A STUDY IN THE MATHEMATICAL THEORY OF THE CONDUCTION OF HEAT.

by

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A thesis presented for the degree of Doctor of Science in the University of Sydney.

PREFACE.

This work represents a study in the application of the Laplace Transformation method to the Theory of Conduction of Heat; with a few exceptions indicated in footnotes, the derivation by this method of all the results is new. In Chapters II, VI, VIII, and X which contain collections of results, some of these are classical and given for completeness, and some are new. Almost all the results of Chapters I, III, IV, VII, and IX are believed to be new.

None of this work has been submitted for any degree, and it is entirely my own with the exception of Chapters I and X, which have been written for publication in collaboration with Professor Carslaw. These are included here since they form an essential part of the whole scheme; the problems considered were solved independently and published jointly.

The parts of this thesis which the referee may deem suitable will be published as soon as possible. Chapters I, VII, and X, and portion of Chapter IV have already been published, and Chapter III, and portions of Chapters VIII and IX are in the press.

It is my pleasure to acknowledge my great indebtedness to Professor Carslaw who not only aroused my interest in the subject, but in the course of a frequent correspondence extending over several years has been most generous with advice and criticism.

I am also indebted to Miss M. E. Clarke for her assistance with the computations of Chapter V and for the preparation of the typescript.

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CONTENTS.

INTRODUCTION.

1. When this work was begun operational methods and their developments by Jeffreys and Bromwich had been applied extensively in various branches of applied mathematics, usually in order to solve partial differential equations in two variables, one of them the time, by transforming them into ordinary linear differential equations. The Laplace transformation procedure, due to Bateman, Doetsch and van der Pol, in a form then being developed by Professor Carslaw, seemed to me much the most attractive approach to these methods. Much of the work hitherto done had been of a didactic and expository character, intended to exhibit the power of the method by the solution of problems to which it was well suited, rather than its capabilities when applied to more difficult problems arising in any one field.

The subsequent chapters represent a systematic study in the application of the method to problems in Conduction of Heat covering practically the whole range of the subject. The problems considered have to a large extent been chosen because of their practical importance and with a view to subsequent numerical computation. Results of one such calculation are given in Chapter V and an extensive numerical study of Chapter IX §§1 - 4 has been completed. A certain amount of selection has been necessary and this has usually taken the form of a bias towards results in cylindrical coordinates, these seem to me the most interesting and important both practically and theoretically.

2. The following sketch is intended merely to indicate the point of view taken in the sequel; in Chapter I the complete solution of a problem is given to illustrate the method, a fuller solution of simpler problems is given in Carslaw and Jaeger, [Phil. Mag. (7) XXVI (1938) 473].

We have to solve the partial differential equation

$$\nabla^2 v - \frac{1}{\kappa}\frac{\partial v}{\partial t} = 0 \qquad \text{.........} \quad (1)$$

in some given region, with a boundary condition at the surface of the region, and with a given value, $v_0(x,y,z)$, of $v(x,y,z,t)$ when $t = 0$.

We multiply the differential equation and its boundary condition by e^{-pt}, where p is a positive constant*, and integrate with respect to t from 0 to ∞. Then, writing

$$\bar{v} = \int_0^\infty e^{-pt}\, v\, dt$$

for the Laplace transform of v, and making certain assumptions, the equation (1) becomes

$$\nabla^2 \bar{v} - \frac{p}{\kappa}\bar{v} = -\frac{1}{\kappa} v_0(x,y,z) , \qquad \text{........} \quad (2)$$

which has to be solved with known boundary conditions.

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* p may of course be complex with $R(p) > 0$. The above statement derives from work on ordinary linear differential equations where the complex variable is not needed and Professor Carslaw and I have retained it throughout our work to keep partial differential equations as much as possible on the same footing. We use a symbol λ for the complex variable of the inversion theorem.

The equation (2) is called the "subsidiary equation" corresponding to the given differential equation and initial conditions, and by solving it we obtain $\bar{v}(x,y,z,p)$.

From $\bar{v}(x,y,z,p)$ we obtain $v(x,y,z,t)$ by the use of the Inversion Theorem for the Laplace transformation* which gives

$$v(x,y,z,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\,\bar{v}(x,y,z,\lambda)\,d\lambda \qquad \ldots\ldots (3)$$

provided $\bar{v}(x,y,z,\lambda)$ is analytic on and to the right of the line $R(\lambda) = \gamma$ and that certain conditions on either $\bar{v}(x,y,z,\lambda)$ or $v(x,y,z,t)$ are satisfied.

These conditions need not be considered here since assumptions have also been made in deriving (2) so that even if it were proved that $\bar{v}(x,y,z,\lambda)$ satisfied the complete conditions for the Inversion Theorem it would still not follow that (3) is the solution of the differential equation and its initial and boundary conditions.

It should be remarked that the conditions on $v(x,y,z,t)$ for the truth of the Inversion Theorem are extremely broad: roughly if we assume *à priori* that there exists a solution of the problem which is of exponential type in t both the assumptions made in deriving (2) and the application of the Inversion Theorem will be justified so that (3) will be the solution of the problem. This basis should be adequate for most purposes of applied mathematics. If it is desired to make the solution completely rigorous it must be verified that the solution $v(x,y,z,t)$ obtained in this way does in fact

* Cf. Doetsch, *Theorie und Anwendung der Laplace-Transformation* (Berlin 1937) *Kap.* 6. Churchill, *Math. Zeits.* 42 (1937) 567; *Math. Ann.* 114 (1937) 591.

satisfy the differential equation and boundary conditions. This may be done in different ways, e.g. by direct verification on the final form of the solution, or by direct verification on the line integral of type (3). For one-variable problems in Conduction of Heat the process of Chapter I § 9 may be used, the verification of the solution of the one-variable problems of Chapters II, III, IV is discussed in Appendix III.

The point of view adopted is thus that the deriving of a solution by the process leading to (3) may be regarded as satisfactory for problems in applied mathematics, but that if it is desired to make the solution completely rigorous verification of the final solution would have to be made.

3. The problems considered divide sharply into two parts; those of Chapters I to V which involve one independent space variable, and those of Chapters VI to X which involve more than one.

For problems involving one space variable the method of procedure is well known and corresponds closely to that of the operational methods in Goldstein's classical paper*. Those considered here are relatively complicated problems of practical importance, and for these the verification of the solutions is discussed in Appendix I.

Problems involving more than one space variable have been very little studied, in these cases the "subsidiary equation" is a partial differential equation and may be treated in a variety of ways. For this reason the work of Chapters VI to X is largely exploratory. Three different methods are developed and each is applied to give complete results for a set of important problems: (i) the obvioud method of treatment by direct separation of variables in the subsidiary equation is studied systematically in Chapter VI; (ii) a new method for certain classes of problem is given in Chapter VII and developments suggested by this method in Chapters VIII and IX; (iii) the Green's functions for point sources for regions bounded by surfaces of the cylindrical coordinate system are determined in Chapter X.

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* Proc. London Math. Soc. (2), 34 (1932) 51.

4. Parts of this work are reproductions of papers already in press. For this reason the Chapters are self-contained and each has its own paragraphing. There are also some minor changes in notation. Short introductions are prefixed to the Chapters indicating briefly their scope.

CHAPTER I

Introduction.

This Chapter contains the detailed solution of a single problem, it is given as an illustration of the method. In all problems subsequently considered the procedure is the same, a solution is obtained formally as a line integral $(\gamma - i\infty, \gamma + i\infty)$, this is transformed into a real infinite integral by use of the contour of Figure 1 or into an infinite series by use of the contour of Figure 3, according to whether the integrand has a branch point or a line of poles. To make this transformation rigorous it is necessary to show that the integral round the circle Γ of Figures 1 or 3 vanishes in the limit as its radius tends to infinity. In cases in which $\bar{v}(\lambda)$ has a line of poles the radius must tend to infinity through a sequence of values which avoid these poles. A discussion of the vanishing of the integral over Γ for problems of Chapters II - IV is given in Appendix III. The matter has also been discussed elsewhere.*

Since the method of obtaining the solution as a line integral is regarded as purely formal, it must be verified that this solution does in fact satisfy the differential equation and boundary conditions.

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* Carslaw and Jaeger, Proc. London Math. Soc. (2), 46 (1940) 361-388. §§ 2. 3.

For one-variable problems in Conduction of Heat the method of Chapter I, § 9 is available, its application to problems of Chapters II - IV is discussed in Appendix I.

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A PROBLEM IN CONDUCTION OF HEAT

BY H. S. CARSLAW AND J. C. JAEGER

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1. The problem of the conduction of heat in a solid sphere with a concentric core of a different material, the surface kept at a constant temperature, and the initial temperature of the whole zero, has already been solved in these *Proceedings**.

This note deals with the corresponding problem where the solid is bounded internally by the sphere $r = a$, while from $r = a$ to $r = b$ it is of one material and from $r = b$ to infinity it is of another. The surface $r = a$ is kept at a constant temperature v_0 and the initial temperature of the whole is zero.

We use the Laplace transformation method†, but the solution can be obtained in a similar way by the path P of the paper referred to above. In §§ 2 and 3 a solution in the form of a line integral is obtained by a formal use of the Laplace transformation. In § 10 it is verified that this solution satisfies the differential equation and boundary conditions. The line integral is transformed into a real infinite integral in §§ 4–8.

2. Let the temperature, conductivity, specific heat and density in $a<r<b$ be v_1, K_1, c_1 and ρ_1; and in $r>b$ let them be v_2, K_2, c_2 and ρ_2.

Put $$\kappa_1 = K_1/c_1\rho_1 \text{ and } \kappa_2 = K_2/c_2\rho_2 .$$

Then, writing $u_1 = v_1 r$ and $u_2 = v_2 r$, we have to solve the following equations‡.

$$\left.\begin{array}{lll}
(1) & \dfrac{\partial u_1}{\partial t} = \kappa_1 \dfrac{\partial^2 u_1}{\partial r^2}, \ (a<r<b); \qquad \dfrac{\partial u_2}{\partial t} = \kappa_2 \dfrac{\partial^2 u}{\partial r^2}, (r>b), & (1') \\
(2) & u_1 = v_0 a, \text{ when } r = a; \qquad u_2 \to 0 \text{ when } r \to \infty, & (2') \\
(3) & u_1 = u_2, \text{ when } r = b, & \\
(4) & K_1\left(\dfrac{\partial u_1}{\partial r} - \dfrac{u_1}{r}\right) = K_2\left(\dfrac{\partial u_2}{\partial r} - \dfrac{u_2}{r}\right), \text{ when } r = b, & (4')
\end{array}\right\} t>0$$

$$(5) \quad u_1 = 0 \text{ when } t = 0, \ (a<r<b); \qquad u_2 = 0 \text{ when } t = 0, \ (r>b). \quad (5')$$

* Carslaw, "The cooling of a solid sphere with a concentric core of a different material", *Proc. Cambridge Phil. Soc.* 20 (1921), 399–410. Bromwich, "Symbolical methods in conduction of heat", *Proc. Cambridge Phil. Soc.* 20 (1921), 411–27. In the first of these papers contour integration is used; in the second Heaviside's operational calculus as developed by Bromwich.

† Carslaw, *Math. Gaz.* 22 (1938), 264–80. Carslaw and Jaeger, *Phil. Mag.* (7), 26 (1938), 473. Carslaw and Jaeger, *Bull. American Math. Soc.* (in the Press). Also the work of Doetsch quoted on p. 396 and papers by Churchill, Lowan and others.

‡ Carslaw, *Conduction of heat*, 2nd ed. (Macmillan, 1921), § 64. We quote this book below as *C.H.*

Multiply (1) by e^{-pt} $(p>0)$ and integrate with regard to t from 0 to ∞. Then

$$\int_0^\infty e^{-pt}\frac{\partial u_1}{\partial t}\,dt = \kappa_1\int_0^\infty e^{-pt}\frac{\partial^2 u_1}{\partial r^2}\,dr. \tag{6}$$

But
$$\int_0^\infty e^{-pt}\frac{\partial u_1}{\partial t}\,dt = \Big[e^{-pt}u_1\Big]_0^\infty + p\int_0^\infty e^{-pt}u_1\,dt$$
$$= p\int_0^\infty e^{-pt}u_1\,dt,$$

by (5). We write*
$$\bar{u}_1 = \int_0^\infty e^{-pt}u_1\,dt. \tag{7}$$

Assuming that we may change the order of integration and differentiation on the right-hand of (6), we thus obtain the "subsidiary equation"

$$\frac{d^2\bar{u}_1}{dr^2} - q_1^2\bar{u}_1 = 0, \quad (a<r<b), \tag{8}$$

where $q_1^2 = p/\kappa_1$. Similarly from (1′) and (5′), we have

$$\frac{d^2\bar{u}_2}{dr^2} - q_2^2\bar{u}_2 = 0, \quad (r>b), \tag{8'}$$

where
$$q_2^2 = p/\kappa_2 \quad\text{and}\quad \bar{u}_2 = \int_0^\infty e^{-pt}u_2\,dt.$$

Also, from (2) and (2′), we obtain

$$\bar{u}_1 = v_0 a/p, \quad\text{when}\quad r=a, \tag{9}$$

and
$$\bar{u}_2 \to 0, \quad\text{when}\quad r\to\infty, \tag{9'}$$

assuming that
$$\lim_{r\to a}\int_0^\infty e^{-pt}u_1\,dt = \int_0^\infty e^{-pt}\lim_{r\to a}u_1\,dt$$

and
$$\lim_{r\to\infty}\int_0^\infty e^{-pt}u_2\,dt = \int_0^\infty e^{-pt}\lim_{r\to\infty}u_2\,dt.$$

Similarly from (3) and (4), we have

$$\bar{u}_1 = \bar{u}_2, \quad\text{when}\quad r=b, \tag{10}$$

and
$$K_1\left(\frac{d\bar{u}_1}{dr} - \frac{\bar{u}_1}{r}\right) = K_2\left(\frac{d\bar{u}_2}{dr} - \frac{\bar{u}_2}{r}\right), \quad\text{when}\quad r=b. \tag{11}$$

Our problem is thus reduced to finding $\bar{u}_1$ and $\bar{u}_2$ from (8), (8′), (9), (9′), (10) and (11).

We obtain u_1 and u_2 from

$$\bar{u}_1 = \int_0^\infty e^{-pt}u_1\,dt \quad\text{and}\quad \bar{u}_2 = \int_0^\infty e^{-pt}u_2\,dt.$$

* We use this bar notation throughout for the Laplace transform. Thus

$$\bar{u}(p) = \int_0^\infty e^{-pt}u(t)\,dt.$$

3. As solutions of equations (8) and (8′) with (9′) of § 2, we take

$$\bar{u}_1 = A \sinh q_1(b-r) + B \cosh q_1(b-r),$$
$$\bar{u}_2 = Ce^{-q_2(r-b)},$$

and determine A, B and C from (9), (10) and (11).

We are thus led to

$$\bar{u}_1 = \frac{v_0 a}{p} \frac{K_1 q_1 \cosh q_1(b-r) + (K_2 q_2 + K) \sinh q_1(b-r)}{K_1 q_1 \cosh q_1(b-a) + (K_2 q_2 + K) \sinh q_1(b-a)} \tag{1}$$

and
$$\bar{u}_2 = \frac{v_0 a}{p} \frac{K_1 q_1 e^{-q_2(r-b)}}{K_1 q_1 \cosh q_1(b-a) + (K_2 q_2 + K) \sinh q_1(b-a)}, \tag{2}$$

where for brevity we have written

$$K = (K_2 - K_1)/b. \tag{3}$$

Then, from the inversion theorem* for the Laplace transformation,

$$u_1 = \frac{v_0 a}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{K_1 \mu_1 \cosh \mu_1(b-r) + (K_2 \mu_2 + K) \sinh \mu_1(b-r)}{K_1 \mu_1 \cosh \mu_1(b-a) + (K_2 \mu_2 + K) \sinh \mu_1(b-a)} \frac{d\lambda}{\lambda} \tag{4}$$

and

$$u_2 = \frac{K_1 v_0 a}{2i\pi \sqrt{\kappa_1}} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{e^{-\mu_2(r-b)}}{K_1 \mu_1 \cosh \mu_1(b-a) + (K_2 \mu_2 + K) \sinh \mu_1(b-a)} \frac{d\lambda}{\sqrt{\lambda}} \tag{5}$$

where we have written

$$\mu_1 = \sqrt{(\lambda/\kappa_1)} \quad \text{and} \quad \mu_2 = \sqrt{(\lambda/\kappa_2)}. \tag{6}$$

4. Consider now the closed circuit in the λ-plane given in Fig. 1. The line AB is at a distance γ from the imaginary axis; the large circle Γ has its centre at the origin and its radius is R. There is a cut along the negative real axis. The circuit is completed by a small circle, centre at the origin, and we take $-\pi < \arg \lambda < \pi$.

We prove in § 5 that within or upon this closed circuit the integrands in § 3, equations (4) and (5), have no poles, and in § 6 that, when the radius of Γ tends to infinity and the radius of the small circle tends to zero, the integrals over BF and CA tend to zero.

Thus we can replace the integrals $\int_{\gamma-i\infty}^{\gamma+i\infty}$ of § 3, equations (4) and (5), by the sum of the integrals over CD, the small circle, and EF, when C and F tend to $-\infty$ and D and E tend to the origin.

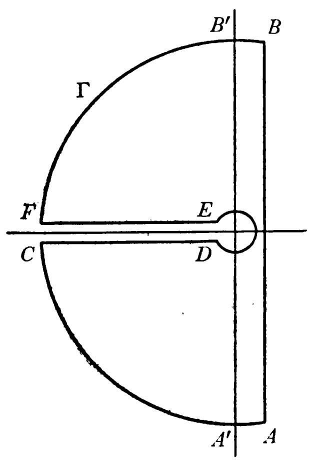

Fig. 1.

In this way we obtain the required solutions in their simplest form.

* See, for example, Doetsch, *Theorie und Anwendung der Laplace-Transformation* (Berlin, 1937), p. 126, Satz 2. Churchill, *Math. Z.* 42 (1937), 569, Theorem 1.

5. Let

$$F(\lambda) = K_1\sqrt{\left(\frac{\lambda}{\kappa_1}\right)}\cosh\sqrt{\left(\frac{\lambda}{\kappa_1}\right)}(b-a)+\left\{K_2\sqrt{\left(\frac{\lambda}{\kappa_2}\right)}+\frac{K_2-K_1}{b}\right\}\sinh\sqrt{\left(\frac{\lambda}{\kappa_1}\right)}(b-a).$$

I. *There is no real positive zero λ_0 of $F(\lambda)$.*

For suppose that there is such a zero. Write

$$l = \frac{K_2\sqrt{\kappa_1}}{K_1\sqrt{\kappa_2}}, \quad m = \frac{K_2-K_1}{bK_1}, \quad \text{and} \quad c = b-a.$$

Then
$$\tanh x = -\frac{x}{lx+cm}$$

must be satisfied by
$$x = c\sqrt{(\lambda_0/\kappa_1)}.$$

We show that this is impossible for any real m.

(i) If $m>0$, then $\quad -x/(lx+cm)<0 \quad$ for $\quad x>0$.

(ii) If $m=0$, then $\quad -x/(lx+cm) = -1/l<0$.

(iii) If $m<0$, the hyperbola $y = -x/(lx+cm)$ has asymptotes

$$x = c\,|\,m\,|/l, \quad \text{and} \quad y = -1/l.$$

Thus, for $x>c\,|\,m\,|/l$, we have $-x/(lx+cm)< -1/l$, while for $0<x<c\,|\,m\,|\,l$ the gradient of $-x/(lx+cm)$ satisfies the relation

$$-\frac{cm}{(lx+cm)}>\frac{1}{c\,|\,m\,|}>1.$$

Therefore in all three cases the graphs of $\tanh x$ and $-x/(lx+cm)$ have no intersection for $x>0$.

II. *There is no real negative zero $-\lambda_0$ of $F(\lambda)$.*

For suppose that there is such a zero. Then we must have

$$\tan x+\frac{x}{cm\pm ilx}=0,$$

where $x = c\sqrt{(\lambda_0/\kappa_1)}$, which is impossible since $l\neq 0$.

III. *There are no complex zeros*.*

If possible, let $\xi+i\eta$ $(\eta \gtrless 0)$ be a root of $F(\lambda)=0$. Then $\xi-i\eta$ is also a root. Let

$$U_1 = \frac{\sinh q_1(r-a)}{\sinh q_1(b-a)}, \quad (a<r<b),$$

where
$$q_1^2 = \frac{\xi+i\eta}{\kappa_1},$$

* Cf. *C.H.* § 106.

and let $$U_2 = e^{-q_2(r-b)}, \quad (r>b),$$

where $$q_2^2 = \frac{\xi+i\eta}{\kappa_2}.$$

Then $$\frac{d^2U_1}{dr^2} - q_1^2 U_1 = 0, \quad (a<r<b), \tag{1}$$

$$\frac{d^2U_2}{dr^2} - q_2^2 U_2 = 0, \quad (r>b), \tag{2}$$

$$U_1 = 0, \quad \text{when} \quad r=a, \tag{3}$$

$$U_1 = U_2, \quad \text{when} \quad r=b, \tag{4}$$

$$K_1 \frac{dU_1}{dr} - K_2 \frac{dU_2}{dr} = \frac{K_1-K_2}{b} U_1, \quad \text{when} \quad r=b. \tag{5}$$

Let U_1', U_2', $q_1'^2$ and $q_2'^2$ be the corresponding expressions for the root $\xi - i\eta$. Then we have

$$(q_1^2 - q_1'^2) \int_a^b U_1 U_1' dr = \int_a^b \left(U_1' \frac{d^2U_1}{dr^2} - U_1 \frac{d^2U_1'}{dr^2} \right) dr,$$

so that $$2i\eta \frac{K_1}{\kappa_1} \int_a^b U_1 U_1' dr = K_1 \left(U_1' \frac{dU_1}{dr} - U_1 \frac{dU_1'}{dr} \right), \quad \text{when} \quad r=b. \tag{6}$$

Similarly

$$2i\eta \frac{K_2}{\kappa_2} \int_b^\infty U_2 U_2' dr = K_2 \left(U_2 \frac{dU_2'}{dr} - U_2' \frac{dU_2}{dr} \right), \quad \text{when} \quad r=b. \tag{7}$$

Adding (6) and (7), and using (4), (5) and the corresponding equations involving U_1' and U_2', we have

$$2i\eta \left\{ \frac{K_1}{\kappa_1} \int_a^b U_1 U_1' dr + \frac{K_2}{\kappa_2} \int_b^\infty U_2 U_2' dr \right\} = 0,$$

which is impossible, since $U_1 U_1'$, $U_2 U_2'$ are positive.

6. The simple theorem given below deals with many cases in which it is required to prove that the integral over the circle Γ of Fig. 1 vanishes as its radius $R \to \infty$.

THEOREM 1. *If* $|f(\lambda, \xi)| < CR^k \exp[-\xi R^{\frac{1}{2}} \cos \frac{1}{2}\theta]$, *when* $\lambda = Re^{i\theta}$, $-\pi < \theta < \pi$, $R > R_0$, *where* ξ *is a parameter, and* R_0, C, *and** $k < \frac{1}{2}$, *are constants independent of* θ *and* ξ, *then*

$$\int e^{\lambda t} f(\lambda, \xi) \frac{d\lambda}{\lambda}$$

taken over the arcs $BB'F$ *and* $AA'C$ *of the circle* Γ *tends to zero as* $R \to \infty$, *if either* $\xi > 0$, $t \geqslant 0$, *or* $\xi \geqslant 0$, $t > 0$.

* If $k < 1$ the result is true for $\xi \geqslant 0$, $t > 0$.

We consider separately the integrals $I_{BB'}$ and $I_{B'F}$ over the arcs BB' and $B'F$. Those over AA' and $A'C$ are treated similarly.

For BB'. Let $\alpha = \cos^{-1}(\gamma/R)$, then

$$|I_{BB'}| < CR^k e^{\gamma t} \int_\alpha^{\frac{1}{2}\pi} d\theta = CR^k e^{\gamma t} \sin^{-1}(\gamma/R).$$

Therefore $\lim_{R\to\infty} |I_{BB'}| = 0$.

For $B'F$. $$|I_{B'F}| < CR^k \int_{\frac{1}{2}\pi}^{\pi} \exp[Rt\cos\theta - \xi R^{\frac{1}{2}} \cos\tfrac{1}{2}\theta]\, d\theta.$$

Thus if $t > 0$, $\xi \geqslant 0$,

$$|I_{B'F}| < CR^k \int_0^{\frac{1}{2}\pi} e^{-Rt\sin\theta}\, d\theta < CR^k \int_0^{\frac{1}{2}\pi} e^{-2Rt\theta/\pi}\, d\theta < \frac{\pi C R^{k-1}}{2t}.$$

And if $t \geqslant 0$, $\xi > 0$,

$$|I_{B'F}| < CR^k \int_{\frac{1}{2}\pi}^{\pi} \exp[-\xi R^{\frac{1}{2}} \cos\tfrac{1}{2}\theta]\, d\theta$$
$$= 2CR^k \int_0^{\frac{1}{4}\pi} \exp[-\xi R^{\frac{1}{2}} \sin\theta]\, d\theta < \frac{\pi C}{\xi} R^{k-\frac{1}{2}}.$$

Therefore in both cases $\lim_{R\to\infty} |I_{B'F}| = 0$, and the theorem is proved.

7. We now show that the multipliers of $(1/\lambda)e^{\lambda t}$ in the integrands of § 3, equations (4) and (5), namely,

$$f(\lambda) = \frac{K_1\mu_1 \cosh\mu_1(b-r) + (K_2\mu_2 + K)\sinh\mu_1(b-r)}{K_1\mu_1\cosh\mu_1(b-a) + (K_2\mu_2+K)\sinh\mu_1(b-a)}, \quad (a<r<b), \qquad (1)$$

and $$g(\lambda) = \frac{\lambda^{\frac{1}{2}} e^{-\mu_2(r-b)}}{K_1\mu_1\cosh\mu_1(b-a) + (K_2\mu_2+K)\sinh\mu_1(b-a)}, \quad (r>b), \qquad (2)$$

satisfy the conditions of Theorem 1. Write $R = \kappa_1\rho$, so that on Γ, for $\pi \geqslant \theta \geqslant 0$,

$$\mu_1 = \rho^{\frac{1}{2}} e^{\frac{1}{2}i\theta}, \quad \mu_2 = (\kappa\rho)^{\frac{1}{2}} e^{\frac{1}{2}i\theta}, \quad \text{where} \quad \kappa = \kappa_1/\kappa_2. \qquad (3)$$

Then on Γ

$$|K_1\mu_1\cosh\mu_1(b-a) + (K_2\mu_2+K)\sinh\mu_1(b-a)|$$
$$= \tfrac{1}{2}|e^{\mu_1(b-a)}[\{K_1 + K_2\kappa^{\frac{1}{2}} + (K_1 - K_2\kappa^{\frac{1}{2}})e^{-2\mu_1(b-a)}\}\mu_1 + K\{1-e^{-2\mu_1(b-a)}\}]|$$
$$\geqslant \tfrac{1}{2}\rho^{\frac{1}{2}} \exp[\rho^{\frac{1}{2}}(b-a)\cos\tfrac{1}{2}\theta]\{|K_1+K_2\kappa^{\frac{1}{2}}| - |K_1 - K_2\kappa^{\frac{1}{2}}| - 2|K|\rho^{-\frac{1}{2}}\}$$
$$> C_1\rho^{\frac{1}{2}}\exp[\rho^{\frac{1}{2}}(b-a)\cos\tfrac{1}{2}\theta], \qquad (4)$$

provided* that ρ is greater than some fixed ρ_1. Also

$$|K_1\mu_1\cosh\mu_1(b-r) + (K_2\mu_2+K)\sinh\mu_1(b-r)|$$
$$\leqslant (K_1\rho^{\frac{1}{2}} + K_2\kappa^{\frac{1}{2}}\rho^{\frac{1}{2}} + |K|)\cosh[\rho^{\frac{1}{2}}(b-r)\cos\tfrac{1}{2}\theta]$$
$$< C_2\rho^{\frac{1}{2}}\exp[\rho^{\frac{1}{2}}(b-r)\cos\tfrac{1}{2}\theta], \quad \text{if} \quad \rho>\rho_2. \qquad (5)$$

Further, $$|\lambda^{\frac{1}{2}}e^{-\mu_2(r-b)}| < C_3\rho^{\frac{1}{2}}\exp[-\rho^{\frac{1}{2}}\kappa^{\frac{1}{2}}(r-b)\cos\tfrac{1}{2}\theta]. \qquad (6)$$

* C, C_1, ... are used for different constants, and ρ_0, ρ_1, ... for fixed values of ρ.

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Therefore $$|f(\lambda)| < C_4 \exp[-\rho^{\frac{1}{2}}(r-a)\cos\tfrac{1}{2}\theta], \quad (\rho > \rho_4), \tag{7}$$
and $$|g(\lambda)| < C_5 \exp[-\rho^{\frac{1}{2}}(\kappa^{\frac{1}{2}}(r-b)+b-a)\cos\tfrac{1}{2}\theta], \quad (\rho > \rho_5). \tag{8}$$
Thus $f(\lambda)$, with $\xi = (r-a)\kappa_1^{-\frac{1}{2}}$, $(b-a)\kappa_1^{-\frac{1}{2}} \geq \xi \geq 0$, and $g(\lambda)$ with
$$\xi = (r-b)\kappa_2^{-\frac{1}{2}} + (b-a)\kappa_1^{-\frac{1}{2}}, \quad \xi \geq (b-a)\kappa_1^{-\frac{1}{2}},$$
satisfy the conditions of Theorem 1, in both cases with $k = 0$. Similarly
$$\left|\frac{\partial f(\lambda)}{\partial r}\right| < C_6\rho^{\frac{1}{2}} \exp[-\rho^{\frac{1}{2}}(r-a)\cos\tfrac{1}{2}\theta], \quad (\rho > \rho_6),$$
and
$$\left|\frac{\partial^2 f(\lambda)}{\partial r^2}\right| < C_7\rho \exp[-\rho^{\frac{1}{2}}(r-a)\cos\tfrac{1}{2}\theta], \quad (\rho > \rho_7), \tag{9}$$
with corresponding results for
$$\left|\frac{\partial g(\lambda)}{\partial r}\right| \quad \text{and} \quad \left|\frac{\partial^2 g(\lambda)}{\partial r^2}\right|.$$
It follows from (7) and (8) by Theorem 1 that, as stated in § 4, the integrals of the integrands of § 3, equations (4) and (5), over BF and CA tend to zero as $R \to \infty$.

8. We can now replace § 3, equation (4), namely,
$$u_1 = \frac{v_0 a}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{K_1\mu_1\cosh\mu_1(b-r)+(K_2\mu_2+K)\sinh\mu_1(b-r)}{K_1\mu_1\cosh\mu_1(b-a)+(K_2\mu_2+K)\sinh\mu_1(b-a)}\frac{d\lambda}{\lambda},$$
by the sum of the integrals over CD, the small circle, and EF of Fig. 1, when the radius of the large circle tends to ∞ and that of the small circle to zero.

From the small circle we obtain
$$v_0 a\frac{K_1+K(b-r)}{K_1+K(b-a)} = v_0 a\frac{K_2+(K_1-K_2)r/b}{K_2+(K_1-K_2)a/b}.$$
From CD, putting* $\lambda = \kappa_1\alpha^2 e^{-i\pi}$ and $\kappa = \sqrt{(\kappa_1/\kappa_2)}$, we have
$$-\frac{v_0 a}{i\pi}\int_0^\infty e^{-\kappa_1\alpha^2 t}\frac{\{K_1\alpha\cos\alpha(b-r)+K\sin\alpha(b-r)\}-iK_2\sqrt{\kappa}\,\alpha\sin\alpha(b-r)}{\{K_1\alpha\cos\alpha(b-a)+K\sin\alpha(b-a)\}-iK_2\sqrt{\kappa}\,\alpha\sin\alpha(b-a)}\frac{d\alpha}{\alpha}.$$
From EF, putting $\lambda = \kappa_1\alpha^2 e^{i\pi}$, we have the conjugate expression. So from CD and EF we obtain
$$-\frac{2v_0 a}{\pi}K_1K_2\sqrt{\kappa}\int_0^\infty e^{-\kappa_1\alpha^2 t}\frac{\alpha\sin\alpha(r-a)\,d\alpha}{[K_1\alpha\cos\alpha(b-a)+K\sin\alpha(b-a)]^2+K_2^2\kappa\alpha^2\sin^2\alpha(b-a)}.$$
Hence finally
$$\frac{v_1}{v_0} = \frac{a}{r}\frac{K_2+(K_1-K_2)r/b}{K_2+(K_1-K_2)a/b}$$
$$-\frac{2K_1K_2\sqrt{\kappa}}{\pi r}\int_0^\infty e^{-\kappa_1\alpha^2 t}\frac{\alpha\sin\alpha(r-a)\,d\alpha}{[K_1\alpha\cos\alpha(b-a)+K\sin\alpha(b-a)]^2+K_2^2\kappa\alpha^2\sin^2\alpha(b-a)}.$$

---

* Here and throughout the sequel it is understood that principal values of square roots are taken.

Also we can replace § 3, equation (5), by the sum of the integrals over CD, the small circle, and EF. The small circle gives

$$\frac{K_1 v_0 a}{K_1 + K(b-a)}.$$

From CD (on putting $\lambda = \kappa_1 \alpha^2 e^{-i\pi}$), we have

$$-\frac{K_1 v_0 a}{i\pi}\int_0^\infty e^{-\kappa_1\alpha^2 t}\frac{e^{i\sqrt{\kappa}\,\alpha(r-b)}\,d\alpha}{[K_1\alpha\cos\alpha(b-a)+K\sin\alpha(b-a)]-iK_2\sqrt{\kappa}\,\alpha\sin\alpha(b-a)},$$

where we have written $\kappa = \kappa_1/\kappa_2$; from EF we obtain the conjugate expression.

So, from CD and EF, we have

$$-\frac{2K_1 v_0 a}{\pi}\int_0^\infty e^{-\kappa_1\alpha^2 t}\frac{(K_1\alpha\cos\alpha(b-a)+K\sin\alpha(b-a))\sin\sqrt{\kappa}\,\alpha(r-b)+K_2\sqrt{\kappa}\,\alpha\sin\alpha(b-a)\cos\sqrt{\kappa}\,\alpha(r-b)}{[K_1\alpha\cos\alpha(b-a)+K\sin\alpha(b-a)]^2+K_2^2\kappa\alpha^2\sin^2\alpha(b-a)}\,d\alpha.$$

Hence finally

$$\frac{v_2}{v_0} = \frac{a}{r}\frac{K_1}{K_2+(K_1-K_2)\,a/b}$$

$$-\frac{2K_1 a}{\pi r}\int_0^\infty e^{-\kappa_1\alpha^2 t}\frac{(K_1\alpha\cos\alpha(b-a)+K\sin\alpha(b-a))\sin\sqrt{\kappa}\,\alpha(r-b)+K_2\sqrt{\kappa}\,\alpha\sin\alpha(b-a)\cos\sqrt{\kappa}\,\alpha(r-b)}{[K_1\alpha\cos\alpha(b-a)+K\sin\alpha(b-a)]^2+K_2^2\kappa\alpha^2\sin^2\alpha(b-a)}\,d\alpha.$$

9. It remains to verify that the solutions given in equations (4) and (5) of § 3 satisfy the differential equations and boundary conditions of § 2. This verification is most easily performed on integrals along the path L' of Fig. 2, which begins at infinity in the direction $\arg\lambda = -\beta$, where $\pi > \beta > \frac{1}{2}\pi$, passes to the right of the origin, keeping all singularities of the integrand to the left, and ends in the direction $\arg\lambda = \beta$. We first have to show that the path L $(\gamma - i\infty, \gamma + i\infty)$ can be deformed into the path L'. This and some other points in the verification process are covered by the following theorem.

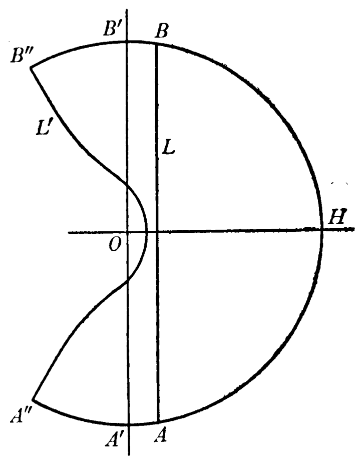

Fig. 2.

THEOREM 2. *If $f(\lambda, \xi)$ is an analytic function of λ on and to the right of the path L', and if*

$$|f(\lambda,\xi)| < CR^k \exp[-\xi R^{\frac{1}{2}}\cos\tfrac{1}{2}\theta], \qquad (1)$$

when $\lambda = Re^{i\theta}, \quad \pi > \theta_0 \geqslant \theta \geqslant 0, \quad R > R_0,$

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where C, $k<1$, R_0, and $\theta_0 > \frac{1}{2}\pi$, are constants, then

I.
$$\int_L e^{\lambda t} f(\lambda,\xi)\frac{d\lambda}{\lambda} = \int_{L'} e^{\lambda t} f(\lambda,\xi)\frac{d\lambda}{\lambda},$$

provided that either $t \geqslant 0$, $\xi > 0$, or $t > 0$, $\xi \geqslant 0$.

II.
$$\int_{L'} e^{\lambda t} f(\lambda,\xi)\frac{d\lambda}{\lambda} \tag{2}$$

is uniformly convergent with respect to t in $t \geqslant 0$ for fixed $\xi > 0$, and with respect to ξ in $\xi \geqslant 0$ for fixed $t > 0$. Also the integral may be differentiated under the integral sign with respect to t in $t \geqslant 0$ for fixed $\xi > 0$, or for fixed $t > 0$ in $\xi \geqslant 0$, and in the latter case the resulting integral is uniformly convergent with respect to ξ in $\xi \geqslant 0$.

III.
$$\lim_{t\to 0}\int_{L'} e^{\lambda t} f(\lambda,\xi)\frac{d\lambda}{\lambda} = 0, \quad \textit{for fixed } \xi > 0.$$

IV. *If, in addition, $\partial f/\partial\xi$ and $\partial^2 f/\partial\xi^2$ satisfy conditions of type* (1), *except that k* ~~*must*~~ need *not be less than* 1, *then*

$$\int_{L'} e^{\lambda t} f(\lambda,\xi)\frac{d\lambda}{\lambda}$$

may be differentiated twice under the integral sign with respect to ξ, in $\xi \geqslant 0$, for fixed $t > 0$.

I. To prove that the path L may be deformed into L' we have to show that the integrals

$$I = \int e^{\lambda t} f(\lambda,\xi)\frac{d\lambda}{\lambda}$$

taken over the arcs $BB'B''$ and $A''A'A$ of the circle of radius R of Fig. 2 tend to zero as $R\to\infty$. Then, since there are no singularities between the paths L and L', the result follows by Cauchy's Theorem.

Over BB'. $\quad |I| < CR^k e^{\gamma t}\sin^{-1}(\gamma/R).$

Over $B'B''$. $\quad |I| < CR^k \exp[-\xi R^{\frac{1}{2}}\cos\frac{1}{2}\beta]\int_{\frac{1}{2}\pi}^{\beta} e^{Rt\cos\theta}\,d\theta.$

Therefore $|I| < \dfrac{\pi C}{2t}R^{k-1}$, if $t>0$, $\xi \geqslant 0$

and $|I| < (\beta - \frac{1}{2}\pi)CR^k\exp[-\xi R^{\frac{1}{2}}\cos\frac{1}{2}\beta]$, if $t \geqslant 0$, $\xi > 0$.

Thus the integral over $BB'B''$, and similarly that over $AA'A''$, tends to zero as $R\to\infty$.

II. On the path L', if $\lambda = \rho e^{i\beta}$, the integrand of (2) is less in modulus than that of

$$C\int^{\infty}\rho^k\exp[+\rho t\cos\beta - \xi\rho^{\frac{1}{2}}\cos\tfrac{1}{2}\beta]\,d\rho,$$

V. <u>If the range of ξ extends to infinity</u>

$$\lim_{\xi\to\infty}\int_{L'} e^{\lambda t} f(\lambda,\xi)\frac{d\lambda}{\lambda} = 0.$$

Proof as for the special case at the foot of p. 18..

and the first part of II follows. Similarly, $\int_{L'} e^{\lambda t} f(\lambda, \xi)\, d\lambda$ is uniformly convergent, and the differentiation under the integral sign is permissible.

III. By II the integral (2) is a continuous function of t in $t \geqslant 0$ for fixed $\xi > 0$. Thus

$$\lim_{t \to 0} \int_{L'} e^{\lambda t} f(\lambda, \xi) \frac{d\lambda}{\lambda} = \int_{L'} f(\lambda, \xi) \frac{d\lambda}{\lambda}.$$

To evaluate the latter integral consider

$$I = \int f(\lambda, \xi) \frac{d\lambda}{\lambda}$$

taken round the arc $B''BHAA''$ of the circle of radius R in Fig. 2. Then

$$|\, I \,| < 2CR^k \exp\left[-\xi R^{\frac{1}{2}} \cos \tfrac{1}{2}\beta\right] \int_0^{\beta} d\theta.$$

Thus if $\xi > 0$, $\lim_{R \to \infty} |\, I \,| = 0$, and, since there are no poles of $f(\lambda, \xi)/\lambda$ in the closed circuit formed by L' and the arc $B''BHAA''$, the result III follows.

IV. If we proceed as in II it follows that

$$\int_{L'} e^{\lambda t} \frac{\partial f(\lambda, \xi)}{\partial \xi} \frac{d\lambda}{\lambda} \quad \text{and} \quad \int_{L'} e^{\lambda t} \frac{\partial^2 f(\lambda, \xi)}{\partial \xi^2} \frac{d\lambda}{\lambda}$$

are uniformly convergent with respect to ξ in $\xi \geqslant 0$ for fixed $t > 0$. Thus (2) may be differentiated twice under the integral sign with respect to ξ.

10. By § 6, equations (7), (8) and (9), it follows that the multipliers of $(1/\lambda)\, e^{\lambda t}$ in the integrands of § 3, equations (4) and (5), satisfy the conditions of Theorem 2.

It follows that the differential equations (1) and (1′), and boundary conditions (5) and (5′) of § 2 are satisfied.

Also § 2, equations (3) and (4) are satisfied since it follows from Theorem 2, II and III, that all quantities involved are continuous at $r = b$.

It remains to prove § 2, equation (2), which by Theorem 2, I may be written

$$\lim_{r \to a} v_1 = \lim_{r \to a} \frac{v_0 a}{2i\pi r} \int_{L'} e^{\lambda t} \frac{K_1 \mu_1 \cosh \mu_1 (b - r) + (K_2 \mu_2 + K) \sinh \mu_1 (b - r)}{K_1 \mu_1 \cosh \mu_1 (b - a) + (K_2 \mu_2 + K) \sinh \mu_1 (b - a)} \frac{d\lambda}{\lambda}.$$

By Theorem 2, II, this integral is uniformly convergent with respect to r in $a \leqslant r \leqslant b$, for fixed $t > 0$, and thus

$$\lim_{r \to a} v_1 = \frac{v_0}{2i\pi} \int_{L'} e^{\lambda t} \frac{d\lambda}{\lambda} = v_0.$$

Finally, the condition (2′) of § 2 may be verified as follows. Let the arc $B''BHAA''$ of the circle of Fig. 2 have fixed radius R so large that by equation (8) of § 7

$$|\, g(\lambda) \,| < C \exp\left[-R^{\frac{1}{2}} \xi \cos \tfrac{1}{2}\beta\right],$$

where

$$\xi = (r - b)\, \kappa_2^{-\frac{1}{2}} + (b - a)\, \kappa_1^{-\frac{1}{2}}.$$

Then by Cauchy's Theorem

$$\int e^{\lambda t} g(\lambda) \frac{d\lambda}{\lambda}$$

over the portion $A''B''$ of L' is equal to the integral over the arc $B''BHAA''$, which is less in modulus than

$$2\beta C \exp\left[Rt - \xi R^{\frac{1}{2}} \cos \tfrac{1}{2}\beta\right],$$

and this tends to zero as $\xi \to \infty$, for $t \geqslant 0$.

Also the integrals over the portions of L' lying outside the circle $B''BHAA''$ are together less in modulus than

$$2C \int_R^\infty \exp\left[-\xi \rho^{\frac{1}{2}} \cos \tfrac{1}{2}\beta\right] d\rho < 4C \int_0^\infty \exp\left[-\xi u \cos \tfrac{1}{2}\beta\right] u\,du$$

$$= \frac{4C}{\xi^2 \cos^2 \frac{1}{2}\beta},$$

and this tends to zero as $\xi \to \infty$.

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CHAPTER II.

INSTANTANEOUS SOURCES IN RODS, CYLINDERS AND SPHERES.

Introduction.

Plane, cylindrical and spherical surface sources are defined in C.H. (Chapter IX). In this Chapter solutions will be obtained for such sources in regions with plane, coaxial cylindrical, and concentric spherical boundaries respectively, with boundary conditions of type

$$k \frac{\partial v}{\partial n} + hv = 0 \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

at the surfaces, where $\frac{\partial}{\partial n}$ denotes differentiation along the outward normal; k and h are constants which will be supposed $\geqslant 0$, but not both to vanish; a brief discussion of the physically unimportant cases in which one of them is negative is given in §§ 9, 10. The form (1) includes the usual special boundary conditions of the theory of conduction of heat; $k = 0$ corresponds to $v = 0$ on the surface, $h = 0$ to no flow of heat over the surface, and $k = 1$ to the usual boundary condition for radiation into a medium at zero. The "k" has been introduced in (1) since it makes the structure of the formulae a little clearer, in particular in the case $k = 0$.

The method consists of adding to the solution "u" for a source of the required type in an infinite medium, a solution "w" of the equation of conduction in the region chosen so that $\lim_{t \to 0} w = 0$ and that $v = u + w$ satisfies the boundary conditions. It is verified in Appendix I by the type of procedure used in Chapter I that w satisfies its differential equation, that $\lim_{t \to 0} w = 0$, and that v satisfies the boundary conditions. Also for $t > 0$ the reduction of the contour integral for v to the series or integral form is justified in Appendix III. Thus for $t > 0$ the solutions are complete and rigorous.

Two further steps are made which are purely formal, (i) a solution for arbitrary initial temperature $f(x)$ is obtained from the source solution, in doing this infinite processes are interchanged without justification, and (ii) t is made to tend to zero in this result in order to obtain formal expansions of $f(x)$ in forms required for subsequent work in Chapter VI. Most of these expansions theorems are known, for example the result of § 1 is a Sturm-Liouville expansion, that of § 3 is a Dini series and the case ~~of~~ $k = 0$ of § 4 is Weber's integral theorem*, the general expansion of § 4 was obtained formally by Goldstein (loc. cit.). The establishing of conditions on the arbitrary functions for the validity of these expansions seems to me a matter for the pure mathematician using the Lebesgue integral.

Green's functions have been studied in this Chapter both because they are of fundamental importance in Conduction of Heat** and because the solutions for them can be verified, only the further step of deriving solutions from them for arbitrary initial temperature $f(x)$ is here regarded as formal. The solutions for arbitrary initial temperature may also be obtained by direct application of the Laplace transformation, but in order to apply the inversion theorem to the solution of the subsidiary equation orders of integration must be interchanged so the solutions obtained in this way are either purely formal or must be completely justified, whereas with the Green's function procedure the first part of the solution is established rigorously and only the last step is regarded as formal or has to be defended.

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* Titchmarsh, Proc. London Math. Soc. (2) 22 (1922) 16.

** C.H. §§ 70 - 93.

1. <u>The rod $0 < x < l$. Instantaneous source of strength Q at $x = x'$ at $t = 0$. Boundary conditions</u> $k_1\frac{\partial v}{\partial x} - h_1 v = 0$ at $x = 0$ <u>and</u> $k_2\frac{\partial v}{\partial x} + h_2 v = 0$ <u>at $x = l$.</u>

We have to solve

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{\kappa}\frac{\partial v}{\partial t}, \quad 0 < x < l, \quad t > 0$$

with
$$k_1\frac{\partial v}{\partial x} - h_1 v = 0, \quad x = 0, \quad t > 0$$
$$k_2\frac{\partial v}{\partial x} + h_2 v = 0, \quad x = l, \quad t > 0,$$

and
$$v = \frac{Q}{2\sqrt{(\pi\kappa t)}}e^{-(x-x')^2/4\kappa t} + w = u + w,$$

where
$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{\kappa}\frac{\partial w}{\partial t}, \quad 0 < x < l, \quad t > 0,$$

and
$$\lim_{t\to 0} w = 0.$$

The subsidiary equation for w is

$$\frac{d^2\bar{w}}{dx^2} - q^2\bar{w} = 0, \quad 0 < x < l,$$

with solution $\bar{w} = A\sinh qx + B\cosh qx$.

Also $\bar{u} = \frac{Q}{2\kappa q}e^{-q|x-x'|}$.

Thus the boundary conditions at $x = 0$ and $x = l$ require

$$k_1 qA - h_1 B = \frac{Q}{2\;q}(h_1 - k_1 q)e^{-qx'}$$

$$A(k_2 q\cosh ql + h_2\sinh ql) + B(k_2 q\sinh ql + h_2\cosh ql) = \frac{Q}{2\kappa q}(k_2 q - h_2)e^{-q(l-x')}.$$

Solving and substituting we find

$$\bar{v} = \frac{Q\left[k_1 q\cosh qx + h_1\sinh qx\right]\left[k_2 q\cosh q(l - x') + h_2\sinh q(l - x')\right]}{\kappa q\left[(k_1 k_2 q^2 + h_1 h_2)\sinh ql + q(k_1 h_2 + k_2 h_1)\cosh ql\right]},$$

when $0 < x < x'$, and when $x' < x < l$ we interchange x and x'.

