\gamma)|^2}, & \tau \geq 0 \\ \gamma(0) = p \end{cases} \tag{3.40}$$

where $\dot{\gamma} = \frac{d}{d\tau}\gamma$. Moreover we have that $y(\gamma(p, \tau)) = \tau$ for all $p \in \Sigma_0$, thus we can write $\gamma(p, y)$ instead of $\gamma(p, \tau)$ since τ and y coincide along γ. Without loss of generality we will consider $\gamma(p,y)$ for $\gamma \geq 0$. We will follow the trajectories until ω is not orthogonal to γ, it is at this point that they are no longer well defined, this is longer than they remain inside N_1 or N. Since we are studying the flow for compact surfaces this is only valid for a finite value of $y = y_{max} > 0$ such that $\gamma(p,y)$ is defined for all $p \in \Sigma_0$ and $y \in [0, y_{max})$, and such that $\omega^T(\gamma(p,y)) \to 0$ as $y \to y_{max}$ at least for some p.

For all points $\bar{y} \in (0, y_{max})$, we set $\Sigma_{\bar{y}} = \{\gamma(p,\bar{\gamma})\}|p \in \Sigma_0\}$. We are able to deduce that $\Sigma_{\bar{y}}$ is a smooth $(n-1)$-dimensional surface contained in the $y = \bar{y}$ hyperplane. This surface is diffeomorphic to Σ_0 under the flow and hence diffeomorphic to S^{n-1}. We are then able to compare two different surfaces by considering their projections on a fixed horizontal n-dimensional hyperplane. We will say that the surfaces Σ_y are shrinking if the projection of Σ_{y_2} is contained in the subset of the hyperplane enclosed by the projection of Σ_{y_1} for any $y_2 \geq y_1$.

Proposition 3.41 (Analogous to Proposition 7.18 [67])**.** *Under the above hypothesis, suppose in addition that $\lambda_1 \geq \alpha \geq 0$ everywhere on N. then*

(i) For any $p \in \Sigma_0$ we have that $|\omega^{\perp}(\gamma(p,y))|$ is bounded away from zero as long as $\gamma(p,y) \in N$. Therefore, any curve $\gamma(p,y)$ is well defined as long as it is contained in N.

(ii) Along any trajectory $\gamma(p,y)$ we have $\frac{d}{dy}\langle \nu, \omega \rangle \geq \alpha$ as long as γ is contained in N.

(iii) The axis of the neck N is approximately equal to ω everywhere. More precisely, any representation of a subset of N as a cylindrical graph of C^1-norm of size $O(\epsilon)$ has an axis $\tilde{\omega}$ such that $1 - \langle \omega, \tilde{\omega} \rangle = O(\epsilon)$.

(iv) If for some $y \geq 0$ we have $v(q) \cdot \omega \geq 0$ for all $q \in \Sigma_{y_1}$, then the surfaces Σ_y are shrinking as long as they are contained in N.

Proof. To prove (ii) we proceed as in the proof of Theorem (3.38). We find that

$$\frac{d}{dy}\langle \nu, \omega \rangle = \sum_{i=1}^{n} \langle \dot{\gamma}, e_i \rangle \langle \nabla_{e_i} \nu, \omega \rangle = \frac{1}{|\omega^T|^2} \sum_{i,j=1}^{n} h_{ij} \langle \omega, e_i \rangle \langle \omega, e_j \rangle \geq \lambda_1 \geq \alpha. \tag{3.42}$$

To prove (iii) we use the fact that $|\langle \nu, \omega \rangle| \leq \epsilon_1$ on Σ_1 by construction , by (ii) we know that $\langle \nu, \omega \rangle$ is non-decreasing and therefore we have that $\langle \nu, \omega \rangle \geq -\epsilon_1$ along any trajectory $\gamma(p,y)$ as long as it stays inside N. Suppose on that $\tilde{\omega}$ is the axis of any cylindrical graph representation of a subset $\tilde{N} \subset N$. Then $|\nu(q) \cdot \tilde{\omega}| = O(\epsilon)$ for every $q \in \tilde{N}$. If $\omega \neq \tilde{\omega}$, let us define

$$v = \omega - \langle \omega, \tilde{\omega} \rangle \tilde{\omega}.$$

Then $|\nu| = \sqrt{1 - \langle \omega, \tilde{\omega} \rangle^2} \neq 0$ and v is orthogonal to $\tilde{\omega}$. On an exact cylinder with axis $\tilde{\omega}$ we can find points where the normal is $\pm\frac{v}{|v|}$. Since $\tilde{N}$ is close to a cylinder, we can find $q \in \tilde{N}$ such that $|v(q) + \frac{v}{|v|} = O(\epsilon)$. Then we have

$$-\epsilon_1 \leq v(q) \cdot \omega = \left(v(q) + \frac{v}{|v|} \right) \cdot \omega - \frac{v}{|v|} \cdot \omega \leq -\sqrt{1 - \langle \omega, \tilde{\omega} \rangle^2} + O(\epsilon,)$$

which shows that $\sqrt{1 - \langle \omega, \tilde{\omega} \rangle^2} = O(\epsilon)$. We can choose the orientation of $\tilde{\omega}$ such that $\langle \omega, \tilde{\omega} \rangle \geq 0$, then the above estimate shows that $\langle \omega, \tilde{\omega} \rangle = 1 - O(\epsilon)$ proving (iii).

Property (i) follows straight from property (iii). It remains to prove (iv). Consider the projections of Σ_y on a horizontal hyperplane. The exterior normal is given by $\nu - \langle \omega, \nu \rangle \omega$ up to a normalising factor. We can see that the horizontal component of $\dot{\gamma}$ points towards the interior of Σ_y provided $\langle \omega, \nu \rangle > 0$ by the following,

$$\begin{aligned} \langle \dot{\gamma}, \nu - \langle \omega, \nu \rangle \omega \rangle &= |\omega^T|^{-2}(\langle \omega^T, \nu - \langle \omega, \nu \rangle \omega \rangle \\ &= |\omega^T|^{-2}(\langle \omega^T, \nu \rangle - \langle \omega, \nu \rangle \langle \omega^T, \omega \rangle) \\ &= -\langle \omega, \nu \rangle. \end{aligned}$$

Also, if $\langle \omega, \nu \rangle > 0$ for some value of y, the same holds for all greater values of y by using (ii). $\square$

To conclude this section we provide one more lemma which will be useful when we study the trajectories of $\gamma(p, y)$ once they leave the neck N. In particular, if all submanifolds Σ_y have a small diameter, then the whole surface foliated by the Σ_y's has large G-curvature.

Lemma 3.43 (Analogous to Lemma 7.19 [67])**.** *Let $c^{\#}, G^{\#}$ be as in Theorem 3.10, and set $\Theta = 1 + (2+\pi)(n-1)c^{\#}$. Let us define the trajectories $\gamma(p, y)$ as in the previous Lemma. Suppose that, for some $0 \leq y_1 < y_2 < y_{\max}$, we have $\lambda_1(\gamma(p, y)) > 0$ for all $y \in [y_1, y_2]$, $p \in \Sigma_0$ and that $\omega \cdot \nu(p) \geq 0$ for all $p \in \Sigma_{y_1}$. Suppose also that Σ_{y_1} has a diamater equal to $2(n-1)/K$ for some $K \geq \Theta G^{\#}$, and that $G(p) \geq K$ for all $p \in \Sigma_{y_1}$. Then we have $G(\gamma(p, y)) \geq K/\Theta$ for all $y \in [y_1, y_2]$, $p \in \Sigma_0$.*

Proof. Using our assumptions and Lemma 3.37 we obtain that $\omega \cdot \nu > 0$ along all trajectories γ for $y \in [y_1, y_2]$. Then by Proposition 3.41(iv) we know that for all $y \in [y_1, y_2]$ the surfaces Σ_y are shrinking. By assumption, Σ_{y_1} is enclosed by an $(n-1)$-dimensional sphere of G-curvature K and of radius $R := (n-1)(n-2)/2K$. Therefore we know we can find a round cylinder with radius R and axis ω which encloses $\cup_{y \in [y_1, y_2]} \Sigma_y$.

Firstly lets consider when $y_2 - y_1 < R$. Then given any $p \in \Sigma_y$ we can find a $p' \in \Sigma_{y_1}$ such that $d(p, p') \leq 2R$. From Theorem 3.10 we obtain

$$G(p) \geq \frac{K}{1 + 2(n-1)c^*}.$$

Now suppose that $y_2 - y_1 \geq R$. Given any $y \in [y_1, y_2]$, let y' be such that $y \in [y', y'+R] \subset [y_1, y_2]$. We take a portion of a cone C having circular section, axis ω, lower and upper basis in the $y = y'$ and $y = y' + R$ hyperplanes respectively. By a suitable choice of the radii $R_1, R_2 \leq R$ of the upper and lower basis we can obtain that C touches $\cup_{y \in [y', y'+R]} \Sigma_y$ from the outside at some point q not lying in the $y = y'$ and $y = y' + R$-planes. Then $G(q)$ is greater than the G curvature of C at q, which is greater than K. Now, given any $p \in \Sigma_y$, it is easy to see that $d(p, q) \leq (2+\pi)R$. It follows that

$$G(p) \geq \frac{K}{1 + (2+\pi)(n-1)c^*}.$$

$\square$

Before we continue onto the next section, we will require the following bound between surgery times.

Theorem 3.44. *Let$\mathcal{M}$ be a G-flow with surgeries. Then the assumption that between two surgery times T_1 and T_2 the curvature must increase from $G*$ to $\alpha G*$ yields the following lower bound*

$$T_2 - T_1 \geq \frac{1}{2n\zeta} \frac{\alpha^2 - 1}{\alpha^2} \frac{1}{G^{*2}}.$$

where ζ is a constant depending only on n.

Proof. We know the evolution equation for G Lemma 3.4(v) is given by

$$\frac{\partial}{\partial t} G = \frac{\partial G}{\partial h_{ij}} (D_i D_j G + h_{ik} h_{jk} G).$$

Using the fact that $\frac{\partial G}{\partial t} \leq C_1 g_{ij}$ for some constant C_1 depending on n yields

$$\frac{\partial}{\partial t} G = C_1 g_{ij} (D_i D_j G + h_{ik} h_{jk} G).$$

We know that for two positive-definite matrices $\operatorname{tr}(AB) \geq 0$, this also tells us that $\operatorname{tr}(A(-B)) \leq 0$.

Now applying the maximum principle we obtain that $D_i D_j G$ is negative and we can bound $g_{ij} h_{ik} h_{jk}$ by $C_2 G^2$ for some constant C_2 depending only on n. Therefore

$$\frac{\partial}{\partial t} G \leq \zeta G^3$$

for some constant ζ depending only on n. A standard comparison results yields the desired estimate

$$T_2 - T_1 \geq \frac{1}{2n\zeta} \frac{\alpha^2 - 1}{\alpha^2} \frac{1}{G_1^2}.$$

$\square$

3.3 The Flow with Surgeries

Now that we have obtained all our results regarding neck detection, we can continue as in Section 2.1 or [67] to prove the following theorem. The proofs of this section will only need minor modification to those presented in Section 8 of [67] now that we have Theorem 3.44.

Theorem 3.45 (Analogous to Theorem 2.55 and Theorem 8.1 [67])**.** *Let $\mathcal{M}_0$ be a smooth closed two-convex hypersurface immersed in $\mathbb{R}^{n+1}$, with $n \geq 3$. Then there exist constants $G_1 < G_2 < G_3$ and a G-flow with surgeries starting from $\mathcal{M}_0$ with the following properties:*

- *Each surgery takes place at a time T_i such that $G_{\max}(T_i-) = G_3$.*
- *After the surgery, all the components of the manifold satisfy $G_{\max}(T_i+) \leq G_2$, except for those diffeomorphic to spheres or to $S^{n+1} \times S^1$, which are neglected afterwards.*
- *Each surgery starts from a cross section of a normal hypersurface neck with mean radius $r(z_o) = \frac{(n-1)(n-2)}{2G}$.*
- *The flow with surgeries terminates after finitely many steps.*

The constants G_i can be any values such that $G_1 \geq \omega_1$,$G_2 = \omega G_1$ and $G_3 = \omega_3 G_2$, with $\omega_i > 1$.

[Section 8 [67]] To prove this theorem we want to apply the Neck Detection Lemma in an iterative way. Given ϵ, k, L the Neck Detection Lemma will give us an η_0, G_0 such that any point (p_0, t_0) with $G(p_0, t_0) \geq G_0$ and $\lambda_1(p_0, t_0) \leq \eta_0 G(p_0, t_0)$ lies at the centre of a (ϵ, k, L)-neck. In particular, any point p in the neck satisfies $G(p, t_0) \approx G(p_0, t_0)$ and $\lambda_1(p, t0) \leq \epsilon G(p, t_0)$. In general η is much smaller than ϵ; thus the information on λ_1 in a general point of the neck is weaker than the hypothesis at the centre p_0.

