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DOCTORAL THESIS

Analysis of Singular Solutions of Certain Painlevé Equations

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A thesis submitted in fulfilment of the requirements
for the degree of Doctor of Philosophy

in the

School of Mathematics and Statistics

May 11, 2018

Declaration of Authorship

I, Michael Twiton, declare that this thesis titled, "Analysis of Singular Solutions of Certain Painlevé Equations" and the work presented in it are my own. I confirm that:

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- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

“In order to solve this differential equation you look at it till a solution occurs to you.”

George Pólya

THE UNIVERSITY OF SYDNEY

Abstract

Faculty of Science
School of Mathematics and Statistics

Doctor of Philosophy

Analysis of Singular Solutions of Certain Painlevé Equations

by Michael Twiton

The six Painlevé equations can be described as the boundary between the non-integrable- and the trivially integrable-systems. Ever since their discovery they have found numerous applications in mathematics and physics. The solutions of the Painlevé equations are, in most cases, highly transcendental and hence cannot be expressed in closed form. Asymptotic methods do better, and can establish the behaviour of some of the solutions of the Painlevé equations in the neighbourhood of a singularity, such as the point at infinity. Although the quantitative nature of these neighbourhoods is not initially implied from the asymptotic analysis, some regularity results exist for some of the Painlevé equations. In this research, we will present such results for some of the remaining Painlevé equations. In particular, we will provide concrete estimates of the intervals of analyticity of a one-parameter family of solutions of the second Painlevé equation, and estimate the domain of analyticity of a "triply-truncated" solution of the fourth Painlevé equation. In addition we will also deduce the existence of solutions with particular asymptotic behaviour for the discrete Painlevé equations, which are discrete integrable nonlinear systems.

Acknowledgements

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List of Abbreviations

ODE	**O**rdinary **D**ifferential **E**quation(s)
PDE	**P**artial **D**ifferential **E**quation(s)
PP	**P**ainlevé **P**roperty
BT	**B**äcklund **T**ransformation(s)
IVP	**I**nitial **V**alue **P**roblem

List of Symbols

C_w	Weighted continuous function space
H	Heaviside step function
Ai	Airy function of the first kind
Bi	Airy function of the second kind
D_ν	Parabolic cylinder function of order ν
H_n	Hermite polynomial of order n
$\mathrm{P_J}$	Jth Painlevé equation
d-$\mathrm{P_J}$	Jth discrete Painlevé equation
a_n^{*m}	Convolution power
Π_-	$\{(x,y) : x \geq 0, y = -\sqrt{x/6}\}$
Π_0^+	$\{(x,y) : x \geq 0, y = \sqrt{x/6}\}$
Π_1^+	$\{(x,y) : x \geq 0, y = \sqrt{x/2}\}$
Π_0^-	$\{(x,y) : x \geq 0, y = -\sqrt{x/6}\}$
Π_1^+	$\{(x,y) : x \geq 0, y = -\sqrt{x/2}\}$
Δ	Forward difference operator

Dedicated to Yinon, Sarah, and Ariel

Chapter 1

Introduction

Differential equations abound in the sciences, particularly in mathematics and physics, and as such, their solutions are of high value. However, with the exception of linear differential equations, there are very few methods of solving differential equations explicitly, and one often has to settle for approximations instead. In this work we will focus on a particular set of ordinary differential equations, known as the *Painlevé equations*. These were obtained in the search for differential equations of second order with the property that all singularities of the solutions whose location depends on the integration constants are poles (this is known as the Painlevé property). Our main objective is to provide concrete estimates on the domains of regularity of certain solutions of various Painlevé equations. We also discuss discrete Painlevé equations, which are non-autonomous discrete integrable systems, and their connection to the standard Painlevé equations.

1.1 Singularities in Differential Equations

When solving differential equations, singularities are a common occurrence. We discuss the difference between singularities in linear ODEs compared with nonlinear ones. In particular, we describe the situation in which singularities appear in the solution even though the ODE itself is regular.

1.1.1 The Linear Case

Consider an $n \times n$ linear system of first order differential equations

$$\mathbf{y}'(x) = A(x)\mathbf{y}(x) + \mathbf{b}(x), \tag{1.1a}$$

with initial conditions

$$\mathbf{y}'(x_0) = \mathbf{y}_0, \tag{1.1b}$$

where $A : \Omega \to \mathbb{C}^{n\times n}$ and $\mathbf{b} : \Omega \to \mathbb{C}^n$ are analytic in a simply connected domain $\Omega \subseteq \mathbb{C}$. Solutions to such systems are as well-behaved as the equations themselves in the following sense [Tes12]:

Theorem 1.1. *Given functions $A : \Omega \to \mathbb{C}^{n\times n}$ and $b : \Omega \to \mathbb{C}^n$, analytic in Ω, the IVP of Equations (1.1) has a unique solution defined on all of Ω.*

Since scalar equations of order ≥ 2 can be recast as a linear system of first order equations, we also have

Corollary 1.2. *Equations of the form*

$$y^{(n)}(x) = \sum_{k=0}^{n-1} a_k(x) y^{(k)}(x) + b(x),$$

where $a_1, \ldots, a_{n-1}, b : \Omega \to \mathbb{C}$ *are analytic in a simply connected domain* $\Omega \subseteq \mathbb{C}$ *admit unique solutions, defined on all of* Ω*, to IVPs with initial data*

$$\begin{aligned} y(x_0) &= y_0, \\ y'(x_0) &= y_0', \\ &\vdots \\ y^{(n-1)}(x_0) &= y_0^{(n-1)}. \end{aligned}$$

It follows from the last two results that solutions to linear problems can develop singularities only at points in which the equation itself is singular.

Example 1.3. *The general solution to the equation*

$$y' + \frac{1}{x^2} y = 0,$$

is

$$y(x) = Ce^{1/x},$$

where C *is an integration constant, and (for* $C \neq 0$*) possesses an essential singularity at the origin.*

However, a point of singularity of the equation does not necessarily imply the singularity of the solutions.

Example 1.4. *The Cauchy-Euler equation*

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{3}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{3}{x^2} y = 0,$$

is singular at the point $x = 0$*. However, its general solution is*

$$y(x) = C_1 x + C_2 x^3,$$

for arbitrary constants C_1*,* C_2*, which is analytic everywhere.*

1.1.2 The Nonlinear Case

The standard existence and uniqueness theorem for ODEs is the Picard-Lindelöf theorem, formulated below for second order equations.

