Geometry of Integrable Lattice Equations and their Reductions

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A thesis submitted to fulfil requirements for
the degree of Doctor of Philosophy.

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March 2018

Statement of Originality

This is to certify that to the best of my knowledge, the content of this thesis is my own work. This thesis has not been submitted for any degree or other purposes. I certify that the intellectual content of this thesis is the product of my own work and that all the assistance received in preparing this thesis and sources have been acknowledged.

Matthew Nolan

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Acknowledgments

I would like to take this opportunity to acknowledge all of the people who have made this thesis possible. During the course of my candidature I have had the opportunity to interact with many mathematicians from all over the world. Each of these meetings has helped introduce me to the world of mathematics, and I apologise to anyone I do not mention explicitly.

Firstly and most importantly, I would like to thank Prof. Nalini Joshi for her guidance over the last few years. Her supervision has taken me from a fresh undergraduate to where I am today and I am immensely grateful. In addition, I would like to thank Dr. Sarah Lobb for her assistance at the beginning of my candidature. Her advice and ever–present support were invaluable in my development. I would also like to thank Dr. Robby Marangell for his very willing support and direction.

I would like to give special thanks to my colleagues Pieter Roffelson, Steven Luu, and James Gregory, both for the exchange of ideas and for the much needed friendship. Finally, I would like my friends and family for keeping me together throughout this experience. In particular my parents for their undying support, and Annie, for being a constant source of support and joy during my final year.

Abstract

Modern research into discrete integrable systems has provided new insights into a wide variety of fields, including generalisations of special functions, orthogonal polynomials and dynamical systems theory. In this thesis, we extend one of the most productive insights in this area to higher dimensions. In particular, we show how to apply ideas from resolution of singularities and birational geometry to discrete systems in higher dimensions.

The most widely studied setting for these ideas lies in spaces of dimension two. By blowing up at certain points to resolve singularities found in maps on surfaces, new surfaces are constructed on which the map becomes an isomorphism, a so-called *space of initial conditions*. This has led to new developments in the field, including the discovery of new examples of integrable maps by Sakai [92] with solutions that have unexpectedly rich properties.

On the other hand, this geometric approach has never been applied to integrable partial difference equations (often called lattice equations), which share other properties with the maps in dimension two. In this thesis, we overcome this gap.

In particular, we examine spaces of initial conditions for integrable lattice equations, which are members of the equations classified by Adler et al [5], known as ABS equations. By explicitly calculating the induced map on their resolved initial value spaces, we find transformations to new lattice equations, and hence find novel reductions to discrete Painlevé equations. We also show that an equation arising from the geometry of ABS equations is satisfied by the coefficients of a cluster algebra associated with a form of the discrete mKdV.

Chapter 1: Introduction

The aim of this chapter is to introduce the topics underlying this thesis. It provides a historical context of the field and an overview of major results found by others. All results in this thesis lie in the field of integrable systems, a field possessing the continuous Painlevé equations at its core. For this reason, we begin our journey by introducing the Painlevé equations and using them to build to major areas covered by this thesis.

1.1 The Painlevé Equations

The Painlevé equations arose in the study of special functions defined as solutions of differential equations. Linear ordinary differential equations are able to define a function as their solutions do not have singularities depending on constants of integration. Toward the end of the 19th century, Painlevé's school sought to find nonlinear differential equations which share this property. By finding equations with the so called *Painlevé property* (that is, that all movable singularities in the solution are poles) Poincaré and Fuchs showed that any first order ordinary differential equation (ODE) with this property can be transformed into the Weierstrass or Riccati equation, which can be solved in terms of previously known functions.

In [82, 83], to search for second order ordinary differential equations with this property, equations for $y(z)$ of the form

$$y'' = F(z, y, y'),$$

were considered (where F is a rational function [55]). It was found that (up to certain transformations) such equations can be transformed into one of 50 canonical forms. Of these, 44 could be solved in terms of previously known functions. The remaining six have been come to be known as the Painlevé equations, and are given by

$$y'' = 6\,y^2 + z, \tag{$\mathrm{P_I}$}$$

$$y'' = 2\,y^3 + z\,y + \alpha, \tag{$\mathrm{P_{II}}$}$$

$$z\,y\,y'' = z\,y'^2 - y\,y' + \delta\,z + \beta\,y + \alpha\,y^3 + \gamma\,z\,y^4, \tag{$\mathrm{P_{III}}$}$$

$$y\,y'' = \frac{1}{2}\,y'^2 + \beta + 2\,(z^2 - \alpha)\,y^2 + 4\,z\,y^3 + \frac{3}{2}\,y^4, \tag{$\mathrm{P_{IV}}$}$$

$$\begin{aligned} y'' = {} & \left(\frac{1}{2\,y} + \frac{1}{y-1}\right) y'^2 - \frac{y'}{z} + \frac{(y-1)^2}{z^2}\left(\alpha\,y + \frac{\beta}{y}\right) \\ & + \gamma\,\frac{y}{z} + \delta\,\frac{y\,(y+1)}{y-1}, \end{aligned} \tag{$\mathrm{P_V}$}$$

$$\begin{aligned} y'' = {} & \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-z}\right) y'^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{y-z}\right) \\ & + \frac{y\,(y-1)\,(y-z)}{z^2\,(z-1)^2}\left(\alpha + \beta\,\frac{z}{y^2} + \gamma\,\frac{z-1}{(y-1)^2} + \delta\,\frac{z\,(z-1)}{(y-z)^2}\right), \end{aligned} \tag{$\mathrm{P_{VI}}$}$$

for complex constants α, β, γ, δ. The most general form of $\mathrm{P_{VI}}$ was added to Painlevé's list by Gambier in [31]. Just as many special functions are defined as the solutions of linear functions, the Painlevé equations define new special functions called the *Painlevé transcendents*. These solutions are known to be transcendental (that is, cannot be expressed in terms of earlier known functions) and therefore define truly new functions.

These equations are known to possess interesting symmetries, some of which give rise to to *Bäcklund transformations*. These are transformations which take one solution of an equation to a solution of the same equation, with different parameter values. For example, if $y(z;\alpha)$ is a solution of $\mathrm{P_{II}}$ with the parameter α, it is known that

$$y(z;\alpha \pm 1) = -y - \frac{2\alpha \pm 1}{2y^2 \pm 2y' + z}, \tag{1.1}$$

is a solution of $\mathrm{P_{II}}$ with the parameter $\alpha \pm 1$.

By eliminating the derivative y' from (1.1), we arrive at the purely discrete equation [56]

$$\frac{\alpha + \frac{1}{2}}{y_{\alpha+1} + y_\alpha} + \frac{\alpha - \frac{1}{2}}{y_\alpha + y_{\alpha-1}} + 2\,y_\alpha^2 + z = 0, \tag{1.2}$$

where $y_\alpha = y(z;\alpha)$. This equation may now be considered a discrete evolution equation with α as the independent variable. We call equations of this type ordinary difference equations or OΔEs. Additionally, under an appropriate limit this discrete equation (1.2) reduces back to the continuous Painlevé I equation ($\mathrm{P_I}$). We call the equation (1.2) a *discrete Painlevé equation.*

1.2 Discrete Painlevé Equations

In the past, discrete Painlevé equations were considered to be discrete equations which possessed some continuum limit to a continuous Painlevé equation (as in the previous example). However, with the discovery of several non-autonomous integrable discrete equations which do not have a limit to a continuous Painlevé equation this definition was in need of improvement. Several attempts were made to find a discrete equivalent to the Painlevé property, including *singularity confinement* [39, 57] which will be explored geometrically in the following chapters. In short, singularity confinement is the phenomenon by which a system is able to recover dependence on initial conditions when passing over a singularity.

The earliest appearances of discrete Painlevé equations were not however in the context of Painlevé systems. The first such example appeared in [94], as the following additive (or d–discrete) equation

$$x_{n+1} + x_n + x_{n-1} = \frac{t_n}{x_n} + 1, \quad t_n = t_0 + d\,n, \quad (n \in \mathbb{Z}). \tag{1.3}$$

It was not related to Painlevé equations until the 1990s, when in [17] the continuum limit $d \to 0$ was calculated in the study of two-dimensional quantum gravity. The result was the first Painlevé equation $y'' = 6y^2 + z$, earning (1.3) the name $d\mathrm{P_I}$. Inspired by this study, in [24], Fokas, Its, and Kitaev showed that there exist more discrete equations which are integrable in the sense of possessing a Lax pair (a pair of linear operators corresponding to a dynamical system which can be used to find solutions), and that some reduce to Painlevé equations under appropriate continuum limits.

This opened the hunt for new Painlevé equations and conjecture on what a discrete equivalent of the Painlevé property should be. The first candidate, *singularity confinement*, was proposed independently by Grammaticos, Ramani, and Papageorgiou [39], and Joshi in [57]. This inspired a flood of discoveries of new discrete Painlevé equations via the deautonomisation of *QRT mappings*.

1.2.1 The QRT Mappings

In [86, 85], Quispel, Roberts, and Thompson developed a family of ordinary discrete mappings now known as the QRT mappings, which are solvable in terms of elliptic functions. These mappings take the form

$$x_{n+1} = \frac{f_1(x_n) - x_{n-1}\, f_2(x_n)}{f_2(x_n) - x_{n-1}\, f_3(x_n)}, \tag{1.4}$$

where each of the f_i are quartic polynomials expressed in terms of 12 parameters,

$$\begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} = \begin{pmatrix} \alpha_0\, x^2 + \beta_0\, x + \gamma_0 \\ \beta_0\, x^2 + \epsilon_0\, x + \zeta_0 \\ \gamma_0\, x^2 + \zeta_0\, x + \mu_0 \end{pmatrix} \times \begin{pmatrix} \alpha_1\, x^2 + \beta_1\, x + \gamma_1 \\ \beta_1\, x^2 + \epsilon_1\, x + \zeta_1 \\ \gamma_1\, x^2 + \zeta_1\, x + \mu_1 \end{pmatrix}. \tag{1.5}$$

The mapping (1.4) is known to have an invariant given by

$$K(x,y) = \frac{\alpha_0 x^2 y^2 + \beta_0 xy(x+y) + \gamma_0(x^2+y^2) + \epsilon_0 xy + \zeta_0(x+y) + \mu_0}{\alpha_1 x^2 y^2 + \beta_1 xy(x+y) + \gamma_1(x^2+y^2) + \epsilon_1 xy + \zeta_1(x+y) + \mu_1}. \tag{1.6}$$

This is an invariant in the sense that if we start with $K_1 = K(x_n, x_{n-1})$ and compute $K_2 = K(x_{n+1}, x_n)$ we find $K_1 = K_2$ if x_{n+1}, x_n, x_{n-1} satisfy (1.4).

In general, deautonomising the parameters will not result in an integrable mapping. However, utilising the belief that integrable mappings should be confining, and therefore deautonomising with the condition that the mapping remain confining, many integrable nonautonomous integrable mappings were found. This vastly expanded the number of discrete Painlevé equations known [88].

In response to this immense equation zoo, there was a need to develop a theory to unify the concept of discrete Painlevé equations. Following in the footsteps of Okamoto's work on the continuous Painlevé equations [79], in [92] Sakai developed the algebro-geometric framework for discrete Painlevé equations that exists today. This is explored in depth in Chapter 2. In Chapter 4, we apply techniques from algebraic geometry to partial difference equations for the first time. For a complete treatment of QRT maps, and details of the background theory of elliptic surfaces on which they are based, we direct the reader to Duistermaat's book [22].

1.3 Partial Difference Equations

For many decades there has been active interest in integrable partial difference equations and their solutions. Analogous to the Painlevé case, the Bäcklund transformations of partial differential equations (PDEs) give rise to higher dimensional discrete equations, or partial difference equations (PΔEs). These equations on lattices possess many interesting properties of their own. In fact, since they possess continuum limits to PDEs and reductions to discrete Painlevé equations, they can be considered as the most fundamental of these objects.

In [33, 34] the Korteweg-de Vries equation (known to model many physical wave phenomena, including shallow water waves and hydromagnetic waves in cold plasma [65, 32, 104]) was considered. It is given by

$$w_t - 6\,w\,w_x + w_{xxx} = 0. \tag{1.7}$$

A general method to solve this equation was given (now known as inverse scattering), and it was shown to possess soliton, or solitary wave solutions. This method was soon extended to the nonlinear Schrodinger equation, the sine-Gordon equation, and others. This also led to the concept of the Lax pair of a system being formalised in [67]. The famous potential Korteweg–de Vries (KdV) equation, given by

$$u_t - 3\,u_x^2 + u_{xxx} = 0, \tag{1.8}$$

is related to (1.7) by $w = u_x$. For some parameter λ, the Bäcklund transformation between two solutions u_1 and u_2 of (1.8) is given by the system [102]

$$(u_1 + u_2)_x = \frac{\lambda}{2} + \frac{1}{2}(u_1 - u_2)^2, \tag{1.9a}$$

$$(u_1 - u_2)_t = 3(u_{1\,x}^2 - u_{2\,x}^2) - (u_1 - u_2)_{xxx}. \tag{1.9b}$$

Such Bäcklund transformations have played an important role in the development of soliton theory. Using a given $(N-1)$–soliton solution u_1, integrating (1.9a) will give the N–soliton solution u_2.

An extremely important (and beautiful) property of Bäcklund transformations is the *Bianchi permutability property*, see [12]. Given two Bäcklund transformations $u \overset{\alpha}{\mapsto} \widetilde{u}$ and $u \overset{\beta}{\mapsto} \widehat{u}$, using parameters α and β respectively, there exists a unique $\widehat{\widetilde{u}} = \widetilde{\widehat{u}}$ resulting from the composition of these two Bäcklund transformations. In other words, the following commuting diagram holds:

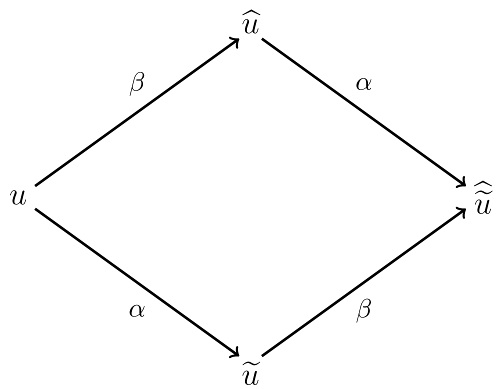

Fig. 1.1: Commuting Bäcklund transformations

Using the Bianchi permutability property of the Bäcklund transformation (1.9a), the composition of the transforms $u \overset{\alpha}{\mapsto} \widetilde{u}$ and $u \overset{\beta}{\mapsto} \widehat{u}$ gives

$$2(u + \widetilde{u})_x = \alpha + (u - \widetilde{u})^2, \tag{1.10a}$$

$$2(u + \widehat{u})_x = \beta + (u - \widehat{u})^2, \tag{1.10b}$$

$$2(\widehat{u} + \widehat{\widetilde{u}})_x = \alpha + (\widehat{u} - \widehat{\widetilde{u}})^2, \tag{1.10c}$$

$$2(\widetilde{u} + \widehat{\widetilde{u}})_x = \beta + (\widetilde{u} - \widehat{\widetilde{u}})^2. \tag{1.10d}$$

We can eliminate the derivatives from this system by computing (1.10a)-(1.10c)+(1.10d)-(1.10b) (see [46]). This eliminates the derivatives from the system and yields the following purely discrete equation

$$(u - \widehat{\widetilde{u}})(\widetilde{u} - \widehat{u}) + \beta - \alpha = 0. \tag{1.11}$$

Taking a continuum limit of (1.11), we arrive back at the continuous potential KdV equation (1.8).

Similar to the Painlevé case, the permutability property allows us to consider (1.11) as a discrete evolution equation over a square lattice, composed of individual quadrilaterals (or quads) such as Figure 1.1. The equation (1.11) is an example of an equation on quad-graphs, a so-called *quad-equation*. See Figure 1.2.

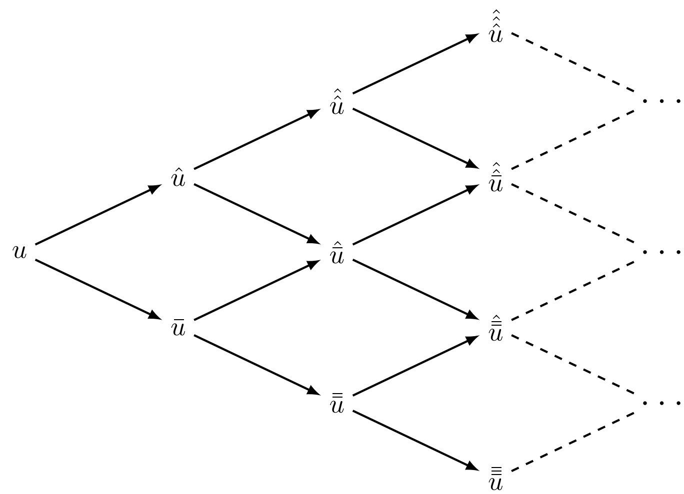

Fig. 1.2: Quadrilateral lattice generated by Bäcklund transformations.

By introducing a third lattice direction with corresponding parameter γ, (1.11) can be consistently applied to the faces of the resulting cubic lattice. This property is called the *consistency around the cube*. A Lax pair can be procedurally computed directly from this consistency, and is hence a strong indicator of integrability. We will see lattice equations and what it means for them to be integrable in more detail in Chapter 3.

1.3.1 Reductions of Lattices

The Ablowitz-Ramani-Segur conjecture [3] states that a nonlinear PDE is integrable in the sense of the inverse scattering method if all nonlinear ODEs obtained by reduction have the Painlevé property. Perhaps the most famous example of such a reduction is that of the modified KdV equation,

$$v_t - 6v^2 v_x + v_{xxx} = 0. \tag{1.12}$$

Introducing the variable $z = x(3t)^{-1/3}$ and the dependent variable $y = (3t)^{1/3} v$ [93], after a single integration we find

$$y'' = 2y^3 + z\,y + \alpha, \tag{1.13}$$

where α is the constant of integration. This is precisely $\mathrm{P_{II}}$. With the discovery of integrable lattice equations, the natural questions arose: What could be a discrete equivalent of such a reduction, and are lattice equations similarly related to discrete Painlevé equations via reduction?

The reduction of integrable lattice equations turns out to be a remarkably rich area, dating back to similarity reductions of integrable lattices in the early 1990s [72]. A purely discrete approach was developed by Nijhoff and others in [75], where reductions were found by introducing an appropriate variable and choosing paths in the lattice. In [96] certain classes of symmetric solutions of the lattice Boussinesq equation were studied, which led to an explicit reduction. An elliptic Painlevé equation was found as the non-autonomous reduction of a lattice equation in [87].

Recently, Joshi, Nakazono, and Shi [59] were able to find reductions of lattice equations by imposing the equation on an $n-$cube and taking a reduction of the resulting $n-$dimensional lattice. Conversely, in [60, 61] the authors construct from τ-functions of Painlevé systems a lattice on which ABS quad-equations appear. In [44] the authors presented a class of reductions of Möbius type for Q-type lattice equations as given in the ABS classification (which we cover in Chapter 3). In this thesis, we focus on the method of so-called *staircase* reductions, first described in [40], which we will cover in detail in Chapter 3.

Today there exists in both continuous and discrete settings a plethora of examples of partial differential and partial difference equations known to reduce to continuous and discrete Painlevé equations respectively, showing the deep relationship between integrable systems of different dimensions. In Chapter 5 we provide novel reductions of several lattice equations to the discrete Painlevé equation $q\mathrm{P_{VI}}$.

1.4 Cluster Algebras

In [25, 27], Fomin and Zelevinsky introduced cluster algebras as a tool in Lie theory, though they have since gained a life of their own. Connections have been found in broad areas of mathematics from topology and tropical geometry to integrable systems. In [28], fundamental structural results for finite type cluster algebras were used to prove the periodicity conjecture for Dynkin diagrams. For a detailed introduction to this rich area we direct the reader to Williams' 2014 article [103]. Of particular interest to us is the natural (and profitable) presence of cluster algebras within the field of discrete integrable systems [105].

In short, a cluster algebra is generated by a directed graph Q called a *quiver*, with vertices labelled with variables $\mathbf{x} = (x_1, x_2, x_3, \dots)$ which we call *cluster variables*, and $\mathbf{y} = (y_1, y_2, y_3, \dots)$, which we call *coefficients*. The triple $(Q, \mathbf{x}, \mathbf{y})$ is called a *seed*. Suppose $\lambda_{i,j}$ is the number of arrows from the vertex i to j, such that $\lambda_{i,j} = -\lambda_{j,i}$. The quiver defines an operation called *mutation* in the following way.

Let $\mu_k : (Q, \mathbf{x}, \mathbf{y}) \to (Q', \mathbf{x}', \mathbf{y}')$ be the mutation at the vertex k of the quiver Q. The mutation gives a new quiver Q' by the following three operations on Q.

1. For all i, j such that $\lambda_{i,k} > 0, \lambda_{k,j} > 0$, add $\lambda_{i,k}\lambda_{k,j}$ arrows from i to j.

2. Reverse the direction of all directed arrows which have edges at k.

3. Remove any 2-cycles appearing in the resulting quiver.

The new coefficients $\mathbf{y}' = (y'_1, y'_2, y'_3, \dots)$ are defined as:

$$y'_k = y_k^{-1}, \tag{1.14a}$$

$$y'_i = y_i(y_k^{-1} + 1)^{-\lambda_{k,i}}, \qquad (\lambda_{k,i} > 0), \tag{1.14b}$$

$$y'_i = y_i(y_k + 1)^{\lambda_{i,k}}, \qquad (\lambda_{i,k} > 0), \tag{1.14c}$$

$$y'_i = y_i, \qquad (\lambda_{i,k} = 0). \tag{1.14d}$$

The new cluster variables $\mathbf{x}' = (x'_1, x'_2, x'_3, \dots)$ are defined as:

$$x'_k = \frac{1}{(y_k + 1)\, x_k} \left(\prod_{\lambda_{k,j} > 0} x_j^{\lambda_{k,j}} + y_k \prod_{\lambda_{j,k} > 0} x_j^{\lambda_{j,k}} \right),$$

$$x'_i = x_i, \qquad (i \neq k).$$

Given an initial seed $(Q, \mathbf{x}, \mathbf{y})$, the cluster algebra $\mathcal{A}(Q, \mathbf{x}, \mathbf{y})$ is the subalgebra generated by the cluster variables from all seeds obtainable by mutations of the initial seed $(Q, \mathbf{x}, \mathbf{y})$. Notably, any cluster variable in $\mathcal{A}(Q, \mathbf{x}, \mathbf{y})$ be expressed in terms of Laurent polynomials of the initial seed [26]. The finite type cluster algebras (those with finitely many seeds) possess a classification parallel to the Cartan–Killing classification of complex simple Lie algebras, and can hence be classified by Dynkin diagrams [27].

By appropriately labelling the vertices of a mutation periodic quiver [29], many discrete integrable systems have been shown to be satisfied by cluster variables and coefficients of particular cluster algebras [80, 52, 81, 11]. Since any cluster variable can be found as a Laurent polynomials of the initial seed, showing that a discrete integrable system arises from cluster algebra mutations is sufficient to prove that it possesses the Laurent property [30, 80]. In Chapter 5, we show for the first time that the coefficients of the cluster algebra generated by a quiver associated with the lattice mKdV equation can satisfy an equation we find arising from transformations of ABS systems.

1.5 Outline of Thesis

This thesis aims to provide novel insights into the solutions of integrable lattice equations by drawing a connection between their birational geometry and their iteration. Chapter 1 serves as a review chapter, briefly introducing the objects of study we will be exploring through algebraic geometry in future chapters.

Chapter 2 serves as a detailed review of the algebraic geometry of discrete Painlevé equations as introduced by Sakai in [92]. We show how to find a so-called resolved space of initial conditions for integrable mappings, and how discrete Painlevé equations arise from translations in affine root systems. We will see how integrability can be seen on the level of the induced map on the Picard group. Chapter 3 introduces in more detail the concept of integrable lattice equations, their history, and what it means for an equation over a lattice to be integrable.

In Chapters 4 and 5 we apply tools from birational geometry to perform resolution of singularities for lattice equations for the first time. In Chapter 4, by studying the iteration map induced by an integrable lattice equation and resolving the codimension-2 singular subvarieties that appear, we show how to build a space of initial conditions on which the map is everywhere regular.

In Chapter 5 we associate to each quad in the lattice such a space of initial conditions and consider the evolution of this system on the *dual graph*. We show how this naturally leads to new transformations of lattice equations, and provide several novel examples of transformations of ABS equations. Using the results of this chapter, we find new reductions of ABS equations to the discrete Painlevé equation commonly called $q\mathrm{P}_{\mathrm{VI}}$. Finally, we prove that the coefficients of a cluster algebra first introduced in [81] can satisfy an equation we find as the transformation of several ABS equations.

Chapter 6 is a discussion of the results and possible future directions of research.

Chapter 2: Painlevé Equations and Algebraic Geometry of Integrable Mappings

2.1 A Brief History of Painlevé Equations

In this chapter, we review key ideas in the development and integrability of discrete Painlevé equations, and view them from the perspective of their geometry. This point of view is one which has shown itself over recent decades to be an extremely profitable perspective to take when examining integrable mappings.

The continuous Painlevé equations arise from two independent sources. They first arose from Painlevé's study of second order differential equations with what we now call the Painlevé property. That is, all movable singularities are poles. This, with the addition of the most general form of Painlevé VI added by Gambier, gave a list of six canonical second order differential equations with the Painlevé property which were not solvable in terms of any previously known functions [82, 83, 31]. The Painlevé equations also independently arise independently from the study of monodromy preserving deformations of linear differential equations, and from similarity reduction of PDEs solvable by inverse scattering. Discrete Painlevé equations attracted attention with the discovery of scaling limits to the Painlevé differential equations. The concept of singularity confinement was introduced as a potential discrete analogue of the Painlevé property [39], and using it as an integrability indicator on non-autonomous QRT mappings [86, 85] discrete Painlevé equations were found systematically [89]. It is worth noting that the discrete Painlevé equations are in fact the more fundamental class, with continuous equations being degenerate forms of the more general discrete systems - the differential operator is itself the continuum limit of a discrete operator.

A major development in the study of the Painlevé equations was taken by Gérard and Okamoto [35, 79, 37, 36] in the 1970s. Their geometric description of the Painlevé equations relies on foliation theory and vector bundles.

Definition 2.1. *Let* $(\mathcal{E}, \pi, \mathcal{B})$ *be a vector bundle, where* $\mathcal{B}$ *is the base space,* $\mathcal{E}$ *is the total space and* π *is the projection operator:* $\mathcal{E} \mapsto \mathcal{B}$*. Let* Φ *be a foliation and* Δ *a holomorphic differential system on* $\mathcal{E}$*, such that the following properties hold.*

- *Each leaf of* Φ *corresponds to a solution of* Δ*.*
- *The leaves of* Φ *are transversal to the fibres of* $\mathcal{E}$*.*
- *For each path* p *in the base* $\mathcal{B}$ *and each point* $X \in E$ *such that* $(X) \in p$*, the path* p *can be lifted into the leaf of* Φ *containing point* X*.*

What is now known as the space of initial conditions consists of the leaves of the above foliation.

This work was extended by Sakai in [92] to give a full decription of Painlevé equations in both continuous and discrete settings through the classification of rational surfaces. We will give an outline of this classification in Section 2.4.

The geometric view has provided many results beyond this classification. Duistermaat and Joshi used the space of initial conditions to study the asymptotics of the first Painlevé equation [21]. Takenawa [95] showed how to use the space of initial conditions to calculate the algebraic entropy of maps. Algebraic geometry has also been used to find special solutions of Painlevé equations. Not only does this algebro-geometric approach yield a classification, but also solutions and analytical tools for future study.

In this chapter we will elaborate upon the theory surrounding this algebro-geometric approach by introducing the notions of blow-ups, root systems and Weyl groups, and the Picard lattice, and how they are applied in the study of discrete Painlevé equations.

2.2 Blowing Up

Throughout this thesis, when confronted with problems of resolution of singularities, we use *blow-ups*. The name *blow-up* is inspired by blowing up an image in photography. Most procedures for resolution of singularities work by successively blowing up singular points until they become smooth. Consequently, we can use blow-ups to resolve the singularities of rational maps. Blow-ups are extremely fundamental in birational gemoetry. In fact, the Cremona group [18, 41], the group of birational automorphisms of the plane, is generated by blow-ups.

We will be considering singularities that may occur at infinity. To handle such situations, it is convenient to work in complex projective space. It is common to use homogeneous coordinates in such a space. However, affine coordinates are also convenient for certain operations. In the following examples we demonstrate the construction with diagrams drawn in $\mathbb{R}^n$, though throughout this thesis the base field we work in is $\mathbb{C}$.

Modern algebraic geometry treats blowing up as an intrinsic operation on an algebraic variety. We will proceed by first giving a geometric construction which will lead to the formal definition at the end. Understanding blow-ups is a crucial first step in understanding the geometric underpinning of discrete Painlevé equations.

Consider the curve C given by

$$y^2 - x^3 = 0. \tag{2.1}$$

This curve has a cusp singularity at the origin. In order to resolve this singularity, we view the curve C as a vertical projection of the smooth space curve C' with the parameterisation $(x, y, z) = (t^2, t^3, t)$, with the cusp resulting from the curve being tangent to the z–axis at $t = 0$, see Figure 2.1.

The challenge then is to find a procedure such that in general when given a singular curve C, we are able to find a smooth curve C' lying on a surface in a higher dimension whose projection back to the original space gives C.

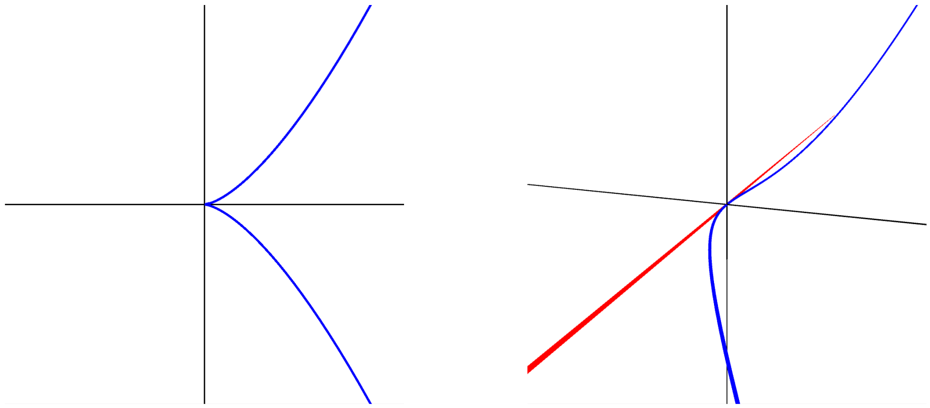

Fig. 2.1: The same curve C' as seen from different angles, causing an apparent cusp in the left image.

Take the surface X' given by $y - x\,z = 0$, containing a curve C' such that its projection onto $z = 0$ is C, see the example Figure 2.2.

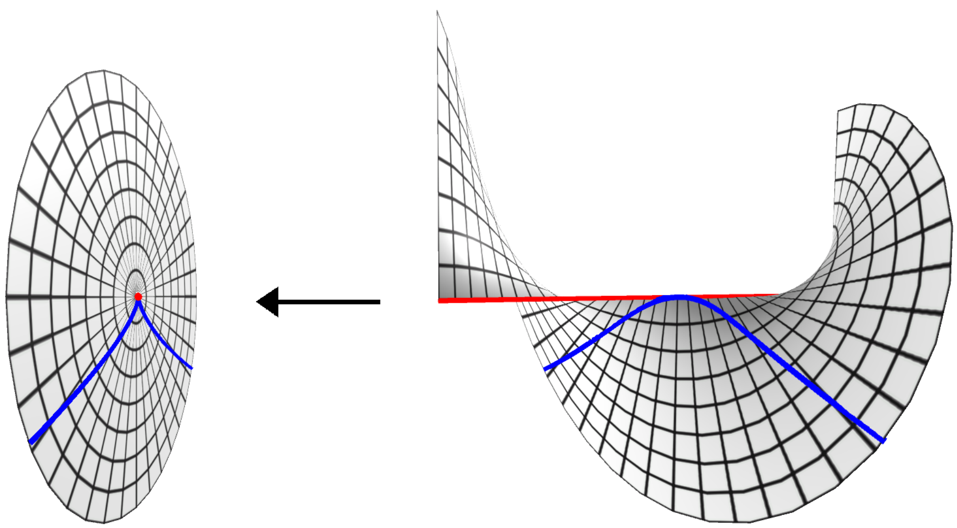

Fig. 2.2: The projection of X' onto $\mathbb{R}^2$.