However, we can let η_0 play the role of ϵ in a further application of the lemma. Namely we can find $\eta_0' G_0'$ such that any point (p_0, t_0) with $G(p_0, t_0) \geq G_0'$ and $\lambda_1(p_0, t_0) \leq \eta_0' G(p_0, t_0)$ lies at the centre of an $(\eta_0, 1, L)$-neck. We can choose $G_0' \geq G_0$. Then any point p of the $(\eta_0, 1, L)$-neck centred at p_0 will satisfy $G(p, t_0) \geq HG0$ and $\lambda_1(p, t_0) \leq \eta_0 G(p, t_0)$, thus is is the centre of an (ϵ, k, L)-neck.

Here we define how to choose our parameters for the surgery procedure depending only on the initial manifold $\mathcal{M}_0$. This choice of parameters is very similar to those described by Huisken and Sinestrari in Section 8 of [67].

(G1) (Choice of the neck parameters) We have defined a surgery procedure on (ϵ_0, k_0)-hypersurface necks in normal form of length L, where ϵ_0 must be suitably small, $k_0 \geq 2$ is any integer, and $L \geq 10 + 8\Lambda$, where Λ is the length parameter in surgery. We also assume that $L \geq 20 + 8\Lambda$ and that ϵ_0 is small enough so that, if $\mathcal{N}$ is a normal $(\epsilon_0, 1)$-hypersurface neck of length $2L$ then the G-curvature at any two points of $\mathcal{N}$ can differ by a factor of most 2.

(G2) (Summary of known parameters) We define $c^{\#}, H^{\#}$ as in Theorem 3.10, $d^{\#}$ as in Lemma 3.21 and Θ as in Lemma 3.43.

(G3) (First application of the Neck Detection Lemma) Choose η_0, K_0 such that if (p_0, t_0) satisfies

$$G(p, t_0) \geq K_0,\ \lambda_1(p, t_0) \leq \eta_0 G(p, t_0), \tag{3.46}$$

and if $\hat{\mathcal{P}}(p,t,L',\theta')$ does not contain surgeries for some $L' \in [L/4, L],\ \theta' \in [d^{\#}/1600, d^{\#}]$, then $\hat{\mathcal{P}}(p,t_0,L'\theta')$ is a shrinking neck and (p,t_0) lies at the centre of a normal (ϵ_0,k_0)-hypersurface neck of length at least $2L'-2$. We also require that η_0, K_0 are such that if (p_0,t_0) satisfies 3.46 and in addition $H(p,t_0) \geq 5k$, where K is the maximum of the G-curvature in the regions inserted in the surgeries, then the conclusions of Lemma 3.36 apply.

Finally, we also require that η_0, K_0 are such that Lemma 3.37 can be applied to the parabolic neighbourhood $\hat{\mathcal{P}}(p,t,L',\theta')$.

(G4) (Second application of the Neck Detection Lemma) Next we set $\epsilon_1 = (n-1)\eta_0/2$. We apply Corollary 3.32(ii) to find η_1, K_1 such that if (p,t_0) satisfies

$$H(p,t_0) \geq K_1,\ \lambda_1(p,t_0) \leq \eta_1 H(p,t_0) \tag{3.47}$$

and the parabolic neighbourhood $\hat{\mathcal{P}}(p,t_0,10,d^{\#}/1600)$ does not contain surgeries, then (p,t_0) lies at the centre of a cylindrical graph of length 5 and C^1-norm less than ϵ_1,. We will then choose η_1, K_1 such that $K_1 \geq K_0, K_1 \geq G^{\#}$ and $\eta_1 \leq \eta_0$.

(G5) (Application of Pinching Theorem 3.38 Now we choose γ_0 such that if $G(p,t_0) > \gamma_0 G^{\#}$ and $\lambda_1(p,t_0) > \eta_1 G(p,t_0)$ then either $\lambda_1 > \eta_1 G$ everywhere on $\mathcal{M}_{t_0}$ or there exists q such that $\lambda_1(q,t_0) \leq \eta_1 G(q,t_0)$ and such that $G(q',t_0) \geq G(p,t_0)/\gamma_0$ for all q' with $d_{t_0}(q',p) \leq d_{t_0}(q,p)$.

(G6) (Third application of the Neck Detection Lemma) Let us set

$$\theta_2 = (10^4 n^5 \zeta \Theta^2 \gamma_0^2)^{-1},$$

where ζ is the constant from Theorem 3.44. Then let us choose K_2, η_2 such that if $G(p,t_0) \geq K_2$, if $\lambda_1(p,t_0) \leq \eta_2 G(p,t_0)$ and if $\hat{\mathcal{P}}(p,t_0,10,\theta_2)$ does not contain surgeries, then (p,t_0) lies on a cylindrical graph of length 5 and C^1-norm less than ϵ_1. We also require $K_2 \geq K_1$.

(G7) We finally define G_1 to be any value such that $G_1 \geq 4\Theta K_2$, and then G_2, G_3 by

$$G_2 = 10\gamma_0 G_1,\ G_3 = 10 G_2.$$

To have a definitive value of these constants, one can simply pick $G_1 = 4\Theta K_2$. However, it is useful to note that the $G_i's$ can be also chosen arbitrarily large.

All of the constants chosen only depend on the parameters describing the initial surface. In proving the main theorem, we will define the surgery algorithm such that the following properties are satisfied

(S) Each surgery is performed on a normal (ϵ_0,k_0)-hypersurface neck, The surgery is performed at times T_i such that $G_{\max}(T_i) = H_3$. After the surgeries are performed, and suitable components whose topology is known are removed, we have $G_{\max}(T_i+) \leq G_2$. In addition, all surgeries satisfy properties (s1)-(s3) with $K^* = G_1$.

The proof of Theorem 3.45 will consist of a finite induction procedure. We start with a G-flow starting from our initial manifold $\mathcal{M}_0$, either smooth or with surgeries satisfying (S), defined up to some time t_0 such that $\max_{\mathcal{M}_{t_0}} G = G_3$. We then show we can perform a finite number of surgeries at time t_0 which also satisfy (S). We can then conclude that such a flow much terminate after a finite number of steps.

Before we go on to prove Theorem 3.45, we will need to obtain a lower bound between surgery times, using the property (S) and applying Theorem 3.44 to our case when $G_3 = 10G_2$ and $G_2 = 10\gamma_0 G_1$ we obtain that

$$T_{k+1} - T_k \geq \frac{10^2 - 1}{10^2} \frac{1}{2n\zeta G_2^2} > \frac{49}{10^4 n\zeta \gamma_0^2 G_1^2}. \tag{3.48}$$

Theorem 3.49 (Analogous to Theorem 2.56 and Theorem 8.2 [67])**.** *[Neck Continuation Theorem] Suppose that $\mathcal{M}_t$ with $t \in [0, t_0]$, is a Brendle-Huisken G-flow with surgeries satisfying (S), and let $\max_{\mathcal{M}_{t_0}} G \geq G_3$. Moreover, let p_0 be such that*

$$G(p_0, t_0) \geq 10G_1, \quad \lambda_1(p_0, t_0) < \eta_1 G(p_0, t_0), \tag{3.50}$$

where η_1, G_1 are as defined in (G1)-(G7). Then (p_0, t_0) lies on some (ϵ_0, k_0)-hypersurface neck N_0 in normal form, which either covers the whole component of $\mathcal{M}_{t_0}$ including p_0 or has a boundary consisting of two cross sections Σ_1, Σ_2, each of which satisfies either of the two following properties:

(i) Σ has mean radius $\frac{(n-1)(n-2)}{G}$

(ii) The cross-section of Σ is the boundary of a region D, diffeomorphic to a disc where the curvature is at least G/Θ. The region D lies after the cross-section Σ and is disjoint from N_0.

Proof. Take p_0 such that 3.50 is satisfied.

$$G(p_0, t_0) \geq 10K_1 \geq 10K_0, \quad \lambda(p_0, t_0) \leq \eta_1 G(p_0, t_0) \leq \eta_0 G(p_0, t_0).$$

Therefore at (p_0, t_0) we can apply neck detection at the ϵ_1-level (G3) and at the ϵ_0-level (G2).

Let us consider the ϵ_0-level first. We know that previous surgeries occurred on necks with curvature close to G_1 and thus $K = 2G_1$ is a bound from above for the curvature in the regions modified by surgeries. It follows that we can apply Lemma 3.36 with $K = 2G_1$. Since $G(p_0, t_0) \geq 10G_1$, we know from definition (G2) and Lemma 3.36 we can ensure that $\hat{\mathcal{P}}(p_0, t_0, L, d^{\#})$ does not contain surgeries and that (p_0, t_0) lies at the centre of a normal (ϵ_0, k_0)-hypersurface neck containing p_0. If N_0 covers the whole manifold then the proof is complete. Otherwise we need to show that starting from p_0, we can move in both directions and find a cross section of N_0 which satisfies either (i) or (ii).

Let z be a parameter of the neck in its normal parametrization. We choose it in such a way that the cross section containing p_0 corresponds to $z = 0$. Without loss of generality we follow the neck in the direction of increasing z, the argument is the same in the other direction. If there is a cross section with average radius $\frac{(n-1)(n-2)}{G_1}$ again we would be done. So instead we assume that no such cross section exists, i.e. $r(z) < \frac{(n-1)(n-2)}{G_1}$ for all $z \in [0, z_{\max}]$, where $z_{\max}$ is the last section of the neck. This also implies $G > G_1/4$ everywhere, until the last section of the neck. We need to show that in this case the neck is bordered by a disc.

How do we approach this problem? Well the Neck Detection Lemma ensures the neck can be continued as long as

(i) G is sufficiently large.

(ii) $\frac{\lambda_1}{G}$ is sufficiently small.

(iii) A suitable backward parabolic neighbourhood is surgery free.

Now since the neck must end one of these properties must fail. However the first cannot fail since $G > \frac{G_1}{4}$. If the second fails then the neck must close up until it ends with a convex cap. If the third is violated then we will use Proposition 3.37 to conclude the neck is bordered by a disc inserted by a previous surgery.

To add some rigour, we define a closed subset $\Omega \subset N_0$ of our neck as follows. We say a point $p \in \Omega$ if

(Ω1) $\lambda_1(p, t_0) \leq \eta_0 G(p, t_0)$

(Ω2) The backward parabolic neighbourhood $P\left(p, t_0, \frac{(n-1)(n-2)}{2G}L, \frac{(n-1)^2(n-2)^2}{(20G_1)^2}d^{\#}\right)$ is surgery free.

We want to show that points of Ω satisfy the hypothesis of the Neck Detection Lemma, therefore the neck cannot end as long as it contains such points. It will follow that the last part of N_0 does not contain points of Ω, and this will be later exploited to infer information on the last part of the neck.

Firstly using Lemma 3.36, a point that satisfies (Ω1) but not (Ω2) is necessarily such that $G(p, t_0) < 10G_1$. In particular, our starting point belongs to Ω. Moreover we recall that all points $p \in N_0$ on the side where $z \geq 0$ satisfy $G(p, t_0) \geq \frac{G_1}{4}$.

Therefore

$$\frac{(n-1)^2(n-2)^2}{(80)^2 G(p, t_0)} \leq \frac{(n-1)^2(n-2)^2}{(20G_1)^2}$$
$$\Rightarrow \hat{\mathcal{P}}\left(p, t_0, L, \frac{d^{\#}}{40^2}\right) \subset \mathcal{P}\left(p, t_0, \frac{(n-1)(n-2)}{2G(p, t_0)}L, \frac{(n-1)^2(n-2)^2}{(20G_1)^2}d^{\#}\right).$$

Therefore (G2) tells us that any $p \in \Omega$ lies at the centre of a normal (ϵ_0, k_0)-hypersurface neck of length $2L - 2$. Thus since the neck ends when $z = z_{\max}$, at least the sections with $z \in [z_{\max} - L + 1, z_{\max}]$ do not contain any point of Ω.

Let us define z^* to be the maximal value of z with the following property; the cross section of z^* contains a point $p_1 \in \Omega$, while there are no points of Ω for $z \in [z^*, z^* + 10]$. We can then consider the following cases

(a) There exists one point p_2 with $z \in [z^*.z^* + 10]$ satisfying (Ω1).

(b) All points $z \in [z^*, z^* + 10]$ do not satisfy (Ω1).