From the Inversion Theorem we have, writing[+] $\mu = \sqrt{(\lambda/\kappa)}$,

$$v = \frac{Q}{2i\pi\kappa}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}}{\mu}\ \frac{[k_1\mu\cosh\mu x + h_1\sinh\mu x][k_2\mu\cosh\mu(\ell - x') + h_2\sinh\mu(\ell - x')]}{(k_1k_2\mu^2 + h_1h_2)\sinh\mu\ell + \mu(k_1h_2 + k_2h_1)\cosh\mu\ell}\,d\lambda \quad \ldots\ldots\ (1)$$

If $h_1 = h_2 = 0$, $\lambda = 0$ is a pole of the integrand of (1), not otherwise. The other poles of the integrand are the zeros of

$$(k_1k_2\mu^2 + h_1h_2)\sinh\mu\ell + \mu(k_1h_2 + k_2h_1)\cosh\mu\ell\,, \quad \ldots\ldots\ (2)$$

this has no real and no complex zeros[x]. The imaginary zeros are the roots of

$$(k_1k_2\beta^2 - h_1h_2)\sin\beta\ell = \beta(k_1h_2 + k_2h_1)\cos\beta\ell\,. \quad \ldots\ldots\ (3)$$

If the positive roots of (2) are $\beta_1, \beta_2, \ldots$ the poles of the integrand of (1) are $\lambda = -\kappa\beta_n^2$, $n = 1, 2, \ldots$.

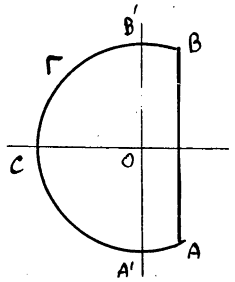

Fig. 3.

To evaluate the integral (1) consider the integral of its integrand taken round the contour of Fig. 3 consisting of the line AB to the right of the imaginary axis and distant γ from it, and portion of the circle whose radius R takes a sequence of values avoiding the zeros of (2); such a sequence would be $\left\{\kappa(n + \tfrac{1}{2})^2\pi^2/\ell^2\right\}$. Then it can be shown that as the radius R tends to infinity in this way the integral over Γ tends to zero and thus $\int_{\gamma-i\infty}^{\gamma+i\infty}$ may be replaced by $2\pi i$ times the sum of the residues of the integrand at the poles within the contour

---

[+] This notation will be used throughout the sequel.

[x] The proof is similar to I § 5.

Now $\left[\frac{d}{d\lambda}\left\{\mu(k_1k_2\mu^2 + h_1h_2)\sinh\mu\ell + \mu^2(k_1h_2 + k_2h_1)\cosh\mu\ell\right\}\right]_{\lambda=-\kappa\beta_n^2}$

$$= \frac{\cos\beta_n\ell}{2\kappa(h_1h_2 - k_1k_2\beta_n^2)}\left\{(k_1^2\beta_n^2 + h_1^2)\left[\ell(k_2^2\beta_n^2 + h_2^2) + k_2h_2\right] + k_1h_1(k_2^2\beta_n^2 + h_2^2)\right\}.$$

Using this result (1) gives

$$v = Q\sum_{n=1}^{\infty} Z_n(x)Z_n(x')e^{-\kappa\beta_n^2 t}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (4)$$

where $$Z_n(x) = \frac{\left[2(k_2^2\beta_n^2 + h_2^2)\right]^{\frac{1}{2}}(k_1\beta_n\cos\beta_n x + h_1\sin\beta_n x)}{\left\{(k_1^2\beta_n^2 + h_1^2)\left[\ell(k_2^2\beta_n^2 + h_2^2) + k_2h_2\right] + k_1h_1(k_2^2\beta_n^2 + h_2^2)\right\}^{\frac{1}{2}}}, \quad \ldots \ (5)$$

and (3), being symmetrical in x and x', holds for $x \gtrless x'$.

If $h_1 = h_2 = 0$ there is in addition a pole at $\lambda = 0$ which gives an additional term $\frac{Q}{\ell}$.

Putting $Q = f(x')dx'$ and integrating from 0 to ℓ we have a formal solution for initial temperature $f(x)$

$$v = \sum_{n=1}^{\infty} Z_n(x)\, e^{-\kappa\beta_n^2 t}\int_0^{\ell} Z_n(x')f(x')dx', \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (6)$$

where if $h_1 = h_2 = 0$ a term $\frac{1}{\ell}\int_0^{\ell} f(x')dx'$ is to be added to (6).

Letting $t \to 0$ we have the formal expansion

$$f(x) = \sum_{n=1}^{\infty} Z_n(x)\int_0^{\ell} Z_n(x')f(x')dx', \quad \ldots\ldots\ldots\ldots\ldots\ldots \ (7)$$

where if $h_1 = h_2 = 0$ a term $\frac{1}{\ell}\int_0^{\ell} f(x')dx'$ is to be added to (7).

2. Semi-infinite rod $x > 0$. Instantaneous source of strength Q at $x = x'$ at $t = 0$. Boundary condition** $k\frac{\partial v}{\partial x} - hv = 0$ at $x = 0$.

As in § 1 we have

$$\bar{v} = Ae^{-qx} + \frac{Q}{2\kappa q}e^{-q|x-x'|}$$

where $k\frac{d\bar{v}}{dx} - h\bar{v} = 0$.

Hence we find

$$\bar{v} = \frac{Q}{\kappa q}\cdot\frac{kq\cosh qx + h\sinh qx}{kq + h}e^{-qx'}, \quad \text{when } x < x',$$

and when $x > x'$ we interchange x and x'.

From the Inversion Theorem

$$v = \frac{Q}{2\pi i\kappa}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t - \mu x'}\,\frac{k\mu\cosh\mu x + h\sinh\mu x}{k\mu + h}\,\frac{d\lambda}{\mu}. \qquad \ldots\ldots\ldots (1)$$

Using the contour of Fig. I with $\lambda = \kappa u^2 e^{i\pi}$ on EF and $\lambda = \kappa u^2 e^{-i\pi}$ on CD, (1) becomes*

$$v = \frac{2Q}{\pi}\int_0^\infty e^{-\kappa u^2 t}\,\frac{(ku\cos ux + h\sin ux)(ku\cos ux' + h\sin ux')}{k^2u^2 + h^2}\,du$$

$$= \frac{2Q}{\pi}\int_0^\infty e^{-\kappa u^2 t}\,Z(u,x)Z(u,x')\,du, \qquad \ldots\ldots\ldots (2)$$

where $Z(u,x) = \dfrac{ku\cos ux + h\sin ux}{\sqrt{(k^2u^2 + h^2)}}$. $\qquad \ldots\ldots\ldots (3)$

For initial temperature $f(x)$ we have formally

$$v = \frac{2}{\pi}\int_0^\infty e^{-\kappa u^2 t}\,Z(u,x)\,du\int_0^\infty Z(u,x')f(x')\,dx', \qquad \ldots\ldots\ldots (4)$$

and so, letting $t \to 0$, $f(x) = \dfrac{2}{\pi}\displaystyle\int_0^\infty Z(u,x)\,du\int_0^\infty Z(u,x')f(x')\,dx'$, ...(5)

* This may alternatively be reduced to the usual form: cf. Carslaw and Jaeger, Phil. Mag. (7), 26 (1938) p. 473, § 4.

** In regions extending to infinity solutions are to be bounded at infinity unless otherwise stated.

3. <u>The cylinder $0 \le r < a$. An instantaneous cylindrical surface source of strength Q over $r = r'$ at $t = 0$. Boundary condition $k\frac{\partial v}{\partial r} + hv = 0$ when $r = a$.</u>

Here we have to solve

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = \frac{1}{\kappa}\frac{\partial v}{\partial t}, \quad 0 \leqslant r < a, \quad t > 0$$

with $k\frac{\partial v}{\partial r} + hv = 0, \quad r = a, \quad t > 0$

and $v = u + w$,

where $u = \frac{Q}{4\pi\kappa t} e^{-\frac{r^2+r'^2}{4\kappa t}} I_0\left(\frac{rr'}{2\kappa t}\right)$.

and $\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} = \frac{1}{\kappa}\frac{\partial w}{\partial t}, \quad 0 \leqslant r < a, \quad t > 0$,

and $\lim_{t \to 0} w = 0, \quad 0 \leqslant r < a$.

Now
$$\left.\begin{aligned}\bar{u} &= \frac{Q}{2\pi\kappa} I_0(qr')K_0(qr), \quad r > r' \\ &= \frac{Q}{2\pi\kappa} I_0(qr)K_0(qr'), \quad r < r'.\end{aligned}\right\}$$

Also if $\bar{w} = A\, I_0(qr)$, the boundary condition when $r = a$ requires

$$A\left\{kq\, I_0'(qa) + hI_0(qa)\right\} = -\frac{Q}{2\pi\kappa} I_0(qr')\left\{kq\, K_0'(qa) + h\, K_0(qa)\right\}$$

and hence

$$\bar{v} = \bar{u} + \bar{w} = \frac{Q}{2\pi\kappa}\, \frac{K_0(qr)\left[kq\, I_0'(qa) + h\, I_0(qa)\right] - I_0(qr)\left[kq\, K_0'(qa) + h\, K_0(qa)\right]}{kq\, I_0'(qa) + h\, I_0(qa)}\, I_0(qr') \quad \ldots (1)$$

when $r > r'$, and when $r < r'$ we interchange r and r'.

The Inversion Theorem gives

$$v = \frac{Q}{4\pi^2 i\kappa}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t}\, I_0(\mu r')\, \frac{K_0(\mu r)\left[k\mu I_0'(\mu a) + h\, I_0(\mu a)\right] - I_0(\mu r)\left[k\mu K_0'(\mu a) + hK_0(\mu a)\right]}{k\mu\, I_0'(\mu a) + h\, I_0(\mu a)}\, d\lambda \quad \ldots (2)$$

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* A large number of problems on Conduction of Heat in cylinders is given in Carslaw and Jaeger, "On some problems in Conduction of Heat with circular symmetry", Proc. Lond. Math. Soc., (2), 46 (1940) ~~in press~~. The verification of the solutions is discussed there and the question of the position of the poles of $\bar{v}(\lambda)$. § 5 below incorporates several of these results.

The poles of the integrand of (2) are at $\lambda = -\kappa\alpha_s^2$ where $\pm\alpha_s$, $s = 1, 2, \ldots$ are the positive roots[*] of

$$k\alpha J_0'(a\alpha) + h\,J_0(a\alpha) = 0\,. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

If $h = 0$ there is also a pole at $\lambda = 0$.

Evaluating the residues at these poles we find

$$v = \frac{Q}{\pi a^2} \sum_{s=1}^{\infty} e^{-\kappa\alpha_s^2 t} \frac{J_0(r\alpha_s)\,J_0(r'\alpha_s)}{J_0^2(a\alpha_s) + J_1^2(a\alpha_s)}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

where if $h = 0$ an additional term $\dfrac{Q}{\pi a^2}$ is to be added to the right hand side.

Putting $Q = 2\pi r' f(r')$ and integrating with respect to r' from 0 to a we have the formal solution for initial temperature $f(r)$

$$v = \frac{2}{a^2} \sum_{s=1}^{\infty} e^{-\kappa\alpha_s^2 t} \frac{J_0(r\alpha_s)}{J_0^2(a\alpha_s) + J_1^2(a\alpha_s)} \int_0^a r' f(r')\,J_0(r'\alpha_s)\,dr'\,, \quad \ldots (5)$$

where if $h = 0$ a term $\dfrac{2}{a^2}\displaystyle\int_0^a r' f(r')\,dr'$ is to be added to the right hand side.

Putting $t = 0$ we have the expansion

$$f(r) = \frac{2}{a^2} \sum_{s=1}^{\infty} \frac{J_0(r\alpha_s)}{J_0^2(a\alpha_s) + J_1^2(a\alpha_s)} \int_0^a r' f(r')\,J_0(r'\alpha_s)\,dr'\,, \qquad \ldots\ldots (6)$$

where if $h = 0$ a term $\dfrac{2}{a^2}\displaystyle\int_0^a r' f(r')\,dr'$ is to be added to the right hand side.

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[*] All real and simple. Watson, Theory of Bessel Functions, §§ 15.23, 15.25. This work will be cited throughout the sequel as W.B.F.

4. The region bounded internally by the cylinder $r = a$. An instantaneous cylindrical surface source of strength Q over $r = r'$ at $t = 0$. The boundary condition at $r = a$, $k\frac{\partial v}{\partial r} - hv = 0$.

Here as in § 3 we find

$$\bar{v} = \frac{Q}{2\pi\kappa}\;\frac{I_0(qr)\left[kq\,K_0'(qa) - h\,K_0(qa)\right] - K_0(qr)\left[kq\,I_0'(qa) - h\,I_0(qa)\right]}{kq\,K_0'(qa) - h\,K_0(qa)}K_0(qr'), \qquad \text{.. (1)}$$

when $r < r'$.

Hence

$$v = \frac{Q}{4\pi^2 i\kappa}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\,\frac{I_0(\mu r)\left[k\mu K_0'(\mu a) - h\,K_0(\mu a)\right] - K_0(\mu r)\left[k\mu I_0'(\mu a) - h\,I_0(\mu a)\right]}{k\mu K_0'(\mu a) - h\,K_0(\mu a)}\,K_0(\mu r')\,d\lambda \qquad \text{...... (2)}$$

$$= \frac{Q}{2\pi}\int_0^\infty e^{-\kappa u^2 t}\,C(u,r)C(u,r')u\,du\,, \qquad \text{...... (3)}$$

where
$$C(u,r) = \frac{J_0(ur)\left[kuY_1(au) + hY_0(au)\right] - Y_0(ur)\left[kuJ_1(au) + hJ_0(au)\right]}{\left\{\left[kuJ_1(au) + hJ_0(au)\right]^2 + \left[kuY_1(au) + hY_0(au)\right]^2\right\}^{\frac{1}{2}}} \qquad \text{..(4)}$$

Hence for initial temperature $f(r)$ we have formally

$$v = \int_0^\infty e^{-\kappa u^2 t}\,C(u,r)u\,du\int_a^\infty C(u,r')f(r')r'dr'\,, \qquad \text{...... (5)}$$

and putting $t = 0$

$$f(r) = \int_0^\infty C(u,r)u\,du\int_a^\infty C(u,r')f(r')r'dr'\,. \qquad \text{...... (6)}$$

5. <u>The hollow cylinder $a < r < b$. An instantaneous cylindrical surface source of strength Q over $r = r'$ at $t = 0$. The boundary conditions</u>

$$k_1 \frac{\partial v}{\partial r} - h_1 v = 0 , \quad r = a, \quad t > 0$$

$$k_2 \frac{\partial v}{\partial r} + h_2 v = 0 , \quad r = b, \quad t > 0.$$

Let as usual $v = u + q$, where

$$\left.\begin{aligned} \bar{u} &= \frac{Q}{2\pi\kappa} I_o(qr')K_o(qr) , \quad r > r' \\ &= \frac{Q}{2\pi\kappa} I_o(qr)K_o(qr') , \quad r < r' , \end{aligned}\right\}$$

and $\bar{w} = AI_o(qr) + BK_o(qr)$.

Then the boundary conditions require

$$A\left[k_1 q I_o'(qa) - h_1 I_o(qa)\right] + B\left[k_1 q K_o'(qa) - h_1 K_o(qa)\right] = -\frac{Q}{2\pi\kappa} K_o(qr')\left[k_1 q I_o'(qa) - h_1 I_o(qa)\right]$$

$$A\left[k_2 q I_o'(qb) + h_2 I_o(qb)\right] + B\left[k_2 q K_o'(qb) + h_2 K_o(qb)\right] = -\frac{Q}{2\pi\kappa} I_o(qr')\left[k_2 q K_o'(qb) + h_2 K_o(qb)\right]$$

Solving and substituting we find, for $a < r < r'$,

$$\bar{v} = \frac{Q}{2\pi\kappa\Delta(q)} \left\{ I_o(qr)\left[k_1 q K_o'(qa) - h_1 K_o(qa)\right] - K_o(qr)\left[k_1 q I_o'(qa) - h_1 I_o(qa)\right] \right\}$$

$$\times \left\{ I_o(qr')\left[k_2 q K_o'(qb) + h_2 K_o(qb)\right] - K_o(qr')\left[k_2 q I_o'(qb) + h_2 I_o(qb)\right] \right\} ,$$

where
$$\Delta(q) = \left[k_1 q I_o'(qa) - h_1 I_o(qa)\right]\left[k_2 q K_o'(bq) + h_2 K_o(bq)\right] - \left[k_2 q I_o'(bq) + h_2 I_o(bq)\right]\left[k_1 q K_o'(aq) - h_1 K_o(aq)\right] . \quad \ldots (1)$$

The poles of the integrand of v are the zeros of $\Delta(\mu)$ i.e. are at $\lambda = -\kappa\alpha_s^2$, where the α_s are the positive roots* of

$$\left[k_1 \alpha J_o'(a\alpha) - h_1 J_o(a\alpha)\right]\left[k_2 \alpha Y_o'(b\alpha) + h_2 Y_o(b\alpha)\right] - \left[k_1 \alpha Y_o'(a\alpha) - h_1 Y_o(a\alpha)\right]\left[k_2 \alpha J_o'(b\alpha) + h_2 J_o(b\alpha)\right] = 0 \quad \ldots (2)$$

---

* The roots are all real and simple, cf. Carslaw and Jaeger, <u>loc. cit.</u>

If $h_1 = h_2 = 0$, $\lambda = 0$ is also a pole of the integrand. Evaluating the residues at these poles we have finally

$$v = \frac{Q}{2\pi} \sum_{s=1}^{\infty} e^{-\kappa \alpha_s^2 t} F(r,\alpha_s) F(r',\alpha_s) , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where

$$F(r,\alpha_s) = \frac{\pi \alpha_s \left[k_1 \alpha_s J_0'(a\alpha_s) - h_1 J_0(a\alpha_s)\right]}{2^{\frac{1}{2}} \left\{ (k_2^2\alpha_s^2 + h_2^2)\left[k_1\alpha_s J_0'(a\alpha_s) - h_1 J_0(a\alpha_s)\right]^2 - (k_1^2\alpha_s^2 + h_1^2)\left[k_2\alpha_s J_0'(b\alpha_s) + h_2 J_0(b\alpha_s)\right]^2 \right\}^{\frac{1}{2}}}$$

$$\times \left\{ J_0(r\alpha_s)\left[k_2\ {}_s Y_0'(b\alpha_s) + h_2 Y_0(b\alpha_s)\right] - Y_0(r\alpha_s)\left[k_2\ {}_s J_0'(b\alpha_s) + h_2 J_0(b\alpha_s)\right] \right\}$$

and, if $h_1 = h_2 = 0$, an additional term $\dfrac{Q}{\pi(b^2 - a^2)}$ is to be added to the right hand side of (3).

For initial temperature $f(r')$ we have formally

$$v = \sum_{s=1}^{\infty} e^{-\kappa \alpha_s^2 t} F(r,\alpha_s) \int_a^b r' f(r') F(r',\alpha_s) dr' \qquad \ldots\ldots\ldots\ldots (4)$$

and, if $h_1 = h_2 = 0$, an additional term

$$\frac{2}{b^2 - a^2} \int_a^b r' f(r') dr'$$

is to be added to the right hand side of (4).

Putting $t = 0$ we find

$$f(r) = \sum_{s=1}^{\infty} F(r,\alpha_s) \int_a^b r' f(r') F(r',\alpha_s) dr' , \qquad \ldots\ldots\ldots (5)$$

where, if $h_1 = h_2 = 0$, an additional term $\dfrac{2}{b^2 - a^2} \displaystyle\int_a^b r' f(r') dr'$ is to be added to the right hand side of (5).

6. <u>The sphere $0 \leq r < a$. Instantaneous spherical surface source of strength Q at $r = r'$ at $t = 0$. Boundary condition</u> $k\frac{\partial v}{\partial r} + hv = 0$, <u>when $r = a$.</u>

Putting $v = u/r$ the equations for u are

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial r^2}, \quad 0 \leq r < a, \quad t > 0 \\ &k\frac{\partial u}{\partial r} + (h - \frac{k}{a})u = 0, \quad r = a, \quad t > 0. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (1)$$

As usual, let $u = u_1 + w$,

where $$u_1 = \frac{Q}{8\pi r'(\pi\kappa t)^{\frac{1}{2}}}\left[e^{-(r-r')^2/4\kappa t} - e^{-(r+r')^2/4\kappa t}\right],$$

and thus $$\bar{u}_1 = \frac{Q}{8\pi r'\kappa q}\left[e^{-q|r-r'|} - e^{-q(r+r')}\right],$$

and where w is to satisfy

$$\frac{\partial w}{\partial t} = \kappa\frac{\partial^2 w}{\partial r^2}, \quad 0 \leq r < a, \quad t > 0,$$

with $\lim_{t \to 0} w = 0$.

The subsidiary equation for w is

$$\frac{d^2\bar{w}}{dr^2} - q^2\bar{w} = 0,$$

and a solution of this which makes $\bar{w}$ finite when $r = 0$ is

$$\bar{w} = A \sinh qr.$$

Substituting in (1) this gives

$$A = -\frac{Q}{8\pi r'\kappa q}\,\frac{\left\{a(h - kq) - k\right\}\left\{e^{-q(a-r')} - e^{-q(a+r')}\right\}}{akq\cosh qa + (ah - k)\sinh qa}.$$

Hence we have for $r' < r < a$

$$\bar{v} = \frac{Q}{4\pi r'\kappa q} \frac{\sinh qr'\left[(ah-k)\sinh q(a-r) + kaq\cosh q(a-r)\right]}{akq\cosh qa + (ah-k)\sinh qa}. \quad \ldots\ldots (2)$$

Thus by the inversion theorem

$$V = \frac{Q}{8\pi^2 i r'\kappa} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \frac{\sinh \mu r'\left[(ah-k)\sinh \mu(a-r) + ka\mu\cosh \mu(a-r)\right]}{ak\mu\cosh \mu a + (ah-k)\sinh \mu a} \frac{d\lambda}{\mu}. \quad \ldots\ldots (3)$$

Let $\xi_1, \xi_2, \ldots$ be the positive roots of the equation

$$ak\xi\cos\xi a + (ah-k)\sin\xi a = 0\,; \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

it is known that the roots of (4) are all real and simple*.

Then using the contour of Fig. 3 it is found that the line integral in (3) equals $2\pi i$ times the sum of the residues at the poles $\lambda = -\kappa\xi_s^2$, $s = 1, 2, \ldots$ There is no pole at $\lambda = 0$ unless $h = 0$.

Now

$$\left[\frac{d}{d\lambda}\left\{ak\mu^2\cosh\mu a + \mu(ah-k)\sinh\mu a\right\}\right]_{\lambda = -\kappa\xi_s^2}$$

$$= \frac{a\cos a\xi_s}{2\kappa(ah-k)}\left[a^2k^2\xi_s^2 + ah(ah-k)\right]$$

Hence finally

$$v = \frac{u}{r} = \frac{Q}{2\pi arr'} \sum_{s=1}^{\infty} \frac{(ah-1)^2 + a^2\xi_s^2}{a^2\xi_s^2 + ah(ah-1)} \sin r\xi_s \sin r'\xi_s\, e^{-\kappa\xi_s^2 t}, \qquad (5)$$

and if $h = 0$ a term $\dfrac{3Q}{4\pi a^3}$ is to be added to the right hand side of (5).

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* C.H., p. 137.

7. The spherical shell $a < r < b$. Instantaneous spherical surface source of strength Q at $r = r'$ at $t = 0$. Boundary conditions $k_1 \frac{\partial v}{\partial r} - h_1 v = 0$ at $r = a$, and $k_2 \frac{\partial v}{\partial r} + h_2 v = 0$, when $r = b$.

Putting $v = u/r$ we have to solve

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial r^2}, \quad a < r < b, \quad t > 0$$

$$k_1 \frac{\partial u}{\partial r} - (h_1 + \frac{k_1}{a})u = 0, \quad r = a, \quad t > 0$$

$$k_2 \frac{\partial u}{\partial r} + (h_2 - \frac{k_2}{b})u = 0, \quad r = b, \quad t > 0.$$

Putting $u = u_1 + w$,

where $u_1 = \frac{Q}{8\pi r'(\pi\kappa t)^{\frac{1}{2}}} \left[e^{-(r-r')^2/4\kappa t} - e^{-(r+r')^2/4\kappa t}\right]$

We have $\bar{u}_1 = \frac{Q}{8\pi r'\kappa q} \left[e^{-q|r-r'|} - e^{-q(r+r')}\right]$

$$\bar{w} = A \sinh qr + B \cosh qr$$

and the boundary conditions require

$$A\left\{bk_2 q \cosh qb + (bh_2 - k_2)\sinh qb\right\} + B\left\{bk_2 q \sinh qb + (bh_2 - k_2)\cosh qb\right\}$$
$$= -\frac{Q}{8\pi r'\kappa q}(bh_2 - k_2 - k_2 qb)\left[e^{-q(b-r')} - e^{-q(b+r')}\right]$$

$$A\left\{ak_1 q \cosh qa - (ah_1 + k_1)\sinh qa\right\} + B\left\{ak_1 q \sinh qa - (ah_1 + k_1)\cosh qa\right\}$$
$$= -\frac{Q}{8\pi r'\kappa q}\left\{(ak_1 q - ah_1 - k_1)e^{-q(r'-a)} + (ak_1 q + ah_1 + k_1)e^{-q(r'+a)}\right\}.$$

Solving for A and B and substituting we find that when $a < r < r'$

$$\bar{u} = \frac{Q}{4\pi r'\kappa q\,\Delta(q)}\left\{ak_1 q \cosh q(r-a) + G \sinh q(r-a)\right\}$$
$$\times \left\{bk_2 q \cosh q(b-r') + H \sinh q(b-r')\right\}, \quad \ldots (1)$$

where $G = ah_1 + k_1$, $H = bh_2 - k_2$, ...(2)

and $\Delta(q) = \left[abk_1k_2q^2 + HG\right] \sinh(b-a)q + q\left[ak_1H + bk_2G\right] \cosh(b-a)q$, (3)

and when $r' < r < b$ we interchange r and r' in (1).

The poles of $\overline{u}(\lambda)$ are $\lambda = -\kappa\xi_s^2$, where $\pm\xi_s$, $s = 1, 2, \ldots$ are the roots of

$$\left[GH - abk_1k_2 \quad {}^2\right] \sin(b-a) \quad + \left[ak_1H + bk_2G\right] \cos(b-a) \quad = 0. \qquad (4)$$

Also if both h_1 and h_2 are zero $\lambda = 0$ is a pole.

Now

$$\frac{d}{d\lambda}\left[\mu\,\Delta(\mu)\right]_{\lambda=-\kappa\xi_s^2} = \frac{(b-a)(a^2k_1^2\xi_s^2+G^2)(b^2k_2^2\xi_s^2+H^2)+(Hak_1+Gbk_2)(GH+abk_1k_2\xi_s^2)}{2\kappa\,(GH-abk_1k_2\,\xi_s^2)}\cos(b-a)\xi_s$$

Using this result we have finally using the inversion theorem with the contour of Fig. 3,

$$v = \frac{Q}{2\pi rr'} \sum_{s=1}^{\infty} e^{-\kappa\xi_s^2 t} R_s(r)R_s(r') , \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

where

$$R_s(r) = \frac{(H^2 + b^2k_2^2\xi_s^2)^{\frac{1}{2}}\left[G \sin\xi_s(r-a) + ak_1\,\xi_s \cos\xi_s(r-a)\right]}{\left\{(b-a)(a^2k_1^2\xi_s^2+G^2)(b^2k_2^2\xi_s^2+H^2)+(Hak_1+Gbk_2)(GH+abk_1k_2\xi_s^2)\right\}^{\frac{1}{2}}}$$

... (6)

If $h_1 = h_2 = 0$ a term $\dfrac{3Q}{4\pi(b^3-a^3)}$ is to be added to the right hand side of (6).

8. Instantaneous spherical surface source of strength Q over $r = r'$ at $t = 0$ in the region bounded internally by $r = a$. Boundary condition $k\dfrac{\partial v}{\partial r} - hv = 0$, when $r = a$.

Putting $v = u/r$ the equations for u are

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial r^2}, \quad r > a, \quad t > 0 \\ &k\frac{\partial u}{\partial r} - \left(h + \frac{k}{a}\right)u = 0, \quad r = a, \quad t > 0. \end{aligned}\right\}$$

Let as before $u = u_1 + w$,

where $$u_1 = \frac{Q}{8\pi r'(\pi\kappa t)^{\frac{1}{2}}}\left[e^{-(r-r')^2/4\kappa t} - e^{-(r+r')^2/4\kappa t}\right]$$

and $$\bar{u}_1 = \frac{Q}{8\pi r'\kappa q}\left[e^{-q|r-r'|} - e^{-q(r+r')}\right].$$

Then a solution of the subsidiary equation for w, finite as $r \to \infty$ is $\bar{w} = Ae^{-qr}$, and the boundary condition at $r = a$ requires

$$A(akq + ah + k)e^{-qa} = \frac{Q}{8\pi r'\kappa q}\left\{(akq - ah - k)e^{-q(r'-a)} + (akq + ah + k)e^{-q(a+r')}\right\}.$$

Thus

$$\bar{u} = \frac{Q}{8\pi r'\kappa q}\left\{e^{-q|r-r'|} + e^{-q(r+r'-2a)} - \frac{2(ah+k)}{akq + ah + k}\,e^{-q(r+r'-2a)}\right\}. \qquad (1)$$

And[*]

$$v = \frac{Q}{8\pi r r'(\pi\kappa t)^{\frac{1}{2}}}\left\{e^{-(r-r')^2/4\kappa t} + e^{-(r+r'-2a)^2/4\kappa t} - 2\,\frac{ah+k}{ak}\int_0^\infty \exp\left[-\frac{ah+k}{ak}\xi - \frac{(r+r'-2a+\xi)^2}{4\kappa t}\right]d\xi\right\}$$

$$= \frac{Q}{8\pi r r'(\pi\kappa t)^{\frac{1}{2}}}\left\{\begin{aligned} &e^{-(r-r')^2/4\kappa t} + e^{-(r+r'-2a)^2/4\kappa t} \\ &- \frac{ah+k}{ak}(4\pi\kappa t)^{\frac{1}{2}}\exp\left[\kappa t\left(\frac{ah+k}{ak}\right)^2 + (r+r'-2a)\,\frac{ah+k}{ak}\right] \\ &\times \ \operatorname{erfc}\left\{\frac{r+r'-2a}{2\sqrt{(\kappa t)}} + \frac{ah+k}{ak}\sqrt{(\kappa t)}\right\} \end{aligned}\right\}$$

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[*] For the reduction see Carslaw and Jaeger, Phil. Mag. (7), XXVI (1938) 473, § 4.

9. It has hitherto been supposed that none of the constants in the boundary conditions are negative. If either or both of h_1/k_1 and h_2/k_2 (or if there is only one bounding surface, h/k) is negative the boundary condition is of no physical interest since it implies that the solid gains heat at this surface at a rate proportional to its temperature, but the effect on mathematics* of the solution is worth brief consideration.

Consider the problem of § 1 : if both h_1/k_1 and h_2/k_2 were positive the expression § 1 (2), namely

$$(k_1k_2\mu^2 + h_1h_2)\sinh\mu\ell + \mu(k_1h_2 + k_2h_1)\cosh\mu\ell$$

had no real zero (save $\mu = 0$ which did not give rise to a pole of the integrand of § 1 (1)).

If $h_1 = h_2 = 0$, $\lambda = 0$ was a pole of the integrand of § 1 (1).

If either or both of h_1/k_1 and h_2/k_2 are negative there may in addition be poles of the integrand of § 1 (1) for real positive values of λ. The exact results are as follows:

(i) if both h_1/k_1 and h_2/k_2 are negative there are either one or two positive zeros of § 1 (2).

(ii) if h_1/k_1 is negative, say $= -H$, writing $h_2' = h_2/k_2$ we find that

(iia) if $H_1 > h_2'$ there is one positive zero of § 1 (2)

(iib) if $H_1 < h_2'$ there are one or no positive zeros according as $\frac{1}{H_1} - \frac{1}{h_2'} < \ell$ or $> \ell$.

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* It may be remarked that Lowan in several papers [e.g. Bull. Amer. Math. Soc. 44 (1938) 125, ibid 45 (1939) 310] has used boundary conditions of this type without specifying the sign of the constants, which sometimes have to be taken $\geqslant 0$, and sometimes $\leqslant 0$ to correspond with the physical boundary conditions of Conduction of Heat.

Since the path $(\gamma - i\infty, \gamma + i\infty)$ of the Inversion Theorem is to have all the singularities of the integrand to its left, a positive pole $\lambda = \lambda_1$ of the integrand will give rise to a term proportional to $e^{\lambda_1 t}$, i.e. exponentially increasing.

10. If we consider the problem of § 3 with negative h/k we obtain the well-known results for Dini series.

It is known* that if h/k is negative the equation § 3 (3)

$$k\alpha J_0'(a\alpha) + h\, J_0(a\alpha) = 0$$

has a pair of pure imaginary zeros, say $\pm i\beta$. Then the integrand of § 3 (2) has a real positive zero $\kappa\beta^2$ in addition to the real negative zeros $-\kappa\alpha_s^2$, $s = 1, 2, \ldots$

The pole $\lambda = \kappa\beta^2$ gives a contribution to the result of

$$\frac{Q}{\pi a^2} e^{\kappa\beta^2 t} \frac{I_0(r\beta)\; I_0(r'\beta)}{I_0^2(a\beta) - I_1^2(a\beta)} .$$

Combining this result with those of § 3 we obtain the expansion** for $f(r')$

$$f(r) = \frac{2}{a^2} \sum_{s=1}^{\infty} \frac{J_0(r\alpha_s)}{J_0^2(a\alpha_s) + J_1^2(a\alpha_s)} \int_0^a r' f(r')\, J_0(r'\alpha_s) dr' + B_0(r) ,$$

where $B_0(r) = 0$ if $h/k > 0$

$$= \frac{2}{a^2} \int_0^a r' f(r') dr' \quad \text{if} \quad h/k = 0$$

$$= \frac{2}{a^2} \frac{I_0(\beta r)}{I_0^2(a\beta) - I_1^2(a\beta)} \int_0^a I_0(\beta r') r' f(r') dr' , \quad \text{if} \quad h/k < 0.$$

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* W.B.F. § 15.25

** W.B.F. § 18.3

CHAPTER III.

HEAT CONDUCTION IN COMPOSITE CIRCULAR CYLINDERS.

The conduction of heat in composite linear and spherical solids has been extensively studied. The object of this Chapter is to present some results for circular cylinders of two materials; it is to be regarded as a sequel to two papers, A problem in conduction of heat*, and Some two dimensional problems in conduction of heat with circular symmetry+, which will be cited as I and II, respectively. The method used here is that of the Laplace transformation as developed in the paper I, which may be consulted for a fuller exposition; the solutions given here are formal, the verification that they satisfy the given differential equation and boundary conditions is discussed in Appendix I.

The algebra is complicated, but is greatly simplified by the systematic use of cylinder functions; the principal results used are collected in § 2.

In § 3 the hollow composite cylinder with zero initial temperature and boundaries kept at constant temperatures for $t > 0$ is discussed; in § 4 the cylindrical surface source in the hollow cylinder with boundaries kept at zero; in § 5 the corresponding results for the solid composite cylinder are given without proof. A brief discussion of the region outside a circular cylinder is given in § 6.

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* Carslaw and Jaeger, Proc. Cambridge Phil. Soc., 35 (1939) 394.

\+ Carslaw and Jaeger, Proc. London Math. Soc. (2), 46 (1940) 361.

2. It is convenient to introduce functions defined as follows:

$$D(x,y) = I_0(x)K_0(y) - K_0(x)I_0(y), \tag{1}$$

$$D_{r,s}(x,y) = \frac{\partial^{r+s}}{\partial x^r \, \partial y^s} D(x,y) \,, \tag{2}$$

where for brevity $D_1(x,y)$ will be written for $D_{1,0}(x,y)$.

These functions are connected with the cylinder functions

$$C(x,y) = J_0(x)Y_0(y) - Y_0(x)J_0(y) \tag{3}$$

$$C_{r,s}(x,y) = \frac{\partial^{r+s}}{\partial x^r \, \partial y^s} C(x,y) \tag{4}$$

by the relations

$$D(ix,iy) = -\tfrac{1}{2}\pi C(x,y) \tag{5}$$

$$i^{r+s} D_{r,s}(ix,iy) = -\tfrac{1}{2}\pi C_{r,s}(x,y) \,. \tag{6}$$

The following properties are trivial

$$D_{r+2,s}(x,y) + \frac{1}{x} D_{r+1,s}(x,y) - D_{r,s}(x,y) = 0 \tag{7}$$

$$D_{r,s+2}(x,y) + \frac{1}{y} D_{r,s+1}(x,y) - D_{r,s}(x,y) = 0 \tag{8}$$

$$D(x,y) = - D(y,x) \tag{9}$$

$$D_{0,1}(x,y) = - D_1(y,x) \tag{10}$$

$$D_1(x,x) = \frac{1}{x} \tag{11}$$

$$D(x,y)D_1(x,z) - D(x,z)D_1(x,y) = \frac{1}{x} D(z,y) \tag{12}$$

$$D(x,y)D_{1,1}(x,z) - D_{0,1}(x,z)D_1(x,y) = \frac{1}{x} D_1(z,y) \tag{13}$$

$$D(x,y)D_{1,1}(x,y) - D_{0,1}(x,y)D_1(x,y) = \frac{1}{xy} \tag{14}$$

$$I_0(x)D_1(x,y) - I_0'(x)D(x,y) = \frac{1}{x} I_0(y) \tag{15}$$

$$I_0(x)D_1(y,z) - I_0(z)D_1(y,x) = I_0'(y)D(x,z) \tag{16}$$

$$I_0(x)D(y,z) + I_0(y)D(z,x) + I_0(z)D(x,y) = 0 \tag{17}$$

Equations (15), (16), (17) are valid also if the I are replaced by K.

3. <u>Hollow cylinder of one material from $r = a$ to $r = b$ and of another from $r = b$ to $r = c$. The initial temperature of the whole zero. The surface $r = a$ kept at V_o, and $r = c$ at zero, for $t > 0$.</u>

Let the temperature, conductivity, specific heat, and density in $a < r < b$ be v_1, K_1, c_1, and ρ_1, and let $\kappa_1 = K_1/c_1\rho_1$; let the corresponding quantities in $b < r < c$ be v_2, K_2, c_2, ρ_2 and κ_2.

Then we have to solve

$$\kappa_1\left(\frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r}\frac{\partial v_1}{\partial r}\right) = \frac{\partial v_1}{\partial t}, \qquad a < r < b, \quad t > 0 \tag{18}$$

$$\kappa_2\left(\frac{\partial^2 v_2}{\partial r^2} + \frac{1}{r}\frac{\partial v_2}{\partial r}\right) = \frac{\partial v_2}{\partial t}, \qquad b < r < c, \quad t > 0 \tag{19}$$

with
$$v_1 = v_2, \qquad r = b, \quad t > 0 \tag{20}$$

$$K_1\frac{\partial v_1}{\partial r} = K_2\frac{\partial v_2}{\partial r}, \qquad r = b, \quad t > 0 \tag{21}$$

$$v_1 = V_o, \qquad r = a, \quad t > 0 \tag{22}$$

$$v_2 = 0, \qquad r = c, \quad t > 0 \tag{23}$$

$$v_1 = 0 \text{ when } t = 0, \ (a < r < b); \quad v_2 = 0 \text{ when } t = 0, \ (b < r < c). \tag{24}$$

Multiplying by e^{-pt}, $p > 0$, integrating with respect to t from 0 to ∞, and writing

$$\bar{v}_1 = \int_0^\infty e^{-pt} v_1 \, dt, \qquad \bar{v}_2 = \int_0^\infty e^{-pt} v_2 \, dt,$$

we obtain the subsidiary equations

$$\frac{d^2\bar{v}_1}{dr^2} + \frac{1}{r}\frac{d\bar{v}_1}{dr} - q_1^2\bar{v}_1 = 0, \qquad a < r < b \tag{25}$$

$$\frac{d^2\bar{v}_2}{dr^2} + \frac{1}{r}\frac{d\bar{v}_2}{dr} - q_2^2\bar{v}_2 = 0, \qquad b < r < c \tag{26}$$

to be solved with
$$\bar{v}_1 = V_o/p, \qquad r = a \tag{27}$$

$$\bar{v}_2 = 0, \qquad r = c \tag{28}$$

$$\bar{v}_1 = \bar{v}_2, \qquad r = b \tag{29}$$

$$K_1\frac{d\bar{v}_1}{dr} = K_2\frac{d\bar{v}_2}{dr}, \qquad r = b \tag{30}$$

where $q_1^2 = p/\kappa_1$, $q_2^2 = p/\kappa_2$. (31)

The general solution of (25) which satisfies (27) is

$$\bar{v}_1 = \frac{V_0}{p}\frac{D(rq_1,bq_1)}{D(aq_1,bq_1)} + A\ D(rq_1,aq_1) ,$$

and the general solution of (26) which satisfies (28) is

$$\bar{v}_2 = B\ D(rq_2,cq_2) .$$

Equations (29) and (30) then require

$$\left.\begin{aligned} &A\ D(bq_1,cq_1) - B\ D(bq_2,cq_2) = 0 \\ &K_1q_1A\ D_1(bq_1,aq_1) - K_2q_2B\ D_1(bq_2,cq_2) = -\frac{K_1V_0}{bp\ D(aq_1,bq_1)} . \end{aligned}\right\}$$

Solving for A and B and substituting, we obtain finally

$$\bar{v}_1 = \frac{V_0}{p\Delta(p)}\left\{K_1q_1D_1(bq_1,rq_1)D(bq_2,cq_2) - K_2q_2D(bq_1,rq_1)D_1(bq_2,cq_2)\right\} \tag{32}$$

$$\bar{v}_2 = \frac{K_1V_0}{bp\ \Delta(p)}D(rq_2,cq_2) , \tag{33}$$

where

$$\Delta(p) = K_1q_1D_1(bq_1,aq_1)D(bq_2,cq_2) - K_2q_2D(bq_1,aq_1)D_1(bq_2,cq_2) . \tag{34}$$

To evaluate v_2 we have from the inversion theorem for the Laplace Transformation:

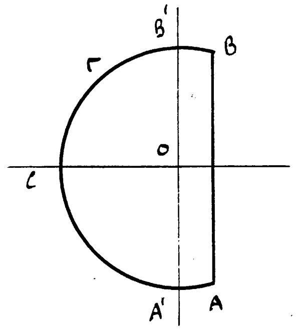
$$v_2 = \frac{K_1V_0}{2\pi ib}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}\ D(r\mu_2,c\mu_2)}{\lambda\Delta(\lambda)}\,d\lambda , \tag{35}$$

where $\mu_1 = \sqrt{(\lambda/\kappa_1)}$, and $\mu_2 = \sqrt{(\lambda/\kappa_2)}$. (36)

The integrand of (35) has simple poles at $\lambda = 0$ and $\lambda = -\kappa_1\alpha_s^2$, where $\pm\alpha_s$, $s = 1, 2, \ldots$, are the roots of

$$K_1C_1(b\alpha,a\alpha)C(\kappa b\alpha,\kappa c\alpha) - \kappa K_2C(b\alpha,a\alpha)C_1(\kappa b\alpha,\kappa c\alpha) = 0 , \tag{37}$$

where $\kappa = \sqrt{(\kappa_1/\kappa_2)}$. (38)



Consider the integral of the integrand of (35) taken round the contour of Fig. 3 consisting of the line AB distant γ from the imaginary axis and portion of a circle Γ which does not pass through any pole of the integrand. The integral over Γ tends to zero* as its radius tends to infinity through a sequence of values which avoid poles of the integrand. Thus the $\int_{\gamma - i\infty}^{\gamma + i\infty}$ in (35) may be replaced by $2\pi i$ times the sum of the residues at the poles of its integrand.

The residue at $\lambda = 0$ is

$$\frac{b \log(r/c)}{K_1 \log(b/c) - K_2 \log(b/a)} . \tag{39}$$

To evaluate the residue at $\lambda = -\kappa_1 \alpha_s^2$ we have

$$\begin{aligned} 2\lambda \frac{d\Delta(\lambda)}{d\lambda} = {} & b(K_1\mu_1^2 - K_2\mu_2^2) D(b\mu_1, a\mu_1) D(b\mu_2, c\mu_2) \\ & + b\mu_1\mu_2 (K_1 - K_2) D_1(b\mu_1, a\mu_1) D_1(b\mu_2, c\mu_2) \\ & + c\mu_2 \left\{ K_1\mu_1 D_{o,1}(b\mu_2, c\mu_2) D_1(b\mu_1 a\mu_1) - K_2\mu_2 D(b\mu_1, a\mu_1) D_{1,1}(b\mu_2, c\mu_2) \right\} \\ & + a\mu_1 \left\{ K_1\mu_1 D(b\mu_2, c\mu_2) D_{1,1}(b\mu_1, a\mu_1) - K_2\mu_2 D_{o,1}(b\mu_1, a\mu_1) D_1(b\mu_2, c\mu_2) \right\} \end{aligned} \tag{40}$$

Now, when $\lambda = -\kappa_1 \alpha_s^2$

$$\frac{K_2\mu_2 D_1(b\mu_2, c\mu_2)}{K_1\mu_1 D_1(b\mu_1, a\mu_1)} = \frac{D(b\mu_2, c\mu_2)}{D(b\mu_1, a\mu_1)} = \frac{C(\kappa b\alpha_s, \kappa c\alpha_s)}{C(b\alpha_s, a\alpha_s)} = \frac{K_2 \kappa C_1(\kappa b\alpha_s, \kappa c\alpha_s)}{K_1 C_1(b\alpha_s, a\alpha_s)} = \sigma, \text{ say.} \tag{41}$$

---

* For details of the method of proof see II, § 2.

Introducing this in (4) and using (14), we obtain

$$\left[\lambda \frac{d\Delta(\lambda)}{d\lambda}\right]_{\lambda=-\kappa_1\alpha_s^2} = \frac{1}{2}\Big[b(K_1\mu_1^2 - K_2\mu_2^2)\sigma D^2(b\mu_1, a\mu_1) + b(K_1/K_2)\mu_1^2(K_1 - K_2)D_1^2(b\mu_1, a\mu_1)\sigma$$

$$+ \frac{K_1\sigma}{b} - \frac{K_2}{b\sigma}\Big]_{\lambda=-\kappa_1\alpha_s^2}$$

$$= \frac{1}{2bK_2F(\alpha_s)C(\kappa b\alpha_s, \kappa c\alpha_s)C(b\alpha_s, a\alpha_s)}, \tag{42}$$

where

$$\frac{1}{F(\alpha_s)} = C^2(\kappa b\alpha_s, \kappa c\alpha_s)\Big\{\frac{1}{4}K_2b^2\pi^2(K_2\kappa^2 - K_1)\alpha_s^2C^2(b\alpha_s, a\alpha_s)$$

$$+ \frac{1}{4}\pi^2b^2K_1(K_1 - K_2)\alpha_s^2C_1^2(b\alpha_s, a\alpha_s) + K_1K_2\Big\}$$

$$- K_2^2C^2(b\alpha_s, a\alpha_s) . \tag{43}$$

Thus the residue of the integrand of (35) at $\lambda = -\kappa\alpha_s^2$ is

$$-\pi bK_2e^{-\kappa_1\alpha_s^2t}\, C(\kappa r\alpha_s, \kappa c\alpha_s)C(\kappa b\alpha_s, \kappa c\alpha_s)C(b\alpha_s, a\alpha_s)F(\alpha_s) . \tag{44}$$

From (39) and (44) we have finally

$$v_2 = V_o\frac{K_1 \log(r/c)}{K_1 \log(b/c) - K_2 \log(b/a)}$$

$$-\pi K_1K_2V_o \sum_{s=1}^{\infty} e^{-\kappa_1\alpha_s^2t}\, C(\kappa r\alpha_s, \kappa c\alpha_s)C(\kappa b\alpha_s, \kappa c\alpha_s)C(b\alpha_s, a\alpha_s)F(\alpha_s) . \tag{45}$$

In the same way we find from (32), using (12),

$$v_1 = V_o\frac{K_1 \log(b/c) - K_2 \log(b/r)}{K_1 \log(b/c) - K_2 \log(b/a)}$$

$$-\pi K_1K_2V_o \sum_{s=1}^{\infty} e^{-\kappa_1\alpha_s^2t}\, C(r\alpha_s, a\alpha_s)C^2(\kappa b\alpha_s, \kappa c\alpha_s)F(\alpha_s) . \tag{46}$$

For the same problem but with <u>$r = a$ maintained at zero and</u>

<u>$r = c$ at V_1 for $t > 0$</u> we find in place of (32) and (33)

$$\bar{v}_1 = -\frac{K_2V_1}{bp\,\Delta(p)}D(rq_1, aq_1)$$

$$\bar{v}_2 = \frac{V_1}{p\,\Delta(p)}\Big\{K_1q_1D(bq_2, rq_2)D_1(bq_1, aq_1) - K_2q_2D(bq_1, aq_1)D_1(bq_2, rq_2)\Big\} ,$$

where $\Delta(p)$ is given by (34). Hence

$$v_1 = -V_1 \frac{K_2 \log(r/a)}{K_1 \log(b/c) - K_2 \log(b/a)}$$
$$+\pi K_2^2 V_1 \sum_{s=1}^{\infty} e^{-\kappa_1 \alpha_s^2 t} \; C(r\alpha_s, a\alpha_s) C(\kappa b\alpha_s, \kappa c\alpha_s) C(b\alpha_s, a\alpha_s) F(\alpha_s) \qquad (47)$$

$$v_2 = V_1 \frac{K_1 \log(b/r) - K_2 \log(b/a)}{K_1 \log(b/c) - K_2 \log(b/a)}$$
$$+\pi K_2^2 V_1 \sum_{s=1}^{\infty} e^{-\kappa_1 \alpha_s^2 t} \; C(\kappa r\alpha_s, \kappa c\alpha_s) C^2(b\alpha_s, a\alpha_s) F(\alpha_s) \, . \qquad (48)$$

where the α_s are the roots of (37).