Writing C' in terms of the coordinates $(x', y') = (x, z) = (x, y/x)$ on $X' \simeq \mathbb{C}^2$, we find

$$x'^2\,(y'^2 - x') = 0. \tag{2.2}$$

This is a blow-up of the curve C with the origin as centre.

We call this the *total transform* of C. The factor x'^2 gives the *exceptional line* e given by $x' = 0$, the preimage of the singularity b (the origin) in $\mathbb{C}^2$. The parabola $y'^2 - x' = 0$ is C', the *strict transform* of C. Gluing X' at $z = +\infty$ and $z = -\infty$, we have the following definition.

Definition 2.2. *Take a point $b : (x, y) = (\alpha, \beta)$ lying on a curve C in $\mathbb{C}^2$, let $[\xi; \eta]$ be homogeneous coordinates on $\mathbb{P}^1$, and define the space X' by*

$$\{((x, y), [\xi : \eta]) \;\mid (x - \alpha)\, \eta - (y - \beta)\, \xi = 0\,\} \subset \mathbb{C}^2 \times \mathbb{P}^1.$$

The map $\pi : X' \to \mathbb{C}^2$ is a birational mapping which is an isomorphism away from the exceptional line $e = \pi^{-1}(b)$. The strict transform of C is the closure of $\pi^{-1}(C \setminus b)$.

In practice, this means if $(x, y) = (\alpha, \beta)$ is a point b in $\mathbb{A}^2$, we view a blow-up as the change of variables to affine charts (x_{11}, y_{11}) and (x_{12}, y_{12}), where $x_{11} = x - \alpha$, $y_{11} = (y - \beta)/(x - \alpha)$, and $x_{12} = (x - \alpha)/(y - \beta)$, $y_{12} = y - \beta$. Away from b this is a holomorphic mapping, but has the effect of replacing the point b by an exceptional line e.

2.3 Resolved Space of Initial Conditions for an Ordinary Difference Equation

Consider the autonomous ordinary difference equation

$$x_{n+1}\, x_{n-1} = \frac{a\, b\, (x_n - c)(x_n - d)}{(x_n - a)(x_n - b)}, \tag{2.3}$$

where n is the discrete valued independent variable, x is a function of n, and $a,\ b,\ c,\ d$ are constant. This is an example of a multiplicative QRT map. Nonautonomous q-difference versions of this equation are known to be discrete equivalents of $\mathrm{P_{III}}$ and $\mathrm{P_{VI}}$ [38, 92], and are therefore often called $q\mathrm{P_{III}}$ and $q\mathrm{P_{VI}}$.

With generic initial data (x_0, x_1), we can use (2.3) to find x_2. From x_1 and x_2 we can find x_3, and this process will continue indefinitely in both the positive and negative directions to form the sequence $\{x_n\}_{n\in\mathbb{Z}}$. Since iterations can become unbounded on the domain (for example if $x_n = a$), to understand the behaviour near infinity we choose to work in a projective space by compactifying $\mathbb{C}^2$ so that the initial conditions (x_0, x_1) are taken to be in $\mathbb{P}^1 \times \mathbb{P}^1$.

Remark 2.1. *There are an infinite number of ways to compactify $\mathbb{C}^2$ using Hirzebruch surfaces. A Hirzebruch surface Σ_i is a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$, containing some C such that $C\cdot C$, the self-intersection of C, is equal to $-i$. All Hirzebruch surfaces can be obtained from each other via a sequence of blow-ups and blow-downs.*

The surface Σ_0 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. By convention, we use $\mathbb{P}^1 \times \mathbb{P}^1$ for discrete equations, and $\mathbb{P}^2$ for continuous.

First, reformulate (2.3) as a system of first order equations for u_n, v_n such that

$$u_n = x_{2n-1},$$

$$v_n = x_{2n},$$

so that the equation (2.3) induces a map ψ_n,

$$\psi_n : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1,$$

$$(u_n, v_n) \longmapsto (u_{n+1}, v_{n+1}).$$

Take generic initial conditions $(u_0, v_0) \in \mathbb{P}^1 \times \mathbb{P}^1$. In the forward direction the map ψ_0 gives

$$u_1 = \frac{a\,b}{u_0}\frac{(v_0 - c)(v_0 - d)}{(v_0 - a)(v_0 - b)}, \tag{2.4a}$$

$$v_1 = \frac{a\,b}{v_0}\frac{(u_1 - c)(u_1 - d)}{(u_1 - a)(u_1 - b)}. \tag{2.4b}$$

Consider the line in $\mathbb{P}^1 \times \mathbb{P}^1$ such that $v_0 = c$, with generic u_0. Under (2.3) this line is mapped to

$$u_1 = 0, \tag{2.5a}$$

$$v_1 = d. \tag{2.5b}$$

The map has taken a line to a point, losing apparently losing one of the degrees of freedom of the initial conditions (u_0). Similarly, the backwards map ψ_0^{-1} should take the point $(u_1, v_1) = (0, d)$ back to the line $v_0 = -c$. However simply putting this point into the equation gives an indeterminate $\frac{0}{0}$.

We call points where such indeterminacies occur *base points*. To circumvent this problem, we want to find a space of initial conditions on which map induced by (2.3) is free of base points. To achieve this, for each pair of points u_n, v_n we blow up $\mathbb{P}^1 \times \mathbb{P}^1$ everywhere base points appear, to find the resulting resolved surface X_n.

The equation lifted to X_n gives a map $\phi_n : X_n \to X_{n+1}$, where X_n is the surface resulting from the resolution of singularities for the pair u_n, v_n, see Figure 2.3. In this case the surface does not change between iterations since the equation's parameters are autonomous. In the case of discrete Painlevé equations the parameters are nonautonomous and hence these surfaces X_n will not coincide, however for generic n the blow-up procedure remains the same as the one outlined here.

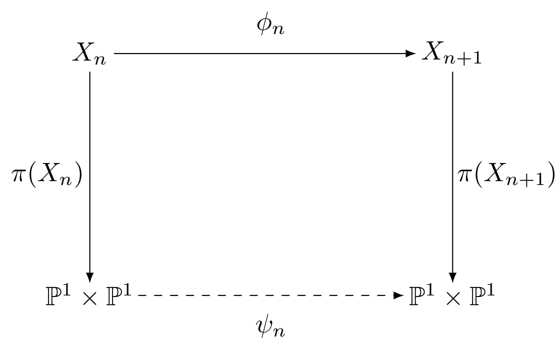

Fig. 2.3: Commuting Diagram

Each surface X_n is a leaf of the fibration as described in Definition 2.1, what is now widely known as the space of initial conditions.

For ease of notation in this section we use the so-called "bar" notation, where $u := u_n$, $\underline{u} := u_{n-1}$, $\bar{u} = u_{n+1}$, $\bar{\bar{u}} := u_{n+2}$, etc. Using this notation, we find the forward and backward steps of (2.3) yield the system

$$\bar{u} = \frac{a\,b\,(v-c)(v-d)}{u\,(v-a)(v-b)}, \tag{2.6a}$$

$$\underline{v} = \frac{a\,b\,(u-c)(u-d)}{v\,(u-a)(u-b)}. \tag{2.6b}$$

Instead of working in homogeneous coordinates, we cover $\mathbb{P}^1 \times \mathbb{P}^1$ in four affine charts using the variables u, v, $U = \frac{1}{u}$, and $V = \frac{1}{v}$. That is,

$$(u, v) \tag{2.7a}$$

$$(U, v) = \left(\frac{1}{u}, v\right) \tag{2.7b}$$

$$(u, V) = \left(u, \frac{1}{v}\right) \tag{2.7c}$$

$$(U, V) = \left(\frac{1}{u}, \frac{1}{v}\right) \tag{2.7d}$$

Looking in the first chart, we see the expression for $\bar{u}$ (2.6a) becomes indeterminate $\frac{0}{0}$ when $(u, v) = (0, c)$. We call this base point b_1 and blow up to remove the indeterminacy. We repeat this procedure until all such indeterminacies are removed.

2.3.1 Blow-up of b_1

Blowing up b_1 using the procedure from Definition 2.2, we find the exceptional line e_1 covered by the charts $(u_{11}, v_{11}) \cup (u_{12}, v_{12})$, where the first subscript denotes the number of blow-ups so far performed, and the second subscript indicates chart. These are

$$(u_{11}, v_{11}) = \left(u, \frac{v-c}{u}\right),$$

$$(u_{12}, v_{12}) = \left(\frac{u}{v-c}, v-c\right).$$

Writing $\bar{u}$ in terms of (u_{11}, v_{11}) and (u_{12}, v_{12}), we find

$$\bar{u} = \frac{a\, b\, v_{11}(u_{11}\, v_{11} + c - d)}{(u_{11}\, v_{11} + c - a)(u_{11}\, v_{11} + c - b)},$$

$$\bar{u} = \frac{a\, b\, (v_{12} + c - d)}{u_{12}\, (v_{12} + c - a)(v_{12} + c - b)}.$$

When looking for new base points after a blow-up, we only need to look on the resulting exceptional line, since all other base points already exist in the pre-blown-up space. Resolving these points in the coordinates of the exceptional line adds unnecessary complexity. Looking on the exceptional line e_1 parameterised by $u_{11} = 0$ and $v_{12} = 0$, we find

$$\bar{u} = \frac{a\, b\, v_{11}(c-d)}{(c-a)(c-b)}, \quad \bar{u} = \frac{a\, b\, (c-d)}{u_{12}\, (c-a)(c-b)}.$$

Since there are no new base points appearing on e_1, we blow-up another appearing in the original charts. If a new base point appears on the exceptional line e_i, we must blow it up in the chart of the ith blow-up. Continuing in the chart (u, v), we can see a new base point b_2 where $(u, v) = (0, d)$.

2.3.2 Blow-up of b_2

Blowing up b_2, we find the exceptional line e_2 covered by the charts $(u_{21}, v_{21}) \cup (u_{22}, v_{22})$. These are

$$(u_{21}, v_{21}) = \left(u, \frac{v-d}{u}\right),$$
$$(u_{22}, v_{22}) = \left(\frac{u}{v-d}, v-d\right).$$

Writing $\bar{u}$ in terms of (u_{21}, v_{21}) and (u_{22}, v_{22}), we find

$$\bar{u} = \frac{a\,b\,(u_{21}\,v_{21} + d - c)\,v_{21}}{(u_{21}\,v_{21} + d - a)(u_{21}\,v_{21} + d - b)},$$
$$\bar{u} = \frac{a\,b\,(v_{22} + d - c)}{u_{22}(v_{22} + d - a)(v_{22} + d - b)}.$$

Looking on the exceptional line e_2 parameterised by $u_{21} = 0$ or $v_{22} = 0$, we find

$$\bar{u} = \frac{a\,b\,(d-c)\,v_{21}}{(d-a)(d-b)}, \quad \bar{u} = \frac{a\,b\,(d-c)}{u_{22}(d-a)(d-b)}.$$

Since there are no new base points appearing on e_2, we blow-up another appearing in the original charts. Looking in the chart (U, v) (the affine chart where u becomes unbounded), we have

$$\bar{u} = \frac{a\,b\,U(v-c)(v-d)}{(v-a)(v-b)}.$$

Here we can see a new base point b_3 where $(U, v) = (0, a)$.

2.3.3 Blow-up of b_3

Blowing up b_3, we find the exceptional line e_3 covered by the charts $(U_{31}, v_{31}) \cup (U_{32}, v_{32})$. These are

$$(U_{31}, v_{31}) = \left(U, \frac{v-a}{U}\right),$$

$$(U_{32}, v_{32}) = \left(\frac{U}{v-a}, v-a\right).$$

Writing $\bar{u}$ in terms of (U_{31}, v_{31}) and (U_{32}, v_{32}), we find

$$\bar{u} = \frac{a\,b\,(U_{31}\,v_{31} + a - c)(U_{31}\,v_{31} + a - d)}{v_{31}(U_{31}\,v_{31} + a - b)},$$

$$\bar{u} = \frac{a\,b\,U_{32}(v_{32} + a - c)(v_{32} + a - d)}{(v_{32} + a - b)}.$$

Looking on the exceptional line e_3 parameterised by $U_{31} = 0$ or $v_{32} = 0$, we find

$$\bar{u} = \frac{ab\,(a-c)(a-d)}{v_{31}(a-b)}, \quad \bar{u} = \frac{abU_{32}(a-c)(a-d)}{(a-b)}.$$

Since there are no new base points appearing on e_3, we blow-up another appearing in the original charts. Looking in the chart (U, v), we find

$$\bar{u} = \frac{abU(v-c)(v-d)}{(v-a)(v-b)}.$$

Here we can see a new base point b_4 where $(U, v) = (0, b)$.

2.3.4 Blow-up of b_4

Blowing up b_4, we find the exceptional line e_3 covered by the charts $(U_{41}, v_{41}) \cup (U_{42}, v_{42})$. These are

$$(U_{41}, v_{41}) = \left(U, \frac{v-b}{U}\right),$$

$$(U_{42}, v_{42}) = \left(\frac{U}{v-b}, v-b\right).$$

Writing $\bar{u}$ in terms of (U_{41}, v_{41}) and (U_{32}, v_{32}), we find

$$\bar{u} = \frac{a\,b\,(U_{41}\,v_{41} + b - c)(U_{41}\,v_{41} + b - d)}{(U_{41}\,v_{41} + b - a)\,v_{41}},$$

$$\bar{u} = \frac{abU_{42}(v_{42} + b - c)(v_{42} + b - d)}{v_{42} + b - a}.$$

Looking on the exceptional line e_4 parameterised by $U_{41} = 0$ or $v_{42} = 0$, we find

$$\bar{u} = \frac{ab\,(b-c)(b-d)}{(b-a)v_{41}}, \quad \bar{u} = \frac{abU_{42}(b-c)(b-d)}{b-a}.$$

There are no new base points appearing on e_4.

The initial conditions u, v can also be used to iterate in the backwards direction. Therefore, to find a truly resolved space of initial conditions, we must also blow up at any base points where $\underline{v}$ (2.6b) is undefined.

Repeating the above procedure procedure for $\underline{v}$, we find four more base points,

$$b_5: \ (u,v) = (-c, 0),$$

$$b_6: \ (u,v) = (-d, 0),$$

$$b_7: \ (u,V) = (a, 0),$$

$$b_8: \ (u,V) = (b, 0).$$

Each of these is resolved after a single blow-up. Thus, after 8 blow-ups of $\mathbb{P}^1 \times \mathbb{P}^1$, we have found a surface X, represented diagrammatically in Figure 2.4.

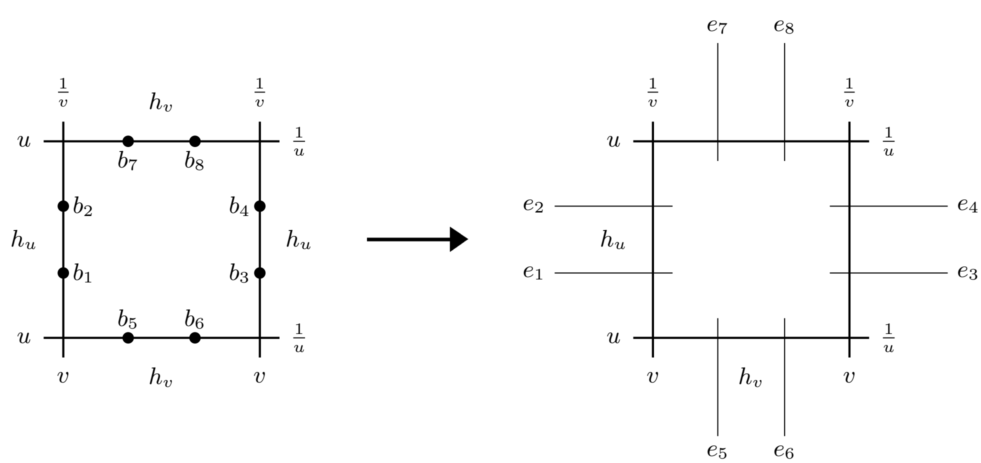

Fig. 2.4: Surface X resulting from the resolution of (2.3) on $\mathbb{P}^1 \times \mathbb{P}^1$, with 8 exceptional lines e_i, $i \in \{1, \ldots, 8\}$.

We now take a brief excursion into root systems and symmetry groups, which form the basis of the geometric theory of discrete Painlevé equations. Once we understand these tools we will return to the example of (2.3).

2.4 Root Systems and Symmetry Groups

Sakai's key insight in [92] was showing how the Painlevé equations arise from translations from an affine Weyl group corresponding to a resolved surface of initial conditions. In this section we give an introduction to the tools necessary for understanding the geometric theory of Painlevé equations. Using these tools, we take the example of the affine Weyl group $A_2^{(1)}$ and show how this gives rise to a discrete Painlevé equation. We follow the comprehensive 2017 review paper by Kajiwara, Noumi, and Yamada [63].

2.4.1 Root Systems

The fundamental object for this section is the *root system* [54].

Definition 2.3. *Consider a finite dimensional vector space V with an inner product denoted by $\langle \cdot, \cdot \rangle$. A root system $\Phi \subset V$ is a finite collection of vectors (or 'roots') that satisfy the following conditions:*

1. *The roots span V.*
2. *The only scalar multiples of a root $\alpha \in \Phi$ that are also in Φ are $-\alpha$ and α.*
3. *Given any two distinct roots α, β, the set Φ also contains the element*

$$\sigma_\alpha(\beta) = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha,$$

the reflection of β in the hyperplane orthogonal to α.

Example 2.1. Consider the root system A_2. Suppose the vector space $V = \mathbb{R}^2$, and start with the roots $\alpha = (2, 0)$ and $\beta = (-1, \sqrt{3})$. We call these the *simple* roots. For any root system Φ there are a number of choices for which roots are the simple roots, but the set of simple roots Δ must satisfy the following properties:

1. Δ must be a basis for vector space spanned by Φ.

2. Every element $\beta \in \Phi$ can be expressed as the sum $\beta = \sum_{\alpha\in\Phi} c_\alpha \alpha$ such that the coefficients c_α are integers which are all nonpositive or all nonnegative.

The set Δ is called the *basis* for Φ.

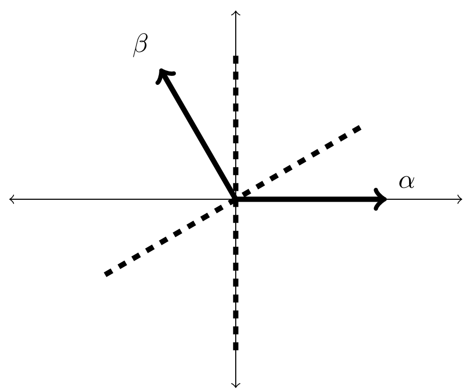

Fig. 2.5: Simple roots α and β of A_2.

Since A_2 is closed under reflection, reflecting α and β in the lines orthogonal to each of them (represented in Figure 2.5 by dotted lines) we find $\sigma_\alpha(\alpha) = -\alpha$, $\sigma_\beta(\beta) = -\beta$, $\sigma_\alpha(\beta) = \sigma_\beta(\alpha) = \alpha+\beta$, and $\sigma_\alpha(-\beta) = \sigma_\beta(-\alpha) = \sigma_{\alpha+\beta}(\alpha+\beta) = -\alpha-\beta$, shown in Figure 2.6.

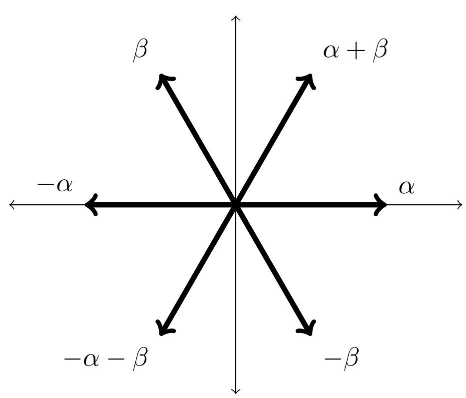

Fig. 2.6: The vectors of the root system A_2.

Definition 2.4. *The Weyl group $W(\Phi)$ of a root system $\Phi \subset V$ is the group of transformations of V generated by the reflections σ_α, with $\alpha \in \Phi$.*

In the case of A_2, the Weyl group is the symmetry group of an equilateral triangle. Note that the Weyl group is not the full symmetry group of the A_2 root system, which has the symmetry group of a hexagon. The set of reflection hyperplanes divides V into disconnected components, which we call *Weyl chambers*. Each Weyl group element permutes the Weyl chambers.

Accompanying the idea of root systems is their corresponding *root lattice*. A lattice in a space V is a discrete subgroup $U \subset V$ which spans V. One example of such a lattice is $\mathbb{Z}^2 \subset \mathbb{R}^2$. In subsequent chapters we study discrete integrable systems over this lattice called *quad-equations*.

Definition 2.5. *If $\Phi \subset V$ is a root system, then $Q(\Phi) := \mathbb{Z}\Phi$ (the additive group spanned by Φ) is called the root lattice* [22].

The proof that $Q(\Phi)$ is a lattice is beyond the scope of this section, but it depends on the following two facts about root systems:

1. If Φ is a root system under both the inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$, then $\langle\cdot,\cdot\rangle_1 = c\langle\cdot,\cdot\rangle_2$ for some $c \in \mathbb{R}$.
2. If $\langle\alpha,\alpha\rangle \in \mathbb{Q}$ for some $\alpha \in \Phi$, then $\langle\beta,\gamma\rangle \in \mathbb{Q}$ for all $\beta, \gamma \in \Phi$.

For the remainder of this chapter we are interested in root systems with the property that the inner product can be normalised so that for any root α, we have $\langle\alpha,\alpha\rangle = 2$. In this case we can neglect the dual root system, which is spanned by the coroots $\alpha^\vee = \dfrac{2}{\langle\alpha,\alpha\rangle}\alpha = \alpha$.

In the case of A_2, the root lattice $Q(A_2)$ is a triangular lattice as shown in see Figure 2.7.

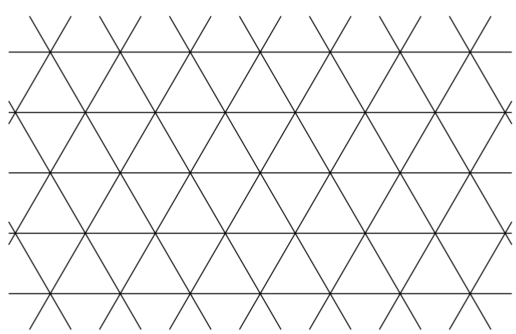

Fig. 2.7: Root lattice $Q(A_2)$.

In addition to root systems, there are also *affine* root systems. A transformation $f : V \to \mathbb{C}$ is said to be affine if it is of the form

$$f(x) = \langle a, x \rangle + b, \tag{2.8}$$

for some $a \in V$ and $b \in \mathbb{C}$.

Definition 2.6. *An affine root system over a vector space V is a root system over the vector space F of affine functions on V.*

The space F is equipped with the inner product $\langle \cdot, \cdot \rangle_F$ such that for any $f, g \in F$, $\langle f, g \rangle_F = \langle \nabla f, \nabla g \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product on V. The dimension of F is $\dim V + 1$.

Root systems and affine root systems are classified by Cartan matrices and Dynkin diagrams. Suppose we have a root system R with simple roots $\{\alpha_0, \ldots, \alpha_n\}$, with an inner product normalised such that for any R, we have $\langle \alpha_i, \alpha_i \rangle = 2$ whenever $i = j$. Define a matrix A, such that

$$A = (a_{ij})_{i,j=0,\ldots,n} = (\langle \alpha_i, \alpha_i \rangle)_{i,j=0,\ldots,n}. \tag{2.9}$$

A is a Cartan matrix corresponding to R. The Cartan matrices are classified by Dynkin diagrams, and we can therefore do the same for root systems. Each node of a root system's associated Dynkin diagram corresponds to one of the simple roots. As an example, see Figure 2.8 for the Dynkin diagram associated with A_2.

Fig. 2.8: Dynkin diagram of A_2.

Proposition 2.1. *Suppose $\Phi \subset V$ is a root system. For each $\alpha \in \Phi$ and $r \in \mathbb{Z}$, define an affine linear function*

$$f_{\alpha,r}(x) = \langle \alpha, x \rangle + r. \tag{2.10}$$

The set $S(\Phi)$ of all such functions $f_{\alpha,r}$ is an affine root system.

A rigorous proof of this can be found in [69].

We call $S(\Phi)$ the affine root system associated with Φ. By setting $f_{\alpha,r}(x) = 0$, it is clear that there is a one-to-one correspondence between the elements of $S(\Phi)$ and the reflection hyperplanes in $Q(\Phi)$. These hyperplanes divide V into disconnected components called *alcoves*. This naturally leads us the *affine Weyl group* $\widetilde{W}(S(\Phi))$. The affine Weyl group is generated by the set of reflections in the Weyl group, together with a Dynkin diagram automorphism π (a function which permutes the nodes of a Dynkin diagram, and therefore permutes the simple roots of $S(\Phi)$).

In the cases we concern ourselves with, the affine Weyl group is generated by reflections from the Weyl group paired with translations [54].

Definition 2.7. *The null root of an affine root system A with simple roots $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ is a vector $\delta = \sum_{i=0}^{n} c_i\,\alpha_i$, $c_i \in \mathbb{Z}$ such that $\langle \alpha_i, \delta \rangle = 0$ for any α_i.*

Returning to the example of A_2, the affine root system $S(A_2)$ associated with A_2 is called $A_2^{(1)}$. Up to 90° rotation, in this case the lattice formed by the reflection hyperplanes corresponding to the elements of $A_2^{(1)}$ matches the lines in $Q(A_2)$, shown in Figure 2.7. $A_2^{(1)}$ is generated by the three simple roots $\alpha_0, \alpha_1, \alpha_2$ and the corresponding affine Weyl group $\widetilde{W}(A_2^{(1)})$ is generated by the reflections $w_i = w_{\alpha_i}$, where w_{α_i} is the reflection about the root α_i. The null root is $\delta = \alpha_0 + \alpha_1 + \alpha_2$.

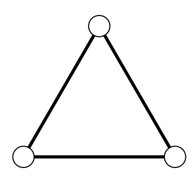

Fig. 2.9: Dynkin diagram of $A_2^{(1)}$.

2.4.2 Translations

As stated above, it was shown in [92] that discrete Painlevé equations arise from the translations of an affine Weyl group. We will now show that translations are generated by the composition of the reflections which generate an affine Weyl group.

Definition 2.8. *For each $\alpha \in V_0$ where $V_0 = \{x \in V \mid \langle \delta, x \rangle = 0\}$, the corresponding Kac translation*

$T_\alpha : V \to V$ is [62]

$$T_\alpha(x) = x + \langle \delta, x \rangle \, \alpha - \left(\frac{1}{2} \langle \alpha, \alpha \rangle \, \langle \delta, x \rangle + \langle \alpha, x \rangle \right) \delta. \tag{2.11}$$

Kac translations can be constructed as the composition of reflections from the affine Weyl group.

By explicit calculation, we find that the action of the fundamental reflections of the Weyl group on the simple roots of $A_2^{(1)}$ are shown in Table 2.1 [77].

	α_0	α_1	α_2
w_0	$-\alpha_0$	$\alpha_0 + \alpha_1$	$\alpha_0 + \alpha_2$
w_1	$\alpha_0 + \alpha_1$	$-\alpha_1$	$\alpha_1 + \alpha_2$
w_2	$\alpha_0 + \alpha_2$	$\alpha_1 + \alpha_2$	$-\alpha_2$

Tab. 2.1: Action of the fundamental reflections on simple roots

We also find that the Kac translation associated with α_1 acts on the simple roots by

$$T_{\alpha_1}[\alpha_0, \alpha_1, \alpha_2] = [2\alpha_0 + \alpha_1 + \alpha_2, -2\alpha_0 - \alpha_1 - 2\alpha_2, \alpha_0 + \alpha_1 + 2\alpha_2]. \tag{2.12}$$

By sequentially applying the reflections w_0, w_1, w_2, we find

$$T_{\alpha_1} w_1 \, w_0 \, w_2 \, w_0 [\alpha_0, \alpha_1, \alpha_2] = [\alpha_0, \alpha_1, \alpha_2]. \tag{2.13}$$

Therefore, $T_{\alpha_1} = w_0 \, w_2 \, w_0 \, w_1$. It may be possible to find T_{α_1} as a different but equivalent decomposition.

Similarly, consider the translation

$$S[\alpha_0, \alpha_1, \alpha_2] = [\alpha_0 + \delta, \alpha_1 - \delta, \alpha_2] = [2\alpha_0 + \alpha_1 + \alpha_2, -\alpha_0 - \alpha_2, \alpha_2]. \tag{2.14}$$

Following the same procedure we find

$$S \, w_1 \, w_2 [\alpha_0, \alpha_1, \alpha_2] = [\alpha_1, \alpha_2, \alpha_0], \tag{2.15}$$

and therefore $S \, w_1 \, w_2 = \pi$, where $\pi(\alpha_i) = \alpha_{i+1}$. This permutes the simple roots and hence is a Dynkin diagram automorphism.

2.4.3 The Picard Group

The final tool we will need in our algebraic toolbox is the Picard group. Intuitively, the elements of the Picard group are equivalence classes of codimension-1 subvarieties of a space.

The QRT mappings are defined in $\mathbb{P}^1 \times \mathbb{P}^1$ by a pencil of bidegree $(2, 2)$ curves, whose intersection multiplicity is 8. Since the discrete Painlevé equations can be thought of as deautonomisations of QRT mappings, we expect the resolution of discrete Painlevé equations in $\mathbb{P}^1 \times \mathbb{P}^1$ to require 8 blow-ups, generating a surface X. This was the case in Section 2.3.

The Picard group of the surface X is generated by the basis $\{h_u, h_v, e_1, e_2, \ldots, e_8\}$, with an operation denoted $+$. The curves h_u and h_v represent the equivalence classes of lines of constant u and v respectively, and e_i represents the exceptional line resulting from the blow-up of the base point b_i. In general, an element of the Picard lattice

$$x = d_1\, h_u + d_2\, h_v - m_1\, e_1 - \cdots - m_8\, e_8, \tag{2.16}$$

corresponds to the class of curves of bidegree (d_1, d_2) in $\mathbb{P}^1 \times \mathbb{P}^1$ passing through the base points b_i with intersection of multiplicity $\geq m_i$.

The Picard group $\text{Pic}(X)$ of a surface X is the group of isomorphism classes of invertible sheaves on X. For our purposes it is sufficient to understand the Picard group as a group whose elements represent the curves on X. Therefore, we consider the Picard group to be

$$\text{Pic}(X) = \mathbb{Z}h_u \oplus \mathbb{Z}h_v \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_8.$$

The Picard group is equipped with a bilinear product called the *intersection* product, which satisfies the following relations.

$$h_u \cdot h_u = h_v \cdot h_v = h_u \cdot e_i = h_v \cdot e_i = 0, \tag{2.17a}$$

$$h_u \cdot h_v = h_v \cdot h_u = 1, \tag{2.17b}$$

$$e_i \cdot e_j = -\delta_{ij}. \tag{2.17c}$$

Most of these relations can be understood with intuition, with the exception of negative self intersection from line 3. Line 1 reads parallel lines do not meet in $\mathbb{P}^1 \times \mathbb{P}^1$, and generic lines do not pass through the base points. Line 2 simply tells us that horizontal and vertical lines will intersect at a point. Castelnuovos contraction theorem states that for any curve C isomorphic to $\mathbb{P}^1$ with $C \cdot C = -1$, there exists a morphism which smoothly contracts C to a point [41]. Therefore, we can think of negative self intersection as an indication of a blow-up.