First let us consider case (a). This will be the case where we can find points which have been modified by previous surgeries and we can apply Proposition 3.37. However we first need to check that the hypothesis of Proposition 3.37 is satisfied. By definition, p_2 does not satisfy (Ω2), that is

$$\mathcal{P}\left(p_2, t_0, \frac{(n-1)(n-2)}{2G(p_2, t_0)}L, \frac{(n-1)^2(n-2)^2}{(20G_1)^2}d^{\#}\right) \tag{3.51}$$

is modified by some surgery. Recall that, by $(G1)$, G can vary at most by a factor of 2 in the part of the neck containing p_1 and p_2. Therefore we have

$$G(p_2, t_0) \geq \frac{G(p_1, t_0)}{2}$$

and

$$\begin{aligned} d_{g(t_0)}(p_1, p_2) < 2(\pi G_0)\frac{(n-1)(n-2)}{2G(p_2, t_0)} &< \frac{(n-1)(n-2)}{8G(p_2, t_0)} \\ \Rightarrow \mathcal{P}\left(p_2, t_0, \frac{(n-1)(n-2)}{2G(p_2, t_0)}\frac{L}{4}, \frac{(n-1)^2(n-2)^2}{(20G_1)^2}d^{\#}\right) \\ \subset \mathcal{P}\left(p_1, t_0, \frac{(n-1)(n-2)}{2G(p_1, t_0)}L, \frac{(n-1)^2(n-2)^2}{(20G_1)^2}d^{\#}\right). \end{aligned}$$

The neighbourhood on the right hand side does not contain surgeries because $p_1 \in \Omega$, this forces the neighbourhood on the left hand side to also not contain surgeries. Using continuity we can replace the L in 3.51 with a suitable $L' \in [L/4, L]$ to obtain a neighbourhood which is not modified by surgeries, but is adjacent to a surgery on the side of increasing z.

If we set $\Theta' = \frac{G(p_2,t_0)^2}{(10G_1)^2}d^{\#}$ we can denote such a neighbourhood as $\hat{\mathcal{P}}(p_2, t_0, L', \Theta')$.

Since $\frac{G_1}{4} \leq G(p_2, t_0) \leq 10G_1$, we have that $\frac{d^{\#}}{40}^2 \leq \Theta' \leq d^{\#}$. Using $(G2)$, we can apply Lemma 3.37 to conclude that (p_2, t_0) lies in a hypersurface neck N bounded on one side by a disc D. The same lemma tells us that the G-curvature on $N \cup D$ is strictly less than $10G_1$. The hypersurface neck N can be combined with N_0 to form a unique neck. The side bordered by D must be in the direction of increasing z, otherwise N should include all of the neck N_0 which is impossible since N_0 contains p_0 which satisfies $G(p_0, t_0) \geq 10G_1$. Thus the theorem is proved in this case.

Now we consider case (b). Assume that all points in N_0 with $z \in [z^*, z^* + 10]$ satisfy $\lambda_1 > \eta_0 G$. We will show that this convexity property suffices to ensure the neck begins to close-up. Here we will be required to use the fact that p_0 lies on an ϵ_1-neck. We continue by proving the $(z^* + 10)$ cross section bounds a region which is convex and diffeomorphic to a disc.

Before using our information on the region $z \in [z^*, z^* + 10]$, we have to go back to the starting point p_0 of our neck on the $z = 0$ section. Using the property that $\lambda_1(p_0, t_0) \leq \eta_1 G(p_0, t_0)$ we know p_0 lies on a cylindrical region with parametrisation ϵ_1 finer than ϵ_0 Rather than p_0, it will be useful to consider the last part of the neck with this property, i.e. the largest such z, then we will also know that $\lambda_1 \geq \eta_1 G$ at that point.

More precisely let $\bar{z} \in [0, z^*]$ be the largest value of z such that the corresponding cross section contains a point $\bar{q}$ with $\lambda_1 \leq \eta_1 G$. We claim that $\hat{\mathcal{P}}(\bar{q}, t_0, 10, \frac{d^{\#}}{1600})$ does not contain surgeries. From our definitions we can deduce that there exists a point $q \in \Omega$ with z coordinate in $[\bar{z} - 10, \bar{z}]$.

Then it is clear that

$$\hat{\mathcal{P}}\left(\bar{q}, t_0, 10, \frac{d^{\#}}{1600}\right) \subset \mathcal{P}\left(q, t_0, \frac{(n-1)(n-2)}{2G(q, t_0)}L, \frac{(n-1)^2(n-2)^2}{(20G_1)^2}d^{\#}\right).$$

which does not contain surgeries, by definition of Ω. Then $(G3)$ tells us that there exists a region $\Gamma \in N_0$ centered at $\bar{q}$ which can be written as a cylindrical graph with C^1-norm less than ϵ_1.

We can now reuse some of our analysis from the last part of Section 3.1. We let ω be a unit normal vector parallel to the axis of Γ. Moreover we assume that ω is parallel to the y-axis, where we have set $y = x_{n+1}$. We normalise y such that $F(\bar{q}, t_0)$ lies on the $\{y = 0\}$-hyperplane. For any $p \in \Sigma_0$ consider the curve $y \to \gamma(y, p)$ which solves 3.40 for $y \geq 0$. Denote by $y_{\max}$ the supremum, of the values for which $\gamma(y, p)$ is defined for all $p \in \Sigma_0$ and set $\Sigma_y := \{\gamma(y, p) | p \in \Sigma_0\}$ for $0 \leq y \leq y_{\max}$. Also given $0 \leq y_1 < y_2 < y_{\max}$, set

$$\Sigma(y_1, y_2) = \cup\{\Sigma_y, y_1 \leq y \leq y_2\}.$$

Let us denote by N_0' the part of N_0 corresponding to $z \in [\bar{z}, z^* + 10]$. The $z = \bar{z}$ cross section contains the point $\bar{q}$ and so is very close to Σ_0. By definition of $\bar{z}$, we have $\lambda_1 \geq \eta_1 G > 0$ on the part of N_0' with $z \in [\bar{z}, z^*]$. In the part containing $z \in [z^*, z^* + 10]$ we have the stronger convexity property $\lambda_1 \geq \eta_0 G > 0$. Therefore, N_0' is a convex region. Then by Proposition 3.41, the axis of N_0 is approximately ω everywhere. Moreover the trajectories of (3.40) are defined as long as they are contained in N_0'. It follows that there exists a smallest value $y' < y_{max}$ such that $\gamma(y', p) \in \partial N_0'$ for some $p \in \Sigma_0$. By construction, we have $|\langle \nu(p), \omega \rangle| \leq \epsilon_1$ for all $p \in \Sigma_0$, since $\Sigma_0 \subset \Gamma$. Recalling (3.42), we know that along curves γ we have $\frac{d}{dy}\langle \nu, \omega \rangle \geq \lambda_1 > 0$, which implies that $\langle \nu(p), \omega \rangle \geq -\epsilon_1$ for all $p \in \Sigma(0, y')$.

Now we exploit the property that $\lambda_1 \geq \eta_0 H$ on the cross sections of N_0 corresponding to $z \in [z^*, z^* + 10]$. Let us set $R^* = r(^*)$ to denote the mean radius of the z^*-section and let $G^* = \frac{(n-1)(n-2)}{2r^*}$. By assumption we have that $G^* > \frac{G^*}{2}$. Since the axis of the neck is close to ω we can assume that the y-coordinate is almost constant on each cross section. The y-coordinate on the $z = z^*$ and $z = z^* + 10$ section differ by approximately $10r^*$ due to our normalisation $\bar{F}$. It follows that the points of $\Sigma(y' - 5r^*, y')$, thus from 3.42, we know that along any curve γ with $y \in [y' - 5r^*, y']$

$$\frac{d}{dy}\langle \nu, \omega \rangle = \frac{1}{|\omega^T|^2} \sum_{i,j=1} h_{ij} \langle \omega, e_i \rangle \langle \omega, e_j \rangle \geq \frac{1}{|\omega^T|^2} \eta_0 G |\omega^T|^2 \geq \eta_0 \frac{G^*}{2}.$$

Thus for any $p' \in \Sigma_{y'}$, i.e. $p' = \gamma(y', p)$ for some $p \in \Sigma_0$, we obtain

$$\langle \nu(p), \omega \rangle = \langle \nu(\gamma(y' - 5r^*, p)), \omega \rangle + \int_{y'-5r^*}^{y'} \frac{d}{dy}\langle \nu, \omega \rangle dy \geq -\epsilon_1 + 5r^* \frac{\eta_0 G^*}{2} > 4\epsilon_1.$$

The positivity of $\langle \nu, \omega \rangle$ on $\Sigma_{y'}$ means that the neck is closing up as the value of z increases. The idea is to show after $\Sigma_{y'}$ our surface is a convex cap. To show this we need to look at the curves $\gamma(y, p)$ for $y > y'$. The region swept out by these curves is no longer a neck as y grows. Nevertheless the curves continue to be well defined until some value $y_{\max}$, which by definition is the first value that $\nu(\gamma(y, p)) \to \pm\omega$ for some p as $y \to y_{\max}$. We are guaranteed that such a value exists due to the compactness of our surface and that all our curves converge to the same point as $y \to y_{\max}$.

We need to show that for all $y \in [y', y_{\max}]$ the following properties hold along all trajectories of (3.40):

(i) $|\langle \nu, \omega \rangle| < 1$

(ii) $\lambda_1 > 0$

(iii) $G \geq \frac{G_1}{4}$

(iv) $\langle \nu, \omega \rangle > \epsilon_1$.

By definition we know that (i) holds in $y \in [y', y_{\max}]$. The other inequalities hold for y close to y', if they did not hold in the whole interval then there exists a smallest $y \in (y', y_{\max})$ where it becomes an equalities, we denote this value by $y^{\#}$.

Firstly we know that (iv) holds for $y = y^{\#}$, moreover since (ii) holds for $y \in [y', y^{\#})$ we know that $\nu \cdot \omega$ is increasing along any trajectory of (3.40) on this y interval. Thus (iv) still holds at $y = y^{\#}$. On the other hand since (iv) golds in $y \in [y', y^{\#}]$ implies that (iii) holds at $y^{\#}$. In fact $\Sigma_{y'}$ has diameter less than $\frac{4(n-1)(n-2)}{G_1}$, whilst by (G3) and (G6) we know that $G_1 > 4\Theta G^{\#}$ and so we can apply Lemma 3.43.

Now suppose that (ii) fails at $y = y^{\#}$, then there exists a point $p^{\#} \in \Sigma_{y^{\#}}$ such that $\lambda_1(p^{\#}) = 0$. By the definition of θ_2 in (G5) we know that

$$\theta_2 \frac{(n-1)^2(n-2)^2}{(2G(p^{\#}, t_0))^2} \leq \theta_2 \frac{16(n-1)^2(n-2)^2\Theta^2}{4G_1^2} < \frac{4}{10^4 n \zeta \gamma_0^2 G_1^2}.$$

Recalling our estimate between surgery times (3.48), we see that the backward parabolic neighbourhood centered at $\hat{\mathcal{P}}(p^{\#}, t_0, 10, \theta_2)$ does not contain surgeries. By (G5), we deduce that a portion of the surface around $p^{\#}$ can be written as a cylindrical graph with C^1-norm less than ϵ_1. Then we denote the axis of this graph by $\tilde{\omega}$, however this $\tilde{\omega}$ must be different to ω , otherwise we would have a contradiction with (iv). Now let us define $v = \omega - \langle \omega, \tilde{\omega} \rangle \tilde{\omega}$, then since v is orthogonal to $\tilde{\omega}$ we can find a point $q^{\#}$ close to $p^{\#}$ such that $\left| v(q^{\#}) + \frac{v}{|v|} \right| \leq \epsilon_1$. But from (iv) we know that

$$4\epsilon_1 < \langle v(q^{\#}), \omega \rangle = \left\langle \left(v(q^{\#}) + \frac{v}{|v|} \right), \omega \right\rangle - \left\langle \frac{v}{|v|}, \omega \right\rangle \leq \epsilon_1 - \sqrt{1 - \langle \omega, \tilde{\omega} \rangle^2},$$

which gives us a contradiction. Therefore (i)-(iv) hold for any value of $y < y_{\max}$.

Now that we know there exists at least a trajectory γ^* of (3.40) such that $\gamma^*(y) \to p^*$ as $y \to y_{\max}$ for some $p^* \in \mathcal{M}_{t_0}$ such that $\langle \nu(p^*), \omega \rangle = 1$. Moreover we know by (iv) that $\langle \nu, \omega \rangle$ cannot tend to -1. Let us define

$$\Sigma_{y_{\max}} = \{ \lim_{y \to y_{\max}} \gamma(y, p | p \in \Sigma_0) \}.$$

We want to show that $\Sigma_{y_{\max}}$ reduces to a single point p^*, this implies that all trajectories of γ tend to the same point p^* as $y \to y_{\max}$ and shows that after the neck region we are left with a convex cap. To see this observe that $\nu(p^*) \cdot \omega = 1$ and so the tangent plane to $\mathcal{M}_{t_0}$ at p^* is the plane $y = y_{\max}$. Since the second fundamental form is positive definite at p^*, locally $\mathcal{M}$ lies below the plane $y = y_{\max}$. On the other hand, $\Sigma_{y_{\max}}$ is the limit of the convex surfaces Σ_y and so is also convex. And so we obtain a contradiction unless $\Sigma_{y_{\max}}$ consists only of the point p^*. This completes the proof.