Theorem 1.5 (Theorem 2.2 of [Tes12])**.** *Consider the IVP*

$$y''(x) = f\left(x, y(x), y'(x)\right), \quad y(x_0) = y_0, y'(x_0) = y_1. \tag{1.2}$$

Suppose that f *is continuous in* x*, and locally Lipschitz continuous in* y *and* y'*, uniformly with respect to* x*. Then there exists a unique solution of the IVP (1.2) in some neighbourhood of* x_0*.*

The Picard-Lindelöf theorem is of a local nature. As such, it does not allow one to predict where singularities will occur. What can be said about singularities in nonlinear equations? Let us begin with the simplest form of nonlinearity – a square.

Example 1.6. *The seemingly regular equation*

$$y' = y^2,$$

has the general solution

$$y(x) = \frac{1}{C - x},$$

which has a pole at the point $x = C$*.*

Equations can have a more complicated distribution of singularities: for instance, the equation

$$y'' + 4y^3y' + y = 0,$$

admits solutions with an accumulation point of algebraic singularities [Smi53]. Furthermore, Table 1.1, taken from [KJH97] displays a wide variety of types of singularities. These new seemingly unpredictable phenomena call for a definition, given below.

	Equation	General Solution	Singularity Type
1.	$y' + y^2 = 0$	$y = (z - z_0)^{-1}$	simple pole
2.	$2yy' = 1$	$y = \sqrt{z - z_0}$	branch point
3.	$y'' + y'^2 = 0$	$y = \ln(z - z_0) + k$	logarithmic branch point
4.	$yy'' + y'^2(y/y' - 1) = 0$	$y = k\exp\left[(z - z_0)^{-1}\right]$	isolated essential singularity
5.	$(1 + y^2)y'' + (1 - 2y)y'^2 = 0$	$y = \tan\left(\ln\left(k(z - z_0)\right)\right)$	nonisolated essential singularity
6.	$\left(y'' + y^3y'\right)^2 = y^2y'^2\left(4y' + y^4\right)$	$y = k\tan\left[k^3(z - z_0)\right]$ or $y = \left[\frac{4/3}{(z-z_0)}\right]^{1/3}$	pole or branch point

TABLE 1.1: Various types of singular behaviours.

Definition 1.7. *Singularities whose position depends on the integration constants are called* ***spontaneous-*** *or* ***movable-*** *singularities.*

Some questions now arise naturally, among them

- What can be then said, about the nature of singularities in nonlinear equations?
- Is there a way to "minimise", in some sense, the singular nature of the solutions?

1.2 The Painlevé Equations

Asking for no movable singularities at all in solutions of nonlinear ODEs results in losing many important and applicable examples. S. Kowalevski found in [Kow90] that the requirement of the solutions being single valued resulted in complete integrability for the case of a spinning top. Continuing in this philosophy, a system all of the movable singularities of its solutions are poles is said to have the "*Painlevé Property*" (PP), named after P. Painlevé.

It was shown by H. Poincaré and L. Fuchs [Fuc84] that for first order equations with the PP of the form

$$y' = F(x, y),$$

where F is analytic with respect to x and rational with respect to y can be reduced to a Riccati equation of the form

$$\frac{\mathrm{d}u}{\mathrm{d}z} = a_0(z) + a_1(z)u + a_2(z)z^2,$$

which in turn, can be linearised (cf. [Inc56, Cha94, Pic84]).

The classification of second order equations with the PP of the form

$$y'' = F(x, y, y'),$$

where F is analytic with respect to x and rational with respect to y, y' led to many new results, some of which are still being explored today. For instance, E. Picard found that in the case of equations of order two or higher movable essential singularities can occur [Pic89].

Painlevé [Pai97], Gambier [Gam10] and Fuchs [Fuc05] proved that there are fifty canonical such equations, amongst which the six referred to as the Painlevé equations $\mathrm{P}_{\mathrm{I}}, \ldots, \mathrm{P}_{\mathrm{VI}}$ give rise to new transcendental functions. These six equations are listed in Table 1.2.

Indeed, the solutions of eleven other equations from the list can be expressed in terms of solutions of the Painlevé equations, while the remaining thirty-three equations are solvable in terms of linear equations, or elliptic functions.

Index	Equation
I	$y'' = 6y^2 - x$
II	$y'' = 2y^3 + xy + a$
III	$y'' = \frac{(y')^2}{y} - \frac{y'}{x} + \frac{d}{y} + \frac{b}{x} + a\frac{y^2}{x} + cy^3$
IV	$y'' = \frac{(y')^2}{2y} + \frac{b}{y} + 2(x^2 - a)y + 4xy^2 + \frac{3}{2}y^3$
V	$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{1}{x}y' + \frac{(y-1)^2}{x^2}\left(ay + \frac{b}{y}\right) + c\frac{y}{x} + d\frac{y(y+1)}{y-1}$
VI	$y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)(y')^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y'$ $+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left(a + b\frac{x}{y^2} + c\frac{x-1}{(y-1)^2} + d\frac{x(x-1)}{(y-x)^2}\right)$

TABLE 1.2: The Painlevé equations $\mathrm{P}_{\mathrm{I}}, \mathrm{P}_{\mathrm{II}}, \mathrm{P}_{\mathrm{III}}, \mathrm{P}_{\mathrm{IV}}, \mathrm{P}_{\mathrm{V}}$ and P_{VI} where y depends on x, the primes denote differentiation with respect to x and the roman letters a, b, c, d refer to constant parameters.

Remark 1.8. *There is a subtle nuance in the tests used widely for the Painlevé property [KJH97]. These rely on being able to expand the solutions in Laurent series around an arbitrary point (not equal to a fixed singularity of the equation). It follows from the definition of the PP that these are convergent Laurent series with non-zero radius of convergence. However,* *the Painlevé property actually requires more than this local result. It requires that none of these poles merge with each other to produce more complicated singularities, as illustrated in Example 5 of Table 1.1.*

1.2.1 The First Painleve̋ Equation

The first Painlevé equation

$$y'' = 6y^2 - x, \tag{1.3}$$

is the simplest of the list, and it has discrete scaling symmetries given by

$$x \mapsto \omega x,$$
$$y \mapsto \omega^3 y,$$

where ω is any fifth root of unity. These symmetries partition the complex x-plane into five sectors, each of angle $2\pi/5$, which play a role in the asymptotic study of Equation (1.3).

Boutroux's work

In his 1913 paper [Bou13] Boutroux proved that the general solution of Equation (1.3) has its poles lying on a deformed lattice near infinity, as can be seen in Figure 1.1.