The solution for the cylinder with $r = a$ maintained at V_0 and $r = c$ at V_1 for $t > 0$ is obtained by adding (46) to (47) and (45) to (48).

4. Hollow cylinder of one material from $r = a$ to $r = b$ and of another from $r = b$ to $r = c$. The surfaces $r = a$ and $r = c$ kept at zero for $t > 0$. An instantaneous cylindrical surface source* of strength Q at $t = 0$ over $r = r'$, $a < r' < b$.

Here in the notation of § 3 we have to solve (18), (19), (20), (21)

with $\quad v_1 = 0 \,, \quad$ when $r = a, \; t > 0 \,, \qquad (49)$

$\quad v_2 = 0 \,, \quad$ when $r = c, \; t > 0 \,, \qquad (50)$

and $\quad v_1 = u_1 + w_1 \,, \qquad (51)$

where $\quad u_1 = \dfrac{Q}{4\pi\kappa_1 t} \exp\left[-(r^2 + r'^2)/4\kappa_1 t\right] I_0\left(\dfrac{rr'}{2\kappa_1 t}\right) \qquad (52)$

and w_1 satisfies

$$\kappa_1\left(\frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r}\frac{\partial w_1}{\partial r}\right) = \frac{\partial w_1}{\partial t}, \quad a < r < b \,, \quad t > 0 \qquad (53)$$

and $\quad \lim_{t \to 0} w_1 = 0 \,, \quad a < r < b \qquad (54)$

It is known that+

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* Cf. II, § 10.

+ The result follows from Watson, Theory of Bessel Functions, § 13.7.

$$\left.\begin{aligned} \bar{u}_1 &= \frac{Q}{2\pi\kappa_1} I_0(r'q_1)K_0(rq_1) , \qquad r > r' \\ &= \frac{Q}{2\pi\kappa_1} K_0(r'q_1)I_0(rq_1) , \qquad r < r' . \end{aligned}\right\} \tag{55}$$

The subsidiary equations, formed as in § 3, are

$$\frac{d^2\bar{v}_2}{dr^2} + \frac{1}{r}\frac{d\bar{v}_2}{dr} - q_2^2\bar{v}_2 = 0 , \qquad b < r < c \tag{56}$$

$$\frac{d^2\bar{w}_1}{dr^2} + \frac{1}{r}\frac{d\bar{w}_1}{dr} - q_1^2\bar{w}_1 = 0 , \qquad a < r < b \tag{57}$$

$$\bar{v}_2 = 0 , \qquad r = c \tag{58}$$

$$\bar{v}_1 = 0 , \qquad r = a \tag{59}$$

$$\bar{v}_1 = \bar{v}_2 , \qquad r = b \tag{60}$$

$$K_1 \frac{d\bar{v}_1}{dr} = K_2 \frac{d\bar{v}_2}{dr} , \qquad r = b , \tag{61}$$

where q_1 and q_2 are defined in (31).

The solution of (56) satisfying (58) is

$$\bar{v}_2 = CD(rq_2, cq_2) . \tag{62}$$

As solution of (57) we take

$$\bar{w}_1 = AD(rq_1, bq_1) + BD(rq_1, aq_1) \tag{63}$$

Then, using the value (55) of $\bar{u}_1$, (59) requires

$$AD(aq_1, bq_1) + \frac{Q}{2\pi\kappa_1} I_0(aq_1)K_0(r'q_1) = 0,$$

and (60) and (61) give

$$\left.\begin{aligned} CD(bq_2, cq_2) - BD(bq_1, aq_1) &= \frac{Q}{2\pi\kappa_1} I_0(q_1 r')K_0(bq_1) \\ K_2 q_2 CD_1(bq_2, cq_2) - K_1 q_1 BD_1(bq_1, aq_1) &= \frac{K_1 q_1 Q}{2\pi\kappa_1} I_0(r'q_1)K_0'(bq_1) + \frac{K_1 A}{b} . \end{aligned}\right\}$$

Solving for A, B, C and substituting in (62) and (63) we obtain on reduction

$$\bar{v}_2 = \frac{QK_1}{2\pi\kappa_1 b\,\Delta(p)} D(r'q_1, aq_1)D(rq_2, cq_2), \tag{64}$$

where $\Delta(p)$ is given by (26),

$$\bar{v}_1 = \frac{Q}{2\pi\kappa_1\Delta(p)} D(rq_1, aq_1)\Big[K_1q_1D(bq_2, cq_2)D_1(bq_1, r'q_1) - K_2q_2D_1(bq_2, cq_2)D(bq_1, r'q_1)\Big],$$

when $a < r < r' < b$, (65)

and when $a < r' < r < b$ we interchange r and r' in (65).

From (64) and the Inversion Theorem for the Laplace Transformation

$$v_2 = \frac{QK_2}{4\pi^2 i\kappa_1 b}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\, D(r'\mu_1, a\mu_1)D(r\mu_2, c\mu_2)\frac{d\lambda}{\Delta(\lambda)}, \qquad (66)$$

where as before $\mu_1 = \sqrt{(\lambda/\kappa_1)}$, $\mu_2 = \sqrt{(\lambda/\kappa_2)}$.

The integrand of (66) is a single-valued function of λ with simple poles at $\lambda = -\kappa_1\alpha_s^2$ where the α_s are the roots of (37).

Then as in § 3 it follows that the line integral in (66) may be replaced by $2\pi i$ times the sum of the residues at these poles. These are immediately evaluated on using (42) and we obtain

$$v_2 = -\tfrac{1}{4}\pi QK_1K_2\sum_{s=1}^{\infty}\alpha_s^2 e^{-\kappa_1\alpha_s^2 t}F(\alpha_s)C(b\alpha_s, a\alpha_s)C(r'\alpha_s, a\alpha_s)C(\kappa b\alpha_s, \kappa c\alpha_s)C(\kappa r\alpha_s, \kappa c\alpha_s) \qquad (67)$$

Similarly, from (65), we obtain, when $a < r < r'$,

$$v_1 = \tfrac{1}{8}\pi^2 bK_2Q\sum_{s=1}^{\infty}\alpha_s^3 e^{-\kappa_1\alpha_s^2 t}F(\alpha_s)C(b\alpha_s, a\alpha_s)C(r\alpha_s, a\alpha_s)C(\kappa b\alpha_s, \kappa c\alpha_s)$$
$$\times\Big\{K_1C(\kappa b\alpha_s, \kappa c\alpha_s)C_1(b\alpha_s, r'\alpha_s) - K_2\kappa C_1(\kappa b\alpha_s, \kappa c\alpha_s)C(b\alpha_s, r'\alpha_s)\Big\}$$

This becomes, on using (41),

$$v_1 = -\tfrac{1}{4}\pi K_1K_2Q\sum_{s=1}^{\infty}\alpha_s^2 e^{-\kappa_1\alpha_s^2 t}F(\alpha_s)C(r\alpha_s, a\alpha_s)C(r'\alpha_s, a\alpha_s)C^2(\kappa b\alpha_s, \kappa c\alpha_s), \qquad (68)$$

and this, being symmetrical in r and r', holds for both $a < r < r'$ and $r' < r < b$.

The solution of the problem of an <u>instantaneous cylindrical surface source of strength Q at r', $b < r' < c$,</u> is obtained in the same way.

We find

$$\bar{v}_1 = \frac{QK_2}{2\pi\kappa_2 b\,\Delta(p)} D(r'q_2, cq_2)D(rq_1, aq_1) \tag{69}$$

$$\bar{v}_2 = \frac{QD(rq_2, cq_2)}{2\pi\kappa_2\,\Delta(p)}\left[K_2 q_2 D(bq_1, aq_1)D_1(bq_2, r'q_2) - K_1 q_1 D_1(bq_1, aq_1)D(bq_2, r'q_2)\right],$$

$$\text{when} \quad c > r > r' > b, \tag{70}$$

where $\Delta(p)$ is given by (34).

Then from the Inversion Theorem we obtain

$$v_1 = -\frac{\pi QK_2^2\kappa_1}{4\kappa_2}\sum_{s=1}^{\infty} e^{-\kappa_1\alpha_s^2 t}\alpha_s^2 F(\alpha_s)C(b\alpha_s, a\alpha_s)C(r\alpha_s, a\alpha_s)C(\kappa r'\alpha_s, \kappa c\alpha_s)C(\kappa b\alpha_s, \kappa c\alpha_s) \tag{71}$$

$$v_2 = -\frac{\pi QK_2^2\kappa_1}{4\kappa_2}\sum_{s=1}^{\infty}\alpha_s^2 e^{-\kappa_1\alpha_s^2 t}F(\alpha_s)C^2(b\alpha_s, a\alpha_s)C(\kappa r\alpha_s, \kappa c\alpha_s)C(\kappa r'\alpha_s, \kappa c\alpha_s). \tag{72}$$

If we put $Q = 2\pi r' f(r')dr'$ in (67), (68), (71), (72) and integrate with respect to r' from a to c, we obtain formally the solution of the problem in which the initial temperature of the cylinder is $f(r)$ and the surfaces $r = a$ and $r = c$ are kept at zero for $t > 0$. Assuming that $f(r)$ is such that the orders of integration and summation may be interchanged we have

$$v_1 = -\tfrac{1}{2}\pi^2 K_1 K_2 \sum_{s=1}^{\infty}\alpha_s^2 e^{-\kappa_1\alpha_s^2 t}F(\alpha_s)C(r\alpha_s, a\alpha_s)C(\kappa b\alpha_s, \kappa c\alpha_s)E(\alpha_s) \tag{73}$$

$$v_2 = -\tfrac{1}{2}\pi^2 K_1 K_2 \sum_{s=1}^{\infty}\alpha_s^2 e^{-\kappa_1\alpha_s^2 t}F(\alpha_s)C(b\alpha_s, a\alpha_s)C(\kappa r\alpha_s, \kappa c\alpha_s)E(\alpha_s), \tag{74}$$

where

$$E(\alpha_s) = C(\kappa b\alpha_s, \kappa c\alpha_s)\int_a^b r'f(r')C(r'\alpha_s, a\alpha_s)dr' + \frac{K_2\kappa_1}{K_1\kappa_2}C(b\alpha_s, a\alpha_s)\int_b^c r'f(r')C(\kappa r'\alpha_s, \kappa c\alpha_s)dr' \tag{75}$$

5. The composite solid cylinder of one material, $v_1, K_1, c_1, \rho_1, \kappa_1$, for $0 \leqslant r < a$ and of another, $v_2, K_2, c_2, \rho_2, \kappa_2$, for $a < r < b$.

The results corresponding to those of §§ 3 and 4 are as follows:

(i) Initial temperature zero. The surface $r = b$ kept at V_0 for $t > 0$.

$$v_1 = V_0 + \pi K_2^2 V_0 \sum_{s=1}^{\infty} e^{-\kappa_1 \alpha_s^2 t} J_0(r\alpha_s) J_0(a\alpha_s) C(\kappa a\alpha_s, \kappa b\alpha_s) F_1(\alpha_s) , \tag{76}$$

$$v_2 = V_0 + \pi K_2^2 V_0 \sum_{s=1}^{\infty} e^{-\kappa_1 \alpha_s^2 t} J_0^2(a\alpha_s) C(\kappa r\alpha_s, \kappa b\alpha_s) F_1(\alpha_s) , \tag{77}$$

where $\pm\alpha_s$, $s = 1, 2, \ldots$, are the roots of

$$K_1 J_0'(a\alpha) C(\kappa a\alpha, \kappa b\alpha) - \kappa K_2 J_0(a\alpha) C_1(\kappa a\alpha, \kappa b\alpha) = 0 , \tag{78}$$

where $\kappa = \sqrt{(\kappa_1/\kappa_2)}$,

and

$$1/F_1(\alpha_s) = \frac{1}{4}\pi^2 a^2 \alpha_s^2 C^2(\kappa a\alpha_s, \kappa b\alpha_s)\Big[K_2(K_2\kappa^2 - K_1)J_0^2(a\alpha_s) + K_1(K_1 - K_2)J_0'^2(a\alpha_s)\Big] - K_2^2 J_0^2(a\alpha_s). \tag{79}$$

(ii) <u>The surface $r = b$ kept at zero for $t > 0$. Instantaneous cylindrical surface source of strength Q at $t = 0$ over $r = r'$, $0 < r' < a$.</u>

$$v_1 = -\frac{1}{4}\pi Q K_1 K_2 \sum_{s=1}^{\infty} \alpha_s^2 e^{-\kappa_1 \alpha_s^2 t} J_0(r'\alpha_s) J_0(r\alpha_s) C^2(\kappa a\alpha_s, \kappa b\alpha_s) F_1(\alpha_s) \tag{80}$$

$$v_2 = -\frac{1}{4}\pi Q K_1 K_2 \sum_{s=1}^{\infty} \alpha_s^2 e^{-\kappa_1 \alpha_s^2 t} J_0(r'\alpha_s) J_0(a\alpha_s) C(\kappa a\alpha_s, \kappa b\alpha_s) C(\kappa r\alpha_s, \kappa b\alpha_s) F_1(\alpha_s), \tag{81}$$

where $F_1(\alpha_s)$ and the α_s are given in (79) and (78).

(iii) <u>The surface $r = b$ kept at zero for $t > 0$. Instantaneous cylindrical surface source of strength Q at $t = 0$ over $r = r'$, $a < r' < b$.</u>

$$v_1 = -\frac{\pi Q K_2^2 \kappa_1}{4\kappa_2} \sum_{s=1}^{\infty} \alpha_s^2 e^{-\kappa_1 \alpha_s^2 t} J_0(a\alpha_s) J_0(r\alpha_s) C(\kappa r'\alpha_s, \kappa b\alpha_s) C(\kappa a\alpha_s, \kappa b\alpha_s) F_1(\alpha_s) \tag{82}$$

$$v_2 = -\frac{\pi Q K_2^2 \kappa_1}{4\kappa_2} \sum_{s=1}^{\infty} \alpha_s^2 e^{-\kappa_1 \alpha_s^2 t} J_0^2(a\alpha_s) C(\kappa r\alpha_s, \kappa b\alpha_s) C(\kappa r'\alpha_s, \kappa b\alpha_s) F_1(\alpha_s) \tag{83}$$

(iv) <u>The surface $r = b$ kept at zero for $t > 0$. The initial temperature of the solid $f(r)$.</u> Here the formal solution is

$$v_1 = -\frac{1}{2}\pi^2 K_1 K_2 \sum_{s=1}^{\infty} \alpha_s^2 e^{-\kappa_1 \alpha_s^2 t} J_0(r\alpha_s) C(\kappa a\alpha_s, \kappa b\alpha_s) F_1(\alpha_s) G_1(\alpha_s) \tag{84}$$

$$v_2 = -\frac{1}{2}\pi^2 K_1 K_2 \sum_{s=1}^{\infty} \alpha_s^2 e^{-\kappa_1 \alpha_s^2 t} J_0(a\alpha_s) C(\kappa r\alpha_s, \kappa b\alpha_s) F_1(\alpha_s) G_1(\alpha_s) , \tag{85}$$

where $$g_1(\alpha_s) = C(\kappa a\alpha_s, \kappa b\alpha_s)\int_0^a r'f(r')J_0(r'\alpha_s)dr'$$
$$+\frac{K_2\kappa_1}{K_1\kappa_2}J_0(a\alpha_s)\int_a^b r'f(r')C(\kappa r'\alpha_s, \kappa b\alpha_s)dr' . \qquad (86)$$

6. <u>The region bounded internally by the cylinder $r = a$. From $r = a$ to $r = b$ the solid is of one material, $v_1, \rho_1, c_1, K_1, \kappa_1$, and for $r > b$ of another $v_2, \rho_2, c_2, K_2, \kappa_2$.</u>

(i) <u>The surface $r = a$ maintained at temperature V_0 for $t > 0$; the initial temperature of the whole zero.</u>

Proceeding as in § 3, the Laplace Transforms of the temperatures in $a < r < b$ and $r > b$ respectively are found to be

$$\bar{v}_1 = \frac{V_0}{p\Delta(p)}\left\{K_1q_1K_0(bq_2)D_1(bq_1, rq_1) - K_2q_2K_0'(bq_2)D(bq_1, rq_1)\right\} \qquad (87)$$

$$\bar{v}_2 = \frac{K_1V_0}{pb\,\Delta(p)}K_0(rq_2) , \qquad (88)$$

where $$\Delta(p) = K_1q_1K_0(bq_2)D_1(bq_1, aq_1) - K_2q_2K_0'(bq_2)D(bq_1, aq_1), \qquad (89)$$

and $$q_1 = \sqrt{(p/\kappa_1)}, \quad q_2 = \sqrt{(p/\kappa_2)}.$$

v_1 and v_2 are determined from these by the use of the Inversion Theorem. This gives

$$v_2 = \frac{K_1V_0}{2\pi ib}\int_{\gamma - i\infty}^{\gamma + i\infty}\frac{e^{\lambda t}}{\lambda\,\Delta(\lambda)}K_0(r\mu_2)d\lambda \qquad (90)$$

where $\mu_1 = \sqrt{(\lambda/\kappa_1)}$, $\mu_2 = \sqrt{(\lambda/\kappa_2)}$.

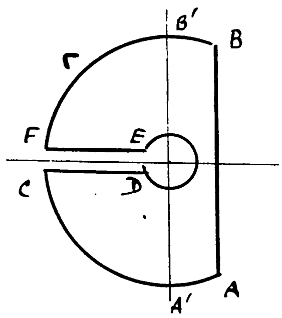

Fig. 1.

The integrand of (90) has a branch point at the origin so we use the contour of Fig. 2 consisting of the line AB distant γ from the imaginary axis, arcs AA'C and BB'F of a circle Γ whose radius R will tend to ∞, the lines CD and EF on which $\arg\lambda$ equals $-\pi$ and π respectively, and a small circle about the origin whose radius ϵ will tend to zero. The integral over Γ tends to zero as the radius tends to infinity*. Also there are no poles of the integrand within the contour. Thus the line integral in (90) may be replaced by the limits of the integrals over CD, EF, and the small circle as $R \to \infty$ and $\epsilon \to 0$.

The small circle gives V_0.

On CD we put $\lambda = \kappa_1 u^2 e^{-i\pi}$ and on EF, $\lambda = \kappa_1 u^2 e^{i\pi}$ and we obtain finally

$$+\frac{4K_1V_0}{\pi^2 b}\int_0^\infty \frac{e^{-\kappa_1 u^2 t}\,du}{u^2\{\phi^2(u)+\psi^2(u)\}}\left[K_1C_1(bu,au)C(\kappa bu,\kappa ru) - K_2\kappa C(bu,au)C_1(\kappa bu,\kappa ru)\right] \tag{91}$$

where
$$\left.\begin{aligned}\phi(u) &= K_1J_0(\kappa bu)C_1(bu,au) - \kappa K_2J_0'(\kappa bu)C(bu,au)\\ \psi(u) &= K_1Y_0(\kappa bu)C_1(bu,au) - \kappa K_2Y_0'(\kappa bu)C(bu,au)\end{aligned}\right\} \tag{92}$$

and $\kappa = \sqrt{(\kappa_1/\kappa_2)}$.

Similarly we obtain

$$v_1 = V_0 + \frac{8K_1K_2V_0}{\pi^3 b^2}\int_0^\infty e^{-\kappa_1 u^2 t}\,\frac{C(ru,au)}{\phi^2(u)+\psi^2(u)}\frac{du}{u^3}\,. \tag{93}$$

(ii) <u>The surface $r = a$ maintained at zero for $t > 0$. An instantaneous cylindrical surface source of strength Q at $t = 0$ over $r = r'$, $a < r' < b$.</u>

$$v_1 = \frac{2K_1K_2Q}{\pi^3 b^2}\int_0^\infty e^{-\kappa_1 u^2 t}\,\frac{C(r'u,au)C(ru,au)}{\phi^2(u)+\psi^2(u)}\frac{du}{u} \tag{94}$$

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* For the method of proof and a more detailed discussion of the procedure of this section, see I.

$$v_2 = \frac{K_1 Q}{b\pi^2} \int_0^\infty e^{-\kappa_1 u^2 t} \frac{C(r'u, au)\,\chi(r,u)}{\phi^2(u) + \psi^2(u)}, \tag{95}$$

where $\chi(r,u) = K_1 C(\kappa bu, \kappa ru) C_1(bu, au) - K_2 \kappa C_1(\kappa bu, \kappa ru) C(bu, au)$, (96)

and $\phi(u)$ and $\psi(u)$ are defined in (92).

(iii) <u>The surface $r = a$ maintained at zero for $t > 0$. An instantaneous cylindrical surface source of strength Q at $t = 0$ over $r = r'$, $r' > b$.</u>

$$v_1 = \frac{K_2 \kappa_1 Q}{\pi^2 b\, \kappa_2} \int_0^\infty e^{-\kappa_1 u^2 t} \frac{C(ru, au)\,\chi(r',u)}{\phi^2(u) + \psi^2(u)}\, du \tag{97}$$

$$v_2 = \frac{\kappa_1 Q}{2\pi \kappa_2} \int_0^\infty e^{-\kappa_1 u^2 t} \frac{\chi(r,u)\,\chi(r'u)}{\phi^2(u) + \psi^2(u)}\, u\,du \tag{98}$$

(iv) <u>The surface $r = a$ maintained at zero for $t > 0$. The initial temperature of the whole $f(r)$.</u> It is assumed that $f(r)$ is such that the orders of integration may be interchanged.

$$v_1 = \frac{4K_1 K_2}{\pi^2 b^2} \int_0^\infty e^{-\kappa_1 u^2 t} \frac{C(ru, au)}{\phi^2(u) + \psi^2(u)}\, g(u)\, \frac{du}{u} \tag{99}$$

$$v_2 = \frac{2K_1}{\pi b} \int_0^\infty e^{-\kappa_1 u^2 t} \frac{\chi(r,u)}{\phi^2(u) + \psi^2(u)}\, g(u)\,du, \tag{100}$$

where

$$g(u) = \int_a^b r' f(r') C(r'u, au)\,dr' + \frac{\pi b \kappa_1}{2K_1 \kappa_2}\, u \int_b^\infty r' f(r')\, \chi(r',u)\,dr'. \tag{101}$$

CHAPTER IV.

SOME MORE GENERAL BOUNDARY CONDITIONS OF PRACTICAL IMPORTANCE.

The usual boundary conditions of the mathematical theory of Conduction of Heat are of type

$$k\frac{\partial v}{\partial n} - hv = 0 ,$$

i.e. constant surface temperature, no flow of heat, or radiation. In many practical problems more general boundary conditions are encountered which may easily be solved by the Laplace transformation method. Some of these are studied here.

The first type of problem considered is that in which heat is transferred from a surface by contact with known mass of well stirred fluid whihh may itself be heated or cooled in some way; if the stirring is perfect the effect is that of perfect conductivity and it is to emphasize the source of the problems that the phrase "perfectly conducting fluid" is used in the enunciations.

The problems considered are all one-dimensional and are intended as a preliminary study to two-dimensional problems of engineering interest.

The second type of problem is that of prescribed flux over a boundary, this boundary condition is not of much interest in conduction of heat, but is of importance in the theory of the flow of compressible liquids through porous media. The constant line source iniated at $t = 0$ in infinite medium is discussed by this method in § 7 and in § 8 a problem illustrating the method of treatment of the constant line source in a region with coaxial cylindrical boundaries.

1. <u>The region bounded internally by the cylinder $r = a$ is initially at unit temperature. The region $r < a$ contains perfectly conducting fluid of density ρ' and specific heat c' initially at unit temperature. For $t > 0$ fluid is removed at a rate Q c.c. per second per unit length and replaced by fluid at zero.</u>

We have to solve

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = \frac{1}{\kappa}\frac{\partial v}{\partial t}, \qquad r > a, \quad t > 0$$

with $\pi a^2 \rho' c' \dfrac{\partial v}{\partial t} = 2\pi a K \dfrac{\partial v}{\partial r} - Q\rho' c' v, \quad r = a, \quad t > 0,$

and $v = 1, \quad r \geqslant a, \quad t = 0.$

The subsidiary equation is

$$\frac{d^2\bar{v}}{dr^2} + \frac{1}{r}\frac{d\bar{v}}{dr} - q^2\bar{v} = -\frac{1}{\kappa}, \qquad r > a$$

with $\pi a^2\rho' c' p\bar{v} - \pi a^2 \rho' c' = 2\pi a K \dfrac{d\bar{v}}{dr} - Q\rho' c'\bar{v}$, when $r = a$.

Thus $$\bar{v} = \frac{1}{p} - \frac{K_0(qr)}{p\left\{(1 + \alpha a^2 q^2) K_0(qa) - \beta a q K_0'(qa)\right\}} \qquad \ldots\ldots\ldots (1)$$

where $\alpha = \pi\kappa/Q, \quad \beta = 2\pi K/\rho' c' Q.$

Therefore

$$v = 1 - \frac{1}{2\pi i}\int_0^\infty \left\{\frac{e^{-\kappa u^2 t}\, K_0(iur)\, 2du}{u\left\{(1 - \alpha a^2 u^2) K_0(iau) - i\beta a u K_0'(iua)\right\}} - \text{Conjugate}\right\}$$

$$= 1 - \frac{2}{\pi}\int_0^\infty \frac{e^{-\kappa u^2 t}\,du}{u}\,\frac{(1 - \alpha a^2 u^2)C_0(ur, ua) + \beta a u\left[J_0(ur)Y_1(ua) - Y_0(ur)J_1(ua)\right]}{\left[(1 - \alpha a^2 u^2)J_0(ua) + \beta a u J_1(au)\right]^2 + \left[(1 - \alpha a^2 u^2)Y_0(ua) + \beta a u Y_1(au)\right]^2}$$

$$\ldots\ldots (2)$$

2. The region bounded externally by the cylinder $r = a$ is initially at zero. At $r = a$ it is in contact with mass M per unit length of perfectly conducting fluid of specific heat c' initially at zero. For $t > 0$ fluid is removed at the mass rate m per unit length per unit time and replaced by fluid at V. There is no external loss of heat from the mass M.

We have to solve

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = \frac{1}{\kappa}\frac{\partial v}{\partial t}, \qquad 0 \leqslant r < a, \quad t > 0$$

with $Mc'\dfrac{\partial v}{\partial t} = -2\pi a K\dfrac{\partial v}{\partial r} + mc'(V - v), \quad r = a, \quad t > 0$ .... (1)

and $v = 0$ when $t = 0, \quad 0 \leqslant r \leqslant a.$

The subsidiary equation is

$$\frac{d^2\bar{v}}{dr^2} + \frac{1}{r}\frac{d\bar{v}}{dr} - q^2\bar{v} = 0, \qquad 0 \leqslant r < a$$

with $Mc'p\bar{v} = -2\pi a K\dfrac{d\bar{v}}{dr} + \dfrac{mc'V}{p} - mc'\bar{v}, \quad$ when $r = a.$

Thus $\bar{v} = \dfrac{VI_0(qr)}{p\left\{(1 + Ap)\,I_0(qa) + kaq\,I_1(aq)\right\}}$ . ... (2)

where $A = M/m, \quad k = 2\pi K/mc'$.

Thus $$v = \frac{V}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\lambda t}\,d\lambda\, I_0(\mu r)}{\lambda\left\{(1 + A\lambda)I_0(\mu a) + ka\mu I_1(\mu a)\right\}}.$$

The integrand has a simple pole at $\lambda = 0$, residue 1, and simple poles at $\lambda = -\kappa\alpha_s^2$ where $\pm\alpha_s$, $s = 1, 2, \ldots$ are the roots of

$$(1 - \kappa A\alpha^2)\,J_0(a\alpha) - ka\,\alpha J_1(a\alpha) = 0. \qquad \text{.................. (3)}$$

The residue at the pole $\lambda = -\kappa\alpha_s^2$ is

$$-\frac{2k\,e^{-\kappa\alpha_s^2 t}\,J_0(r\alpha_s)}{J_0(a\alpha_s)\left\{k\alpha_s^2(ka^2 + 2\kappa A) + (1 - \kappa A\alpha_s^2)^2\right\}}$$

Thus

$$v = V - 2kV \sum_{s=1}^{\infty} \frac{e^{-\kappa\alpha_s^2 t} J_0(r\alpha_s)}{J_0(a\alpha_s)\left\{k\alpha_s^2(ka^2 + 2\kappa A) + (1 - \kappa A\alpha_s^2)\right\}} . \qquad \text{......} (4)$$

If in addition heat is supplied to the mass M at rate H per unit length per unit time, V on the right hand side of (4) is to be replaced by $V + H/mc'$.

3. <u>The region bounded externally by the cylinder $r = a$ is initially at unit temperature. At $t = 0$ it is put in contact with mass M per unit length of perfectly conducting fluid of specific heat c' at zero temperature. The mass M loses heat at a rate proportional to its temperature.</u>

We have to solve

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = \frac{1}{\kappa}\frac{\partial v}{\partial t}, \qquad 0 \leqslant r < a, \quad t > 0$$

with $$Mc'\frac{\partial v}{\partial t} = -2\pi a K\frac{\partial v}{\partial r} - kv, \quad r = a, \quad t > 0.$$

The subsidiary equation is

$$\frac{d^2\bar{v}}{dr^2} + \frac{1}{r}\frac{d\bar{v}}{dr} - q^2\bar{v} = -\frac{1}{\kappa}, \quad 0 \leqslant r < a,$$

with $$(Mc'p + k)\bar{v} + 2\pi a K\frac{d\bar{v}}{dr} = 0, \quad \text{when } r = a.$$

Thus $$\bar{v} = \frac{1}{p} - \frac{(k + Mc'p)\, I_0(qr)}{p\left\{(Mc'p + k)\, I_0(qa) + 2\pi a q\, I_1(qa)\right\}} . \qquad \text{............} (1)$$

$$v = 1 - \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{(k + Mc'\lambda)\, I_0(\mu r)\, e^{\lambda t}\, d\lambda}{\lambda\left\{(Mc'\lambda + k)\, I_0(\mu a) + 2\pi a\mu I_1(\mu a)\right\}}$$

the integrand has a simple pole at $\lambda = 0$ of unit residue and simple poles at $\lambda = -\kappa\alpha_s^2$, where $\pm\alpha_s$, $s = 1, \ldots$ are the roots of

$$(k - Mc'\kappa\alpha^2)\, J_0(a\alpha) - 2\pi a\alpha\, J_1(a\alpha) = 0. \qquad \text{..............} (2)$$

Now

$$\left[\lambda \frac{d}{d\lambda}\left\{(Mc'\lambda + k)I_0(\mu a) + 2\pi h\mu a I_1(\mu a)\right\}\right]_{\lambda=-\kappa\alpha_s^2} = -\frac{J_0(\alpha_s a)}{4\pi}\left\{(k - Mc'\kappa\alpha_s^2)^2 + 4\pi\alpha_s^2(\kappa Mc' + \pi a^2)\right\}$$

Thus

$$v = -4\pi \sum_{s=1}^{\infty} \frac{e^{-\kappa\alpha_s^2 t}(k - Mc'\kappa\alpha_s^2)\, J_0(r\alpha_s)}{\left\{(k - Mc'\kappa\alpha_s^2)^2 + 4\pi\alpha_s^2(\kappa Mc' + \pi a^2)\right\} J_0(a\alpha_s)} \qquad \text{.......} \; (3)$$

4. <u>The hollow cylinder $a < r < b$ with zero initial temperature and boundary conditions of type § 2 (1) at $r = a$ and $r = b$.</u>

The general boundary condition of this type is

$$k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v = k_4 , \qquad \text{..............................} (1)$$

we shall take this at $r = a$, and

$$k_1' \frac{\partial v}{\partial t} + k_2' \frac{\partial v}{\partial r} + k_3' v = k_4' , \qquad \text{..............................} (2)$$

when $r = b$. It will be assumed that the constants k have values arising from a physical problem, otherwise there may be additional terms in the solution which increase exponentially with the time.*

We have to solve

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = \frac{1}{\kappa}\frac{\partial v}{\partial t}, \qquad a < r < b, \qquad t > 0$$

with $v = 0$, when $t = 0$, $a \leqslant r \leqslant b$, and boundary conditions (1) and (2).

The subsidiary equations are

$$\frac{d^2\bar{v}}{dr^2} + \frac{1}{r}\frac{d\bar{v}}{dr} - q^2\bar{v} = 0 , \qquad a < r < b ,$$

---

* In practical problems the signs of the constants are such that the zeros of $\Delta(\mu)$ are all at negative values of λ. It is possible to have positive zeros but the cases in which they arise are artificial. Cf. Chap. 2, §§ 9 and 10.

with $(k_1 p + k_3)\bar{v} + k_2 \frac{d\bar{v}}{dr} = k_4/p$, $\qquad r = a$

$(k_1' p + k_3')\bar{v} + k_2' \frac{d\bar{v}}{dr} = k_4'/p$, $\qquad r = b$.

The solution is

$$p\bar{v}\,\Delta(q) = k_4\Big\{\Big[(k_1'p+k_3')K_o(qb) - k_2'qK_1(qb)\Big]I_o(qr) - \Big[(k_1'p+k_3')I_o(qb)+k_2'qI_1(qb)\Big]K_o(qr)\Big\}$$
$$- k_4'\Big\{\Big[(k_1p+k_3)K_o(qa) - k_2qK_1(qa)\Big]I_o(qr) - \Big[(k_1p+k_3)I_o(qa)+k_2qI_1(qa)\Big]K_o(qr)\Big\}$$

where

$$\Delta(q) = \Big[(k_1p+k_3)I_o(qa)+k_2qI_1(qa)\Big]\Big[(k_1'p+k_3')K_o(qb) - k_2'qK_1(qb)\Big]$$
$$- \Big[(k_1'p+k_3')I_o(qb)+k_2'qI_1(qb)\Big]\Big[(k_1p+k_3)K_o(qa) - k_2qK_1(qa)\Big]$$

The integrand of $v(\lambda)$ has a simple pole at $\lambda = 0$ of residue

$$\frac{ak_4(-k_2'+bk_3' \log r/b) - bk_4'(-k_2+ak_3 \log r/a)}{abk_3k_3' \log(a/b) - ak_3k_2' + bk_2k_3'} ,$$

though for special values of the constants, e.g. $k_3 = k_3' = 0$, it may have a double one. It also has ~~a~~ simple poles at $\lambda = \pm\kappa\alpha_s^2$, where $\pm\alpha_s$, $s = 1, 2, \ldots$ are the roots of

$$\Big[(k_3 - k_1\kappa\alpha^2)J_o(\alpha a) - k_2\alpha J_1(\alpha a)\Big]\Big[(k_3' - k_1'\kappa\alpha^2)Y_o(\alpha b) - k_2'\alpha Y_1(\alpha b)\Big]$$
$$- \Big[(k_3' - k_1'\kappa\alpha^2)J_o(\alpha b) - k_2'\alpha J_1(\alpha b)\Big]\Big[(k_3 - k_1\kappa\alpha^2)Y_o(\alpha a) - k_2\alpha Y_1(\alpha a)\Big] = 0 \qquad \ldots (3)$$

Also

$$\Big[\lambda\frac{d}{d\lambda}\Delta(\mu)\Big]_{\lambda=-\kappa\alpha_s^2} = -\frac{\Big[PJ_o(a\alpha_s) - k_2\alpha_sJ_1(a\alpha_s)\Big]^2(P'^2 + k_2'Q'\alpha_s^2) - \Big[P'J_o(b\alpha_s) - k_2'\alpha_sJ_1(b\alpha_s)\Big]^2(P^2 + k_2Q\alpha_s^2)}{2\Big[P'J_o(b\alpha_s) - k_2'\alpha_sJ_1(b\alpha_s)\Big]\Big[PJ_o(a\alpha_s) - k_2\alpha_sJ_1(a\alpha_s)\Big]}$$

where $P = k_3 - \kappa k_1\alpha_s^2$, $P' = k_3' - \kappa k_1'\alpha_s^2$, $Q = k_2 + 2\kappa k_1/a$, $Q' = k_2' + 2k_1'\kappa/b$.

$\ldots\ldots\ldots$ (4)

Thus finally

$$v = \frac{ak_4(-k_2' + bk_3' \log r/b) - bk_4'(-k_2 + ak_3 \log r/a)}{abk_3k_3' \log(a/b) - ak_3k_2' + bk_2k_3'}$$

$$+ \pi \sum_{s=1}^{\infty} \frac{e^{-\kappa\alpha_s^2 t}\left[P'J_0(b\alpha_s) - k_2'\alpha_s J_1(b\alpha_s)\right]\left[PJ_0(a\alpha_s) - k_2\alpha_s J_1(a\alpha_s)\right]}{\left[PJ_0(a\alpha_s) - k_2\alpha_s J_1(a\alpha_s)\right]^2(P'^2 + k_2'Q'\alpha_s^2) - \left[P'J_0(b\alpha_s) - k_2'\alpha_s J_1(b\alpha_s)\right]^2(P^2 + k_2Q\alpha_s^2)}$$

$$\times \left\{ \begin{array}{l} k_4\left[J_0(\alpha_s r)\left\{P'Y_0(\alpha_s b) - k_2'\alpha_s Y_1(\alpha_s b)\right\} - Y_0(\alpha_s r)\left\{P'J_0(\alpha_s b) - k_2'\alpha_s J_1(\alpha_s b)\right\}\right] \\ - k_4'\left[J_0(\alpha_s r)\left\{PY_0(\alpha_s a) - k_2\alpha_s Y_1(\alpha_s a)\right\} - Y_0(\alpha_s r)\left\{PJ_0(\alpha_s a) - k_2\alpha_s J_1(\alpha_s a)\right\}\right] \end{array} \right.$$

.... (5)

5. Examples of some importance in the theory of § 4 are:

(i) <u>The region $r < a$ is perfectly conducting, of density ρ' and specific heat c'. Heat is supplied to it at rate H per unit time per unit length for $t > 0$. The region $a < r < b$ has conductivity K, specific heat c, density ρ. The boundary condition at $r = b$ is $k\frac{\partial v}{\partial r} + hv = 0$. The initial temperature of the whole zero.</u>

Here the boundary condition at $r = a$ is

$$\pi a^2 \rho' c' \frac{\partial v}{\partial t} = 2\pi a K \frac{\partial v}{\partial r} + H .$$

Thus in the notation of § 4 (1)

$$k_1 = \pi a^2 \rho' c' , \quad k_2 = -2\pi a K , \quad k_3 = 0 , \quad k_4 = H$$

$$k_1' = 0, \quad k_2' = k , \quad k_3' = h , \quad k_4' = 0 .$$

This problem may be regarded as an approximation to the case of an insulated wire heated by electric current.

(ii) The region bounded externally by the cylinder $r = a$ contains perfectly conducting fluid of density ρ' and specific heat c'. The region $a < r < b$ is a conductor, K, ρ, c. At $r = b$ the boundary condition is $k\frac{\partial v}{\partial r} + hv = 0$. The initial temperature of the whole is zero. For $t > 0$ volume Q of fluid is removed per unit length of the interior and replaced by fluid at V.

Here the boundary condition at $r = a$ is

$$\pi a^2 \rho' c' \frac{\partial v}{\partial t} = 2\pi a K \frac{\partial v}{\partial r} + Q\rho' c'(V - v).$$

So in the notation of § (4)

$$k_1 = \pi a^2 \rho' c', \quad k_2 = -2\pi a K, \quad k_3 = Q\rho' c', \quad k_4 = Q\rho' c' V$$

$$k_1' = 0, \quad k_2' = k, \quad k_3' = h, \quad k_4' = 0.$$

Problems of this type occur in the theory of hot water systems.

(iii) The region $a < r < b$. Zero initial temperature. Constant flux F over $r = a$ for $t > 0$. No flow of heat over $r = b$.

Here the boundary condition at $r = a$ is

$$-K\frac{\partial v}{\partial r} = F.$$

So in § (4) (1)

$$k_1 = 0, \quad k_2 = -K, \quad k_3 = 0, \quad k_4 = F,$$

and $\quad k_1' = 0, \quad k_3' = 0, \quad k_4' = 0.$

Problems involving prescribed flux over a boundary do not arise much in Conduction of Heat; they are of importance in the theory of flow of compressible fluids through porous media where the same differential equation occurs.

(iv) A one-dimensional heat interchanger: The region $r < a$ contains mass M' per unit length of perfectly conducting fluid, ρ', c', which is removed at rate m' per unit length per unit time and replaced by fluid at V. The region $r > b$ contains M'' per unit length of perfectly conducting fluid, ρ'', c'', which is removed at rate m'' per unit length per unit time and replaced by fluid at zero. The region $a < r < b$ is a conducting medium K, ρ, c. The initial temperature of the whole zero.

The boundary conditions are

$$M'c'\frac{\partial v}{\partial t} = 2\pi a K\frac{\partial v}{\partial r} + m'c'(V - v) , \qquad r = a$$

$$M''c''\frac{\partial v}{\partial t} = 2\pi b K\frac{\partial v}{\partial r} - m''c''v , \qquad r = b .$$

So in § (4), (1) and (2), we have

$$k_1 = M'c' , \quad k_2 = 2\pi a K , \quad k_3 = m'c' , \quad k_4 = m'c'V$$

$$k_1' = M''c'' , \quad k_2' = 2\pi b K , \quad k_3' = m''c'' , \quad k_4' = 0.$$

6. Problems on flow of heat in the region bounded internally by a cylinder $r = a$ with general boundary conditions of type § 4 (1) at $r = a$ may lead to a difficulty, absent in that of § 1, sufficiently illustrated by the following problem:

The region bounded internally by a cylinder $r = a$ is initially at unit temperature. For $t > 0$ constant flow of heat Q from the region $r > a$ over $r = a$.

We have to solve

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - q^2 v = 0 , \qquad r > a , \quad t > 0$$

with $\quad 2\pi r K\frac{\partial v}{\partial r} = Q , \qquad r = a , \quad t > 0 ,$

and $\quad v = 1 , \quad r > a , \quad t = 0.$

The subsidiary equation is

$$\frac{d^2\bar{v}}{dr^2} + \frac{1}{r}\frac{d\bar{v}}{dr} - q^2\bar{v} = -\frac{1}{\kappa}, \qquad r > a$$

with $$2\pi a K \frac{d\bar{v}}{dr} = \frac{Q}{p}.$$

Hence $$\bar{v} = \frac{1}{p} + \frac{Q K_o(qr)}{2\pi a K qp K_o'(qa)} .$$

Thus $$v = 1 - \frac{Q}{4\pi^2 iaK}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t} K_o(\mu r)d\lambda}{\lambda\mu K_1(\mu a)} . \qquad \ldots\ldots\ldots\ldots (1)$$

The flux at any radius r is in the same way

$$F = -K\frac{\partial v}{\partial r} = -\frac{Q}{4\pi^2 ia}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t} K_1(\mu r)d\lambda}{\lambda K_1(\mu a)} \qquad \ldots\ldots\ldots\ldots (2)$$

Proceeding in the usual way using the contour of Fig. 1, equation (2) gives

$$F = -\frac{Q}{2\pi r} - \frac{Q}{\pi^2 a}\int_0^\infty e^{-\kappa u^2 t}\,\frac{J_1(ur)Y_1(ua) - Y_1(ur)J_1(ua)}{u\left[J_1^2(ua) + Y_1^2(ua)\right]}\,du \qquad \ldots. (3)$$

The integral of (1), however, cannot be evaluated using the contour of Fig. 1. It can be expressed as a loop integral about the origin, $\int_{-\infty}^{(o+)}$, but this cannot be contracted onto the real axis. We may proceed as follows: In the usual way, using the contour of Fig. 1, we find

$$\frac{1}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t} K_o(\mu r)d\lambda}{\mu K_1(\mu a)} = -\frac{2\kappa}{\pi}\int_0^\infty e^{-\kappa u^2 t}\,\frac{J_o(ur)Y_1(ua) - Y_o(ur)J_1(ua)}{J_1^2(ua) + Y_1^2(ua)}\,du \qquad .. (4$$

Integrating this with respect to t from 0 to t we obtain

$$\frac{1}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{(e^{\lambda t} - 1)K_o(\mu r)d\lambda}{\lambda\mu K_1(\mu a)} = \frac{2}{\pi}\int_0^\infty \frac{(e^{-\kappa u^2 t} - 1)\left[J_o(ur)Y_1(ua) - Y_o(ur)J_1(ua)\right]du}{u^2\left[J_1^2(ua) + Y_1^2(ua)\right]}$$

$$\ldots.. (5)$$

Now it follows, by integrating round a contour consisting of the line $R(z) = \gamma$ and portion of a circle of radius $R \to \infty$ in the right hand half plane, that when $r \geqslant a$

$$\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{K_0(\mu r)d\lambda}{\lambda \mu K_1(\mu a)} = 0 \; . \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Thus from (1), (5), and (6)

$$v = 1 - \frac{Q}{\pi^2 aK} \int_0^\infty \frac{(e^{-\kappa u^2 t} - 1)\left[J_0(ur)Y_1(ua) - Y_0(ur)J_1(ua)\right] du}{u^2\left[J_1^2(ua) + Y_1^2(ua)\right]} \qquad \ldots\ldots (7)$$

The flux for any radius $r > a$ is

$$F = - K \frac{\partial v}{\partial r} = - \frac{Q}{\pi^2 a} \int_0^\infty \frac{(e^{-\kappa u^2 t} - 1)\left[J_1(ur)Y_1(ua) - Y_1(ur)J_1(ua)\right] du}{u\left[J_1^2(ua) + Y_1^2(ua)\right]} \qquad \ldots (8)$$

This agrees with (3) since

$$\int_0^\infty \frac{J_1(ur)Y_1(ua) - Y_1(ur)J_1(ua)}{u\left[J_1^2(ua) + Y_1^2(ua)\right]} du = - \frac{\pi}{2} \frac{a}{r} , \qquad r > a , \qquad \ldots\ldots\ldots (9)$$

as may be seen by letting $t \to 0$ in (3). The result is easily proved directly*.

7. The difficulty of § 6 is also encountered in treating by this method the problem of a constant line source in infinite medium.

Suppose we have a <u>constant line source at the origin in infinite medium for $t > 0$. The initial temperature zero. The flow of heat from the line source Q per unit length per unit time.</u>

We have to solve

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = \frac{1}{\kappa}\frac{\partial v}{\partial t} , \qquad r > 0, \quad t > 0$$

with $\lim_{r \to 0} \left[- 2\pi r K \frac{\partial v}{\partial r} \right] = Q.$

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* Titchmarsh, Proc. London Math. Soc. (2), 22 (1922) 15.

The subsidiary equation is

$$\frac{d^2\bar{v}}{dr^2} + \frac{1}{r}\frac{d\bar{v}}{dr} - q^2\bar{v} = 0 , \qquad r > 0 .$$

with $\lim_{r \to 0} \left(- 2\pi r K \frac{d\bar{v}}{dr}\right) = \frac{Q}{p}$.

The solution is $\bar{v} = \frac{Q}{2\pi K}\frac{K_0(qr)}{p}$ .... (1)

and the flux is given by $\bar{F} = \frac{Q}{2\pi K}\frac{K_1(qr)}{q}$ ....(2)

From (2) we find in the usual way, using Fig. 1,

$$F = \frac{Q}{2\pi r} - \frac{Q}{2\pi}\int_0^\infty e^{-\kappa u^2 t} J_1(ur)du$$

$$= \frac{Q}{2\pi r} - \frac{Q}{4\pi}\frac{1}{\sqrt{\pi\kappa t}} e^{-r^2/8\kappa t} I_{\frac{1}{2}}(r^2/8\kappa t)$$

$$= \frac{Q}{2\pi r} e^{-r^2/4\kappa t} .$$

From (1) we have

$$v = \frac{Q}{4\pi^2 iK}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{K_0(\mu r)e^{\lambda t}\, d\lambda}{\lambda} \qquad \text{................... (3)}$$

and this may be expressed as a loop integral $(-\infty, 0+)$ but the loop cannot be contracted about the origin. But

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} K_0(\mu r)d\lambda = \kappa\int_0^\infty e^{-\kappa u^2 t} J_0(ur)u\, du \qquad \text{.......... (4)}$$

$$= \frac{1}{2t} e^{-r^2/4\kappa t} \qquad \text{............ (5)}$$

Integrating (5) with respect to t from 0 to t we have as in § 6 for $r > 0$

$$v = \frac{Q}{4\pi K}\int_0^t \frac{e^{-r^2/4\kappa t}\, dt}{t} . \qquad \text{.............................. (6)}$$

Or from (4) we have as in § 6

$$v = -\frac{Q}{2\pi K}\int_0^\infty (e^{-\kappa u^2 t} - 1)\, J_0(ur)\,\frac{du}{u} \; . \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

All these might have been derived by integrating the solution for the instantaneous line source.