□

We can now prove the main theorem.

Proof of Theorem 3.45. Consider a flow defined on $[0, t_0]$, which is smooth or has had surgeries satisfying (S) at times before t_0. Assume that t_0 is the first time after the last surgery such that $G_{\max}(t_0) = G_3$. We show that we can perform a finite number of surgeries on $\mathcal{M}_{t_0}$, which satisfy (S) and that after these surgeries the maximum curvature drops such that $G_{\max} \leq G_2$ everywhere except for regions diffeomorphic to S^n or $S^{n-1} \times S^1$ which will be discarded after the surgery.

Let us consider any point p_0 such that $G(p_0, t_0) \geq G_2$, we first deal with the case for which $\lambda_1(p_0, t_0) \leq \eta_1 G(p_0, t_0)$, then we can apply the Neck Continuation Theorem to see that p_0 belongs to a neck N_0. Let us denote by $\mathcal{A}$ the region of the neck N_0 together with one or two discs as occurring in case (ii) of the theorem. Then we deduce that $\mathcal{A}$ contains the point p_0 and has one of the following structures.

(a) It has two boundary components and is diffeomorphic to $S^{n-1} \times [-1, 1]$.

(b) It has one boundary component and is diffeomorphic to a disc.

(c) It contains no boundary and coincides with the connected component of $\mathcal{M}$ containing p_0 and is diffeomorphic to S^n or $S^{n-1} \times S^1$.

In all of these cases $\partial\mathcal{A}$ (if non-empty) consists of one of two cross sections of the neck N_0 with mean radius equal to $\frac{(n-1)(n-2)}{G}$ and so with $G \approx \frac{G_1}{2}$.

If instead $\lambda_1(p_0, t_0) > \eta_1 G(p_0, t_0)$ we apply Theorem 3.38 to find a point q_0 such that $\lambda_1(q_0, t_0) \leq \eta_1 G(q_0, t_0)$ and that $G(q, t_0) \geq G(p_0, t_0)\gamma_0$ for all q such that $d_{t_0}(q, p_0) \leq d_{t_0}(q_0, p_0)$. In particular we have that $G(q_0, t_0) \geq \frac{G_2}{\gamma_0} \geq 10G_1$. Then we can proceed as in the first case and define a region $\mathcal{A}$ containing the point q_0 consisting of a neck with a possible union of one or two discs at the ends. Moreover we claim that $p_0 \in \mathcal{A}$, if this were not the case then any path from p_0 to q_0 must intersect the boundary of $\mathcal{A}$. At the points of $\partial\mathcal{A}$ however, we know G is close to $\frac{G}{2}$. But we also know that along the geodesic from p_0 to q_0 we have $G > \frac{G(p_0,t_0)}{\gamma_0} \geq 10G_1$ which yields a contradiction.

In both cases we are able to define regions of $\mathcal{A}$ containing p_0. We continue to do this all over our surface until we cover all points with curvature larger than G_2. However we need to make sure that any two regions defined in such a way are disjoint, otherwise the surgeries would interfere with each other. To show this, recall that $\partial\mathcal{A}$ consists of cross sections of a neck with mean radius equal to $\frac{(n-1)(n-2)}{G_1}$. This means that, if we meet one such cross section in the application of the Neck Continuation Theorem we stop as we have achieved property (i) of Theorem 3.49. Therefore our two regions can at most overlap on boundary points.

The area of any such regions is bounded by a fixed multiple of $(G_1)^{-n}$. Therefore , we find a finite collection $\mathcal{A}, \mathcal{A}', \ldots \mathcal{A}^{(k)}$ which covers all points of $\mathcal{M}_{t_0}$ with $G > G_2$.

After having identified the regions with large curvature, we proceed with the surgeries. The $\mathcal{A}^{(i)}$'s with no boundary components are diffeomorphic to S^n or $S^{n-1} \times S^1$ and can be discarded. In the other ones we proceed to do surgery near each boundary component as follows: We know that any such component is a cross section of a neck with mean radius $\frac{(n-1)(n-2)}{G_1}$. Such a cross section surely exists by continuity because the contains the point p_0, or q_0, where the curvature is at least $G_2 \geq 10G_1$. We then perform a standard surgery centred at the cross section $\Sigma^{(i)}$. If the boundary $\partial\mathcal{A}^{(i)}$ has two components we proceed with surgery on both sides. Notice that these surgeries performed on different regions are independent from each other because the $\mathcal{A}^{(i)}$'s can touch only at boundary points, while the surgeries are performed at cross sections inside the interior $\mathcal{A}^{(i)}$'s, where the mean radius is half the one on the boundary.

In both the cases, the surgeries created a connected component diffeomorphic to a sphere which includes all points of $\mathcal{A}^{(i)}$ with $G > G_2$. Such a component is neglected when we continue with the flow and so the maximum curvature has decreased to below G_2 after this procedure.

It is easy to see that surgeries performed in such a manner satisfy (S) and $(g1) - (g3)$ with $K^* = G_1$ and $r^* = \frac{(n-1)(n-2)}{2G_1}$. This construction ensures that each surgery takes place on a cross section with mean radius $2r^*$. Moreover each surgery is essential from removing a component of the surface when the $G_{\max} > G_2$.

Afterwards we can restart the flow and continue until we reach a time when $G_{\max} = G_3$ and repeat the procedure. There can only be a finite number of surgery times, because the area of the surface is decreasing under smooth flow and each surgery decreases the area by a fixed multiple of $(G_1)^{-n}$. This implies that the whole surface is removed in the surgery procedure as pieces are identified as diffeomorphic to S^n or $S^{n-1} \times S^1$. $\square$

Chapter 4

Level-Set Construction for Brendle-Huisken G-Flow

4.1 The Level-Set Equation for Brendle-Huisken G-Flow

Like in the mean curvature flow case, we begin by considering a smooth function $u = u(x,t)$ such that $Du = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n})$ does not vanish on some open subset O of $\mathbb{R}^n \times [0, T)$.

Assume further that each level-set of u smoothly evolves according to the G-flow. We focus our attention onto any one such level-set, and consider its zero sets given by

$$\Gamma_t = \{x \in \mathbb{R}^n \mid u(x,t) = 0\}$$

for $t \geq 0$. Let $\nu = \nu(x,t) = \frac{Du}{|Du|}$ be the unit normal vector to $\{\Gamma_t\}_{t \geq 0}$ evolving according to the evolution equation

$$\frac{\partial F}{\partial t} = -G\nu. \tag{4.1}$$

Take $e_i = \frac{\partial F}{\partial x_i}$ then

$$\begin{aligned}
\langle D_t \nu, e_i \rangle &= -\left\langle \nu, D_t \frac{\partial F}{\partial x_i} \right\rangle \\
&= -\left\langle \nu, \frac{\partial}{\partial x_i} \frac{\partial F}{\partial t} \right\rangle \\
&= -\left\langle \nu, \frac{\partial}{\partial x_i} (-G\nu) \right\rangle \\
&= \frac{\partial G}{\partial x_i} \\
&= D_{e_i} G.
\end{aligned}$$

Now (4.1) also implies that

$$\begin{aligned}
D_t \nu &= D_{-G\nu} \nu \\
\Rightarrow \ \langle D_\nu \nu, e_i \rangle &= -G^{-1} D_{e_i} G.
\end{aligned}$$

Looking at $\tilde{u}(x,t) = 0$. For some fixed $s \geq 0$, we obtain that

$$\begin{aligned} 0 &= \frac{d}{ds}\tilde{u}(x(s), s) \\ &= -G\nu D\tilde{u}(x(s), s) + \tilde{u}_t(x(s), s) \end{aligned}$$

Setting s equal to t we obtain

$$\begin{aligned} \tilde{u}_t &= G\nu D\tilde{u} \\ \tilde{u}_t &= |D\tilde{u}|G \end{aligned} \tag{4.2}$$

Now arguing as in the mean curvature case we make the transformation $\tilde{u}(x(t),t) \to u(x) - t = 0$ to see that

$$|Du|G = 1 \tag{4.3}$$

So

$$\begin{aligned} &\log(G) + \log|Du| = 0 \\ \Rightarrow\ & D\log(G) + D\log|Du| = 0. \end{aligned} \tag{4.4}$$

Since $D\log(G) = G^{-1}DG$ we obtain that

$$D\log|Du| + D_\nu \log G \cdot \nu = D_\nu \nu \tag{4.5}$$

where

$$\begin{aligned} D_\nu \log G &= -\frac{1}{G}D_{-G\nu}\log G \\ &= \left(\frac{1}{G}\right)_t \end{aligned}$$

this lets us express (4.5) as

$$D\log|Du| - D_\nu \nu = -\left(\frac{1}{G}\right)_t \cdot \nu \tag{4.6}$$

Plugging in $\nu = \frac{Du}{|Du|}$ we evaluate $D_\nu \nu$,

$$\begin{aligned} D_\nu \nu &= \frac{D_\nu(Du)|Du| - (Du)D_\nu|Du|}{|Du|^2} \\ &= \frac{D_{\sum \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}}(Du)(Du) - (Du)D_{\sum \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}}|Du|}{|Du|^3}. \end{aligned}$$

Now,

$$\begin{aligned} \frac{\partial}{\partial x_i}(Du) &= \left(\frac{\partial^2 y}{\partial x_1 \partial x_i}, \frac{\partial^2 u}{\partial x_2 \partial x_i}, \dots, \frac{\partial^2 u}{\partial x_n \partial x_i}\right) \text{ and} \\ \frac{\partial}{\partial x_i}|Du| &= \frac{\frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_1 \partial x_i} + \frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_2 \partial x_i} + \cdots + \frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_1 n \partial x_i}}{|Du|} \\ &= \frac{\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial x_i}}{|Du|}. \end{aligned}$$

Therefore

$$D_\nu \nu = \frac{\frac{\partial u}{\partial x_i}\left(\frac{\partial^2 u}{\partial x_1 \partial x_i}, \cdots \frac{\partial^2 u}{\partial x_n \partial x_i}\right)}{|Du|^2} - \frac{\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial x_i}\left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)}{|Du|^4}.$$

Now,

$$D\log|Du| = \frac{D|Du|}{|Du|} = \frac{\left(\frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_1 \partial x_i}, \ldots, \frac{\partial u}{\partial x_i}\frac{\partial^2 u}{\partial x_n \partial x_i}\right)}{|Du|^2}.$$

And we obtain

$$\Rightarrow \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} Du = -\left(\frac{1}{G}\right)_t \cdot \nu \tag{4.7}$$

Plugging $\nu = \frac{Du}{|Du|}$ we obtain the following level-set equation

$$|Du|\Delta_\infty u = -\left(\frac{1}{G}\right)_t. \tag{4.8}$$

The PDE $\Delta_\infty u$ is known as the infinity Laplacian and was first derived by Aronsson [8] [9] [10] and [11] as the governing equation for the so-called absolute minimizer u of the L^∞ variational problem of minimizing

$$I_\infty[v] := \text{ess} - \text{supp} U|Dv|. \tag{4.9}$$

It is a highly degenerate and highly nonlinear elliptic PDE and has been studied in detail by Spruck, Jensen, Arronson, Smart, Barron and Savin among others [31] [32] [69], [12]. To overcome the difficulties of this PDE Jensen [69] used the weak solutions of Crandall and Lions [24], also known as viscosity solutions, in conjunction with some arguments using integration by parts to show that (4.9) is the unique viscosity solution of the infinity Laplace equation. Solutions to the infinity Laplacian are also known as infinity harmonic functions.

Moreover in 2008 Evans and Savin were able to prove $C^{1,\alpha}$ regularity for dimension 2 and proposed a method for $n \geq 3$, however their result depends on conjectured gradient estimates [31]. More recently in 2011 Evans and Smart were able to show that an infinity harmonic function are everywhere differentiable [32]. However it still remains an open problem to determine if infinity harmonic functions are necessarily continuously differentiable for dimensions $n \geq 3$.

However at this time we are unable to obtain the level-set equation for G-flow explicitly and cannot apply Lauer's method in Appendix 5.4.

Remark 4.10. *For any extrinsic flow we can complete this process and obtain a similar result. Suppose we have have a flow evolving by*

$$\frac{\partial}{\partial t}F(p,t) = -\mathcal{A}\nu$$

then

$$|Du|\Delta_\infty u = -\left(\frac{1}{\mathcal{A}}\right)_t$$

will guarantee that each level-set of u will evolve by $\mathcal{A}$, in regions where u is smooth and $|Du|$ non-vanishing.

Chapter 5

Appendix

5.1 The Maximum Principle

In this section of the appendix we discuss a very important tool used regularly in Geometric Analysis, The Maximum Principle. The sources for the arguments and proofs presented here are [43] and [88].