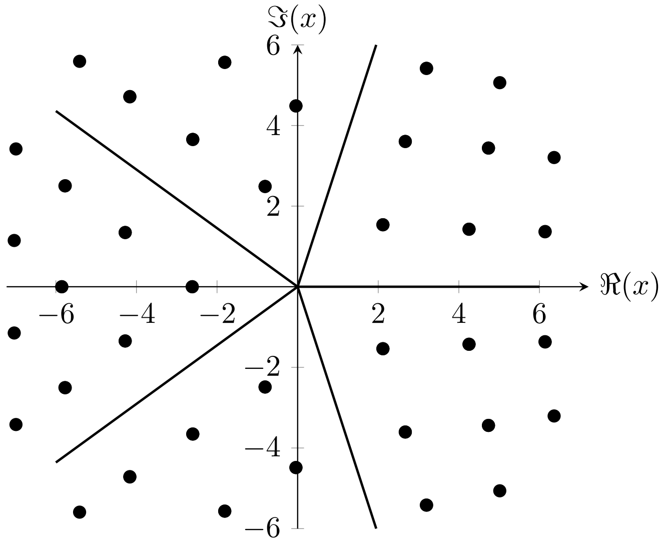

FIGURE 1.1: Illustration of the pole distribution of the general solution of P_I. The dots represent poles of the Padé approximant of the solution y, of P_I of order $[80/80]$ with initial values $y(0) = y'(0) = 0$ (see Appendix A.2).

Boutroux also proved the existence of solutions which have no poles of large modulus in certain sectors of angle $4\pi/5$ in the complex x plane, which he called *intégrales tronquées* (see Figure 1.2).

He further proved the existence of *intégrales tritronquées* – solutions which are free of poles of large modulus in sectors of angle $8\pi/5$ (Figure 1.3). The way Boutroux obtained these results, starting from the P_I variant

$$y'' = 6y^2 - 6x,$$

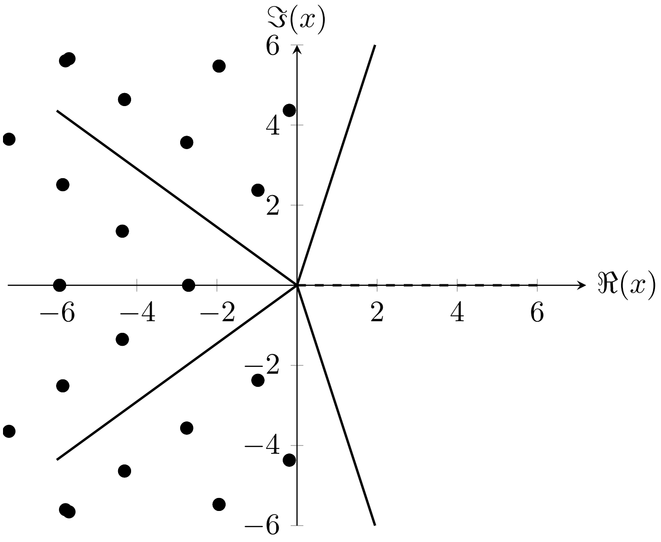

FIGURE 1.2: Illustration of the pole distribution of a tronquée solution of P_I. The dots represent poles of the Padé approximant of the solution y, of P_I of order $[100/100]$ with initial values $y(0) = 0.16$ and $y'(0) \approx 0.2193934573994778462$ (see Appendix A.2).

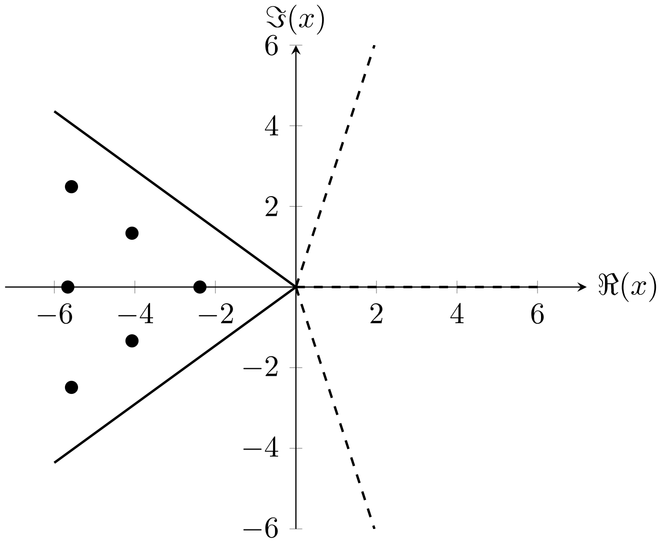

FIGURE 1.3: Illustration of the pole distribution of a tritronquée solution of P_I. The dots represent poles of the Padé approximant of the solution y, of P_I of order $[100/100]$ with initial values $y(0) = -0.1875543083404949$ and $y'(0) = -0.3049055602612289$ (see Appendix A.2).

was to use the change of variables, given by

$$y = \sqrt{x}Y, \tag{1.4a}$$

$$X = \frac{4}{5}x^{\frac{5}{4}}. \tag{1.4b}$$

The new coordinates defined in Equations (1.4) are known as *Boutroux's coordinates* for $\mathrm{P_I}$. They result in the ODE

$$Y'' + \frac{Y'}{X} - \frac{4}{25}\frac{Y}{X^2} = 6Y^2 - 6, \tag{1.5}$$

which has the asymptotic limit

$$Y'' = 6Y^2 - 6. \tag{1.6}$$

Equation (1.6) can be solved by means of the Weierstraß $\wp$-function. Therefore, the movable poles of a solution of $\mathrm{P_I}$, for large x, are arranged in a deformed lattice. Boutroux proceeded by proving the existence of solutions admitting the asymptotic behaviour

$$y(x) \sim \left(\frac{x}{6}\right)^{1/2},$$

for large x in sectors of angle $8\pi/5$. Note that this does not initially tell us anything about the behaviour of such solutions in the finite domain.

Joshi and Kitaev's Work

In their 2001 paper [JK01], N. Joshi and A.V. Kitaev studied the behaviour of the *real* tritronquée solution $Y(x)$ (that is which maps real numbers to real numbers) in the finite domain. They began with modern existence proofs for the tronquée and tritronquée solutions of $\mathrm{P_I}$, based on Wasow's [Was02] theorem (Theorem 12.1, also stated in Section 2.2). The authors then proceed to study the real tritronquée solution of $\mathrm{P_I}$, establishing a characterisation of it based on its behaviour in the finite domain. Numerical estimates of the location of the real pole, closest to the origin, of the tritronquée solution are calculated also, by means of a one-parameter family of solutions tangent to the parabolic branch

$$\Pi_- := \left\{ y = -\sqrt{\frac{x}{6}} \right\}, \tag{1.7}$$

resulting in an estimated interval of analyticity for the tritronquée solution $Y(x)$.