Defining an inner product $\langle x, y \rangle = -x \cdot y$, we can think of $\mathrm{Pic}(X)$ as a 10 dimensional vector space, with the natural basis $\{h_u, h_v, e_1, \ldots, e_8\}$. From Hartshorne [41], we know that the anticanonical divisor of $\mathbb{P}^1 \times \mathbb{P}^1$ blown up at 8 points is given by

$$-K_X = 2h_u + 2h_v - \sum_{i=1}^{8} e_i. \tag{2.18}$$

It turns out that if we find the set of irreducible divisors D, that is all $D_i \in \mathrm{Pic}X$ such that $D_i \cdot D_i = -2$, then $-K_X$ uniquely decomposes as a linear combination the elements of D. In fact, D generates an affine root system with null root $\delta \equiv -K_X$. In the case of the example from the previous section, we have

$$D_0 = h_u - e_1 - e_2,$$

$$D_1 = h_u - e_3 - e_4,$$

$$D_2 = h_v - e_5 - e_6,$$

$$D_3 = h_v - e_7 - e_8,$$

see Figure 2.4. What's more, the orthogonal complement of D (the set of all $\alpha_i \in \mathrm{Pic}(X)$ such that $\alpha_i \cdot D_j = 0 \ \forall \ i, j$) forms another affine root system corresponding to the symmetry group of the equation [92].

Finally, we can put the pieces together to see a discrete Painlevé equation arising from translations of an affine root system.

2.5 Surface Theory to Discrete Painlevé Equations

Suppose we have a surface X_n resulting from 8 blow-ups of $\mathbb{P}^1 \times \mathbb{P}^1$, with symmetry group $\widetilde{W}(A_2^{(1)})$. In this section we will find a discrete Painlevé equation arising from the action of translations from

$\widetilde{W}(A_2^{(1)})$.

If we have a map ϕ' acting on $\mathrm{Pic}(X_n)$, then we can consider an equivalent Cremona transformation acting on the surface X_n [20]. We then wish to realise the group $\widetilde{W}(A_2^{(1)})$ as a group of Cremona transformations on $\mathbb{P}^1 \times \mathbb{P}^1$, with four (nonautonomous) parameters b_1, b_2, b_3, b_4.

Since $\langle \alpha_i, \alpha_i \rangle = -\alpha_i \cdot \alpha_i = 2$, each α_i gives rise to a reflection w_i through the hyperplane perpendicular to α_i such that for every $x \in \mathrm{Pic}(X_n)$,

$$w_i(x) = x + (x \cdot \alpha_i)\, \alpha_i. \tag{2.19}$$

Since all α_i, D_j are perpendicular, $w_i(D_j) = D_j$. We also know from the previous section that the root system $A_2^{(1)}$ is spanned by some $\{\alpha_0, \alpha_1, \alpha_2\} \subset \mathrm{Pic}(X_n)$, and its orthogonal complement $E_6^{(1)}$ is spanned by some $\{D_0, D_1, D_2, D_3, D_4, D_5, D_6\} \subset \mathrm{Pic}(X_n)$. We deduce a configuration of base points such that after completing the resolution as in Section 2.3, the inaccessible curves form $E_6^{(1)}$.

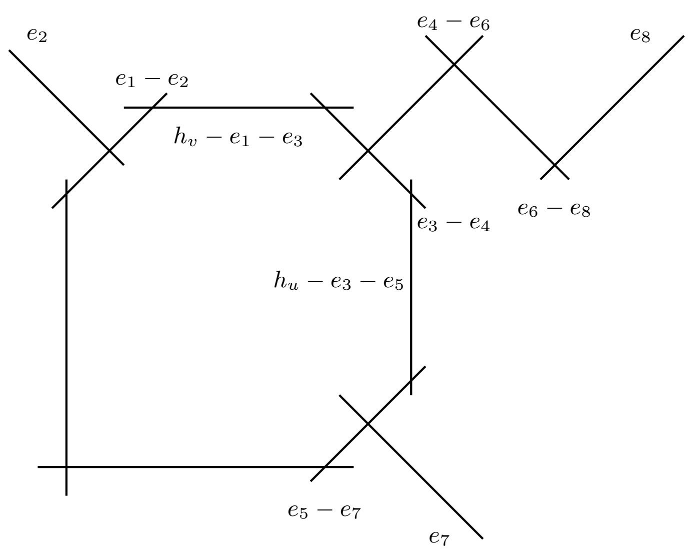

Fig. 2.10: Surface X obtained by 8 blow-ups of $\mathbb{P}^1 \times \mathbb{P}^1$, chosen to give the desired symmetry group $\widetilde{W}(A_2^{(1)})$.

From this we see that one possibility for the simple roots of $E_6^{(1)}$ in terms of the basis of $\mathrm{Pic}(X_n)$ is

$$D_0 := e_1 - e_2,$$

$$D_1 := h_v - e_1 - e_3,$$
$$D_2 := e_3 - e_4,$$
$$D_3 := h_u - e_3 - e_5,$$
$$D_4 := e_5 - e_7,$$
$$D_5 := e_4 - e_6,$$
$$D_6 := e_6 - e_8,$$

and hence

$$\alpha_0 = h_v - e_5 - e_7,$$
$$\alpha_1 = h_u - e_1 - e_2,$$
$$\alpha_2 = h_u + h_v - e_3 - e_4 - e_6 - e_8.$$

Using (2.19) and (2.21), we can calculate the action of w_i on the Picard lattice. In particular,

$$\begin{aligned} w_1(h_v) &= h_v + (h_v \cdot \alpha_0)\,\alpha_0, \\ &= h_v + (h_v \cdot (h_u - e_1 - e_2))\,(h_u - e_1 - e_2), \\ &= h_u + h_v - e_1 - e_2. \end{aligned} \tag{2.22}$$

To realise w_1 as a Cremona transformation on $\mathbb{P}^1 \times \mathbb{P}^1$, we start by finding a sequence of blow-ups and blow-downs which results in the action $h_v \mapsto h_u + h_v - e_1 - e_2$, then proving that the result does in fact correspond to w_1. Following this procedure, we find the actions of the elements of $\widetilde{W}(A_2^{(1)})$ are

$$w_0 : (u, v, b_1, b_2, b_3, b_4) \longmapsto \left(u - \frac{b_1}{v}, v, -b_1, b_2 - b_1, b_3, b_1 + b_4\right), \tag{2.23a}$$

$$w_1 : (u, v, b_1, b_2, b_3, b_4) \longmapsto \left(u, \frac{u\,v - b_2}{u}, b_1 - b_2, -b_2, b_3, b_4 - b_2\right), \tag{2.23b}$$

$$\begin{aligned} w_2 : (u, v, b_1, b_2, b_3, b_4) \longmapsto \Bigg(&\frac{-b_3^2 + b_3\,u - b_4 + u\,(u+v)}{b_3 + u + v}, \frac{b_3^2 + b_3\,v + b_4 + v\,(u+v)}{b_3 + u + v}, \\ &b_1 + b_3^2 + b_4, b_2 - b_3^2 - b_4, b_3, b_1 - b_3^2\Bigg), \end{aligned} \tag{2.23c}$$

$$\pi : (u, v, b_1, b_2, b_3, b_4) \longmapsto \Big(-u - v - b_3, u, -b_2, -b_4 - b_3^2, b_3, b_1 - b_3^2\Big). \tag{2.23d}$$

Explicit calculations can be found in [53].

Consider the translation (2.12), found to be $w_0\, w_2\, w_0\, w_1$. Suppose the map ϕ_1' acts on $\mathrm{Pic}(X)$ such that $\phi_1' = w_0\, w_2\, w_0\, w_1$. Since this is a translation we expect the equivalent map on X_n to be an automorphism, and hence we want to find the Cremona transformation of $\mathbb{P}^1 \times \mathbb{P}^1$ corresponding to ϕ_1'. If we define

$$(\bar{u}, \bar{v}, \bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4) = \phi_1(u, v, b_1, b_2, b_3, b_4), \tag{2.24}$$

then by iteratively applying (2.23), and solving the linear difference equations which arise for b_1, b_2, b_3, b_4, we find $b_1 = b_2 = 0$, $b_3 = \textit{constant}$ and $b_4 = -b_3^2$, and therefore

$$(\bar{u}, \bar{v}, \bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4) = (u, v, 0, 0, b_3, -b_3^2), \tag{2.25}$$

a trivial case.

Now, suppose the map ϕ_2' acts on the Picard lattice, such that $\phi_2' = w_1\, w_2\, \pi^2$. Using Table 2.1, a straightforward calculation shows

$$\begin{aligned} w_1\, w_2\, \pi^2[\alpha_0, \alpha_1, \alpha_2] &= w_1\, w_2\, \pi[\alpha_1, \alpha_2, \alpha_0], \\ &= w_1\, w_2[\alpha_2, \alpha_0, \alpha_1], \\ &= w_1[-\alpha_2, \alpha_0 + \alpha_2, \alpha_1 + \alpha_2], \\ &= [-\alpha_1 - \alpha_2, \alpha_0 + 2\alpha_1 + \alpha_2, \alpha_2], \\ &= [\alpha_0 - \delta, \alpha_1 + \delta, \alpha_2], \end{aligned}$$

and hence this is also a translation in $\widetilde{W}(A_2^{(1)})$. Using (2.23), we find that the translation $w_1\, w_2\, \pi^2 \in \widetilde{W}(A_2^{(1)})$ gives

$$u_{n+1} = -u_n - v_{n+1} + \frac{(b_3^2 + b_4 + C_1 - C_2)\, n - C_2 + b_3^2 + b_4}{v_{n+1}} - b_3, \tag{2.26a}$$

$$v_{n+1} = -u_n - v_n + \frac{(b_3^2 + b_4 + C_1 - C_2)\, n - C_2}{u_n} - b_3, \tag{2.26b}$$

for constants C_1, C_2, b_3, b_4. Upon the substitutions

$$\begin{aligned} b_3 &= \gamma, \\ b_4 &= -\alpha + 2c - \gamma^2, \\ C_1 &= -\alpha + \beta - c, \\ C_2 &= \beta + c, \end{aligned}$$

we find

$$u_{n+1} + v_{n+1} + u_n = \frac{2\alpha\,(n+1) + \beta - c}{v_{n+1}} + \gamma, \tag{2.27a}$$

$$v_{n+1} + u_n + v_n = \frac{2\alpha\, n + \beta + c}{u_n} + \gamma. \tag{2.27b}$$

If we also define a function x_n such that $u_n = x_{2n}$, $v_n = x_{2n-1}$, we have

$$x_{n+1} + x_n + x_{n-1} = \frac{\alpha\, n + \beta + c(-1)^n}{x_n} + \gamma. \tag{2.28}$$

In the case $c = 0, \gamma = 1$, this is the equation (1.3) discovered by Shohat in [94]. The equation (2.28) is sometimes known as the asymmetric first discrete Painlevé equation. However, due to its origin from surface theory, it can also be referred to as $d\mathrm{P}(E_6^{(1)})$.

2.6 Painlevé Equations to Surface Theory

In the same way as a map on the Picard group can give rise to a discrete Painlevé equation (2.26), it is similarly fruitful to consider a discrete Painlevé equation as inducing a map ϕ' which acts on elements of the Picard group of the corresponding resolved surface.

Consider again the example from Section 2.3. By blowing up at 8 base points, we found a surface X (Figure 2.4) on which the map induced by (2.3) induces an automorphism. Recall that the irreducible divisors on this surface are

$$D_0 = h_u - e_1 - e_2, \tag{2.29a}$$

$$D_1 = h_u - e_3 - e_4, \tag{2.29b}$$

$$D_2 = h_v - e_5 - e_6, \tag{2.29c}$$

$$D_3 = h_v - e_7 - e_8. \tag{2.29d}$$

Using the intersection form we are able to compute the corresponding Cartan matrix

$$A = (-D_i \cdot D_j)_{i,j=0,\ldots,3} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}.$$

This is the Cartan matrix of type $A_3^{(1)}$, with the orthogonal complement in Pic(X) giving the corresponding symmetry group $D_5^{(1)}$.

One advantage of considering the induced mapping on the Picard group is that it is a linear mapping. Therefore by determining how ϕ' acts on each of the basis elements $\{h_u, h_v, e_1, \ldots, e_8\}$ we can compute $\phi'(x)$ for any $x \in \text{Pic}(X)$. We can do this directly by using (2.3) to compute the image of each of the $e_1, \ldots, e_8$ and a representative generic h_u and h_v, and finding the degree of the image and the intersection with the exceptional lines. While possible, this can be very computationally difficult.

Since ϕ' is an isomorphism it must preserve the inner product (and thus the intersection form), and hence the intersection number. For any pair of curves C_1, C_2 on the resolved surface X, we know $\phi'(C_1) \cdot \phi'(C_2) = C_1 \cdot C_2$. Therefore, the action of ϕ' on the set of curves with self-intersection -2 (2.29) must be a simple permutation.

By finding the image of each of the D_i under ϕ' then considering the exceptional lines e_j at the end of a sequence of blow-ups, we can find the action of ϕ' on the Picard lattice using the linearity of ϕ'.

For example, D_0 is parameterised by $(u, v) = (0, v)$, and upon substituting $u = 0$ into (2.6) we find that the action of ϕ' on D_0 is $D_0 \mapsto (\frac{1}{u}, v) = (0, \frac{ab}{v})$, which is D_1. By a similar process, we find the cycles

$$D_0 \longmapsto D_1 \longmapsto D_0, \tag{2.30a}$$

$$D_2 \longmapsto D_3 \longmapsto D_2. \tag{2.30b}$$

The image of h_u and h_v are complicated, so we first consider the images of e_j. Calculating the image of e_5 under the action of (2.6), we find that

$$\phi'(e_5) = h_v - e_6.$$

Similarly, calculating the image of e_6 we find $\phi'(e_6) = h_v - e_5$. Substituting this into $\phi'(D_2) = D_3$, we find

$$\phi'(h_v - e_5 - e_6) = h_v - e_7 - e_8,$$

$$\phi'(h_v) - \phi'(e_5) - \phi'(e_6) = h_v - e_7 - e_8,$$
$$\phi'(h_v) = h_v + \phi'(e_6) + \phi'(e_5) - e_7 - e_8,$$
$$\Longrightarrow \phi'(h_v) = 2h_u + h_v - e_5 - e_6 - e_7 - e_8.$$

Similarly for h_u, after finding $\phi'(e_1)$ and $\phi'(e_2)$ and substituting into $\phi'(D_0) = D_1$, we find

$$\phi'(h_u) = 5h_u + 2h_v - e_1 - e_2 - e_3 - e_4 - 2e_5 - 2e_6 - 2e_7 - 2e_8.$$

Using the resulting expressions we computed for $\phi'(h_u)$, $\phi'(h_v)$, and each $\phi'(e_j)$, we have the linear transformation on the basis of $\mathrm{Pic}(X)$ generated by the equation (2.6).

$$\phi' \begin{pmatrix} h_u \\ h_v \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \end{pmatrix} = \begin{pmatrix} 5 & 2 & -1 & -1 & -1 & -1 & -2 & -2 & -2 & -2 \\ 2 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 2 & 1 & 0 & -1 & 0 & 0 & -1 & -1 & -1 & -1 \\ 2 & 1 & -1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 \\ 2 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ 2 & 1 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} h_u \\ h_v \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \end{pmatrix}. \tag{2.31}$$

As we shall see, this matrix contains all the necessary information on the integrability of (2.3).

2.6.1 Algebraic Entropy

One popular definition of integrability of mappings is *algebraic entropy*. In general, the composition of n degree d maps is degree d^n. However, if common factors can be eliminated without changing the map on generic points this can lower the degree of the iterates. It was observed that for systems with invariants there is sufficient factorisation for the growth to become polynomial [23, 16, 47, 100]. In [10] the authors introduced the notion of *algebraic entropy* as a characterising number of a discrete system with rational evolution.

Definition 2.9. *Given a map ψ, let the sequence d_n be the degree of the successive iterates ψ^n of ψ. The limit*

$$\epsilon := \lim_{n\to\infty} \frac{1}{n} \log d_n,$$

is always defined. The quantity ϵ is called the algebraic entropy of the map.

In the cases where d_n has polynomial dependence on n, the algebraic entropy is zero. As these cases permit invariants and low dynamical complexity, vanishing algebraic entropy is strongly linked with integrability.

The algebraic tools of this chapter can be used to compute the algebraic entropy directly. In [95], a method is provided to calculate the algebraic entropy of a mapping on a plane with a simple idea: Take a line of generic intial data and consider the growth of degrees of the line under iteration of the map. In $\mathrm{Pic}(X)$, a straight line has the representation $h_u + h_v$. The bidegree of the image of the line under the mapping is the coefficients of h_u and h_v of the image in the Picard lattice, $\phi'(h_u + h_v)$. Using the linear mapping on $\mathrm{Pic}(X)$, we find that the entropy is given by

$$\epsilon = \log(\max(\lambda_n)), \tag{2.32}$$

where $\{\lambda_n\}_{n\in\mathbb{Z}}$ are the eigenvalues of ϕ' (2.31). In the particular example (2.31), as expected we find $\epsilon = \log(1) = 0$.

Chapter 3: Discrete Lattice Systems and the ABS Classification

Lattice systems are to maps what partial differential equations are to ordinary differential equations, and form a well-established and important field in the theory of integrable systems. A great deal of progress has been made in the solutions of integrable lattice equations, including the discovery of $N-$soliton solutions. For selected examples, see [91, 8, 71, 70].

In [76, 14] it was shown that in particular, integrable systems on *quad-graphs* (that is, cellular decompositions of a surface whose cells are quadrilateral) are fundamental, and explain many useful results in the field. For this reason, throughout this thesis we will focus on integrable systems on quad-graphs. These are equations relating the values of the solution on the vertices of an elementary quadrilateral or quad, that is a quadrilateral of minimal size on the lattice. This chapter will be dedicated to exploring the background of integrable lattice equations on quad-graphs, and give recent developments in the field. In particular, we cover the ABS classification [5, 6] a classification of integrable systems on quad-graphs (up to certain assumptions).

3.1 Notation

To begin, we must review notation we will use in the study of lattice equations.

Consider a square lattice $\mathbb{Z}^2$, letting l and m be the independent variables serving as coordinates, and suppose we have the dependent variable $x_{l,m}$ as a function of l and m on the vertices, see Figure 3.1.

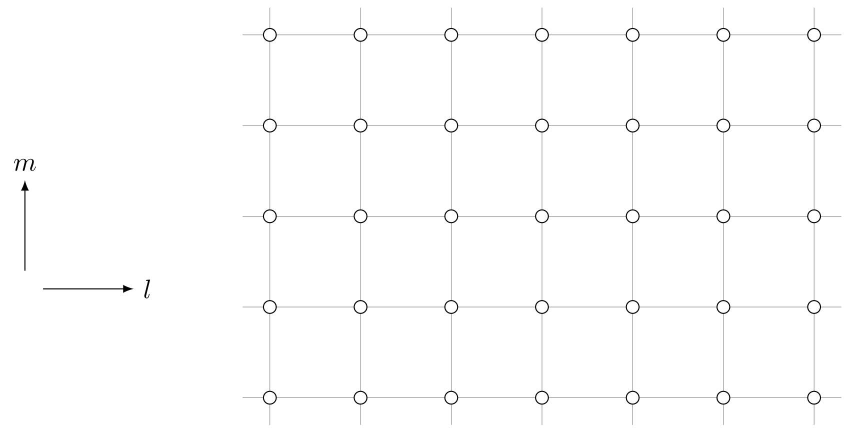

Fig. 3.1: Square lattice on the plane, with vertices labelled with dots.

Throughout this thesis we use the following notation. For some point in the lattice with coordinates l, m, we define

$$x := x_{l,m},$$

$$\bar{x} := x_{l+1,m},$$

$$\hat{x} := x_{l,m+1},$$

$$\hat{\bar{x}} := x_{l+1,m+1},$$

and so on. In this setting quad-equations will take the form

$$Q(x, \bar{x}, \hat{x}, \hat{\bar{x}}; \alpha, \beta) = 0, \tag{3.1}$$

where α, β are parameters corresponding to the two lattice directions, and $x, \bar{x}, \hat{x}, \hat{\bar{x}}$ correspond to the four vertices of a generic quadrilateral in the lattice, as shown in Figure 3.2.

The parameters α, β are related to the origin many integrable lattice equations have in terms of Bäcklund transformations of integrable partial differential equations as introduced in Chapter 1.

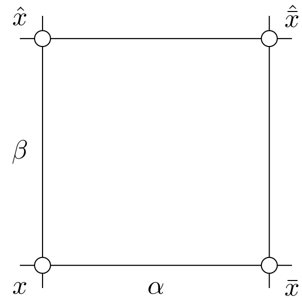

Fig. 3.2: Elementary quadrilateral with bar-shift notation.

Throughout this thesis we also use notation on a generic quadrilateral such that for the dependent variable $x_{l,m}$ at a generic point l, m, we say

$$x := x_{l,m},$$

$$u := x_{l+1,m},$$

$$v := x_{l,m+1},$$

$$y := x_{l+1,m+1},$$

see Figure 3.3. Throughout this thesis we use both standards, with the choice of which depending on clarity.

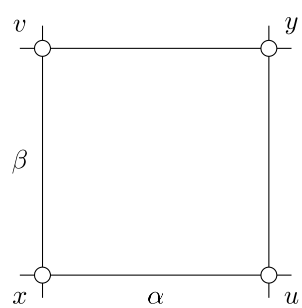

Fig. 3.3: Elementary quadrilateral with vertices x, u, v, y.

3.2 Initial Conditions

In this thesis we focus on lattice equations in two dimensions, so in this section we introduce initial conditions for lattice equations specifically in two dimensions. However, the concepts discussed here generalise to any higher dimension.

In order for a quad-equation to define an evolution over a lattice it is necessary to have some sensible initial conditions. In the case of lattice equations it is possible for initial conditions to be overdetermined and hence cause incompatibilities, such as the initial conditions labelled (2) in Figure 3.4. In order to avoid such incompatibilities we consider initial conditions lying along a diagonal staircase. We consider regular staircases such as the staircase labelled (1) (that is, staircases with steps of constant height and width). We call (1) a $(2, 1)$–staircase.

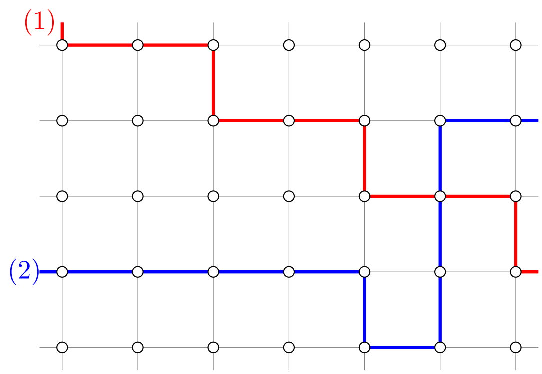

Fig. 3.4: Sets of initial conditions.

Definition 3.1. *An (m, n)–staircase of initial conditions is a set of initial conditions for a lattice equation which lie on a staircase of the form of (1) in Figure 3.4, with step width m and height n. Such choices of initial conditions allow iteration over the lattice in the directions perpendicular to the staircase [98].*

In order to use a staircase of initial conditions to iterate over the whole lattice the staircase must extend infinitely far in both directions. However, using a finite staircase it is still possible to iterate over a finite region. For example, using a (1,1)–staircase of initial conditions with only 5 vertices it is possible to solve over a 3×3 vertex region of the lattice. We see such a configuration in Chapter 5.

3.3 Integrability

There exist many different views on what the definition of integrability should be even in continuous systems, some of which have directly inspired analogous definitions for discrete systems such as the existence of a Lax pair [67], and possessing an infinite series of conservation laws [68]. There also exist many indicators of integrability which are strictly discrete in nature. We provide a brief outline of a few such definitions here.

1. *Singularity Confinement*

Singularity confinement was considered as a discrete equivalent to the Painlevé property, analysing the singularity structure of an equation. We say that a discrete equation confines singularities or is *confining* if movable singularities in the solution are cancelled out after a certain number of iterations. For example, consider the map

$$x_{n+1} + x_{n-1} = x_n + \frac{1}{x_n}. \tag{3.2}$$

If for some N the iteration x_N vanishes, then we find x_{N+1} and x_{N+2} become unbounded, and hence further iterations become indeterminate as we find an expression of the form $\infty - \infty$. We therefore consider $x_N = 0$ as a singularity of the mapping [90]. In general we consider a singularity as where the value of some x_{n+1} does not depend on x_{n-1}. In such a situation the equation has lost dependence on a degree of freedom of the initial conditions, and we consider confinement therefore as the recovery of such a recovery of this degree of freedom.

If instead we introduce a quantity ϵ such that $|\epsilon| \ll 1$ and taking $x_N = \epsilon$, by iterating (3.2) and taking the limit as $\epsilon \to 0$, instead of indeterminacies we find $x_{N+3} = 0$ and $x_{N+4} = x_{N-1}$, and hence we say the system has 'recovered' its lost degree of freedom.

This property has a similar definition for lattice equations. Consider a quad-equation for the function $x_{l,m}$

$$Q(x_{l,m}, x_{l+1,m}, x_{l,m+1}, x_{l+1,m+1}; \alpha, \beta) = 0, \tag{3.3}$$

with initial conditions $x_{l-1,m+1}$, $x_{l-1,m}$, $x_{l-1,m-1}$, $x_{l,m-1}$, $x_{l+1,m-1}$, such that using (3.3) to iterate over the lattice we find that at $x_{l,m}$ the system loses dependence on the initial condition $x_{l-1,m-1}$, see Figure 3.5. If the system recovers this lost degree of freedom at $x_{l+1,m+1}$, we again say the system is confining. As we saw in Chapter 1, this property is not sufficient for integrability [47]. We explore this property further in subsequent chapters.

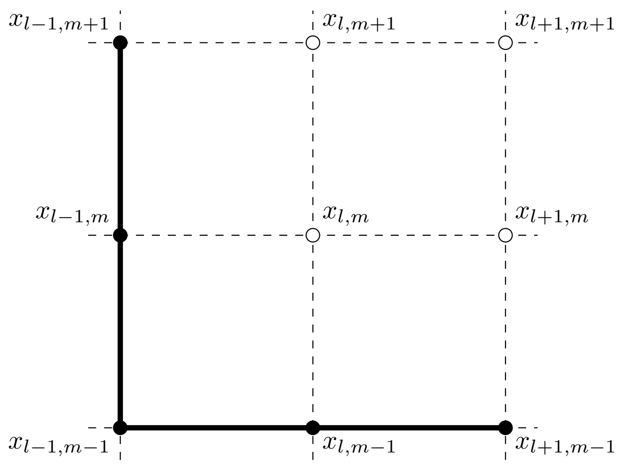

Fig. 3.5: 3×3 section of $\mathbb{Z}^2$, with initial values marked in bold.

2. *Algebraic Entropy*

Recall from Chapter 2 the algebraic entropy ϵ of a map, defined as

$$\epsilon := \lim_{n\to\infty} \frac{1}{n} \log(d_n), \tag{3.4}$$

where d_n is the degree of the nth iterate. The sequence $\frac{1}{n}\log(d_n)$ always possesses a limit as $n \to \infty$, and hence a map always has a well defined algebraic entropy. Vanishing entropy is considered a strong indicator of integrability [10, 78, 95].

In the case of quad-equations, the algebraic entropy is defined separately in each of the four diagonal directions of iteration (corresponding to the four corners of a quadrilateral), and these values do not necessarily coincide. In the case of integrable quad-equations, the algebraic entropy vanishes in all four directions [101].

To find the algebraic entropy of a lattice equation, take a regular (1,1)–staircase as initial conditions, as in Figure 3.6. Define the sequence of degrees $d^{(n)}_{\pm\pm}$ (where the subscript denotes direction of iteration) to be the degree of the iterates along successive diagonals parallel to the staircase, see Figure 3.6.

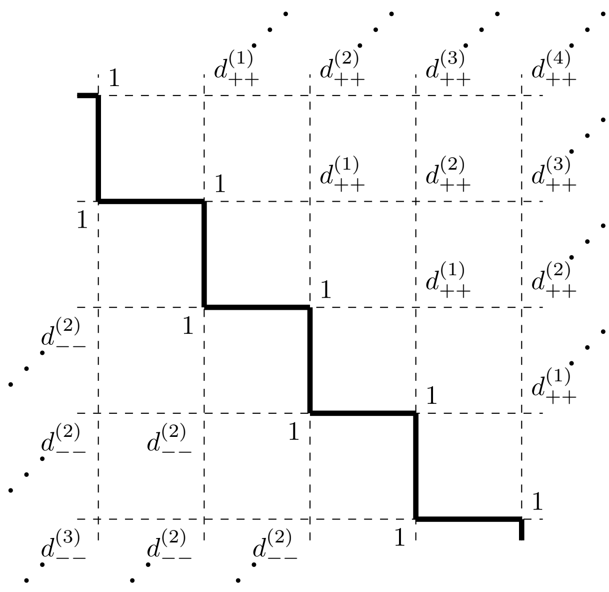

Fig. 3.6: Sequence of degrees used to calculate the entropy of a map in the positive direction of both independent variables, labels indicate degree of iterate.

Now, just as with maps, the algebraic entropy in each of the four directions is defined as

$$\epsilon_{\pm\pm} = \lim_{n\to\infty} \frac{1}{n} \log(d^{(n)}_{\pm\pm}), \tag{3.5}$$

and vanishing entropy indicates polynomial growth.

3. *Multidimensional Consistency*

Multidimensional consistency is the property that several copies of an equation may be imposed simultaneously on a higher dimensional lattice, such that no inconsistency or multi-valuedness occurs.

In the case of a quad-equation of the form (3.1) where Q is multilinear in each of the vertices $x, \bar{x}, \hat{x}, \hat{\bar{x}}$, it is possible to solve uniquely for any of the vertices in terms of the other 3. It is then simple to test multidimensional consistency (in this case consistency around a cube).

Consistency is a key concept in the study of integrable systems, and a key step was taken in [14] where it was shown that not only does multidimensional consistency imply the existence of a Lax pair and zero curvature representation, but that they can be found algorithmically using only the equation and the consistency property.

To begin, introduce a new lattice direction associated with the independent variable n, with associated parameter γ. Now considering $x_{l,m,n}$ as the dependent variable depending on l, m, n such that $\tilde{x} = x_{l,m,n+1}$ and imposing copies of the equation (3.1) on each elementary quadrilateral of the 3-dimensional lattice, we have the system

$$Q(x, \bar{x}, \hat{x}, \hat{\bar{x}}; \alpha, \beta) = 0, \tag{3.6a}$$

$$Q(x, \bar{x}, \tilde{x}, \tilde{\bar{x}}; \alpha, \gamma) = 0, \tag{3.6b}$$

$$Q(x, \hat{x}, \tilde{x}, \hat{\tilde{x}}; \beta, \gamma) = 0. \tag{3.6c}$$

Now, given initial data x, $\bar{x}$, $\hat{x}$, $\tilde{x}$, there are three ways to calculate the value of $\hat{\tilde{\bar{x}}}$ depending on the order of application of (3.6), or different paths taken over the cube to $\hat{\tilde{\bar{x}}}$ from the initial data. The equation (3.1) is said to be multidimensionally consistent if the value of $\hat{\tilde{\bar{x}}}$ is independent of path taken.