5.1.1 Weak Maximum Principle

Definition 5.1. *An elliptic differential operator is of the form,*

$Lu = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u$, where $a^{ij} = a^{ji} \geq 0$. We will also denote by λ the smallest eigenvalue of a^{ij}. Moreover we will assume $\frac{|b^i|}{\lambda} \leq C < \infty$.

Theorem 5.2 (Theorem 3.1 [43])**.** *(Weak Maximum Principle) Let L be elliptic in the bounded domain Ω. Suppose that*

$$Lu \geq 0 \ (\leq 0) \ c = 0, \ in \ \Omega$$

with $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$. Then the maximum of $u \in \bar{\Omega}$ is achieved on $\partial\Omega$, that is

$$\sup_{\Omega} u = \sup_{\partial\Omega} u \quad \left(\inf_{\Omega} u = \inf_{\partial\Omega} u \right).$$

The conclusion will remain true if $\frac{|b^i|}{\lambda}$ is only locally bounded in Ω, if for example $a^{ij}, b^i \in C^0(\Omega)$. If u is not assumed continuous in $\bar{\Omega}$, then the conclusion can be replaced by

$$\sup_{\Omega} u = \lim_{x \to \partial\Omega} \sup u \quad \left(\inf_{\Omega} u = \lim_{x \to \partial\Omega} \inf u \right)$$

Proof. If $Lu > 0$ in Ω then the strong maximum principle holds, i.e. u cannot achieve an interior maximum in $\bar{\Omega}$. If there was such a point x_0, $Du(x_0) = 0$ and the matrix $D^2u(x_0) < 0$, but the matrix $[a^{ij}(x)] > 0$ since L is elliptic. By the definition of elliptic function this leaves us with $Lu(x_0) = a^{ij}(x_0)D_{ij}u(x_0) \leq 0$, a contradiction.

By our assumption we know that $\frac{|b^i|}{\lambda} \leq C$ and since $a^{11} \geq \lambda$, there is a sufficiently large γ for which

$$Le^{\gamma x_1} = (\gamma^2 a^{11} + \gamma b^1)e^{\gamma x_1} \geq \lambda(\gamma^2 - \gamma b_0)e^{\gamma x_1} > 0.$$

Hence for any $\epsilon > 0$, $L(u + \epsilon e^{\gamma x_1}) > 0$ in Ω s.t.

$$\sup_{\Omega}(u + \epsilon e^{\gamma x_1}) = \sup_{\partial\Omega}(u + \epsilon e^{\gamma x_1})$$

by the above. Letting $\epsilon \to 0$ finishes the proof. $\square$

If we assume more generally that $c \leq 0$ in Ω and consider the subset $\Omega^+ \subset \Omega$ where $u > 0$, we see that if $Lu \geq 0$ in Ω, then $L_0 u = a^{ij} D_{ij} u + b^i D_i u \geq -cu \geq 0$ in Ω^+. And hence the maximum of u on $\bar{\Omega^+}$ mst be achieved on $\partial\bar{\Omega^+}$ and hence also on $\partial\Omega$. Thus we can obtain the following corollary, where we let $u^+ = max(u, 0)$ and $u^- = min(u, o)$:

Corollary 5.3 (Corollary 3.2 [43])**.** *Let L be elliptic in the bounded domain Ω. Suppose that in Ω, $Lu \geq 0$ (≤ 0), $c \leq 0$, with $u \in C^0(\bar{\Omega})$. Then*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ \quad \left(\inf_{\Omega} u \geq \inf_{\partial\Omega} u^-\right).$$

If $Lu = 0$ in Ω, then

$$\sup_{\Omega} |u| = \sup_{\Omega} |u|.$$

In this corollary, we cannot relax the condition that $c \leq 0$. This is clear from the existence of positive eigenvalues for the problem $\Delta u + \lambda u = 0$, $u = 0$ on $\partial\Omega$.

Theorem 5.4 (Comparison Principle)**.** *Let L be elliptic in Ω with $c \leq 0$ in Ω. Suppose that u and v are functions in $C^2(\Omega) \cap C^0(\Omega)$, satisfying $Lu = Lv$ in Ω, $u = v$ on $\partial\Omega$. Then $u = v$ in Ω. If $Lu \geq Lv$ in ω and $u \leq v$ on $\partial\Omega$ then $u \leq v$ in Ω.*

5.1.2 Strong Maximum Principle

Lemma 5.5 (Lemma 3.4 [43])**.** *Suppose that L is uniformly elliptic, $c = 0$ and $L \geq 0$ in Ω. Let $x_0 \in \partial\Omega$ such that:*

- *u is continuous at x_0.*
- *$u(x_0) > u(x)$ $\forall$ $x \in \Omega$.*
- *$\partial\Omega$ satisfies an interior sphere condition at x_0.*

Then the outer normal derivative f u at x_0, if it exists, satisfies the strict inequality

$$\frac{\partial u}{\partial \nu}(x_0) > 0.$$

If $c \leq 0$ and $\frac{c}{\lambda}$ is bounded, then the same conclusion holds provided $u(x_0) \geq 0$. Moreover if $u(x_0) = 0$ the same conclusion holds irrespective of the sign of c.

Proof. Since Ω satisfies the interior sphere condition at x_0, there exists a ball $B = B_R(y) \subset \Omega$ with $x_o \in \partial B$.

For $0 < \rho < R$ we introduce the function $v = e^{\alpha r^2} - e^{\alpha R^2}$ where $r = |x - y| > \rho$ and α is a constant yet to be determined. We can then obtain,

$$\begin{aligned} Lv(x) &\geq e^{\alpha r^2}[4\alpha^2 a^{ij}(x_i - y_i)(x_j - y_j) - 2\alpha(a^{ii} + b^i(x_i - y_i))] + cv \\ &\geq e^{-\alpha r^2}[4\alpha^2 \lambda(x) r^2 - 2\alpha(a^{ii} + |b|r) + c], \quad b = (b^1, ..., b^n). \end{aligned}$$

By assumption $\frac{a^{ii}}{\lambda}$, $\frac{|b|}{\lambda}$ and $\frac{c}{\lambda}$ are all bounded. Hence α may be chosen large enough such that $Lv \geq 0$ through the annular region $A = B_R(y) - B_\rho(y)$. Since $u - u(x_0) < 0$ on $\partial B_\rho(y)$ there exists $\epsilon > 0$ such that $u - u(x_0) + \epsilon v \leq 0$ on $\partial B_\rho(y)$. Thus we have $L(u - u(x_0) + \epsilon v) = Lu - L(u(x_0)) + L(\epsilon v) \geq -cu(x_0) \geq 0$ in A, and $u - u(x_0) + \epsilon v \leq 0$ on ∂A. The weak maximum principle now implies that $u - u(x_0) + \epsilon v \leq 0$ through A.

Taking the normal derivative at x_0, we obtain as required,

$$\frac{\partial u}{\partial \nu}(x_0) \geq -\epsilon \frac{\partial v}{\partial \nu}(x_0) = -\epsilon v^{'}(R) > 0.$$

For c of arbitrary sign if $u(x_0) = 0$ the preceding argument remains valid if L is replaced everywhere by $L - c^+$. □

Theorem 5.6 (Theorem 3.5 [43])**.** *[Strong Maximum Principle] Let L be uniformly elliptic, $c = 0$ add $Lu \geq 0$ (≤ 0) in a domain Ω which is not necessarily bounded. Then if u achieves its max (min) in the interior of Ω, u is constant. If $c \leq 0$ and $\frac{c}{\lambda}$ is bounded, then u cannot achieve a non-negative max (non positive min) in the interior of Ω unless it is constant.*

Proof. Arguing by contradiction we assume that u is non-constant and achieves its maximum $M \geq 0$ in the interior of Ω, then the set Ω^- on which $u < M$ satisfies $\Omega^- \subset \Omega$ and $\partial\Omega^- \cap \Omega \neq \emptyset$. Let x_0 be a point in Ω^- that is closer to $\partial\Omega^-$ than to $\partial\Omega$ and consider the largest ball $B \subset \Omega^-$ having x_0 as centre. Then $u(y) = M$ for some point $y \in \partial B$ whilst $u < M$ in B. The Lemma 5.5 implies that $Du(y) \neq 0$, which is impossible at the interior max y. □

If $c < 0$ at some point, then the constant of Theorem 5.6 is obviously zero. Also, if $u = 0$ at an interior max (min), then it follows from the proof of the theorem that $u \equiv 0$ irrespective of the sign of c.

5.1.3 Tensor Maximum Principle

For this section we will denote by N a compact manifold with metric $g = \{g_{ij}\}$ and V a vector bundle over N.

Lemma 5.7 (Section 4 [49])**.** *Suppose $\frac{\partial f}{\partial t} = \Delta f + \phi(f)$. Let $s(f)$ be a convex function on the bundle invariant under parallel translation whose level curves $s(f) \leq c$ are preserved by the ODE $\frac{\partial f}{\partial t} = \phi(f)$. Then the inequality $s(f) \leq c$ is preserved by the PDE for any constant c. Furthermore if $s(f) \leq c$ at one point at time $t = 0$, then $s(f) < c$ everywhere on M for all $t > 0$.*

Proof. Let h be a function on M with $s(f) \leq h \leq c$. If $s(f) < c$ at some point p we can make $h < c$ at that point. Then we solve the system for the pair (f, h)

$$\frac{\partial f}{\partial t} = \Delta f + \phi(f), \quad \frac{\partial h}{\partial t} = \Delta h.$$

The set $X = \{(f, h) | s(f) \leq h\}$ is closed and convex since if $s(f_1) \leq h_1$ and $s(f_2) \leq h_2$,

$$s\left(\frac{f_1 + f_2}{2}\right) \leq \frac{s(f_1) + s(f_2)}{2} \leq \frac{h_1 + h_2}{2},$$

and X is invariant under parallel translation. Therefore X is preserved,, and $s(f) \leq c$. If $h < c$ at one point at $t = 0$, then $h < c$ everywhere for $t > 0$ by the strong maximum principle, so $s(f) < c$ for $t > 0$. □

Lemma 5.8 (Lemma 8.2 [49]). *Let M be a symmetric bilinear form on V. Suppose M satisfies a heat equation $\frac{\partial M}{\partial t} = \Delta M + \phi(M)$, where the matrix $\phi(M) \geq 0$ for all $M \geq 0$. Then if $M \geq 0$ at time $t = 0$, it remains so for $t \geq 0$. Moreover there exists an interval $0 < t < \delta$ on which the rank of M is constant and the null space of M is invariant under parallel translation ,invariant in time and also lies in the null space of $\phi(M)$.*

Proof. M is a symmetric bilinear form on V, i.e. $M = M_{ij} e_i^* \otimes ej^*$, where e_i forms an orthonormal basis for V. Thus the convex cone $M \geq 0$ is invariant under parallel translation, since angles and lengths preserved we know orthonormal basis preserved and $\sum M_{ij}$ stays non-negative. And if $\phi(M) \geq 0$ then the ODE $\frac{dM}{dt} = \phi(M)$ preserves the cone $M \geq 0$. Hence so does the PDE. Let $m_1 \leq m_2 \leq \cdots m_n$ be the eigenvalues of M. Then $m_1 + \cdots + m_k$ is a concave function of M and is invariant under parallel translation, since

$$m_1 + \cdots + m_k = \inf\{\text{tr}M|P : P \subset C \text{ is a subset of dimension } k\}.$$

Note that dim $M \geq k \iff m_1 + \cdots + m_k = 0$. If $m_1 + \cdots + m_k > 0$ at one point at $t = 0$, by Lemma 5.7 it will be greater than 0 everywhere at subsequent times. It follows that the rank M remains constant on some interval $0 < t < \delta$ (rank finite and discrete so this is clear).

Let v be any smooth section of V in *null* M on $0 < t < \delta$. Then

$$0 = \frac{\partial}{\partial t}\left(M_{\alpha\beta} v^\alpha v^\beta\right) = \left(\frac{\partial}{\partial t} M_{\alpha\beta}\right) v^\alpha v^\beta + 2M_{\alpha\beta} v^\alpha \frac{\partial v^\beta}{\partial t}.$$

Since $M_{\alpha\beta} v^\alpha = 0$ the last term disappears and we obtain,

$$0 = \frac{\partial}{\partial t}(M_{\alpha\beta} v^\alpha v^\beta) = (\frac{\partial}{\partial t} M_{\alpha\beta}) v^\alpha v^\beta.$$

Also

$$\begin{aligned} 0 = \Delta(M_{\alpha\beta} v^\alpha v^\beta) = (\Delta M_{\alpha\beta}) v^\alpha v^\beta + 4g^{kl} D_k M_{\alpha\beta} v^\alpha D_l v^\beta \\ + 2M_{\alpha\beta} g^{kl} D_k v^\alpha D_l v^\beta + 2M_{\alpha\beta} v^\alpha \Delta v^\beta. \end{aligned} \tag{5.9}$$

and again the last term disappears. Lastly

$$0 = D_k(M_{\alpha\beta} v^\alpha) = (D_K M_{\alpha\beta}) v^\alpha + M_{\alpha\beta} D_k v^\alpha. \tag{5.10}$$

Now (5.9) and (5.10) give us,

$$(\Delta M_{\alpha\beta}) v^\alpha v^\beta = 2g^{kl} M_{\alpha\beta} D_k v^\alpha D_l v^\beta.$$

Substituting into the evolution equation, we obtain,

$$2M_{\alpha\beta} g^{kl} D_k v^\alpha D_l v^\beta + \phi(M) v^\alpha v^\beta = 0.$$

Since $M \geq 0$ and $\phi(M) \geq 0$, we must have $v \in \phi(M)$ and $D_k v^\alpha \in M$ for all k. This shows that $M \subseteq (\phi(M))$. Since $D_k v^\alpha \in M$ this implies that M is invariant under parallel translation. i.e. Suppose that $f_1 v_1 + \cdots + f_k v_k = 0$ we want to show that $D_\gamma(f_1 v_1 + \cdots + f_k v_k) = 0$ with $f_1(0) = 1$ and $f_i(0) = 0$ for all $i \geq 2$.