The asymptotic expansion

$$Y(x) \sim -\sqrt{\frac{x}{6}} - \frac{1}{48x^2} + \mathcal{O}\left(x^{-9/2}\right),$$

tells us that the graph of $Y(x)$ eventually lies below the parabolic branch Π_- (see Figure 1.4). The authors proved that a finite point of intersection with Π_- implies infinitely many following ones, which must, according the asymptotic behaviour, have a finite accumulation point. The meromorphic nature of solutions of Equation (1.3) then implies that the functions $Y(x)$ and $-\sqrt{\frac{x}{6}}$ coincide in an interval, which is a contradiction (see Figure 1.5).

Figure 1.6 shows a portion of the graph of $Y(x)$. The reasoning above implies that the interval of existence of $Y(x)$ contains the positive real semi-axis, and that the graph of $Y(x)$ remains below Π_-. This was the first proof that the tritronquée solution of $\mathrm{P_I}$ is pole free on finite region containing a semi-axis.

Remark 1.9. *Dubrovin conjectured in [DGK09] that the tritronquée solution $Y(x)$ is pole-free in the sector $|\arg(x)| < 4\pi/5$, and it was proven in [CHT+14].*

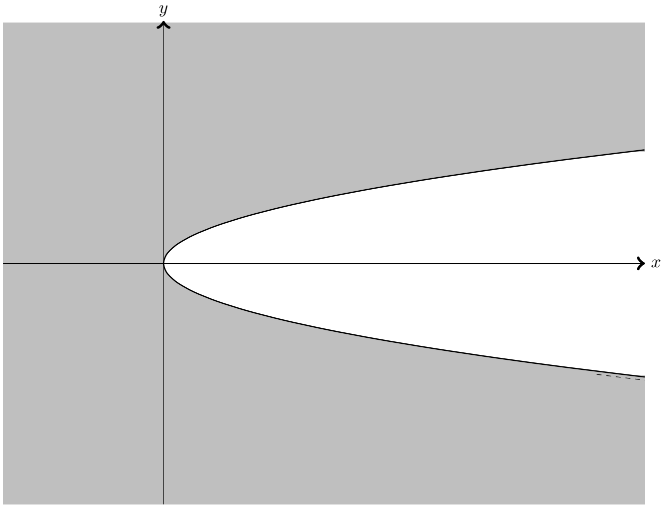

FIGURE 1.4: Convexity plot of Equation (1.3): solutions are convex in the region $6y^2 - x > 0$ (shaded), and concave in the region $6y^2 - x < 0$. The tritronquée solution $Y(x)$ is plotted for large x (dashed), displaying its asymptotic behaviour.

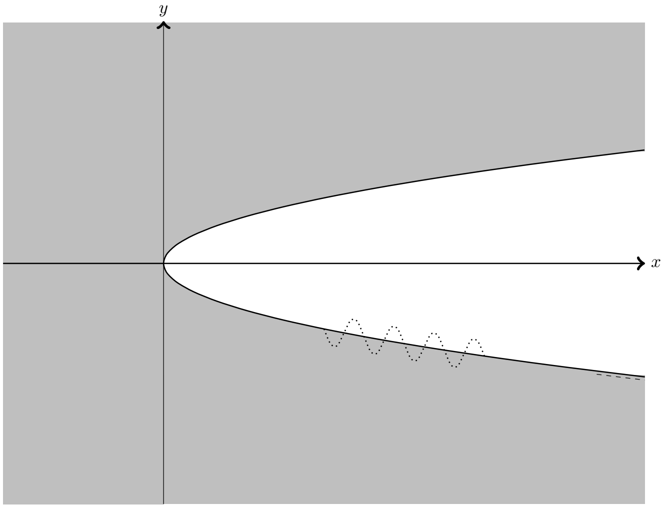

FIGURE 1.5: Portions of the graphs of the tritronquée solution $Y(x)$ (dashed), and of a solution oscillating about the parabolic branch Π_- (dotted).

1.2.2 The Second Painlevé Equation

Boutroux also discussed the second Painlevé equation

$$y'' = 2y^3 + xy + a, \tag{1.8}$$

which has discrete scaling symmetries given by

$$\begin{aligned} x &\mapsto \omega x, \\ y &\mapsto \omega^2 y, \end{aligned}$$

where ω is any third root of unity. The six sectors

$$S_k = \left\{ x : \frac{2\pi k}{6} < \arg x < \frac{2\pi(k+1)}{6} \right\}, \quad k = 0, 1, 2, 3, 4, 5.$$

play a role analogous to the five sectors of $\mathrm{P_I}$ above, in the sense that Boutroux had demonstrated, for each sector S_k, the existence of solutions having no poles of large modulus in it. These sectors come about as domains of validity of certain asymptotic series expansions of $\mathrm{P_{II}}$, and their boundaries consist of what is known as Stokes- or anti-Stokes- lines.

The pole distribution of a truncated solution, such as mentioned above, appears in Figure 1.8. More recently, using the the complex WKB method [Fok06] all simply- and multiply- truncated solutions of $\mathrm{P_{II}}$ have been classified. This classification has also been done using singular submanifolds of a certain complex manifold [Nov12].

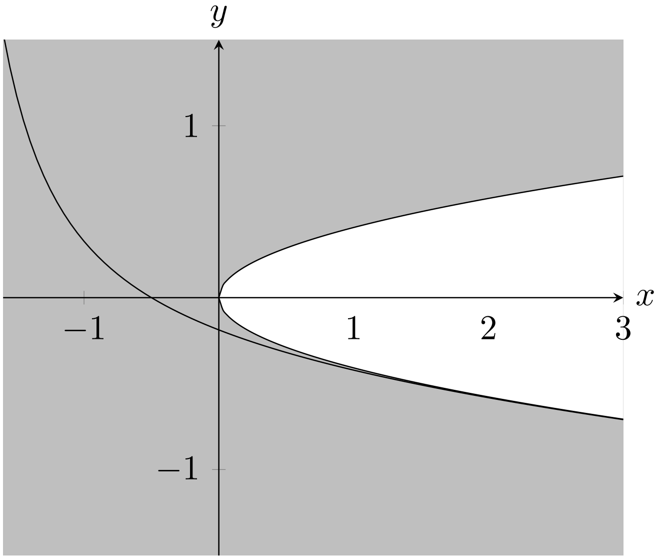

FIGURE 1.6: Plot of the tritronquée solution of Equation (1.3), which is real on the real line.