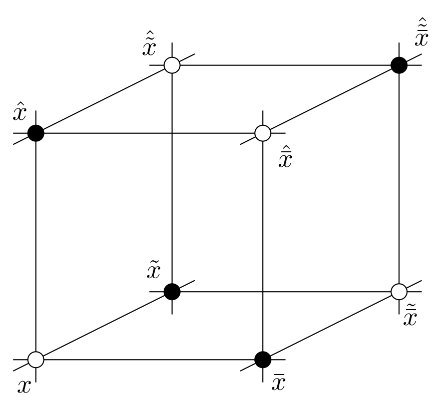

Fig. 3.7: Elementary cube

As an example, consider the cross ratio equation, given by

$$\frac{(x-\hat{x})(\bar{x}-\hat{\bar{x}})}{(x-\bar{x})(\hat{x}-\hat{\bar{x}})}=\frac{\beta}{\alpha}, \tag{3.7}$$

or in the form (3.1),

$$\alpha\,(x-\hat{x})(\bar{x}-\hat{\bar{x}})-\beta\,(x-\bar{x})(\hat{x}-\hat{\bar{x}})=0. \tag{3.8}$$

Solving for the vertex $\hat{\bar{x}}$ gives

$$\hat{\bar{x}}=\frac{\alpha\,\bar{x}\,(x-\hat{x})-\beta\,\hat{x}\,(x-\bar{x})}{\alpha\,(x-\hat{x})-\beta\,(x-\bar{x})}. \tag{3.9}$$

Introducing a third lattice direction with corresponding independent variable n and parameter γ also gives

$$\tilde{\bar{x}}=\frac{\alpha\,\bar{x}\,(x-\tilde{x})-\gamma\,\tilde{x}\,(x-\bar{x})}{\alpha\,(x-\tilde{x})-\gamma\,(x-\bar{x})}, \tag{3.10}$$

and

$$\hat{\tilde{x}}=\frac{\beta\,\hat{x}\,(x-\tilde{x})-\gamma\,\tilde{x}\,(x-\hat{x})}{\beta\,(x-\tilde{x})-\gamma\,(x-\hat{x})}, \tag{3.11}$$

Now, to find $\hat{\tilde{\bar{x}}}$, shift (3.9) in the n direction and substitute in values for $\tilde{\bar{x}}$ and $\hat{\tilde{x}}$ from (3.10) and (3.11) respectively. This gives

$$\hat{\tilde{\bar{x}}}=\frac{(\alpha-\gamma)\,(\tilde{x}-\hat{x})\,\bar{x}-(\beta-\gamma)\,(\tilde{x}-\bar{x})\,\hat{x}}{(\alpha-\gamma)\,(\tilde{x}-\hat{x})-(\beta-\gamma)\,(\tilde{x}-\bar{x})}, \tag{3.12}$$

which is invariant under cyclic permutation of lattice directions, and therefore (3.7) is multidimensionally consistent.

Note that $\hat{\tilde{\bar{x}}}$ does not depend on x, only the vertices spanning a tetrahedron in the elementary cube marked with black circles in Figure 3.7. This is called the *tetrahedron property*. A similar relation holds for the white vertices of the cube.

3.4 Famous Examples of Integrable Quad-Equations

The study of interable lattice equations is relatively new, with the earliest examples appearing in the 1970s and 1980s in the context of discretising known PDEs with soliton solutions [1, 2, 50, 51, 19, 84]. Many examples are two-dimensional, however lattice equations do exist in dimensions 3 and above. Through examples this thesis has already introduced some integrable lattice equations ((1.11), (3.7)), and in this section we introduce a few more famous examples of integrable lattice equations to demonstrate the depth of this field of study, and stoke the reader's interest. A more comprehensive list can be found in [73].

Example 3.1. In Chapter 1, we saw the lattice KdV equation (1.11) appear from Bäcklund transformations of the KdV equation. In fact, there are many examples of lattice equations in the KdV family. Possibly the most famous example of an integrable lattice equation is the *lattice potential Kortweg-de Vries* (KdV) equation, first presented in [74] in the form

$$(p - q + \hat{u} - \bar{u})(p + q - \hat{\bar{u}} + u) = p^2 - q^2, \tag{3.13}$$

where $p, q \in \mathbb{C}$ are parameters. Related to (3.13) by a Miura transformation is the *lattice modified KdV* equation, given by

$$p(v\,\hat{v} - \bar{v}\,\hat{\bar{v}}) - q(v\,\bar{v} - \hat{v}\,\hat{\bar{v}}) = 0, \tag{3.14}$$

where $v_{l,m}$ is the dependent variable. Also in this list is the cross ratio equation (3.7). This equation is also known as the *lattice Schwarzian KdV* equation, or more recently as Q1 as we will explain in the next section.

Example 3.2. Another famous example is the discrete *sine-Gordon* equation, given by

$$p\,\sin(\theta - \bar{\theta} - \hat{\theta} + \hat{\bar{\theta}}) - q\,\sin(\theta + \bar{\theta} + \hat{\theta} + \hat{\bar{\theta}}) = 0, \tag{3.15}$$

where $\theta_{l,m}$ is the dependent variable. It was first presented by Hirota in [50].

Example 3.3. In [4] Adler found a quad-equation resulting as the nonlinear superposition principle for Bäcklund transformations of the Krichever-Novikov equation [66]. It was given in a different form in [45] as

$$p(u\,\bar{u} - \hat{u}\,\hat{\bar{u}}) - q(u\,\hat{u} - \bar{u}\,\hat{\bar{u}}) + p\,q\,r(1 + u\,\bar{u}\,\hat{u}\,\hat{\bar{u}}) = 0. \tag{3.16}$$

Solutions to this equation were first found in [7], using a non-trivial seed solution and Bäcklund transformations.

Example 3.4. It is also possible to have an integrable lattice *system* with several dependent variables. Consider the following system of equations on a quad, often called the lattice Boussinesq equation [48].

$$B1 \equiv \tilde{w} - u\,\tilde{u} + v = 0,$$

$$B2 \equiv \hat{w} - u\,\hat{u} + v = 0,$$

$$B3 \equiv w - u\,\hat{\tilde{u}} + \hat{\tilde{v}} + \frac{p-q}{\hat{u}-\tilde{u}} = 0.$$

This is a multicomponent map, such that at each vertex position there are the values u, v and w. After taking the hat and tilde shifts respectively, one finds

$$\hat{\tilde{u}} = \frac{\tilde{v}-\hat{v}}{\tilde{u}-\hat{u}},$$

$$\hat{\tilde{v}} = \frac{\tilde{u}\,\hat{v}-\hat{u}\,\tilde{v}}{\tilde{u}-\hat{u}}.$$

This system is integrable in the sense of multidimensional consistency. Thus three dimensional consistency also allows the addition of a new direction $\bar{u}$, with a corresponding parameter r. The new equations are

$$\bar{w} - u\,\bar{u} + v = 0,$$

$$w - u\,\hat{\bar{u}} + \hat{\bar{v}} + \frac{r-q}{\hat{u}-\bar{u}} = 0,$$

$$w - u\,\tilde{\bar{u}} + \tilde{\bar{v}} + \frac{r-p}{\tilde{u}-\bar{u}} = 0.$$

Thus we may have many dependent variables on any $n-$dimensional square lattice. In Chapter 5 we find such a system of quad-equations through the geometry of the ABS equations.

3.5 The ABS Classification

The ABS list is a complete (up to Möbius transformation) list of 9 integrable lattice equations of the form (3.1) with the following properties:

1. Linearity in each argument.
2. Possessing the symmetry group of the square.
3. Multidimensional consistency.
4. Tetrahedron property.

The classification was first performed in [5], and subsequently refined in [15], where the tetrahedron property was replaced by non-degeneracy conditions. The result is a list of 9 equations (up to Möbius transformation) in three groups, named H1-H3, A1-A2, Q1-Q4.

Some of these were already well known, for example the cross ratio equation (3.7) is a particular case of Q1, and the lattice KdV equation (1.11) is H1. Note that the lattice potential KdV (1.11) is also multidimensionally consistent and possesses the tetrahedron property. Both (1.11) and (3.7) also each possess the symmetry of a square, as

$$Q(x,u,v,y;\alpha,\beta) = \sigma_1\, Q(u,x,y,v;\alpha,\beta) = \sigma_2\, Q(x,v,u,y;\beta,\alpha), \tag{3.17}$$

where $\sigma_1, \sigma_2 = \pm 1$.

However, the classification also produced some new examples. The full list is given in the following theorem.

Theorem 3.1. *Up to common Möbius transformations of the variables x, u, v, y and point transformation of the parameters α, β, the list of three dimensionally consistent equations on quad-graphs with the properties of multi-linearity, symmetry of the square and the tetrahedron property is exhausted by the following.*

List A:

$$\alpha\,(x+v)\,(u+y) - \beta\,(x+u)\,(v+y) - \delta^2\,\alpha\,\beta\,(\alpha-\beta) = 0. \tag{A1}$$

$$(\beta^2-\alpha^2)\,(x\,u\,v\,y+1) + \beta\,(\alpha^2-1)\,(x\,v+u\,y) - \alpha\,(\beta^2-1)(x\,u+v\,y) = 0. \tag{A2}$$

List H:

$$(x-y)\,(u-v) + \beta - \alpha = 0. \tag{H1}$$

$$(x-y)\,(u-v) + (\beta-\alpha)(x+u+v+y) + \beta^2 - \alpha^2 = 0. \tag{H2}$$

$$\alpha\,(x\,u+v\,y) - \beta(x\,v+u\,y) + \delta\,(\alpha-\beta) = 0. \tag{H3}$$

List Q:

$$\alpha\,(x-v)\,(u-y) - \beta\,(x-u)(v-y) + \delta^2\,\alpha\,\beta\,(\alpha-\beta) = 0. \tag{Q1}$$

$$\begin{aligned}\alpha(x-v)\,(u-y) - \beta(x-u)\,(v-y) + \alpha\,\beta\,(\alpha-\beta)\,(x+u+v+y)\\ -\alpha\,\beta\,(\alpha-\beta)(\alpha^2-\alpha\,\beta+\beta^2) = 0.\end{aligned} \tag{Q2}$$

$$\begin{aligned}(\beta^2-\alpha^2)\,(x\,y+u\,v) + \beta\,(\alpha^2-1)\,(x\,u+v\,y) - \alpha\,(\beta^2-1)\,(x\,v+u\,y)\\ -\delta^2\,(\alpha^2-\beta^2)\,(\alpha^2-1)\,(\beta^2-1)/(4\,\alpha\,\beta) = 0.\end{aligned} \tag{Q3}$$

$$\begin{aligned}a_0\,x\,u\,v\,y + a_1\,(x\,u\,v+u\,v\,y+x\,v\,y+x\,u\,y) + a_2\,(x\,y+u\,v)\\ +\bar{a}_2\,(x\,u+v\,y) + \tilde{a}_2\,(x\,v+u\,y) + a_3\,(x+u+v+y) + a_4 = 0,\end{aligned} \tag{Q4}$$

where $r(x) = 4\,x^3 - g_2\,x - g_3$, $a^2 = r(\alpha)$, $b^2 = r(\beta)$, *and the coefficients* a_i *are given by*

$$\begin{aligned}a_0 &= a+b, \quad a_1 = -\beta\,a-\alpha\,b, \quad a_2 = \beta^2\,a+\alpha^2\,b,\\ \bar{a}_2 &= \frac{a\,b\,(a+b)}{2(\alpha-\beta)} + \beta^2\,a - \left(2\alpha^2-\frac{g_2}{4}\right)b,\\ \tilde{a}_2 &= \frac{a\,b\,(a+b)}{2(\beta-\alpha)} + \alpha^2\,b - \left(2\beta^2-\frac{g_2}{4}\right)a,\\ a_3 &= \frac{g_3}{2}a_0 - \frac{g_2}{4}a_1, \quad a_4 = \frac{g_2^2}{16}a_0 - g_3\,a_1.\end{aligned}$$

3.5.1 The Classification

In this section we summarise the ABS classification as given in [5], and demonstrate how the base varieties in the space of initial conditions are related to this classification. We will discuss these base varieties and their resolution further in following chapters.

In [5], it was shown that for any quad-equation $Q(x,u,v,y;\alpha,\beta)=0$ satisfying the tetrahedron property and multidimensional consistency, then defining $g(x,u;\alpha,\beta)$ such that

$$g(x,u;\alpha,\beta) = Q\,Q_{yv} - Q_y\,Q_v, \tag{3.18a}$$

$$g(x,v;\beta,\alpha) = Q\,Q_{yu} - Q_y\,Q_u, \tag{3.18b}$$

for some antisymmetric $k(\alpha, \beta) = -k(\beta, \alpha)$ it is possible to write $g(x, u; \alpha, \beta)$ in the form

$$g(x, u; \alpha, \beta) = k(\alpha, \beta)\, h(x, u; \alpha), \tag{3.19}$$

such that the discriminant of $h(x, u; \alpha)$

$$r(x) = h_u^2 - 2\, h\, h_u, \tag{3.20}$$

does not depend on the parameter α. In fact, it turns out that this property (that g factorises as in (3.19) and the polynomial r does not depend on parameters) is not only necessary, but an almost sufficient condition for three-dimensional consistency and the tetrahedron property. The list of functions with this property is made up of twelve items, of which only two fail on consistency. We will refer to this property as the property (**R**).

The biquadratics are directly related to the singularity structure of a quad-equation:

Proposition 3.1. *Consider a multilinear quad-equation of the form*

$$Q(x, u, v, y; \alpha, \beta) = 0.$$

This equation becomes indeterminate where the biquadratics vanish.

Proof. This fact was originally proved in [6], but we include the proof here in our context for completeness.

Rewriting the generic equation $Q(x, u, v, y; \alpha, \beta) = 0$ to isolate dependence on the variable y we find

$$\begin{aligned} 0 &= Q(x, u, v, y) = yQ_y + [Q - yQ_y] \\ \Rightarrow y &= \frac{yQ_y - Q}{Q_y}, \end{aligned} \tag{3.21}$$

where, due to the linear dependence of Q on y, neither the numerator nor the denominator depend on y. A singularity occurs if and only if both numerator and denominator simultaneously vanish. That is,

$$yQ_y - Q = 0,$$

$$Q_y = 0.$$

Firstly, we prove that on singularities, the biquadratics must vanish. Isolating dependence on v, we can write the above equations as

$$v[yQ_{vy} - Q_v] - [vyQ_{vy} - yQ_y - vQ_v + Q] = 0,$$
$$v[Q_{vy}] - [vQ_{vy} - Q_y] = 0.$$

We can combine these to eliminate the dependence on v:

$$v[Q_{vy}][yQ_{vy} - Q_v] - [Q_{vy}][vyQ_{vy} - yQ_y - vQ_v + Q] = 0,$$
$$v[Q_{vy}][yQ_{vy} - Q_v] - [vQ_{vy} - Q_y][yQ_{vy} - Q_v] = 0.$$
$$\Rightarrow QQ_{vy} - Q_vQ_y = 0.$$

This is exactly that the biquadratic in x, u vanishes. Similarly, we could have eliminated the dependence on u to get the condition that the biquadratic in x, v must vanish:

$$QQ_{uy} - Q_uQ_y = 0.$$

Secondly, we must show that if the biquadratics vanish, then we have a singularity. Suppose we have the vanishing biquadratic

$$QQ_{vy} - Q_vQ_y = 0.$$

If $Q_v \neq 0$, then

$$Q_y = \frac{QQ_{vy}}{Q_v} = 0,$$

and

$$yQ_y - Q = Q\left[\frac{yQ_{vy}}{Q_v} - 1\right] = 0,$$

and since we know $Q = 0$, we have a singularity. If $Q_v = 0$, then we may replace v in the above by u or x. If $Q_x = Q_u = Q_v = 0$, then y is constant and $Q_y = 0$.

□

Each biquadratic defines a 2-dimensional surface in the 3-dimensional space of x, u, v, and it is at the intersection of these surfaces that we encounter singularities, or base varieties. This result has important implications for the results of future chapters.

Example 3.5. Consider the equation H2 from the ABS list.

$$(x-y)\,(u-v)+(\beta-\alpha)(x+u+v+y)+\beta^2-\alpha^2=0. \tag{3.22}$$

This equation has the corresponding biquadratic function g as defined in (3.18) given by

$$g(x,u;\alpha,\beta)=2\,(\beta-\alpha)\,(x+u+\alpha), \tag{3.23a}$$

$$g(x,v;\beta,\alpha)=2\,(\alpha-\beta)\,(x+u+\beta). \tag{3.23b}$$

Taking $k(\alpha,\beta)=2\,(\beta-\alpha)$, then we have the polynomials

$$h(x,u;\alpha)=x+u+\alpha, \tag{3.24a}$$

$$h(x,v;\beta)=x+v+\beta. \tag{3.24b}$$

Taking the discriminant $r(x)=h_u^2-2\,h\,h_{uu}$, we find $r(x)=1$.

Solving for the vertex y, we find

$$y=\frac{x(u-v)+(\beta-\alpha)(x+u+v)+\beta^2-\alpha^2}{u-v+\alpha-\beta}. \tag{3.25}$$

If (x,u,v) lie on the line defined by the equations

$$x+u+\alpha=x+v+\beta=0, \tag{3.26}$$

then (3.25) becomes $y=\frac{0}{0}$, and hence y is undefined. In the next chapter we show how to resolve such singularities for quad-equations.

3.5.2 The Synthesis

To generate the list of quad-equations $Q(x,u,v,y;\alpha,\beta)=0$ with the properties of the ABS equations, we start by generating a list of candidate polynomials which could arise as the discriminant $r(x)$ associated with some quad-equation in the sense of the property (**R**), then reverse engineer all quad-equations satisfying the assumptions of the ABS classification.

Due to the symmetry and multilinearity conditions on Q, we know only two cases are possible.

$$Q=a_0\,x\,u\,v\,y+a_1\,(x\,u\,v+u\,v\,y+x\,v\,y+x\,u\,y)+a_2(x\,y+u\,v)$$

$$+\,\bar{a}_2(x\,u+v\,y)+\tilde{a}_2(x\,v+u\,y)+a_3(x+u+v+y)+a_4, \tag{3.27}$$

$$Q=a_1(x\,u\,v+u\,v\,y-x\,v\,y-x\,u\,y)+a_2(x\,y-u\,v)$$
$$+\,a_3(x-u-v+y). \tag{3.28}$$

where

$$a_i(\beta,\alpha)=\epsilon\,a_i(\alpha,\beta),\quad \tilde{a}_2(\alpha,\beta)=\epsilon\,\bar{a}_2(\beta,\alpha),\quad \epsilon=\pm 1.$$

In fact, in the case (3.28) we additionally find we must have $a_1=a_2=a_3=0$, which is the trivial case, violating assumption. We therefore discard this and focus our attention on (3.27). For equations of this form we have

$$h(x,u;\alpha)=b_0\,x^2u^2+b_1\,x\,u\,(x+u)+b_2\,(x^2+u^2)+\hat{b}_2\,x\,u+b_3(x+u)+b_4, \tag{3.29a}$$

$$r(x)=c_0\,x^4+c_1\,x^3+c_2\,x^2+c_1\,x+c_4, \tag{3.29b}$$

where $b_i=b_i(\alpha)$.

Given a polynomial $r(x)$ of degree 4, using Möbius transformations one can bring it into one of 6 canonical forms, depending on its distribution of roots, including any shifted to infinity via Möbius transformation. Therefore, all quad-equations (up to Möbius transformation) satisfying the assumptions of the ABS classification will possess a discriminant belonging to the following list:

- $r(x)=0$,
- $r(x)=1$ (r has one quadruple zero),
- $r(x)=x$ (r has one simple zero and one triple zero),
- $r(x)=x^2$ (r has two double zeroes),
- $r(x)=x^2-1$ (r has two simple zeroes and one double zero),
- $r(x)=4x^3-g_2\,x-g_3$, $\Delta=g_2^3-27g_3^2=0$ (r has four simple zeroes).

All that remains to complete the classification is to find all quad-equations of the form (3.27) which have a discriminant belonging to this list and eliminating those which don't satisfy all assumptions of the ABS classification.

Proposition 3.2. *For a given polynomial $r(x)$ of (at most) degree 4 in one of the canonical forms above, the symmetric biquadratic polynomials $h(x,u)$ with $r(x)$ as their discriminant are exhaused by the following list:*

$$r(x)=0: \qquad h=\frac{1}{\alpha}(x-u)^2, \tag{q0}$$

$$h=(\gamma_0\, x\, u+\gamma_1\,(x+u)+\gamma_2)^2, \tag{h1}$$

$$r(x)=1: \qquad h=\frac{1}{2\,\alpha}(x-u)^2-\frac{\alpha}{2}, \tag{q1}$$

$$h=\gamma_0\,(x+u)^2+\gamma_1\,(x+u)+\gamma_2)^2,\ \ \gamma_1^2-4\,\gamma_0\gamma_2=1, \tag{h2}$$

$$r(x)=x: \qquad h=\frac{1}{4\,\alpha}(x-u)^2-\frac{\alpha}{2}(x+u)+\frac{\alpha^3}{4}, \tag{q2}$$

$$r(x)=x^2: \qquad h=\gamma_0\, x^2\, u^2+\gamma_1\, x\, u+\gamma_2,\ \ \gamma_1^2-4\gamma_0\,\gamma_2=1 \tag{h3}$$

$$r(x)=x^2-\delta^2: \quad h=\frac{\alpha}{1-\alpha^2}(x^2+u^2)-\frac{1+\alpha^2}{1-\alpha^2}x\, u+\delta^2\frac{1-\alpha^2}{4\,\alpha}, \tag{q3}$$

$$r(x)=4x^3-g_2\, x-g_3: \quad h=\frac{1}{\sqrt{r(\alpha)}}\Bigg(\bigg(x\, u+\alpha\,(x+u)+\frac{g_2}{4}\bigg)^2 \\ -(x+u+\alpha)(4\,\alpha\, x\, u-g_3)\Bigg). \tag{q4}$$

Proof. To prove this we simply need to solve the system of the form

$$b_1^2-4\, b_0\, b_1=c_0,$$

$$2\, b_1\,(\hat{b}_2-2\, b_2)-4\, b_0\, b_3=c_1,$$

$$\hat{b}_2^2-4\, b_2^2-2\, b_1\, b_3-4\, b_0\, b_3=c_2,$$

$$2\, b_3(\hat{b}_2-2\, b_2)-4\, b_1\, b_4=c_3,$$

$$b_3^2-4\, b_2\, b_4=c_4,$$

where b_i are the coefficients of $h(x,u)$, and c_i are the coefficients of $r(x)$ as in (3.29a), (3.29b), which can be done by straightforward analysis. □

The presence of the term $\sqrt{r(\alpha)}$ in (q4) clearly shows the presence of elliptic curves at play in these equations. We can now reconstruct all polynomials Q for each h. From (3.27) we have

$$g(x,u;\alpha,\beta)=(\bar{a}_2\, a_0-a_1^2)x^2u^2+(a_1(\bar{a}_2-\tilde{a}_2)+a_0\, a_3-a_1\, a_2)x\, u\,(x+u) \\ +(a_1\, a_3-a_2\,\tilde{a}_2)(x^2+u^2)+(\bar{a}_2^2-\tilde{a}_2^2+a_0\, a_4-a_2^2)x\, u$$

$$+ (a_3(\bar{a}_2 - \tilde{a}_2) + a_1\, a_4 - a_2\, a_3)(x+u) + \bar{a}_2\, a_4 - a_3^2. \tag{3.30}$$

Using (3.19) and (3.29a), then denoting $b_i = b_i(\alpha)$, $b'_i = b_i(\beta)$, $k = k(\alpha, \beta)$, we have the following system for the unknowns a_i:

$$\begin{aligned} \bar{a}_2\, a_0 - a_1^2 &= k\, b_0, & \tilde{a}_2\, a_0 - a_1^2 &= -k\, b'_0, \\ a_1(\bar{a}_2 - \tilde{a}_2) + a_0\, a_3 - a_1\, a_2 &= k\, b_1, & a_1\,(\tilde{a}_2 - \bar{a}_2) + a_0\, a_3 - a_1\, a_2 &= -k\, b'_1, \\ a_1\, a_3 - a_2\, \tilde{a}_2 &= k\, b_2, & a_1\, a_3 - a_2\, \bar{a}_2 &= -k\, b'_2, \\ \bar{a}_2^2 - \tilde{a}_2^2 + a_0\, a_4 - a_2^2 &= k\, \hat{b}_2, & \tilde{a}_2^2 - \bar{a}_2^2 + a_0\, a_4 - a_2^2 &= -k\, \hat{b}_2, \\ a_3(\bar{a}_2 - \tilde{a}_2) + a_1\, a_4 - a_2\, a_3 &= k\, b_3, & a_3(\tilde{a}_2 - \bar{a}_2) + a_1\, a_4 - a_2\, a_3 &= -k\, b'_3, \\ \bar{a}_2\, a_4 - a_3^2 &= k\, b_4, & \tilde{a}_2\, a_4 - a_3^2 &= -k\, b'_4. \end{aligned}$$

Consider the cases (q0), (q1), (q2), (q3), (q4). Solving the above system we find the polynomials Q corresponding the the equations $\mathrm{Q1}_{\delta=0}$, $\mathrm{Q1}_{\delta=1}$, Q2, Q3, and Q4, respectively. A straightforward test shows that all these functions are consistent around the cube and possess the tetrahedron property.

Taking the cases (h1), (h2), (h3) we find the equations $\mathrm{A1}_{\delta=0}$, $\mathrm{A1}_{\delta=1}$, H2, $\mathrm{H3}_{\delta=1}$, A2, all of which satisfy consistency around the cube and the tetrahedron property. We also find the polynomials

$$Q = (x-y)(u-v) + k_1(\alpha, \beta), \tag{$\widehat{\mathrm{H1}}$}$$

$$Q = \frac{1 + k_2(\alpha,\beta)}{2}(x\,u + v\,y) - \frac{1 - k_2(\alpha,\beta)}{2}(x\,v + u\,y), \tag{$\widehat{\mathrm{H3}}_0$}$$

where $k_i(\alpha, \beta) = -k_i(\beta, \alpha)$. These equations satisfy the property (**R**) regardless of the choice of k_i, but are not in general consistent around the cube. Requiring that $\widehat{\mathrm{H1}}$, $\mathrm{H3}_0$ satisfy this consistency condition we find the equations H1 and $\mathrm{H3}_{\delta=0}$ belonging to the ABS list.

Thus we have found a list of all multilinear quad-equations (up to Möbius transformation) with the symmetry of the square which possess the property (**R**). By requiring that all these equations also be consistent around the cube, we have found a complete list of integrable quad-equations (up to Möbius transformation) satisfying the assumptions of the ABS classification given above (multilinearity, symmetry of the square, multidimensional consistency, tetrahedron property).

3.6 Reductions

We will now introduce the idea of *staircase reductions* of lattice equations, first given in [40]. There exist a plethora of examples of staircase reductions to discrete Painlevé equations. In Chapter 5, by studying the induced map of a lattice equation over multiple steps we are led to new reductions of lattice equations related to staircase reductions.

Consider the lattice $\mathbb{Z}^2$ with coordinates l, m, and a dependent variable $x_{l,m}$. To perform a reduction over such a lattice, we perform what is commonly referred to as a *staircase reduction* [40, 99]. In its most basic form, we make the assumption that taking some n_1 steps in the l–direction will give the same value of the solution as taking n_2 steps in the m–direction. More explicitly, for some $n_1, n_2 \in \mathbb{Z}$ we impose on the solution $x_{l,m}$ the condition

$$x_{l+n_1,m} \equiv x_{l,m+n_2}.$$

This condition allows us to define a new function $x_N = x_{l,m}$, such that $N = n_2\, l + n_1\, m$. Substituting this into the equation we wish to reduce, we now have an ordinary difference equation for x_N. We call this reduction an (n_1, n_2)–reduction.

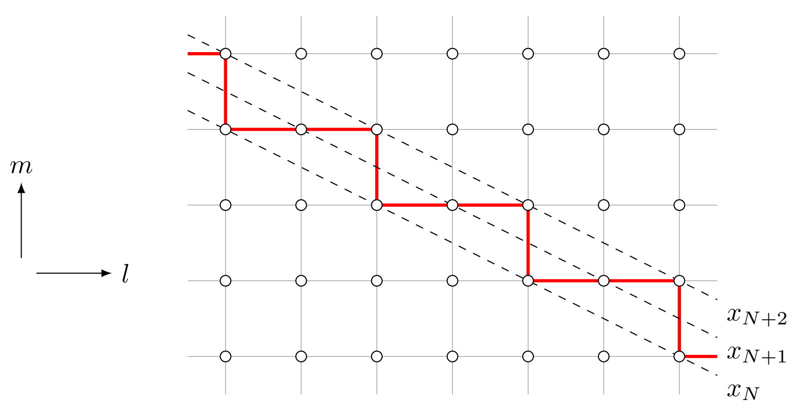

Fig. 3.8: An example (2,1)–reduction.

Taking a regular staircase of vertices in the lattice with step width n_1 and height n_2, all the vertices along the upper corners will have the common value x_N for some N, and similarly for the lower corners. See the example Figure 3.8.

Example 3.6. Consider the ABS equation Q1 over a lattice $x_{l,m}$, given by

$$\alpha\,(x_{l,m} - x_{l,m+1})\,(x_{l+1,m} - x_{l+1,m+1}) - \beta\,(x_{l,m} - x_{l+1,m})\,(x_{l,m+1} - x_{l+1,m+1}) + \delta^2\alpha\,\beta\,(\alpha - \beta) = 0.$$

Following [58], we perform a (2,1)-reduction by imposing $x_{l,m+1} \equiv x_{l+2,m}$, and hence we can define the dependent variable $u_n = u_{l+2m} = x_{l,m}$ governed by the ordinary difference equation

$$\alpha\,(u_n - u_{n+2})\,(u_{n+1} - u_{n+3}) - \beta\,(u_n - u_{n+1})\,(u_{n+2} - u_{n+3}) + \delta^2\alpha\,\beta\,(\alpha - \beta) = 0. \tag{3.31}$$

Simplifying this expression with the change of variables $w_n = u_{n+2} - u_{n+1}$ and defining the parameter $\gamma := \delta^2\alpha\,\beta\,(\alpha - \beta)$, we find

$$\alpha\,(w_n + w_{n-1})\,(w_{n+1} + w_n) - \beta\,w_{n-1}w_{n+1} + \gamma = 0. \tag{3.32}$$

Solving for w_{n+1} we can see that this is a QRT mapping, since we have

$$w_{n+1} = \frac{(-\gamma - \alpha\,w_n^2) - w_{n-1}\,(\alpha\,w_n)}{(\alpha\,w_n) - w_{n-1}(\beta - \alpha)}. \tag{3.33}$$

Using the expression (1.6) we can immediately see that the invariant is

$$K = \frac{\alpha(\alpha + \beta)(w_n + w_{n-1})^2 + \beta\,(\alpha + \beta)\,w_n w_{n-1} - \alpha\,\gamma}{(w_n + w_{n-1})(\beta\,w_n w_{n-1}) + \gamma)}. \tag{3.34}$$

Many examples have been found of reductions of integrable lattice equations to integrable ordinary difference equations, and by deautonomising the result, discrete Painlevé equations [43, 58].

In the remainder of the thesis, we use tools from birational geometry outlined in the previous chapter on integrable lattice equations. By building a space of initial conditions on which the induced map becomes in some sense an isomorphism, we find new information about solutions and reductions of integrable lattice equations, and the relationship between the singular points of Painlevé equations and the iteration map of integrable lattice equations.

Chapter 4: Resolution of Singularities for Quad-Equations

In Chapter 2 we introduced the idea of resolution of singular points in the plane via blow-ups, then applied this technique in the study of discrete Painlevé equations. In the case of birational maps with spaces of initial conditions of dimension $d > 2$ the map may become undefined along varieties within that space. We call such singularities *singular* or *base* varieties.

It is very natural to be interested in such singularities. Not only are they closely related to singularity confinement (as we will show in this chapter), in [9] such singularities were used to give boundary conditions on a section of a lattice that were not overdetermined, and hence obtain exact solutions of lattices that were constructed on a regular singularity-bounded strip.

In this chapter we extend these ideas of resolution of singularities to varieties of higher dimension, discussing the process of resolution of higher dimensional subvarieties and framing singularity confinement of lattice equations in this context.

4.1 Blowing Up Along Submanifolds

Blowing up along subvarieties which are not only point sets has already been shown to be a fruitful area of research within integrable systems. In [97], the authors construct a birational representation of Weyl groups of a class which contains the types $A_n^{(1)}$, $D_n^{(1)}$, and $E_n^{(1)}$, important in the study of the geometry of discrete Painlevé equations.