As $t \to \infty$ the expressions (6) and (7), like § 6 (7), tend to infinity. There are no logarithm steady state terms as in bounded regions.

8. Problems involving constant line sources initiated at $t = 0$ in bounded regions are easily dealt with in the same way. As an example consider the following:

The region $a < r < b$ is initially at zero temperature. At $t = 0$ a constant line source of strength $2\pi KQ$ is established at the point $r = r'$, $\theta = 0$. There is no flow of heat over the boundaries.

We require a solution of

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = \frac{1}{\kappa}\frac{\partial v}{\partial t}\,, \qquad t > 0, \quad a < r < b\,, \qquad \ldots\ldots \quad (1)$$

such that $v = 0$, when $t = 0$, $a < r < b$, ... (2)

and $\frac{\partial v}{\partial r} = 0$, when $r = a$, $t > 0$, ... (3)

$\frac{\partial v}{\partial r} = 0$, when $r = b$, $t > 0$, ... (4)

$$\lim_{R \to 0} R\frac{\partial v}{\partial R} = Q\,, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

where $R = \left\{r^2 + r'^2 - 2rr'\cos\theta\right\}^{\frac{1}{2}}$, ... (6)

and in (5) $\frac{\partial}{\partial R}$ is written for a differentiation along the normal to the circle R = constant.

The subsidiary equation is

$$\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{v}}{\partial \theta^2} - q^2 \bar{v} = 0 \ , \qquad \text{(7)}$$

where $\quad q^2 = p/k \quad .$ $\qquad$ (8)

This is to be solved with

$$\frac{\partial \bar{v}}{\partial r} = 0 \ , \quad \text{when} \quad r = a \ , \qquad \text{(9)}$$

$$\frac{\partial \bar{v}}{\partial r} = 0 \ , \quad \text{when} \quad r = b \ , \qquad \text{(10)}$$

$$\lim_{R \to 0} R \frac{\partial \bar{v}}{\partial R} = -\frac{Q}{p} \qquad \text{(11)}$$

A solution of (7) satisfying (11) (the solution for a line source in an infinite region) is

$$-\frac{Q}{p} K_0(qR) \qquad \text{(12)}$$

$$= \begin{cases} -\dfrac{Q}{p} \displaystyle\sum_{n=0}^{\infty} \epsilon_n I_n(qr) K_n(qr') \cos n\theta \ , & r < r' \ , \qquad \text{(13)} \\[2ex] -\dfrac{Q}{p} \displaystyle\sum_{n=0}^{\infty} \epsilon_n I_n(qr') K_n(qr) \cos n\theta \ , & r > r' \ , \qquad \text{(14)} \end{cases}$$

where $\quad \epsilon_n = 1 \quad$ if $\quad n = 0 \quad$ and $\quad \epsilon_n = 2 \quad$ if $\quad n > 0 \ .$ $\qquad$ (15)

Thus we take as the general solution of (7) and (11)

$$\bar{v} = -\frac{Q}{p} \sum_{n=0}^{\infty} \epsilon_n \left\{ I_n(qr) K_n(qr') + A_n(q) I_n(qr) + B_n(q) K_n(qr) \right\} \cos n\theta, \quad r < r' \qquad \text{(16)}$$

$$\bar{v} = -\frac{Q}{p} \sum_{n=0}^{\infty} \epsilon_n \left\{ I_n(qr') K_n(qr) + A_n(q) I_n(qr) + B_n(q) K_n(qr) \right\} \cos n\theta, \quad r > r', \qquad \text{(17)}$$

and determine $A_n(q)$ and $B_n(q)$ so that these satisfy (9) and (10).

This requires

$$\left. \begin{aligned} A_n(q) I_n'(qa) + B_n(q) K_n'(qa) &= - I_n'(qa) K_n(qr') \\ A_n(q) I_n'(qb) + B_n(q) K_n'(qb) &= - K_n'(qb) I_n(qr') \ . \end{aligned} \right\} \quad n \geqslant 0$$

Solving and substituting in (15) we obtain when $r < r'$

$$\bar{v} = -\frac{Q}{p}\sum_{n=0}^{\infty} \frac{\{I_n(qr)K_n'(qa) - K_n(qr)I_n'(qa)\}\{I_n(qr')K_n'(qb) - K_n(qr')I_n'(qb)\}}{I_n'(qa)K_n'(qb) - K_n'(qa)I_n'(qb)} \epsilon_n \cos n\theta \qquad \text{........ (18)}$$

and when $r > r'$ we have to interchange r and r' in (18).

Applying the Inversion Theorem to the terms of (18) we obtain,

when $a < r < r'$,

$$v = -\frac{Q}{2\kappa i}\sum_{n=0}^{\infty} \epsilon_n \cos n\theta \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \frac{[I_n(\mu r)K_n'(\mu a) - K_n(\mu r)I_n'(\mu a)][I_n(\mu r')K_n'(\mu b) - K_n(\mu r')I_n'(\mu b)]}{I_n'(\mu a)K_n'(\mu b) - K_n'(\mu a)I_n'(\mu b)} \frac{d\lambda}{\lambda} \qquad \text{.... (19)}$$

where $\mu = \sqrt{(\lambda/\kappa)}$.

The integrands of (19) are all single valued functions of λ, so, using the contour of Fig. 3, the line integrals become $2i\pi$ times the sum of the residues at their poles.

The term involving $n = 0$ has a double pole at the origin with residue

$$\frac{1}{b^2 - a^2}\left\{2\kappa t + \frac{r^2 + r'^2}{2} + a^2 \log\frac{a}{r} + b^2 \log\frac{b}{r'} + \frac{a^2 b^2}{b^2 - a^2}\log\frac{b}{a} - \frac{3}{4}(a^2 + b^2)\right\}. \qquad (20)$$

A term involving $n > 0$ has a simple pole at the origin with residue

$$\frac{(r^{2n} + a^{2n})(r'^{2n} + b^{2n})}{2nr^n r'^n(b^{2n} - a^{2n})}. \qquad \text{........................ (21)}$$

Finally a term involving $n \geq 0$ has simple poles at $\lambda = -\kappa\alpha_{n,s}^2$, where $\pm\alpha_{n,s}$, $n = 0, 1, \ldots$, $s = 0, 1, \ldots$, are the zeros (all real and simple) of

$$J_n'(\alpha a)Y_n'(\alpha b) - J_n'(\alpha b)Y_n'(\alpha a) = 0. \qquad \text{.................. (22)}$$

The residue of the n^{th} integrand at the pole $\lambda = -\kappa\alpha_{n,s}^2$ is

$$\frac{\frac{1}{2}\pi^2 a^2 b^2 \alpha^2_{n,s}\, J'^2_n(\alpha_{n,s}a)\, U_n(r\alpha_{n,s})\, U_n(r'\alpha_{n,s})}{a^2(n^2-b^2\alpha^2_{n,s})J'^2_n(a\alpha_{n,s}) - b^2(n^2-a^2\alpha^2_{n,s})J'^2_n(b\alpha_{n,s})}\, e^{-\kappa\alpha^2_{n,s}t}, \quad \ldots\ldots (23)$$

where $U_n(r\alpha_{n,s}) = J_n(r\alpha_{n,s})Y'_n(b\alpha_{n,s}) - Y_n(r\alpha_{n,s})J'_n(b\alpha_{n,s})$ . .... (24)

Adding these results we have finally

$$v = -\frac{Q}{b^2-a^2}\left\{2\kappa t + \tfrac{1}{2}(r^2+r'^2) - \tfrac{3}{4}(a^2+b^2) + a^2\log\frac{a}{r} + b^2\log\frac{b}{r'} + \frac{a^2b^2}{b^2-a^2}\log\frac{b}{a}\right\}$$

$$- Q\sum_{n=1}^{\infty}\frac{(r^{2n}+a^{2n})(r'^{2n}+b^{2n})}{nr^n r'^n(b^{2n}-a^{2n})}\cos n\theta$$

$$-\tfrac{1}{2}Q\pi^2a^2b^2\sum_{n=0}^{\infty}\epsilon_n\cos n\theta\sum_{s=1}^{\infty}\frac{e^{-\kappa\alpha^2_{n,s}t}\,\alpha^2_{n,s}\,J'^2_n(a\alpha_{n,s})U_n(r\alpha_{n,s})U_n(r'\alpha_{n,s})}{a^2(n^2-b^2\alpha^2_{n,s})J'^2_n(a\alpha_{n,s}) - b^2(n^2-a^2\alpha^2_{n,s})J'^2_n(b\alpha_{n,s})},$$

.... (25)

when $a < r < r'$, and to obtain the result for $r' < r < b$ we have to interchange r and r'. The second term may of course be expressed as a series of logarithms, corresponding to an infinite series of images.

CHAPTER V.

A NUMERICAL STUDY[+] OF HEAT FLOW IN THE REGION BOUNDED INTERNALLY BY A CIRCULAR CYLINDER.

The problem of heat flow in the region bounded internally by the cylinder $r = a$, with unit initial temperature and $r = a$ kept at zero for $t > 0$, has been considered by many writers[*]. It has practical interest in connection with the cooling of deep mines and Smith (loc. cit.) has considered it from this point of view and derived a solution for the case in which the temperature at $r = a$ is a step function of the time; he has obtained a solution in a form suitable for computation by means of an empirical numerical approximation which makes it difficult to determine the error in the final results. The object of this chapter is (i) to give numerical values for the solution sufficient for practical purposes, these are given in § 1, and in § 2 are compared with the obvious physical approximations, the cooling of a solid bounded by a plane or bounded internally by a sphere, (ii) to consider the problem of the cooling of

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[+] The importance of the problem was suggested to me by Professor Carslaw after correspondence with Professor Ingersoll.

[*] Nicholson, Proc. Roy. Soc., A, 100 (1921) 226. Goldstein, Proc. London Math. Soc. (2), 34 (1931) 53 (operational methods). Smith, Journ. App. Phys. 8 (1937) 441 (Carslaw's contour integral method). Carslaw and Jaeger, Phil. Mag. (7) 27 (1938) 473 (Laplace Transformation). Titchmarsh, Theory of Fourier Integrals, § 10.10. All these authors save Smith consider the related problem in which the initial temperature is zero and $r = a$ is kept at unity for $t > 0$; the result of the present problem is obtained by subtracting from unity the result of this type with $r = a$ kept at -1.

the region bounded internally by a cylinder, and initially at unit temperature, by forced convection of air in the region $r < a$. This gives rise to a boundary condition of "radiation" type, $\frac{\partial v}{\partial r} - h\,v = 0$ at $r = a$. Numerical values of h for forced convection are given in Fishenden and Saunders*. The solution is given in § 3 and a comparison with the corresponding problems for plane and spherical boundaries in § 4.

Another type of boundary condition which may be of some practical importance has been discussed in Chapter IV, § 1.

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* The Calculation of Heat Transmission. London, H.M. Stationery Office.

1. The region bounded internally by the cylinder $r = a$. The initial temperature V_0. The surface $r = a$ kept at zero for $t > 0$. To find the flux at $r = a$.

It is known [cf. Carslaw and Jaeger, loc. cit.] that the temperature at any point in this problem is given by

$$v = -\frac{2V_0}{\pi}\int_0^\infty e^{-\kappa u^2 t}\,\frac{J_0(ur)Y_0(ua) - Y_0(ur)J_0(ua)}{J_0^2(ua) + Y_0^2(ua)}\,\frac{du}{u}.$$

Thus the flux from the solid at $r = a$ is

$$F = K\left[\frac{\partial v}{\partial r}\right]_{r=a}$$

$$= \frac{4KV_0}{a\pi^2}\int_0^\infty \frac{e^{-\kappa u^2 t}\,du}{u\left[J_0^2(au) + Y_0^2(au)\right]}.$$

Hence

$$\frac{F}{KV_0} = \frac{4}{a\pi^2}\int_0^\infty \frac{e^{-\alpha x^2}\,dx}{x\left[J_0^2(x) + Y_0^2(x)\right]} \qquad \text{.................. (1)}$$

where $\alpha = \kappa t/a^2$. .... (2)

Thus to find the flux we need the values of

$$I(\alpha) = \int_0^\infty \frac{e^{-\alpha x^2}\,dx}{x\left[J_0^2(x) + Y_0^2(x)\right]}. \qquad \text{.................. (3)}$$

This is most easily put in a form suitable for numerical integration by integration by parts. Let

$$\bar{\psi}(x) = \frac{\pi}{2} + \arg H_0^{(1)}(x), \qquad \text{.................... (4)}$$

then $\bar{\psi}(0) = 0$, ... (5)

and $$\frac{d}{dx}\,\bar{\psi}(x) = \frac{2}{\pi x\left[J_0^2(x) + Y_0^2(x)\right]}. \qquad \text{.................... (6)}$$

also, for large x, $\bar{\psi}(x) = x + \frac{\pi}{4} - \frac{1}{8x} + O(\frac{1}{x^2})$ . ... (7)

Substituting the result (6) in (3) and integrating by parts we obtain

$$I(\alpha) = \pi\alpha \int_0^\infty x e^{-\alpha x^2} \bar{\psi}(x) \, dx . \qquad \text{.................... (8)}$$

This form is convenient for large values of α . For moderate values it is better to remove the first terms of its asymptotic expansion (7) from $\bar{\psi}(x)$ and to integrate the terms involving these separately. This gives

$$I(\alpha) = \pi\alpha \left\{ \frac{\pi}{8\alpha} + \frac{1}{4\alpha}\sqrt{\frac{\pi}{\alpha}} - \frac{1}{16}\sqrt{\frac{\pi}{\alpha}} \right\} + \pi\alpha \int_0^\infty e^{-\alpha x^2} \left\{ x\bar{\psi}(x) + \frac{1}{8} - \frac{\pi x}{4} - x^2 \right\} dx \qquad \text{...... (9)}$$

For small values of α the following series expansion may be derived following Goldstein (loc. cit.)

$$I(\alpha) = \frac{\pi^2}{4} \left\{ .5642\,\alpha^{-\frac{1}{2}} + .5 - .1410\,\alpha^{\frac{1}{2}} + .125\,\alpha - .1469\,\alpha^{3/2} + .203\,\alpha^2 - .315\,\alpha^{5/2} + .536\,\alpha^3 - \dots \right\} \qquad \text{....... (10)}$$

this will give four place accuracy up to $\alpha = 0.1$.

Numerical values of $I(\alpha)$ are given* in Fig. IV.

If the surface temperature is not constant the results may easily be obtained by numerical integration using Duhamel's theorem.

For the case in which the surface temperature is a step function of the time Smith (loc. cit.) has given formulae involving a function $G(\alpha)$ given by

$$G(\alpha) = \frac{4\alpha}{\pi} I(\alpha)$$

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* It is proposed to publish a four-place table covering the range of importance in practice. This is not yet complete.

so that numerical values for this case can easily be obtained from the results of this section.

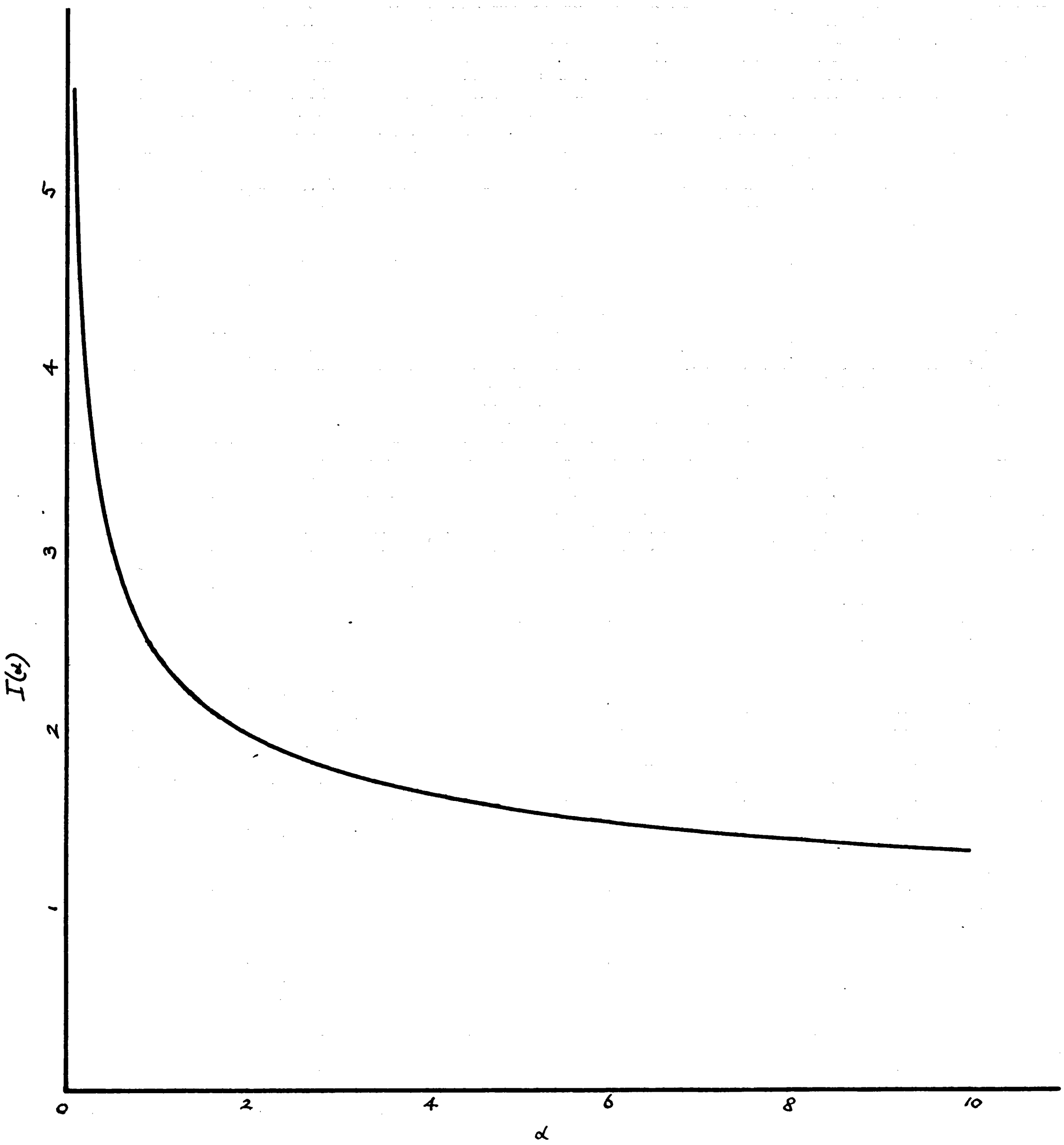

Fig. 4.

2. It is perhaps of some interest to compare the results for the region bounded internally by a cylinder with the obvious physical approximations: the regions bounded internally by a sphere and bounded by a plane respectively. The results for the two latter cases are well known:

For the sphere (radius a) $\dfrac{F}{KV_0} = \dfrac{1}{\sqrt{(\pi\kappa t)}} + \dfrac{1}{a}$ .... (1)

For the plane $\dfrac{F}{KV_0} = \dfrac{1}{\sqrt{(\pi\kappa t)}}$ .... (2)

For the cylinder (radius a) $\dfrac{F}{KV_0} = \dfrac{4}{a\pi^2} I(\alpha)$ .... (3)

These are compared in Figure 5 for constants of the order of those occurring in mining practice, namely, $\kappa = .01$, $a = 150$; the units used are c.g.s. throughout. It is seen that for these values the results for the cylinder lie throughout a little less than midway between the sphere and plane results, which provides an easy qualitative guide to their behaviour.

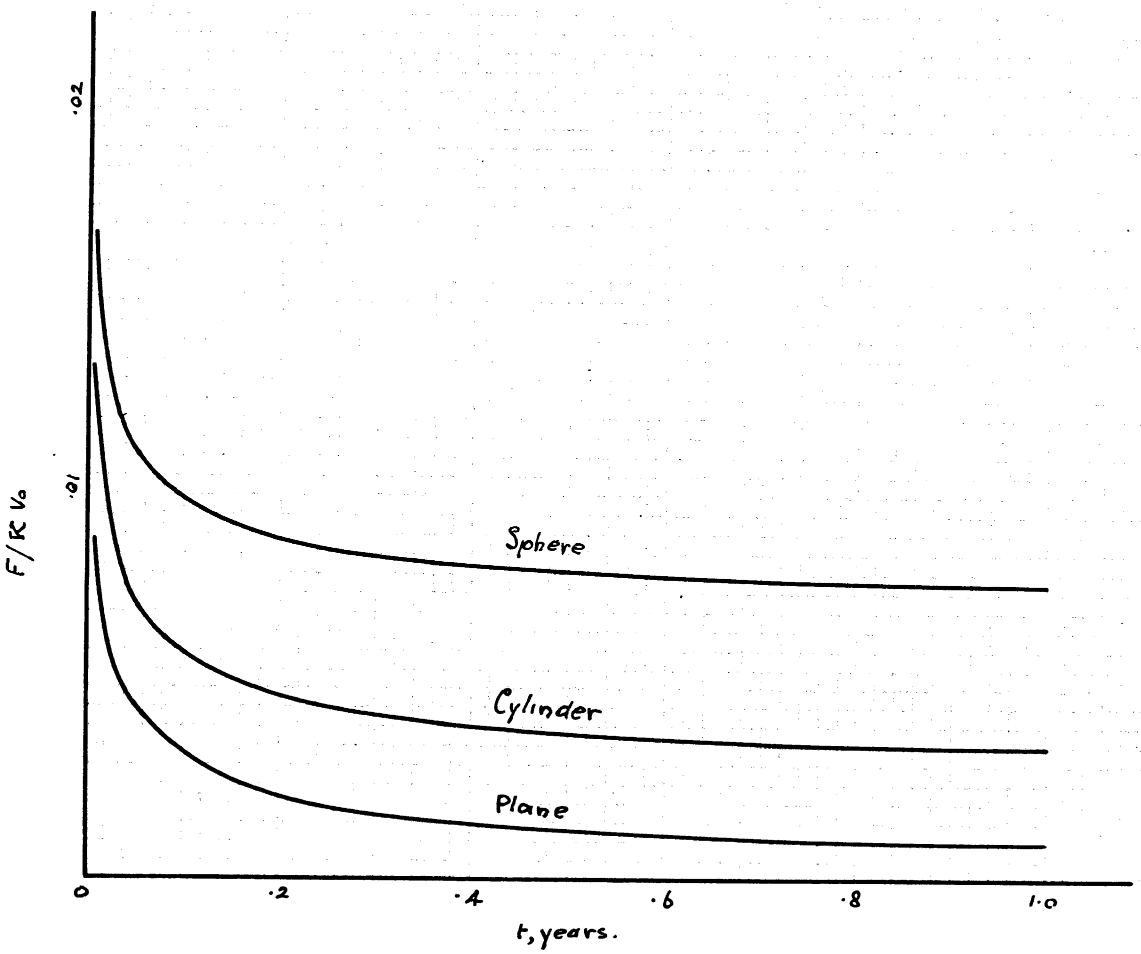

Fig. 5.

3. Solid bounded internally by $r = a$. Radiation at $r = a$ into a medium at zero. The initial temperature V_0. To find the flux over $r = a$.

The boundary condition at $r = a$ is

$$\frac{\partial v}{\partial r} - hv = 0.$$

The solution giving the temperature at any point is*

$$v = -\frac{2hV_0}{\pi}\int_0^\infty e^{-\kappa u^2 t}\,\frac{J_0(ur)\left[uY_1(ua)+hY_0(ua)\right] - Y_0(ur)\left[uJ_1(ua)+hJ_0(ua)\right]}{\left[uJ_1(ua)+hJ_0(ua)\right]^2+\left[uY_1(ua)+hY_0(ua)\right]^2}\,\frac{du}{u} \qquad \ldots (1)$$

Thus the flux from the solid over $r = a$ is

$$\frac{F}{KV_0} = \frac{4h^2 a}{\pi^2}\, I(b,\alpha) , \qquad \ldots\ldots (2)$$

where $\alpha = \kappa t/a^2$, $b = ah$, $\qquad \ldots\ldots (3)$

and
$$I(b,\alpha) = \int_0^\infty \frac{e^{-\alpha x^2}}{\left[xJ_1(x) + bJ_0(x)\right]^2 + \left[xY_1(x) + bY_0(x)\right]^2}\,\frac{dx}{x} \qquad \ldots\ldots (4)$$

Letting $t \to 0$ and $r \to a$ in (1) we obtain

$$I(b,0) = \frac{\pi^2}{4b} \qquad \ldots\ldots (5)$$

and therefore

$$\frac{F}{KV_0} \to h \text{ as } t \to 0. \qquad \ldots\ldots (6)$$

It may be remarked that the surface temperature of the solid is

$$v_s = \frac{4hV_0}{\pi^2 a}\, I(b,\alpha) \qquad \ldots\ldots (7)$$

so that when $I(b,\alpha)$ is known the surface temperature is also known.

---

* This may be obtained from Chapter II, § 4. A direct derivation is given in Carslaw and Jaeger, Proc. London Math. Soc. (2), 46 (1940), 361. ~~(in press)~~.

To bring (4) to a form suitable for numerical integration we proceed as in § 1.

Let $\varphi(x) = \frac{\pi}{2} + \arg\left[x\, H_1^{(1)}(x) + b\, H_o^{(1)}(x)\right]$, ... (8)

Then as $x \to 0$, $\varphi(x) \to 0$

and, for large x, $\varphi(x) = x - \frac{\pi}{4} + (b + \frac{3}{8})\frac{1}{x} + 0(\frac{1}{x^2})$.

Then, integrating by parts as in § 1, we find

$$I(b,\alpha) = \pi \int_o^{\infty} \left\{ \frac{\alpha}{x^2 + b^2} + \frac{1}{(x^2 + b^2)^2} \right\} x e^{-\alpha x^2} \varphi(x)\, dx \qquad \text{........ (9)}$$

$$= \pi \int_o^{\infty} e^{-\alpha x^2} \left\{ \frac{\alpha}{x^2+b^2} + \frac{1}{(x^2+b^2)^2} \right\} \left[x\,\varphi(x) - x^2 + \frac{1}{4}\pi x - (b+\frac{3}{8})\right] dx$$

$$+ \frac{\pi^{3/2}}{16\, b^2}\left[(8b + 3)\sqrt{\alpha} - \sqrt{\pi}\right] + \frac{\pi^2}{4b^3} e^{\alpha b^2} (\frac{3}{8} + b + b^2)\, \mathrm{erfc}(b\sqrt{\alpha}) \qquad \text{.. (8)}$$

In Figure 6 are given curves of $I(b,\alpha)$ against α for $b = .2, 1, 5. 10.$

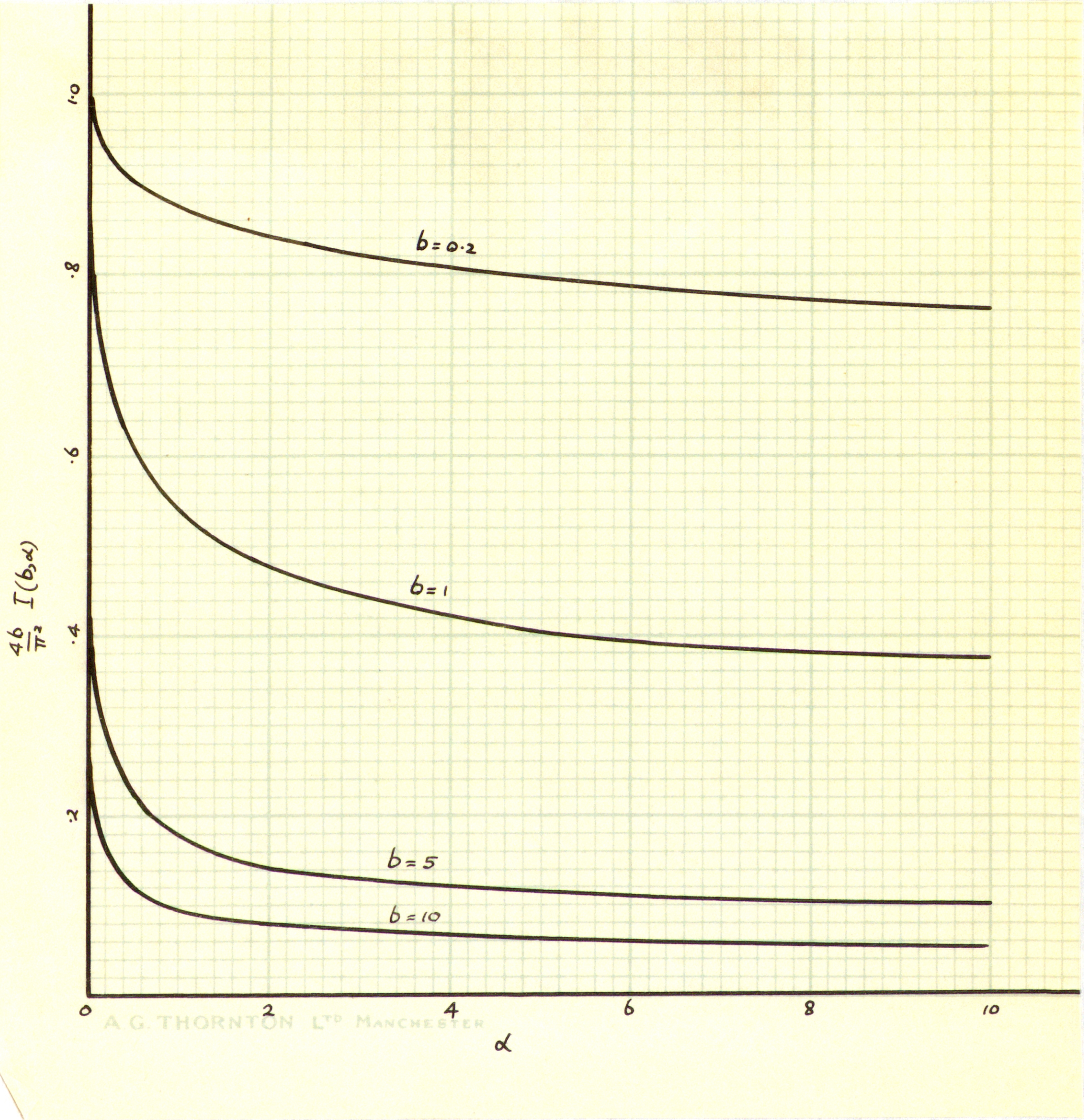
1.0
.8
.6
.4
.2
$\frac{4b}{\pi^2} I(b,\alpha)$
b = 0.2
b = 1
b = 5
b = 10
0
2
4
6
8
10
α
A G THORNTON L^TD MANCHESTER

Fig. 6

4. In view of the impossibility of giving complete results for the problem of § 3, it seems worth while comparing the values for the cylinder with those for the plane and the sphere of the same radius. The results are:

For the plane $$\frac{F}{hKV_0} = e^{\kappa t h^2} \operatorname{erfc} h\sqrt{(\kappa t)} \qquad \ldots\ldots (1)$$

For the cylinder $$\frac{F}{hKV_0} = \frac{4ha}{\pi^2} I(b, \varkappa) \qquad \ldots\ldots (2)$$

For the sphere $$\frac{F}{hKV_0} = \frac{1}{1+ah} + \frac{ah}{1+ah} e^{\kappa t(h+1/a)^2} \operatorname{erfc}(h+\tfrac{1}{a})\sqrt{(\kappa t)} \qquad \ldots\ldots (3)$$

A table from which $e^{x^2} \operatorname{erfc} x$ may be obtained is given in Pearson, Tables for Statisticians and Biometricians, Vol. 2, p. 11.

Since $e^{x^2} \operatorname{erfc} x = \frac{1}{x\sqrt{\pi}} + O(\frac{1}{x^3})$, (1) and (3) may be replaced by

$$\frac{F}{hKV_0} = \frac{1}{h\sqrt{(\pi\kappa t)}} \quad \text{and} \quad \frac{1}{1 + ah} + \frac{a^2 h}{(ah+1)^2\sqrt{(\pi\kappa t)}}$$

if $h\sqrt{(\kappa t)}$ is sufficiently large.

In Figure 7 the values calculated from (1), (2), and (3) are compared from the case $a = 150$, $\kappa = .01$. Curves I, II, III are results for the sphere, cylinder and plane corresponding to $b = .2$ ($h = .00133$), and Curves IV, V, VI those corresponding to $b = 1$ ($h = .00667$). As in § 2 the values for the plane and sphere provide only a rough qualitative approximation to those for the cylinder.

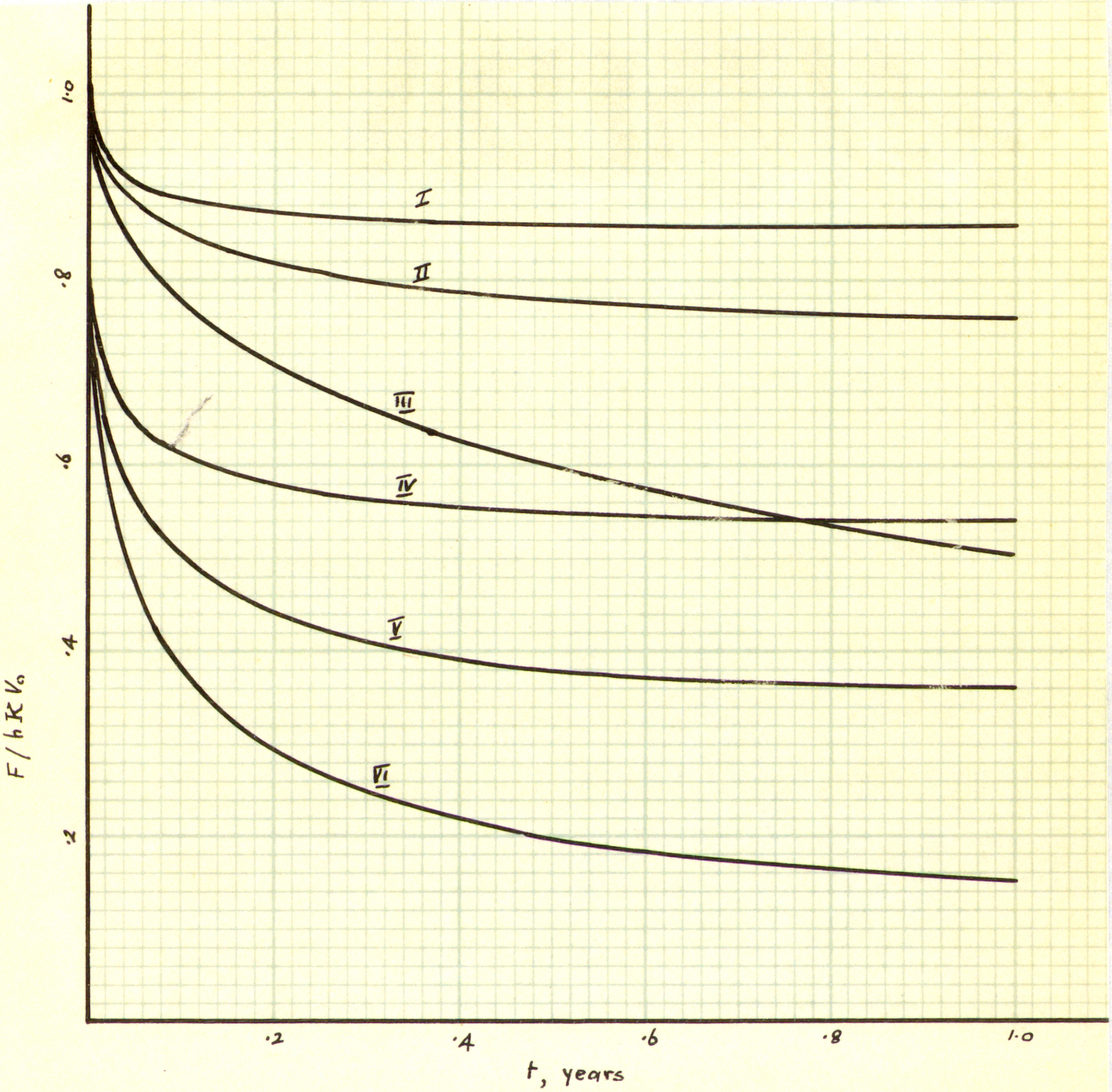

Fig. 7.

CHAPTER VI.

THE REGIONS BOUNDED INTERNALLY AND EXTERNALLY BY FINITE CIRCULAR CYLINDERS WITH ZERO INITIAL TEMPERATURE AND PRESCRIBED SURFACE CONDITIONS.

Introduction.

In this Chapter will be given a complete set of formal solutions for solids bounded by coaxial cylinders and planes perpendicular to the axis with zero initial temperatures, prescribed temperatures at some surfaces, and at the others boundary conditions of the type used in Chapter II. The surface temperatures will be assumed to be expansible in the forms developed in Chapter II; when the regions extend to infinity the conditions for these, e.g. Fourier's or Weber's integral theorems, are extremely restrictive; a different method which avoids this difficulty is developed in Chapters VII and VIII. The results for the finite cylinder are classical, but are given here for completeness, the use of the general boundary conditions of Chapter II enables a very large collection of results to be given compactly.

This Chapter is intended as a study in the most obvious method of applying the Laplace procedure to two variable problems, namely that of separation of variables in the subsidiary equation. The general results are formal, but it may be remarked that the method is superior to the classical ones in that by using the results of Chapter II the possibility of omitting isolated terms (such as the first term of a Dini series*) is obviated. Results deduced by the transformation of line integrals using the contours of Figures 1 or 3 (e.g. the deductions of § 2 (8) from § 2 (6)) may be justified as in Appendix III.

The surface temperatures will be assumed throughout to be independent of time; the extension to the case in which they are known functions of the time is trivial.

In § 1 a collection of the expansions of Chapter II is made for convenient reference.

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* Cf. C.H., p. 118, II; Goldstein, Proc. London Math. Soc. (2) 34 p. 83.

1. It will be assumed that the arbitrary functions occurring can be expanded in one of the following forms discussed in Chapter II.

(i) That appropriate to the region $0 < x < \ell$ with boundary conditions $k_1 \frac{\partial v}{\partial x} - h_1 v = 0$ when $x = 0$, and $k_2 \frac{\partial v}{\partial x} + h_2 v = 0$ when $x = \ell$.

$$f(x) = \sum_{n=1}^{\infty} Z_n(x) \int_0^{\ell} Z_n(x')f(x')dx' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\text{where} \quad Z_n(x) = \frac{\left[2(k_2^2\beta_m^2+h_2^2)\right]^{\frac{1}{2}} (k_1\beta_m \cos\beta_m x + h_1 \sin\beta_m x)}{\left\{(k_1^2\beta_m^2+h_1^2)\left[\ell(k_2^2\beta_n^2+h_2^2)+k_2h_2\right] + k_1h_1(k_2^2\beta_n^2+h_2^2)\right\}^{\frac{1}{2}}} \quad \ldots (2)$$

and the β_n are the roots of

$$(k_1k_2\beta^2 - h_1h_2)\sin\beta\ell - \beta(k_1h_2 + k_2h_1)\cos\beta\ell = 0, \quad \ldots (3)$$

and if $h_1 = h_2 = 0$ a term $\frac{1}{\ell}\int_0^{\ell} f(x')dx'$ is to be added to the right hand side of (1).

(ii) That appropriate to the region $x > 0$ with boundary condition $k\frac{\partial v}{\partial x} - hv = 0$ at $x = 0$.

$$f(x) = \frac{2}{\pi}\int_0^{\infty} Z(u,x)du \int_0^{\infty} Z(u,x')f(x')dx' \quad \ldots\ldots\ldots\ldots (4)$$

$$\text{where} \quad Z(u,x) = \frac{ku\cos ux + h\sin ux}{\sqrt{(k^2u^2 + h^2)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

(iii) That appropriate to the region $0 \leqslant r < a$ with boundary condition $k\frac{\partial v}{\partial r} + hv = 0$ at $r = a$.

$$f(r) = \frac{2}{a^2}\sum_{s=1}^{\infty} \frac{J_0(r\alpha_s)}{J_0^2(a\alpha_s) + J_1^2(a\alpha_s)} \int_0^a r'f(r')\,J_0(r'\alpha_s)dr', \quad \ldots (6)$$

where the α_s are the roots of

$$k\alpha J_0'(a\alpha) + hJ_0(a\alpha) = 0\,, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

and if $h = 0$ a term $\dfrac{2}{a^2}\displaystyle\int_0^a r'f(r')dr'$ is to be added to the right hand side of (6).

(iv) <u>That appropriate to the region $r > a$ with boundary condition</u> $k\dfrac{\partial v}{\partial r} - hv = 0$ <u>at $r = a$.</u>

$$f(r) = \int_0^\infty C(u,r)u\,du \int_a^\infty C(u,r')\,f(r')r'dr'\,, \qquad \ldots\ldots\ldots\ldots \quad (8)$$

$$\text{where}\quad C(u,r) = \frac{J_0(ur)\left[kuY_1(au) + hY_0(au)\right] - Y_0(ur)\left[kuJ_1(au) + hJ_0(au)\right]}{\left\{\left[kuJ_1(au) + hJ_0(au)\right]^2 + \left[kuY_1(au) + hY_0(au)\right]^2\right\}^{\frac{1}{2}}} \qquad (9)$$

(v) <u>That appropriate to the region $a < r < b$ with boundary conditions</u> $k_1\dfrac{\partial v}{\partial r} - h_1v = 0$ <u>at $r = a$ and</u> $k_2\dfrac{\partial v}{\partial r} + h_2v = 0$ <u>at $r = b$.</u>

$$f(r) = \sum_{s=1}^{\infty} F(r,\alpha_s)\int_a^b r'f(r')F(r',\alpha_s)dr'\,, \qquad \ldots\ldots\ldots\ldots \quad (10)$$

$$\text{where}\quad F(r,\alpha_s) = \frac{\pi\alpha_s\left[k_1\alpha_sJ_0'(a\alpha_s) - h_1J_0(a\alpha_s)\right]}{2^{\frac{1}{2}}\left\{(k_2^2\alpha_s^2+h_2^2)\left[k_1\alpha_sJ_0'(a\alpha_s)-h_1J_0(a\alpha_s)\right]^2 - (k_1^2\alpha_s^2+h_1^2)\left[k_2\alpha_sJ_0'(b\alpha_s)+h_2J_0(b\alpha_s)\right]^2\right\}^{\frac{1}{2}}}$$

$$\times\left\{J_0(r\alpha_s)\left[k_2\alpha_sY_0'(b\alpha_s)+h_2Y_0(b\alpha_s)\right] - Y_0(r\alpha_s)\left[k_2\alpha_sJ_0'(b\alpha_s)+h_2J_0(b\alpha_s)\right]\right\} \quad \ldots (11)$$

and the α_s are the roots of

$$\left[k_1\alpha J_0'(a\alpha) - h_1J_0(a\alpha)\right]\left[k_2\alpha Y_0'(b\alpha) + h_2Y_0(b\alpha)\right] - \left[k_1\alpha Y_0'(a\alpha) - h_1Y_0(a\alpha)\right]\left[k_2\alpha J_0'(b\alpha) + h_2J_0(b\alpha)\right] = 0\,,$$

and if $h_1 = h_2 = 0$ an additional term $\dfrac{2}{b^2 - a^2}\displaystyle\int_a^b r'f(r')dr'$ is to be added to the right hand side of (10).

2. <u>The finite cylinder $0 \leq r < a$, $0 < z < l$. The surface $r = a$ kept at $f(z)$ for $t > 0$. Boundary conditions</u> $k_1 \frac{\partial v}{\partial z} - h_1 v = 0$, <u>$z = 0$, $t > 0$, and</u> $k_2 \frac{\partial v}{\partial z} + h_2 v = 0$, <u>$z = l$, $t > 0$.</u>

We have to solve

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} = \frac{1}{\kappa}\frac{\partial v}{\partial t}, \quad 0 \leq r < a, \quad 0 < z < l, \quad t > 0$$

with $v = 0$, when $t = 0$, $0 \leq r < a$, $0 < z < l$

$$k_1 \frac{\partial v}{\partial z} - h_1 v = 0, \quad z = 0, \quad 0 \leq r < a, \quad t > 0,$$

$$k_2 \frac{\partial v}{\partial z} + h_2 v = 0, \quad z = l, \quad 0 \leq r < a, \quad t > 0,$$

and $v = f(z)$, when $r = a$, $0 < z < l$.

The subsidiary equation is

$$\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}}{\partial r} + \frac{\partial^2 \bar{v}}{\partial z^2} - q^2 \bar{v} = 0, \quad 0 \leq r < a, \quad 0 < z < l, \qquad \ldots\ldots (1)$$

to be solved with $\bar{v} = \frac{1}{p} f(z)$, $r = a$, $0 < z < l$, ... (2)

$$k_1 \frac{\partial \bar{v}}{\partial z} - h_1 \bar{v} = 0, \quad z = 0, \quad 0 \leq r < a, \qquad \ldots\ldots\ldots (3)$$

$$k_2 \frac{\partial \bar{v}}{\partial z} + h_2 \bar{v} = 0, \quad z = l, \quad 0 \leq r < a. \qquad \ldots\ldots\ldots (4)$$

We assume that $f(z)$ can be expanded in a series of type § 1 (1). A solution of (1) satisfying (3) and (4) and finite when $r = 0$ is

$$Z_n(z)\ I_0\left[r\sqrt{(q^2 + \beta_n^2)}\right],$$

where the $Z_n(z)$ and β_n are defined in § 1 (2) and (3).

Thus a solution of (1) and the boundary conditions (2), (3) and (4) is

$$\bar{v} = \sum_{n=1}^{\infty} \frac{1}{p} \frac{Z_n(z) I_0\left[r\sqrt{(q^2+\beta_n^2)}\right]}{I_0\left[a\sqrt{(q^2+\beta_n^2)}\right]} \int_0^l f(z') Z_n(z') dz', \qquad \ldots\ldots\ldots (5)$$

where if $h_1 = h_2$ a term $\frac{1}{lp} \frac{I_0(qr)}{I_0(aq)} \int_0^l f(z')dz'$ is to be added to the right hand side of (5).

To determine v from (5) we require

$$I = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\lambda t}}{\lambda} \frac{I_0\left[r\sqrt{(\frac{\lambda}{\kappa} + \beta_m^2)}\right]}{I_0\left[a\sqrt{(\frac{\lambda}{\kappa} + \beta_m^2)}\right]} d\lambda \qquad \text{................ (6)}$$

The integrand has a simple pole at $\lambda = 0$ of residue

$$I_0(r\beta_m)/I_0(a\beta_m) \, .$$

and simple poles at $\lambda = -\kappa(\beta_m^2 + \alpha_s^2)$, where α_s, $s = 1, 2, \ldots$ are the positive roots of

$$J_0(a\alpha) = 0 \, . \qquad \text{........................... (7)}$$

The residue at the pole $\lambda = -\kappa(\beta_m^2 + \alpha_s^2)$ is

$$\frac{2\alpha_s}{a(\beta_m^2 + \alpha_s^2)} \frac{J_0(r\alpha_s)}{J_0'(a\alpha_s)} e^{-\kappa(\beta_m^2 + \alpha_s^2)t} \, .$$

Thus

$$I = \frac{I_0(r\beta_m)}{I_0(a\beta_m)} + \frac{2}{a}\sum_{s=1}^{\infty} \frac{\alpha_s}{\beta_m^2 + \alpha_s^2} \frac{J_0(r\alpha_s)}{J_0'(a\alpha_s)} e^{-\kappa(\beta_m^2 + \alpha_s^2)t} \, . \qquad \text{...... (8)}$$

Using this result in (5) we have finally

$$v = \sum_{m=1}^{\infty}\left\{\frac{I_0(r\beta_m)}{I_0(a\beta_m)} + \frac{2}{a}\sum_{s=1}^{\infty}\frac{\alpha_s}{\beta_m^2 + \alpha_s^2}\frac{J_0(r\alpha_s)}{J_0'(a\alpha_s)} e^{-\kappa(\beta_m^2 + \alpha_s^2)t}\right\} Z_m(z)\int_0^{\ell} Z_m(z')f(z')dz' \qquad \text{... (9)}$$

where if $h_1 = h_2 = 0$ a term

$$\frac{1}{\ell}\left\{1 + \frac{2}{a}\sum_{s=1}^{\infty}\frac{J_0(r\alpha_s)}{\alpha_s J_0'(a\alpha_s)} e^{-\kappa\alpha_s^2 t}\right\}\int_0^{\ell} f(z')dz'$$

is to be added to the right hand side of (9).

3. The region $0 < z < l$, $r > a$. The surface $r = a$ kept at $f(z)$ for $t > 0$. Boundary conditions $k_1 \frac{\partial v}{\partial z} - h_1 v = 0$, $z = 0$, $t > 0$, and $k_2 \frac{\partial v}{\partial z} + h_2 v = 0$, $z = l$, $t > 0$.

Proceeding as in § 2 we find

$$\bar{v} = \sum_{n=1}^{\infty} \frac{Z_n(z) K_0\left[r\sqrt{(q^2+\beta_n^2)}\right]}{p\, K_0\left[a\sqrt{(q^2+\beta_n^2)}\right]} \int_0^l f(z')Z_n(z')dz' \qquad \ldots\ldots\ldots\ldots (1)$$

where the $Z_n(z')$ and the β_n are defined in § 1 (2) and (3).