In 1979, M. J. Ablowitz, M. D. Kruskal and H. Segur [AKS79] have first identified a one-parameter family of solutions of Equation (1.8) with $a = 0$, admitting the asymptotic behaviour

$$y(x) \sim k\,\mathrm{Ai}(x), \quad x \to +\infty. \tag{1.9}$$

Here the parameter k is real, and Ai and Bi are the standard basis of the solution space of the ODE

$$w'' = xw,$$

with asymptotic behaviour

$$\mathrm{Ai}(x) \sim \frac{\mathrm{e}^{-\frac{2}{3}x^{3/2}}}{2\pi^{1/2}x^{1/4}}, \quad x \to +\infty,$$
$$\mathrm{Bi}(x) \sim \frac{\mathrm{e}^{\frac{2}{3}x^{3/2}}}{\pi^{1/2}x^{1/4}}, \quad x \to +\infty.$$

The behaviour (1.9) came about out of their studies of self-similar solutions of the modified Korteweg-de Vries (MKdV) PDE

$$v_t - 6v^2 v_x + v_{xxx} = 0.$$

The authors have managed to relate the asymptotic behaviour of Equation (1.9) with other asymptotic behaviours, depending on the value of $|k|$:

- In the case where $|k| < 1$, we have that
  $$y(x) \sim d(k)|x|^{-\frac{1}{4}} \sin\left(\frac{2}{3}|x|^{\frac{3}{2}} - \frac{3}{4}d^2(k)\ln|x| - c(k)\right), \quad x \to -\infty, \tag{1.10}$$
  where numerical observations suggested that
  $$d^2(k) = -\frac{1}{\pi}\ln\left(1-k^2\right). \tag{1.11}$$
- In the case where $|k| = 1$, we have that
  $$y(x) \sim \operatorname{sgn}(k)\left(\left|\frac{x}{2}\right|^{\frac{1}{2}} - \frac{1}{2}|2x|^{-5/2} + \mathcal{O}\left(|x|^{-\frac{11}{2}}\right)\right), \quad x \to -\infty. \tag{1.12}$$
- In the case where $|k| > 1$, the authors have found that the solution has a simple pole $x_0 = x_0(k)$ somewhere on the real axis:
  $$y(x) \sim \operatorname{sgn}(k)\left(\frac{1}{x-x_0} - \frac{x_0}{6}(x-x_0) + \mathcal{O}\left((x-x_0)^2\right)\right), \quad x \to x_0^+. \tag{1.13}$$

Remark 1.10. *The conjectured formula (1.11) was proven by P. A. Clarkson and J. B. McLeod in [CM88]. An exact formula for the phase term, $c(k)$, of Equation (1.10) was given for the first time by H. Segur and M. J. Ablowtiz [SA81], and was re-derived by P. A. Deift and X. Zhou, using Riemann-Hilbert methods [DZ95]. The formula for the phase term is*

$$c(k) = \frac{3}{2}d^2(k)\ln 2 - \ln\Gamma\left(\frac{i}{2}d^2(k)\right) - \frac{\pi}{2}\operatorname{sgn}(k) - \frac{\pi}{4},$$

with $d^2(k)$ given by Equation (1.11).

In their 1980 paper [HM80], S. P. Hastings and J. B. McLeod studied the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - xy = 2y|y|^{\alpha}, \tag{1.14}$$

over the real line, with α being a positive quantity [1]. They have found that the solutions of Equation (1.14) satisfying the boundary condition

$$\lim_{x\to+\infty} y(x) = 0,$$

---

[1] The parameter α used here should not be confused with the parameter of $\mathrm{P_{II}}$ in its traditional form.

are precisely the solutions y_k of the family of integral equations [2]

$$y_k(x) = k\,\mathrm{Ai}(x) + 2\pi \int_x^{+\infty} \{\mathrm{Ai}(x)\,\mathrm{Bi}(t) - \mathrm{Bi}(x)\,\mathrm{Ai}(t)\} y_k(t)|y_k(t)|^{\alpha} \mathrm{d}t, \tag{1.15}$$

which may obtained via the method of variation of parameters.

Moreover, the authors have shown the existence of a unique $k^* = k^*(\alpha) > 0$ with the property that solutions y_k with $k > k^*$ are positive over their interval of existence, which is bounded below, whilst solutions y_k with $0 < k < k^*$ cross the x-axis at some point. The case $k = k^*$ gives a non-vanishing solution which is continuously defined over the entire real line (see Figure 1.7).

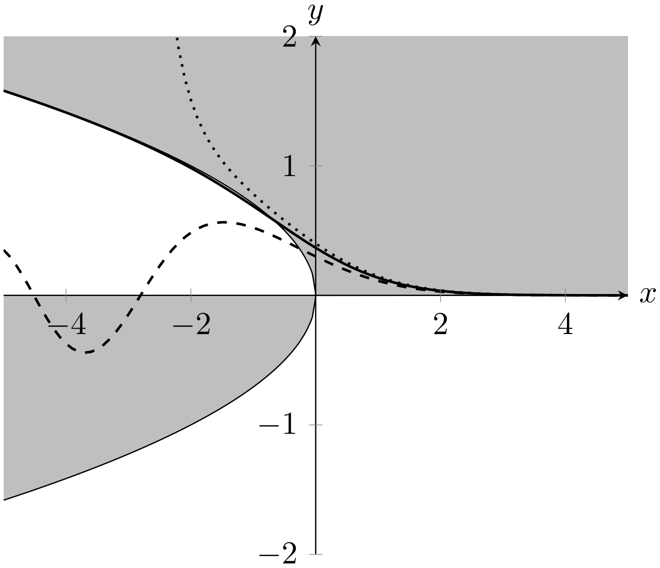

FIGURE 1.7: Convexity plot of Equation (1.14) with $\alpha = 2$: solutions are convex in the shaded region, and concave in the white region. Graphs of solutions y_k with $k < k^*$ (dashed), $k = k^*$ (solid) and $k > k^*$ (dotted), obtained via shooting methods, are plotted as well.

When $\alpha = 2$, Equations (1.10), (1.12) and (1.13) suggest that $k^*(2) = 1$. This is truly the case, as was shown in [HM80]. In addition, when $\alpha = 2$, the solutions of Equations (1.14),(1.15) are solutions of the second Painlevé equation. The solution $y_{k^*(2)}$ is known as the Hastings-McLeod solution, and it is characterised by its boundary behaviour

$$\begin{aligned} y(x) &\to 0, & x &\to +\infty, \\ y(x) &\sim \sqrt{\frac{-x}{2}}, & x &\to -\infty. \end{aligned}$$

To the best of our knowledge, a rigorous proof regarding the existence and uniqueness of solutions of Equation (1.15) appears not to be complete in the literature. We will thus provide a functional-analytic proof for the special case $\alpha = 2$ in Section 2.1.