$D_\gamma(f_1 v_1 + \cdots + f_k v_k) = f_1^{'} v_1 + f D_\gamma v_1 + \cdots = 0$, letting $f_i D_\gamma v_i = a_{ij} v_j$ we obtain $\sum_i f_i^{'} v_i + \sum_{i,j} a_{ij} v_j = 0$ this reduces to the first order ODE $f_i^{'} + \sum_j a_{ij} = 0$ which can be solved, proving our assertion.

Now to see that M is also invariant in time, note first that Δv^α lies in M, since it is invariant under parallel translation. Then

$$\begin{aligned} 0 = g^{kl} D_k (M_{\alpha\beta} D_l v^\alpha) &= g^{kl} D_k M_{\alpha\beta} D_l v^\alpha + M_{\alpha\beta} \Delta v^\alpha \\ \Rightarrow\ g^{kl} D_k M_{\alpha\beta} D_l v^\alpha &= 0. \end{aligned}$$

Then

$$0 = \Delta(M_{\alpha\beta} v^\alpha) = \Delta M_{\alpha\beta} v^\alpha + 2 g^{kl} D_k M_{\alpha\beta} D_l v^\alpha + M_{\alpha\beta} \Delta v^\alpha$$

and hence $(\Delta M_{\alpha\beta}) v^\alpha = 0$. Then

$$0 = \frac{\partial}{\partial t}(M_{\alpha\beta} v^\alpha) = M_{\alpha\beta} \frac{\partial v^\alpha}{\partial t} + (\Delta M_{\alpha\beta} + \phi(M)_{\alpha\beta}) v^\alpha.$$

Now $M \subseteq \phi(M)$, so $\phi(M)_{\alpha\beta} v^\alpha = 0$ also. Thus $M_{\alpha\beta} \frac{\partial v^\alpha}{\partial t} = 0$, and $\frac{\partial v}{\partial t}$ lies in M whenever v does. This shows M is invariant in time. $\square$

5.2 Hamilton's Harnack Inequality

One of the main estimates used in understanding the long time behaviour of solutions to mean curvature flow, Ricci flow and other types of geometric flows is the differential Harnack estimate. We take a look at the Harnack inequality as derived by Hamilton in [50], and look at what form this takes in the strictly convex and ancient case. The ancient case will be of particular interest to us for the section on ancient solutions to mean curvature flow in Section 1.5.

Theorem 5.11 (Theorem 1.1 [50]). *For any weakly convex solution to the mean curvature flow with $t > 0$ we have*

$$\frac{\partial H}{\partial t} + \frac{H}{2t} + 2\langle \nabla H, V\rangle + H(V,V) \geq 0$$

for all tangent vectors V.

Proof. For a proof refer to [50]. □

Corollary 5.12. *For any weakly convex solution to the mean curvature flow with $t > 0$ we have*

$$H(p_2,t_2) \geq H(p_1,t_1)\sqrt{\frac{t_2}{t_1}}\exp(-\Delta/4)$$

for any two points $p_1, p_2 \in \mathcal{M}$ and times t_1, t_2 with $0 < t_1 < t_2$, with

$$\Delta = \inf \int \left|\frac{d\gamma}{dt}\right|_{\mathcal{M}} dt$$

where the infimum is taken over all $\gamma : [t_1, t_2] \to \mathbb{R}^n$ with $\gamma(t) \in X(t)_{\mathcal{M}}$, $\gamma(t_i) = x_i$ and $\left|\frac{d\gamma}{dt}\right|_{\mathcal{M}} = \frac{d\gamma}{dt} - \left\langle \frac{d\gamma}{dt}\nu\right\rangle \nu$ is the length of its component tangent to the surface $\mathcal{M}$.

Proof. Move along any path $\gamma(t) = F(X(t),t)$ and in the Harnack inequality, let $V = \frac{1}{2}\frac{dX}{dt}$ to obtain

$$\frac{\partial H}{\partial t} + \frac{H}{2t} + \left\langle \nabla H, \frac{dX}{dt}\right\rangle + \frac{1}{4}H\left(\frac{dX}{dt}, \frac{dX}{dt}\right) \geq 0.$$

Moreover we know that

$$\frac{dH}{dt} = D_t H + \left\langle \nabla H, \frac{dX}{dt}\right\rangle$$
$$\Rightarrow \frac{dH}{dt} \geq -\frac{1}{4}\left(\frac{dX}{dt}, \frac{dX}{dt}\right) - \frac{H}{2t}.$$

If our surface is convex have the following identity

$$H(V,V) \leq H|V|^2.$$

This lets us obtain

$$\Rightarrow \frac{d}{dt}\log(H) \geq -\frac{1}{4}\left|\frac{dX}{dt}\right| - \frac{1}{2t}$$
$$\Rightarrow \log\frac{H(p_2,t_2)}{H(x_1,t_1)} \geq -\frac{1}{2}\log\left(\frac{t_2}{t_1}\right) - \frac{1}{4}\int\left|\frac{d\gamma}{dt}\right|_{\mathcal{M}} dt$$

and so the result follows after exponentiating. □

Corollary 5.13. *For strictly convex solutions to the mean curvature flow, $h_{ij} > 0$, Hamilton's differential Harnack estimate takes the form*

$$\frac{\partial H}{\partial t} - \frac{H}{2t} - \frac{|\nabla H|^2}{H} \geq 0.$$

Proof. To obtain this expression, we make a clever choice for V. We pick $V = -\frac{\nabla H}{H}$ and plug it into Theorem 5.11. $\square$

Corollary 5.14. *For ancient, strictly convex solutions to the mean curvature flow, Hamilton's differential Harnack estimate takes the form*

$$\frac{\partial H}{\partial t} - \frac{|\nabla H|^2}{H} \geq 0.$$

Proof. In Corollary 1.101 replace t with $t - t_0$ and take the limit as $t_0 \to -\infty$. $\square$

Corollary 5.15. *For any strictly convex ancient solution to the mean curvature flow, $t < 0$ we have*

$$H(p_1, t_1) \leq H(p_2, t_2) \exp\left(\frac{diam_I^2(\mathcal{M}_{t_1})}{4(t_2 - t_1)}\right).$$

for any points $p_1, p_2 \in \mathcal{M}$ and $t_1 < t_2 < 0$.

Proof. As before take any path $\gamma(t) = F(X(t), t)$, from Corollary 5.14 we know that

$$\frac{\partial H}{\partial t} - \frac{|\nabla H|^2}{H} \geq 0$$

and again we know that

$$\frac{dH}{dt} = D_t H + \left\langle \nabla H, \frac{dX}{dt} \right\rangle.$$

Putting these together we see that

$$\frac{dH}{dt} \geq -\left\langle \nabla H, \frac{dX}{dt} \right\rangle + \frac{|\nabla H|^2}{H}$$

Dividing both sides by H and completing the square

$$\begin{aligned}\frac{1}{H}\frac{dH}{dt} &\geq -\left|\frac{dX}{dt}\right|\left|\frac{\nabla H}{H}\right| + \frac{|\nabla H|^2}{H^2} \\ &\geq \left(\frac{|\nabla H|}{H} - \frac{1}{2}\left|\frac{dX}{dt}\right|\right)^2 - \frac{1}{4}\left|\frac{dX}{dt}\right|^2\end{aligned}$$

And hence

$$\begin{aligned}\frac{d}{dt}\log(H) &\geq -\frac{1}{4}\left|\frac{dX}{dt}\right|^2 \\ \Rightarrow \log\left(\frac{H(p_2, t_2)}{p_1, t_1}\right) &\geq -\frac{1}{4}(t_2 - t_1)\Delta\end{aligned}$$

and the result follows after exponentiating and the definition of $\text{diam}_I^2(\mathcal{M}_{t_1})$. $\square$

5.3 Neck Detection and Construction

5.3.1 Uniqueness and Existence of Necks

In this section of the appendix we are dealing with mean curvature flow with surgeries of two-convex hypersurfaces [67]. The main focus is to expand on the discussion in Section 2.12 we will do so using the arguments presented by Hamilton in [46].

Firstly we establish how the Neck Detection Lemma allows us to detect necks where the cross sections will be diffeomorphic to S^{n-1}, refer to Lemma 2.46.

We then show how we are able to glue these cross sections together with full control on their parametrisation - for this we will show we can use a harmonic spherical parametrisation [46]. We then introduce the notion of a normal and maximal necks, this allows us to obtain uniqueness, existence and overlapping properties for normal parametrisations on (ϵ, k)-cylindrical hypersurface necks. Lastly given a neck $N : S^{n-1} \times [a, b] \to \mathcal{M}$ we show that in the case that either $a = \infty$ or $b = \infty$ that this forces them to both to be ∞ and we are left with a solid tube $S^{n-1} \times S^1$.

Lemma 5.16 (Lemma 2.46 and Lemma 7.4 [67]). *Let $\mathcal{M}_t$, $t \in [0, T)$ be a mean curvature flow with surgeries as defined in [67]. Starting from an initial manifold $\mathcal{M}_t \in C(R, \alpha)$ for some R, α. Let $\epsilon, \theta, L > 0$ and $k \geq k_0 \geq 2$ be given. Then we can find η_0, H_0 with the following property. Suppose that $p_0 \in \mathcal{M}$ and $t \in [0, T)$ are such that*

(ND1) $H(p_0, t_0) \geq H_0$, $\frac{\lambda_1(p_0,t_0)}{H(p_0,t_0)} \leq \eta_0$

(ND2) The neighbourhood $\hat{\mathcal{P}}(p_0, t_0, L, \theta)$ does not contain surgeries.

Then

(i) The neighbourhood $\hat{\mathcal{P}}(p_0, t_0, L\theta)$ is an $(\epsilon, k_0 - 1, L, \theta)$-shrinking curvature neck;

(ii) The neighbourhood $\hat{\mathcal{P}}(p_0, t_0, L-1, \theta/2)$ is an $(\epsilon, k, L-1, \theta/2)$ shrinking curvature neck.

The constant η_0 depends on α, ϵ, k, L and θ), whilst $H_0 = h_0 R^{-1}$, where h_0 depends on α, ϵ, k, L and θ.

We can combine Lemma 5.16 with the following proposition found in [46] C3.2, to find that there is a closed cross section with tightly pinched Riemannian curvature. This tells us that there is some diffeomorphism of this cross section to that of a standard sphere S^{n-1}, [64].

Proposition 5.17 (Proposition 3.5 [67]). *Let $k \geq 1$. For all $L \geq 10$ there exists $\epsilon(n, L) > 0$ and $c(n, L)$ such that at any point $p \in \mathcal{M}$ which lies at the centre of a (ϵ, k, L) extrinsic curvature neck with $0 < \epsilon \leq \epsilon(n, L)$ has a neighbourhood which after appropriate rescaling can be written as a cylindrical function $u : S^{n-1} \times [-(L-1), (L-1)] \to \mathbb{R}$ over some standard cylinder in $\mathbb{R}^{n+1}$, satisfying*

$$||u||_{C^{k+2}} \leq c(n, L)\epsilon$$

Proof. The proof of the above can be found in [67] Proposition 3.5. □

Once we know these cross sections are (ϵ, k) spherical by Proposition 5.17, we can obtain a harmonic spherical parametrisation, Theorem C1.1 in [46].

Definition 5.18. *A* spherical parametrisation *of $\mathcal{M}^n$ is a local diffeomorphism $P : S^n \to \mathcal{M}^n$ of the sphere to $\mathcal{M}$.*

Definition 5.19. *A* harmonic spherical parametrisation *is of the form $P^* = PF$ where we want*

$$F(S^n, \bar{g}) \to (S^n, g)$$

to be harmonic from the standard metric $\bar{g}$ to the pull-back metric g.