When performing a blow-up in a higher dimension, the core idea is the same as before. A blow-up is, in effect, a geometric transformation which replaces a subvariety U of V with all the directions pointing out of U. To perform a blow-up centred at a *subvariety* b in $\mathbb{C}^n$, we use the following definition.

Definition 4.1. *To blow up a codimension-k subvariety b of* $\mathbb{C}^n$*, let b be the locus of the equations* $x_1 = x_2 = \cdots = x_k = 0$ *and* $[y_1 : y_2 : \cdots : y_k]$ *be homogeneous coordinates on* $\mathbb{P}^{k-1}$*. The blow-up is the pullback of the map* $\pi : \tilde{\mathbb{C}}^n \longrightarrow \mathbb{C}^3$*, where* $\tilde{\mathbb{C}}^n$ *is given by*

$$\{((x_1, x_2, \ldots, x_k), [y_1 : y_2 : \cdots : y_k])|x_i\, y_j - x_j\, y_i = 0\ \forall\ i, j\} \subset \mathbb{C}^n \times \mathbb{P}^{k-1}.$$

The subvariety b is called the centre of the blow-up. A blow-up is an isomorphism everywhere away from b, and maps the centre to an exceptional locus isomorphic to $\mathbb{P}^{k-1}$*.*

Notice that taking this definition such that $n = k = 2$, we naturally find the definition of the blow-up of a point in $\mathbb{C}^2$ given in Chapter 2.

To motivate this process, we begin by considering the p roblem of parameterising a surface. In many cases it is immediately apparent what such a parameterisation might be (e.g. a sphere, plane, etc) but in many more cases, particularly where a surface may not be everywhere smooth, finding such a parameterisation may be significantly more difficult. By blowing up a non-smooth variety to a smoother, simpler one, a parameterisation for the original surface becomes more apparent.

Example 4.1. Consider the cone given by

$$x_1^2 + x_2^2 - x_3^2 = 0. \tag{4.1}$$

This surface is not smooth at the origin, so we blow up with centre $(0, 0, 0)$. Since this is a codimension-3 subvariety (a point), we use the map $\pi : \tilde{\mathbb{C}}^3 \rightarrow \mathbb{C}^3$, where $\tilde{\mathbb{C}}^3$ is the surface described by

$$\{((x_1, x_2, x_3), [y_1 : y_2 : y_3])|x_i\, y_j - x_j\, y_i = 0\ \forall\ i, j\} \subset \mathbb{C}^3 \times \mathbb{P}^2.$$

This is an isomorphism everywhere away from the origin, but replaces the origin with an *exceptional* *plane* isomorphic to $\mathbb{P}^2$. We therefore consider the blow-up in three charts, corresponding to the three affine charts of $\mathbb{P}^2$.

The equations defining this surface are

$$x_1\, y_2 = x_2\, y_1, \tag{4.2a}$$

$$x_1\, y_3 = x_3\, y_1, \tag{4.2b}$$

$$x_2\, y_3 = x_3\, y_2. \tag{4.2c}$$

Since $[y_1 : y_2 : y_3] = [y_1/y_3 : y_2/y_3 : 1] = [y_1/y_2 : 1 : y_3/y_1] = [1 : y_2/y_1 : y_3/y_1]$, in the first blow-up chart we have affine coordinates

$$x'_1 := \frac{y_1}{y_3} = \frac{x_1}{x_3}, \quad x'_2 := \frac{y_2}{y_3} = \frac{x_2}{x_3}, \tag{4.3}$$

and therefore in this chart the blow-up is given by the coordinate transform

$$x_1 = x'_1\, x'_3, \tag{4.4a}$$

$$x_2 = x'_2\, x'_3, \tag{4.4b}$$

$$x_3 = x'_3. \tag{4.4c}$$

Substituting this into (4.1) we find

$$x'^2_3(x'^2_1 + x'^2_2 - 1) = 0. \tag{4.5}$$

The resulting variety has two components. The exceptional plane x'_3=0, and the cylinder $x'^2_1 + x'^2_2 = 1$, which are smooth (see Figure 4.1). Similar results follow for the other two charts.

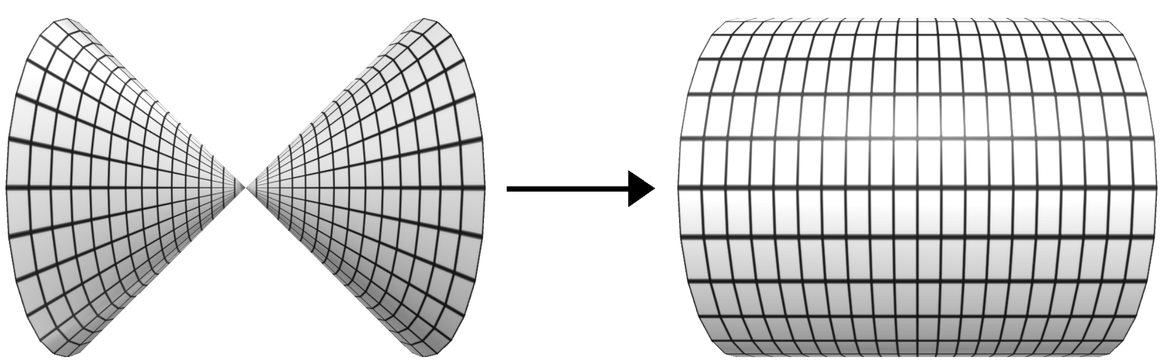

Fig. 4.1: Cones blown up to a cylinder.

The cylinder $x_1'^2 + x_2'^2 = 1$ is parameterised in 3-space by

$$(x_1', x_2', x_3) = (\cos(u), \sin(u), v), \tag{4.6}$$

where $u \in [-\pi, \pi]$, $v \in (-\infty, \infty)$. Using (4.4), this immediately gives the parameterisation of the cone

$$(x_1, x_2, x_3) = (v\,\cos(u), v\,\sin(u), v). \tag{4.7}$$

Example 4.2. Consider the Whitney umbrella, the surface X defined by

$$x^2 - y^2\,z = 0. \tag{4.8}$$

Notice that this surface is again singular at the origin, see Figure 4.2.

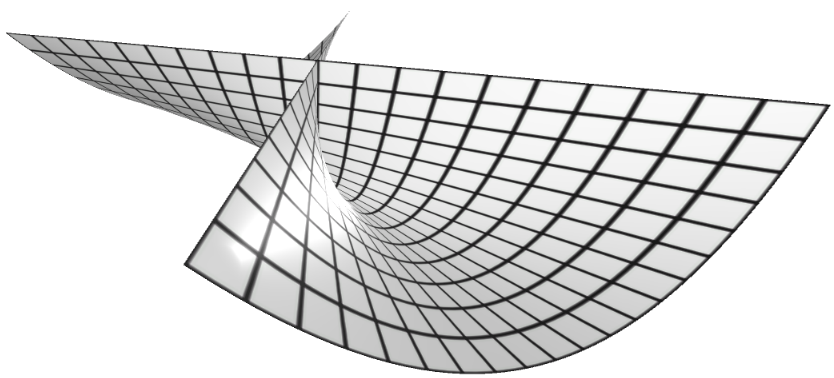

Fig. 4.2: Whitney umbrella.

We may be tempted to once again blow up at the origin as with (4.1) in the previous example. However using (4.4) we find

$$z'^2\,(x'^2 - y'^2\,z') = 0, \tag{4.9}$$

yielding the strict transform

$$x'^2 - y'^2\,z = 0, \tag{4.10}$$

and exceptional component $z^2 = 0$. In this case, after blowing up the strict transform is exactly the same as before. The singularity was not improved after blowing up and hence the origin was too small a centre [42]. In this case, the entire positive z-axis (of which the origin is a part) is a pinch singularity. Choosing instead the (codimension-2) z-axis as centre, the total transform of X is the pullback of the map to $\mathbb{C}^3$ from

$$\{((x, y, z), [\xi, \eta])| x\,\eta - y\,\xi = 0\} \subset \mathbb{C}^3 \times \mathbb{P}^1.$$

We consider this space in two affine charts corresponding to the affine charts of $\mathbb{P}^1$. In the chart parameterised by $x' := \xi/\eta$ we have

$$x' = \frac{\xi}{\eta} = \frac{x}{y}, \tag{4.11}$$

and hence in this chart the blow-up is given by the change of coordinates

$$(x, y, z) = (x'\,y', y', z'). \tag{4.12}$$

Substituting into (4.8) we find

$$y'^2\,(x'^2 - z') = 0, \tag{4.13}$$

and hence the strict transform is

$$x'^2 - z' = 0. \tag{4.14}$$

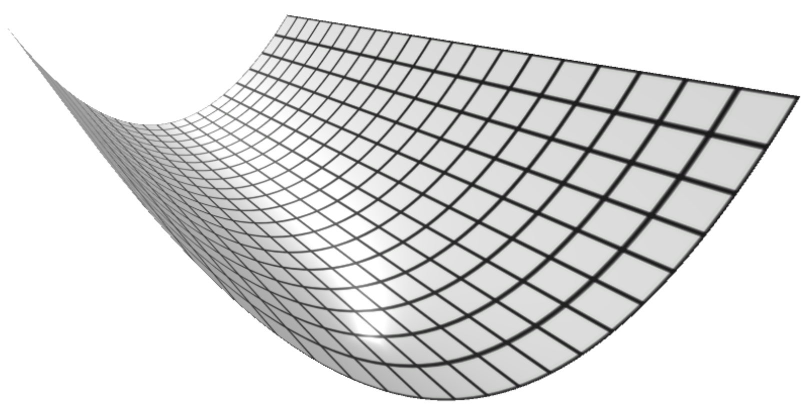

Fig. 4.3: Strict transform of Figure 4.2.

By inspection, this surface has the parameterisation

$$(x', y', z') = (v, u, v^2), \tag{4.15}$$

where $u \in (-\infty, \infty)$, $v \in (-\infty, \infty)$, and hence using (4.12) we have found a parameterisation of the Whitney umbrella (4.8),

$$(x, y, z) = (u\,v, u, v^2). \tag{4.16}$$

These parameterisations were used to generate Figures 4.1 and 4.2. These are simple examples, and the smooth parameterising manifolds were found after a single step. In other examples it may take many steps, but in characteristic zero it is always possible to resolve singular varieties after successive blow-ups in this way [49]. For further information on the resolution of polynomials of arbitrary dimension, we direct the reader towards [64, 13].

Recall that in Example 3.5 we saw that looking at H2 on a single quadrilateral with vertices x, u, v, y, the map which gives the value of the vertex y in terms of the other 3 becomes undefined when x, u, v lie on the line

$$x + u + \alpha = x + v + \beta = 0.$$

This line is a singularity of the map $(x, u, v) \mapsto y$ in the same sense as the base points in Chapter 2, a codimension-2 subvariety on which the map becomes undefined.

The remainder of this chapter is dedicated to the discussion of such singularities on a single quadrilateral, and their resolution with blow-ups. New difficulties arise when blowing up higher dimensional base varieties, for example finding the correct choice of centre (as shown in Example 4.2). In the next section we motivate the resolution of quad-equations with singularity confinement, then explicitly resolve all singularities which appear when solving for one vertex on a generic quadrilateral for several examples.

4.2 Singular Varieties of $\mathrm{H3}_{\delta=0}$

Consider the ABS equation $\mathrm{H3}_{\delta=0}$. Solving for the vertex y and setting $\delta = 0$, this equation takes the form

$$y = x\frac{\alpha\,u - \beta\,v}{\beta\,u - \alpha\,v}. \tag{4.17}$$

Consider this equation over a lattice $(l,m)\in\mathbb{Z}^2$ with dependent variable $x_{l,m}\in\mathbb{C}\cup\{\infty\}$, and take as initial conditions $x_{l,m+2}$, $x_{l,m+1}$, $x_{l,m}$, $x_{l+1,m}$, $x_{l+2,m}$. This allows the equation to be solved over a 3×3 section of the lattice, see Figure 4.4.

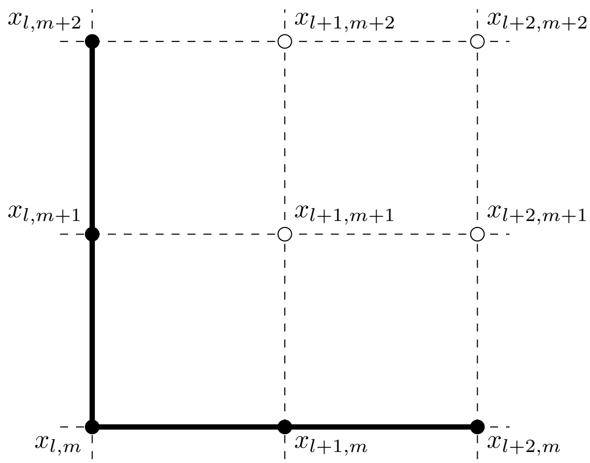

Fig. 4.4: 3×3 section of $\mathbb{Z}^2$, with initial values marked in bold.

Suppose $x_{l,m}\neq 0$, $\alpha\, x_{l+1,m}=\beta\, x_{l,m+1}$ and $x_{l+1,m}, x_{l,m+1}\neq 0$. For all bounded $x_{l,m}$ we have $x_{l+1,m+1}=0$, and hence for subsequent iterations the solution has apparently lost a degree of freedom corresponding to $x_{l,m}$. The number of degrees of freedom of the system has dropped from 4 to 3. Solving for $x_{l+2,m+1}$ and $x_{l+1,m+2}$ we find

$$x_{l+2,m+1}=x_{l+1,m}\frac{\alpha\, x_{l+2,m}-\beta\, x_{l+1,m+1}}{\beta\, x_{l+2,m}-\alpha\, x_{l+1,m+1}}=\frac{\alpha}{\beta}\, x_{l+1,m}, \tag{4.18a}$$

$$x_{l+1,m+2}=x_{l,m+1}\frac{\alpha\, x_{l+1,m+1}-\beta\, x_{l,m+2}}{\beta\, x_{l+1,m+1}-\alpha\, x_{l,m+2}}=\frac{\beta}{\alpha}\, x_{l,m+1}. \tag{4.18b}$$

Finally, solving for $x_{l+2,m+2}$ we find

$$\begin{aligned} x_{l+2,m+2} &= x_{l+1,m+1}\frac{\alpha\, x_{l+2,m+1}-\beta\, x_{l+1,m+2}}{\beta\, x_{l+2,m+1}-\alpha\, x_{l+1,m+2}}, \\ &= \frac{x_{l+1,m+1}}{\alpha\,\beta}\,\frac{\alpha^3\, x_{l+1,m}-\beta^3\, x_{l,m+1}}{\alpha\, x_{l+1,m}-\beta\, x_{l,m+1}}, \end{aligned} \tag{4.19}$$

which is undetermined, since $x_{l+1,m+1}=0$ and $\alpha\, x_{l+1,m}-\beta\, x_{l,m+1}=0$. The iteration has landed on a singular variety, and $x_{l+2,m+2}$ is undefined.

However, if instead we start by taking $\alpha\, x_{l+1,m}-\beta\, x_{l,m+1}=\epsilon$ and at the end of the iterations performed above take the limit as $\epsilon \to 0$, a cancellation occurs and we now find

$$x_{l+2,m+2} = \frac{x_{l,m}\, x_{l+2,m}\, x_{l,m+2}}{x_{l,m}\,(x_{l+2,m} + x_{l,m+2} - x_{l+2,m}\, x_{l,m+2})}. \tag{4.20}$$

The fact that these factorisations occur is related to the polynomial growth of the solution and vanishing algebraic entropy [10]. Note that by passing over a quadrilateral which is locally indeterminate, the solution at $x_{l+2,m+2}$ has in a sense recovered the lost degree of freedom, such that under iteration the number of degrees of freedom of the solution has followed the apparent sequence $4 \to 3 \to 3 \to 4$.

From the perspective of an observer who knows only the values of $x_{l+1,m+1}, x_{l+2,m+1}, x_{l+1,m+2}$, divining the value of $x_{l+2,m+2}$ is not possible in this case since there are infinitely many distinct solutions for which the values of the vertices $x_{l+1,m+1}, x_{l+2,m+1}, x_{l+1,m+2}$ lie on this base line. In order to overcome this problem, we want to find initial conditions such that such solutions become distinguishable and this ambiguity is avoided. We achieve this by blowing up wherever such a singular variety could occur.

We demonstrate this idea by repeating the example of $\mathrm{H3}_{\delta=0}$ from this perspective. On a generic quad with vertices x, u, v, y solving for y, the equation has a base line b given by

$$x = \beta\, u - \alpha\, v = 0. \tag{4.21}$$

Hence, we blow up with the line b as centre.

The singularity b is a codimension–2 base variety, so it is blown up to an exceptional plane e isomorphic to $b \times \mathbb{P}^1$, where b is the curve defined by (4.21). Taking $[\xi : \eta]$ as the projective coordinate on $\mathbb{P}^1$ and defining $x' := \frac{\xi}{\eta}$, then from (4.21), in this chart the blow-up is given by the change of variables

$$x' = \frac{x}{\beta\, u - \alpha\, v}, \tag{4.22a}$$

$$u' = \beta\, u - \alpha\, v, \tag{4.22b}$$

$$v' = v, \tag{4.22c}$$

and therefore,

$$x = x'\, u', \tag{4.23a}$$

$$u = \frac{1}{\beta}(u' + \alpha\, v'), \tag{4.23b}$$

$$v = v'. \tag{4.23c}$$

We say that the variable x' *parametrises* the exceptional plane. Substituting into (4.17), we find

$$y = x' \left(\frac{\alpha}{\beta}\, u' + \left(\frac{\alpha^2 - \beta^2}{\beta} \right) v' \right), \tag{4.24}$$

In this example we encountered the base line b on the quad where $x = x_{l+1,m+1}, u = x_{l+2,m+1}, v = x_{l+1,m+2}, y = x_{l+2,m+2}$, and we therefore have

$$x_{l+2,m+2} = x'_{l+1,m+1} \left(\frac{\alpha}{\beta}\, x'_{l+2,m+1} + \left(\frac{\alpha^2 - \beta^2}{\beta} \right) x'_{l+1,m+2} \right). \tag{4.25}$$

In this chart the equation is nowhere singular. Since $x_{l+1,m+1}$, $x_{l+2,m+1}$, $x_{l+1,m+2}$ lie on b, then $x_{l+2,m+2}$ lies on the image of the exceptional plane e resulting from the blow-up of b.

When $x_{l+1,m+1}$, $x_{l+2,m+1}$, $x_{l+1,m+2}$ lie on b, then the relation $\alpha\, x_{l+1,m} = \beta\, x_{l,m+1}$ also holds. We can use (4.17) to calculate the values of $x'_{l+1,m+1}$, $x'_{l+2,m+1}$, $x'_{l+1,m+2}$ on the exceptional plane e in terms of the initial conditions $x_{l,m+2}$, $x_{l,m+1}$, $x_{l,m}$, $x_{l+1,m}$, $x_{l+2,m}$ to find

$$x'_{l+1,m+1} = \frac{\beta\, x_{l,m}\, x_{l+2,m}\, x_{l,m+2}}{(\alpha^2 - \beta^2) x_{l+1,m} (x_{l,m}(x_{l+2,m+x_{l,m+2}}) - x_{l+2,m}\, x_{l,m+2})},$$

$$x'_{l+2,m+1} = 0,$$

$$x'_{l+1,m+2} = x_{l+1,m}.$$

Therefore,

$$x_{l+2,m+2} = \frac{x_{l,m}\, x_{l+2,m}\, x_{l,m+2}}{x_{l,m}(x_{l+2,m} + x_{l,m+2}) - x_{l+2,m}\, x_{l,m+2}}. \tag{4.26}$$

In this setting, the value of $x_{l,m}$ is carried across the singularity to $x_{l+2,m+2}$ via the value of $x'_{l+1,m+1}$, the variable parameterising the exceptional plane e resulting from the blow-up of the line b given by (4.21).

By choosing an appropriate compact space of initial conditions (to include where the solution may become unbounded) and using blow-ups to successively resolve all singular varieties which may appear, we create a 'space of initial conditions' for each quadrilateral such that no singularities or ambiguities persist. In the next chapter we consider the relationship between these resolved space of initial conditions on different quads. For the remainder of this chapter, we perform the explicit resolution of several examples of ABS equations.

4.3 Resolution of Singularities on One Quadrilateral

In this section we cover the complete resolution of several examples from the ABS list on a generic quadrilateral, chosen to highlight new challenges which can arise in higher dimensions and how to overcome them. To begin, consider a quad-equation $Q = 0$ on a generic quadrilateral with vertices x, u, v, y, and take the vertices x, u, v as initial conditions solving for the vertex y. In order to include cases where solutions become unbounded, we compactify $\mathbb{C}^3$ to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

There are an infinite number of valid compactifications of $\mathbb{C}^3$. We have also performed resolutions of initial value spaces of lattice equations in $\mathbb{P}^3$, but for the purpose of clarity in this thesis we work in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. For ease of notation we denote the Cartesian product of N copies of $\mathbb{P}^1$ as $(\mathbb{P}^1)^N$.

To compactify to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, we take each vertex x, u, v in the initial conditions and generate the three coordinates $[x : 1]$, $[u : 1]$, $[v : 1]$, such that

$$([x : 1], [u : 1], [v : 1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 = (\mathbb{P}^1)^3.$$

Using shorthand such that $[x : 1] = [1 : X]$, we can perform a change of variables $X = 1/x$ and examine the region where X vanishes. This is equivalent to considering where x becomes unbounded.

4.3.1 Resolution of $\mathrm{H3}_{\delta=0}$

Consider a map ψ from $(\mathbb{P}^1)^3$ to $(\mathbb{P}^1)^3$ mapping the vertices (v, x, u) to (v, y, u) according to $\mathrm{H3}_{\delta=0}$.

$$\psi : (\mathbb{P}^1)^3 \to (\mathbb{P}^1)^3, \quad ([v : 1], [x : 1], [u : 1]) \longmapsto ([v : 1], [y : 1], [u : 1]). \tag{4.27}$$

This map takes the 3-dimensional space of initial conditions for solving for y to the 3-dimensional space of initial conditions for solving for x, with the inverse map

$$\psi^{-1} : (\mathbb{P}^1)^3 \to (\mathbb{P}^1)^3, \quad ([v : 1], [y : 1], [u : 1]) \longmapsto ([v : 1], [x : 1], [u : 1]),$$

see Figure 4.5.

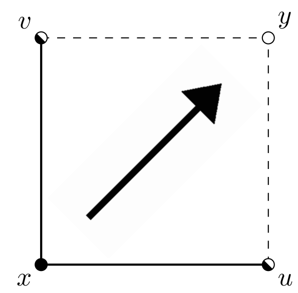

Fig. 4.5: Iteration on one quadrilateral $\psi : (\mathbb{P}^1)^3 \to (\mathbb{P}^1)^3$.

We find that ψ is undefined along the four base lines given by

$$u = v = 0, \tag{b_1}$$

$$x = \alpha\, v - \beta\, u = 0, \tag{b_2}$$

$$X = \alpha\, V - \beta\, U = 0, \tag{b_3}$$

$$U = V = 0. \tag{b_4}$$

Blowing up along these base varieties (and any new base varieties which appear on the resulting exceptional planes) we build a space $\mathcal{X}$ such that the map lifted to $\mathcal{X}$ is everywhere well defined.

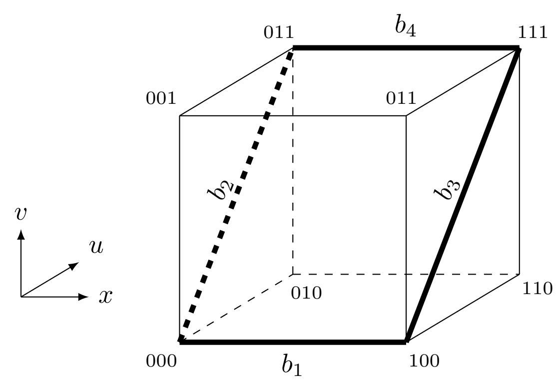

Fig. 4.6: Base varieties of $\mathrm{H3}_{\delta=0}$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, base varieties drawn in bold.

We name the 8 affine charts such that 000 has affine coordinates (x, u, v), 100 has (X, u, v), 010 has (x, U, v), 001 has (x, u, V), 110 has (X, U, v), 101 has (X, u, V), 011 has (x, U, V), and 111 has (X, U, V).

By resolving the base varieties we find in each affine chart we resolve all base varieties which appear in this space.

Blow-up of b_1

In the first affine chart (x, u, v), we have

$$y = x\frac{\alpha\, u - \beta\, v}{\beta\, u - \alpha\, v}, \tag{4.28}$$

and we see the base varieties

$$u = v = 0, \tag{b_1}$$

$$x = \alpha\, v - \beta\, u = 0. \tag{b_2}$$

In this chart we see the base lines b_1 and b_2, and their intersection at the origin. Consider the blow-up of this intersection. By replacing (x, u, v) with $(x', x'\,u', x'\,v')$, $(x'\,u', u', u'\,v')$, or $(x'\,v', u'\,v', v')$, we have

$$y = x'\frac{\alpha\, u' - \beta\, v'}{\beta\, u' - \alpha\, v'}, \tag{4.29a}$$

$$y = x'\,u'\frac{\alpha - \beta\, v'}{\beta - \alpha\, v'}, \tag{4.29b}$$

$$y = x'\,v'\frac{\alpha\, u' - \beta}{\beta\, u' - \alpha}, \tag{4.29c}$$

respectively. This has not improved the nature of the singularities. Indeed, in the first chart the equation is left completely unchanged. Instead we must blow up with the entire base line as centre.

To blow up b_1, we use the change of variables

$$\begin{cases} x^{(11)} &= x, \\ u^{(11)} &= \dfrac{u}{v}, \\ v^{(11)} &= v, \end{cases} \qquad \Longrightarrow \qquad \begin{cases} x &= x^{(11)}, \\ u &= u^{(11)}\, v^{(11)}, \\ v &= v^{(11)}, \end{cases} \tag{4.30}$$

and

$$\begin{cases} x^{(12)} &= x, \\ u^{(12)} &= u, \\ v^{(12)} &= \dfrac{v}{u}, \end{cases} \qquad \Longrightarrow \qquad \begin{cases} x &= x^{(12)}, \\ u &= u^{(12)}, \\ v &= u^{(12)}\, v^{(12)}, \end{cases} \tag{4.31}$$

where the superscript indicates the blow-up chart. Applying (4.30) and (4.31) to (4.28), we find

$$y = x^{(11)} \frac{\alpha\, u^{(11)} - \beta}{\beta\, u^{(11)} - \alpha}, \tag{4.32a}$$

$$y = x^{(12)} \frac{\alpha - \beta\, v^{(12)}}{\beta - \alpha\, v^{(12)}}, \tag{4.32b}$$

respectively. In both charts b_1 is resolved with this blow-up, but b_2 persists as $x^{(11)} = \beta\, u^{(11)} - \alpha = 0$. Blowing up along b_2 using the changes of variables

$$\begin{cases} x^{(21)} &= \dfrac{x^{(11)}}{\beta\, u^{(11)} - \alpha}, \\ u^{(21)} &= \beta\, u^{(11)} - \alpha, \\ v^{(21)} &= v^{(11)}, \end{cases} \qquad \Longrightarrow \qquad \begin{cases} x^{(11)} &= x^{(21)}\, u^{(21)}, \\ u^{(11)} &= \dfrac{u^{(21)} + \alpha}{\beta}, \\ v^{(11)} &= v^{(21)}, \end{cases} \tag{4.33}$$

and

$$\begin{cases} x^{(22)} &= x^{(11)}, \\ u^{(22)} &= \dfrac{\beta\, u^{(11)} - \alpha}{x^{(11)}}, \\ v^{(22)} &= v^{(11)}, \end{cases} \qquad \Longrightarrow \qquad \begin{cases} x^{(11)} &= x^{(22)}, \\ u^{(11)} &= \dfrac{x^{(22)}\, u^{(22)} + \alpha}{\beta}, \\ v^{(11)} &= v^{(22)}, \end{cases} \tag{4.34}$$

and applying them to (4.32a), we find respectively

$$y = \frac{x^{(21)}(\alpha\, u^{(21)} + \alpha^2 - \beta^2)}{\beta}, \tag{4.35a}$$

$$y = \frac{\alpha\, x^{(22)}\, u^{(22)} + \alpha^2 - \beta^2}{\beta\, u^{(22)}}. \tag{4.35b}$$

The equation is now fully resolved in this chart. This space now contains two exceptional planes. Specifically, e_2 and e_1^*, the lift of e_1 after blowing up along b_2. In the chart (4.33), e_1^* is the region given by $\{v^{(21)} = 0\}$, and e_2 by $\{u^{(21)} = 0\}$, and hence they intersect along a line where $x^{(21)}$ is free. Similarly in the chart (4.34), e_1^* is the region given by $\{v^{(22)} = 0\}$, and e_2 by $\{x^{(22)} = 0\}$, and hence they intersect along a line where $u^{(22)} = \frac{1}{x^{(21)}}$ is free. Alternatively, we could have chosen to resolve first b_2, then b_1. While this might change the exact definitions of the coordinates, the blow-up structure is preserved.

Similarly, in the chart $(X, U, V) = (1/x, 1/u, 1/v)$, we have

$$y = \frac{1}{X}\frac{\alpha V - \beta U}{\beta V - \alpha U}, \tag{4.36}$$

where we see the base lines b_3 and b_4.

$$X = \alpha V - \beta U = 0, \tag{b_3}$$

$$U = V = 0. \tag{b_4}$$

There is a neat symmetry in this equation where the Möbius transformation $x \to 1/x$, $u \to 1/u$, $v \to 1/v$, $y \to 1/y$ leaves the equation unchanged, so the singularity structure in this chart appears the same as the first.

To blow up b_4, we use the change of variables

$$\begin{cases} X^{(41)} &= X, \\ U^{(41)} &= \dfrac{U}{V}, \\ V^{(41)} &= V, \end{cases} \qquad \Longrightarrow \qquad \begin{cases} X &= X^{(41)}, \\ U &= U^{(41)}\,V^{(41)}, \\ V &= V^{(41)}, \end{cases} \tag{4.37}$$

and

$$\begin{cases} X^{(42)} &= X, \\ U^{(42)} &= U, \\ V^{(42)} &= \dfrac{V}{U}, \end{cases} \qquad \Longrightarrow \qquad \begin{cases} X &= X^{(42)}, \\ U &= U^{(42)}, \\ V &= U^{(42)}\,V^{(42)}, \end{cases} \tag{4.38}$$

Applying (4.37) and (4.38) to (4.36), we find

$$y = \frac{1}{X^{(41)}}\frac{\beta U^{(41)} - \alpha}{\alpha U^{(41)} - \beta}, \tag{4.39a}$$

$$y = \frac{1}{X^{(42)}}\frac{\alpha V^{(42)} - \beta}{\beta V^{(42)} - \alpha}, \tag{4.39b}$$

respectively. In both charts b_4 is resolved with this blow-up, but as expected due to symmetry b_3

persists as $X^{(41)} = \beta\, U^{(41)} - \alpha = 0$. Blowing up along b_3 using the changes of variables

$$\begin{cases} X^{(31)} &= \dfrac{X^{(41)}}{\beta\, U^{(41)} - \alpha}, \\ U^{(31)} &= \beta\, U^{(41)} - \alpha, \\ V^{(31)} &= V^{(41)}, \end{cases} \qquad \Longrightarrow \qquad \begin{cases} X^{(41)} &= X^{(31)}\, U^{(31)}, \\ U^{(41)} &= \dfrac{U^{(31)} + \alpha}{\beta}, \\ V^{(41)} &= V^{(31)}, \end{cases} \tag{4.40}$$

and

$$\begin{cases} X^{(32)} &= X^{(41)}, \\ U^{(32)} &= \dfrac{\beta\, U^{(41)} - \alpha}{X^{(41)}}, \\ V^{(32)} &= V^{(41)}, \end{cases} \qquad \Longrightarrow \qquad \begin{cases} X^{(41)} &= X^{(32)}, \\ U^{(41)} &= \dfrac{X^{(32)}\, U^{(32)} + \alpha}{\beta}, \\ V^{(41)} &= V^{(32)}, \end{cases} \tag{4.41}$$

and applying them to (4.39a), we find respectively

$$y = \frac{1}{X^{(31)}} \frac{\beta}{\alpha\, U^{(31)} + \alpha^2 - \beta^2}, \tag{4.42a}$$

$$y = \frac{\beta\, U^{(32)}}{\alpha\, X^{(32)}\, U^{(32)} + \alpha^2 - \beta^2}. \tag{4.42b}$$

Thus the equation is fully resolved. The result remains the same regardless of choice of affine chart to blow up in. In the next section we consider the map ϕ (the induced map ψ lifted to the resolved space $\mathcal{X}$), including the image of the exceptional planes e_1, e_2 e_3, and e_4.