An extension of Dubrovin's conjecture to $\mathrm{P_{II}}$, made by V.Y. Novokshenov in [Nov14] has been proven for the Hastings-McLeod solution in [HXZ16]. In other words, the solution was proved to have no poles in the region $\arg x \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right] \cup \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$ (see

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[2] Equation (2.1) of [HM80] is in fact missing the factor of π, which is the reciprocal of the Wronskian of the Airy functions Ai and Bi.

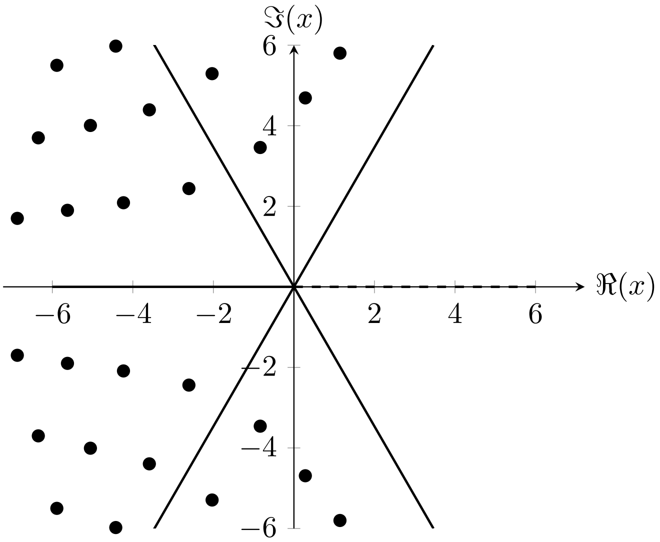

FIGURE 1.8: Illustration of the pole distribution of a tronquée solution of P_{II}. The dots represent poles of the Padé approximant of the solution y, of P_{I} of order $[60/60]$ with initial values $y(0) = 0.1$ and $y'(0) = -0.07348321929513198102$ (see Appendix A.2).

Figure 1.9). This was done via the construction of quasi-solutions approximating the Hastings-McLeod solution in various domains. Triply truncated solutions exist for P_{II} as well [Nov12], however, they are no longer real-valued over the real line.

1.2.3 The Fourth Painlevé Equation

In their 1992 paper [CM92] P.A. Clarkson and J.B. McLeod studied the boundary value problem for the fourth Painlevé equation

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2} = 3\eta^5 + 2\xi\eta^3 + \left(\frac{1}{4}\xi^2 - \nu - \frac{1}{2}\right)\eta, \tag{1.16a}$$

$$\lim_{\xi\to+\infty} \eta\left(\xi\right) = 0. \tag{1.16b}$$

In particular, it is proved that the BVP (1.16) admits solutions $\eta = \eta_k(\xi;\nu)$, with

$$\eta_k(\xi;\nu) \sim kD_\nu(\xi), \quad \text{as} \quad \xi \to +\infty, \tag{1.17}$$

for some nonzero k, where D_ν denotes the *parabolic cylinder function* [DLMF] . Moreover, any nonzero solution of Equations (1.16) is in fact of the form η_k for some $k \neq 0$ (the case $k = 0$ corresponds to the constant solution $\eta(\xi) \equiv 0$). An important result of [CM92] is the following theorem.

Theorem 1.11 ([CM92], Theorem 4.2). *If $|k| < k^*_\nu$, for some k^*_ν, the solution asymptotic to $kD_\nu(\xi)$ exists for all ξ, has the same number of zeros as $D_\nu(\xi)$ and as $\xi \to -\infty$*

$$\eta_k(\xi;\nu) \sim \kappa_n D_n(\xi), \qquad \text{if } \nu = n \in \mathbb{Z}^+, \tag{1.18a}$$

$$\eta_k(\xi;\nu) \sim \pm\left(-\frac{1}{6}\xi\right)^{1/2} + d\left(-\xi\right)^{-1/2}\sin\phi(\xi), \qquad \text{if } \nu \notin \mathbb{Z}^+, \tag{1.18b}$$

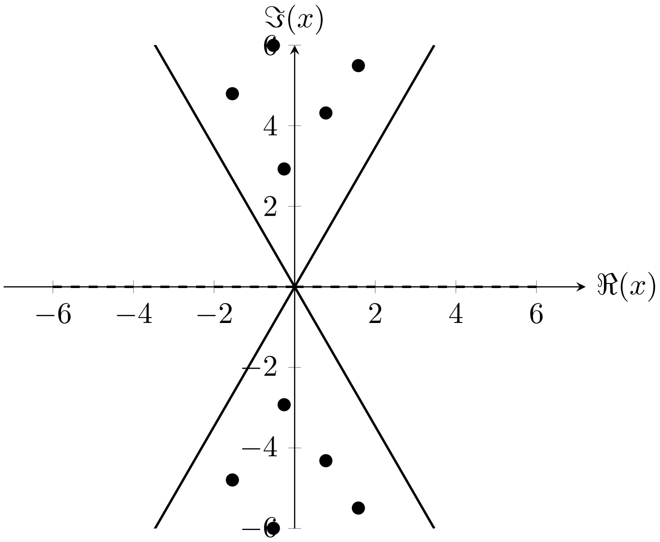

FIGURE 1.9: Illustration of the pole distribution of the Hastings-McLeod solution of P_{II}. The dots represent poles of the Padé approximant of the solution y, of P_I of order $[60/60]$ with initial values $y(0) = 0.3670615515480784$ and $y'(0) = y'(0) = -0.2953721054475501$ (see Appendix A.2).

with

$$\phi(\xi) = \frac{\xi^2}{2\sqrt{3}} + \frac{1}{2}\sqrt{3}\left(\nu + \frac{1}{2}\right)\ln|\xi| + c + \mathcal{O}\left(\xi^{-2}\right),$$

and where the constants κ_n, d, c are dependent on k. If $|k| = k_\nu^*$, then as $\xi \to -\infty$

$$\eta_k(\xi;\nu) \sim \operatorname{sgn}(k)\left(-\frac{1}{2}\xi\right)^{1/2}, \tag{1.19}$$

and if $|k| > k_\nu^*$, then $\eta_k(\xi;\nu)$ has a pole at a finite point ξ_0, dependent on k, so

$$\eta_k(\xi;\nu) \sim \operatorname{sgn}(k)(\xi - \xi_0)^{-1}, \quad \text{as } \xi \downarrow \xi_0.$$