To determine v we require

$$I = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\lambda t}}{\lambda} \frac{K_0\left[r\sqrt{(\frac{\lambda}{\kappa} + \beta_n^2)}\right]}{K_0\left[a\sqrt{(\frac{\lambda}{\kappa} + \beta_n^2)}\right]} d\lambda \qquad \ldots\ldots\ldots\ldots (2)$$

This has a simple pole at $\lambda = 0$ of residue $K_0(r\beta_n)/K_0(a\beta_n)$.

Also it has a branch point at $\lambda = -\kappa\beta_n^2$, taking the usual contour about this point and putting $\lambda = -\kappa\beta_n^2 + \kappa u^2 e^{-i\pi}$ on CD and $\lambda = -\kappa\beta_n^2 + \kappa u^2 e^{i\pi}$ on EF we obtain (the small circle about the point $-\kappa\beta_n^2$ gives zero)

$$I = \frac{e^{-\kappa\beta_n^2 t}}{2\pi i} \int_0^{\infty} \frac{e^{-\kappa u^2 t}\, 2u du}{u^2 + \beta_n^2} \left\{ \frac{J_0(ru) - iY_0(ru)}{J_0(au) - iY_0(au)} - \frac{J_0(ru) + iY_0(ru)}{J_0(au) + iY_0(au)} \right\} + \frac{K_0(r\beta_n)}{K_0(a\beta_n)}$$

$$= \frac{K_0(r\beta_n)}{K_0(a\beta_n)} + \frac{2}{\pi} e^{-\kappa\beta_n^2 t} \int_0^{\infty} \frac{e^{-\kappa u^2 t}\, u du}{u^2 + \beta_n^2} \frac{C_0(ru, au)}{J_0^2(au) + Y_0^2(au)}, \qquad \ldots\ldots\ldots\ldots (3)$$

where $C_0(x,y) = J_0(x)Y_0(y) - Y_0(x)J_0(y)$.)

Using this result in (1) we have

$$v = \sum_{n=1}^{\infty} \left\{ \frac{K_0(r\beta_n)}{K_0(a\beta_n)} + \frac{2}{\pi} e^{-\kappa\beta_n^2 t} \int_0^{\infty} \frac{e^{-\kappa u^2 t}\, u du}{u^2 + \beta_n^2} \frac{C_0(ru, au)}{J_0^2(au) + Y_0^2(au)} \right\} Z_n(z) \int_0^l f(z')Z_n(z')dz'$$

$$\ldots (5)$$

and if $h_1 = h_2 = 0$ a term

$$\frac{1}{l} \left\{ 1 + \frac{2}{\pi} \int_0^{\infty} e^{-\kappa u^2 t} \frac{C_0(ru, au)}{J_0^2(au) + Y_0^2(au)} u du \right\} \int_0^l f(z')dz' \quad \text{is to be}$$

added to the right hand side of (5).

4. <u>The hollow cylinder $a < r < b$. The surface $r = a$ kept at $f(z)$ for $t > 0$. Boundary conditions</u> $k\frac{\partial v}{\partial r} + hv = 0$, <u>when $r = b$, $t > 0$;</u> $k_1 \frac{\partial v}{\partial z} - h_1 v = 0$, <u>$z = 0$, $t > 0$; and</u> $k_2 \frac{\partial v}{\partial z} + h_2 v = 0$, <u>$z = \ell$, $t > 0$.</u>

As in § 2 we obtain

$$\bar{v} = \sum_{n=1}^{\infty} \frac{I_0(q_n r)\left[kq_n K_0'(q_n b) + hK_0(q_n b)\right] - K_0\left[(q_n r)\right]\left[kq_n I_0'(q_n b) + hI_0(q_n b)\right]}{I_0(q_n a)\left[kq_n K_0'(q_n b) + hK_0(q_n b)\right] - K_0\left[(q_n a)\right]\left[kq_n I_0'(q_n b) + hI_0(q_n b)\right]}$$

$$\times \frac{Z_n(z)}{p} \int_0^{\ell} f(z')Z_n(z')\,dz' \qquad \ldots\ldots\ldots\ldots\ldots \text{(1)}$$

where $Z_n(z)$ and β_n are defined in § 1 (2) and (3) and

$$q_n = \sqrt{(q^2 + \beta_n^2)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

To find v we require

$$B(\beta_n, r, t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\lambda t}}{\lambda} \frac{I_0(r\mu_n)\left[k\mu_n K_0'(b\mu_n) + hK_0(b\mu_n)\right] - K_0(r\mu_n)\left[k\mu_n I_0'(b\mu_n) + hI_0(b\mu_n)\right]}{I_0(a\mu_n)\left[k\mu_n K_0'(b\mu_n) + hK_0(b\mu_n)\right] - K_0(a\mu_n)\left[k\mu_n I_0'(b\mu_n) + hI_0(b\mu_n)\right]}\,d\lambda$$

$$\ldots\ldots\ldots \text{(3)}$$

where $\mu_n = \sqrt{(\frac{\lambda}{\kappa} + \beta_n^2)}$.

This has a simple pole at $\lambda = 0$ of residue

$$\frac{I_0(r\beta_n)\left[k\beta_n K_1(b\beta_n) - hK_0(b\beta_n)\right] + K_0(r\beta_n)\left[k\beta_n I_1(b\beta_n) + hI_0(b\beta_n)\right]}{I_0(a\beta_n)\left[k\beta_n K_1(b\beta_n) - hK_0(b\beta_n)\right] + K_0(a\beta_n)\left[k\beta_n I_1(b\beta_n) + hI_0(b\beta_n)\right]}, \qquad \text{(4)}$$

and simple poles at $\lambda = -\kappa\beta_n^2 - \kappa\alpha_s^2$, where α_s, $s = 1, 2, \ldots$ are the positive zeros of

$$J_0(a\alpha)\left[k\alpha Y_0'(b\alpha) + hY_0(b\alpha)\right] - Y_0(a\alpha)\left[k\alpha J_0'(b\alpha) + hJ_0(b\alpha)\right] = 0. \qquad \text{(5)}$$

The residue at the pole $\lambda = -\kappa\beta_n^2 - \kappa\alpha_s^2$ is

$$\pi e^{-\kappa(\beta_n^2+\alpha_s^2)t} \frac{\alpha_s^2\left[k\alpha_s J_1(b\alpha_s) - hJ_0(b\alpha_s)\right]^2 C_0(r\alpha_s, a\alpha_s)}{(\alpha_s^2+\beta_n^2)\left\{(k^2\alpha_s^2+h^2)J_0^2(a\alpha_s) - \left[k\alpha_s J_1(b\alpha_s) - hJ_0(b\alpha_s)\right]^2\right\}}, \quad (6)$$

where $C_0(x,y) = J_0(x)Y_0(y) - Y_0(x)J_0(y)$.

Thus (5) gives

$$B(\beta_n,r,t) = \frac{I_0(r\beta_n)\left[k\beta_n K_1(b\beta_n) - hK_0(b\beta_n)\right] + K_0(r\beta_n)\left[k\beta_m I_1(b\beta_n) + hI_0(b\beta_m)\right]}{I_0(a\beta_n)\left[k\beta_n K_1(b\beta_n) - hK_0(b\beta_n)\right] + K_0(a\beta_m)\left[k\beta_m I_1(b\beta_m) + hI_0(b\beta_m)\right]}$$

$$+ \pi \sum_{s=1}^{\infty} \frac{e^{-\kappa(\beta_n^2+\alpha_s^2)t}\alpha_s^2\left[k\alpha_s J_1(b\alpha_s) - hJ_0(b\alpha_s)\right]^2 C_0(r\alpha_s, a\alpha_s)}{(\alpha_s^2+\beta_n^2)\left\{(k^2\alpha_s^2+h^2)J_0^2(a\alpha_s) - \left[k\alpha_s J_1(b\alpha_s) - hJ_0(b\alpha_s)\right]^2\right\}}. \quad (7)$$

And with this value of $B(\beta_n,r,t)$ we have from (1)

$$v = \sum_{n=1}^{\infty} B(\beta_n,r,t)Z_n(z)\int_0^{\ell} f(z')Z_m(z')dz', \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (8)$$

and if $h_1 = h_2 = 0$ we have to add to the right hand side of (8) a term

$$\left\{\frac{k - bh\log(r/b)}{k - bh\log(a/b)} + \pi\sum_{s=1}^{\infty} e^{-\kappa\alpha_s^2 t}\frac{\left[k\alpha_s J_1(b\alpha_s) - hJ_0(b\alpha_s)\right]^2 C_0(r\alpha_s, a\alpha_s)}{(k^2\alpha_s^2+h^2)J_0^2(a\alpha_s) - \left[k\alpha_s J_1(b\alpha_s) - hJ_0(b\alpha_s)\right]^2}\right\}$$

$$\times \frac{1}{\ell}\int_0^{\ell} z'f(z')dz'. \quad \ldots\ldots\ldots\ldots \quad (9)$$

5. <u>The semi-infinite cylinder $0 \le r < a$, $z > 0$. The surface $r = a$ kept at $f(z)$ for $t > 0$.</u> <u>Boundary condition</u> $k_1 \frac{\partial v}{\partial z} - h_1 v = 0$, <u>$z = 0$, $t > 0$.</u>

We assume $f(z)$ can be expressed in the form § 1(4).

Proceeding as in § 2 we find

$$\bar{v} = \frac{2}{p\pi} \int_0^\infty \frac{I_0\left[r\sqrt{(q^2+u^2)}\right]}{I_0\left[a\sqrt{(q^2+u^2)}\right]} Z(u,z)du \int_0^\infty Z(u,z')f(z')dz' . \qquad (1)$$

Using the result § 2 (8) this gives

$$v = \frac{2}{\pi}\int_0^\infty \left\{ \frac{I_0(ru)}{I_0(au)} + \frac{2}{a} \sum_{s=1}^{\infty} \frac{\alpha_s}{u^2+\alpha_s^2} \frac{J_0(r u)}{J_0'(au)} e^{-\kappa(\alpha_s^2+u^2)t} \right\} Z(u,z)du \int_0^\infty Z(u,z')f(z')dz', \quad \ldots (2)$$

where the α_s are the positive roots of $J_0(a\alpha) = 0$.

6. <u>The region $r > a$, $z > 0$. The surface $r = a$ kept at $f(z)$ for $t > 0$.</u> <u>Boundary condition</u> $k_1 \frac{\partial v}{\partial z} - h_1 v = 0$, <u>$z = 0$, $r > a$, $t > 0$.</u>

Assuming that $f(z)$ can be expanded in to form § 1 (4) we find as in § 5

$$\bar{v} = \frac{2}{p\pi} \int_0^\infty \frac{K_0\left[r\sqrt{(q^2+u^2)}\right]}{K_0\left[a\sqrt{(q^2+u^2)}\right]} Z(u,z)du \int_0^\infty Z(u,z')f(z')dz' . \quad \ldots\ldots (1)$$

Then using the result § 3 (3) we have

$$v = \frac{2}{\pi}\int_0^\infty \left\{ \frac{K_0(ru)}{K_0(au)} + \frac{2}{\pi} e^{-\kappa u^2 t} \int_0^\infty \frac{e^{-\kappa u'^2 t}\, u'du'}{u^2+u'^2} \frac{C_0(ru',au')}{J_0^2(au')+Y_0^2(au')} \right\} Z(u,z)du \int_0^\infty Z(u,z')f(z')dz' \quad \ldots (2)$$

7. <u>The hollow semi-infinite cylinder $a < r < b$, $z > 0$. The surface $r = a$ kept at $f(z)$ for $t > 0$.</u> <u>Boundary conditions</u> $k_1 \frac{\partial v}{\partial z} - h_1 v = 0$, <u>when $z = 0$, $a < r < b$, $t > 0$,</u> and $k\frac{\partial v}{\partial r} + hv = 0$, <u>when $r = b$, $z = 0$, $t > 0$.</u>

Assuming that $f(z)$ can be expanded in the form § 1(4) we have as in § 5

$$\bar{v} = \frac{2}{p\pi}\int_0^\infty \frac{I_0(q'r)\left[kq'K_0'(q'b) + hK_0(q'b)\right] - K_0(q'r)\left[kq'I_0'(q'b) + hI_0(q'b)\right]}{I_0(q'a)\left[kq'K_0'(q'b) + hK_0(q'b)\right] - K_0(q'a)\left[kq'I_0'(q'b) + hI_0(q'b)\right]}$$

$$\times Z(u,z)du \int_0^\infty Z(u,z')f(z')dz' , \qquad \ldots (1)$$

where $q' = \sqrt{(q^2 + u^2)}$.

Then it follows as in § 4 that

$$v = \frac{2}{\pi}\int_0^\infty B(u,r,t)Z(u,z)du \int_0^\infty Z(u,z')f(z')dz' , \qquad \ldots\ldots\ldots\ldots (2)$$

where

$$B(u,r,t) = \frac{I_0(ru)\left[kuK_1(bu) - hK_0(bu)\right] + K_0(ru)\left[kuI_1(bu) + hI_0(bu)\right]}{I_0(au)\left[kuK_1(bu) - hK_0(bu)\right] + K_0(au)\left[kuI_1(bu) + hI_0(bu)\right]}$$

$$+\pi \sum_{s=1}^\infty \frac{e^{-\kappa(u^2+\alpha_s^2)}\alpha_s^2\left[k\alpha_s J_1(b\alpha_s) - hJ_0(b\alpha_s)\right]^2 C_0(r\alpha_s, a\alpha_s)}{(\alpha_s^2+u^2)\left\{(k^2\alpha_s^2+h^2)J_0^2(a\alpha_s) - \left[k\alpha_s J_1(b\alpha_s) - hJ_0(b\alpha_s)\right]^2\right\}}$$

and the α_s are the positive roots of § 4 (5).

8. Finite cylinder $0 \leq r < a$, $0 < z < l$. $z = 0$ maintained at $f(r)$ for $t > 0$. Boundary conditions $k_1 \frac{\partial v}{\partial z} + h_1 v = 0$, $z = l$, $0 \leq r < a$, $t > 0$, and $k\frac{\partial v}{\partial z} + hv = 0$, when $r = a$, $0 < z < l$, $t > 0$.

We assume that $f(r)$ can be expanded in the series § 1 (6)

$$f(r) = \frac{2}{a^2} \sum_{s=1}^\infty \frac{J_0(r\alpha_s)}{J_0^2(a\alpha_s) + J_0'^2(a\alpha_s)} \int_0^a r'f(r')J_0(r'\alpha_s)dr' , \qquad \ldots (1)$$

where the α_s are the positive roots of

$$k\alpha J_0'(a\alpha) + h J_0(a\alpha) = 0 , \qquad \ldots\ldots\ldots\ldots (2)$$

and a term $\frac{2}{a^2}\int_0^a r'f(r')dr'$ is to be added to the right hand side of (2) in the case $h = 0$.

Here the subsidiary equation is

$$\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}}{\partial r} + \frac{\partial^2 \bar{v}}{\partial z^2} - q^2 \bar{v} = 0 , \quad 0 \leq r < a, \quad 0 < z < l, \qquad \ldots\ldots (3)$$

to be solved with $\bar{v} = \frac{1}{p} f(r)$, when $z = 0$, $0 \leq r < a$ ..... (4)

$$k_1 \frac{\partial \bar{v}}{\partial z} + h_1 \bar{v} = 0 , \text{ when } z = l, \quad 0 \leq r < a \qquad \ldots\ldots (5)$$

$$k \frac{\partial \bar{v}}{\partial r} + h\bar{v} = 0 , \quad \text{when } r = a, \quad 0 < z < l. \qquad \ldots\ldots (6)$$

A solution of (3) which satisfies (5) and (6) and is finite when $r = 0$ is

$$J_0(r\alpha_s)\left[k_1 q_s \cosh q_s(l - z) + h_1 \sinh q_s(l - z)\right] , \qquad \ldots\ldots (7)$$

where $q_s = \sqrt{(q^2 + \alpha_s^2)}$,

and α_s is the root of

$$k\alpha J_0'(a\alpha) + h\, J_0(a\alpha) = 0.$$

Thus a solution of (3) satisfying (4), (5), and (6) is

$$\bar{v} = \frac{2}{pa^2} \sum_{s=1}^{\infty} \frac{k_1 q_s \cosh q_s(l-z) + h_1 \sinh q_s(l-z)}{k_1 \cosh q_s l + h_1 \sinh q_s l} \cdot \frac{J_0(r\alpha_s)}{J_0^2(a\alpha_s) + J_1^2(a\alpha_s)} \int_0^a r' f(r') J_0(r'\alpha_s) dr' \qquad \ldots (8)$$

where the summation is over the positive roots of (2), and if $h = 0$ a term

$$\frac{2}{pa^2} \cdot \frac{k_1 q \cosh q(l - z) + h_1 \sinh q(l - z)}{k_1 q \cosh ql + h_1 \sinh ql} \int_0^a r' f(r') dr'$$

is to be added to the right hand side of (8).

To determine v from (8) we require

$$A(\alpha_s, z, t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\lambda t}}{\lambda} \frac{k_1 \mu_s \cosh \mu_s(l - z) + h_1 \sinh \mu_s(l - z)}{k_1 \mu_s \cosh \mu_s l + h_1 \sinh \mu_s l} d\lambda , \qquad (9)$$

where $\mu_s = \sqrt{(\frac{\lambda}{\kappa} + \alpha_s^2)}$.

The integrand of (9) has simple poles at $\lambda = 0$ and at $\lambda = -\kappa(\beta_n^2 + \alpha_s^2)$, where β_n, $n = 1,2,\ldots,$ are the positive roots of

$$k_1 \beta \cos \beta \ell + h_1 \sin \beta \ell = 0. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Evaluating the residues at these (9) becomes

$$A(\alpha_s, z, t) = \frac{k_1\alpha_s \cosh(\ell - z)\alpha_s + h_1 \sinh(\ell - z)\alpha_s}{k_1\alpha_s \cosh \ell\alpha_s + h_1 \sinh \ell\alpha_s} - \sum_{m=1}^{\infty} e^{-\kappa(\alpha_s^2+\beta_m^2)t} \frac{2\beta_m(h_1^2 + k_1^2\beta_m^2)\sin \beta_m z}{(\alpha_s^2+\beta_m^2)(k_1h_1 + \ell h_1^2 + \ell k_1^2\beta_m^2)} \qquad \ldots. (11)$$

And using this result in the terms of (8) we have finally

$$v = \frac{2}{a^2} \sum_{s=1}^{\infty} A(\alpha_s, z, t) \frac{J_0(r\alpha_s)}{J_0^2(a\alpha_s) + J_1^2(a\alpha_s)} \int_0^a r'f(r')J_0(r'\alpha_s)dr' , \qquad \ldots (12)$$

where if $h = 0$ a term

$$\frac{2}{a^2} \left\{ 1 - \sum_{m=1}^{\infty} e^{-\kappa\beta_m^2 t} \frac{2(h_1^2 + k_1\beta_m^2)\sin \beta_m z}{\beta_m(k_1h_1 + \ell h_1^2 + \ell k_1^2 \beta_m^2)} \right\} \int_0^a r'f(r')dr'$$

is to be added to the right hand side of (12).

9. The region $r > a$, $0 < z < \ell$. $z = 0$ maintained at $f(r)$ for $t > 0$. Boundary conditions $k_1 \frac{\partial v}{\partial z} + h_1 v = 0$, when $z = \ell$, $r > a$, $t > 0$, and $k \frac{\partial v}{\partial z} - hv = 0$, when $r = a$, $0 < z < \ell$, . $t > 0$.

We assume that $f(r)$ can be expanded in the form § 1 (8), namely

$$f(r) = \int_0^{\infty} C(u,r)udu \int_0^{\infty} C(u,r')f(r')r'dr' , \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

where $C(u,r)$ is defined in § 1 (9).

Then as in § 8 we find

$$\bar{v} = \frac{1}{p} \int_0^{\infty} \frac{k_1q' \cosh q'(\ell - z) + h_1 \sinh q'(\ell - z)}{k_1q' \cosh q'\ell + h_1 \sinh q'\ell} C(u,r)udu \int_a^{\infty} C(u,r')r'dr', \qquad \ldots (2)$$

where $q' = \sqrt{(q^2 + u^2)}$.

Then proceeding as in § 8 we have

$$v = \int_0^\infty \left\{ \frac{k_1 u \cosh u(\ell - z) + h_1 \sinh u(\ell - z)}{k_1 u \cosh \ell u + h_1 \sinh \ell u} - \sum_{m=1}^{\infty} e^{-\kappa(u^2+\beta_m^2)t} \frac{2\beta_m(h_1^2 + k_1^2\beta_m^2)\sin \beta_m z}{(\beta_m^2+u^2)(k_1h_1+\ell h_1^2+\ell k_1^2\beta_m^2)} \right\}$$

$$\times\, C(u,r)u\,du \int_a^\infty C(u,r')f(r')r'dr' , \qquad \ldots\ldots\ldots\ldots\ (3)$$

where the β_m are the positive roots of

$$k_1\beta \cos \beta\ell + h_1 \sin \beta\ell = 0.$$

10. <u>The hollow cylinder $a < r < b$, $0 < z < \ell$. $z = 0$ maintained at $f(r)$ for $t > 0$. Boundary conditions</u> $k_1\frac{\partial v}{\partial r} - h_1 v = 0$, <u>when $r = a$, $0 < z < \ell$, $t > 0$;</u> $k_2 \frac{\partial v}{\partial r} + h_2 v = 0$, <u>when $r = b$, $0 < z < \ell$, $t > 0$;</u> $k \frac{\partial v}{\partial z} + hv = 0$ <u>when $z = \ell$, $a < r < b$, $t > 0$.</u>

We assume that $f(r)$ can be expanded in the form § 1 (10),

$$f(r) = \sum_{s=1}^{\infty} F(r,\alpha_s) \int_a^b r'f(r')F(r',\alpha_s)dr' , \qquad \ldots\ldots\ldots\ldots\ (1)$$

where $F(r,\alpha_s)$ and the α_s are defined in § 1 (11) and (12).

Then as in § 8 we find

$$\bar{v} = \frac{1}{p} \sum_{s=1}^{\infty} \frac{k_1 q_s \cosh q_s(\ell-z) + h_1 \sinh q_s(\ell-z)}{k_1 q_s \cosh q_s\ell + h_1 \sinh q_s\ell} F(r,\alpha_s) \int_a^b r'f(r')F(r',\alpha_s)dr' \qquad \ldots\ (2)$$

where $q_s = \sqrt{(q^2 + \alpha_s^2)}$, and if $h_1 = h_2 = 0$ an additional term

$$\frac{2}{p(b^2 - a^2)} \frac{k_1 q \cosh q(\ell - z) + h_1 \sinh q(\ell - z)}{k_1 q \cosh q\ell + h_1 \sinh q\ell} \int_a^b r'f(r')dr'$$

is to be added to the right hand side of (2).

Hence using the results of § 8 we have

$$v = \sum_{s=1}^{\infty} A(\alpha_s, z, t) F(r, \alpha_s) \int_a^b r' f(r') dr' \, F(r', \alpha_s) \quad \ldots\ldots\ldots\ldots\ldots (3)$$

where $A(\alpha_s, z, t)$ is defined in § 8 (11) except that k_1 and h_1 there are to be replaced by k and h, and the values of α_s to be used in this result are the roots of § 1 (12). If $h_1 = h_2 = 0$ an extra term

$$\frac{2}{b^2 - a^2} A(0, z, t) \int_0^a r' f(r') dr'$$

is to be added to the right hand side of (3).

11. Semi-infinite cylinder $z > 0$, $0 \leq r < a$. $z = 0$ maintained at $f(r)$ for $0 \leq r < a$, $t > 0$. Boundary conditions $k \frac{\partial v}{\partial r} + hv = 0$, when $r = a$, $z > 0$, $t > 0$.

As in § 8 we assume that $f(r)$ can be expanded in the series § 1 (6)

$$f(r) = \frac{2}{a^2} \sum_{s=1}^{\infty} \frac{J_0(r\alpha_s)}{J_0^2(a\alpha_s) + J_1^2(a\alpha_s)} \int_0^a J_0(r'\alpha_s) r' f(r') dr' , \quad \ldots\ldots (1)$$

where the α_s are the positive roots of

$$k\alpha J_0'(a\alpha) + hJ_0(a\alpha) = 0 , \quad \ldots\ldots\ldots\ldots\ldots (2)$$

and a term $\frac{2}{a^2} \int_0^a r' f(r') dr'$ is to be added if $h = 0$.

The subsidiary equation is

$$\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{v}}{\partial r} + \frac{\partial^2 \bar{v}}{\partial z^2} - q^2 \bar{v} = 0 , \quad 0 \leq r < a, \quad z > 0, \quad \ldots\ldots (3)$$

with $\bar{v} = \frac{1}{p} f(r)$, when $z = 0$, $0 \leq r < a$, $\ldots\ldots\ldots\ldots$ (4)

and $k \frac{\partial \bar{v}}{\partial r} + h\bar{v} = 0$, when $r = a$, $z > 0$. $\ldots\ldots\ldots\ldots$ (5)

A solution of (3) satisfying (5) and finite when $r = 0$, and when $z \to \infty$, is

$$e^{-q_s z} J_0(r\alpha_s) ,$$

where the α_s is a root of (2), and $q_s = \sqrt{(q^2 + \alpha_s^2)}$.

Thus the solution of (3), (4), and (5) is

$$\bar{v} = \frac{2}{pa^2} \sum_{s=1}^{\infty} \frac{e^{-q_s z} J_0(r\alpha_s)}{J_0^2(a\alpha_s) + J_1^2(a\alpha_s)} \int_0^a r'f(r')J_0(r'\alpha_s)dr' , \quad \ldots\ldots\ldots (6)$$

and if $h = 0$ a term $\frac{2}{pa^2} e^{-qz} \int_0^a r'f(r')dr'$ is to be added to the right hand side of (6).

To determine v we require

$$I = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\lambda t - z\sqrt{(\frac{\lambda}{\kappa} + \alpha_s^2)}} d\lambda}{\lambda} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

The integrand of (7) has a simple pole at $\lambda = 0$ of residue $e^{-z\alpha_s}$. Also it has a branch point at $\lambda = -\kappa\alpha_s^2$. Deforming the contour about this in the usual way and putting $\lambda = -\kappa\alpha_s^2 + \kappa u^2 e^{-i\pi}$ on CD and $\lambda = -\kappa\alpha_s^2 + \kappa u^2 e^{i\pi}$ on EF we obtain

$$I = e^{-z\alpha_s} - \frac{2e^{-\kappa\alpha_s^2 t}}{\pi} \int_0^{\infty} e^{-\kappa u^2 t} \frac{u \sin zu}{u^2 + \alpha_s^2} du \quad \ldots\ldots\ldots\ldots\ldots (8)$$

$$= e^{-z\alpha_s} + \tfrac{1}{2} e^{z\alpha_s} \operatorname{erfc}\left[\alpha_s\sqrt{\kappa t} + \frac{z}{2\sqrt{\kappa t}}\right] - \tfrac{1}{2} e^{-z\alpha_s} \operatorname{erfc}\left[\alpha_s\sqrt{\kappa t} - \frac{z}{2\sqrt{\kappa t}}\right] \quad \ldots\ldots (9)$$

Using the value (8) in (6) we find

$$v = \frac{2}{a^2} \sum_{s=1}^{\infty} \left\{ e^{-z\alpha_s} - \frac{2}{\pi} e^{-\kappa\alpha_s^2 t} \int_0^{\infty} e^{-\kappa u^2 t} \frac{u \sin zu}{u^2 + \alpha_s^2} du \right\} \frac{J_0(r\alpha_s)}{J_0^2(a\alpha_s) + J_1^2(a\alpha_s)} \int_0^a r'f(r')J_0(r'\alpha_s)dr' \quad \ldots (10)$$

where if $h = 0$ a term

$$\frac{2}{a^2} \left\{ 1 - \frac{2}{\pi} \int_0^{\infty} e^{-\kappa u^2 t} \frac{\sin zu \, du}{u} \right\} \int_0^a r'f(r')dr'$$

$$= \frac{2}{a^2} \operatorname{erfc} \frac{z}{2\sqrt{\kappa t}} \int_0^a r'f(r')dr'$$

has to be added to the right hand side of (10).

12. The region $z > 0$, $r > a$. $z = 0$ maintained at $f(r)$ for $r > a$, $t > 0$. Boundary condition $k\frac{\partial v}{\partial r} + hv = 0$, when $r = a$, $z > 0$, $t > 0$.

We assume that $f(r)$ can be expressed in the form § 1 (8), namely

$$f(r) = \int_0^\infty C(u,r)u\,du \int_a^\infty C(u,r')\,f(r')r'dr' \qquad \ldots\ldots\ldots\ldots (1)$$

where $C(u,r)$ is defined in § 1 (9). Then as in § 11 we find

$$\bar{v} = \frac{1}{p}\int_0^\infty e^{-q'z}\,C(u,r)u\,du \int_a^\infty C(u,r')f(r')r'dr' \qquad \ldots\ldots\ldots\ldots (2)$$

where $q' = \sqrt{(q^2 + u^2)}$.

Then using the result § 11 (8) it follows that

$$v = \int_0^\infty \left\{ e^{-zu} - \frac{2}{\pi} e^{-\kappa u^2 t} \int_0^\infty e^{-\kappa u'^2 t}\,\frac{u' \sin zu'}{u^2 + u'^2}\,du' \right\} C(u,r)u\,du \int_a^\infty C(u,r')f(r')r'dr', \qquad \ldots (3)$$

or using § 11 (9) the integral in the bracket may be expressed in terms of error functions.

13. The semi-infinite hollow cylinder $z > 0$, $a < r < b$. $z = 0$ maintained at $f(r)$ for $t > 0$. Boundary conditions $k_1\frac{\partial v}{\partial r} - h_1 v = 0$, when $r = a$, $t > 0$; $k_2\frac{\partial v}{\partial r} + h_2 v = 0$, when $r = b$, $z > 0$, $t > 0$.

We assume that $f(r)$ can be expanded in the series § 1 (10)

$$f(r) = \sum_{s=1}^{\infty} F(r,\alpha_s) \int_a^b r'f(r')F(r',\alpha_s)dr' \qquad \ldots\ldots\ldots (1)$$

where the α_s are the roots of § 1 (12) and if $h_1 = h_2 = 0$ an additional term $\frac{2}{b^2 - a^2}\int_a^b r'f(r')dr'$ is to be added to the right hand side of (1)

Then as in § 11 we obtain

$$\bar{v} = \frac{1}{p} \sum_{s=1}^{\infty} e^{-q_s z} F(r,\alpha_s) \int_a^b r'f(r')F(r',\alpha_s)dr' , \qquad \ldots\ldots\ldots\ldots (2)$$

where $q_s = \sqrt{(q^2 + \alpha_s^2)}$, and if $h_1 = h_2 = 0$ an extra term

$$\frac{2}{(b^2 - a^2)p} e^{-qz} \int_a^b r'f(r')dr'$$

is to be added to the right hand side of (2).

As in § 11 we have finally

$$v = \sum_{s=1}^{\infty} \left\{ e^{-z\alpha_s} - \frac{2}{\pi} e^{-\kappa\alpha_s^2 t} \int_0^{\infty} e^{-\kappa u^2 t} \frac{u \sin zu}{u^2 + \alpha_s^2} du \right\} F(r,\alpha_s) \int_a^b r'f(r')F(r',\alpha_s)dr' \quad \ldots (3)$$

where if $h_1 = h_2 = 0$ an additional term

$$\frac{2}{b^2 - a^2} \left\{ 1 - \frac{2}{\pi} \int_0^{\infty} e^{-\kappa u^2 t} \frac{\sin zu \, du}{u} \right\} \int_a^b r'f(r')dr'$$

is to be added to the right hand of (3).

The infinite integrals in the brackets may be expressed in terms of error functions as in § 11 (9).

CHAPTER VII.

THE SOLUTION OF BOUNDARY VALUE PROBLEMS BY A DOUBLE LAPLACE TRANSFORMATION.

J. C. Jaeger.

1. The Laplace Transformation has most frequently been used to transform a linear partial differential equation with two independent variables into an ordinary differential equation, the "subsidiary equation", from the solution of which that of the original equation is deduced. If it is applied to an equation with more than two independent variables the subsidiary equation is also a partial differential equation and is usually solved by classical methods. The object of this note is to point out that partial differential equations in which the range of two (or more) independent variables is $(0, \infty)$ may easily be handled by simultaneous Laplace Transformations in these variables. The point of view is that of Chapter I in which the Laplace Transformation method has been regarded as purely formal, and the solution as subject to verification. The method is related to that of Doetsch[+] for equations of elliptic type but assumes no theoretical basis. In § 2 a simple two variable problem is discussed to illustrate the method and in §§ 3, 4 and 5 two new three variable problems of a type to which it is well adapted.

~~* Carslaw and Jaeger, Proc. Cambridge Phil. Soc., in press; Proc. London Math. Soc., August, 1939; Bull. Amer. Math. Soc., 45 (1939) 407.~~

+ Doetsch; Theorie und Anwendungen der Laplace-Transformation, (Berlin 1937), Chapter 22.

1. We consider a function $v(x,t)$ in the range $x > 0$, $t > 0$.

Let $u_o(x) = v(x,0)$, $u_1(x) = \left[\frac{\partial v}{\partial t}\right]_{t=0}$

$w_o(t) = v(0,t)$, $w_1(t) = \left[\frac{\partial v}{\partial x}\right]_{x=0}$

The Laplace Transform with respect to t will be denoted by a "bar", thus

$$\overline{f}(p) = \int_0^\infty e^{-pt} f(t)dt ,$$

and that with respect to x by a capital letter, thus

$$G(p') = \int_0^\infty e^{-p'x} g(x)\, dx ,$$

and the double Laplace Transform by both

$$\overline{V}(p',p) = \int_0^\infty\int_0^\infty e^{-pt-p'x} v(x,t)\, dx\, dt ,$$

p and p' are supposed to have real parts sufficiently large to ensure convergence. This notation indicates compactly the variables occurring; if there is another independent variable y it will occur throughout.

Then subject to suitable restrictions on v we have

$$\int_0^\infty\int_0^\infty e^{-pt-p'x} \frac{\partial v}{\partial t}\, dx\, dt = p\overline{V} - U_o \tag{1}$$

$$\int_0^\infty\int_0^\infty e^{-pt-p'x} \frac{\partial^2 v}{\partial t^2}\, dx\, dt = p^2\overline{V} - pU_o - U_1 \tag{2}$$

$$\int_0^\infty\int_0^\infty e^{-pt-p'x} \frac{\partial v}{\partial x}\, dx\, dt = p'\overline{V} - \overline{w}_o \tag{3}$$

$$\int_0^\infty\int_0^\infty e^{-pt-p'x} \frac{\partial^2 v}{\partial x^2}\, dx\, dt = p'^2\overline{V} - p'\overline{w}_o - \overline{w}_1 \tag{4}$$

From a given differential equation and boundary conditions we obtain a subsidiary equation and boundary conditions by multiplying by $\exp[-pt-p'x]$, integrating with respect to x and t from 0 to ∞, and using (1) to (4). Solving this equation we obtain $\bar{V}(p',p)$.

To derive $v(x,t)$ from $\bar{V}(p',p)$ we assume an inversion theorem which may be derived formally from Fourier's integral theorem in several variables.

$$v(x,t) = -\frac{1}{4\pi^2}\int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda \int_{\gamma'-i\infty}^{\gamma'+i\infty} e^{\lambda t + \lambda' x}\,\bar{V}(\lambda',\lambda)\,d\lambda' \qquad (5)$$

where $\pi > \arg\lambda > -\pi$, and $\pi > \arg\lambda' > -\pi$, provided that:

$$\left.\begin{array}{l}\bar{V}(\lambda',\lambda) \text{ is bounded in some half planes } R(\lambda') > \alpha',\ R(\lambda) > \alpha, \\ \text{and } \gamma > \alpha,\ \gamma' > \alpha'\end{array}\right\} \qquad (6)$$

Other conditions in addition to (6) will of course be necessary for the truth of the inversion theorem (5), but neither these nor the assumptions involved in (1) to (4) need be discussed since the whole process of determining $v(x,t)$ as a double contour integral of type (5) is regarded as purely formal. It has then to be verified that this solution does satisfy the differential equation and boundary conditions. This is done in the case of the equation of conduction of heat by transforming the paths of integration in (5) to paths beginning in the third and ending in the second quadrant and the verification is then performed on these new integrals. The whole process follows the lines of the one-variable case* and will not be given here. The contour integrals of type (5) are evaluated by a deformation of the contour precisely as in the one-variable case, this will be done without comment, the method and justification having been given previously ~~in the papers referred to.~~

2. Heat conduction in the region $x > 0$. The initial temperature zero. The end $x = 0$ kept at unit temperature for $t > 0$.

We have to solve

* ~~Carslaw and Jaeger, loc. cit.~~ Cf. Chapter I.

$$\frac{\partial^2 v}{\partial x^2} - \frac{1}{\kappa}\frac{\partial v}{\partial t} = 0\,, \qquad t > 0,\quad x > 0 \tag{1}$$

with* $\quad u_0(x) = 0\,, \qquad x > 0 \qquad (2)$

$\qquad w_0(t) = 1\,, \qquad t > 0 \qquad (3)$

$\qquad w_1(t)$ unknown.

By §1 (1) and §1 (4) the subsidiary equation is

$$(p'^2 - \frac{p}{\kappa})\bar{v} = \frac{p'}{p} + \bar{w}_1$$

and so $\bar{v}(p', p) = \dfrac{p' + p\bar{w}_1(p)}{p(p'^2 - p/\kappa)}$.

For $\bar{v}(\lambda', \lambda)$ to satisfy §1 (6) we must have+ $\bar{w}(\lambda) = -(\lambda\kappa)^{-\frac{1}{2}}$ and thus

$$\bar{v}(\lambda', \lambda) = \frac{1}{\lambda[\lambda' + \sqrt{(\lambda/\kappa)}]}\,.$$

Hence by the inversion theorem §1 (5)

$$x(x,t) = -\frac{1}{4\pi^2}\int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda \int_{\gamma'-i\infty}^{\gamma'+i\infty} \frac{e^{\lambda t + \lambda' x}\, d\lambda'}{\lambda[\lambda' + \sqrt{(\lambda/\kappa)}]} \tag{4}$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t - x\sqrt{(\lambda/\kappa)}}\,\frac{d\lambda}{\lambda}$$

$$= 1 - \frac{2}{\sqrt{\pi}}\int_0^{x/2\sqrt{(\kappa t)}} e^{-\xi^2}\, d\xi$$

3. <u>Heat conduction in the region $x > 0,\ 0 < y < b$; $y = b$ Maintained at unit temperature for $t > 0$; $y = 0$ and $x = 0$ Maintained at zero for $t > 0$; the initial temperature zero.</u>

We have to solve

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* Boundary conditions stated shortly as in (2) and (3) are to be understood as:

$\qquad$ for fixed $x > 0$, $\lim\limits_{t \to 0} v(x,t) = 0$, $\qquad (2')$

$\qquad$ for fixed $t > 0$, $\lim\limits_{x \to 0} v(x,t) = 1$. $\qquad (3')$

+ From this result the flux of heat at the origin may be obtained directly.

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - \frac{1}{\kappa}\,\frac{\partial v}{\partial t} = 0\,, \qquad t > 0,\ \ x > 0,\ \ 0 < y < b$$

with $\quad u_0(y,x) = 0\,, \qquad x > 0\,,\ \ 0 < y < b$

$\quad w_0(y,t) = 0\,, \qquad t > 0\,,\ \ 0 < y < b$

$\quad w_1(y,t)$ unknown

$\quad v(y,x,t) = 0\,,$ when $y = 0,\ \ x > 0,\ \ t > 0$

$\quad v(y,x,t) = 1\,,$ when $y = b,\ \ x > 0,\ \ t > 0.$

The subsidiary equation is

$$\frac{d^2\overline{V}}{dy^2} + q^2\overline{V} = \overline{w}_1(y),$$

where $\quad q^2 = p'^2 - p/\kappa\quad .$ (1)

This is to be solved with $\overline{V} = \dfrac{1}{pp'}$ when $y = b$

$\qquad\qquad\qquad\qquad\quad = 0$ when $y = 0.$

The solution is

$$\overline{V}(y,p',p) = \frac{\sin qy}{pp'\sin qb} - \frac{\sin qy}{q\sin qb}\int_y^b \sin q(b-y')\overline{w}_1(y')dy'$$
$$- \frac{\sin q(b-y)}{q\sin qb}\int_0^y \sin qy'\overline{w}_1(y')dy'\ . \qquad (2)$$

Now $\sin b\sqrt{(\lambda'^2 - \frac{\Delta}{\kappa})}$, qua function of λ', has zeros at $\lambda' = \pm\sqrt{(\frac{n^2\pi^2}{b^2} + \frac{\Delta}{\kappa})}$, $n = 0,1,\ldots$ so to satisfy (6) $\S$ 1 the numerator of $\overline{V}(y,\lambda',\lambda)$ must vanish at $\lambda' = +\sqrt{(\frac{n^2\pi^2}{b^2} + \frac{\Delta}{\kappa})}$, $n = 1,2,\ldots$, that is

$$\frac{(-)^n n\pi}{\lambda b(\frac{n^2\pi^2}{b^2} + \frac{\Delta}{\kappa})^{\frac{1}{2}}} + \int_0^b \sin\frac{n\pi y'}{b}\,\overline{w}_1(y')dy' = 0, \qquad n = 1,2,\ldots\ . \qquad (3)$$

The residue of $e^{\lambda' x}\,\overline{V}(y,\lambda',\lambda)$, qua function of λ' at the pole $\lambda' = -\sqrt{(\frac{n^2\pi^2}{b^2} + \frac{\Delta}{\kappa})}$ is

$$\frac{\exp\left[-x\sqrt{(\frac{n^2\pi^2}{b^2} + \frac{\Delta}{\kappa})}\right]}{b(\frac{n^2\pi^2}{b^2} + \frac{\Delta}{\kappa})^{\frac{1}{2}}}\sin\frac{n\pi y}{b}\left\{\frac{(-)^n n\pi}{\lambda b(\frac{n^2\pi^2}{b^2} + \frac{\Delta}{\kappa})^{\frac{1}{2}}} - \int_0^b \sin\frac{n\pi y'}{b}\,\overline{w}_1(y')dy'\right\}\ .$$

Using (3) in this gives for the result of the λ'-integration in $\S$ 1 (5)

$$\frac{\sinh y\sqrt{(\lambda/\kappa)}}{\lambda \sinh b\sqrt{(\lambda/\kappa)}} + \sum_{n=1}^{\infty} \frac{2\kappa n\pi(-)^n}{\lambda(\kappa n^2\pi^2 + \lambda b^2)} \sin\frac{n\pi y}{b} \exp\left[-x\left(\frac{n^2\pi^2}{b^2} + \frac{\lambda}{\kappa}\right)^{\frac{1}{2}}\right] ,$$

where the first term comes from the pole $\lambda' = 0$.

For the λ- integration we require

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\exp\left[\lambda t - x\left(\frac{\lambda}{\kappa} + \alpha^2\right)^{\frac{1}{2}}\right] d\lambda}{\lambda\left(\frac{\lambda}{\kappa} + \alpha^2\right)}$$

$$= \frac{1}{\alpha^2} e^{-\alpha x} - \frac{1}{\alpha^2} e^{-\kappa\alpha^2 t} + \frac{2}{\pi}\int_0^{\infty} e^{-\kappa t(\alpha^2 + u^2)} \sin ux \frac{du}{u(u^2 + \alpha^2)} , \qquad \ldots . (4)$$

and $\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}}{\lambda} \frac{\sinh y\sqrt{(\lambda/\kappa)}}{\sinh b\sqrt{(\lambda/\kappa)}} d\lambda = \frac{y}{b} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-)^n}{n} e^{-\kappa n^2\pi^2 t/b^2} \sin\frac{n\pi y}{b}$.

Using these results we have finally

$$v(y,x,t) = \frac{y}{b} + \frac{2}{\pi}\sum_{n=1}^{\infty} e^{-n\pi x/b} \frac{(-)^n}{n} \sin\frac{n\pi y}{b}$$

$$+ \frac{4}{b^2}\sum_{n=1}^{\infty} n(-)^n \sin\frac{n\pi y}{b}\int_0^{\infty} e^{-\kappa t(u^2 + n^2\pi^2/b^2)} \frac{\sin ux}{u(u^2 + n^2\pi^2/b^2)} du. \qquad (5)$$

Clearly problems in which the temperature of $y = b$ is a function of x or t may be dealt with in the same way provided the function satisfies fairly wide conditions.

4. <u>Heat conduction in the region $x > 0$, $0 < y < b$; the initial temperature unity; the surface maintained at zero for $t > 0$.</u>

We have to solve

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - \frac{1}{\kappa}\frac{\partial v}{\partial t} = 0 , \quad t > 0 , x > 0 , 0 < y < b$$

with $u_0(y,x) = 1$, $x > 0$, $0 < y < b$

$w_0(y,t) = 0$, $t > 0$, $0 < y < b$

$w_1(y,t)$ unknown,

$v(y,x,t) = 0$ when $y = 0$ and $y = b$ for $x > 0$ and $t > 0$.

The subsidiary equation is

$$\frac{d^2\overline{V}}{dy^2} + q^2\overline{V} = \overline{w}_1(y) - \frac{1}{\kappa p'} ,$$

where $q^2 = p'^2 - p/\kappa$.

This is to be solved with $\overline{V} = 0$, when $y = 0$ and $y = b$. The solution is

$$\overline{V}(y,p',p) = -\frac{\sin qb - \sin qy - \sin q(b-y)}{\kappa p' q^2 \sin qb}$$

$$- \frac{\sin qy}{q \sin qb}\int_y^b \sin q(b-y')\overline{w}_1(y')dy' - \frac{\sin q(b-y)}{q \sin qb}\int_0^y \sin qy'\overline{w}_1(y')dy', \quad (1)$$

The zeros of $\sin b(\lambda'^2 - \frac{\lambda}{\kappa})^{\frac{1}{2}}$, qua function of λ', are at $\lambda' = \pm(\frac{n^2\pi^2}{b^2} + \frac{\lambda}{\kappa})^{\frac{1}{2}}$, $n = 0,1,2,\ldots$, and thus, in order that §1 (6) may be satisfied, we must have

$$-\frac{b\left[1-(-)^n\right]}{n\pi\kappa(\frac{n^2\pi^2}{b^2}+\frac{\lambda}{\kappa})^{\frac{1}{2}}} + \int_0^b \sin\frac{n\pi y'}{b}\overline{w}_1(y')dy' = 0 , \quad n = 1,2,\ldots . \quad (2)$$

The residue of $e^{\lambda' x}\overline{V}(y,\lambda',\lambda)$ at the pole $\lambda' = -(\frac{n^2\pi^2}{b^2}+\frac{\lambda}{\kappa})^{\frac{1}{2}}$ is

$$-\frac{1}{b}(\frac{n^2\pi^2}{b^2}+\frac{\lambda}{\kappa})^{-\frac{1}{2}}\exp\left[-x(\frac{n^2\pi^2}{b^2}+\frac{\lambda}{\kappa})\right]\sin\frac{n\pi y}{b}\left\{\frac{b\left[1-(-)^n\right]}{\kappa n\pi(\frac{n^2\pi^2}{b^2}+\frac{\lambda}{\kappa})^{\frac{1}{2}}} + \int_0^b \sin\frac{n\pi y'}{b}\overline{w}_1(y')dy'\right\}$$

Using (2) in this result of the λ'- integration in §1 (5) is

$$-\frac{4b^2}{\pi}\sum_{r=0}^{\infty}\frac{1}{(2r+1)\left[\kappa(2r+1)^2\pi^2+\lambda b^2\right]}\exp\left[-x(\frac{(2r+1)^2\pi^2}{b^2}+\frac{\lambda}{\kappa})^{\frac{1}{2}}\right]\sin\frac{(2r+1)\pi y}{b}$$

$$+\frac{\sinh b(\lambda/\kappa)^{\frac{1}{2}} - \sinh y(\lambda/\kappa)^{\frac{1}{2}} - \sinh(b-y)(\lambda/\kappa)^{\frac{1}{2}}}{\lambda \sinh b(\lambda/\kappa)^{\frac{1}{2}}} ,$$

where the last term has been derived from the pole $\lambda' = 0$; there are no poles at $\lambda' = \pm\sqrt{(\lambda/\kappa)}$.

Carrying our the λ- integration in the usual way we have finally

$$v(y,x,t) = \frac{8}{\pi^2}\sum_{r=0}^{\infty} \frac{\sin(2r+1)\pi y/b}{(2r+1)} \int_0^{\infty} \frac{\sin ux}{u} \exp\left[-\kappa t(u^2 + (2r+1)^2 \pi^2/b^2)\right] du. \tag{3}$$

The same method applies to cases in which the initial temperature $u_o(y,x)$ is a function of x and y under fairly general conditions on the function.

As an example let $u_o(y,x) = x$, for $x > 0$, $0 < y < b$.

The only change in $\overline{V}(y,p',p)$ is that the first term of (1) is replaced by

$$\frac{-\sin bq + \sin qy + \sin q(b-y)}{\kappa p'^2 q^2 \sin bq}.$$

Proceeding in the same way we find that the only pole of $\overline{V}(y,\lambda',\lambda)$ qua function of λ' is $\lambda' = 0$, where the residue is

$$x\,\frac{\sinh b(\lambda/\kappa)^{\frac{1}{2}} - \sinh y(\lambda/\kappa)^{\frac{1}{2}} - \sinh(b-y)(\lambda/\kappa)^{\frac{1}{2}}}{\lambda \sinh b(\lambda/\kappa)^{\frac{1}{2}}}.$$

Performing the λ- integration of (§1)(5) we obtain finally

$$v(y,x,t) = \frac{4x}{\pi} \sum_{r=0}^{\infty} \frac{1}{2r+1} e^{-\kappa\pi^2(2r+1)^2 t/b^2} \sin\frac{(2r+1)\pi y}{b}. \tag{4}$$

5. Heat conduction in the semi-infinite cylinder $x > 0,\ 0 \leq y < a$. $y = a$ maintained at unit temperature for $x > 0$ and $t > 0$; $x = 0$ maintained at zero for $0 \leq y < a$, $t > 0$; the initial temperature zero.