In this example, all base varieties were resolved by blowing up once. However, in some examples we find new base varieties on the resulting exceptional plane after blowing up, much like in the case of Painlevé equations where base points appear on exceptional lines.

4.3.2 Resolution of $\mathrm{H3}_{\delta=1}$

Consider now the equation $\mathrm{H3}_{\delta=1}$, that is

$$\alpha\,(x\,u + v\,y) - \beta\,(x\,v + u\,y) + \alpha^2 - \beta^2 = 0. \tag{4.43}$$

Solving for the vertex y, we have

$$y = \frac{x(\alpha\,u - \beta\,v) + \alpha^2 - \beta^2}{\beta\,u - \alpha\,v}. \tag{4.44}$$

Compactifying the initial conditions for this quadrilateral x, u, v to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ we find 3 singular lines, given by

$$\alpha X + u = \beta X + v = 0, \qquad (b_1)$$

$$X = \alpha u - \beta v = 0, \qquad (b_2)$$

$$U = V = 0, \qquad (b_3)$$

see Figure 4.7.

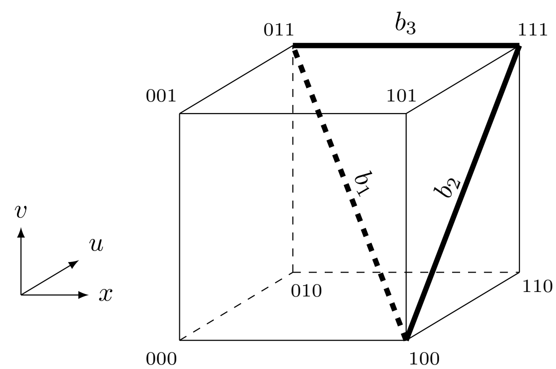

Fig. 4.7: Base varieties of $\mathrm{H3}_{\delta=0}$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, base varieties drawn in bold.

We proceed by blowing up in each of the three charts where these intersections are visible.

Resolution in the chart 111

In this affine chart we have coordinates (X, U, V), and hence

$$y = \frac{\beta U - \alpha V - X U V (\alpha^2 - \beta^2)}{X (\beta V - \alpha U)}. \qquad (4.45)$$

Here we see the base varieties

$$\alpha X U + 1 = \beta X V + 1 = 0, \qquad (b_1)$$

$$X = \beta U - \alpha V = 0, \qquad (b_2)$$

$$U = V = 0. \qquad (b_3)$$

In this chart we will only resolve b_2 and b_3, and leave b_1 to be resolved in other charts.

For starters, we blow up b_3 with the change of variables

$$\begin{cases} X^{(31)} &= X, \\ U^{(31)} &= \dfrac{U}{V}, \\ V^{(31)} &= V, \end{cases} \implies \begin{cases} X &= X^{(31)}, \\ U &= U^{(31)}\,V^{(31)}, \\ V &= V^{(31)}, \end{cases} \tag{4.46}$$

and

$$\begin{cases} X^{(32)} &= X, \\ U^{(32)} &= U, \\ V^{(32)} &= \dfrac{V}{U}, \end{cases} \implies \begin{cases} X &= X^{(32)}, \\ U &= U^{(32)}, \\ V &= U^{(32)}\,V^{(32)}, \end{cases} \tag{4.47}$$

where the superscript indicates the blow-up chart. This gives, respectively,

$$y = \frac{\beta\,U^{(31)} - \alpha - X^{(31)}\,U^{(31)}\,V^{(31)}\,(\alpha^2 - \beta^2)}{X^{(31)}\,(\beta - \alpha\,U^{(31)})}, \tag{4.48a}$$

$$y = \frac{\beta - \alpha\,V^{(32)} - X^{(32)}\,U^{(32)}\,V^{(32)}\,(\alpha^2 - \beta^2)}{X^{(32)}\,(\beta\,V^{(32)} - \alpha)}. \tag{4.48b}$$

After this blow-up we see the strict transforms of b_1 and b_2,

$$\beta\,X^{(31)}\,V^{(31)} + 1 = \alpha\,U^{(31)} - \beta = 0, \tag{b_1}$$

$$X^{(31)} = \beta\,U^{(31)} - \alpha = 0. \tag{b_2}$$

Blowing up b_2 in the chart $(X^{(31)}, U^{(31)}, V^{(31)})$ with

$$\begin{cases} X^{(21)} &= \dfrac{X^{(31)}}{\beta\,U^{(31)} - \alpha}, \\ U^{(21)} &= \beta\,U^{(31)} - \alpha, \\ V^{(21)} &= V^{(31)}, \end{cases} \implies \begin{cases} X^{(31)} &= X^{(21)}\,U^{(21)}, \\ U^{(31)} &= \dfrac{1}{\beta}\left(U^{(21)} + \alpha\right), \\ V^{(31)} &= V^{(21)}, \end{cases} \tag{4.49}$$

and

$$\begin{cases} X^{(22)} &= X^{(31)}, \\ U^{(22)} &= \dfrac{\beta\,U^{(31)} - \alpha}{X^{(31)}}, \\ V^{(22)} &= V^{(31)}, \end{cases} \implies \begin{cases} X^{(31)} &= X^{(22)}, \\ U^{(31)} &= \dfrac{1}{\beta}\left(X^{(22)}\,U^{(22)} + \alpha\right), \\ V^{(31)} &= V^{(22)}, \end{cases} \tag{4.50}$$

gives the expressions

$$y = \frac{\beta - \alpha(\alpha^2-\beta^2)\, X^{(21)}\, V^{(21)} - (\alpha^2-\beta^2)\, X^{(21)}\, U^{(21)}\, V^{(21)}}{X^{(21)}\left(\alpha\, U^{(21)} + \alpha^2 - \beta^2\right)}, \tag{4.51a}$$

$$y = \frac{\beta\, U^{(22)} - \alpha\,(\alpha^2-\beta^2)\, V^{(22)} - (\alpha^2-\beta^2)\, X^{(22)}\, U^{(22)}\, V^{(22)}}{\alpha\, X^{(22)}\, U^{(22)} + \alpha^2 - \beta^2}. \tag{4.51b}$$

All that is left to resolve in this chart is the section of b_1 visible in this chart. We choose to resolve this in other affine charts.

Resolution in the chart 011

If we define the variables $x^{(31)} := (X^{(31)})^{-1}$, $x^{(32)} := (X^{(32)})^{-1}$, then blowing up b_3 in this chart is computationally equivalent to changing chart to find

$$y = \frac{x^{(31)}(\beta\, U^{(31)} - \alpha) - U^{(31)}\, V^{(31)}\,(\alpha^2-\beta^2)}{\beta - \alpha\, U^{(31)}}, \tag{4.52a}$$

$$y = \frac{x^{(32)}(\beta - \alpha\, V^{(32)}) - U^{(32)}\, V^{(32)}\,(\alpha^2-\beta^2)}{\beta\, V^{(32)} - \alpha}. \tag{4.52b}$$

In this chart all that remains is the resolution of b_1, which in this chart is described by

$$x^{(31)} + \beta\, V^{(31)} = \alpha\, U^{(31)} - \beta = 0. \tag{b_1}$$

Using the changes of variables

$$\begin{cases} x^{(11)} &= \dfrac{x^{(31)} + \beta\, V^{(31)}}{\alpha\, U^{(31)} - \beta}, \\ U^{(11)} &= \alpha\, U^{(31)} - \beta, \\ V^{(11)} &= V^{(31)}, \end{cases} \implies \begin{cases} x^{(31)} &= x^{(11)}\, U^{(11)} - \beta\, V^{(11)}, \\ U^{(31)} &= \dfrac{1}{\alpha}\left(U^{(11)} + \beta\right), \\ V^{(31)} &= V^{(11)}, \end{cases} \tag{4.53}$$

and

$$\begin{cases} x^{(12)} &= x^{(31)} + \beta\, V^{(31)}, \\ U^{(12)} &= \dfrac{\alpha\, U^{(31)} - \beta}{x^{(31)} + \beta\, V^{(31)}}, \\ V^{(12)} &= V^{(31)}, \end{cases} \implies \begin{cases} x^{(31)} &= x^{(12)}, \\ U^{(31)} &= \dfrac{1}{\alpha}\left(x^{(12)}\, U^{(12)} + \beta\right), \\ V^{(31)} &= V^{(12)}, \end{cases} \tag{4.54}$$

we find, respectively,

$$y = \frac{1}{\alpha}\left(\alpha^2 V^{(11)} - \beta\, x^{(11)}\, U^{(11)} - (\alpha^2-\beta^2) x^{(11)}\right), \tag{4.55a}$$

$$y = \frac{\beta\, x^{(12)}\, U^{(12)} - \alpha^2 U^{(12)} V^{(12)} - (\alpha^2 - \beta^2)}{\alpha\, U^{(12)}}. \tag{4.55b}$$

There are no more singular varieties visible in this chart. We finish by resolving in the chart 100.

Resolution in the chart 100

Starting in this chart we have

$$y = \frac{\alpha\, u - \beta\, v - X(\alpha^2 - \beta^2)}{X(\beta\, u - \alpha\, v)}, \tag{4.56}$$

and we see the base varieties b_1 and b_2, described by

$$\alpha\, X + u = \beta\, X + v = 0, \tag{b_1}$$

$$X = \alpha\, u - \beta\, v = 0. \tag{b_2}$$

As before, blowing up b_1 then b_2, the equation is again fully resolved.

4.3.3 Resolution of Q1

We will now consider the resolution of Q1, which is significantly more complicated. It involves many features not seen in the previous example, including new base varieties appearing after blowing up.

Consider the ABS equation Q1, given canonically by

$$\alpha\,(x - v)(u - y) - \beta\,(x - u)(v - y) + \delta^2 \alpha\, \beta(\alpha - \beta) = 0. \tag{4.57}$$

Solving for the vertex y we find

$$y = \frac{x\,(\alpha\, u - \beta\, v) - (\alpha - \beta)\, u\, v + \delta^2 \alpha\, \beta\,(\alpha - \beta)}{(\alpha - \beta)\, x + \beta\, u - \alpha\, v}, \tag{4.58}$$

Compactify the initial conditions x, u, v to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. In this space we find the system has two base lines b_1 and b_2, given in the first affine coordinate chart (x, u, v) by

$$x - u + \alpha\, \delta = x - v + \beta\, \delta = 0, \tag{b_1}$$

$$x - u - \alpha\, \delta = x - v - \beta\, \delta = 0, \tag{b_2}$$

see Figure 4.8. These base lines intersect in the chart where x, u, v become unbounded, so we consider the equation in the chart $(X, U, V) = (1/x, 1/u, 1/v)$, which is of the form

$$y = \frac{\alpha\, V - \beta\, U - (\alpha - \beta)\, X + \delta^2 \alpha\, \beta\, (\alpha - \beta)\, X\, U\, V}{(\alpha - \beta)\, U\, V + \beta\, X\, V - \alpha\, X\, U}. \tag{4.59}$$

In this chart the equations of the base varieties now take the forms

$$X - U - \alpha\, \delta\, X\, U = X - V - \beta\, \delta\, X\, V = 0, \tag{b_1}$$

$$X - U + \alpha\, \delta\, X\, U = X - V + \beta\, \delta\, X\, V = 0. \tag{b_2}$$

We wish to resolve the singularities visible in this chart.

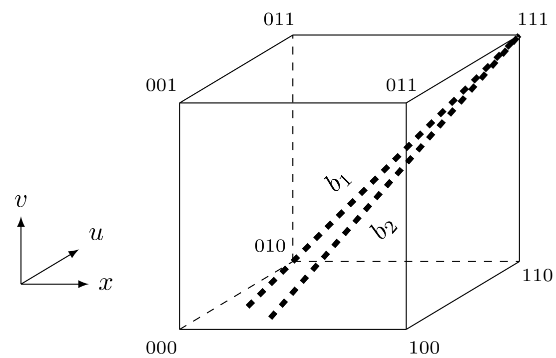

Fig. 4.8: Base varieties of Q1 in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, base varieties drawn in bold.

Unlike in previous examples, the base lines b_1 and b_2 intersect tangentially. Before we can resolve either base variety we have to 'untangle' their intersection, by blowing up at the origin.

We blow up at the origin with the three changes of variables

$$(X^{(01)}, U^{(01)}, V^{(01)}) = \left(X, \frac{U}{X}, \frac{V}{X}\right), \tag{4.60a}$$

$$(X^{(02)}, U^{(02)}, V^{(02)}) = \left(\frac{X}{U}, U, \frac{V}{U}\right), \tag{4.60b}$$

$$(X^{(03)}, U^{(03)}, V^{(03)}) = \left(\frac{X}{V}, \frac{U}{V}, V\right). \tag{4.60c}$$

In all three charts of this blow-up we see the proper transforms of b_1 and b_2, and a new base line b_3.

For example, in the first chart $(X^{(01)}, U^{(01)}, V^{(01)})$, we have the equation

$$y = \frac{(\beta - \alpha) + (\alpha\, V^{(01)} - \beta\, U^{(01)}) + \alpha\, \beta\, \delta^2\, (\alpha - \beta)\, (U^{(01)}\, V^{(01)} (X^{(01)})^2)}{X^{(01)}\, (\beta\, V^{(01)} - \alpha\, U^{(01)} + (\alpha - \beta)\, U^{(01)}\, V^{(01)})}, \tag{4.61}$$

and the three base lines b_1, b_2, b_3 given by

$$1 - U^{(01)} - \alpha\,\delta\,X^{(01)}\,U^{(01)} = 1 - V^{(01)} - \beta\,\delta\,X^{(01)}\,V^{(01)} = 0, \tag{b_1}$$

$$1 - U^{(01)} + \alpha\,\delta\,X^{(01)}\,U^{(01)} = 1 - V^{(01)} + \beta\,\delta\,X^{(01)}\,V^{(01)} = 0, \tag{b_2}$$

$$X^{(01)} = \alpha\,V^{(01)} - \beta\,U^{(01)} + \beta - \alpha = 0. \tag{b_3}$$

The base varieties b_1, b_2, and b_3 intersect at $(X^{(01)}, U^{(01)}, V^{(01)}) = (0, 1, 1)$. Therefore, changing chart to

$$(X^{(01)}, u^{(01)}, v^{(01)}) = (X^{(01)}, \frac{1}{U^{(01)}}, \frac{1}{V^{(01)}}), \tag{4.62}$$

the intersection is still visible and we have the map

$$y = \frac{(\beta-\alpha)\,u^{(01)}\,v^{(01)} + (\alpha\,u^{(01)} - \beta\,v^{(01)}) + \alpha\,\beta\,\delta^2\,(\alpha-\beta)\,(X^{(01)})^2}{X^{(01)}\,(\beta\,u^{(01)} - \alpha\,v^{(01)} + \alpha - \beta)}, \tag{4.63}$$

with the base varieties

$$\alpha\,\delta\,X^{(01)} - u^{(01)} + 1 = \beta\,\delta\,X^{(01)} - v^{(01)} + 1, \tag{b_1}$$

$$\alpha\,\delta\,X^{(01)} + u^{(01)} - 1 = \beta\,\delta\,X^{(01)} + v^{(01)} - 1, \tag{b_2}$$

$$X^{(01)} = (\alpha-\beta)\,u^{(01)}\,v^{(01)} - \alpha\,u^{(01)} + \beta\,v^{(01)} = 0. \tag{b_3}$$

We proceed by blowing up b_1, b_2, b_3 in sequence, along with any new base varieties which appear.

Blow-up of b_1

To blow up b_1, we use the changes of variables

$$\begin{cases} x^{(11)} &= X^{(01)}, \\ u^{(11)} &= \alpha\,\delta\,X^{(01)} + u^{(01)} - 1, \\ v^{(11)} &= \dfrac{\beta\,\delta\,X^{(01)} + v^{(01)} - 1}{\alpha\,\delta\,X^{(01)} + u^{(01)} - 1}, \end{cases} \implies \begin{cases} X^{(01)} &= x^{(11)}, \\ u^{(01)} &= 1 + u^{(11)} - \alpha\,\delta\,x^{(11)}, \\ v^{(01)} &= 1 + u^{(11)}\,v^{(11)} - \beta\,\delta\,x^{(11)}, \end{cases} \tag{4.64}$$

and

$$\begin{cases} x^{(12)} &= X^{(01)}, \\ u^{(12)} &= \dfrac{\alpha\,\delta\,X^{(01)} + u^{(01)} - 1}{\beta\,\delta\,X^{(01)} + v^{(01)} - 1}, \\ v^{(12)} &= \beta\,\delta\,X^{(01)} + v^{(01)} - 1, \end{cases} \implies \begin{cases} X^{(01)} &= x^{(12)}, \\ u^{(01)} &= 1 + u^{(12)}\,v^{(12)} - \alpha\,\delta\,x^{(12)}, \\ v^{(01)} &= 1 + v^{(12)} - \beta\,\delta\,x^{(12)}. \end{cases} \tag{4.65}$$

The exceptional plane e_1 is $\{u^{(11)} = 0\} \cap \{v^{(12)} = 0\}$.

Performing the substitutions (4.64) and (4.65) in (4.62) give

$$y = \frac{\beta - \alpha\, v^{(11)} - (\alpha-\beta)\,(u^{(11)}\, v^{(11)} - \beta\,\delta\, x^{(11)} - \alpha\,\delta\, x^{(11)}\, v^{(11)})}{x^{(11)}(\beta - \alpha\, v^{(11)})}, \tag{4.66a}$$

$$y = \frac{\alpha - \beta\, u^{(12)} + (\alpha-\beta)\,(u^{(12)}\, v^{(12)} - \alpha\,\delta\, x^{(12)} - \beta\,\delta\, x^{(12)}\, u^{(12)})}{x^{(12)}\,(\alpha - \beta\, u^{(12)})}, \tag{4.66b}$$

respectively. The singularity b_1 is now fully resolved, and the proper transforms of b_2 and b_3 remain. In the first chart $(x^{(11)}, u^{(11)}, v^{(11)})$, they are given by

$$2\,\alpha\,\delta\, x^{(11)} - u^{(11)} = \alpha\, v^{(11)} - \beta = 0, \tag{b_2}$$

$$x^{(11)} = (\alpha-\beta)\, u^{(11)}\, v^{(11)} + \alpha\, v^{(11)} - \beta = 0. \tag{b_3}$$

We now blow up b_2 in this chart.

Blow-up of b_2

To blow up b_2, we use the changes of variables

$$\begin{cases} x^{(21)} &= 2\,\alpha\,\delta\, x^{(11)} - u^{(11)}, \\ u^{(21)} &= u^{(11)}, \\ v^{(21)} &= \dfrac{\alpha\, v^{(11)} - \beta}{2\,\alpha\,\delta\, x^{(11)} - u^{(11)}}, \end{cases} \implies \begin{cases} x^{(11)} &= \dfrac{x^{(21)} + u^{(21)}}{2\,\alpha\,\delta}, \\ u^{(11)} &= u^{(21)}, \\ v^{(11)} &= \dfrac{\beta + x^{(21)}\, v^{(21)}}{\alpha}, \end{cases} \tag{4.67}$$

and

$$\begin{cases} x^{(22)} &= \dfrac{2\,\alpha\,\delta\, x^{(11)} - u^{(11)}}{\alpha\, v^{(11)} - \beta}, \\ u^{(22)} &= u^{(11)}, \\ v^{(22)} &= \alpha\, v^{(11)} - \beta, \end{cases} \implies \begin{cases} x^{(11)} &= \dfrac{x^{(22)}\, v^{(22)} + u^{(22)}}{2\,\alpha\,\delta}, \\ u^{(11)} &= u^{(22)}, \\ v^{(11)} &= \dfrac{v^{(22)} + \beta}{\alpha}. \end{cases} \tag{4.68}$$

The exceptional plane e_2 is $\{x^{(21)} = 0\} \cap \{v^{(22)} = 0\}$.

Making the substitutions (4.67) and (4.68) in (4.66a), we now find

$$y = \frac{\delta\,((\alpha-\beta)\, v^{(21)}\,(u^{(21)} - x^{(21)}) + 2\,\alpha\, v^{(21)} - 2\,\beta\,(\alpha-\beta))}{v^{(21)}\,(u^{(21)} + x^{(21)})} \tag{4.69a}$$

$$y = \frac{\delta\,(2\,\alpha + (\alpha-\beta)\,(u^{(22)} - v^{(22)}\, x^{(22)}) - 2\,(\alpha+\beta)\,\beta\, x^{(22)})}{u^{(22)} + v^{(22)}\, x^{(22)}}, \tag{4.69b}$$

respectively. The singularity b_2 is now fully resolved, and only the proper transform of b_3 remains. In the chart $(x^{(21)}, u^{(21)}, v^{(21)})$, it is given by

$$x^{(21)} + u^{(21)} = \beta\,(\alpha-\beta) - \alpha\,v^{(21)} + (\alpha-\beta)\,x^{(21)}\,v^{(21)} = 0. \tag{b_3}$$

Blow-up of b_3

We blow up b_3 in the chart $(x^{(21)}, u^{(21)}, v^{(21)})$ with the changes of variables

$$\begin{cases} x^{(31)} &= x^{(21)} + u^{(21)}, \\ u^{(31)} &= u^{(21)}, \\ v^{(31)} &= \dfrac{\beta\,(\alpha-\beta) - (\alpha + (\alpha-\beta)\,x^{(21)})\,v^{(21)}}{x^{(21)} + u^{(21)}}, \end{cases}$$

$$\implies \begin{cases} x^{(21)} &= x^{(31)} - u^{(31)}, \\ u^{(21)} &= u^{(31)}, \\ v^{(21)} &= \dfrac{x^{(31)}\,v^{(31)}}{\alpha + (\alpha-\beta)\,(u^{(31)} - x^{(31)})}, \end{cases} \tag{4.70}$$

and

$$\begin{cases} x^{(32)} &= \dfrac{x^{(21)} + u^{(21)}}{\beta\,(\alpha-\beta) - (\alpha + (\alpha-\beta)\,x^{(21)})\,v^{(21)}}, \\ u^{(32)} &= u^{(21)}, \\ v^{(32)} &= \beta\,(\alpha-\beta) - (\alpha + (\alpha-\beta)\,x^{(21)})\,v^{(21)}, \end{cases}$$

$$\implies \begin{cases} x^{(21)} &= x^{(32)}\,v^{(32)} - u^{(32)}, \\ u^{(21)} &= u^{(32)}, \\ v^{(21)} &= \dfrac{x^{(32)}\,v^{(32)} - \beta\,(\alpha-\beta)}{-\alpha + (\alpha-\beta)\,(x^{(32)}\,v^{(32)} - u^{(32)})}. \end{cases} \tag{4.71}$$

The exceptional plane e_3 is $\{x^{(31)} = 0\} \cap \{v^{(32)} = 0\}$. The base line b_3 is now fully resolved.

In the chart $(x^{(32)}, u^{(32)}, v^{(32)})$, we find two new base lines we call b_4 and b_5, given by

$$x^{(32)} = (\alpha-\beta)\,u^{(32)} + \alpha = 0, \tag{b_4}$$

$$(\alpha-\beta)\,u^{(32)} - \beta\,(\alpha-\beta)^2\,x^{(32)} + \alpha = v^{(32)} - \beta\,(\alpha-\beta) = 0. \tag{b_5}$$

In the chart $(x^{(31)}, u^{(31)}, v^{(31)})$ only b_5 is visible, and hence we blow up b_4 and b_5 in the second chart.

Blow-up of b_4

We blow up b_4 in the chart $(x^{(32)}, u^{(32)}, v^{(32)})$ with the changes of variables

$$\begin{cases} x^{(41)} &= x^{(32)}, \\ u^{(41)} &= \dfrac{(\alpha-\beta)\,u^{(32)}+\alpha}{x^{(32)}}, \\ v^{(41)} &= v^{(32)}, \end{cases} \quad \Longrightarrow \quad \begin{cases} x^{(32)} &= x^{(41)}, \\ u^{(32)} &= \dfrac{\alpha - x^{(41)}\,u^{(41)}}{\alpha-\beta}, \\ v^{(32)} &= v^{(41)}, \end{cases} \tag{4.72}$$

and

$$\begin{cases} x^{(42)} &= \dfrac{x^{(32)}}{(\alpha-\beta)\,u^{(32)}+\alpha}, \\ u^{(42)} &= (\alpha-\beta)\,u^{(32)}+\alpha, \\ v^{(42)} &= v^{(32)}, \end{cases} \quad \Longrightarrow \quad \begin{cases} x^{(32)} &= x^{(42)}\,u^{(42)}, \\ u^{(32)} &= \dfrac{u^{(42)}-\alpha}{\alpha-\beta}, \\ v^{(32)} &= v^{(42)}, \end{cases} \tag{4.73}$$

The exceptional plane e_4 is $\{x^{(41)} = 0\} \cap \{u^{(42)} = 0\}$.

The base line b_4 is now fully resolved, and all that remains is the total transform of b_5, given in the chart $(x^{(42)}, u^{(42)}, v^{(42)})$ by

$$\beta\,(\alpha-\beta)^2\,x^{(42)} = v^{(42)} - \beta\,(\alpha-\beta) = 0. \tag{b_5}$$

Blow-up of b_5

We blow up b_5 in the chart $(x^{(42)}, u^{(42)}, v^{(42)})$ with the changes of variables

$$\begin{cases} x^{(51)} &= \beta\,(\alpha-\beta)^2\,x^{(42)}, \\ u^{(51)} &= u^{(42)}, \\ v^{(51)} &= \dfrac{v^{(42)}-\beta\,(\alpha-\beta)}{\beta\,(\alpha-\beta)^2\,x^{(42)}}, \end{cases} \quad \Longrightarrow \quad \begin{cases} x^{(42)} &= \dfrac{x^{(51)}+1}{\beta\,(\alpha-\beta)^2}, \\ u^{(42)} &= u^{(51)}, \\ v^{(42)} &= x^{(51)}\,v^{(51)} + \beta\,(\alpha-\beta), \end{cases} \tag{4.74}$$

and

$$\begin{cases} x^{(52)} &= \dfrac{\beta\,(\alpha-\beta)^2\,x^{(42)}}{v^{(42)}-\beta\,(\alpha-\beta)}, \\ u^{(52)} &= u^{(42)}, \\ v^{(52)} &= v^{(42)} - \beta\,(\alpha-\beta), \end{cases} \quad \Longrightarrow \quad \begin{cases} x^{(42)} &= \dfrac{x^{(52)}\,v^{(52)}+1}{\beta\,(\alpha-\beta)^2}, \\ u^{(42)} &= u^{(52)}, \\ v^{(42)} &= v^{(52)} | \beta\,(\alpha-\beta), \end{cases} \tag{4.75}$$

The exceptional plane e_5 is $\{x^{(51)} = 0\} \cap \{v^{(52)} = 0\}$.

Making the substitutions (4.74) and (4.75), we find

$$y = \frac{\delta\,(\beta-\alpha)\,(2\,\beta\,(\alpha-\beta) + x^{(51)}\,v^{(51)} + v^{(51)})}{v^{(51)}(x^{(51)}+1)}, \tag{4.76a}$$

$$y = \frac{\delta\,(\beta-\alpha)\,(x^{(52)}\,v^{(52)} + 2\,\beta\,(\alpha-\beta)\,x^{(52)} + 1)}{x^{(52)}\,v^{(52)} + 1}, \tag{4.76b}$$

and hence the map is now fully resolved.

4.4 Induced Map on a Quadrilateral

Now consider a map of the form (4.27), lifted to a resolved space $\mathcal{X}$ as discussed in the previous chapter.

On a generic quadrilateral with vertices x, u, v, y, compactifying $\mathbb{C}^3$ to $(\mathbb{P}^1)^3 = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, we have a map

$$\psi : (\mathbb{P}^1)^3 \to (\mathbb{P}^1)^3, \quad ([v:1],[x:1],[u:1]) \longmapsto ([v:1],[y:1],[u:1]),$$

and its inverse

$$\psi^{-1} : (\mathbb{P}^1)^3 \to (\mathbb{P}^1)^3, \quad ([v:1],[y:1],[u:1]) \longmapsto ([v:1],[x:1],[u:1]).$$

Upon resolving singularities for the maps ψ and ψ^{-1}, we have the resolved maps ϕ and ϕ^{-1}, with corresponding resolved spaces $\mathcal{X}$ and $\mathcal{Y}$, such that the maps $\phi(\mathcal{X}) = \mathcal{Y}$ and $\phi^{-1}(\mathcal{Y}) = \mathcal{X}$ are everywhere well defined. The natural question arises: what are the images of exceptional planes under these maps?

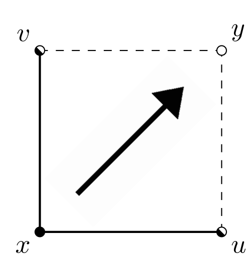

Fig. 4.9: The map $\psi : (\mathbb{P}^1)^3 \to (\mathbb{P}^1)^3$.

Due to the symmetry of the ABS equations, the map ϕ becomes self inverse and an automorphism on the resolved space $\mathcal{X}$, so that for any point $p \in \mathcal{X}$ we have $\phi(\phi(p)) = p$. As an example of this construction, consider again the model example of $\mathrm{H3}_{\delta=0}$. In this case the map works on the space we found in the Section 4.3.1 by blowing up the base varieties shown in Figure 4.6.

Recall that solving for the vertex y, we find

$$y = x\frac{\alpha\, u - \beta\, v}{\beta\, u - \alpha\, v}. \tag{4.77}$$

Consider the exceptional plane e_1, from the resolution of the line b_1 where $u = v = 0$. The vertices u and v are shared by the forward and backward iterations, and calculating the forward iteration under ϕ, we see that $\phi(e_1) = e_1$. In a similar fashion, we find $\phi(e_4) = e_4$.

Now, suppose the initial conditions of a quadrilateral lie on the surface S_1 given by

$$\alpha\, u - \beta\, v = 0. \tag{4.78}$$

We choose this surface since, on the level of the unresolved space, the forward iteration map ψ blows it down to the base line b_2 for the inverse step. For this reason, we expect that on the level of the resolved space it must be taken to at least some region of the exceptional plane e_2.

In the resolved space of initial conditions of the inverse step, the exceptional plane e_2 resulting from the blow-up of b_2 is parameterised by

$$\alpha\, u = \beta\, v, \tag{4.79a}$$

$$y^{(21)} = \frac{y}{\alpha\, u - \beta\, v}. \tag{4.79b}$$

We know from the equation (4.77) which defines this map that in terms of x, u, v, the variable $y^{(21)}$ (which parameterises e_2 for the resolved backward iteration map) is given by

$$y^{(21)} = \frac{y}{\alpha\, u - \beta\, v} = \frac{x}{\alpha\, v - \beta\, u}. \tag{4.80}$$

The image of S_1 (4.78) covers all values of $y^{(21)}$, and hence

$$\phi(S_1) = e_2. \tag{4.81}$$

Equivalently, we find that $\phi(e_2) = S_1$.

Finally, we follow a similar argument with another surface S_2 also blown down by the equation, given by

$$\beta\, u - \alpha\, v = 0. \tag{S_2}$$

On the level of the unresolved space, this plane is blown down to the base line b_3, and here we find that on the level of the resolved space $\phi(S_2) = e_3$, and hence $\phi(e_3) = S_2$.