Theorem 5.20 (Theorem 1.1 [46])**.** *If there exists a geometrically (ϵ, k) spherical parametrization of $\mathcal{M}$, then there also exists a harmonic spherical parametrization. If $n \geq 3$ it is unique up to rotation.*

Remark 5.21. *For $n = 2$ it is unique up to a conformal transformation, and hence unique up to a rotation if we also require that the centre of mass of the pull-back metric g on $S^n \subset \mathbb{R}^{n+1}$ lies at the origin 0. This makes the $n = 2$ case more complicated to deal with.*

This theorem from [46] improves on our parametrisation by giving us a harmonic one. This makes the parametrisation rigid and close to the standard parametrisation of the sphere in angular directions, the only freedom left now is the rigid rotation of the standard S^{n-1} in each cross section of the neck. That is, the z coordinate does not matter, we will have the same rotation.

To obtain a unique z-coordinate along the neck, we can use the Implicit Function Theorem to make the cross sections of constant mean curvature and then label them by the volume between them, this is shown in the proof of the next lemma. Since this is an elliptic equation we can get our cross sections even closer to the standard round sphere in higher norms than the first cross sections we found at the beginning. To do so we first need to define a normal neck.

Definition 5.22. *A topological neck N in a manifold $\mathcal{M}$ is a local diffeomorphism of a cylinder into $\mathcal{M}$*

$$N : S^{n-1} \times [a, b] \to (\mathcal{M}, g)$$

The neck is called normal *if it satisfies the following conditions:*

(i) Each cross section $\Sigma_z = N(S^{n-1} \times \{z\}) \subset (\mathcal{M}, g)$ has constant mean curvature.

(ii) The restriction of N to each $S^{n-1} \times \{z\}$ equipped with the standard metric is a harmonic map to Σ_z equipped with the metric induced by g, and

(iii) in case $n = 3$ only, the centre of mass of the pull-back of g on $S^2 \times \{z\}$ considered as a subset of $\mathbb{R}^3 \times \{z\}$ lies at the origin $0 \times \{z\}$.

(iv) The volume of any subcylinder with respect to the pullback of g is given by

$$Vol(S^{n-1} \times [v, w], g) = \sigma_{n-1} \int_v^w r(z)^n dz.$$

(v) For any Killing vector field $\bar{V}$ on $S^{n-1} \times \{z\}$ we have that

$$\int_{S^{n-1} \times \{z\}} \bar{g}(\bar{V}, U) d\mu = 0$$

where U is the unit normal vector field to Σ_z in $(\mathcal{M}, g)$ and $d\mu$ is the measure of the metric $\bar{g}$ on the standard cylinder.

The following lemma and proof from [46] C2.1 tells us how to fit all the cross sections together with complete control on their parametrisation.

Lemma 5.23 (Lemma 2.1 [46])**.** *There exists (ϵ, k) so that if N_1 and N_2 are necks in the same manifold $\mathcal{M}$ and are both normal and geometrically (ϵ, k) cylindrical. Then if there exists a diffeomorphism F of the cylinders such that $N_2 = FN_1$, then F is an isometry in the standard metrics on the cylinders.*

Proof. For any smooth constant mean curvature hypersurface, there exists a unique one-parameter family of nearby constant mean curvature hypersurfaces by the Implicit Function Theorem. The map F takes an end of one cylinder to an end of the other. Since these constant mean curvature hypersurfaces agree under F, so do all the nearly ones; and we can pursue this all the way from one end to the other. Referring to the Definition 2.45 condition (i) guarantees that F preserves the foliation by horizontal spheres.

Given the foliation, condition (ii) together with the geometric closeness to the standard metric makes F act by isometry on each horizontal sphere $S^{n-1} \times \{z\}$.

Condition (iv) forces the vertical height functions z to differ by an isometry of $\mathbb{R}$.

Lastly condition (v) ensures that the possible rotations in the harmonic spherical parametrisation of each individual cross section are glued together in such a way that there is only one rotation of the standard S^{n-1} left to choose for the whole neck; because by parts (i),(ii),(iv) we are dealing with a map of the cylinder to itself which preserves the height and acts on each horizontal sphere by rotation, and if it is perpendicular to the rotations it must be constant. □

It is this rigidity of the parametrisation along the neck that ensures that we are not just somehow diffeomorphic to $S^{n-1} \times [a, b]$ in the neck, but also extremely close (up to rescaling) to the standard metric and parametrisation of the cylinder. In particular this ensures that there is a diffeomorphism unique up to a rotation and close to an isometry between the two cross sections at the ends of a neck.

We now have uniqueness. For existence of normal necks refer to Theorem C2.2 in [46].

5.3.2 Overlapping Properties

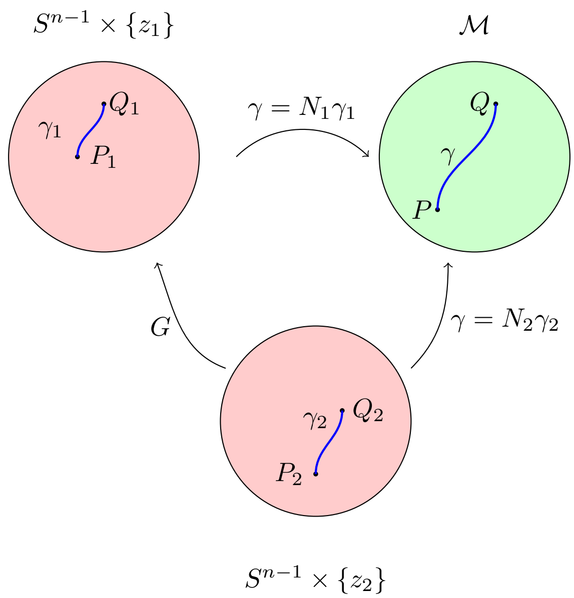
Next we combine normal necks which are cylindrical enough and overlap more than a little bit near the ends into a single neck. Unfortunately Lemma 5.23 is not enough. It tells us that if a diffeomorphism exists then we have isometry, but it does not guarantee the existence of this diffeomorphism F. The next theorem and proof from [46] C2.4 will guarantee the existence of such a diffeomorphism and give us the overlapping properties we require.

Theorem 5.24 (Theorem 2.4 [46])**.** *For and $\delta > 0$ we can choose $\epsilon > 0$ and k with the following property. If N_1 and N_2 are two normal necks in the same manifold M which are both geometrically (ϵ, k) cylindrical, and if there is any point P_1 in the domain cylinder of N_1 at standard distance at least δ from the ends whose imagine in M is also in the image of N_2, then there exists a normal neck N which is also geometrically (ϵ, k) cylindrical, and there exist diffeomorphisms F_1 and F_2 such that $N_1 = NF_1$ and $N_2 = NF_2$, provided $n \geq 3$*

Proof. If $n \geq 3$ then the cylinder $S^{n-1} \times [a, b]$ is simply connected. Let $P_2 \in S^{n-1} \times \{z_2\}$ be a point in the cylinder N_2 whose image $P = N_2 P_2$ in $\mathcal{M}$ is the same as the image $P = N_1 P_1$ of the given $P_1 \in S^{n-1} \times \{z_1\}$. We claim that we can find a map

$$G : S^{n-1} \times \{z_2\} \to S^{n-1} \times \{z_1\}$$

such that $N_1 G = N_2$ and $GP_2 = P_1$. To see this we take any path γ_2 from P_2 to any point $Q_2 \in S^{n-1} \times \{z_2\}$. Let $\gamma = N_2\gamma_2$ be its projection in $\mathcal{M}$, we then lift γ to a path γ_1 in the first cylinder with $\gamma = N_1\gamma_1$. The point P_1 is well in the interior, so we can lift this path until we reach a point Q_1 with $N_1Q_1 = Q = N_2Q_2$.



The only case where this would fail would be if γ_1 ran into the boundary of the first cylinder. But we claim this won't happen as γ_1 is nearly horizontal. The metric $(\mathcal{M}, g)$ will pull back onto metrics (N_1, g_1) and (N_2, g_2), both of which are close to the standard metrics $\bar{g}_1$ and $\bar{g}_2$ on the two cylinders.

The horizontal spheres on the standard cylinders are where the Ricci curvatures of the product metric are all $n-1$, while in the vertical direction they are 0. For $k \geq 0$ the curvatures of g_1 are close to $\bar{g}_1$ and g_2 are close to those of $\bar{g}_2$. The Ricci curvature in the direction of γ_2 is close to $n-1$ since it is in S^{n-1}, and the Ricci curvature of g_1 in the direction of γ_1 is equal to that of g_2 in γ_2. Therefore γ_1 is close to horizontal. As long as the path γ_2 is not too long and (ϵ, k) are chosen well enough, the path γ_1 cannot exit the cylinder since its length is about the same. Since S^{n-1} is simply connected the map G taking Q_2 to Q_1 is uniquely defined by this process and the choice of P_1 and P_2.

The image of $S^{n-1} \times \{z_2\}$ under the map G will be another constant mean curvature sphere as locally G extends to an isometry from g_2 to g_1, this new constant mean curvature sphere will be nearly horizontal and pass through P_1. Applying the Inverse Function Theorem, we know that such spheres are unique.

This tells us that the image of $S^{n-1} \times \{z_2\}$ under G is exactly the sphere $S^{n-1} \times \{z_1\}$, so that γ_1 stayed exactly horizontal.

It remains to check whether the orientations of the normal bundles in the cylinders to the two spheres agree in their images in $\mathcal{M}$. If they do not we can flip one of the cylinders and continue the argument. Then the sphere $S^{n-1} \times \{z_2 + \mu\}$ will map to the sphere $S^{n-1} \times \{z_1 + \mu\}$ under the obvious extension of G using similar lifts, for μ near 0 and hence for μ in some interval. This process lets us patch our cylinders together using G, which must be an isometry from $\bar{g}_2$ to $\bar{g}_1$ using the Lemma 5.23. $\square$

5.3.3 Maximal Normal Necks

Lastly we will define a maximal neck and show that all our (ϵ, k)-cylindrical geometric necks can be classified as either a maximal normal neck of finite length or that our manifold $\mathcal{M}$ is diffeomorphic to a quotient of $S^{n-1} \times \mathbb{R}$.

Definition 5.25. *An (ϵ, k)-cylindrical hypersurface neck N is a* maximal normal (ϵ, k)-cylindrical hypersurface neck *if N is normal and if whenever N^* is another such normal neck with $N = N^*F$ for some diffeomorphism F then the map F is onto.*

We finish by showing a result from [46] C2.5. We will show that we can classify our necks as finite maximal normal necks or $S^{n-1} \times S^1$.

Theorem 5.26 (Theorem 2.5 [46])**.** *For any $\delta > 0$ we can choose $\epsilon > 0$ and k so that any normal neck defined on a cylinder of length at least 3δ which is geometrically (ϵ, k) cylindrical is contained in a maximal normal (ϵ, k) neck; or else the target manifold M is diffeomorphic to a quotient of $S^{n-1} \times \mathbb{R}$ by a group of isometries in the standard metric.*

Proof. Since the neck N has a domain cylinder of standard length at least 3δ, a point P in the middle has standard distance at least δ from either end. If there is any other normal neck N^* which is geometrically (ϵ, k) cylindrical with $N = N^*F$ for some F, then Theorem 5.24 allows us to extend the definition of N to a longer cylinder, and this extension $\bar{N}$ is unique, and now $N^* = \bar{N}\bar{F}$ for a map $\bar{F}$.

Take the largest extension $\tilde{N}$ if N. It will be defined on $S^{n-1} \times B^1$ for some interval $B^1 \subset \mathbb{R}$. If B^1 is of the form $[a, b]$ with $-\infty < a < b < \infty$ we have a maximal (ϵ, k) neck. If we have an interval $(a, b], (a, b]$ or (a, b) with $-\infty < a < b < \infty$, we have enough bounds to extend the neck to the endpoints, so the original was not the largest.

If $a = \infty$ but $b < \infty$ or vice-versa, then there must be two points P_1 and P_2 in the domain cylinder at different heights z_1 and z_2 with the same image in $\mathcal{M}$, because $\mathcal{M}$ has a finite volume and N is clearly a local isometry so there must be considerable overlap. In fact we can make P_1 and P_2 at least δ from the finite end. Then Theorem 5.24 shows that the neck N must repeat itself, so both $a = \infty$ and $b = \infty$. □

Remark 5.27. *When we detect $S^{n-1} \times S^1$ we haven't glued together the cross sections $S^{n-1} \times \{a\}$ and $S^{n-1} \times \{b\}$, this is a more complicated case. What has happened is we have detected a return to the same cross section in $\mathcal{M}$, and due to uniqueness of these cross sections Lemma 5.23 no twisting/rotation can occur and we return with the same orientation.*

Remark 5.28. *Given a cylinder $S^{n-1} \times [a, b]$ it is possible to glue the ends together $S^{n-1} \times [a, b]/\varphi$ where φ is an orientation reversing homeomorphism $\varphi : S^{n-1} \times \{a\} \to S^{n-1} \times \{b\}$ such that this structure is topologically equivalent to $S^{n-1} \times S^1$. Regardless of the rotation of the cross sections at the ends $S^{n-1} \times \{a\}$ and $S^{n-1} \times \{b\}$.*

We can verify this as follows. We can think of this as a two-step process. We want choose an orientation of $S^{n-1} \times [a, b]$ such that we have an orientable manifold in the end. Suppose we want to glue $\{p\} \times \{a\}$ to $\{q\} \times \{b\}$. Then a small neighbourhood of $\{p\} \times \{a\}$ in $S^n \times \{a\}$ should be identified with a small neighbourhood of $\{q\} \times \{b\}$ in $S^{n-1} \times \{b\}$. These are two oriented discs, and we identify them by any orientation reversing homeomorphism. Then the resulting identification space is an oriented manifold with boundary. The boundary is a $(n-1)$-sphere. Hence glue to this an oriented ball, again identifying the boundary spheres by any orientation reversing homeomorphism. The result is homeomorphic with $S^{n-1} \times S^1$.