Theorem 1.11 has been proved in the case in which the parameter ν is an integer [BCHM92], and numerical evidence of its validity for some of the non-integer values of ν can be found in [BCH93]. Moreover, connection formulae have been established for P_{IV} transcendents. The connection formulae were expressed erroneously in [BCHM92, LW13] and correctly in [BCH93, TW94], with the first rigorous proof given in [IK98] using the isomonodromy and Riemann-Hilbert methods. In greater detail, the change of variables

$$\eta(\xi) = 2^{-\frac{3}{4}}\sqrt{w(x)}, \quad x = \frac{\xi}{\sqrt{2}},$$

transforms Equation (1.16a) into

$$w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4xw^2 + 2\left(x^2 - (2\nu+1)\right)w,$$

which is P_{IV} with parameter values $a = 2\nu+1$ and $b = 0$. The family $\{\eta_k\}_{k\in\mathbb{R}}$ transforms into a family $\{w_k\}_{k\in\mathbb{R}}$ with the asymptotic behaviour

$$w_k(x) \sim k^2 2^{3/2} D_\nu^2\left(\sqrt{2}x\right), x \to +\infty.$$

In the case where $\nu \notin \mathbb{Z}$ and

$$0 < k^2 2\sqrt{2\pi}\Gamma\left(\alpha + \frac{1}{2}\right) < 1,$$

we have that

$$w_k(x) = -\frac{2x}{3} + 2\sqrt{2}a\cos\left(\frac{x^2}{\sqrt{3}} - \sqrt{3}a^2\ln\left(2\sqrt{3}x^2\right) + \phi\right) + \mathcal{O}\left((-x)^{-1/4}\ln(-x)\right), \tag{1.20}$$

as $x \to -\infty$, with

$$a^2 = -\frac{1}{2\sqrt{3}\pi}\ln\left(1 - |s_-|^2\right), \quad a > 0, \tag{1.21a}$$

$$\phi = -\frac{3\pi}{4} - \frac{2\pi}{3}\alpha - \arg\Gamma\left(-i\sqrt{3}a^2\right) - \arg s_-, \tag{1.21b}$$

$$s_- = 1 - \frac{2(2\pi)^{3/2}\mathrm{e}^{-i\pi\alpha}}{\Gamma\left(\frac{1}{2} - \alpha\right)}k^2. \tag{1.21c}$$

In the case where $\alpha = 0$, the value

$$k^2 = \frac{\Gamma\left(\frac{1}{2}\right)}{2(2\pi)^{3/2}} = \frac{1}{4\sqrt{2\pi}}.$$

results in a solution which has no oscillatory term in Equation (1.20). The pole distribution of this solution has been studied numerically in [RF13]. We will see such non-oscillatory behaviour in Section 2.2.

Additional studies of the asymptotic behaviours near infinity of P_{IV} transcendents have also been carried out [Kap98]. Using the notation of Equation (1.16a), they entail, among other things, the following families

- One solution with $\eta(\xi) \sim \sqrt{-\xi/6}$ as $\xi \to -\infty$.
- One-parameter family of solutions with $\eta(\xi) \sim \sqrt{-\xi/2}$ as $\xi \to -\infty$.
- One-parameter family of solutions with $\eta(\xi) \sim C/\sqrt{-\xi}$ as $\xi \to -\infty$, for some constant C.

We will see how some of these solutions arise, starting with a power series ansatz in Section 2.2. Furthermore, we will provide estimates on the intervals of existence of some of these solutions using classical methods such as estimations of integrals.

Unlike the previous Painlevé equations, estimates on the interval of existence of distinguished solutions of the fourth Painlevé equation are not known (There are however continuity results over rays in the complex plane of the independent variable [Fok06]). Furthermore, a deep understanding of the distribution of poles of the fourth Painlevé transcendents is currently lacking. Further asymptotics of other Painlevé equations can be found in [Fok06].

1.3 Discrete Painlevé Equations

In this section we will discuss discrete equations, and in particular introduce the discrete Painlevé equations, which are discrete analogues of the continuous Painlevé equations of Section 1.2. Some well-known scalar discrete Painlevé equations are presented in Table 1.3, taken from [GR04].

Name	Equation
d-$\mathrm{P_I}$	$x_{n+1} + x_n + x_{n-1} = \dfrac{z_n}{x_n} + 1$
d-$\mathrm{P_{II}}$	$x_{n+1} + x_{n-1} = \dfrac{z_n x_n + a}{1 - x_n^2}$
q-$\mathrm{P_{III}}$	$x_{n+1} x_{n-1} = \dfrac{(x_n - a q_n)(x_n - b q_n)}{(1 - c x_n)(1 - x_n/c)}$
d-$\mathrm{P_{IV}}$	$(x_{n+1} + x_n)(x_n + x_{n-1}) = \dfrac{(x_n^2 - a^2)(x_n^2 - b^2)}{(x_n - z_n)^2 - c^2}$
q-$\mathrm{P_V}$	$(x_{n+1} x_n - 1)(x_n x_{n-1} - 1) = \dfrac{(x_n - a)(x_n - 1/a)(x_n - b)(x_n - 1/b)}{(1 - c x_n q_n)(1 - x_n q_n/c)}$
d-$\mathrm{P_V}$	$\dfrac{(x_n + x_{n+1} - z_n - z_{n+1})(x_n + x_{n-1} - z_n - z_{n-1})}{(x_n + x_{n+1})(x_n + x_{n-1}))}$ $= \dfrac{(x_n - z_n - a)(x_n - z_n + a)(x_n - z_n - b)(x_n - z_n + b)}{(x_n - c)(x_n + c)(x_n - d)(x_n + d)}$
q-$\mathrm{P_{VI}}$	$\dfrac{(x_n x_{n+1} - q_n q_{n+1})(x_n x_{n-1} - q_n q_{n-1})}{(x_n x_{n+1} - 1)(x_n x_{n-1} - 1)}$ $= \dfrac{(x_n - a q_n)(x_n - q_n/a)(x_n - b q_n)(x_n - q_n/b)}{(x_n - c)(x_n - 1/c)(x_n - d)(x_n - 1/d)}$

TABLE 1.3: A selected list of scalar discrete Painlevé equations. Here $z_n = \alpha n + \beta$, $q_n = q_0 \lambda^n$ and $a, b, c, d, \alpha, \beta, \lambda$ are constants.

1.3.1 General Discrete Equations

Classical examples of discrete equations are the linear difference equations of the form

$$x_n = \sum_{k=1}^{N} a_k x_{n-k},$$

where n, the independent variable, is an integer, $a_1, \dots, a_N$ are complex coefficients, and the sequence $(x_n)_{n=0}^{\infty}$ of complex numbers plays the role of the dependent variable.

In the case of linear difference equations it is readily seen, by iteration, that the solutions exist for all n if the coefficients are well defined. Moreover, in the general case, the solution can be expressed in closed form using the power ansatz $x_n = \lambda^n$ and the linear superposition principle. However, in the nonlinear case, movable singularities may occur, as the example below illustrates.