We have to solve

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{1}{y}\frac{\partial v}{\partial y} - \frac{1}{\kappa}\frac{\partial v}{\partial t} = 0\,, \quad t > 0,\ x > 0,\ 0 \leq y < a$$

with $u_0(y,x) = 0$, $\quad x > 0$, $\quad 0 \leq y < a$

$w_0(y,t) = 0$, $\quad t > 0$, $\quad 0 \leq y < a$

$w_1(y,t)$ unknown

$v(y,x,t) = 1$, when $y = a$, $x > 0$, $t > 0$.

The subsidiary equation is

$$\frac{d^2\bar{v}}{dy^2} + \frac{1}{y}\frac{d\bar{v}}{dy} + q^2\bar{v} = \bar{w}_1(y) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1)$$

where $q^2 = p'^2 - p/\kappa$.

This is to be solved with $\bar{v} = \frac{1}{pp'}$ when $y = a$.

A particular integral of the equation (1) is

$$\frac{\pi\left[C_0(qa,qy)\right]}{2J_0(qa)}\int_0^y y'J_0(qy')\bar{w}_1(y')dy' + \frac{\pi J_0(qy)}{2J_0(qa)}\int_y^a \left[C_0(qa,qy')\right]y'\bar{w}_1(y')dy'\,,$$

where $C_0(x,y) = J_0(x)Y_0(y) - Y_0(x)J_0(y)$.

So the solution of (1) and its boundary conditions is

$$\bar{V}(y,p',p) = \frac{J_0(qy)}{pp'J_0(qa)} + \frac{\pi C_0(qa,qy)}{2J_0(qa)}\int_0^y J_0(qy')\bar{w}_1(y')y'dy'$$

$$+ \frac{\pi J_0(qy)}{2J_0(qa)}\int^a C_0(qa,qy')\bar{w}_1(y')y'dy'\,. \qquad \ldots\ldots \ (2)$$

$J_0\left[a(\lambda'^2 - \lambda/\kappa)^{\frac{1}{2}}\right]$, *qua* function of λ', has zeros at $\lambda' = \pm\sqrt{(\alpha_s^2 + \frac{\lambda}{\kappa})}$, where $\pm\alpha_s$, $s = 1, 2, \ldots,$ are the roots of

$$J_0(a\alpha) = 0 \,. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

To satisfy § 1 (6) the numerator of $\bar{V}(y,\lambda',\lambda)$ must vanish for $\lambda' = +\sqrt{(\alpha_s^2 + \frac{\lambda}{\kappa})}$, $s = 1, 2, \ldots,$ i.e. we must have

$$\frac{J_0(y\alpha_s)}{\lambda\sqrt{\alpha_s^2 + \frac{\lambda}{\kappa}}} - \frac{J_0(y\alpha_s)}{a\alpha_s\, J_0'(a\alpha_s)} \int_0^a J_0(y'\alpha_s)\bar{w}_1(y')y'dy' = 0\,, \quad s = 1,2,\ldots \qquad (4)$$

The residue of $\bar{V}(y,\lambda',\lambda)$ *qua* function of λ' at the pole $\lambda' = -\sqrt{(\alpha_s^2 + \frac{\lambda}{\kappa})}$ is

$$\frac{\alpha_s \exp\left[-x\sqrt{(\alpha_s^2+\frac{\lambda}{\kappa})}\right]}{a(\alpha_s^2 + \lambda/\kappa)^{\frac{1}{2}}\, J_0'(a\alpha_s)} \left\{\frac{J_0(y\alpha_s)}{\lambda(\alpha_s^2 + \lambda/\kappa)^{\frac{1}{2}}} + \frac{J_0(y\alpha_s)}{a\alpha_s\, J_0'(a\alpha_s)} \int_0^a J_0(y'\alpha_s)\bar{w}_1(y')y'dy'\right.$$

Using (4) in this and adding the residue at $\lambda' = 0$ gives for the result of the λ'-integration in § 1 (5)

$$\frac{I_0\left[y(\lambda/\kappa)^{\frac{1}{2}}\right]}{\lambda I_0\left[a(\lambda/\kappa)^{\frac{1}{2}}\right]} + \sum_{s=1}^{\infty} \frac{2\alpha_s\, J_0(y\alpha_s)\, \exp\left[-x\sqrt{(\alpha_s^2 + \lambda/\kappa)}\right]}{a\lambda(\alpha_s^2 + \lambda/\kappa)\, J_0'(a\alpha_s)} \,. \qquad \ldots\ldots\ldots \quad (5)$$

For the λ-integration we use § 3 (4) and the result

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t}}{\lambda}\, \frac{I_0\left[y(\lambda/\kappa)^{\frac{1}{2}}\right]}{I_0\left[a(\lambda/\kappa)^{\frac{1}{2}}\right]}\, d\lambda = 1 + \frac{2}{a}\sum_{s=1}^{\infty} \frac{J_0(y\alpha_s)\, e^{-\kappa\alpha_s^2 t}}{a\alpha_s\, J_0'(a\alpha_s)}\,, \qquad \ldots\ldots \quad (6)$$

where the α_s are the roots of (3). Thus we obtain finally

$$v(y,x,t) = 1 + \frac{2}{a}\sum_{s=1}^{\infty} \frac{e^{-x\alpha_s}\, J_0(y\alpha_s)}{\alpha_s\, J_0'(a\alpha_s)} + \frac{4}{\pi a}\sum_{s=1}^{\infty} \frac{\alpha_s J_0(y\alpha_s)}{J_0'(a\alpha_s)} \int_0^{\infty} \frac{e^{-\kappa t(\alpha_s^2+u^2)}\,\sin xu\, du}{u(u^2+\alpha_s^2)}$$

$$\ldots\ldots \quad (7)$$

CHAPTER VIII.

THE REGIONS BOUNDED BY A CIRCULAR CYLINDER AND PLANES PERPENDICULAR TO ITS AXIS WITH CONSTANT SURFACE TEMPERATURES.

In Chapter VI the temperatures of solids bounded by coaxial cylinders and planes perpendicular to the axis with given surface conditions and zero initial temperature were discussed by the obvious method of separating variables in the subsidiary equation. In cases in which the temperature $f(x)$ of a surface extending to infinity was prescribed, the conditions on $f(x)$ would in fact be very restrictive, and in particular the important case $f(x) = \text{Constant}$ cannot be solved in this way. In Chapter VII a method was developed for dealing with some problems involving semi-infinite solids and was found to be applicable to problems in which the surface or initial temperatures were not so restricted. The form of the results of that Chapter suggests an alternative method of separating the variables in the subsidiary equation. In this Chapter formal solutions will be given in this way for the semi-infinite cylindrical regions $z > 0, \ 0 \leqslant r < a$; $r > a, \ 0 < z < \ell$; $z > 0, \ r > a$ with constant surface temperatures. Problems with radiation boundary conditions

$$k \frac{\partial v}{\partial n} + hv = k_1$$

at the surfaces, k, h, k_1 being constants, may be dealt with in the same way. The same method applies to problems which can be solved by the methods of Chapter VI such as the finite cylinder with constant surface temperature. The two methods give different forms for the steady state solutions but the transient terms are identical.

The problems solved constitute a complete set for the regions bounded by the cylinder $r = a$, and planes $z = 0$ and $z = \ell$ with constant surface temperatures and zero initial temperature. In addition in §§ 5 and 6 two problems with constant initial temperature are solved directly; this seems worth while doing in view of the importance of these problems, the solutions can of course be derived from the other results of this Chapter.

1. The problem of Chapter VII, § 5. Heat conduction in the semi-infinite cylinder $z > 0$, $0 \leq r < a$. $r = a$ maintained at unit temperature for $z > 0$ and $t > 0$; $z = 0$ maintained at zero for $0 \leq r < a$, $t > 0$; the initial temperature zero.

The subsidiary equation is

$$\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}}{\partial r} + \frac{\partial^2 \bar{v}}{\partial z^2} - q^2 \bar{v} = 0, \qquad 0 \leq r < a, \quad z > 0 \qquad \ldots\ldots (1)$$

to be solved with

$$\bar{v} = 1/p, \quad r = a, \quad z > 0 \qquad \ldots\ldots (2)$$

$$= 0, \quad z = 0, \quad 0 \leq r < a. \qquad \ldots\ldots (3)$$

We seek a solution of the type

$$\bar{v} = \frac{1}{p}\,\frac{I_0(qr)}{I_0(qa)} + \sum_{n=1}^{\infty} a_n J_0(r\alpha_n) e^{-z\sqrt{(q^2+\alpha_n^2)}} \qquad \ldots\ldots (4)$$

where the α_n, $n = 1, 2, \ldots$ are the positive roots of

$$J_0(a\alpha_n) = 0. \qquad \ldots\ldots (5)$$

The first term is the solution of (1) and (2) for the infinite cylinder, the second a series of correcting terms which satisfy (1) and vanish when $r = a$; the coefficients a_n are to be chosen so that (4) satisfies (3). This requires

$$0 = \frac{1}{p}\,\frac{I_0(qr)}{I_0(qa)} + \sum_{n=1}^{\infty} a_n J_0(r\alpha_n).$$

Thus $$a_n = -\frac{2\alpha_n}{ap(\alpha_n^2+q^2)\,J_1(a\alpha_n)}$$

and so $$\bar{v} = \frac{1}{p}\,\frac{I_0(qr)}{I_0(qa)} - \frac{2}{a}\sum_{n=1}^{\infty} \frac{\alpha_n}{p(\alpha_n^2+q^2)}\,\frac{J_0(r\alpha_n)}{J_1(a\alpha_n)}\, e^{-z\sqrt{(\alpha_n^2+q^2)}}. \qquad \ldots (6)$$

This is in agreement with Chapter VII § 5 (5) except for the change of notation and the value of v follows as before from Chapter VII § 5 (4) and Chapter VII § 5 (6)

$$v = 1 + \frac{2}{a}\sum_{n=1}^{\infty} \frac{J_0(r\alpha_n)e^{-z\alpha_n}}{\alpha_n J_0'(a\alpha_n)} + \frac{4}{\pi a}\sum_{n=1}^{\infty} \frac{\alpha_n J_0(r\alpha_n)}{J_0'(a\alpha_n)} \int_0^{\infty} \frac{e^{-\kappa t(\alpha_n^2+u^2)}\sin zu\,du}{u(u^2+\alpha_n^2)} \quad \ldots\ldots (7)$$

2. __The region $r > a$, $0 < z < l$. $z = l$ kept at V for $t > 0$. The other surfaces at zero. The initial temperature zero.__

The subsidiary equation is

$$\frac{\partial^2\bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{v}}{\partial r} + \frac{\partial^2\bar{v}}{\partial z^2} - q^2\bar{v} = 0, \quad r > a, \quad 0 < z < l \quad \ldots\ldots (1)$$

with $\bar{v} = 1/p$, when $z = l$, $r > a$ .... (2)

$= 0$, $z = 0$, $r > a$ .... (3)

$= 0$, $0 < z < l$, $r = a$. .... (4)

Choose $$\bar{v} = \frac{1}{p}\frac{\sinh qz}{\sinh ql} + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi z}{l} K_0(rq_n), \quad \ldots\ldots (5)$$

where $q_n = \sqrt{(q^2 + n^2\pi^2/l^2)}$.

This satisfies (1), (2) and (3) and in order that it may satisfy (4) we require

$$\frac{1}{p}\frac{\sinh qz}{\sinh ql} + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi z}{l} K_0(aq_n) = 0.$$

Hence $$a_n = \frac{2n\pi(-)^n}{pl^2 q_n^2 K_0(aq_n)},$$

and $$\bar{v} = \frac{1}{p}\frac{\sinh qz}{\sinh ql} + \frac{2\pi}{pl^2}\sum \frac{(-)^n n K_0(rq_n)}{q_n^2 K_0(aq_n)} \sin n\pi z/l. \quad \ldots\ldots (6)$$

Now

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}}{\lambda\left[\frac{\lambda}{\kappa}+\frac{n^2\pi^2}{\ell^2}\right]}\;\frac{K_0\left[r\sqrt{\left(\frac{n^2\pi^2}{\ell^2}+\frac{\lambda}{\kappa}\right)}\right]}{K_0\left[a\sqrt{\left(\frac{n^2\pi^2}{\ell^2}+\frac{\lambda}{\kappa}\right)}\right]}\,d\lambda$$

$$= -\frac{\ell^2}{n^2\pi^2}e^{-\kappa n^2\pi^2 t/\ell^2}+\frac{\ell^2}{n^2\pi^2}\frac{K_0(n\pi r/\ell)}{K_0(n\pi a/\ell)}-\frac{2}{\pi}e^{-\kappa n^2\pi^2 t/\ell^2}\int_0^\infty\frac{e^{-\kappa u^2 t}\,C_0(ur,ua)\,du}{u(u^2+n^2\pi^2/\ell^2)\left[J_0^2(ua)+Y_0^2(ua)\right]} \qquad \text{....... (7)}$$

And we have finally

$$v=\frac{z}{\ell}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-)^n K_0(n\pi r/\ell)}{n\,K_0(n\pi a/\ell)}\sin\frac{n\pi z}{\ell}$$

$$-\frac{4}{\ell^2}\sum_{n=1}^{\infty}(-)^n e^{-\kappa n^2\pi^2 t/\ell^2}\sin\frac{n\pi z}{\ell}\int_0^\infty\frac{e^{-\kappa u^2 t}\,C_0(ur,ua)\,du}{u(u^2+n^2\pi^2/\ell^2)\left[J_0^2(ua)+Y_0^2(ua)\right]}\,. \qquad \text{...... (8)}$$

3. <u>The region $r > a$, $z > 0$. $z = 0$ kept at unity for $t > 0$, $r > a$. $r = a$ at zero for $z > 0$, $t > 0$. The initial temperature zero.</u>

The subsidiary equation is

$$\frac{\partial^2\bar v}{\partial r^2}+\frac{1}{r}\frac{\partial\bar v}{\partial r}+\frac{\partial^2\bar v}{\partial z^2}-q^2\bar v=0\,,\quad r>a,\quad z>0 \qquad \text{......... (1)}$$

$$\text{with}\quad \bar v = 1/p\,,\quad z=0,\quad r>a \qquad \text{......... (2)}$$

$$\quad\;\; =0\,,\quad r=a\,,\quad z>0. \qquad \text{......... (3)}$$

Let $\bar v=\frac{1}{p}e^{-qz}+\int_0^\infty f(u)\sin uz\,K_0\left[r(q^2+u^2)^{\frac12}\right]du.$

This satisfies (1) and (2) and (3) requires

$$\frac{1}{p}e^{-qz}+\int_0^\infty f(u)\sin uz\,K_0\left[a(q^2+u^2)^{\frac12}\right]du=0\,.$$

Hence, by Fourier's integral theorem

$$f(u')=-\frac{2u'}{\pi p(u'^2+q^2)\,K_0\left[a(u'^2+q^2)^{\frac12}\right]}\,.$$

Thus

$$\bar{v} = \frac{1}{p} e^{-qz} - \frac{2}{\pi p} \int_0^\infty \frac{u \sin uz}{u^2 + q^2} \frac{K_0\left[r(u^2+q^2)^{\frac{1}{2}}\right]}{K_0\left[a(u^2+q^2)^{\frac{1}{2}}\right]} du . \qquad \ldots\ldots\ldots\ldots (4)$$

And, using § 2 (6)

$$v = 1 - \frac{2}{\pi} \int_0^\infty \frac{\sin uz\, K_0(ur)}{u\, K_0(ua)} du$$

$$+ \frac{4}{\pi^2} \int_0^\infty e^{-\kappa u^2 t} u \sin uz\, du \int_0^\infty \frac{e^{-\kappa u'^2 t}\; C_0(u'r, u'a) du'}{u'(u'^2+u^2)\left[J_0^2(u'a)+Y_0^2(u'a)\right]} . \qquad (5)$$

4. <u>The region $r > a$, $z > 0$. $r = a$ kept at unity for $z > 0$, $t > 0$. $z = 0$ kept at zero for $r > a$, $t > 0$. The initial temperature zero.</u>

The subsidiary equation is

$$\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}}{\partial r} + \frac{\partial^2 \bar{v}}{\partial z^2} - q^2 \bar{v} = 0 , \quad r > a , \quad z > 0 , \qquad \ldots\ldots\ldots\ldots (1)$$

with

$$\bar{v} = 1/p , \quad r = a, \quad z > 0 \qquad \ldots\ldots\ldots\ldots (2)$$

$$\bar{v} = 0 , \quad z = 0, \quad r > a. \qquad \ldots\ldots\ldots\ldots (3)$$

Let

$$\bar{v} = \frac{1}{p}\frac{K_0(qr)}{K_0(qa)} + \int_0^\infty f(u)\, C_0(ru, au)\, e^{-z(q^2+u^2)^{\frac{1}{2}}} du , \qquad \ldots\ldots\ldots\ldots (4)$$

this satisifes (1) and (2); (3) requires

$$\frac{1}{p}\frac{K_0(qr)}{K_0(qa)} + \int_0^\infty f(u)\, C_0(ru, au)\, du = 0 , \quad r > a. \qquad \ldots\ldots (5)$$

Now by II § 4 (6) (Weber's integral theorem)

$$K_0(qr) = \int_0^\infty \frac{C_0(ur, ua) u\, du}{J_0^2(ua)+Y_0^2(ua)} \int_0^\infty C_0(ur', ua)\, K_0(qr') r' dr' .$$

Thus (5) gives

$$f(u) = -\frac{u}{pK_0(qa)\left[J_0^2(ua) + Y_0^2(ua)\right]} \int_a^\infty C_0(ur', ua)\, K_0(qr')r'dr'$$

$$= \frac{2u}{\pi p(u^2+q^2)\left[J_0^2(ua)+Y_0^2(ua)\right]}.$$

And we have

$$\bar{v} = \frac{1}{p}\frac{K_0(qr)}{K_0(qa)} + \frac{2}{\pi p}\int_0^\infty e^{-z(q^2+u^2)^{\frac{1}{2}}} \frac{C_0(ur, ua)u\,du}{(u^2+q^2)\left[J_0^2(ua)+Y_0^2(ua)\right]} \qquad \ldots\ldots (6)$$

And thus, using VII § 3 (4)

$$v = 1 + \frac{2}{\pi}\int_0^\infty \frac{e^{-uz}\, C_0(ru, au)}{J_0^2(au)+Y_0^2(au)}\frac{du}{u}$$

$$+ \frac{4}{\pi^2}\int_0^\infty \frac{C_0(ru, au)u\,du}{J_0^2(au)+Y_0^2(au)}\int_0^\infty e^{-\kappa t(u^2+u'^2)}\frac{\sin u'z\, du'}{u'(u^2+u'^2)} \qquad \ldots\ldots (7)$$

5. <u>Heat conduction in the region $0 < z < \ell$, $r > a$. The surfaces kept at zero. The initial temperature unity.</u>

We have to solve

$$\frac{\partial^2\bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{v}}{\partial r} + \frac{\partial^2\bar{v}}{\partial z^2} - q^2\bar{v} = -\frac{1}{\kappa}, \quad r > a, \quad 0 < z < \ell \qquad \ldots\ldots\ldots (1)$$

with $\bar{v} = 0$, when $z = 0$ and $z = \ell$, $r > a$ $\qquad \ldots\ldots\ldots (2)$

$\bar{v} = 0$, $\quad r = a$, $\quad 0 < z < \ell$. $\qquad \ldots\ldots\ldots (3)$

Seek a solution

$$\bar{v} = \frac{\sinh q\ell - \sinh qz - \sinh q(\ell - z)}{p \sinh q\ell} + \sum_{n=1}^\infty a_n \sin\frac{n\pi z}{\ell} K_0(rq_n),$$

where $q_n = \sqrt{(q^2 + n^2\pi^2/\ell^2)}$.

This satisfies (1) and (2), and (3) requires

$$\frac{\sinh q\ell - \sinh qz - \sinh q(\ell - z)}{p \sinh q\ell} + \sum_{n=1}^{\infty} a_n \sin \frac{n\pi z}{\ell} K_0(aq_n) = 0.$$

Hence

$$a_n = \frac{2q^2\left[(-)^n - 1\right]}{n\pi p\left[q^2 + n^2\pi^2/\ell^2\right]K_0(aq_n)} ,$$

and

$$\bar{v} = \frac{\sinh q\ell - \sinh qz - \sinh q(\ell - z)}{p \sinh q\ell} + \frac{2}{\pi\kappa} \sum_{n=1}^{\infty} \frac{\left[(-)^n - 1\right]K_0(rq_n)}{n\left[q^2 + n^2\pi^2/\ell^2\right]K_0(aq_n)} \sin \frac{n\pi z}{\ell} .$$

.... (4)

Therefore

$$v = -\frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi z/\ell}{(2n+1)} e^{-\kappa(2n+1)^2\pi^2 t/\ell^2} \int_0^{\infty} \frac{e^{-\kappa u^2 t}\, C_0(ur, ua)\, du}{\left[J_0^2(ua) + Y_0^2(ua)\right]u} \quad .. (5)$$

6. <u>The region $z > 0$, $r > a$. Initial temperature unity. Surfaces kept at zero for $t > 0$.</u>

The fact that the solution for this case follows from the solutions for constant boundary temperature and zero initial temperature suggests a combination of the solutions of §§ 3 and 4. Thus

$$\bar{v} = \frac{1}{p} - \frac{1}{p}\frac{K_0(qr)}{K_0(qa)} - \frac{1}{p} e^{-qz}$$
$$+ \frac{2}{\pi p} \int_0^{\infty} \frac{u \sin uz}{u^2 + q^2} \frac{K_0\left[r(u^2+q^2)^{\frac{1}{2}}\right]}{K_0\left[a(u^2+q^2)^{\frac{1}{2}}\right]} du - \frac{2}{\pi p} \int_0^{\infty} \frac{e^{-z(q^2+u^2)^{\frac{1}{2}}} C_0(ur, ua) u\, du}{(u^2+q^2)\left[J_0^2(ua) + Y_0^2(ua)\right]}$$

.... (1)

satisfies the subsidiary equation and its boundary conditions. Hence we have the solution

$$v = -1 + \frac{2}{\pi} \int_0^\infty \frac{\sin uz \, K_0(ur)}{u \, K_0(ua)} du - \frac{2}{\pi} \int_0^\infty \frac{e^{-uz} \, C_0(ru, au) du}{\left[J_0^2(au) + Y_0^2(au)\right] u}$$

$$- \frac{4}{\pi^2} \int_0^\infty \frac{C_0(ur, ua)}{u\left[J_0^2(ua) + Y_0^2(ua)\right]} du \int_0^\infty \frac{e^{-kt(u^2+u'^2)} \sin u'z}{u'} du' \, . \qquad \ldots (2)$$

$$= -\frac{4}{\pi^2} \int_0^\infty \frac{C_0(ur, ua) \, du}{u\left[J_0^2(ua) + Y_0^2(ua)\right]} \int_0^\infty e^{-kt(u^2+u'^2)} \frac{\sin u'z}{u'} du' \, , \qquad \ldots\ldots (3)$$

since the first three terms of (2) are the steady state solution which must be zero.

7. <u>The finite cylinder $0 \leq r < a$, $0 < z < \ell$. Initial temperature zero. $r = a$ kept at unity for $t > 0$. $z = 0$ and $z = \ell$ kept at zero for $t > 0$.</u>

The subsidiary equation is

$$\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}}{\partial r} + \frac{\partial^2 \bar{v}}{\partial z^2} - q^2 \bar{v} = 0 \,, \quad 0 \leq r < a, \quad 0 < z \quad \ell \,. \qquad \ldots\ldots (1)$$

with $\bar{v} = 1/p$, $r = a$, $0 < z < \ell$ $\ldots\ldots$ (2)

$\bar{v} = 0$, when $z = 0$ and $z = \ell$, $0 \leq r < a$. $\ldots\ldots$ (3)

We choose

$$\bar{v} = \frac{I_0(qr)}{p\, I_0(qa)} + \sum_{s=1}^{\infty} A_s J_0(r\alpha_s) \cosh\, (\tfrac{1}{2}\ell - z) q_s \,, \qquad \ldots\ldots (4)$$

where* $q_s = \sqrt{(q^2 + \alpha_s^2)}$. This satisfies (1) and (2), and (3) requires

$$\frac{I_0(qr)}{p\, I_0(qa)} + \sum_{s=1}^{\infty} A_s J_0(r\alpha_s) \cosh \tfrac{1}{2}\ell q_s = 0 \,.$$

Thus $\quad A_s \cosh \tfrac{1}{2}\ell q_s = - \dfrac{2\alpha_s}{ap\, J_1(a\alpha_s)(q^2 + \alpha_s^2)}$,

and, substituting in (4),

$$\bar{v} = \frac{I_0(qr)}{p\, I_0(qa)} - \frac{2}{a} \sum_{s=1}^{\infty} \frac{\alpha_s J_0(r\alpha_s)}{p\, J_1(a\alpha_s)(q^2 + \alpha_s^2)} \; \frac{\cosh(\tfrac{1}{2}\ell - z) q_s}{\cosh \tfrac{1}{2}\ell q_s} \,. \qquad \ldots\ldots (5)$$

Then, using the result

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\lambda t}\, d\lambda}{\lambda(k^2 + \lambda/\kappa)} \; \frac{\cosh(\tfrac{1}{2}\ell - z)(k^2 + \lambda/\kappa)^{\frac{1}{2}}}{\cosh \tfrac{1}{2}\ell (k^2 + \lambda/\kappa)^{\frac{1}{2}}}$$

$$= \frac{\cosh k(\tfrac{1}{2}\ell - z)}{k^2 \cosh \tfrac{1}{2}\ell\, \alpha_s} - \frac{1}{k^2} e^{-\kappa k^2 t} + \frac{4}{\pi} e^{-\kappa k^2 t} \sum_{n=0}^{\infty} \frac{e^{-\kappa(2n+1)^2 \pi^2 t/\ell^2} \sin(2n+1)\pi z/\ell}{(2n+1)\left[k^2 + (2n+1)^2 \pi^2/\ell^2\right]} \,. \qquad \ldots (6)$$

we have finally

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* The α_s are the positive roots of $J_0(a\alpha)$.

$$v = 1 - \frac{2}{a} \sum_{s=1}^{\infty} \frac{J_0(r\alpha_s)}{\alpha_s J_1(a\alpha_s)} \frac{\cosh(\frac{1}{2}\ell - z)\alpha_s}{\cosh \frac{1}{2}\ell\alpha_s}$$

$$- \frac{8}{a\pi} \sum_{s=1}^{\infty} \frac{\alpha_s J_0(r\alpha_s)}{J_1(a\alpha_s)} e^{-\kappa\alpha_s^2 t} \sum_{n=0}^{\infty} \frac{e^{-\kappa(2n+1)^2\pi^2 t/\ell^2} \sin(2n+1)\pi z/\ell}{(2n+1)\left[\alpha_s^2 + (2n+1)^2\pi^2/\ell^2\right]} \quad \ldots\ldots (7)$$

The transient terms of (7) are the same as those of Chapter VI, § 2 (9), but the steady state terms are obtained in a different form.

8. <u>The finite cylinder $0 \leq r < a$, $0 < z < \ell$. Initial temperature zero. $z = \ell$ kept at unity for $t > 0$. The other surfaces kept at zero for $t > 0$.</u>

The subsidiary equation is

$$\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}}{\partial r} + \frac{\partial^2 \bar{v}}{\partial z^2} - q^2 \bar{v} = 0 , \quad 0 \leq r < a , \quad 0 < z < \ell , \quad \ldots\ldots (1)$$

with $\bar{v} = 1/p , \quad z = \ell , \quad 0 \leq r < a$ .... (2)

$\bar{v} = 0 , \quad z = 0 , \quad 0 \leq r < a$ .... (3)

$\bar{v} = 0 , \quad r = a , \quad 0 < z < \ell .$ .... (4)

We seek a solution of type

$$\bar{v} = \frac{\sinh qz}{p \sinh q\ell} + \sum_{n=1}^{\infty} a_n \sin \frac{n\pi z}{\ell} I_0(rq_n) , \quad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

where $q_n = \sqrt{(q^2 + n^2\pi^2/\ell^2)}$.

This satisfies (1), (2), and (3), and in order that (4) may be satisfied we require

$$\frac{\sinh qz}{p \sinh q\ell} + \sum_{n=1}^{\infty} a_n \sin \frac{n\pi z}{\ell} I_0(aq_n) = 0.$$

Thus $a_n = \dfrac{2n\pi(-)^n}{p\ell^2 q_n^2 I_0(aq_n)}$,

and $$\bar{v} = \frac{\sinh qz}{p \sinh q\ell} + \frac{2\pi}{p\ell^2} \sum_{n=1}^{\infty} \frac{n(-)^n I_0(rq_n)}{q_n^2 I_0(aq_n)} \sin \frac{n\pi z}{\ell} . \quad \ldots\ldots\ldots\ldots (6)$$

Therefore finally

$$v = \frac{z}{l} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-)^n I_0(n\pi r/l)}{n\, I_0(n\pi a/l)} \sin\frac{n\pi z}{l} - \frac{4\pi}{a} \sum_{n=1}^{\infty} n(-)^n \sin\frac{n\pi z}{l} \sum_{s=1}^{\infty} \frac{J_0(r\alpha_s)e^{-\kappa t(\alpha_s^2 + n^2\pi^2/l^2)}}{\alpha_s(l^2\alpha_s^2 + n^2\pi^2)J_0'(a\alpha_s)} \qquad \ldots\ldots (7)$$

where the α_s are the positive roots of $J_0(a\alpha) = 0$.

The transient terms of (7) agree with those of Chapter VI, § 8 (12).

9. <u>The semi-infinite cylinder $0 \leq r < a$, $z > 0$. The initial temperature zero. $z = 0$ kept at unity for $t > 0$, $0 \leq r < a$. $r = a$ kept at zero for $z > 0$, $t > 0$.</u>

The subsidiary equation is

$$\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}}{\partial r} + \frac{\partial^2 \bar{v}}{\partial z^2} - q^2\bar{v} = 0, \quad 0 \leq r < a, \quad z > 0, \qquad \ldots\ldots\ldots (1)$$

to be solved with

$$\bar{v} = 1/p, \quad z = 0, \quad 0 \leq r < a \qquad \ldots\ldots\ldots (2)$$

$$\bar{v} = 0, \quad r = a, \quad z > 0. \qquad \ldots\ldots\ldots (3)$$

Here we choose

$$\bar{v} = \frac{1}{p}e^{-qz} + \int_0^{\infty} I_0(rq')\sin uz\, f(u)\, du, \qquad \ldots\ldots\ldots (4)$$

where $q' = \sqrt{(q^2 + u^2)}$.

This satisfies (1) and (2), and (3) requires

$$\frac{1}{p}e^{-qz} + \int_0^{\infty} I_0(aq')\sin uz\, f(u)\, du = 0.$$

Thus $$f(u) = -\frac{2u}{\pi p q'^2\, I_0(aq')}$$

and $$\bar{v} = \frac{1}{p}e^{-qz} - \frac{2}{\pi p}\int_0^{\infty} \frac{u \sin uz}{q^2 + u^2}\frac{I_0(rq')}{I_0(aq')}\, du. \qquad \ldots\ldots\ldots (5)$$

Therefore

$$v = 1 - \frac{2}{\pi}\int_0^\infty \frac{I_0(ur)\sin uz\,du}{u\,I_0(ua)} + \frac{4}{\pi a}\sum_{s=1}^{\infty}\frac{J_0(r\alpha_s)}{\alpha_s J_0'(a\alpha_s)}\int_0^\infty \frac{e^{-\kappa t(\alpha_s^2+u^2)}\,u\sin uz\,du}{(\alpha_s^2+u^2)}, \quad \ldots (6)$$

where the α_s are the positive roots of $J_0(a\alpha) = 0$.

The transient terms of (6) agree with those of Chapter VI, § 11 (10).

10. <u>The region $r > a$, $0 < z < l$. $r = a$ kept at unity for $t > 0$, $0 < z < l$. The other surfaces at zero. The initial temperature zero.</u>

The subsidiary equation is

$$\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}}{\partial r} + \frac{\partial^2 \bar{v}}{\partial z^2} - q^2\bar{v} = 0, \quad r > a, \quad 0 < z < l, \quad \ldots\ldots (1)$$

to be solved with

$$\bar{v} = 1/p, \quad \text{when } r = a, \quad 0 < z < l \quad \ldots\ldots\ldots\ldots (2)$$

$$\bar{v} = 0, \quad \text{when } z = 0 \text{ and } z = l, \quad r > a. \quad \ldots\ldots\ldots\ldots (3)$$

We choose

$$\bar{v} = \frac{K_0(qr)}{p\,K_0(qa)} + \int_0^\infty C_0(ru,au)\cosh(\tfrac{1}{2}l - z)(q^2+u^2)^{\frac{1}{2}} f(u)\,du. \quad \ldots (4)$$

This satisfies (1) and (2), and (3) requires

$$\frac{K_0(qr)}{p\,K_0(qa)} + \int_0^\infty C_0(ru,au)\cosh \tfrac{1}{2}l\,(q^2+u^2)^{\frac{1}{2}} f(u)\,du = 0.$$

Thus, by Chapter II, § 4 (6),

$$f(u)\cosh \tfrac{1}{2}l(q^2+u^2)^{\frac{1}{2}} = -\frac{u}{p\,K_0(qa)\left[J_0^2(ua)+Y_0^2(ua)\right]}\int_a^\infty C_0(ur',ua)K_0(qr')r'dr'$$

$$= \frac{2u}{\pi p(q^2+u^2)\left[J_0^2(ua)+Y_0^2(ua)\right]}.$$

Using this result in (4) we obtain

$$\bar{v} = \frac{K_0(qr)}{p\,K_0(qa)} + \int_0^\infty \frac{2\cosh(\tfrac{1}{2}l - z)(q^2+u^2)^{\frac{1}{2}}\,C_0(ru,au)u\,du}{\pi p(q^2+u^2)\left[J_0^2(ua)+Y_0^2(ua)\right]\cosh \tfrac{1}{2}l\,(q^2+u^2)^{\frac{1}{2}}}. \quad \ldots (5)$$

Therefore, using the result § 7 (6), we obtain finally

$$v = 1 + \frac{2}{\pi} \int_0^{\infty} \frac{\cosh(\tfrac{1}{2}\ell - z)u \; C_0(ru, au) \; du}{u \cosh \tfrac{1}{2}\ell u \left[J_0^2(au) + Y_0^2(au)\right]}$$

$$+ \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{\sin(2n+1)\pi z/\ell}{(2n+1)} \int_0^{\infty} \frac{e^{-\kappa u^2 t - \kappa(2n+1)^2\pi^2 t/\ell^2} \; C_0(ur, ua) u \; du}{\left[u^2 + (2n+1)^2\pi^2/\ell^2\right]\left[J_0^2(ua) + Y_0^2(ua)\right]} \quad \ldots\ldots (6)$$

The transient part of (6) is identical with that in the solution Chapter VI, § 3 (5) but the steady state terms are obtained in different forms by the two methods.

CHAPTER IX.

THE WEDGE WITH ZERO INITIAL TEMPERATURE.

It is well known* that the solution of the problem of conduction of heat in a region with prescribed surface temperature and zero initial temperature may be obtained from the steady state solution and the Green's function for the region with boundaries at zero. The Green's function for the region is thus the most fundamental but solutions for problems with zero initial temperature and prescribed surface temperature are of considerable interest and are frequently derived independently.

In Chapter VI a complete set of solutions for the regions bounded by coaxial cylinders and planes perpendicular to the axis was given by the obvious process of separating the variables in the subsidiary equation+. This provides a very simple method of obtaining formally the results which would be obtained by the more sophisticated Green's function procedure**.

The method of Chapter VI depended on the fact that the

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* C.H. § 9.

+ The operational equivalent of this was used by Goldstein, J. Angew. Math. und Mech., 12 (1932), 235.

** C.H., Ch. X, § 80.

subsidiary equation

$$\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}}{\partial r} + \frac{\partial^2 \bar{v}}{\partial z^2} - q^2 \bar{v} = 0$$

had separable solutions of two types

$$\sin mz \; I_o\left[r \sqrt{(q^2 + m^2)}\right] \qquad \text{..........} \; (1)$$

and

$$J_o(\alpha r) \; \sinh \; z \sqrt{(\alpha^2 + q^2)} \qquad \text{..........} \; (2)$$

and by choice of the appropriate one all the boundary value problems could be solved.

In the present chapter problems of the same type for the wedge will be considered. Here the subsidiary equation is

$$\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{v}}{\partial \theta^2} - q^2 \bar{v} = 0 \qquad \text{..........} (3)$$

of which there is only one type of separable solution, namely,

$$\sin m\theta \; I_m(qr) \qquad \text{..........} \; (4)$$

Using solutions of type (4), problems in which the temperature vanishes on planes $\theta = 0$ and $\theta = \theta_o$ can be solved (these are discussed in §§ 1, 2) but not problems in which the temperature is prescribed on these planes.

The method of Chapter VIII which involved the use of the solution of the subsidiary equation for a wider region is not available here but suggests that the steady state solution might be used in the same way. Clearly this process is closely related to the method of obtaining a solution from the Green's function and the steady state solution, but again it gives a fairly easy formal process for getting the solution withoug the use of Green's function.

1. The region $0 < r < a$, $0 < \theta < \theta_0$. Zero initial temperature. The boundaries $\theta = 0$ and $\theta = \theta_0$ kept at zero for $t > 0$. $r = a$ kept at $f(\theta) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi\theta}{\theta_0} d\theta$.

The subsidiary equation is

$$\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{v}}{\partial \theta^2} - q^2 \bar{v} = 0 , \quad 0 < r < a , \quad 0 < \theta < \theta_0 \tag{1}$$

with $\bar{v} = 0$ when $\theta = 0$ and $\theta = \theta_0$, $0 < r < a$ ....(2)

$$\bar{v} = \frac{1}{p} \sum_{n=1}^{\infty} a_n \sin \frac{n\pi\theta}{\theta_0} , \quad 0 < \theta < \theta_0 , \ r = a \quad \text{.............(3)}$$

We shall write throughout

$$s = n\pi/\theta_0 . \quad \text{.................................(4)}$$

A solution of (1) which satisfies (2) is

$$\sin s\theta \, I_s(qr) ,$$

and thus the solution of the subsidiary equation and boundary conditions is

$$\bar{v} = \frac{1}{p} \sum_{n=1}^{\infty} a_n \sin s\theta \frac{I_s(qr)}{I_s(qa)} . \quad \text{....................(5)}$$

To determine v we require

$$I = \frac{1}{2i\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \frac{I_s(\mu r)}{I_s(\mu a)} \frac{d\lambda}{\lambda} , \quad \text{....................(6)}$$

where $\mu = \sqrt{(\lambda/\kappa)}$.

$\frac{1}{\lambda}\frac{I_s(\mu r)}{I_s(\mu a)}$ is a single values function of λ with a simple pole at $\lambda = 0$, and simple poles at $\lambda = -\kappa\alpha_n^2$, where α_n, $n = 1, 2, \ldots$ are the positive roots of $J_s(a\alpha) = 0$.

Evaluating the residues at these poles (6) becomes

$$I = \left(\frac{r}{a}\right)^s + \frac{2}{a} \sum_{n=1}^{\infty} \frac{J_s(r\alpha_n)}{\alpha_n J_s'(a\alpha_n)} e^{-\kappa\alpha_n^2 t} \quad \text{..........(7)}$$

Thus finally

$$v = \sum_{n=1}^{\infty} a_n \sin s\theta \left[\left(\frac{r}{a}\right)^s + \frac{2}{a} \sum_{m=1}^{\infty} \frac{J_s(r\alpha_m)}{\alpha_m J_s'(a\alpha_m)} e^{-\kappa\alpha_m^2 t} \right] \quad \text{........ (8)}$$

2. <u>The region $0 < \theta < \theta_0$, $r > a$. Zero initial temperature. The boundaries $\theta = 0$ and $\theta = \theta_0$ kept at zero for $r > a$, $t > 0$. $r = a$ kept at $f(\theta)$ for $t > 0$.</u>

As in § 1 the solution of the subsidiary equation and boundary conditions is

$$\bar{v} = \frac{1}{p} \sum_{n=1}^{\infty} a_n \sin s\theta \frac{K_s(qr)}{K_s(qa)}, \quad \text{.................. (1)}$$

where $s = n\pi/\theta_0$.

To find v we require

$$I = \frac{1}{2i\pi} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{K_s(\mu r)}{K_s(\mu a)} \frac{d\lambda}{\lambda}. \quad \text{.................. (2)}$$

Since $K_s(z)$, $s > 0$, has no zeros for $|\arg z| \leqslant \pi/2$ we use the path of Fig. 1 and obtain finally

$$I = \left(\frac{a}{r}\right)^s + \frac{2}{\pi} \int_0^{\infty} e^{-\kappa u^2 t} \frac{C_s(ur, ua)\,du}{u\left[J_s^2(au) + Y_s^2(au)\right]}, \quad \text{............ (3)}$$

where $C_s(x,y) = J_s(x)Y_s(y) - J_s(y)Y_s(x)$.

And

$$v = \sum_{n=1}^{\infty} a_n \sin s\theta \left[\left(\frac{a}{r}\right)^s + \frac{2}{\pi} \int_0^{\infty} e^{-\kappa u^2 t} \frac{C_s(ur, au)\,du}{u\left[J_s^2(au) + Y_s^2(au)\right]} \right]. \quad \text{..(4)}$$

For $t = 0$ the terms in the brackets all vanish, cf. Titchmarsh, <u>Proc. London Math. Soc.</u> (2) 22 (1922) 15.

3. <u>The region $0 < r < a$, $0 < \theta < \theta_0$. $\theta = 0$ maintained at zero, $\theta = \theta_0$ at unity, $r = a$ at zero. Zero initial temperature.</u>

As suggested in the Introduction we have first to find the steady temperature temperature: consider

$$v = \frac{\theta}{\theta_0} + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi\theta}{\theta_0}\left(\frac{r}{a}\right)^{n\pi/\theta_0} ,$$

this satisfies the steady state equation and the boundary conditions for $\theta = 0$ and $\theta = \theta_0$. The boundary condition for $r = a$ requires

$$0 = \frac{\theta}{\theta_0} + \sum_{n=1}^{\infty} a_n \sin\frac{n\pi\theta}{\theta_0} .$$

Hence $a_n = \frac{2\,(-)^n}{2\,\pi}$, and the steady state solution is

$$v = \frac{\theta}{\theta_0} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-)^n}{n} \sin\frac{n\pi\theta}{\theta_0}\left(\frac{r}{a}\right)^{n\pi/\theta_0} . \qquad \text{.......... (1)}$$

For the general problem the subsidiary equation is

$$\frac{\partial^2\bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial\bar{v}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\bar{v}}{\partial\theta^2} - q^2\bar{v} = 0 , \quad 0 < r < a, \quad 0 < \theta < \theta_0 , \qquad \text{... (2)}$$

with $\bar{v} = \frac{1}{p}$, when $\theta = \theta_0$, $0 < r < a$ .... (3)

$= 0$, $\theta = 0$, $0 < r < a$ .... (4)

$= 0$, $r = a$, $0 < 0 < \theta_0$ .... (5)

We seek a solution of the type

$$\bar{v} = \frac{\theta}{p\theta_0} + \frac{2}{p\pi} \sum_{n=1}^{\infty} \frac{(-)^n}{n} \sin\frac{n\pi\theta}{\theta_0}\left(\frac{r}{a}\right)^{n\pi/\theta_0} + \sum_{n=1}^{\infty} \sin\frac{n\pi\theta}{\theta_0} \sum_{\alpha} a_{n,\alpha} J_s(\alpha r) , \qquad \text{.. (6)}$$

where the α are the roots of $J_s(\alpha a) = 0$. s is written for $n\pi/\theta_0$ in the orders of Bessel functions..

(6) satisfies the boundary conditions (3), (4) and (5), and to satisfy (2) we require

$$0 = -\frac{\theta}{\theta_0} - \frac{2}{\kappa\pi}\sum_{n=1}^{\infty}\frac{(-)^n}{n}\sin\frac{n\pi\theta}{\theta_0}\left(\frac{r}{a}\right)^{n\pi/\theta_0} - \sum_{n=1}^{\infty}\sin\frac{n\pi\theta}{\theta_0}\sum_{\alpha}a_{n,\alpha}(\alpha^2+q^2)J_s(\alpha r) \quad \ldots\ldots (7)$$

Now $\dfrac{\theta}{\kappa\theta_0} = -\dfrac{2}{\pi\kappa}\displaystyle\sum_{n=1}^{\infty}\frac{(-)^n}{n}\sin\frac{n\pi\theta}{\theta_0}$

so (7) requires

$$0 = \frac{2}{\pi\kappa}\frac{(-)^n}{n}\left[1 - \left(\frac{r}{a}\right)^{r\pi/\theta_0}\right] - \sum_{\alpha}a_{n,\alpha}(\alpha^2+q^2)\,J_s(\alpha r)\,.$$

And hence, multiplying by $rJ_s(\alpha r)$ and integrating with respect to r from 0 to a

$$\frac{2}{\pi\kappa}\frac{(-)^n}{n}\int_0^a\left[1 - \left(\frac{r}{a}\right)^s\right]rJ_s(\alpha r)dr - a_{n,\alpha}(\alpha^2+q^2)\cdot\tfrac{1}{2}a^2\left[J_s'(a)\right]^2 = 0$$

Thus $\quad a_{n,\alpha} = \dfrac{4}{\pi\kappa a^2}\dfrac{(-)^n}{n}\dfrac{1}{(\alpha^2+q^2)\left[J_s'(\alpha a)\right]^2}\displaystyle\int_0^a\left[1-\left(\frac{r}{a}\right)^s\right]r\,J_s(\alpha r)\,dr$

and

$$\bar{v} = \frac{\theta}{p\theta_0} + \frac{2V}{p\pi}\sum_{n=1}^{\infty}\frac{(-)^n}{n}\sin\frac{n\pi\theta}{\theta_0}\left(\frac{r}{a}\right)^{n\pi/\theta_0} + \frac{4}{\pi\kappa a^2}\sum_{n=1}^{\infty}\frac{(-)^n}{n}\sin\frac{n\pi\theta}{\theta_0}$$

$$\cdot\sum_{\alpha}\frac{J_s(\alpha r)}{(\alpha^2+q^2)\left[J_s'(\alpha a)\right]^2}\int_0^a\left[1-\left(\frac{r}{a}\right)^s\right]rJ_s(\alpha r)d\alpha\,.$$

Therefore

$$v = \frac{\theta}{\theta_0} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-)^n}{n}\sin\frac{n\pi\theta}{\theta_0}\left(\frac{r}{a}\right)^{n\pi/\theta_0}$$

$$+\frac{4}{\pi a^2}\sum_{n=1}^{\infty}\frac{(-)^n}{n}\sin\frac{n\pi\theta}{\theta_0}\sum_{\alpha}e^{-\kappa\alpha^2 t}\frac{J_s(\alpha r)}{\left[J_s(\alpha a)\right]^2}\int_0^a\left[1-\left(\frac{r}{a}\right)^s\right]rJ_s(\alpha r)d\alpha\,,$$

which is the result obtained in the usual way by combining (1) and the Green's function.

4. <u>The region $r > 0$, $0 < \theta < \theta_0$. $\theta = 0$ kept at zero and $\theta = \theta_0$ at unity for $t > 0$. Zero initial temperature.</u>

The subsidiary equation is

$$\frac{\partial^2 \bar{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{v}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{v}}{\partial \theta^2} - q^2 \bar{v} = 0 , \quad r > 0 , \quad 0 < \theta < \theta_0 \qquad \ldots\ldots (1)$$

with $\bar{v} = 1/p$ when $\theta = \theta_0$, $r > a$ .... (2)

$= 0$ when $\theta = 0$, $r > a$ .... (3)

The steady state solution is θ/θ_0.

We seek a solution of (1) of type

$$\bar{v} = \frac{\theta}{p\theta_0} + \sum_{n=1}^{\infty} \sin\frac{n\pi\theta}{\theta_0} \int_0^{\infty} f_n(u)\, J_s(ur)\, du , \qquad \ldots\ldots.(4)$$

where s is written for $n\pi/\theta_0$ in the orders of Bessel functions.