What we have done is attached a n-dimensional 1-handle to $S^{n-1} \times [a, b]$ with attaching region in different components of the boundary, and then attached a n-dimensional n-handle. We need only to make sure we attach with the right orientations. From this, we can define a homeomorphism with the standard $S^{n-1} \times S^1$. The manifold will have a natural smooth structure at all points except at corner points, the union of which coincides with the boundary of the handle's base. This structure can be uniquely extended to a smooth structure on the entire manifold. Such extension is called smoothing of corners, refer to [90].

This can go wrong if we fail to choose the right orientation when attaching the 1-handle. For example in dimension 3 when we choose an orientation preserving homeomorphism, then a loop running along the 1-handle and then connecting $\{p\} \times \{a\}$ and $\{q\} \times \{b\}$ in $S^2 \times [a, b]$ would have the neighbourhood of a solid Klein bottle, not a torus.

Now since the ϵ closeness is true even on the spacetime region of the neck we are able to control the diffeomorphism type of the neck in a backward parabolic neighbourhood. Moreover we can also control it in cases where surgery has occurred at an earlier time on a region adjacent to the neck. This is needed in the proof of Lemma 2.52, required to prove the Neck Continuation Theorem, Theorem 2.56.

After we have completed the surgery process as described in section 2.1. we have attached a convex region diffeomorphic to the standard disc to a neck. This allows us to see that after each surgery the surgered region together with the long neck it is attached to is diffeomorphic to a standard disc.

Lastly in the proof of the neck continuation Theorem [67], Huisken and Sinestrari shows that in the case a neck does close, it does so to a standard convex cap diffeomorphic to a disc that is attached in the standard way to the standard neck. This shows that a neck type which ends in both directions will be diffeomorphic to the standard sphere S^n because it consists of the standard cylinder glued to two standard discs without sphere twisting.

5.4 Reconciliation Between the Flow with Surgeries and the Viscosity Solution

This section depends on obtaining a viscosity solution for G-flow which we were unable to do. Once this has been obtained however we can follow the method Joseph Lauer used for mean curvature flow explained in Section 2.2.2 and [71].

As mentioned before the surgery process depends on a surgery parameter G_3. Its role is to initiate a surgery when the maximum of G of the evolving hypersurface becomes G_3, and to control the scale at which each surgery is performed. We are now able to prove that as G_3 goes to infinity the surgery process converges to the level-set flow.

Definition 5.29. *Let $K \subset \mathbb{R}^{n+1}$ be closed and $\{K_t\}_{t\geq 0}$ a one parameter family of closed sets such that the spacetime track $\cup(K_t \times \{t\}) \subset \mathbb{R}^{n+2}$ is closed. Then $\{K_t\}_{t\geq 0}$ is a* viscosity set flow *for K if for every G-flow Σ_t on $[a,b]$ we have $K_a \cap \Sigma_a = \emptyset \Rightarrow K_t \cap \Sigma_t = \emptyset$ for all $t \in [a,b]$.*

Definition 5.30. *The level-set flow of a compact set $K \subset \mathbb{R}^{n+1}$, is the* maximal-viscosity set-flow*. $K \subset \mathbb{R}^{n+1}$ is level flow if for any viscosity set flow $\hat{K}$ we have $\hat{K} \subset K_t$ for all $t \geq 0$.*

The existence of a maximal-viscosity set flow is verified by taking the closure of the union of all viscosity set flows with a given initial data. If K_t is the viscosity set flow of K, we denote by $\hat{K}$ the spacetime track swept out by K_t. That is,

$$\hat{K} = \cup_{t\geq 0} K_t \times \{t\} \subset \mathbb{R}^{n+2}.$$

Let $\Sigma_G \subset \mathbb{R}^{n+2}$ be the spacetime track swept out by the hypersurfaces, and as in previous sections let G_3 denotes our surgery parameter.

We work with regions bounded by the evolving hypersurface. $K \subset \mathbb{R}^{n+1}$ compact domain such that ∂K is a two-convex hypersurface. Then if ∂K_G is G-flow with surgeries we define $K_G \subset \mathbb{R}^{n+2}$ to be the region of spacetime such that $t = T$ time-slice of K_G is the compact domain bounded by $(\partial K_H)_T$.

Before we move on to a statement of the main theorem, we state the Jordan-Brouwer Separation Theorem which will be essential in proving Lemma 5.33.

Theorem 5.31 (Jordan-Brouwer Separation Theorem)**.** *Let $\mathcal{M} \subset \mathbb{R}^m$ be a connected, compact, orientable smooth hypersurface. Its complement $\mathbb{R}^m \backslash \mathcal{M}$ has two connected components, the exterior U_1 and the interior U_2. Moreover the closure of U_2 is a compact manifold with boundary $\partial U_1 = \mathcal{M}$.*

Proof. For a proof of the theorem or more details please refer to [75]. □

Theorem 5.32 (Analogous to Theorem 2.86 and Theorem A [71])**.** *[Main Theorem] Let $K \subset \mathbb{R}^{n+1}$ with $n \geq 3$, be compact with ∂K two-convex. Then for G sufficiently large, let K_G be the result of G-flow with surgeries performed with parameter G_3, and initial condition $(K_G)_0 = K$. Then*

$$\lim_{G\to\infty} K_G = \hat{K}.$$

The key ingredient in proving this theorem is the following lemma.

Lemma 5.33 (Analogous to Lemma 2.87 and Lemma 2.2 [71]). *Given $\epsilon > 0$ there exists $G_0 > 0$ such that if $G \geq G_0$, T a surgery time and $x \in \mathbb{R}^{n+1}$ such that*

$$B_\epsilon(x) \subset (K_G)_T^- \Rightarrow B_\epsilon(x) \subset (K_G)_T^+.$$

where we use $(\partial K_G)_T^-$ and $(\partial K_G)_T^+$ to refer to the pre- and post-surgery hypersurfaces at surgery time T and $(K_G)_T^-$ and $(K_G)_T^+$ to the regions they bound.

Proof. Let K_G be the G-flow with surgery.

As seen in Section 3.3. we know that there exists a finite collection of subsets $\{\mathcal{A}_i\}_{i=1}^m$ which cover the regions of $(\partial K_G)_T^-$ with $G \geq G_3$. Then we know there are three possibilities for the structure of each $\mathcal{A}_i$ depending on the number of boundary components. For completeness we state them again.

(a) It has two boundary components and is diffeomorphic to $S^{n-1} \times [-1, 1]$.

(b) It has one boundary component and is diffeomorphic to a disc.

(c) It contains no boundary and coincides with the connected component of $\mathcal{M}$ containing p_0 and is diffeomorphic to S^n or $S^{n-1} \times S^1$.

As long as it contains at least one boundary component we perform a surgery.

Now we know there exists an embedding $N : S^{n-1} \times [a, b] \to \mathcal{A}_i$ such that each $\Sigma_z = N(S^{n-1} \times \{z\})$ has constant $G = \frac{(n-1)(n-2)}{2r_0}$ everywhere, where r_0 is our mean radius. In general the boundary $\partial\mathcal{A}_i$ will always contain at least one of Σ_a or Σ_b and on those boundary components $G = \frac{G_1}{2}$.

Without loss of generality suppose that $\Sigma_a \subset \partial\mathcal{A}_i$ and pick a point $z_0 \in [a, b]$ sufficiently close to a such that $G = G_1$ on Σ_{z_0} and $a < z_0 - 4\Lambda < z_0 + 4\Lambda < b$ for sufficiently large $\Lambda \geq 10$.

For convenience let $z_0 := 0$. Then we can extend the map N to a local diffeomorphism as stated in Proposition 2.23 to obtain a solid tub

$$T : B_1^n \times [-4\Lambda, 4\Lambda] \to \mathbb{R}^{n+1}.$$

We call it T here to avoid confusion with the G-flow. The surgery procedure then removes our two collars $N(S^{n-1} \times [-3\Lambda, 3\Lambda])$ replacing them with two convex caps contained in $F(B_1^n \times [-3\Lambda, 3\Lambda])$ resulting in a smooth embedded hypersurface.

Now we can apply the Jordan-Brouwer Separation Theorem for hypersurfaces to see that if we have a point $x \in (K_H)_T^- \setminus T(B_1^n \times [-3\Lambda, 3\Lambda])$ then x remains in the interior of the surface after standard surgery.

Since K is ϵ_0-close to a standard tube and $\Lambda \geq 10$ is sufficiently large we can pick our surgery parameter G_3 large enough such that if $G \geq G_3$ then

$$\begin{aligned} B_\epsilon(x) \subset (K_G)_T^- &\Rightarrow B_\epsilon(x) \cap T(B_1^n \times [-3\Lambda, 3\Lambda]) = \emptyset \\ &\Rightarrow B_\epsilon(x) \subset (K_G)_T^-. \end{aligned}$$

Therefore $B_\epsilon(x)$ lies in the region bounded by the hypersurface after surgery. Moreover, at any surgery time T finitely many surgeries occur, however these surgeries do not interfere with each other as the solid tubes associated with each surgery are disjoint.

It remains to prove the discarded components of the surgery procedure do not bound a ball of radius ϵ. Recall that the discarded components take one of the following forms:

(i) No boundary components: Then $\mathcal{A}_i$ is diffeomorphic to S^n or $S^{n-1} \times S^1$ and is discarded.

(ii) One boundary component: Then $\mathcal{A}_i$ is diffeomorphic to a ball. In this case the curvature does not decrease significantly along one direction of the neck and only one standard surgery is performed. After, the surgery the end of the cylinder with high curvature will be diffeomorphic to S^n and discarded.

(iii) Two boundary components: Then surgery occurs on each component and we end up with two capped cylinders and a component diffeomorphic to S^2, which is discarded.

In each case the surgery procedure guarantees that for the discarded components $G \geq \frac{G_1}{2}$. Now suppose that Σ is one such hypersurface and x lies in the region bounded by Σ with $d(x, \Sigma) \geq \epsilon$. Then if $y \in \Sigma$ realises $\mathrm{dist}(x, \Sigma)$ then $G|_y < \frac{n}{d} \leq \frac{n}{\epsilon}$ since $\Sigma \cap \mathrm{int}(B_d(x)) = \emptyset$. This gives a contradiction as long as $H_3 \geq \frac{2n}{\epsilon \omega_1}$, where ω_1 is defined in Theorem 3.45. $\square$

Now we are able to prove Theorem 5.32.

Proof. Given an $\epsilon > 0$ sufficiently small, let $t_\epsilon > 0$ be the time such that

$$\mathrm{dist}(\partial K, \partial K_{t_\epsilon}) = \epsilon.$$

Such a time exists since ∂K is two-convex. Let $\Omega_\epsilon \subset \mathbb{R}^{n+2}$ be the level-set flow K_{t_ϵ}.

We now claim that $\Omega_\epsilon \subset K_G$ for all $G \geq G_3$.

We pick our ϵ large enough depending on G_3 such that at the first surgery time T for K_G, Ω_ϵ has vacated the region affected by surgery, we know such an ϵ exists as the region is two-convex. Now since the distance between the viscosity set flow and G-flow with surgeries is non-decreasing on the interval $[0, T)$ we know that $d((\Omega_\epsilon)_T, (\partial K_G)^-_T) \geq \epsilon$. By applying Lemma 5.33 and the definition of Hausdorff distance we know that $d((\Omega_\epsilon)_T, (\partial K_G)^+_T) \geq \epsilon$. Since $(\partial K_G)^+_T$ is a smooth hypersurface we can repeat this argument for each subsequent surgery time. This proves our claim.

Since $\lim\limits_{\epsilon \to 0} \Omega_\epsilon = \hat{K}$, the claim implies that $\hat{K} \subset \lim\limits_{G \to \infty} K_G$ as the limit of closed sets is closed.

Lastly, since each G-flow with surgeries is also a viscosity set flow for K, we have $\lim\limits_{G \to \infty} K_G \subset \hat{K}$.

$\square$

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