This construction has given us a map ϕ which is an automorphism on the resolved space of initial conditions $\mathcal{X}$ for one quadrilateral. In the next chapter, we associate every individual quad in a larger lattice region with an associated resolved value space, $\mathcal{X}_{l,m}$. This gives us a lattice with vertices labelled with resolved spaces as introduced in this chapter. We consider the evolution of these systems.

Chapter 5: Induced Mapping Between Resolved Spaces of Lattice Equations

In this chapter we extend the investigation of the previous chapter to larger regions on the lattice. We considered the singularities of the induced map of a quad-equation which gives the value of the solution on one vertex of a quadrilateral in terms of the other three, and resolved singularities appearing in this map. Due to the symmetry of the ABS equations this gave an automorphism on the resolved 3-dimensional space.

Recall that for a discrete Painlevé equation with dependent variable x_n, the iterates are pairs of points $u_n := x_{2n}$, $v_n := x_{2n+1}$. Each pair of points u_n, v_n are considered as coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$ and the equation is then thought of as defining a map ψ from $\mathbb{P}^1 \times \mathbb{P}^1$ to itself. Blowing up at points where the map is singular generates a so-called resolved space of initial conditions X_n, on which the map induced by the equation $\phi : X_n \to X_{n+1}$ is everywhere well-defined.

We present a similar approach for quad-equations. Taking three vertices on a square lattice such that it is possible to use a quad-equation $Q = 0$ to solve for a fourth vertex with coordinates l, m, we associate this quad with a corresponding resolved space as defined in the previous chapter. If we choose a preferred direction of iteration (this direction is typically established by the initial conditions, see Definition 3.1) then we can assign to each quad a resolved space of initial conditions $\mathcal{X}_{l,m}$, parameterised by the *two* independent variables l, m.

In this chapter we study the spaces of initial conditions over the lattice. It is no longer the case that we have a simple everywhere well-defined map between any two neighbouring $\mathcal{X}_{l,m}$, but by investigating the relationships between the spaces of initial conditions for each quad we find previously unseen transformations of lattice equations and new reductions of ABS equations to discrete Painlevé equations.

5.1 Spaces of Initial Conditions on the Dual Graph

In this this section we introduce the approach of this chapter, giving defining notation and terminology where necessary, then motivate this approach by reconsidering Section 4.2 from this perspective.

Consider a lattice whose vertices are labelled $x_{l,m}$, and consider a sublattice consisting of 4 adjacent quadrilaterals labelled A, B, C, D as shown in Figure 5.1. Take 5 points of a regular (1,1)–staircase in its interior as initial conditions (see Definition 3.1). We call this staircase I_0.

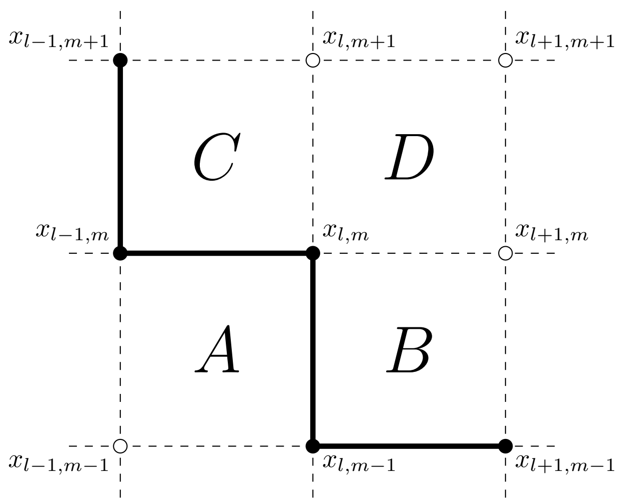

Fig. 5.1: 5-point staircase of initial values I_0 (shown in bold).

These initial conditions can be used to iterate along the diagonal axis in the direction perpendicular to the direction of I_0. Suppose that for a generic quad, in the direction of increasing l, m the equation has a resolved space of initial conditions $\mathcal{X}$, and in the direction of decreasing l, m a resolved space $\mathcal{Y}$.

Recall from Section 4.4 that we have a mapping between $\mathcal{X}$ and $\mathcal{Y}$ that is everywhere well-defined. Therefore, we can equivalently assign $\mathcal{Y}_{l-1,m-1}$ or $\mathcal{X}_{l,m}$, and hence label the vertices of the *dual graph* with a corresponding $\mathcal{X}_{l,m}$. The dual graph is a new lattice with a vertex corresponding to each quad of the original lattice, see Figure 5.2.

Definition 5.1. *The dual graph Z' of a lattice Z is the lattice which has a vertex for every face of Z, and an edge for everywhere two faces in Z are separated by an edge.*

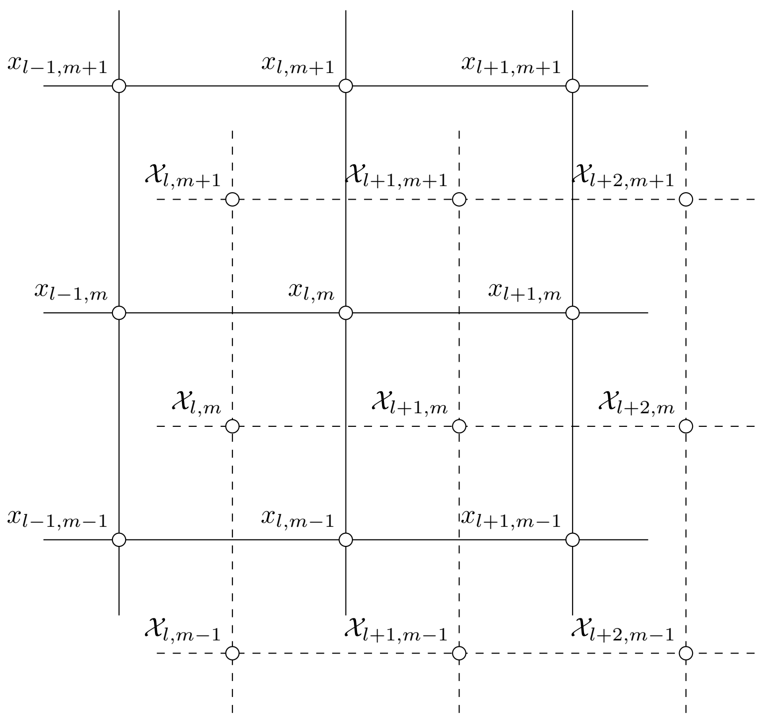

Fig. 5.2: Labelling the dual graph with $\mathcal{X}_{l,m}$.

Note that since in our case the lattice is $\mathbb{Z}^2$, the dual graph $(\mathbb{Z}^2)' \cong \mathbb{Z}^2$.

If, in addition to a staircase of initial conditions, we are given a corresponding staircase of initial conditions on the dual graph we can use the equation and its resolution to iterate over the lattice *without indeterminacies*, so long as no indeterminacies occur due to the iteration of $\mathcal{X}_{l,m}$ on the dual graph. This is not always the case however, as we shall show in the following sections.

In this chapter we study the equation's evolution on this dual graph. We commence by continuing with the example of $\mathrm{H3}_{\delta=0}$.

Example 5.1. Consider the equation $\mathrm{H3}_{\delta=0}$, in the form

$$y = x\frac{\alpha\, u - \beta\, v}{\beta\, u - \alpha\, v}. \tag{5.1}$$

As in Section 4.2, assume

$$x_{l-1,m-1} \neq 0, \tag{5.2a}$$

$$\alpha\, x_{l,m-1} = \beta\, x_{l-1,m}, \tag{5.2b}$$

Recall on the quad A in Figure 5.1 solving for the vertex $x_{l,m}$, we find $x_{l,m} = 0$. From the perspective of the original lattice the equation blows down the system on this quad. We will demonstrate how we can use the dual graph construction of this chapter to transport the lost initial data across this singularity.

Recall from the previous chapter that solving for y on a generic quad with vertices x, u, v, y, we have four base varieties in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$,

$$u = v = 0, \tag{b_1}$$

$$x = \alpha\, v - \beta\, u = 0, \tag{b_2}$$

$$X = \alpha\, V - \beta\, U = 0, \tag{b_3}$$

$$U = V = 0, \tag{b_4}$$

which are each resolved after one blow-up, giving one exceptional plane each.

We saw previously that for generic $x_{l+1,m-1}, x_{l-1,m+1}$ this blow-down is confined in two steps, as the vertex $x_{l+1,m+1}$ recovers the degree of freedom from $x_{l-1,m-1}$. This locally occurs as a blow-up on the quad D, where the solution lands on a base variety as iterating the system for generic $x_{l+1,m-1}$, $x_{l-1,m+1}$, as we find

$$\alpha\, x_{l,m+1} = \beta\, x_{l+1,m}.$$

This is not true for all choices $x_{l+1,m-1}, x_{l-1,m+1}$ however, and we shall now see how exploring the dual graph construction helps us more easily understand this phenomenon.

If in the resolved space of initial conditions for the quadrilateral D the solution lies on the exceptional plane e_2, we know from (4.25) that $x_{l+1,m+1}$ depends on the variable parametrising the exceptional plane e_2 in $\mathcal{X}_{l+1,m+1}$, in the sense of (4.22a). There is an equivalent exceptional plane in the resolved space $\mathcal{X}_{l,m}$ for each quad.

Define the variable $w_{l,m}$ to be this variable parametrising the exceptional plane e_2 in the space $\mathcal{X}_{l,m}$, so that each vertex in the dual graph is labelled with the corresponding $w_{l,m}$. In terms of the variables on the original lattice, from (4.22a) we have

$$w_{l+1,m+1} = \frac{x_{l,m}}{\alpha\, x_{l,m+1} - \beta\, x_{l+1,m}}. \tag{5.3}$$

The natural question now arises: Can we find a scalar lattice equation governing this new variable?

In terms of the initial conditions I_0 (Figure 5.1), we find $w_{l+1,m+1}$ is given by

$$\begin{aligned} w_{l+1,m+1} = {} & x_{l,m}\,(\beta\, x_{l,m} - \alpha\, x_{l-1,m+1})\,(\alpha\, x_{l,m} - \beta\, x_{l+1,m-1})\,/ \\ & (\alpha\, x_{l-1,m}\,(\alpha\, x_{l,m} - \beta\, x_{l-1,m+1})\,(\alpha\, x_{l,m} - \beta\, x_{l+1,m-1}) \\ & -\beta\, x_{l,m-1}\,(\beta\, x_{l,m} - \alpha\, x_{l-1,m+1})\,(\beta\, x_{l,m} - \alpha\, x_{l+1,m-1}))\,. \end{aligned} \tag{5.4}$$

This expression is indeterminate along the base variety b_2 from the quads B and C. Making the following substitutions from the resolution of these base varieties,

$$w_{l+1,m} = \frac{x_{l,m-1}}{\alpha\, x_{l,m} - \beta\, x_{l+1,m-1}} \implies x_{l+1,m-1} = \frac{\alpha\, x_{l,m}\, w_{l+1,m} - x_{l,m-1}}{\beta\, w_{l+1,m}}, \tag{5.5a}$$

$$w_{l,m+1} = \frac{x_{l-1,m}}{\alpha\, x_{l-1,m+1} - \beta\, x_{l,m}} \implies x_{l-1,m+1} = \frac{x_{l-1,m} + \beta\, x_{l,m}\, w_{l,m+1}}{\alpha\, w_{l,m+1}}, \tag{5.5b}$$

we find

$$w_{l+1,m+1} = \frac{x_{l,m}}{\beta\, x_{l-1,m} - \alpha\, x_{l,m-1} + (\beta^2 - \alpha^2)(w_{l,m+1} - w_{l+1,m})\, x_{l,m}}. \tag{5.6}$$

Now this expression is indeterminate where $x_{l,m} = \beta\, x_{l-1,m} - \alpha\, x_{l,m-1} = 0$. To blow up this base variety, we would introduce a variable parametrising the exceptional plane e_2 from $\mathcal{Y}_{l-1,m-1}$,

$$W_{l-1,m-1} = \frac{x_{l,m}}{\alpha\, x_{l,m-1} - \beta\, x_{l-1,m}}. \tag{5.7}$$

However, using (5.1),

$$\frac{x_{l,m}}{\alpha\, x_{l,m-1} - \beta\, x_{l-1,m}} = -\frac{x_{l-1,m-1}}{\alpha\, x_{l-1,m} - \beta\, x_{l,m-1}} \implies W_{l-1,m-1} = -w_{l,m},$$

and hence

$$w_{l+1,m+1} = \frac{w_{l,m}}{(\beta^2-\alpha^2)(w_{l,m+1}-w_{l+1,m})\,w_{l,m}+1}. \tag{5.8}$$

This is an equation purely in terms of $w_{l,m}$. Observe that (5.3) is a Miura transformation, which is not invertible.

From the resolution of singularities of H3 we have found a transformation of a lattice equation to a different lattice system by studying the relationship between resolved spaces on the dual graph. This transformation holds for any choice of initial conditions.

Note that this equation does not possess the complete symmetry of the square and hence is not itself an ABS equation. Additionally, (5.8) is not fully resolved. The base varieties which remain correspond to where the solution passes through base varieties of (5.1) on any *two* of A, B, C at once. If $w_{l+1,m+1}$ is not determined then we cannot use it to determine $x_{l+1,m+1}$, and hence this configuration of initial values is not confining for *all* choices of $x_{l-1,m-1}, x_{l-1,m+1}, x_{l+1,m-1}$.

It is not in general true that the variables parametrising exceptional planes can form their own independent lattice system in this way. For the remainder of this chapter, we use this approach to find several more examples of this type of transformation. This leads us to new reductions to discrete Painlevé equations.

5.1.1 Q3

Consider the ABS equation $\mathrm{Q3}_{\delta=0}$ on a generic quadrilateral with vertices x, u, v, y.

$$(\beta^2-\alpha^2)(x\,y+u\,v)+\beta(\alpha^2-1)(x\,u+v\,y)-\alpha(\beta^2-1)(x\,v+u\,y)=0. \tag{5.9}$$

Solving (5.9) for the vertex y, we find

$$y = \frac{\beta\,(\alpha^2-1)\,x\,u-\alpha\,(\beta^2-1)\,x\,v+(\beta^2-\alpha^2)\,u\,v}{\alpha\,(\beta^2-1)\,u-\beta\,(\alpha^2-1)\,v+(\alpha^2-\beta^2)\,x}. \tag{5.10}$$

In the compactified space of initial conditions $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ we find the base varieties b_1 and b_2, which are given in the first affine chart by

$$\alpha\,x-u=\beta\,x-v=0, \tag{b_1}$$

$$x - \alpha\, u = x - \beta\, v = 0. \tag{b_2}$$

Now, consider $Q3_{\delta=0}$ over the lattice with vertices labelled $x_{l,m}$ as in Figure 5.1, and consider the 3-dimensional space of initial conditions for the quadrilateral D. Both b_1 and b_2 are resolved after a single blow-up.

To resolve the base variety b_2, we perform the change of variables

$$\begin{cases} x^{(11)} &= x, \\ u^{(11)} &= \dfrac{x - \alpha\, u}{x - \beta\, v}, \\ v^{(11)} &= x - \beta\, v, \end{cases} \qquad \Longrightarrow \qquad \begin{cases} x &= x^{(11)}, \\ u &= \dfrac{x^{(11)} - u^{(11)}\, v^{(11)}}{\alpha}, \\ v &= \dfrac{x^{(11)} - v^{(11)}}{\beta}, \end{cases}$$

where the superscript represents the blow-up chart. As in the previous example, we wish to compute the value of $u^{(11)}$ parametrising the exceptional plane e_2 in terms of an earlier iteration. Therefore, we label each vertex on the dual graph with the variable

$$w_{l+1,m+1} := \frac{x_{l,m} - \alpha\, x_{l+1,m}}{x_{l,m} - \beta\, x_{l,m+1}}, \tag{5.11}$$

parametrising the exceptional plane e_2 in the corresponding resolved space. In terms of the 5-point staircase of initial conditions from Figure 5.1 we have

$$\begin{aligned} w_{l+1,m+1} =& ((\alpha^2 - 1)\,(x_{1,m} - \beta\, x_{l,m-1})\,(\beta\, x_{l,m} - \alpha\, x_{l+1,m-1}) \\ & \qquad ((\alpha^2 - \beta^2)\, x_{l-1,m} - \beta\,(\alpha^2 - 1)\, x_{l-1,m+1} + \alpha\,(\beta^2 - 1)\, x_{l,m})) \\ & /((\beta^2 - 1)\,(x_{l,m} - \alpha\, x_{l-1,m})\,(\alpha\, x_{l,m} - \beta\, x_{l-1,m+1}) \\ & \qquad (\beta\,(\alpha^2 - 1)\, x_{l,m} + (\beta^2 - \alpha^2)\, x_{l,m-1} - \alpha\,(\beta^2 - 1)\, x_{l+1,m-1})). \end{aligned} \tag{5.12}$$

We now ask if we can express this relation as a scalar lattice equation for $w_{l,m}$ alone.

For generic $x_{l,m-1}, x_{l+1,m-1}$, we find base varieties for (5.12) corresponding to the base variety b_2 for quads B and C. These occur due to the presence of $x_{l,m+1}$ and $x_{l+1,m}$ in the definition of $w_{l+1,m+1}$, (5.11). That is, (5.12) is undefined where

$$x_{l-1,m} - \alpha\, x_{l,m} = x_{l-1,m} - \beta\, x_{l-1,m+1} = 0, \tag{5.13a}$$

$$x_{l,m-1} - \alpha\, x_{l+1,m-1} = x_{l,m-1} - \beta\, x_{l,m} = 0. \tag{5.13b}$$

We wish to resolve these base varieties. Making the substitutions for $w_{l,m+1}$ and $w_{l+1,m}$ resulting from the blow-up of b_2 on the quads C and B respectively, we find

$$w_{l,m+1} := \frac{x_{l-1,m} - \alpha\, x_{l,m}}{x_{l-1,m} - \beta\, x_{l-1,m+1}} \implies x_{l-1,m+1} = \frac{x_{l-1,m}(w_{l,m+1} - 1) + \alpha\, x_{l,m}}{\beta\, w_{l,m+1}},$$
$$w_{l+1,m} := \frac{x_{l,m-1} - \alpha\, x_{l+1,m-1}}{x_{l,m-1} - \beta\, x_{l,m}} \implies x_{l+1,m-1} = \frac{x_{l,m-1} - x_{l,m-1}\, w_{l+1,m} + \beta\, x_{l,m}\, w_{l,m-1}}{\alpha}.$$

Substituting into (5.12), we find

$$w_{l+1,m+1} = \frac{\alpha^2 - 1}{\beta^2 - 1} \frac{x_{l+1,m+1} - \beta\, x_{l+1,m}}{x_{l+1,m+1} - \alpha\, x_{l,m+1}} \frac{(w_{l+1,m} - 1)\,(\alpha^2 - \beta^2\, w_{l,m+1} + w_{l,m+1} - 1)}{(w_{l,m+1} - 1)\,(\alpha^2 - \beta^2\, w_{l+1,m} + w_{l+1,m} - 1)}.$$

At this point it suffices to recognise from (5.9) that

$$w_{l,m} := \frac{x_{l,m} - \alpha\, x_{l+1,m}}{x_{l,m} - \beta\, x_{l,m+1}} = \frac{\alpha^2 - 1}{\beta^2 - 1} \frac{x_{l+1,m+1} - \beta\, x_{l+1,m}}{x_{l+1,m+1} - \alpha\, x_{l,m+1}},$$

and hence

$$w_{l+1,m+1} = w_{l,m} \frac{(w_{l+1,m} - 1)\,(\alpha^2 - 1 - (\beta^2 - 1)\, w_{l,m+1})}{(w_{l,m+1} - 1)\,(\alpha^2 - 1 - (\beta^2 - 1)\, w_{l+1,m})}. \tag{5.14}$$

Finally, defining the parameter

$$r := \frac{\alpha^2 - 1}{\beta^2 - 1},$$

we find the lattice equation for $w_{l,m}$

$$w_{l+1,m+1} = w_{l,m} \frac{(w_{l+1,m} - 1)\,(w_{l,m+1} - r)}{(w_{l,m+1} - 1)\,(w_{l+1,m} - r)}. \tag{5.15}$$

Since each $w_{l,m}$ was defined in a consistent way, we have found a transformation from $\mathrm{Q3}_{\delta=0}$ (5.9) to a new lattice equation (5.15) for $w_{l,m}$, where

$$w_{l+1,m+1} = \frac{x_{l,m} - \alpha\, x_{l+1,m}}{x_{l,m} - \beta\, x_{l,m+1}}.$$

Following this same procedure using the base variety b_1 leads us to introduce the dependent variable $z_{l,m}$, where

$$z_{l+1,m+1} = \frac{\alpha\, x_{l,m} - x_{l+1,m}}{\beta\, x_{l,m} - x_{l,m+1}}. \tag{5.16}$$

If $x_{l,m}$ follows $\mathrm{Q3}_{\delta=0}$ (5.9) then we find $z_{l,m}$ obeys the lattice equation

$$z_{l+1,m+1} = z_{l,m} \frac{(z_{l+1,m} - \frac{1}{r_1})(z_{l,m+1} - r_1\, r_2)}{(z_{l,m+1} - \frac{1}{r_1})(z_{l+1,m} - r_1\, r_2)}, \tag{5.17}$$

where the parameters r_1 and r_2 are defined as

$$r_1 = \frac{\alpha}{\beta}, \quad r_2 = \frac{\alpha^2 - 1}{\beta^2 - 1}.$$

Note that both (5.15), (5.17) are equations of the form

$$w_{l+1,m+1} = w_{l,m}\frac{(w_{l+1,m} + a)(w_{l,m+1} + b)}{(w_{l+1,m} + b)(w_{l,m+1} + a)}, \tag{5.18}$$

for some constants a, b. This equation is clearly not fully resolved. Taking $(w_{l,m}, w_{l+1,m}, w_{l,m+1})$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ we find 6 base varieties, as $w_{l+1,m+1}$ is undefined when

$$w_{l,m} = w_{l+1,m} - b = 0, \tag{b_1}$$

$$w_{l,m} = w_{l,m+1} - a = 0, \tag{b_2}$$

$$w_{l+1,m} - a = w_{l,m+1} - a = 0, \tag{b_3}$$

$$w_{l+1,m} - b = w_{l,m+1} - b = 0, \tag{b_4}$$

$$W_{l,m} = w_{l+1,m} - a = 0, \tag{b_5}$$

$$W_{l,m} = w_{l,m+1} - b = 0, \tag{b_6}$$

where $W_{l,m} = 1/w_{l,m}$.

As in the case of (5.8), this transformation has broken the symmetry of the square in the horizontal and vertical directions. The symmetry along the diagonal axes is preserved, this equation is left with the symmetry group of a parallelogram.

In [71] a Lax pair was found for (5.18), and it was associated with the lattice mKdV equation $H3_{\delta=0}$ by taking the dependent variable $w_{l,m}$ to be the ratio of the vertices $x_{l,m}$ over diagonals on the lattice ($w_{l,m} = x_{l,m+1}/x_{l+1,m}$). Selected details are provided in Appendix A. Using the method outlined in this chapter using the base varieties b_1 or b_4 for $H3_{\delta=0}$ also yields this transformation.

Recall from Chapter 3 that the ABS equations were classified up to Möbius transformation by some parameter independent polynomial $r(x)$. In the case where $\delta = 0$, the polynomials corresponding to $H3_\delta$ and $Q3_\delta$ coincide, so that $r(x) = x^2$. However, the common characteristic polynomial $r(x)$ is found to not be necessary to give a transformation to a common equation.

For the remainder of this section, we demonstrate several more examples of transformations of lattice equations to (5.18) using this approach. Later in this chapter we show that (5.18) has a reduction to a discrete Painlevé equation often called $q\mathrm{P}_{\mathrm{VI}}$, with surface type $A_3^{(1)}$, thus relating the solution of $q\mathrm{P}_{\mathrm{VI}}$ to several lattice equations.

5.1.2 Q1

Now, consider the ABS equation $\mathrm{Q1}_\delta$. This is characterised by the polynomial $r(x) = 1$ and is given by

$$\alpha\,(x-v)\,(u-y) - \beta\,(x-u)\,(v-y) + \delta^2 \alpha\,\beta\,(\alpha-\beta) = 0. \tag{5.19}$$

Solving (5.19) for the vertex y we find

$$y = \frac{x\,(\alpha\,u - \beta\,v) - (\alpha-\beta)\,u\,v + \delta^2\alpha\,\beta\,(\alpha-\beta)}{(\alpha-\beta)\,x + \beta\,u - \alpha\,v}. \tag{5.20}$$

Recall that in the resolution of this equation we changed chart to where x, u, v become unbounded, giving

$$y = \frac{\alpha\,V - \beta\,U - (\alpha-\beta)\,X + \delta^2\alpha\,\beta\,(\alpha-\beta)\,X\,U\,V}{(\alpha-\beta)\,U\,V + \beta\,X\,V - \alpha\,X\,U}. \tag{5.21}$$

We then blew up at the origin in this chart where the two base lines b_1 and b_2 intersect, and subsequently blew up b_1 and b_2 in this blown up space. Depending on which we choose to resolve first, we find (in terms of X, U, V) the variables parametrising the exceptional planes e_1 and e_2 resulting from the blow-ups of b_1 and b_2 are

$$v^{(11)} = \frac{V\,(X - U - \alpha\,\delta\,X\,U)}{U\,(X - V - \beta\,\delta\,X\,V)}, \tag{5.22a}$$

$$v^{(21)} = \frac{V\,(X - U + \alpha\,\delta\,X\,U)}{U\,(X - V + \beta\,\delta\,X\,V)}. \tag{5.22b}$$

Rewriting these in terms of the original coordinates x, u, v, we find

$$v^{(11)} = \frac{x - u + \alpha\,\delta}{x - v + \beta\,\delta}, \tag{5.23a}$$

$$v^{(21)} = \frac{x - u - \alpha\,\delta}{x - v - \beta\,\delta}, \tag{5.23b}$$

as if we had performed the resolutions in the original (x, u, v) chart.

Considering this equation over a lattice $x_{l,m}$, label the vertices of the dual graph with the variables parametrising e_1 and e_2 for the corresponding quad. That is, label the dual graph with $u_{l,m}$, $v_{l,m}$, such that

$$u_{l+1,m+1} = \frac{x_{l,m} - x_{l+1,m} + \alpha\,\delta}{x_{l,m} - x_{l,m+1} + \beta\,\delta}, \tag{5.24a}$$

$$v_{l+1,m+1} = \frac{x_{l,m} - x_{l+1,m} - \alpha\,\delta}{x_{l,m} - x_{l,m+1} - \beta\,\delta}. \tag{5.24b}$$

In the case $\delta = 0$, $u_{l,m}$ and $v_{l,m}$ coincide.

Calculating $u_{l+1,m+1}$ and $v_{l+1,m+1}$ in terms of the 5-point staircase from Figure 5.1, it is straightforward to show using (5.19) that

$$u_{l+1,m+1} = u_{l,m}\frac{(u_{l+1,m} - 1)(u_{l,m+1} - \frac{\alpha}{\beta})}{(u_{l+1,m} - \frac{\alpha}{\beta})(u_{l,m+1} - 1)}, \tag{5.25a}$$

$$v_{l+1,m+1} = v_{l,m}\frac{(v_{l+1,m} - 1)(v_{l,m+1} - \frac{\alpha}{\beta})}{(v_{l+1,m} - \frac{\alpha}{\beta})(v_{l,m+1} - 1)}. \tag{5.25b}$$

These equations are both particular forms of (5.18). Note the equations governing $u_{l,m}$ and $v_{l,m}$ are identical.

5.1.3 A1

Consider the ABS equation A1$_\delta$, given by

$$\alpha\,(x+v)\,(u+y) - \beta\,(x+u)\,(v+y) - \delta^2\alpha\,\beta\,(\alpha-\beta) = 0, \tag{5.26}$$

or solving for the vertex y,

$$y = \frac{\alpha\,u\,(x+v) - \beta\,v\,(x+u) - \delta^2\alpha\,\beta(\alpha-\beta)}{\beta(x+u) - \alpha(x+v)}. \tag{5.27}$$

We find two base curves for this equation in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. These are the base lines b_1 and b_2, given by

$$x + u + \alpha\,\delta = x + v + \beta\,\delta = 0, \tag{b_1}$$

$$x + u - \alpha\,\delta = x + v - \beta\,\delta = 0. \tag{b_2}$$

Supposing that a lattice with vertices labelled $x_{l,m}$ obeys (5.26), label the vertices of the dual graph with the variables $u_{l,m}$ and $v_{l,m}$, such that

$$u_{l+1,m+1} = \frac{x_{l,m} + x_{l+1,m} + \alpha\,\delta}{x_{l,m} + x_{l,m+1} + \beta\,\delta}, \tag{5.28a}$$

$$v_{l+1,m+1} = \frac{x_{l,m} + x_{l+1,m} - \alpha\,\delta}{x_{l,m} + x_{l,m+1} - \beta\,\delta}, \tag{5.28b}$$

so that $u_{l,m}$ and $v_{l,m}$ parametrise the exceptional planes e_1 and e_2 respectively in the corresponding resolved space $\mathcal{X}_{l,m}$ for each quad.

Unlike in previous examples, we do not find separate lattice equations between exceptional planes for $u_{l,m}$ and $v_{l,m}$. However, we are able to find a *system* of equations for $u_{l,m}, v_{l,m}$.

In terms of the 5-point staircase Figure 5.1, $u_{l+1,m+1}$ is given by

$$\begin{aligned} u_{l+1,m+1} = {} & (\alpha\,(\beta\,\delta + x_{l,m-1} + x_{l,m})\,(x_{l-1,m}\,(\alpha-\beta) + \alpha\,x_{l-1,m+1} - \beta\,x_{l,m}) \\ & (\delta(\alpha-\beta) + x_{l,m} - x_{l+1,m-1}))\,/ \\ & (\beta(\alpha\,\delta + x_{l-1,m} + x_{l,m})(\delta(\beta-\alpha) - x_{l-1,m+1} + x_{l,m}) \\ & (x_{l,m-1}\,(\beta-\alpha) - \alpha\,x_{l,m} + \beta\,x_{l+1,m-1})). \end{aligned} \tag{5.29}$$

This expression is indeterminate along both b_1 and b_2 for the quads B and C. If we now make the substitutions for $x_{l-1,m+1}$ and $x_{l+1,m-1}$ in terms of $v_{l+1,m}$ and $v_{l,m+1}$ from the expressions

$$v_{l,m+1} := \frac{x_{l-1,m} + x_{l,m} - \alpha\,\delta}{x_{l-1,m} + x_{l-1,m+1} - \beta\,\delta}, \qquad v_{l+1,m} := \frac{x_{l,m-1} + x_{l+1,m-1} - \alpha\,\delta}{x_{l,m-1} + x_{l,m} - \beta\,\delta},$$

we find

$$\begin{aligned} u_{l+1,m+1} &= \frac{(x_{l,m} + x_{l,m-1} + \beta\,\delta)\,(v_{l+1,m} - 1)\,(\alpha - \beta\,v_{l,m+1})}{(x_{l,m} + x_{l-1,m} + \alpha\,\delta)\,(v_{l,m+1} - 1)\,(\alpha - \beta\,v_{l+1,m})}, \\ &= \frac{(x_{l-1,m-1} + x_{l,m-1} + \alpha\,\delta)\,(v_{l+1,m} - 1)\,(\alpha - \beta\,v_{l,m+1})}{(x_{l-1,m-1} + x_{l-1,m} + \beta\,\delta)\,(v_{l,m+1} - 1)\,(\alpha - \beta\,v_{l+1,m})}, \qquad (5.30) \\ &= u_{l,m}\frac{(v_{l+1,m} - 1)\,(\alpha - \beta\,v_{l,m+1})}{(v_{l,m+1} - 1)\,(\alpha - \beta\,v_{l+1,m})}. \qquad (5.31) \end{aligned}$$

Performing the same calculation for $v_{l+1,m+1}$ we find the system for $u_{l,m}$, $v_{l,m}$

$$u_{l+1,m+1} = u_{l,m}\frac{(v_{l+1,m} - 1)\,(\beta\,v_{l,m+1} - \alpha)}{(v_{l,m+1} - 1)\,(\beta\,v_{l+1,m} - \alpha)}, \tag{5.32a}$$

$$v_{l+1,m+1} = v_{l,m}\frac{(u_{l+1,m} - 1)\,(\beta\,u_{l,m+1} - \alpha)}{(u_{l,m+1} - 1)\,(\beta\,u_{l+1,m} - \alpha)}. \tag{5.32b}$$

Alternatively, recognising that

$$\frac{\beta\,v_{l,m+1} - \alpha}{\beta\,v_{l+1,m} - \alpha} = \frac{v_{l,m}\,v_{l,m+1}}{u_{l,m}\,u_{l,m+1}}\frac{\beta\,u_{l,m+1} - \alpha}{\beta\,u_{l+1,m} - \alpha}, \tag{5.33}$$

we find

$$u_{l+1,m+1} = v_{l,m}\frac{(v_{l+1,m}-1)\,(\alpha\,u_{l,m+1}^{-1}-\beta)}{(v_{l,m+1}^{-1}-1)\,(\beta\,u_{l+1,m}-\alpha)}, \tag{5.34a}$$

$$v_{l+1,m+1} = u_{l,m}\frac{(u_{l+1,m}-1)\,(\alpha\,v_{l,m+1}^{-1}-\beta)}{(u_{l,m+1}^{-1}-1)\,(\beta\,v_{l+1,m}-\alpha)}. \tag{5.34b}$$

Note that in the case where $u_{l,m} = v_{l,m}$, (5.32) and (5.34) both reduce to

$$u_{l+1,m+1} = u_{l,m}\frac{(u_{l+1,m}-1)\,(u_{l,m+1}-r)}{(u_{l,m+1}-1)\,(u_{l+1,m}-r)}, \tag{5.35}$$

where $r = \alpha/\beta$. This is again a form of (5.18). This may be achieved either by choosing initial conditions such that $u_{l,m} = v_{l,m}$, or by recognising that in the case $\delta = 0$, then $u_{l,m} \equiv v_{l,m}$.