This satisfies (2) and (3), and (1) requires

$$0 = -\frac{\theta}{\kappa\theta_0} - \sum_{n=1}^{\infty} \sin\frac{n\pi\theta}{\theta_0} \int_0^{\infty} (u^2 + q^2)\, J_s(ur)\, f_n(u)\, du . \qquad \ldots\ldots\ldots (5)$$

Now $$\frac{\theta}{\kappa\theta_0} = -\frac{2}{\pi\kappa} \sum_{n=1}^{\infty} \frac{(-)^n}{n} \sin\frac{n\pi\theta}{\theta_0} ,$$

and thus (5) requires

$$\int_0^{\infty} (u^2 + q^2)\, J_s(ur) f_n(u)\, du = \frac{2}{\pi\kappa}\frac{(-)^n}{n} . \qquad \ldots\ldots\ldots\ldots (6)$$

Now $$\int_0^{\infty} \frac{J_s(ur)\, du}{u} = \frac{1}{s}$$

so we may take $f_n(u) = \dfrac{2(-)^n}{\kappa\theta_0 u(u^2 + q^2)}$, and obtain

$$\bar{v} = \frac{\theta}{p\theta_0} + \sum_{n=1}^{\infty} \frac{2(-)^n}{\kappa\,\theta_0} \sin\frac{n\pi\theta}{\theta_0} \int_0^{\infty} \frac{J_s(ur)\, du}{u(u^2 + q^2)} \qquad \ldots\ldots\ldots\ldots\ldots (7)$$

$$v = \frac{\theta}{\theta_0} + \frac{2}{\kappa\theta_0} \sum_{n=1}^{\infty} (-)^n \sin\frac{n\pi\theta}{\theta_0} \int_0^{\infty} e^{-\kappa u^2 t}\, \frac{J_s(ur)\, du}{u} . \qquad \ldots\ldots\ldots\ldots\ldots (8)$$

5. <u>The region $r > a$, $0 < \theta < \theta_o$. $\theta = \theta_o$ kept at unity for $r > a$, $t > 0$. The other surfaces at zero. The initial temperature zero.</u>

From § 4 it is known that for the problem of the wedge $r > 0$, $0 < \theta < \theta_o$, with $\theta = 0$ at zero and $\theta = \theta_o$ at unity for $t > 0$ and zero initial temperature the transform of the temperature on $r = a$ is

$$\frac{\theta}{p\theta_o} + \sum_{n=1}^{\infty} \frac{2(-)^n}{\kappa\theta_o} \sin\frac{n\pi\theta}{\theta_o} \int_0^{\infty} \frac{J_s(ua)\,du}{u(u^2+q^2)}$$

$$\text{i.e.} \quad \sum_{n=1}^{\infty} (-)^n \sin\frac{n\pi\theta}{\theta_o} \left\{ -\frac{2}{\pi np} + \frac{2}{\kappa\theta_o} \int_0^{\infty} \frac{J_s(ua)\,du}{u(u^2+q^2)} \right\}. \qquad \ldots\ldots(1)$$

We seek a solution of the subsidiary equation on $0 < \theta < \theta_o$, $r > a$ which will have the value (1) when $r = a$, and will vanish when $\theta = 0$ and $\theta = \theta_o$. Such a solution will be

$$\sum_{n=1}^{\infty} (-)^n \sin\frac{n\pi\theta}{\theta_o} \left\{ -\frac{2}{\pi np} + \frac{2}{\kappa\theta_o} \int_0^{\infty} \frac{J_s(au)\,du}{u(u^2+q^2)} \right\} \frac{K_s(qr)}{K_s(qa)} . \qquad \ldots\ldots (2)$$

Thus the solution of the subsidiary equation for the whole problem is the difference of (1) and (2), namely

$$\bar{v} = \frac{\theta}{p\theta_o} + \sum_{n=1}^{\infty} \frac{2(-)^n}{\kappa\theta_o} \sin\frac{n\pi\theta}{\theta_o} \int_0^{\infty} \frac{J_s(ur)\,du}{u(u^2+q^2)} + \sum_{n=1} \frac{2(-)^n}{n\pi p} \sin\frac{n\pi\theta}{\theta_o} \frac{K_s(qr)}{K_s(qa)}$$

$$- \frac{2}{\kappa\theta_o} \sum_{n=1}^{\infty} (-)^n \sin\frac{n\pi\theta}{\theta_o} \frac{K_s(qr)}{K_s(qa)} \int_0^{\infty} \frac{J_s(ua)\,du}{u(u^2+q^2)}$$

$$\text{Now} \quad \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\lambda t} K_s(r\sqrt{(\lambda/\kappa)})}{K_s(a\sqrt{(\lambda/\kappa)})} d\lambda = \left(\frac{a}{r}\right)^s + \frac{2}{\pi} \int_0^{\infty} \frac{e^{-\kappa u^2 t}\, C_s(ur, ua)\,du}{u\left[J_s^2(ua) + Y_s^2(ua)\right]}$$

and, for real α,

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\lambda t}\,K_s(r\sqrt{(\lambda/\kappa)})}{(\lambda+\kappa\alpha^2)K_s(a\sqrt{(\lambda/\kappa)})}\,d\lambda$$

$$=\frac{1}{2}\left\{\frac{K_s(ir\alpha)}{K_s(ia\alpha)}+\text{Conjugate}\right\}e^{-\kappa\alpha^2 t}+\frac{2}{\pi}\int_0^\infty\frac{e^{-\kappa t(u^2+\alpha^2)}\;C_s(ur,ua)\,u\,du}{u\left[J_s^2(ua)+Y_s^2(ua)\right](u^2-\alpha^2)}$$

$$=\frac{J_s(r\alpha)J_s(a\alpha)+Y_s(r\alpha)Y_s(a\alpha)}{J_s^2(a\alpha)+Y_s^2(a\alpha)}\,e^{-\kappa\alpha^2 t}+\frac{2}{\pi}\int_0^\infty\frac{u\,e^{-\kappa t(u^2+\alpha^2)}\;C_s(ur,ua)\,du}{u\left[J_s^2(ua)+Y_s^2(ua)\right](u^2-\alpha^2)}$$

Thus

$$v=\frac{\theta}{\theta_0}+\frac{2}{\theta_0}\sum_{n=1}^{\infty}(-)^n\sin\frac{n\pi\theta}{\theta_0}\int_0^\infty e^{-\kappa u^2 t}\,\frac{J_s(ur)\,du}{u}$$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-)^n}{n}\left(\frac{a}{r}\right)^{n\pi/\theta_0}\sin\frac{n\pi\theta}{\theta_0}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-)^n}{n}\sin\frac{n\pi\theta}{\theta_0}\int_0^\infty\frac{e^{-\kappa u^2 t}\;C_s(ur,ua)\,du}{u\left[J_s^2(ua)+Y_s^2(ua)\right]}$$

$$-\frac{2}{\theta_0}\sum_{n=1}^{\infty}(-)^n\sin\frac{n\pi\theta}{\theta_0}\int_0^\infty\frac{e^{-\kappa u^2 t}\;J_s(ua)\left[J_s(ur)J_s(ua)+Y_s(ur)Y_s(ua)\right]\,du}{u\left[J_s^2(ua)+Y_s^2(ua)\right]}$$

$$-\frac{4}{\pi\theta_0}\sum_{n=1}^{\infty}(-)^n\sin\frac{n\pi\theta}{\theta_0}\int_0^\infty\frac{\cancel{e^{-\kappa u^2 t}}\;J_s(ua)\,du}{u}\int_0^\infty\frac{v\,e^{-\kappa v^2 t}\;C_s(vr,va)\,dv}{v\left[J_s^2(va)+Y_s^2(va)\right](v^2-u^2)}$$

CHAPTER X.

THE DETERMINATION OF GREEN'S FUNCTION FOR THE EQUATION OF CONDUCTION OF HEAT IN CYLINDRICAL CO-ORDINATES BY THE LAPLACE TRANSFORMATION.

1. Introduction. The Green's Function of this paper is the temperature (v) at P, (r, θ, z) at time t due to an instantaneous unit source at P', (r', θ', z') at time t = 0, the solid being at zero temperature and the surface kept at zero.

Let $R^2 = r^2 + r'^2 - 2rr' \cos(\theta - \theta')$ (1)

and $u = \frac{1}{8(\pi\kappa t)^{3/2}} e^{-[R^2 + (z - z')^2]/4\kappa t}$ (2)

Put $v = u + w$.

Then in the solid w has to satisfy

$$\frac{\partial w}{\partial t} = \kappa\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2}\right), \quad t > 0, \tag{4}$$

$$\lim_{t \to 0} w = 0, \tag{5}$$

and $w = -u$, on the surface, $t > 0$. (6)

Let the Laplace Transforms of u, v, and w be $\bar{u}$, $\bar{v}$, and $\bar{w}$.

i.e. $\bar{u} = \int_0^\infty e^{-pt} u \, dt$, $(p > 0)$ (7)

and so on.

Then in the solid $\bar{w}$ has to satisfy

$$\frac{\partial^2 \bar{w}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{w}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{w}}{\partial \theta^2} + \frac{\partial^2 \bar{w}}{\partial z^2} - q^2\bar{w} = 0, \tag{8}$$

where $q = \sqrt{(p/\kappa)}$, (9)

and $\bar{w} = -\bar{u}$, on the surface (10)

We know* that

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* The results (12) and (13) are derived from § 13.47 (2) of Watson, Theory of Bessel Functions, by taking $\mu = -\frac{1}{2}$, $\nu = 0$, and $\mu = 0$, $\nu = \frac{1}{2}$ respectively.

$$\bar{u} = \frac{e^{-q\sqrt{[R^2 + (z-z')^2]}}}{4\pi\kappa\sqrt{[R^2 + (z-z')^2]}} \tag{11}$$

$$= \frac{1}{2\pi^2\kappa}\int_0^\infty \cos\xi(z-z')K_0(\eta R)\,d\xi, \quad \text{where } \eta = \sqrt{(\xi^2 + q^2)} \tag{12}$$

$$= \frac{1}{4\pi\kappa}\int_0^\infty e^{-\eta|z-z'|}\,\frac{J_0(\xi R)}{\eta}\,\xi\,d\xi\ . \tag{13}$$

Having fond $\bar{w}$, with the help of these results, and thus $\bar{v}$, we obtain v from the inversion theorem for the Laplace transformation*,

$$v(t) = \frac{1}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\,\bar{v}(\lambda)\,d\lambda \tag{14}$$

for all regions bounded by the surfaces of the cylindrical co-ordinate system.

The corresponding results for two dimensions (line source at (r', θ')) can be deduced from these by integration with regard to z', but they are easily obtained directly, starting with

$$u = \frac{1}{4\pi\kappa t}\,e^{-R^2/4\kappa t} \tag{15}$$

and

$$\bar{u} = \frac{1}{2\pi\kappa}K_0(qR) \tag{16}$$

instead of the above.

2. <u>The region bounded, externally, by $r = a$. Source at $(r', \theta', 0)$.</u>

Here using (12) and the result

$$K_0(\ R) = \sum_{n=-\infty}^{\infty} \cos n(\theta - \theta')\, I_n(\eta r)\, K_n(\eta r'), \qquad r < r'$$

$$= \sum_{n=-\infty}^{\infty} \cos n(\theta - \theta')\, I_n(\eta r')\, K_n(\eta r), \qquad r > r'$$

we find that a value of $\bar{w}$ satisfying (4) and (6) and finite at the origin is

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* As elsewhere [cf. Carslaw and Jaeger, Proc. Cambridge Phil. Soc. 35 (1939) 394 and Proc. London Math. Soc., (2), 46 (1940), 367 ~~in press~~] we regard the use of the Inversion Theorem as formal, and the results as subject to verification.

$$\bar{w} = -\frac{1}{2\pi^2\kappa} \sum_{n=-\infty}^{\infty} \cos n(\theta-\theta') \int_0^{\infty} \cos \xi z \frac{I_n(\eta r)I_n(\eta r')K_n(\eta a)}{I_n(\eta a)} d\xi , \quad (17)$$

where $\eta = \sqrt{(\xi^2 + q^2)} = \sqrt{(\xi^2 + p/\kappa)}$ (18)

The notation (18) will be used throughout the paper.

Writing $F_n(x,y) = I_n(x)K_n(y) - K_n(x)I_n(y)$, (19)

we find from (3), (12) and (17) that

$$\bar{v} = \frac{1}{2\pi^2\kappa} \sum_{n=-\infty}^{\infty} \cos n(\theta - \theta') \int_0^{\infty} \cos \xi z \frac{I_n(\eta r')F_n(\eta a, \eta r)}{I_n(\eta a)} d\xi \quad (20)$$

when $r > r'$, and we interchange r and r' when $r < r'$.

Writing $\mu = \sqrt{(\xi^2 + \lambda/\kappa)}$, we find* that

$$\frac{1}{2i\pi}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t} \frac{I_n(\mu r')F_n(\mu a, \mu r)}{I_n(\mu a)} d\lambda = \frac{2\kappa}{a^2} \sum_{\alpha} e^{-\kappa\xi^2 t} \frac{J_n(\alpha r)J_n(\alpha r')}{[J_n'(\alpha a)]^2} e^{-\kappa\xi^2 t}$$

the summation being taken over the positive roots of $J_n(\alpha a) = 0$.

$$\text{Thus} \quad v = \frac{1}{\pi^2 a^2} \sum_{n=-\infty}^{\infty} \cos n(\theta - \theta') \sum_{\alpha} e^{-\kappa\alpha^2 t} \frac{J_n(\alpha r)J_n(\alpha r')}{[J_n'(\alpha a)]^2} \int_0^{\infty} e^{-\kappa\xi^2 t} \cos \xi z d\xi$$

$$= \frac{e^{-z^2/4\kappa t}}{2\pi a^2 \sqrt{(\pi\kappa t)}} \sum_{n=-\infty}^{\infty} \cos n(\theta-\theta') \sum_{\alpha} e^{-\kappa\alpha^2 t} \frac{J_n(\alpha r)J_n(\alpha r')}{[J_n'(\alpha a)]^2} , \quad (21)$$

and this is valid for $r \gtrless r'$.

3. <u>The region bounded, internally, by $r = a$. Source at $(r', \theta', 0)$.</u>

$$\text{Here} \quad \bar{v} = \frac{1}{2\pi^2\kappa} \sum_{n=-\infty}^{\infty} \cos n(\theta-\theta') \int_0^{\infty} \frac{K_n(\eta r')F_n(\eta r, \eta a)}{K_n(\eta a)} \cos \xi z \, d\xi , \quad r < r', \quad (22)$$

where $F_n(x, y)$ is defined in (19). This gives, for $r \gtrless r'$,

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* The method of evaluating integrals of this type by completing the contour by a large circle to the left of the imaginary axis is assumed known, as also that for the corresponding integral arising from (22) which has a branch point at the origin: cf. Carslaw and Jaeger, <u>loc. cit.</u>

$$v = \frac{1}{2\pi^2} \sum_{n=-\infty}^{\infty} \cos n(\theta - \theta') \int_0^{\infty} \alpha e^{-\kappa\alpha^2 t} \frac{U_n(\alpha r)U_n(\alpha r')}{J_n^2(\alpha a) + Y_n^2(\alpha a)} d\alpha \int_0^{\infty} \cos \xi z \, e^{-\kappa\xi^2 t} d\xi$$

$$= \frac{e^{-z^2/4\kappa t}}{4\pi\sqrt{(\pi\kappa t)}} \sum_{n=-\infty}^{\infty} \cos n(\theta - \theta') \int_0^{\infty} \alpha e^{-\kappa\alpha^2 t} \frac{U_n(\alpha r)U_n(\alpha r')}{J_n^2(\alpha a) + Y_n^2(\alpha a)} d\alpha , \qquad (23)$$

where $U_n(\alpha r) = J_n(\alpha r)Y_n(\alpha a) - J_n(\alpha a)Y_n(\alpha r).$ (24)

4. <u>The region bounded by $r = a$ and $r = b$ $(a < b)$. Source at $(r', \theta', 0)$.</u>

Here $$\bar{v} = \frac{1}{2\pi^2\kappa} \sum_{n=-\infty}^{\infty} \cos n(\theta - \theta') \int_0^{\infty} \frac{F_n(\eta a, \eta r)F_n(\eta b, \eta r')\cos \xi z \, d\xi}{F_n(\eta a, \eta b)} , \quad a < r < r'. \qquad (25)$$

And this gives

$$v = \frac{\sqrt{\pi}}{8\sqrt{(\kappa t)}} e^{-z^2/4\kappa t} \sum_{n=-\infty}^{\infty} \cos n(\theta - \theta') \sum_{\alpha} \alpha^2 e^{-\kappa\alpha^2 t} \frac{J_n^2(\alpha b)U_n(\alpha r)U_n(\alpha r')}{J_n^2(\alpha a) - J_n^2(\alpha b)} , \qquad (26)$$

for $r \gtrless r'$, the summation being over the positive roots of $U_n(\alpha b) = 0$, where $U_n(\alpha r)$ is defined by (24).

3. <u>The region between $z = 0$ and $z = l$. Source at (r', θ', z').</u>

To find $\bar{w}$ for this case we use (13) and thus take

$$\bar{w} = \frac{1}{4\pi\kappa} \int_0^{\infty} \frac{\xi}{\eta} J_0(\xi R)\left[A \sinh \eta z + B \sinh \eta(l - z)\right] d\xi ,$$

where $B \sinh \eta l = - e^{-\eta z'}$

$A \sinh \eta l = - e^{-\eta(l - z')}$.

Then

$$\bar{v} = \frac{1}{2\pi\kappa} \int_0^{\infty} \frac{\xi J_0(\xi R) \sinh \eta(l - z') \sinh \eta z}{\eta \sinh \eta l} d\xi , \qquad 0 < z < z'$$

$$= \frac{1}{2\pi^2 i\kappa} \int_{-i\infty}^{+i\infty} \frac{\xi K_0(\xi R) \sinh(l - z)(q^2 - \xi^2)^{\frac{1}{2}} \sinh z(q^2 - \xi^2)^{\frac{1}{2}} d\xi}{(q^2 - \xi^2)^{\frac{1}{2}} \sinh l(q^2 - \xi^2)^{\frac{1}{2}}}$$

Completing the path by a large circle to the right, the integrand has poles at $\xi = q_m$ when $q_m = \sqrt{(q^2 + m^2\pi^2/l^2)}$, $m = 1, 2, \ldots$.

Therefore we have, for $z \gtrless z'$,

$$\bar{v} = \frac{1}{\pi\kappa \ell} \sum_{n=1}^{\infty} \sin \frac{n\pi z}{\ell} \sin \frac{n\pi z'}{\ell} K_0(q_n R). \tag{27}$$

But $$\int_0^{\infty} e^{-pt}\, e^{-\kappa n^2\pi^2 t/\ell^2}\, e^{-R^2/4\kappa t}\, \frac{dt}{t} = 2\, K_0(q_n R) .$$

Therefore $$v = \frac{2}{\ell} \sum_{n=1}^{\infty} e^{-\kappa n^2\pi^2 t/\ell^2} \sin \frac{n\pi z}{\ell} \sin \frac{n\pi z'}{\ell} \cdot \frac{e^{-R^2/4\kappa t}}{4\pi\kappa t} . \tag{28}$$

6. The region bounded by the cylinder $r = a$, externally, and the planes $z = 0$, $z = \ell$. Source at (r', θ', z').

Here

$$\bar{v} = \frac{1}{\pi\kappa \ell} \sum_{n=1}^{\infty} \sin \frac{n\pi z}{\ell} \sin \frac{n\pi z'}{\ell} \sum_{n=-\infty}^{\infty} \frac{I_n(q_n r) F_n(a q_n, r' q_n)}{I_n(q_n a)} \cos n(\theta - \theta') \tag{29}$$

where $q_n = \sqrt{(q^2 + n^2\pi^2/\ell^2)}$, and $r < r'$.

This gives for $r \gtrless r'$

$$v = \frac{2}{\pi a^2 \ell} \sum_{n=1}^{\infty} e^{-\kappa n^2\pi^2 t/\ell^2} \sin \frac{n\pi z}{\ell} \sin \frac{n\pi z'}{\ell} \sum_{n=-\infty}^{\infty} \cos n(\theta - \theta') \sum_{\alpha} e^{-\kappa\alpha^2 t} \frac{J_n(\alpha r) J_n(\alpha r')}{\left[J_n'(\alpha a)\right]^2} \tag{30}$$

where the α are the roots of $J_n(\alpha a) = 0$.

7. The region bounded by $r = a$, internally, and the planes $z = 0$, $z = \ell$. Source at (r', θ', z').

Here

$$\bar{v} = \frac{1}{\pi\kappa \ell} \sum_{n=1}^{\infty} \sin \frac{n\pi z}{\ell} \sin \frac{n\pi z'}{\ell} \sum_{n=-\infty}^{\infty} \frac{K_n(q_n r') F_n(r q_n, a q_n)}{K_n(q_n a)} \cos n(\theta - \theta'), \tag{31}$$

when $r < r'$.

And $$v = \frac{1}{\pi \ell} \sum_{n=1}^{\infty} e^{-\kappa n^2\pi^2 t/\ell^2} \sin \frac{n\pi z}{\ell} \sin \frac{n\pi z'}{\ell} \sum_{n=-\infty}^{\infty} \cos n(\theta - \theta') \int_0^{\infty} \frac{\alpha e^{-\kappa\alpha^2 t} U_n(\alpha r) U_n(\alpha r')}{J_n^2(\alpha a) + Y_n^2(\alpha a)} d\alpha \tag{32}$$

for $r \gtrless r'$.

8. <u>The region bounded by $r = a$ and $r = b$, $a < b$, and by the planes $z = 0$, $z = \ell$. Source at (r', θ', z').</u>

Here

$$\bar{v} = \frac{1}{\pi\kappa\ell} \sum_{m=1}^{\infty} \sin\frac{m\pi z}{\ell} \sin\frac{m\pi z'}{\ell} \sum_{n=-\infty}^{\infty} \frac{F_n(aq_m, rq_m)F_n(bq_m, r'q_m)}{F_n(aq_m, bq_m)} \cos n(\theta - \theta'), \qquad (33)$$

$a < r < r'$.

$$v = \frac{\pi}{2\ell} \sum_{m=1}^{\infty} e^{-\kappa m^2\pi^2 t/\ell^2} \sin\frac{m\pi z}{\ell} \sin\frac{m\pi z'}{\ell} \sum_{n=-\infty}^{\infty} \cos n(\theta - \theta_0) \sum_{\alpha} e^{-\kappa\alpha^2 t} \frac{\alpha^2 J_n^2(\alpha b) U_n(\alpha r) U_n(\alpha r')}{J_n^2(\alpha a) - J_n^2(\alpha b)}, \qquad (34)$$

for $r \gtrless r'$, the summation being over the positive roots of $U_n(\alpha b) = 0$.

9. <u>The region bounded by the planes $\theta = 0$, $\theta = \theta_0$. Source at $(r', \theta', 0)$.</u>

Here we use a result of Dougall[*]:-

<u>If r, r', and λ are real and positive and</u> $0 < \theta - \theta' < 2\pi$, $r > r'$,

$$K_0(\lambda R) = P\int_{-\infty i}^{\infty i} \frac{\cos\nu(\pi - \theta + \theta')}{\sin\nu\pi} K_\nu(\lambda r) I_\nu(\lambda r')\, i\, d\nu ,$$

where P implies that the principal value at the origin is taken.

Thus using (12) we have when $\theta_0 > \theta > \theta'$, $r > r'$,

$$\bar{u} = \frac{i}{2\pi^2\kappa} \int_0^{\infty} \cos\xi z \left[P\int_{-\infty i}^{\infty i} \frac{\cos\nu(\pi - \theta + \theta')}{\sin\nu\pi} I_\nu(\eta r') K_\nu(\eta r)\, d\nu \right] d\xi .$$

This gives

$$\bar{w} = -\frac{i}{2\pi^2\kappa} \int_0^{\infty} \cos\xi z \left[P\int_{-\infty i}^{\infty i} \frac{\cos\nu(\pi - \theta_0 + \theta')\sin\nu\theta + \cos\nu(\pi - \theta')\sin\nu(\theta_0 - \theta)}{\sin\nu\pi\,\sin\nu\theta_0} I_\nu(\eta r') K_\nu(\eta r)\, d\nu \right] d\xi$$

Since this satisfies the equation for $\bar{w}$ and makes $\bar{u} + \bar{w}$ zero on $\theta = 0$ and $\theta = \theta_0$.

Also we have

$$\bar{v} = -\frac{i}{\pi^2\kappa} \int_0^{\infty} \cos\xi z \left[\int_{-\infty i}^{\infty i} \frac{\sin\nu\theta' \sin\nu(\theta_0 - \theta)}{\sin\nu\theta_0} I_\nu(\eta r') K_\nu(\eta r)\, d\nu \right] d\xi ,$$

since the path can now be completed at the origin.

[*] Gray and Mathews, <u>Treatise on Bessel Functions</u>, Ed. 2, p. 101.

The inner integral is evaluated by completing the path $(-\infty i, \infty i)$ by a large semi-circle in the right hand half plane and evaluating the residues at the poles $n\pi/\theta_0$ of the integrand. We shall write throughout the sequel

$$s = n\pi/\theta_0 \qquad (35)$$

and $\sum_s$ will imply a summation over the values $n = 1$ to ∞. Then we obtain for $r > r'$ and $0 < \theta < \theta_0$

$$\bar{v} = \frac{2}{\pi\kappa\theta_0} \sum_s \sin s\theta \sin s\theta' \int_0^\infty \cos \xi z \, I_s(\eta r') K_s(\eta r) \, d\xi . \qquad (36)$$

But for $r > r'$

$$I_s(qr') K_s(qr) = \kappa \int_0^\infty e^{-pt} dt \int_0^\infty \alpha e^{-\kappa\alpha^2 t} J_s(\alpha r) J_s(\alpha r') \, d\alpha .$$

Hence we find, for $r \gtrless r'$ and $0 < \theta < \theta_0$

$$v = \frac{1}{\theta_0(\pi\kappa t)^{\frac{1}{2}}} e^{-z^2/4\kappa t} \sum_s \sin s\theta \sin s\theta' \int_0^\infty \alpha e^{-\kappa\alpha^2 t} J_s(\alpha r) J_s(\alpha r') d\alpha . \qquad (37)$$

10. The region bounded by $r = a$, externally, and by the planes $\theta = 0$, $\theta = \theta_0$. Source at $(r', \theta', 0)$.

$$\bar{v} = \frac{2}{\pi\kappa\theta_0} \int_0^\infty \cos \xi z \left\{ \sum_s \sin s\theta \sin s\theta' \frac{I_s(\eta r') F_s(\eta a, \eta r)}{I_s(\eta a)} \right\} d\xi , \quad r > r'. \qquad (38)$$

$$v = \frac{2e^{-z^2/4\kappa t}}{a^2\theta_0(\pi\kappa t)^{\frac{1}{2}}} \sum_s \sin s\theta \sin s\theta' \sum_\alpha \frac{J_s(\alpha r) J_s(\alpha r') e^{-\kappa\alpha^2 t}}{\left[J_s'(\alpha a)\right]^2} , \qquad (39)$$

where $\sum_\alpha$ implies a summation over the positive roots of $J_s(\alpha a) = 0$.

11. The region bounded by $r = a$, internally, and by the planes $\theta = 0$, $\theta = \theta_0$. Source at $(r', \theta', 0)$.

$$\bar{v} = \frac{2}{\pi\kappa\theta_0} \int_0^\infty \cos \xi z \left\{ \sum_s \sin s\theta \sin s\theta' \frac{K_s(\eta r') F_s(\eta r, \eta a)}{K_s(\eta a)} \right\} d\xi , \quad r < r'. \qquad (40)$$

$$v = \frac{e^{-z^2/4\kappa t}}{\theta_o(\pi\kappa t)^{\frac{1}{2}}} \sum_s \sin s\theta \sin s\theta' \int_0^\infty \frac{\alpha e^{-\kappa\alpha^2 t} U_s(\alpha r)U_s(\alpha r')d\alpha}{J_s^2(\alpha a) + Y_s^2(\alpha a)}, \qquad (41)$$

where $U_s(\alpha r)$ is defined in (24).

12. <u>The region $a < r < b$, $0 < \theta < \theta_o$. Source at $(r', \theta', 0)$.</u>

$$\bar{v} = \frac{2}{\pi\kappa\theta_o} \int_0^\infty \cos\xi z \left\{ \sum_s \sin s\theta \sin s\theta' \frac{F_s(\eta a, \eta r)F_s(\eta b, \eta r')}{F_s(\eta a, \eta b)} \right\} d\xi, \quad a < r < r'. \qquad (42)$$

$$v = \frac{\pi^2 e^{-z^2/4\kappa t}}{2\theta_o(\pi\kappa t)^{\frac{1}{2}}} \sum_s \sin s\theta \sin s\theta' \sum_\alpha \frac{\alpha^2 e^{-\kappa\alpha^2 t} J_s^2(b\alpha)U_s(\alpha r')}{J_s^2(\alpha a) - J_s^2(\alpha b)}, \qquad (43)$$

the summation in α being over the positive roots of $U_s(\alpha b) = 0$.

13. <u>The region $0 < z < \ell$, $0 < \theta < \theta_o$. Source at (r', θ', z').</u>

$$\bar{v} = \frac{4}{\kappa\ell\theta_o} \sum_{n=1}^\infty \sin\frac{n\pi z}{\ell} \sin\frac{n\pi z'}{\ell} \sum_s \sin s\theta \sin s\theta' I_s(r'q_n)K_s(rq_n), \quad r > r', \qquad (44)$$

where $q_n = \sqrt{(q^2 + n^2\pi^2/\ell^2)}$.

$$v = \frac{4}{\ell\theta_o} \sum_{n=1}^\infty e^{-\kappa n^2\pi^2 t/\ell^2} \sin\frac{n\pi z}{\ell} \sin\frac{n\pi z'}{\ell} \sum_s \sin s\theta \sin s\theta' \int_0^\infty \alpha e^{-\kappa\alpha^2 t} J_s(\alpha r)J_s(\alpha r')d\alpha \qquad (45)$$

14. <u>The region $0 < z < \ell$, $0 < \theta < \theta_o$, $0 \le r < a$. Source at (r', θ', z').</u>

$$\bar{v} = \frac{4}{\kappa\ell\theta_o} \sum_{n=1}^\infty \sin\frac{n\pi z}{\ell} \sin\frac{n\pi z'}{\ell} \sum_s \sin s\theta \sin s\theta' \frac{I_s(r'q_n)F_s(aq_n, rq_n)}{I_s(aq_n)},$$

$$r' < r < a, \qquad (46)$$

$$v = \frac{8}{a^2\ell\theta_o} \sum_{n=1}^\infty e^{-\kappa n^2\pi^2 t/\ell^2} \sin\frac{n\pi z}{\ell} \sin\frac{n\pi z'}{\ell} \sum_s \sin s\theta \sin s\theta' \sum_\alpha \frac{e^{-\kappa\alpha^2 t} J_s(\alpha r)J_s(\alpha r')}{[J_s'(\alpha a)]^2}$$

where the α are the positive roots of $J_s(\alpha a) = 0$.

15. The region $0 < z < \ell$, $0 < \theta < \theta_0$, $r > a$. Source at (r', θ', z').

$$\bar{v} = \frac{4}{\kappa \ell \theta_0} \sum_{n=1}^{\infty} \sin\frac{n\pi z}{\ell} \sin\frac{n\pi z'}{\ell} \sum_s \sin s\theta \sin s\theta' \frac{K_s(r'q_n)F_s(rq_n, aq_n)}{K_s(aq_n)}, \quad a < r < r', \tag{48}$$

$$v = \frac{4}{\ell\theta_0} \sum_{n=1}^{\infty} e^{-\kappa n^2\pi^2 t/\ell^2} \sin\frac{n\pi z}{\ell} \sin\frac{n\pi z'}{\ell} \sum_s \sin s\theta \sin s\theta' \int_0^{\infty} \frac{\alpha e^{-\kappa\alpha^2 t} U_s(\alpha r)U_s(\alpha r')\,d\alpha}{J_s^2(\alpha a) + Y_s^2(\alpha a)} \tag{49}$$

16. The region $0 < z < \ell$, $0 < \theta < \theta_0$, $a < r < b$. Source at (r', θ', z').

$$\bar{v} = \frac{4}{\kappa \ell \theta_0} \sum_{n=1}^{\infty} \sin\frac{n\pi z}{\ell} \sin\frac{n\pi z'}{\ell} \sum_s \sin s\theta \sin s\theta' \frac{F_s(aq_n, rq_n)F_s(bq_n, r'q_n)}{F_s(aq_n, bq_n)},$$

$$a < r < r'. \tag{50}$$

$$v = \frac{2\pi^2}{\ell\theta_0} \sum_{n=1}^{\infty} e^{-\kappa n^2\pi^2 t/\ell^2} \sin\frac{n\pi z}{\ell} \sin\frac{n\pi z'}{\ell} \sum_s \sin s\theta \sin s\theta' \sum_{\alpha} \frac{\alpha^2 e^{-\kappa\alpha^2 t} J_s^2(\alpha b) U_s(\alpha r)U_s(\alpha r')}{J_s^2(\alpha a) - J_s^2(\alpha b)} \tag{51}$$

where the α are the positive roots of $U_s(\alpha b) = 0$.

APPENDIX I.

Verification of the solutions of a number of problems obtained by the Laplace transformation method.

1. It was remarked in the Introduction that one of the advantages of the Laplace transformation method was that it solved very simply certain types of problem under assumptions which might well be regarded as adequate on physical grounds for problems in applied mathematics. To make the solution completely rigorous it is necessary to verify that it does satisfy the conditions of the problem.

For ordinary linear differential equations with constant coefficients a general verification process* can be given covering all cases (with one exception). For partial differential equations no such general process has been found. For problems in Conduction of Heat involving one space variable, such as flow in slabs, cylinders, and spheres, the method of Chapter I is available. A discussion of of this method for the cylinder $0 \leqslant r < a$ with constant surface temperature and zero initial temperature has also been given elsewhere.+ The process is tedious and Theorem 2 of Chapter I was introduced to save as much repetition as possible. It is necessary to prove an order result for each problem from which by Theorem 2 most of the verification can be done; certain details have to be discussed independently for each problem (it may be remarked than an extension of Theorem 2 would cover many of these). To illustrate the method

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* Carslaw and Jaeger, "Operational Methods in Applied Mathematics" (Oxford, in press), § 34 and 35. Doetsch, loc. cit., Chap. 18.
+ Proc. Lond. Math. Soc. 46 (1940) 361 (§ 4).

in this Appendix verifications are given for a complete set of problems[*] for the cylindrical regions $0 \leqslant r < a$, $a < r < b$ and $r > a$ with boundary conditions

$$k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v = k_4 \, ,$$

k_1, k_2, k_3, k_4 being real constants, at a surface. The problems considered are for each region (i) ~~constant surface temperature and~~ zero initial temperature, (ii) ~~constant surface temperature and~~ constant initial temperature, (iii) the instantaneous cylindrical surface source. The complete set of verifications for the general boundary condition above has been given to make it clear that no exceptional cases arise. They include as particular cases Chapter II §§ 3 - 5, Chapter IV §§ 1 - 5, and most of the problems in Proc. London Math. Soc., loc. cit. Some remarks on the remaining problems of Chapter II and those of Chapter III are made in § 11.

2. Notation.

We write

$$f(\lambda) = (k_1\lambda + k_3)I_0(\mu a) + k_2\mu I_1(\mu a) \qquad \ldots\ldots\ (1)$$

$$g(\lambda) = (k_1\lambda + k_3)K_0(\mu a) - k_2\mu K_1(\mu a) \qquad \ldots\ldots\ (2)$$

$$F(\lambda) = (k_1'\lambda + k_3')I_0(\mu b) + k_2'\mu I_1(\mu b) \qquad \ldots\ldots\ (3)$$

$$G(\lambda) = (k_1'\lambda + k_3')K_0(\mu b) - k_2'\mu K_1(\mu b) \qquad \ldots\ldots\ (4)$$

where $\mu = \sqrt{(\lambda/\kappa)}$.

C will be used for any positive constant. $\rho_1, \rho_2, \ldots$ for fixed values of ρ.

[*] Cf. Radial Heat Flow in Circular Cylinders with a General Boundary Condition: Journ. & Proc. Roy. Soc. N.S.W. LXXIV (1940) 342. This paper will be referred to as R.H.F.

3. The region[*] $0 \leqslant r < a$. Zero initial temperature. Boundary Condition at $r = a$,

$$k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v = k_4 , \quad t > 0. \qquad \ldots\ldots\ldots (5)$$

Here

$$\bar{v} = \frac{k_4 I_0(qr)}{p\left\{ (k_1 p + k_3) I_0(aq) + k_2 q I_1(aq) \right\}}, \text{ where } q = \sqrt{(p/\kappa)}.$$

Thus, with the notation of (1), the solution obtained formally by application of the inversion theorem is

$$v = \frac{k_4}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\lambda t}\, I_0(\mu r)\, d\lambda}{f(\lambda)} . \qquad \ldots\ldots\ldots (6)$$

From the asymptotic expansions of the Bessel functions it follows that when

$$\lambda = \kappa \rho e^{i\theta} , \quad \pi > \theta_0 \geqslant \theta \geqslant 0 \qquad \ldots\ldots (7)$$

$$|f(\lambda)| > C \rho^{\alpha} \exp\left[a \rho^{\frac{1}{2}} \cos \tfrac{1}{2}\theta\right], \quad \text{if} \quad \rho > \rho_0 \qquad \ldots\ldots (8)$$

where α is 3/4, 1/4, or − 1/4 according as $k_1 \neq 0$; $k_1 = 0$, $k_2 \neq 0$; or $k_1 = k_2 = 0$; respectively.

Also since[+]

$$|I_0(z)| \leqslant \exp|R(z)| \qquad \ldots\ldots (9)$$

we have when $\lambda = \kappa\rho e^{i\theta}$, $\pi > \theta_0 \geqslant \theta \geqslant 0$

$$|I_0(\mu r)| < C \exp\left[r \rho^{\frac{1}{2}} \cos \tfrac{1}{2}\theta\right]$$

Thus $$\left| \frac{I_0(\mu r)}{f(\lambda)} \right| < C \rho^{-\alpha} \exp\left[-(a-r)\rho^{\frac{1}{2}} \cos \tfrac{1}{2}\theta\right], \quad \rho > \rho_0, \; 0 \leqslant r \leqslant a.$$

$$\ldots\ldots (10)$$

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[*] Chapter IV, § 2. R.H.F. § 2.

[+] W.B.F. § 3.31.

Similarly

$$\left| \frac{\partial}{\partial r} \frac{I_0(\mu r)}{f(\lambda)} \right| < rC\rho^{1-\alpha} \exp\left[-(a-r)\rho^{\frac{1}{2}} \cos \tfrac{1}{2}\theta\right], \quad \rho > \rho_0, \; 0 \leq r \leq a \quad \ldots (11)$$

and

$$\left| \frac{\partial^2}{\partial r^2} \frac{I_0(\mu r)}{f(\lambda)} \right| < C \rho^{1-\alpha} (1 + \tfrac{1}{2}\rho r^2) \exp\left[-(a-r)\rho^{\frac{1}{2}} \cos \tfrac{1}{2}\theta\right], \quad \rho > \rho_0, \; 0 \leq r \leq a \quad \ldots (12)$$

where α has the values 3/4, 1/4 or -1/4.

Thus in all cases the integrand of (6) satisfies the conditions* of Chapter I, Theorem 2.

It follows immediately that

$$v = \frac{k_4}{2\pi i} \int_{L'} \frac{e^{\lambda t} I_0(\mu r) d\lambda}{\lambda f(\lambda)}, \quad \ldots\ldots\ldots\ldots (13)$$

when $t \geq 0$, $0 \leq r < a$ or $t > 0$, $0 \leq r \leq a$,

that $\lim_{t \to 0} v = 0$, for fixed r in $0 \leq r < a$, and that v satisfies its differential equation.

It remains to verify that the boundary condition (5) is satisfied, namely that, for fixed $t > 0$,

$$\lim_{r \to a} \left\{ k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v \right\} = k_4 \quad \ldots\ldots\ldots\ldots (14)$$

We take v in the form (13) and observe that by Theorem 2 we may differentiate under the integral sign with respect to r in $0 \leq r \leq a$ for fixed $t > 0$, and with respect+ to t for fixed $t > 0$ and $0 \leq r \leq a$.

Thus

$$k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v = \frac{k_4}{2\pi i} \int_{L'} \frac{e^{\lambda t} \{(k_1\lambda + k_3) I_0(\mu r) + k_2 \mu I_1(\mu r)\} d\lambda}{\lambda f(\lambda)}$$

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* These are taken to include, here and subsequently, those of IV of that Theorem.

+ This result follows from the proof of Theorem 2, II. It was not given in the original statement of that Theorem.

and by II and IV of Theorem 2 this integral is uniformly convergent with respect to r in $0 \leq r \leq a$ for fixed $t > 0$. Thus

$$\lim_{r \to a} \left(k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v\right) = \frac{k_4}{2\pi i} \int_{L'} \frac{e^{\lambda t} d\lambda}{\lambda} = k_4$$

4. The region* $0 \leq r < a$. Unit initial temperature. Boundary condition at $r = a$

$$k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v = k_4, \quad t > 0. \qquad \text{.........} \quad (15)$$

Here the solution of the subsidiary equation and its boundary conditions is

$$\bar{v} = \frac{1}{p} + \frac{[k_4 - k_3] I_0(qr)}{p\left\{(k_1 p + k_3) I_0(qa) + k_2 q I_1(qa)\right\}} . \qquad \text{.........} \quad (16)$$

And thus+

$$v = 1 + \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\lambda t} [k_4 - k_3] I_0(\mu r) d\lambda}{\lambda\{(k_1\lambda + k_3) I_0(\mu a) + k_2 \mu I_1(\mu a)\}}, \quad t > 0 \quad .. \quad (17)$$

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* Chapter IV, § 3.

+ Applying the inversion theorem to $1/p$ gives (in its usual sense as a principal value integral)

$$\left.\begin{aligned} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\lambda t} d\lambda}{\lambda} &= 1, \quad t > 0 \\ &= \tfrac{1}{2}, \quad t = 0 \\ &= 0, \quad t < 0 \end{aligned}\right\} \qquad \text{.....} \quad (A)$$

The integrand of this line integral is not of the type contemplated in Theorem 2, but the same argument shows that the line integral in (A) may be replaced by the path L' if $t > 0$.

It follows from (10), (11), (12) that in all cases the integrand of the line integral in (17) satisfies the conditions of Theorem 2. It follows that the path can be deformed into L', that v satisfies its differential equation, and that $\lim_{t \to 0} v = 1$.

Also, as in § 3, for fixed $t > 0$,

$$k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v = k_3 + \frac{1}{2\pi i} \int_{L'} \frac{(k_4 - k_3)\left[(k_1\lambda + k_3) I_0(\mu r) + k_2 \mu I_1(\mu r)\right] e^{\lambda t} d\lambda}{\lambda f(\lambda)}$$

and

$$\lim_{r \to a} \left(k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v\right) = k_3 + \frac{1}{2\pi i} \int_{L'} \frac{(k_4 - k_3) e^{\lambda t} d\lambda}{\lambda}$$
$$= k_4 .$$

5. <u>The instantaneous cylindrical surface source at</u> $t = 0$ <u>over</u> $r = r'$ <u>in the region</u> $0 \leqslant r < a$ <u>with boundary condition* at</u> $r = a$

$$k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v = 0 , \quad t > 0. \qquad \text{.......... (18)}$$

The solution for the instantaneous cylindrical surface source at $t = 0$ over $r = r'$ in infinite medium was

$$u = \frac{Q}{4\pi\kappa t} e^{-\frac{r^2 + r'^2}{4\kappa t}} I_0\left(\frac{rr'}{2\kappa t}\right) , \quad t > 0 , \qquad \text{........ (19)}$$

and

$$\left.\begin{aligned} \bar{u} &= \frac{Q}{2\pi\kappa} I_0(qr')K_0(qr) , & r \geqslant r' \\ &= \frac{Q}{2\pi\kappa} I_0(qr)K_0(qr') , & r \leqslant r' \end{aligned}\right\} \qquad \text{........ (20)}$$

The method of solution consisted in determining a function w which satisfied

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{1}{\kappa}\frac{\partial w}{\partial t} = 0 , \quad 0 \leqslant r < a, \quad t > 0 \qquad \text{....... (21)}$$

and

$$\lim_{t \to 0} w = 0 , \quad 0 \leqslant r < a , \qquad \text{....... (22)}$$

and such that $v = u + w$ satisfies the boundary condition (18).

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* R.H.F., § 3.

It is found that

$$w = -\frac{Q}{4\pi^2 i\kappa}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{I_0(\mu r')I_0(\mu r)g(\lambda)e^{\lambda t}\,d\lambda}{f(\lambda)} \quad \ldots\ldots\ldots (23)$$

and

$$v = -\frac{Q}{4\pi^2 i\kappa}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{I_0(\mu r')\left\{I_0(\mu r)g(\lambda) - K_0(\mu r)f(\lambda)\right\}e^{\lambda t}\,d\lambda}{f(\lambda)}, \quad \ldots (24)$$

when $r' \leqslant r < a$.

Now when $\lambda = \kappa\rho\, e^{i\theta}$, $\pi > \theta_0 \geqslant \theta \geqslant 0$,

$$\left|\frac{I_0(\mu r')I_0(\mu r)g(\lambda)}{f(\lambda)}\right| < C\rho^{-1/4}\exp\left[(r+r'-2a)\rho^{\frac{1}{2}}\cos\tfrac{1}{2}\theta\right], \quad \rho > \rho_0 \quad \ldots (25)$$

$$\left|\frac{I_0(\mu r')\left\{I_0(\mu r)g(\lambda) - K_0(\mu r)f(\lambda)\right\}}{f(\lambda)}\right| < C\rho^{-1/2}\exp\left[(r'-r)\rho^{\frac{1}{2}}\cos\tfrac{1}{2}\theta\right],$$

$$r' \leqslant r \leqslant a, \quad \rho > \rho_1, \quad \ldots (26)$$

with similar results for the derivatives.

It follows from (25) and Theorem 2 that w satisfies (21) and (22). Also from (26) that the path of integration in (24) may be deformed into L' and that the integral over L' may be differentiated under the integral sign with respect to r and t for $r' < r \leqslant a$ and fixed $t > 0$. Thus

$$k_1\frac{\partial v}{\partial t} + k_2\frac{\partial v}{\partial r} + k_3 v = -\frac{Q}{4\pi^2 i\kappa}\int_{L'}\frac{I_0(\mu r')\left\{\begin{array}{l}\left[(k_1\lambda+k_3)I_0(\mu r)+k_2\mu I_1(\mu r)\right]g(\lambda)\\ -\left[(k_1\lambda+k_3)K_0(\mu r)-k_2\mu K_1(\mu r)\right]f(\lambda)\end{array}\right\}}{f(\lambda)}e^{\lambda t}d\lambda$$

and the integral is uniformly convergent with respect to r in $r' < r \leqslant a$ for fixed $t > 0$. Therefore

$$\lim_{r\to a}\left(k_1\frac{\partial v}{\partial t} + k_2\frac{\partial v}{\partial r} + k_3 v\right) = 0.$$

Since we have always used the inversion theorem purely formally and not established conditions for its validity, it is necessary to complete the proof to show that the application of the inversion theorem to (20) gives (19).

Consider the region $0 \leq r \leq r'$, we have

$$\frac{Q}{4\pi^2 i\kappa} \int_{\gamma - i\infty}^{\gamma + i\infty} I_0(\mu r) K_0(\mu r') e^{\lambda t}\, d\lambda$$

and on $\lambda = \kappa\rho\, e^{i\theta}$

$$|I_0(\mu r) K_0(\mu r')| < c\,\rho^{-1/4} \exp\left[-(r' - r)\,\rho^{\frac{1}{2}} \cos \tfrac{1}{2}\theta\right], \quad 0 \leq r \leq r', \quad \rho > \rho_0$$

Thus by Theorem 1 (footnote) the integrals over the arcs BB'F and AA'C of Fig. 1 tend to zero as $\rho \to \infty$ for $t > 0$, $0 \leq r \leq r'$.

Therefore

$$\frac{Q}{4\pi^2 i\kappa} \int_{\gamma - i\infty}^{\gamma + i\infty} I_0(\mu r) K_0(\mu r') e^{\lambda t}\, d\lambda$$

$$= -\frac{2\kappa Q}{4\pi^2 i\kappa} \int_0^\infty e^{-\kappa u^2 t}\, u\, J_0(ur)\left[K_0(iur') - K_0(-iur')\right] du$$

$$= \frac{Q}{2\pi} \int_0^\infty e^{-\kappa u^2 t} J_0(ur) J_0(ur')\, u\, du$$

$$= \frac{Q}{4\pi\kappa t} \exp\left(-\frac{r' + r'^2}{4\kappa t}\right) I_0\left(\frac{rr'}{2\kappa t}\right), \quad t > 0, \quad 0 \leq r \leq r'.$$

The proof for the other range is similar.

6. <u>The region $r > a$. Zero initial temperature*. Boundary condition at</u> $r = a$

$$k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v = k_4 \ , \qquad t > 0. \qquad \ldots\ldots\ldots (27)$$

Here
$$v = \frac{k_4}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\lambda t} K_0(\mu r) d\lambda}{\lambda g(\lambda)} \qquad \ldots\ldots\ldots (28)$$

From the asymptotic expansions it follows that for $\lambda = \kappa \rho e^{i\theta}$, $\pi > \theta_0 \geqslant \theta \geqslant 0$

$$\left| \frac{K_0(\mu r)}{g(\lambda)} \right| < C \rho^{\alpha} \exp\left[-(r - a) \rho^{\frac{1}{2}} \cos \tfrac{1}{2}\theta \right] , \qquad \rho > \rho_0 \qquad \ldots (29)$$

where $\alpha = -1, -\frac{1}{2}$ or 0 according as $k_1 \neq 0$; $k_1 = 0, k_2 \neq 0$; or $k_1 = k_2 = 0$ respectively. The derivatives satisfy similar conditions.

Thus in all cases the conditions of Theorem II are satisfied and it follows that the path can be deformed into L', that v satisfies the differential equation, and that $\lim_{t \to 0} v = 0$. It is verified as in § 3 that the boundary condition (27) is satisfied.

The remaining condition

$$\lim_{r \to \infty} v = 0 \ , \quad \text{for fixed } t > 0,$$

follows from Theorem 2, V.

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* R.H.F. § 5.

7. The region $r > a$. Unit initial temperature*. Boundary condition at $r = a$

$$k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v = k_4 , \quad t > 0 . \qquad \ldots\ldots\ldots (30)$$

Here $$\bar{v} = \frac{1}{p} + \frac{(k_4 - k_3)K_0(qr)}{p\left\{(k_1 p + k_3)K_0(qa) - k_2 q\, K_1(qa)\right\}} \qquad \ldots\ldots\ldots (31)$$

and $$v = 1 + \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{(k_4 - k_3)K_0(\mu r)e^{\lambda t}\, d\lambda}{g(\lambda)} , \quad t > 0. \qquad \ldots (32)$$

Now for $\lambda = \kappa\rho\, e^{i\theta}$, $\pi > \theta_0 \geqslant \theta \geqslant 0$,

$$\left|\frac{(k_4 - k_3)K_0(\mu r)}{g(\lambda)}\right| < C\rho^{\alpha} \exp\left[-(r-a)\,\rho^{\frac{1}{2}} \cos \tfrac{1}{2}\theta\right] , \quad \rho > \rho_0$$

where $\alpha = -1, -\frac{1}{2}, 0$ according as $k_1 \neq 0$; $k_1 = 0, k_2 \neq 0$; $k_1 = k_2 = 0$, respectively. The derivatives satisfy similar conditions. Thus the integrand of the line integral in (32) satisfies the conditions of Theorem 2 and it follows as before that the differential equation and boundary condition is satisfied. Also that $\lim_{t \to 0} v = 1$, $r > a$. Also it follows from V, Theorem 2, that

$$\lim_{r \to \infty} v = 1 , \quad \text{for fixed } t > 0.$$

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* Chapter IV, § 1.

8. <u>An instantaneous cylindrical surface source at</u> $t = 0$ <u>over</u> $r = r'$ <u>in the region</u> $r > a$ <u>with boundary condition at</u> $r = a$

$$k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v = 0 , \quad t > 0. \quad \ldots\ldots\ldots (33)$$

Here

$$v = \frac{Q}{4\pi^2 i\kappa} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\lambda t} K_0(\mu r') \{ I_0(\mu r) g(\lambda) - K_0(\mu r) f(\lambda) \} \, d\lambda}{g(\lambda)} ,$$

$$a < r \leqslant r'. \quad \ldots (34)$$

And $$w = - \frac{Q}{4\pi^2 i\kappa} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{K_0(\mu r') K_0(\mu r) f(\lambda) d\lambda}{g(\lambda)} . \quad \ldots\ldots (35)$$

Now when $\lambda = \kappa \rho e^{i\theta}$, $\pi > \theta_0 \geqslant \theta \geqslant 0$,

$$\left| \frac{K_0(\mu r') \{ I_0(\mu r) g(\lambda) - K_0(\mu r) f(\lambda) \}}{g(\lambda)} \right| < C \rho^{-\frac{1}{2}} \exp\left[-(r' - r) \rho^{\frac{1}{2}} \cos \tfrac{1}{2}\theta \right],$$

$$a \leqslant r \leqslant r', \quad \rho > \rho_0, \quad \ldots (36)$$

and $$\left| \frac{K_0(\mu r') K_0(\mu r) f(\lambda)}{g(\lambda)} \right| < C \rho^{-\frac{1}{2}} \exp\left[-(r + r' - 2a) \rho^{\frac{1}{2}} \cos \tfrac{1}{2}\theta \right], \quad \rho > \rho_1$$

$$\ldots (37)$$

with similar results for the derivatives.

Thus the integrands of w and v satisfy the conditions of Theorem 2 and it follows that w satisfies

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} = \frac{1}{\kappa} \frac{\partial w}{\partial t} , \quad r > a, \quad t > 0$$

$$\lim_{t \to 0} w = 0 , \quad r > a$$

and $$\lim_{t \to \infty} w = 0 , \quad t > 0$$

Also, as in § 5, that v satisfies (33)

* R.H.F., § 6.

9. The hollow cylinder[*] $a < r < b$. Zero initial temperature.

Boundary conditions

$$k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v = k_4 , \quad r = a, \quad t > 0 \qquad \ldots\ldots\ldots (38)$$

$$k_1' \frac{\partial v}{\partial t} + k_2' \frac{\partial v}{\partial r} + k_3' v = 0 , \quad r = b, \quad t > 0. \qquad \ldots\ldots\ldots (39)$$

The "k_4'" in 39 is taken zero for shortness. This entails no loss of generality.

Here

$$v = \frac{k_4}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\{G(\lambda) I_o(\mu r) - F(\lambda) K_o(\mu r)\} e^{\lambda t} \, d\lambda}{\lambda \{f(\lambda) G(\lambda) - g(\lambda) F(\lambda)\}} \qquad \ldots\ldots (40)$$

Now on $\lambda = \kappa \rho e^{i\theta}$, $\pi > \theta_o \geqslant \theta \geqslant 0$

$$\left| \frac{G(\lambda) I_o(\mu r) - F(\lambda) K_o(\mu r)}{f(\lambda) G(\lambda) - g(\lambda) F(\lambda)} \right| < C \rho^{\alpha} \exp\left[-(r-a) \rho^{\frac{1}{2}} \cos \tfrac{1}{2}\theta\right], \rho > \rho_s, \ldots (41)$$

where $\alpha = -1, -\frac{1}{2}, 0$ according as $k_1 \neq 0$; $k_1 = 0, k_2 \neq 0$; $k_1 = k_2 = 0$. Thus in all cases the integrand of v satisfies the conditions of Theorem II and it follows that the path of integration may be deformed into L', that v satisfies its differential equation, and that

$\lim_{t \to o} v = 0$, $a < r < b$.

For the boundary conditions at $r = a$ and $r = b$ we have for fixed $t > 0$

$$k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v$$

$$= \frac{k_4}{2\pi i} \int_{L'} \frac{G(\lambda)\{(k_1\lambda + k_3) I_o(\mu r) + k_2 \mu I_1(\mu r)\} - F(\lambda)\{(k_1\lambda + k_3) K_o(\mu r) - k_2 \mu K_1(\mu r)\}}{\lambda \{f(\lambda) G(\lambda) - g(\lambda) F(\lambda)\}} e^{\lambda t} d\lambda$$

and the integral is uniformly convergent with respect to r in

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[*] Chapter IV, § 4. R.H.F., § 4.

$a \leq r \leq b$ for fixed $t > 0$. Thus

$$\lim_{r \to a} \left(k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v\right) = \frac{k_4}{2\pi i} \int_{L'} \frac{e^{\lambda t}\, d\lambda}{\lambda} = k_4 , \quad t > 0.$$

Also $\lim_{r \to b} \left(k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v\right) = 0, \quad t > 0.$

The solution of the corresponding problem with unit initial temperature follows from (36) along the lines of §§ 4, 7.

10. An instantaneous cylindrical surface source at $t = 0$ over $r = r'$ in the hollow cylinder $a < r < b$ with boundary conditions*

$$k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v = 0, \quad r = a, \quad t > 0 \qquad \ldots\ldots (42)$$

$$k_1' \frac{\partial v}{\partial t} + k_2' \frac{\partial v}{\partial r} + k_3' v = 0, \quad r = b, \quad t > 0. \qquad \ldots\ldots (43)$$

Here

$$v = \frac{Q}{4\pi^2 i\kappa} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\{I_0(\mu r)g(\lambda) - K_0(\mu r)f(\lambda)\}\{I_0(\mu r')G(\lambda) - K_0(\mu r')F(\lambda)\}\, e^{\lambda t} d\lambda}{f(\lambda)G(\lambda) - g(\lambda)F(\lambda)},$$

$$a < r \leq r' \qquad \ldots\ldots (44)$$

and if $r' \leq r < b$ we interchange r and r' in (44).

Also

$$w = -\frac{Q}{4\pi^2 i\kappa} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{I_0(\mu r)G(\lambda)[K_0(\mu r')f(\lambda) - I_0(\mu r')g(\lambda)] + K_0(\mu r)f(\lambda)[I_0(\mu r')G(\lambda) - K_0(\mu r')F(\lambda)]}{f(\lambda)G(\lambda) - g(\lambda)F(\lambda)} e^{\lambda t} d\lambda$$

$$\ldots\ldots (45)$$

On $\lambda = \kappa\rho e^{i\theta}$, $\pi > \theta_0 \geq \theta \geq 0$,

$$\left| \frac{[I_0(\mu r)g(\lambda) - K_0(\mu r)f(\lambda)][I_0(\mu r')G(\lambda) - K_0(\mu r')F(\lambda)]}{f(\lambda)G(\lambda) - g(\lambda)F(\lambda)} \right| < C\rho^{-\frac{1}{2}} \exp\left[-(r'-r)\rho^{\frac{1}{2}}\cos\tfrac{1}{2}\theta\right]$$

$$\rho > \rho_0, \quad a \leq r \leq r'$$

$$\ldots\ldots (46)$$

and if $r' \leq r \leq b$ we interchange r and r' in (46).

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* R.H.F. § 4.

Also

$$\left|\frac{I_o(\mu r)G(\lambda)\left[K_o(\mu r')f(\lambda) - I_o(\mu r')g(\lambda)\right] + K_o(\mu r)f(\lambda)\left[I_o(\mu r')G(\lambda) - K_o(\mu r')F(\lambda)\right]}{f(\lambda)G(\lambda) - g(\lambda)F(\lambda)}\right|$$

$$< C\rho^{-\frac{1}{2}} \exp\left[-(2b - r - r')\rho^{\frac{1}{2}} \cos \tfrac{1}{2}\theta\right], \ \rho > \rho_1$$

$$\text{or} \quad < C\rho^{-\frac{1}{2}} \exp\left[-(r + r' - 2a)\rho^{\frac{1}{2}} \cos \tfrac{1}{2}\theta\right], \ \rho > \rho_2 \qquad \ldots\ldots\ldots\ldots\ldots (47)$$

with similar results for the derivatives.

Thus the integrands of (44) and (45) satisfy the conditions of Theorem 2. It follows that the paths of integration may be deformed into L', that w satisfies its differential equation, and that

$$\lim_{t \to 0} w = 0.$$

Finally

$$\left(k_1\frac{\partial v}{\partial t} + k_2\frac{\partial v}{\partial r} + k_3 v\right) = \frac{Q}{4\pi^2 i\kappa}\int_{L'} \frac{\left[(k_1\lambda + k_3)I_o(\mu r) + k_2\mu I_1(\mu r)\right]g(\lambda) - \left[(k_1\lambda + k_3)K_o(\mu r) - k_2\mu K_1(\mu r)\right]f(\lambda)}{f(\lambda)G(\lambda) - g(\lambda)F(\lambda)}$$

$$\times \left\{I_o(\mu r')G(\lambda) - K_o(\mu r')F(\lambda)\right\} e^{\lambda t} d\lambda, \qquad a < r < r',$$

the integral is uniformly convergent with respect to r for $a \leq r < r'$ for fixed $t > 0$. Thus

$$\lim_{r \to a} \left(k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v\right) = 0.$$

Similarly the boundary condition (39) is satisfied.

11. It was remarked in § 1 that the verifications of this Appendix include the instantaneous cylindrical source problems of Chapter II §§ 3 - 5. The corresponding solutions of Chapter II for instantaneous plane and spherical sources may be dealt with in the same way; the work

is simpler as Bessel functions are not involved.

With regard to the problems of Chapter III the only new feature, the division into two media, introduces no difficulties. To illustrate this a brief discussion of the problem of Chapter III, § 3 is given.

On $\lambda = \kappa_1 \rho e^{i\theta}$, $\pi > \theta_0 \geqslant \theta \geqslant 0$ we find from Chapter III § 3 (32) and (3) in the notation of that Chapter

$$\left|\frac{K_1\mu_1 D_1(b\mu_1, r\mu_1) D(b\mu_2, c\mu_2) - K_2\mu_2 D(b\mu_1, r\mu_1) D_1(b\mu_2, c\mu_2)}{\Delta(\lambda)}\right|$$

$$< C \exp\left[\rho^{\frac{1}{2}}(a-r)\cos\tfrac{1}{2}\theta\right], \quad \rho > \rho_0, \quad a \leqslant r \leqslant b$$

and

$$\left|\frac{D(r\mu_2, c\mu_2)}{\Delta(\lambda)}\right| < C \exp\left[\rho^{\frac{1}{2}}\left\{-(b-a) - \kappa(r-b)\right\}\cos\tfrac{1}{2}\theta\right], \quad \rho > 1, \quad b \leqslant r \leqslant c.$$

Thus the integrands of the line integrals for v_1 and v_2 satisfy the conditions of Theorem 2. It follows that

$$\lim_{t \to 0} v_1 = 0, \quad \text{for fixed } r \text{ in } a < r \leqslant b$$

$$\lim_{t \to 0} v_2 = 0, \quad \text{for fixed } r \text{ in } b \leqslant r < c,$$

that v_1 and v_2 satisfy their differential equations, and that the paths L may be deformed into L'. Also by Theorem 2 II and IV

$$v_1 = \frac{V_0}{2\pi i}\int_{L'} \frac{e^{\lambda t}\left\{K_1\mu_1 D_1(b\mu_1, r\mu_1) D(b\mu_2, c\mu_2) - K_2\mu_2 D(b\mu_1, r\mu_1) D_1(b\mu_2, c\mu_2)\right\}}{\lambda\Delta(\lambda)}\, d\lambda$$

and $v_2 = \dfrac{V_0}{2\pi i}\displaystyle\int_{L'} \frac{e^{\lambda t}\, D(r\mu_2, c\mu_2)\, d\lambda}{\lambda\Delta(\lambda)}$

are uniformly convergent with respect to r in $a \leqslant r \leqslant b$ and $b \leqslant r \leqslant c$ respectively for fixed $t > 0$, and their derivatives with respect to r have the same property. Thus the conditions (20), (21), (22), (23) of Chapter III § 3 are satisfied.

APPENDIX II.

1. In this Appendix are discussed properties of the roots of certain equations required in the solution of problems* of Conduction of Heat in solids bounded internally or externally by circular cylinders, with the boundary condition

$$k_1 \frac{\partial v}{\partial t} + k_3 v + k_2 \frac{\partial v}{\partial r} = k_4 \qquad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

at a surface. k_1, k_2, k_3, k_4 are real constants.

In (1) we take $k_1 \geqslant 0$ and if $k_1 = 0$ we take $k_3 \geqslant 0$. This convention is assumed throughout.

It is assumed in this Appendix that two of k_1, k_2, k_3 are not zero; in that case the results are well known. κ, where it occurs, is a real positive constant.

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* Chapter IV, §§ 1 - 5. Also "Radial Heat Flow in Circular Cylinders with a general Boundary Condition", Journal and Proceedings, Royal Society, New South Wales, LXXIV (1940) 342; this paper will be referred to as R.H.F.

2. The roots of the equation[*]

$$(k_3 - k_1\kappa\alpha^2)J_o(a\alpha) - k_2\alpha\, J_1(a\alpha) = 0 \quad \ldots\ldots\ (1)$$

are all real and simple provided

$$k_1 \geqslant 0, \quad k_2 \geqslant 0, \quad k_3 \geqslant 0. \quad \ldots\ldots\ (2)$$

(i) Pure imaginary roots.

A pure imaginary root $\alpha = i\beta$ of (1) is a real zero of

$$(k_3 + k_1\kappa\beta^2)I_o(a\beta) + k_2\beta\, I_1(a\beta) \quad \ldots\ldots\ (3)$$

Now $I_o(z)$ and $I_1(z)$ are both positive for real positive z and thus, taking as in § 1 $k_1 \geqslant 0$, the expression (2) is certainly always positive if $\beta \geqslant 0$, $k_2 \geqslant 0$ and $k_3 \geqslant 0$.

Thus if $k_1 \geqslant 0$, $k_2 \geqslant 0$, $k_3 \geqslant 0$, (3) has no real positive zero. It is an even function so also has no real negative zero. These conditions are satisfied by the physical problems in Chapter IV. If they are not satisfied there may be real zeros of (3).

(ii) Complex roots.

If ξ and η are conjugate complex roots of (1) we have

$$(k_3 - k_1\kappa\xi^2)J_o(a\xi) + k_2\xi\, J_o'(a\xi) = 0$$

$$(k_3 - k_1\kappa\eta^2)J_o(a\eta) + k_2\eta\, J_o'(a\eta) = 0.$$

Thus

$$- k_1\kappa(\xi^2 - \eta^2)J_o(a\xi)J_o(a\eta) + k_2\left\{\xi\, J_o(a\eta)J_o'(a\xi) - \eta\, J_o'(a\eta)J_o(a\xi)\right\} = 0.$$

[*] Chap. IV, §§ 2, 3. R.H.F. §§ 2, 3.

Now[*]

$$a(\eta J_0(a\xi)J_0'(a\eta) - \xi J_0(a\eta)J_0'(a\xi)) = (\xi^2 - \eta^2)\int_0^a x J_0(\xi x)J_0(\eta x)dx$$

Therefore

$$ak_1\kappa(\eta^2 - \xi^2)J_0(a\xi)J_0(a\eta) + k_2(\eta^2 - \xi^2)\int_0^a xJ_0(\xi x)J_0(\eta x)dx = 0.$$

If $k_1 \geqslant 0$ and $k_2 \geqslant 0$ this is impossible and so there can be no complex root.

(iii) From the asymptotic expansions of the Bessel functions it follows that (1) has real roots, situated, for large n, near the points

$$\pm(n\pi - \tfrac{1}{4}\pi), \text{ if } k_1 \neq 0, \text{ or } k_1 = k_2 = 0$$
$$\pm(n\pi + \tfrac{1}{4}\pi), \text{ if } k_1 = 0, \; k_2 \neq 0.$$

(iv) <u>The equation (1) has no repeated roots (except possibly at $\alpha = 0$).</u>

This follows[+] from the fact that

$$y = (1 - Az^2)\, J_0(z) + Bz\, J_0'(z)$$

satisfies a linear second order differential equation, namely,

$$z\left[2AB + B^2 - A + A^2z^2\right]y'' + \left\{3A + B - 2AB - B^2 - 3A^2z^2\right\}y' + z\left\{4A^2 + 4AB + B^2 + A^2z^2\right\}y = 0.$$

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[*] G. & M., p. 69 (23).

[+] G. & M., p. 79. Th. I. A repeated zero is possible also at $Az = \pm\sqrt{(A - 2AB - B^2)}$.

3. To prove that[+]

$$(k_1\lambda + k_3)K_0(\mu a) - k_2\mu K_1(\mu a) \qquad \ldots\ldots\ldots \quad (1)$$

where $\mu = \sqrt{(\lambda/\kappa)}$ has no zero for $-\pi \leqslant \arg\lambda \leqslant \pi$ provided

$$k_1 \geqslant 0, \quad k_2 \leqslant 0, \quad k_3 \geqslant 0. \qquad \ldots\ldots\ldots\ldots \quad (2)$$

This is equivalent to proving that

$$(ak_3 + k_1\kappa z^2/a)K_0(z) + k_2 z K_0'(z) \qquad \ldots\ldots\ldots\ldots \quad (3)$$

has no zeros for $R(z) \geqslant 0$.

(i) The expression (3) has no zero for real positive z if

$$k_1 \geqslant 0, \quad k_3 \geqslant 0, \quad k_2 \leqslant 0.$$

This follows since $K_0(z) > 0$, $K_1(z) > 0$ for real positive z.

(ii) The expression (3) has no complex zero ξ. For if η is the conjugate of ξ we have

$$(ak_3 + k_1\kappa\xi^2/a)K_0(\xi) + k_2\xi K_0'(\xi) = 0$$

$$(ak_3 + k_1\kappa\eta^2/a)K_0(\eta) + k_2\eta K_0'(\eta) = 0$$

Thus $\frac{k_1\kappa}{a}(\xi^2 - \eta^2)K_0(\xi)K_0(\eta) + k_2\left\{\xi K_0'(\xi)K_0(\eta) - \eta K_0'(\eta)K_0(\xi)\right\} = 0$

Now[*] $\xi K_0'(\xi)K_0(\eta) - \eta K_0'(\eta)K_0(\xi) = (\eta^2 - \xi^2)\int_1^\infty xK_0(\xi x)K_0(\eta x)\,dx$

Therefore $(\xi^2 - \eta^2)\frac{k_1\kappa}{a}K_0(\xi)K_0(\eta) - k_2(\xi^2 - \eta^2)\int_1^\infty xK_0(\xi x)K_0(\eta x)\,dx = 0.$

Thus if $k_1 \geqslant 0$, $k_2 \leqslant 0$

we have a contradiction and so no complex zero is possible.

(iii) The expression (3) has no pure imaginary zero $z = iy$, for this implies

$$\tfrac{1}{2}\pi i(ak_3 - k_1\kappa y^2/a)\left\{-J_0(y) + iY_0(y)\right\} + \tfrac{1}{2}\pi i k_2 y\left\{-J_0'(y) + iY_0'(y)\right\} = 0$$

[+] The equation occurs in Chap. IV, § 1 and R.H.F. § 5.

[*] G. & M., p. 70 (30).

i.e. $(ak_3 - k_1\kappa y^2/a)J_0(y) + k_2yJ_0'(y) = 0$

and $(ak_3 - k_1\kappa y^2/a)Y_0(y) + k_2yY_0'(y) = 0.$

It follows that $J_0(y)Y_0'(y) - Y_0(y)J_0'(y) = 0,$

but this is equal to $2/(\pi y)$ and so we have a contradiction.

4. <u>The zeros of</u>+

$$\left[(k_1\kappa z^2 + k_3)I_0(az) + k_2zI_1(az)\right]\left[(k_1'\kappa z^2+k_3')K_0(bz) - k_2'zK_1(bz)\right]$$

$$-\left[(k_1'\kappa z^2+k_3')I_0(bz)+k_2'zI_1(bz)\right]\left[(k_1\kappa z^2+k_3)K_0(az) - k_2zK_1(az)\right],$$

where $b > a$. The k's and k''s are real constants, $\kappa > 0$. .... (1)

(i) <u>The expression (1) has no real positive zeros if</u>

$$k_1 \geqslant 0,\quad k_1' \geqslant 0,\quad k_3 \geqslant 0,\quad k_3' \geqslant 0,\quad k_2 \leqslant 0,\quad k_2' \geqslant 0. \qquad \ldots\ldots (2)$$

(1) may be written

$$(k_1\kappa z^2 + k_3)(k_1'\kappa z^2 + k_3')\left[I_0(az)K_0(bz) - K_0(az)I_0(bz)\right]$$

$$- k_2k_2'z^2\left[I_1(az)K_1(bz) - K_1(az)I_1(bz)\right]$$

$$+ k_2z(k_1'\kappa z^2 + k_3')\left[I_1(az)K_0(bz) + I_0(bz)K_1(az)\right]$$

$$- k_2'z(k_1\kappa z^2 + k_3)\left[I_0(az)K_1(bz) + I_1(bz)K_0(az)\right] \qquad \ldots\ldots\ldots (3)$$

It is known* that $I_0(az)K_0(bz) - I_0(bz)K_0(az)$

and $I_1(az)K_1(bz) - K_1(az)I_1(bz)$

have no real positive zeros and since $b > a$ it follows from the

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+ Chapter IV, §§ 4,5. R.H.F. §§ 4,7.

* G. & M., p. 82, Th. X.

asymptotic expansions that both are negative.

Also $\quad I_1(az)K_0(bz) + K_1(az)I_0(bz)$

and $\quad I_0(az)K_1(bz) + K_0(az)I_1(bz)$

are both positive for real positive z. Thus the terms of (3) have respectively the signs of

$$-(k_1\kappa z^2+k_3)(k_1'\kappa z^2+k_3'), \quad k_2k_2'z^2, \quad k_2z(k_1'\kappa z^2+k_3'), \quad -k_2'z(k_1\kappa z^2+k_3).$$

As in § 1 above we take $k_1 \geqslant 0$, $k_1' \geqslant 0$ (and if either vanishes the corresponding k_3 or $k_3' \geqslant 0$). Then if

$$k_1 \geqslant 0, \quad k_1' \geqslant 0, \quad k_3 \geqslant 0, \quad k_3' \geqslant 0, \quad k_2 \leqslant 0, \quad k_2' \geqslant 0$$

all terms of (3) are $\leqslant 0$ for real positive z and thus there is no real root.

The conditions (2) are satisfied in the cases of physical importance discussed in Chapter IV, § 5 and R.H.F. § 7. If the conditions (2) are not satisfied there may be real positive zeros.

(ii) <u>The expression (1) has no complex zeros if</u>

$$k_1 \geqslant 0, \quad k_1' \geqslant 0, \quad k_2 \leqslant 0, \quad k_2' \geqslant 0.$$

Consider the differential equation containing the parameter z

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dU}{dr}\right) - z^2U = 0, \qquad a < r < b, \qquad \ldots\ldots\ldots\ldots\ldots, \ (4)$$

with boundary conditions

$$(k_1\kappa z^2 + k_3)U + k_2\frac{dU}{dr} = 0\ , \qquad r = a \qquad \ldots\ldots\ldots\ldots\ldots \ (5)$$

$$(k_1'\kappa z^2 + k_3')U + k_2'\frac{dU}{dr} = 0\ , \qquad r = b. \qquad \ldots\ldots\ldots\ldots\ldots \ (6)$$

If $z = \xi$, a zero of (1), a non-zero solution of the differential equation (4) and boundary conditions (5) and (6) is

$$U = \left[(k_1 \kappa \xi^2 + k_3)K_o(a\xi) - k_2\xi K_1(a\xi)\right] I_o(r\xi)$$
$$- \left[(k_1 \kappa \xi^2 + k_3) I_o(a\xi) + k_2 \xi I_1(a\xi)\right] K_o(r\xi)$$

Suppose ξ is a complex zero of (1) and let η be its conjugate, then U^*, the conjugate of U satisfies

$$\frac{1}{r}\frac{d}{dr}(r\frac{dU^*}{dr}) - \eta^2 U^* = 0 , \quad a < r < b, \qquad \ldots\ldots\ldots\ldots (7)$$

with $(k_1 \kappa \eta^2 + k_3)U^* + k_2 \frac{dU^*}{dr} = 0 , \quad r = a \qquad \ldots\ldots\ldots\ldots (8)$

$(k_1' \kappa \eta^2 + k_3')U^* + k_2' \frac{dU^*}{dr} = 0 , \quad r = b \qquad \ldots\ldots\ldots\ldots (9)$

From (4) and (7) we obtain

$$(\eta^2 - \xi^2)\int_a^b rUU^* dr = \int_a^b \left[U \frac{d}{dr}(r\frac{dU^*}{dr}) - U^* \frac{d}{dr}(r\frac{dU}{dr})\right]dr$$

$$= \left[Ur\frac{dU^*}{dr} - U^* r\frac{dU}{dr}\right]_a^b$$

$$= \left[-\frac{Ur(k_1'\kappa\eta^2 + k_3')}{k_2'} + \frac{U^* r(k_1'\kappa\xi^2 + k_3')}{k_2'}\right]_{r=b}$$

$$- \left[-\frac{Ur(k_1\kappa\eta^2 + k_3)}{k_2} + \frac{U^* r(k_1\kappa\xi^2 + k_3)}{k_2}\right]_{r=a}$$

$$= \frac{bk_1'\kappa}{k_2'}(\xi^2 - \eta^2)|U|^2_{r=b} - \frac{ak_1\kappa}{k_2}(\xi^2 - \eta^2)|U|^2_{r=a}$$

i.e. $$(\eta^2 - \xi^2)\left\{\int_a^b r|U|^2 dr + \frac{bk_1'\kappa}{k_2'}|U|^2_{r=b} - \frac{ak_1\kappa}{k_2}|U|^2_{r=a}\right\} = 0$$

Thus taking, as always, $k_1 \geqslant 0$, $k_1' \geqslant 0$, if $k_2 < 0$, $k_2' > 0$ we have a contradiction. The extension to $k_2 \leqslant 0$, $k_2' \geqslant 0$ is trivial.

(iii) The expression (1) has pure imaginary roots $\pm i\alpha_s$, $s = 1, 2, \ldots$ where the α_s are the roots of

$$\left[(k_3 - k_1\kappa\alpha^2)J_0(a\alpha) - k_2\alpha J_1(a\alpha)\right]\left[(k_3' - k_1'\kappa\alpha^2)Y_0(b\alpha) - k_2'\alpha Y_1(b\alpha)\right]$$
$$- \left[(k_3' - k_1'\kappa\alpha^2)J_0(b\alpha) - k_2'\alpha J_1(b\alpha)\right]\left[(k_3 - k_1\kappa\alpha^2)Y_0(a\alpha) - k_2\alpha Y_1(a\alpha)\right] = 0 \qquad \ldots\ldots\ldots (10)$$

From the asymptotic expansions of the Bessel functions it follows that for large s these are near

$$\frac{s}{b-a} \quad \text{or} \quad \frac{(s+\frac{1}{2})\pi}{b-a}$$

depending on which of the k vanish.

(iv) The roots of equation (10) are simple.

At a repeated root of (10) the derivative of the left hand side of (10) will also vanish. This gives on evaluating

$$\left[(k_3' - k_1'\kappa\alpha^2)J_0(b\alpha) - k_2'\alpha J_1(b\alpha)\right]\left\{(k_3 - \kappa k_1\alpha^2)^2 + k_2(k_2 + 2\kappa k_1/a)\alpha^2\right\}^{\frac{1}{2}}$$
$$\pm \left[(k_3 - \kappa k_1\alpha^2)J_0(a\alpha) - k_2\alpha J_1(a\alpha)\right]\left\{(k_3' - \kappa k_1'\alpha^2)^2 + k_2'(k_2' + 2\kappa k_1'/b)\alpha^2\right\}^{\frac{1}{2}} = 0 \qquad \ldots\ldots\ldots (11)$$

Combining this with (10) we obtain

$$\left[(k_3' - \kappa k_1'\alpha^2)Y_0(b\alpha) - k_2'\alpha Y_1(b\alpha)\right]\left\{(k_3 - k_1\kappa\alpha^2)^2 + k_2(k_2 + 2\kappa k_1/a)\alpha^2\right\}^{\frac{1}{2}}$$
$$\pm \left[(k_3 - \kappa k_1\alpha^2)Y_0(a\alpha) - k_2\alpha Y_1(a\alpha)\right]\left\{(k_3' - \kappa k_1'\alpha^2)^2 + k_2'(k_2' + 2\kappa k_1'/b)\alpha^2\right\}^{\frac{1}{2}} \qquad \ldots\ldots (12)$$

Now the expressions in (11) and (12) are linearly independent solutions of a linear second order differential equation so cannot* vanish simultaneously and we have a contradiction.

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* G. & M., p. 80, Th. 2. There may be repeated roots for exceptional values which make the coefficient of the second dericative in this equation vanish.

APPENDIX III.

1. In this Appendix it is verified for the complete set of problems discussed in Chapter IV, R.H.F., and Appendix I that the integrals round the arcs BB'C and AA'C of the circle Γ of Fig. 3, or round the arcs BB'F, AA'C of the circle Γ of Fig. 1, tend to zero as the radii tend to infinity. When Fig. 3 is used the radius is to tend to infinity through a sequence of values avoiding the poles of the integrand; these poles have been determined in Appendix II.

In all cases it has merely to be verified that the integrands of the line integrals for $\mathbf{v}$ satisfy the conditions of Theorem 1, Chapter I. The proof of that theorem still holds if the radius tends to infinity through a sequence of values. The upper arc $\pi \geqslant \theta > 0$ only will be discussed, the lower ones are treated similarly.

2. Lemma. For $\lambda = \kappa(n+\frac{1}{2})^2\,\dfrac{\pi^2}{a^2}\,e^{i\theta}$, $\mu = \sqrt{(\lambda/\kappa)}$, $\pi \geqslant \theta \geqslant 0$.

$$|\cosh(\mu a - \tfrac{1}{4}\pi i)| > C\,\exp\left[(n+\tfrac{1}{2})\pi\cos\tfrac{1}{2}\theta\right] \qquad \ldots\ldots (1)$$

where C is a constant independent of n.

$$\begin{aligned}|\cosh(\mu a - \tfrac{1}{4}\pi i)|^2 &= \left|\cosh\left[(n+\tfrac{1}{2})\pi e^{i\theta/2} - \tfrac{1}{4}\pi i\right]\right|^2 \\ &= \tfrac{1}{2}\left\{\cosh\left[(2n+1)\pi\cos\tfrac{1}{2}\theta\right] + \cos\left[(2n+1)\pi\sin\tfrac{1}{2}\theta - \tfrac{\pi}{2}\right]\right\} \\ &= \tfrac{1}{2}\cosh\left[(2n+1)\pi\cos\tfrac{1}{2}\theta\right]\left\{1 + \sin\left[(2n+1)\pi\sin\tfrac{1}{2}\theta\right]\times\right. \\ &\qquad\left.\times\operatorname{sech}\left[(2n+1)\pi\cos\tfrac{1}{2}\theta\right]\right\}\end{aligned}$$

Now let $\beta = 2 \sin^{-1} \dfrac{2n + \frac{5}{4}}{2n + 1}$

so that $\cos \dfrac{\beta}{2} = \dfrac{\sqrt{n + 7/16}}{2n + 1}$.

(I) $\pi \geqslant \theta \geqslant \beta$

Then $(2n + 1)\pi \geqslant (2n + 1)\pi \sin \dfrac{\theta}{2}$

$\geqslant (2n + 1)\pi \sin \dfrac{\beta}{2}$

$\geqslant (2n + 3/4)\pi$

Therefore $0 \leqslant \sin\left[(2n + 1)\pi \sin \dfrac{\theta}{2}\right] \leqslant \dfrac{1}{\sqrt{2}}$

Thus $|\cosh (\mu a - \frac{1}{4}\pi i)|^2 < C\, e^{(2n+1)\pi \cos \frac{\theta}{2}}$ .... (i)

(II) $\beta \geqslant \theta \geqslant 0$

$\left|\sin\left[(2n+1)\pi \sin \dfrac{\theta}{2}\right] \operatorname{sech} \left[(2n+1)\pi \cos \dfrac{\theta}{2}\right]\right|$

$\leqslant \operatorname{sech}\left[(2n + 1)\pi \cos \dfrac{\theta}{2}\right]$

$\leqslant \operatorname{sech}\left[(2n + 1)\pi \cos \dfrac{\beta}{2}\right]$

$\leqslant \operatorname{sech} \dfrac{\pi\sqrt{7}}{4}$, for every positive integer n .

Thus $\left\{1 + \sin\left[(2n+1)\pi \sin \dfrac{\theta}{2}\right] \operatorname{sech}\left[(2n+1)\pi \cos \dfrac{\theta}{2}\right]\right\}$

$\geqslant 1 - \operatorname{sech} \dfrac{\pi\sqrt{7}}{4}$, for every positive integer.

Therefore $|\cosh(\mu a - \frac{1}{4}\pi i)|^2 > C\, e^{(2n+1)\pi \cos \frac{\theta}{2}}$ .... (ii)

From (i) and (ii) the result follows.

The same argument gives, when $\lambda = \kappa(n + \frac{1}{2})^2 \dfrac{\pi^2}{a^2} e^{i\theta}$,

$$|\cosh(\mu a - \tfrac{3}{4}\pi i)| > C \exp\left[(n + \tfrac{1}{2})\pi \cos \tfrac{1}{2}\theta\right] \quad \text{....... (2)}$$

3. The region $0 \leqslant r < a$. Zero initial temperature. Boundary condition at $r = a$

$$k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v = k_4 , \quad t > 0.$$

Here $v = \dfrac{k_4}{2\pi i} \displaystyle\int_{\gamma - i\infty}^{\gamma + i\infty} \dfrac{e^{\lambda t}\, I_0(\mu r)\, d\lambda}{\lambda\{(k_1\lambda + k_3) I_0(\mu a) + k_2 \mu I_1(\mu a)\}}$ .... (3)

Now, in the notation of Appendix I, § 2,

$$f(\lambda) = (k_1\lambda + k_3)I_0(\mu a) + k_2\mu I_1(\mu a)$$

$$= \frac{2(k_1\lambda + k_3)e^{\frac{1}{4}\pi i}}{(2\pi\mu a)^{\frac{1}{2}}} \cosh(\mu a - \tfrac{1}{4}\pi i) + \frac{2k_2\mu e^{\frac{3}{4}\pi i}}{(2\pi\mu a)^{\frac{1}{2}}} \cosh(\mu a - \tfrac{3}{4}\pi i)$$

+ similar terms $O(\frac{1}{\mu})$ compared with the above.

Thus* if $\lambda = \kappa(n + \frac{1}{2})^2 \frac{\pi^2}{a^2} e^{i\theta}$, $\pi \geqslant \theta > 0$

$$|f(\lambda)| > C\, n^{3/2}\, |\cosh(\mu a - \tfrac{1}{4}\pi i)| > C\, n^{3/2} \exp\left[(n+\tfrac{1}{2})\pi \cos \tfrac{1}{2}\theta\right],$$

$n > n_0$, $k_1 \neq 0$ ...... (4)

or $> Cn^{\frac{1}{2}}|\cosh(\mu a - \frac{3}{4}\pi i)| > C\, n^{\frac{1}{2}} \exp\left[(n + \frac{1}{2})\pi \cos \frac{1}{2}\theta\right]$, $n > n_1$,

if $k_1 = 0$, $k_2 \neq 0$, ... (5)

or $> Cn^{-\frac{1}{2}}|\cosh(\mu a - \frac{3}{4}\pi i)| > Cn^{-\frac{1}{2}} \exp\left[(n + \frac{1}{2})\pi \cos \frac{1}{2}\theta\right]$, $n > n_2$,

if $k_1 = k_2 = 0$, ... (6)

where the results of § 2. (1) and (2) have been used.

Also $|I_0(z)| \leqslant \exp|R(z)|$.

Thus on $\lambda = \kappa(n + \frac{1}{2})^2 \frac{\pi^2}{a^2} e^{i\theta}$

$$\left|\frac{I_0(\mu r)}{f(\lambda)}\right| < C\, n^{\alpha} \exp\left\{(n + \tfrac{1}{2})\pi . \frac{(r - a)}{a} \cos \tfrac{1}{2}\theta\right\},\quad \pi \geqslant \theta > 0,\ 0 \leqslant r \leqslant a,$$

$n > n_3$

where α is $-3/2$, $-\frac{1}{2}$ or $\frac{1}{2}$ according as $k_1 \neq 0$; $k_1 = 0$, $k_2 \neq 0$; or $k_1 = k_2 = 0$.

In all cases the conditions of Theorem I are satisfied and thus the integral over Γ tends to zero as its radius tends to infinity if either

$0 \leqslant r \leqslant a$, $t > 0$

or $0 \leqslant r < a$, $t \geqslant 0$.

* Circles of these radii do not pass through any pole of the integrand of Appendix I § 3 (6). Cf. Appendix II § 2 (iii).

4. The region $0 \leqslant r < a$. Unit initial temperature. Boundary condition at $r = a$

$$k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v = k_4 , \quad t > 0.$$

Here[*]

$$v = 1 + \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{\lambda t} \left[k_4 - k_3\right] I_0(\mu r) d\lambda}{\lambda\left\{(k_1\lambda + k_3) I_0(\mu a) + k_2 \mu I_1(\mu a)\right\}} , \quad t > 0.$$

It follows from the results of § 3 that in all cases the conditions of Theorem I are satisfied.

5. The instantaneous cylindrical surface source over $r = r'$ in $0 \leqslant r < a$ with boundary condition at $r = a$

$$k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v = 0 , \quad t > 0.$$

Here, in the notation of Appendix I, § 2,

$$v = - \frac{Q}{4\pi^2 i\kappa} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{I_0(\mu r') \left\{ I_0(\mu r) g(\lambda) - K_0(\mu r) f(\lambda) \right\} e^{\lambda t} d\lambda}{f(\lambda)} ,$$

$$r' \leqslant r < a.$$

From the asymptotic expansions it follows that, for

$$\lambda = \kappa(n + \tfrac{1}{2})^2 \frac{\pi^2}{a^2} e^{i\theta} , \quad \pi \geqslant \theta \geqslant 0$$

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[*] Appendix I, § 4.

$$\left| I_0(\mu r') \left\{ I_0(\mu r) g(\lambda) - K_0(\mu r) f(\lambda) \right\} \right| < C\, n^{\alpha} \exp\left[(n+\tfrac{1}{2})\pi \frac{(r'-r+a)}{a} \cos \tfrac{1}{2}\theta \right],$$

$$r' \leqslant r < a, \quad n > n_1,$$

where $\alpha = \frac{1}{2}, -\frac{1}{2}, -3/2$ according as $k_1 \neq 0$; $k_1 = 0, k_2 \neq 0$; $k_1 = k_2 = 0$.

Thus, using § 3, (4), (5), (6), we have when

$$\lambda = \kappa (n+\tfrac{1}{2})^2 \frac{\pi^2}{a^2} e^{i\theta}, \quad \pi \geqslant \theta \geqslant 0$$

$$\left| \frac{I_0(\mu r') \left\{ I_0(\mu r) g(\lambda) - K_0(\mu r) f(\lambda) \right.}{f(\lambda)} \right| < \frac{C}{n} \exp \left\{ (n+\tfrac{1}{2})\pi \frac{(r-r')}{a} \cos \tfrac{1}{2}\theta \right\}$$

$$r' \leqslant r < a, \quad n > n_2.$$

Thus the conditions of Theorem I are satisfied for $t > 0$ if $r' \leqslant r \leqslant a$ and similarly they are satisfied if $0 \leqslant r \leqslant r'$.

6. <u>The problems corresponding to those of §§ 3, 4, 5 for the region bounded internally by the cylinder</u> $r = a$

In these cases the order properties proved in Appendix I § 6 (29) and § 8 (36) hold in fact for $\pi \geqslant \theta \geqslant 0$ and it follows immediately that the conditions of Theorem I are satisfied.

7. <u>The hollow cylinder</u> $a < r < b$. <u>Zero initial temperature.</u>

<u>Boundary conditions</u>

$$k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v = k_4, \quad r = a, \quad t > 0$$

$$k_1' \frac{\partial v}{\partial t} + k_2' \frac{\partial v}{\partial r} + k_3' v = 0, \quad r = b, \quad t > 0.$$

Here

$$v = \frac{k_4}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\{G(\lambda)I_0(\mu r) - F(\lambda)K_0(\mu r)\}\, e^{\lambda t}\, d\lambda}{\lambda\{f(\lambda)G(\lambda) - g(\lambda)F(\lambda)\}} \qquad \ldots\ldots (7)$$

It follows from the asymptotic expansions that

$$\begin{aligned} 2\mu\sqrt{(ab)}\,\{f(\lambda)G(\lambda) - g(\lambda)F(\lambda)\} = &- 2(k_1\lambda + k_3)(k_1'\lambda + k_3')\sinh\mu(b-a) \\ &- 2k_2'\mu(k_1\lambda + k_3)\cosh\mu(b-a) \\ &+ 2k_2\mu(k_1'\lambda + k_3')\cosh\mu(b-a) \\ &+ 2k_2k_2'\mu^2\sinh\mu(b-a) \\ &+ \text{corresponding terms } O\left(\tfrac{1}{\mu}\right) \text{ compared} \\ &\quad \text{with the above} \qquad \ldots\ldots\ldots (8) \end{aligned}$$

We take* $\lambda = \kappa(n + \frac{1}{4})^2 \frac{\pi^2}{(b-a)^2} e^{i\theta}$, $\pi \geqslant \theta > 0$ and it follows as in § 2 that

$$|\sinh\mu(b-a)| > C\exp\left[(n + \tfrac{1}{4})\pi\cos\tfrac{1}{2}\theta\right] \qquad \ldots\ldots (9)$$

and

$$|\cosh\mu(b-a)| > C\exp\left[(n + \tfrac{1}{4})\pi\cos\tfrac{1}{2}\theta\right] \qquad \ldots\ldots (10)$$

Thus for $\lambda = \kappa(n + \frac{1}{4})^2 \frac{\pi^2}{(b-a)^2} e^{i\theta}$, $\pi \geqslant \theta > 0$,

$$|f(\lambda)G(\lambda) - g(\lambda)F(\lambda)| > C\, n^{\alpha}\exp\left[(n + \tfrac{1}{4})\pi\cos\tfrac{1}{2}\theta\right], \quad n > n_0 \qquad \ldots (11)$$

$$\begin{aligned} \text{where}\quad \alpha &= 3, \quad k_1 \neq 0,\ k_1' \neq 0 \\ &= 2, \quad k_1 = 0,\ k_2 \neq 0,\ k_1' \neq 0 \\ &= 1, \quad k_1 = k_2 = 0,\ k_1' \neq 0 \ \text{ or } \ k_1 = k_1' = 0,\ k_2 \neq 0,\ k_2' \neq 0 \\ &= 0, \quad k_1 = k_2 = k_1' = 0,\ k_2' \neq 0 \\ &= -1, \quad k_1 = k_2 = k_1' = k_2' = 0. \end{aligned}$$

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* A circle of radius $\kappa(n + \frac{1}{4})^2 \frac{\pi^2}{(b-a)^2}$ will in no case pass through a pole of the integrand of (7). Cf. Appendix II, § 4 .

Also

$$|G(\lambda)I_o(\mu r) - F(\lambda)K_o(\mu r)| < C\, n^{\alpha} \exp\left[(n + \tfrac{1}{4})\left(\frac{b-r}{b-a}\right)\pi \cos \tfrac{1}{2}\theta\right], \quad n > n_1 \quad \dots (12)$$

where $\alpha = 1, 0, -1$ according as $k_1' \neq 0$; $k_1' = 0$, $k_2' \neq 0$; $k_1' = k_2' = 0$.

Thus on $\lambda = \kappa(n + \frac{1}{4})^2 \dfrac{\pi^2}{(b-a)^2} e^{i\theta}$, $\pi \geqslant \theta > 0$

$$\left|\frac{G(\lambda)I_o(\mu r) - F(\lambda)K_o(\mu r)}{f(\lambda)G(\lambda) - g(\lambda)F(\lambda)}\right| < C\, n^{\alpha} \exp\left[(n + \tfrac{1}{4})\frac{(a-r)}{(b-a)}\pi \cos \tfrac{1}{2}\theta\right], \quad n > n_2$$

where α is $-2, -1, 0$.

Thus in all cases the conditions of Theorem I are satisfied.

8. The instantaneous cylindrical surface source over $r = r'$ in the hollow cylinder $a < r < b$ with boundary conditions

$$k_1 \frac{\partial v}{\partial t} + k_2 \frac{\partial v}{\partial r} + k_3 v = 0, \qquad r = a, \quad t > 0$$

$$k_1' \frac{\partial v}{\partial t} + k_2' \frac{\partial v}{\partial r} + k_3' v = 0, \qquad r = b, \quad t > 0.$$

Here

$$v = \frac{Q}{4\pi^2 i\kappa}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{\{I_o(\mu r)g(\lambda) - K_o(\mu r)f(\lambda)\}\{I_o(\mu r')G(\lambda) - K_o(\mu r')F(\lambda)\}e^{\lambda t}d\lambda}{f(\lambda)G(\lambda) - g(\lambda)F(\lambda)}$$

$$a < r \leqslant r'.$$

On $\lambda = \kappa(n + \frac{1}{4})^2 \dfrac{\pi^2}{(b-a)^2} e^{i\theta}$, $\pi \geqslant \theta > 0$

$$\left|\{I_o(\mu r)g(\lambda) - K_o(\mu r)f(\lambda)\}\{I_o(\mu r')G(\lambda) - K_o(\mu r')F(\lambda)\}\right|$$

$$< C\, n^{\alpha} \exp\left\{(n + \tfrac{1}{4})\pi \frac{(b-a+r-r')}{b-a} \cos \tfrac{1}{2}\theta\right\}$$

$$n > n_4$$

$$\text{where } \alpha = 2\,, \quad k_1 \neq 0, \quad k_1' \neq 0$$
$$= 1\,, \quad k_1 = 0, \quad k_1' \neq 0, \quad k_2 \neq 0$$
$$= 0\,, \quad k_1 = k_1' = 0, \quad k_2 \neq 0, \quad k_2' \neq 0; \quad \text{or} \quad k_1 = 0,\ k_1' \neq 0,\ k_2 = 0$$
$$= -1\,, \quad k_1 = k_1' = 0, \quad k_2 = 0, \quad k_2' \neq 0$$
$$= -2\,, \quad k_1 = k_1' = k_2 = k_2' = 0.$$

Thus in all cases

$$\left| \frac{\{ I_0(\mu r) g(\lambda) - K_0(\mu r) f(\lambda) \}\{ I_0(\mu r') G(\lambda) - K_0(\mu r') F(\lambda) \}}{f(\lambda) G(\lambda) - g(\lambda) F(\lambda)} \right|$$

$$< C\, n^{-1} \exp \left\{ (n + \tfrac{1}{4}) \frac{\pi(r - r')}{b - a} \cos \tfrac{1}{2}\theta \right\}, \quad n > n_0\,,$$

and the conditions of Theorem I are satisfied, if $t > 0$ and $a \leqslant r \leqslant r'$, and similarly they are satisfied if $r' \leqslant r \leqslant b$.