5.2 Reduction of ABS Equations

In the previous section we found several transformations of ABS systems to (5.18). In this section we show that this equation possesses a reduction to an ordinary difference equation with surface type $A_3^{(1)}$, which can be deautonomised to (depending on the choice of parameters) the equation $q\mathrm{P}_{\mathrm{III}}$ or $q\mathrm{P}_{\mathrm{VI}}$.

If both the parameters a, b vanish, then the equation becomes the trivial $w_{l+1,m+1} = w_{l,m}$. Therefore, assuming non-zero a, under the trivial scaling $w_{l,m} \mapsto a\,w_{l,m}$ we find

$$w_{l+1,m+1} = w_{l,m}\frac{(w_{l+1,m}+1)\,(w_{l,m+1}+\frac{b}{a})}{(w_{l+1,m}+\frac{b}{a})\,(w_{l,m+1}+1)}. \tag{5.36}$$

Defining a new parameter $\gamma := b/a$ then we have the single parameter equation

$$w_{l+1,m+1} = w_{l,m}\frac{(w_{l+1,m}+1)\,(w_{l,m+1}+\gamma)}{(w_{l+1,m}+\gamma)\,(w_{l,m+1}+1)}. \tag{5.37}$$

Proposition 5.1. *The equation* (5.37) *possesses a periodic reduction to an ordinary difference equation with surface type* $A_3^{(1)}$.

Proof. Take some nonvanishing variable k, and assume that the solution $w_{l,m}$ has a periodicity such that

$$w_{l,m+1} = \frac{k}{w_{l+1,m}}. \tag{5.38}$$

Substituting the condition (5.38) into (5.37), we find

$$\begin{aligned} w_{l+1,m+1} &= w_{l,m}\frac{(w_{l+1,m}+1)\,(w_{l,m+1}+\gamma)}{(w_{l+1,m}+\gamma)\,(w_{l,m+1}+1)}, \\ \frac{k}{w_{l+2,m}} &= w_{l,m}\frac{(w_{l+1,m}+1)\,(\frac{k}{w_{l+1,m}}+\gamma)}{(w_{l+1,m}+\gamma)\,(\frac{k}{w_{l+1,m}}+1)}, \\ \frac{w_{l+2,m}}{k} &= \frac{1}{w_{l,m}}\frac{(w_{l+1,m}+\gamma)\,(k+w_{l+1,m})}{(w_{l+1,m}+1)\,(k+\gamma\, w_{l+1,m})}, \\ w_{l+2,m} &= \frac{k}{\gamma\, w_{l,m}}\frac{(w_{l+1,m}+\gamma)\,(w_{l+1,m}+k)}{(w_{l+1,m}+1)\,(w_{l+1,m}+k/\gamma)}. \end{aligned} \tag{5.39}$$

Since this equation relates only vertices on the lattice with the same m coordinate, we can consider (5.39) as an ordinary difference equation for the variable $y_n = w_{l+1,m}$, where $n = l+1$. Making this substitution we find

$$y_{n+1}\,y_{n-1} = \frac{k}{\gamma}\frac{(y_n+\gamma)\,(y_n+k)}{(y_n+1)\,(y_n+k/\gamma)}. \tag{5.40}$$

This is an ordinary difference equation for y_n of the form (2.3) on which we performed resolution of singularities in Chapter 2. We saw that it has a resolved space of initial conditions of type $A_3^{(1)}$. This equation can be deautonomised to $q\mathrm{P}_{\mathrm{III}}$ [38] or $q\mathrm{P}_{\mathrm{VI}}$ [92]. Thus, we have found reductions from the equations $\mathrm{H3}_{\delta=0}$, $\mathrm{Q3}_{\delta=0}$, $\mathrm{Q1}_{\delta}$, and $\mathrm{A1}_{\delta}$ to the difference equation (5.40). □

These reductions are highly nontrivial. For example, consider this reduction in the case $\mathrm{Q3}_{\delta=0}$ (5.9). We have a lattice with vertices labelled $x_{l,m}$. Following Proposition 5.1, we define a new dependent variable $z_{l,m}$ by

$$z_{l+1,m+1} = -\frac{\beta\,(\alpha\, x_{l,m} - x_{l+1,m})}{\alpha\,(\beta\, x_{l,m} - x_{l,m+1})}. \tag{5.41}$$

This obeys the lattice equation

$$z_{l+1,m+1} = z_{l,m}\frac{(z_{l+1,m}+1)(z_{l,m+1}+\gamma)}{(z_{l,m+1}+1)(z_{l+1,m}+\gamma)}, \tag{5.42}$$

where

$$\gamma = \frac{\beta^2\,(\alpha^2-1)}{\alpha^2\,(\beta^2-1)}.$$

Using (5.38) and (5.41), we perform the reduction by assuming that for some nonzero k, the solution $x_{l,m}$ satisfies a periodic condition such that

$$\frac{\beta\,(\alpha\, x_{l-1,m} - x_{l,m})}{\alpha\,(\beta\, x_{l-1,m} - x_{l-1,m+1})} = k\left(\frac{\alpha\,(\beta\, x_{l,m-1} - x_{l,m})}{\beta\,(\alpha\, x_{l,m-1} - x_{l+1,m-1})}\right)$$

otherwise written

$$k\,\alpha^2(\beta\,x_{l,m-1} - x_{l,m})(\beta\,x_{l-1,m} - x_{l-1,m+1}) = \beta^2(\alpha\,x_{l-1,m} - x_{l,m})(\alpha\,x_{l,m-1} - x_{l+1,m-1}).$$

By Proposition 5.1, this periodicity allows us to define the variable y_n

$$y_n = -\frac{\beta\,(\alpha\,x_{l,m} - x_{l+1,m})}{\alpha\,(\beta\,x_{l,m} - x_{l,m+1})}, \tag{5.43}$$

where $n = l + 1$, so that y_n satisfies

$$y_{n+1}\,y_{n-1} = \frac{k}{\gamma}\frac{(y_n+\gamma)\,(y_n+k)}{(y_n+1)\,(y_n+k/\gamma)}. \tag{5.44}$$

5.3 A Cluster Algebra Associated with ABS Equations

In [81], by performing a (0,0,2)-reduction of a quiver associated with a cluster algebra whose cluster variables satisfy the Hirota-Miwa equation, a quiver Q_{mKdV} was found which generates a cluster algebra $\mathcal{A}(Q_{\mathrm{mKdV}}, \mathbf{x})$ whose cluster variables $\mathbf{x}$ satisfy the bilinear form of the discrete mKdV (5.45), see Figure 5.3.

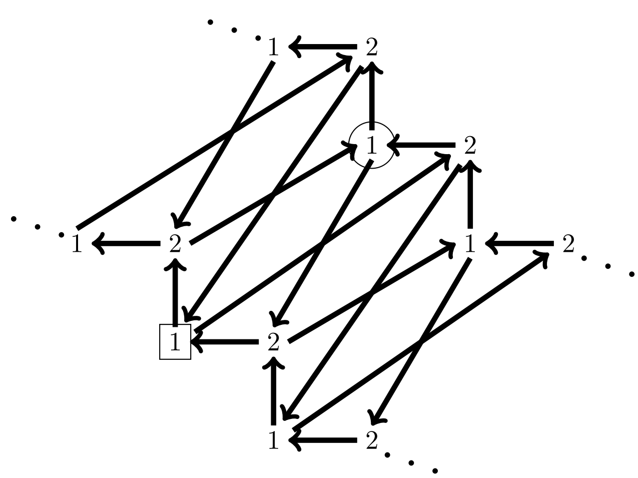

Fig. 5.3: The quiver Q_{mKdV}

The correspondence between the cluster variables $\mathbf{x}$ and the solution of the discrete mKdV is found in the following way:

Label the vertices of the bottom left staircase of Figure 5.3 such that the vertex marked with a box is labelled with $x_{0,0}$, the vertex immediately to the right of it with $x_{1,0}$, the vertex above it with $x_{0,1}$, and so on. Similarly, label the vertex marked with a circle with $w_{0,0}$, and continue with the top right staircase in the same way. Let μ'_i denote the sequential mutation of all vertices marked with an i.

Mutating in the order $\mu'_1, \mu'_2, \mu'_1, \mu'_2 \ldots$, denote $x_{l+1,m+1}$ and $w_{l+1,m+1}$ as the new cluster variables obtained by mutating at $w_{l,m}$ and $x_{l,m}$, respectively. The cluster variables $x_{l,m}$, $w_{l,m}$ satisfy the system of equations

$$w_{l+1,m+1}\, x_{l,m} = x_{l+1,m}\, w_{l,m+1} + w_{l+1,m}\, x_{l,m+1}, \tag{5.45a}$$

$$x_{l+1,m+1}\, w_{l,m} = w_{l+1,m}\, x_{l,m+1} + x_{l+1,m}\, w_{l,m+1}. \tag{5.45b}$$

Nonautonomous equations can be found from cluster algebras with coefficients. Letting $\mathbf{y}$ be the coefficients and labelling the quiver Q_{mKdV} with cluster variables as in the coefficient free case, we find that for the cluster algebra $\mathcal{A}(Q_{\mathrm{mKdV}}, \mathbf{x}, \mathbf{y})$, the cluster variables satisfy

$$w_{l+1,m+1}\, x_{l,m} = a_{l,m}\, x_{l,m+1}\, w_{l+1,m} + b_{l,m}\, w_{l,m+1}\, x_{l+1,m}, \tag{5.46a}$$

$$x_{l+1,m+1}\, w_{l,m} = c_{l,m}\, w_{l,m+1}\, x_{l+1,m} + d_{l,m}\, x_{l,m+1}\, w_{l+1,m}, \tag{5.46b}$$

where $a_{l,m}, b_{l,m}, c_{l,m}, d_{l,m}$ are rational functions of the coefficients $\mathbf{y}$ such that $a_{l,m} + b_{l,m} = c_{l,m} + d_{l,m} = 1$ holds.

It was shown in [81] that the quiver obtained by performing a (2,-2)–reduction to the quiver Q_{mKdV} generates a cluster algebra whose *coefficients* satisfy a nonautonomous form of (5.40), the form of $q\mathrm{P_{VI}}$ found in Proposition 5.1 as a reduction of (5.37). A healthy curiosity leads us to sarch for a labelling of $\mathbf{y}$ such that we can draw a connection between the coefficients of the cluster algebra $\mathcal{A}(Q_{\mathrm{mKdV}}, \mathbf{x}, \mathbf{y})$ and the equation (5.37), which we obtained as a transformation of different ABS equations.

5.3.1 Labelling of the Coefficients $\mathbf{y}$

Analogous to the procedure used to find (5.45), in order to find a correspondence between the coefficients of the cluster algebra $\mathcal{A}(Q_{\mathrm{mKdV}}, \mathbf{x}, \mathbf{y})$ and the solution of (5.37), we must name the elements of $\mathbf{y}$ in such a way that they can be identified as iterates of (5.37).

To vertices of Q_{mKdV} labelled with an $x_{l,m}$, assign the coefficient $u_{l,m}$ if the vertex is marked with a 1 in Figure 5.3, or $\bar{u}_{l,m}$ otherwise. Similarly, to the vertices labelled with a $w_{l,m}$, assign the coefficient $v_{l,m}$ to the vertices marked with a 1, or $\bar{v}_{l,m}$ otherwise.

Upon mutating at all vertices marked with a 1, the new coefficients are labelled $v_{l,m} \to \bar{u}_{l+1,m+1}$, $u_{l,m} \to \bar{v}_{l+1,m+1}$ if the coefficient corresponds to a vertex being mutated, and $\bar{u}_{l,m} \to u_{l,m}$, $\bar{v}_{l,m} \to v_{l,m}$ otherwise.

By the definition of the action of mutation on coefficients (1.14), we find that the coefficients $\mathbf{y}$ satisfy the following system of equations

$$\bar{v}_{l+1,m+1} = \frac{1}{u_{l,m}}, \tag{5.47a}$$

$$u_{l+1,m} = \bar{u}_{l+1,m} \frac{(u_{l,m}+1)(v_{l+1,m-1}+1)}{(u_{l+1,m-1}^{-1}+1)(v_{l,m}^{-1}+1)}, \tag{5.47b}$$

$$\bar{u}_{l+1,m+1} = \frac{1}{v_{l,m}}, \tag{5.47c}$$

$$v_{l+1,m} = \bar{v}_{l+1,m} \frac{(v_{l,m}+1)(u_{l+1,m-1}+1)}{(v_{l+1,m-1}^{-1}+1)(u_{l,m}^{-1}+1)}. \tag{5.47d}$$

Using (5.47a) and (5.47c) to make substitutions into (5.47d) and (5.47b) respectively, we have the following system of equations for $u_{l,m}$, $v_{l,m}$,

$$u_{l+1,m} = \frac{1}{v_{l,m-1}} \frac{(u_{l,m}+1)(v_{l+1,m-1}+1)}{(u_{l+1,m-1}^{-1}+1)(v_{l,m}^{-1}+1)}, \tag{5.48a}$$

$$v_{l+1,m} = \frac{1}{u_{l,m-1}} \frac{(v_{l,m}+1)(u_{l+1,m-1}+1)}{(v_{l+1,m-1}^{-1}+1)(u_{l,m}^{-1}+1)}. \tag{5.48b}$$

Proposition 5.2. *Consider the cluster algebra $\boldsymbol{\mathcal{A}}(Q_{\mathrm{mKdV}}, \boldsymbol{x}, \boldsymbol{y})$. For certain choices of $v_{l,m}$ in the initial seed, then for some parameter γ both $u_{l,m}$ and $v_{l,m}$ satisfy*

$$u_{l+1,m+1} = u_{l,m} \frac{(u_{l+1,m}+\gamma)(u_{l,m+1}+1)}{(u_{l+1,m}+1)(u_{l,m+1}+\gamma)}, \tag{5.49a}$$

$$v_{l+1,m+1} = v_{l,m} \frac{(v_{l+1,m} + \gamma)\,(v_{l,m+1} + 1)}{(v_{l+1,m} + 1)\,(v_{l,m+1} + \gamma)}. \tag{5.49b}$$

Therefore the coefficients $\mathbf{y}$ can satisfy (5.37).

Proof. Rewriting (5.48), we obtain

$$u_{l+1,m} = \frac{u_{l+1,m-1}\, v_{l,m}}{v_{l,m-1}} \frac{(u_{l,m} + 1)\,(v_{l+1,m-1} + 1)}{(u_{l+1,m-1} + 1)\,(v_{l,m} + 1)}, \tag{5.50a}$$

$$v_{l+1,m} = \frac{u_{l,m}\, v_{l+1,m-1}}{u_{l,m-1}} \frac{(v_{l,m} + 1)\,(u_{l+1,m-1} + 1)}{(v_{l+1,m-1} + 1)\,(u_{l,m} + 1)}. \tag{5.50b}$$

Multiplying (5.50a) and (5.50b) together, we find $u_{l,m}$ and $v_{l,m}$ are related by

$$\frac{u_{l+1,m}}{u_{l+1,m-1}} \frac{u_{l,m-1}}{u_{l,m}} = \frac{v_{l+1,m-1}}{v_{l+1,m}} \frac{v_{l,m}}{v_{l,m-1}}.$$

Rearranging, we can define functions in terms of $x_{l,m}$ which are constant with respect to m and l, respectively

$$\lambda(l) := \frac{u_{l+1,m}}{u_{l,m}} \frac{v_{l+1,m}}{v_{l,m}} = \frac{u_{l+1,m-1}}{u_{l,m-1}} \frac{v_{l+1,m-1}}{v_{l,m-1}}, \tag{5.51a}$$

$$\mu(m) := \frac{u_{l+1,m}}{u_{l+1,m-1}} \frac{v_{l+1,m}}{v_{l+1,m+1}} = \frac{u_{l,m}}{u_{l,m-1}} \frac{v_{l,m}}{v_{l,m-1}}. \tag{5.51b}$$

Defining $\gamma_{l,m} := u_{l,m}\, v_{l,m}$, then

$$\frac{\gamma_{l+1,m}}{\gamma_{l,m}} = \lambda(l), \quad \frac{\gamma_{l,m}}{\gamma_{l,m-1}} = \mu(m), \tag{5.52}$$

and hence $\gamma_{l,m} =: L(l)\, M(m)$ for some $L(l)$, $M(m)$. Using (5.50a) and the definition of the non-autonomous parameter $\gamma_{l,m} = u_{l,m}\, v_{l,m}$, we find,

$$\begin{aligned}
u_{l+1,m+1} &= \frac{u_{l+1,m}\, v_{l,m+1}}{v_{l,m}} \frac{(u_{l,m+1} + 1)\,(v_{l+1,m} + 1)}{(u_{l+1,m} + 1)\,(v_{l,m+1} + 1)}, \\
&= \frac{u_{l+1,m}\, \frac{\gamma_{l,m+1}}{u_{l,m+1}}}{\frac{\gamma_{l,m}}{u_{l,m}}} \frac{(u_{l,m+1} + 1)\,(\frac{\gamma_{l+1,m}}{u_{l+1,m}} + 1)}{(u_{l+1,m} + 1)\,(\frac{\gamma_{l,m+1}}{u_{l,m+1}} + 1)}, \\
&= \frac{\gamma_{l,m+1}}{\gamma_{l,m}} \frac{u_{l,m}\, u_{l+1,m}}{u_{l,m+1}} \frac{(u_{l,m+1} + 1)\,(\frac{\gamma_{l+1,m}}{u_{l+1,m}} + 1)}{(u_{l+1,m} + 1)\,(\frac{\gamma_{l,m+1}}{u_{l,m+1}} + 1)}, \\
&= \mu(m+1)\, u_{l,m} \frac{(u_{l,m+1} + 1)\,(u_{l+1,m} + \gamma_{l+1,m})}{(u_{l+1,m} + 1)\,(u_{l,m+1} + \gamma_{l,m+1})},
\end{aligned} \tag{5.53}$$

and similarly,

$$v_{l+1,m+1} = \mu(m+1)\, v_{l,m} \frac{(v_{l,m+1} + 1)\,(v_{l+1,m} + \gamma_{l+1,m})}{(v_{l+1,m} + 1)\,(v_{l,m+1} + \gamma_{l,m+1})}. \tag{5.54}$$

Equations (5.53) and (5.54) are deautonomised versions of (5.37).

If an initial seed is chosen such that $\lambda(l) \equiv \mu(m) \equiv 1$, then γ is constant and hence

$$u_{l+1,m+1} = u_{l,m} \frac{(u_{l,m+1}+1)\,(u_{l+1,m}+\gamma)}{(u_{l+1,m}+1)\,(u_{l,m+1}+\gamma)}, \tag{5.55a}$$

$$v_{l+1,m+1} = v_{l,m} \frac{(v_{l,m+1}+1)\,(v_{l+1,m}+\gamma)}{(v_{l+1,m}+1)\,(v_{l,m+1}+\gamma)}, \tag{5.55b}$$

which are of the desired form (5.49). □

Finally, we present a pleasing corollary connecting the integrability of the system (5.46) to the solutions of ABS equations via the Laurent property.

Corollary 5.1. *Consider the bilinear form of the discrete mKdV with coefficients, given by*

$$w_{l+1,m+1}\, x_{l,m} = a_{l,m}\, x_{l,m+1}\, w_{l+1,m} + b_{l,m}\, w_{l,m+1}\, x_{l+1,m}, \tag{5.56a}$$

$$x_{l+1,m+1}\, w_{l,m} = c_{l,m}\, w_{l,m+1}\, x_{l+1,m} + d_{l,m}\, x_{l,m+1}\, w_{l+1,m}. \tag{5.56b}$$

If $a_{l,m}$ $b_{l,m}$, $c_{l,m}$, $d_{l,m}$ are certain rational functions of solutions of $A1_\delta$, $H3_{\delta=0}$, $Q1_\delta$, or $Q3_{\delta=0}$, the system (5.56) *exhibits the Laurent property.*

Proof. By the definition of mutation, the cluster algebra $\boldsymbol{\mathcal{A}}(Q_{\mathrm{mKdV}}, \boldsymbol{x}, \boldsymbol{y})$ yields the system

$$w_{l+1,m+1}\, x_{l,m} = \frac{1}{1+u_{l,m}} \left(w_{l+1,m}\, x_{l,m+1} + u_{l,m}\, x_{l+1,m}\, w_{l,m+1}\right), \tag{5.57a}$$

$$x_{l+1,m+1}\, w_{l,m} = \frac{1}{1+v_{l,m}} \left(x_{l+1,m}\, w_{l,m+1} + v_{l,m}\, w_{l+1,m}\, x_{l,m+1}\right). \tag{5.57b}$$

The coefficients $a_{l,m}$ $b_{l,m}$, $c_{l,m}$, $d_{l,m}$ from (5.56) are therefore given by

$$a_{l,m} = \frac{1}{1+u_{l,m}}, \tag{5.58a}$$

$$b_{l,m} = \frac{u_{l,m}}{1+u_{l,m}}, \tag{5.58b}$$

$$c_{l,m} = \frac{1}{1+v_{l,m}}, \tag{5.58c}$$

$$d_{l,m} = \frac{v_{l,m}}{1+v_{l,m}}. \tag{5.58d}$$

From Proposition 5.2, we know that choosing an initial seed such that for some constant γ we have $u_{l,m} = \gamma / v_{l,m}$, then $u_{l,m}$ and $v_{l,m}$ satisfy

$$u_{l+1,m+1} = u_{l,m} \frac{(u_{l,m+1}+1)\,(u_{l+1,m}+\gamma)}{(u_{l+1,m}+1)\,(u_{l,m+1}+\gamma)}, \tag{5.59a}$$

$$v_{l+1,m+1} = v_{l,m} \frac{(v_{l,m+1}+1)\,(v_{l+1,m}+\gamma)}{(v_{l+1,m}+1)\,(v_{l,m+1}+\gamma)}. \tag{5.59b}$$

We prove Corollary 5.1 first for $Q3_{\delta=0}$.

Under the trivial scaling $u_{l,m} = -\gamma\, y_{l,m}$, we have

$$y_{l+1,m+1} = y_{l,m} \frac{(y_{l,m+1}-1/\gamma)\,(y_{l+1,m}-1)}{(y_{l+1,m}-1/\gamma)\,(y_{l,m+1}-1)}. \tag{5.60}$$

From Chapter 5, we know that if $\zeta_{l,m}$ satisfies $Q3_{\delta=0}$ (5.9) with α, β such that

$$\gamma = \frac{\beta^2-1}{\alpha^2-1},$$

then (5.60) is satisfied by

$$y_{l,m} = \frac{\zeta_{l,m} - \alpha\,\zeta_{l+1,m}}{\zeta_{l,m} - \beta\,\zeta_{l,m+1}}.$$

Thus, (5.59) is satisfied by

$$u_{l,m} = -\gamma\, y_{l,m} = -\frac{\beta^2-1}{\alpha^2-1}\frac{\zeta_{l,m} - \alpha\,\zeta_{l+1,m}}{\zeta_{l,m} - \beta\,\zeta_{l,m+1}},, \tag{5.61a}$$

$$v_{l,m} = \frac{\gamma}{u_{l,m}} = \frac{-1}{y_{l,m}} = -\frac{\zeta_{l,m} - \beta\,\zeta_{l,m+1}}{\zeta_{l,m} - \alpha\,\zeta_{l+1,m}}. \tag{5.61b}$$

Substituting (5.61) into (5.58), we have found coefficients $a_{l,m}, b_{l,m}, c_{l,m}, d_{l,m}$ as rational functions of solutions of $Q3_{\delta=0}$ such that (5.56) exhibits the Laurent property.

Following the same reasoning, we can construct similar functions of the solutions of $A1_\delta$, $H3_{\delta=0}$, and $Q1_\delta$. □

Chapter 6: Conclusion

In this thesis, we extended the geometric description of initial value spaces for integrable systems to multiple dimensions. This approach was pioneered by Okamoto [79] for the Painlevé equations. Sakai's development [92] of this approach is now famous for his discovery of a large class of discrete Painlevé equations with similar geometric properties. In this thesis, we extended the geometric approach to construct initial value spaces for lattice equations.

In Chapter 1 we reviewed the background and major results from earlier work we would use in the remainder of the thesis. First by reviewing the historical context of Painlevé equations, we were led to discrete Painlevé equations as OΔEs arising from Bäcklund transformations of continuous Painlevé equations. This discussion led us to more general theory of OΔEs and higher dimensional PΔEs, and tying all these objects together under the banner of integrable systems. We also introduced cluster algebras and their natural appearance within the field of discrete integrable systems.

In Chapter 2 we gave a summary of the algebro-geometric theory of discrete Painlevé equations introduced by Sakai in [92]. First, we carried out the resolution of singularities of an ordinary difference equation of surface type $A_3^{(1)}$. Next, we introduced all the geometric tools necessary to understand Sakai's construction. We then considered the other direction, and constructed a discrete Painlevé equation from the action of an affine Weyl group on the Picard lattice realised as Cremona transformations of $\mathbb{P}^1 \times \mathbb{P}^1$. Finally, we showed how this construction is intimately related to the integrability of the mapping via algebraic entropy [95].

In Chapter 3 we introduced integrable lattice equations, and defined concepts and notation such as integrability, reductions, and initial conditions in this higher dimensional discrete setting. We reviewed the classification and synthesis of the ABS lattice equations and their reduction to discrete Painlevé equations.

In Chapter 4 we showed how to use blow-ups to resolve singularities of surfaces, including where they may be singular along some $n > 0$ dimensional subvariety. Using these techniques we performed the resolution of singularities of lattice equations for the first time. Performing this resolution for several examples from the ABS list, we saw how varying parameters affects the blow-up structure. The Q-class created some difficulties in their resolution which were explained and overcome in Section 4.3, where we demonstrated the example of Q1. Using these resolutions of singularities we constructed spaces on which the mapping on a single quadrilateral is everywhere well defined.

In Chapter 5 we expanded on the results of Chapter 4. Labelling every quad with an associated space of initial conditions $\mathcal{X}_{l,m}$, we found transformations to new lattice equations by considering the evolution of the resolved spaces of initial conditions labelling vertices of the dual graph. In particular, we found transformations of the ABS equations $\mathrm{A1}_\delta$, $\mathrm{H3}_{\delta=0}$, $\mathrm{Q1}_\delta$, or $\mathrm{Q3}_{\delta=0}$ to the quad-equation (5.18), given by

$$w_{l+1,m+1} = w_{l,m}\frac{(w_{l+1,m}+a)(w_{l,m+1}+b)}{(w_{l+1,m}+b)(w_{l,m+1}+a)}.$$

We showed that this equation possesses a periodic reduction to a discrete Painlevé equation ($q\mathrm{P}_{\mathrm{VI}}$), and hence we found reductions of the original ABS equations to this discrete Painlevé equation.

Finally, we showed that the coefficients of a cluster algebra introduced in [81] can satisfy (5.18). From this we used the properties of cluster algebras to prove results about the solutions of these ABS equations.

Using this approach on lattice equations poses many new challenges. As seen in Section 4.3.3, due to the higher dimension of the singular sets the resolution of quad equations can be more complicated than the resolution of points in the plane. However in this thesis, by introducing a framework for the resolution of singularities of lattice equations, we have been able to find transformations of several integrable lattice equations to the lattice equation (5.18), and hence a reduction to the discrete Painlevé equation with surface type $A_3^{(1)}$. Additionally, we were able to use these results to draw a connection from the solutions of ABS equations to the coefficients of a cluster algebra.

We have hence provided novel insights into the reductions of lattice equations to discrete Painlevé equations.

An interesting and largely untapped research direction may be to study if integrable lattice equations can arise from the action of affine Weyl groups, as we saw is the case for O∆Es in Chapter 2. In particular, since the equation (5.18) gives an O∆E with surface type $A_3^{(1)}$ from a straightforward periodic reduction, does (5.18) arise from the action of the affine Weyl group $\widetilde{W}(D_5^{(1)})$ as its reduction does, and can it be connected to a surface of type $A_3^{(1)}$? This would open a broad direction of research in the area of partial difference equations.

Appendix

A Lax Pair for (5.18)

In [71], (5.37) appeared as a transformation of the lattice mKdV, or $H3_{\delta=0}$. The equation in the form

$$w_{l+1,m+1} = w_{l,m} \frac{(w_{l+1,m} - \frac{a}{b})(w_{l,m+1} - \frac{b}{a})}{(w_{l+1,m} - \frac{b}{a})(w_{l,m+1} - \frac{a}{b})}, \tag{6.1}$$

appears as the compatibility condition of the Lax pair

$$\psi_{l,m+1} = \psi_{l+1,m}\, w_{l,m} + (b - a\, w_{l,m})\, \psi_{l,m}, \tag{6.2a}$$

$$\psi_{l+1,m+1} = \frac{a^2 - b^2}{a - b\, w_{l,m}} \psi_{l,m+1} + \lambda\, \frac{a\, w_{l,m} - b}{a - b\, w}\, \psi_{l,m}, \tag{6.2b}$$

where $\psi_{l,m}$ is a scalar function and λ is a spectral parameter.

It was shown that it is possible to write $w_{l,m}$ in terms of $f_{l,m}$, $g_{l,m}$ as

$$w_{l,m} = \frac{f_{l+1,m}\, g_{l,m+1}}{f_{l,m+1}\, g_{l+1,m}}, \tag{6.3}$$

where f and g both satisfy

$$(a + b)\, f_{l-1,m+1}\, f_{l+1,m} + (a - b)\, f_{l-1,m}\, f_{l+1,m+1} = 2\, a\, f_{l,m}\, f_{l,m+1}, \tag{6.4a}$$

$$(a + b)\, f_{l+1,m-1}\, f_{l,m+1} + (b - a)\, f_{l,m-1}\, f_{l+1,m+1} = 2\, b\, f_{l,m}\, f_{l+1,m}. \tag{6.4b}$$

From (6.4) it is possible to derive the following 5-point lattice equation

$$(p - q)^2\, f_{l-1,m-1}\, f_{l+1,m+1} - (p + q)^2\, f_{l+1,m-1}\, f_{l-1,m+1} + 4\, a\, b\, f^2 = 0. \tag{6.5}$$

This is Hirota's discrete-time Toda equation [50].

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