Example 1.12. *The difference equation*

$$x_{n+1} = x_n - \frac{x_n^2 + 1}{2x_n},$$

with the initial value $x_0 = 1$, leads to $x_1 = 0$, and consequentially to x_2 being undefined.

1.3.2 The Method of Singularity Confinement

The singularity of Example 1.12 is not avoidable. At the early 1990s A. Ramani, B. Grammaticos and V. Papageorgiou [GRP91, RGP94] presented a method that allows one to "iterate through" singularities in some equations. This method is known as *singularity confinement* , and we will illustrate it with an example, taken from [GR04].

Example 1.13. *Consider the equation*

$$x_{n+1} + x_n + x_{n-1} = a_n + \frac{b_n}{x_n}, \tag{1.22}$$

and suppose that for some n, x_n is regular while $x_{n+1} = 0$. If we take ϵ to be small, and set $x_{n+1} = \epsilon$, we get

$$x_{n+2} = \frac{b_{n+1}}{\epsilon} + a_{n+1} - x_n + \mathcal{O}(\epsilon),$$

and

$$x_{n+3} = -\frac{b_{n+1}}{\epsilon} + a_{n+2} - a_{n+1} + x_n + \mathcal{O}(\epsilon),$$

which are singular as $\epsilon \to 0$ for nonzero b_{n+1}. The next term, x_{n+4} diverges, unless $a_{n+3} - a_{n+2} = 0$, rendering the sequence a_n constant, in which case

$$x_{n+4} = \frac{b_{n+1} - b_{n+2} - b_{n+3}}{b_{n+1}}\epsilon + \mathcal{O}(\epsilon^2),$$

is finite as $\epsilon \to 0$. In order to avoid the scenario where x_{n+5} is undefined, we obtain the relation

$$b_{n+1} - b_{n+2} - b_{n+3} + b_{n+4} = 0,$$

whose solution is $b_n = \alpha n + \beta + \gamma(-1)^n$. Thus, the method had reduced Equation (1.22) into a four-parameter family of equations, one of them being

$$x_{n+1} + x_n + x_{n-1} = a + \frac{z_n}{x_n},$$

which is a discrete form of P_I.

Aside from Example 1.13, one historical approach to obtaining discrete Painlevé equations was to start with some general functional form, and apply the method of singularity confinement in order to obtain constraints on its parameters. This method is now known not to be sufficient for integrability and further tests have been developed to identify integrable discrete equations. We mention for example the counterexample given by [HV98].

The discrete Painlevé equations can be thought of as a "boundary" between the integrable- and non-integrable- discrete systems of order two, in the sense that any simpler system is trivially solvable (usually linearisable) - while more complicated equations are not integrable.

The continuum limits allow one to study the Painlevé equations themselves via their discrete counterparts: for example, the discrete Painlevé equations are integrable discretisations of the continuous ones, creating potential methods for the numerical study of the classical Painlevé equations.

The first instance of a discrete, non-autonomous integrable system appears in the work of E. Laguerre [Lag85], albeit the system he had discussed was of order greater than two.

In Chapter 3 we will explore the connection between the discrete first Painlevé equation in the form

$$\text{d-P}_{\text{I}}: \quad w_{n+1} + w_n + w_{n-1} = -2t + \frac{\gamma_n}{w_n},$$

and the fourth Painlevé equation P_{IV}. In particular we will use known asymptotic behaviours for P_{IV} in the case where the independent variable turns large, in order to derive asymptotic properties of d-P_{I} when its parameter turns large.

1.4 Outline of the Thesis

In Section 2.1 of this thesis, we will provide, for the first time, a rigorous proof of the existence and uniqueness of a one-parameter family of solutions of the second Painlevé equation, as stated in [HM80]. This reference asserts that a certain integral equation can be solved by iteration, but does not explicitly provide the proof. We provide the missing proof. In particular, we prove in Theorem 2.2 that this integral equation has a unique fixed point in an appropriate space of functions by applying a fixed point theorem.

In Section 2.2 we prove Theorem 1.11, which gives the existence and uniqueness of a family of solutions $\{\eta_k(\xi, -1/2)\}\}_{k\in\mathbb{R}}$ of the fourth Painlevé equation.

In Section 2.2 we will formulate, for the first time, a rigorous existence and uniqueness proof for the family of solutions $\{\eta_k(\xi, -1/2)\}\}_{k\in\mathbb{R}}$ of the fourth Painlevé equation stated in Theorem 1.11. In addition we demonstrate the existence of a one-parameter family of monotonically decreasing solutions of P_{IV}, defined over the ray $[0, +\infty)$. We then prove the existence of tronquée and tritronquée solutions for P_{IV}, using Wasow's theorem (Theorem 2.5). Although solutions of P_{IV} with no poles in certain directions in the complex plane have been found before [Fok06], our proof of the existence and uniqueness of the tritronquée solution of P_{IV} is the first time a solution is shown to be free of poles of large modulus in a half-plane. This is one of the major results of this thesis. Next, we study a solution which has the same leading asymptotic behaviour as one of the members of the family $\{\eta_k(\xi, -1/2)\}\}_{k\in\mathbb{R}}$. In particular we study the solution's behaviour on the real axis using classical methods, as was done for the tritronquée solution of the first Painlevé equation in [JK01], and deduce, for the first time, its extended existence over an interval of the form $(\alpha, +\infty)$ for a constant α which is estimated numerically. This is another major result of the thesis.

In Section 2.3 we use the Hamiltonian structure of the fourth Painlevé equation and study its behaviour near the equilibrium points of its limiting autonomous equation. In particular, after applying a certain change of variables, the methods used in [DJ11], which rely on a Hamiltonian formulation of P_{I}, become available. Using them we obtain an additional existence proof of some of the truncated solutions of the fourth Painlevé equation.

In Section 2.4 we provide the Mathematica code that has been used in order to numerically compute the solutions of certain IVPs of P_{IV} in Section 2.2.

Finally, in Chapter 3 we discuss the discrete first Painlevé equation, and its relationship with the fourth Painlevé equation. In particular, we deduce new asymptotic behaviours of the discrete first Painlevé equation in the limit as a parameter approaches infinity.

1.4.1 Outline of Main Results

In this section, we collect and outline the main results of the thesis.

Our main original results fall into three parts. The first part concerns the existence of physically important solutions on the real line for the second and fourth Painlevé equations. Although such solutions have been studied before, in detail for P_{II} and in lesser detail for P_{IV}, certain important aspects of the study were missing the literature. We fill such gaps. In particular, we provide proofs of existence and uniqueness for bounded solutions with certain asymptotic behaviours on the real line, described for P_{II} by Hastings and McLeod [HM80] and for P_{IV} by [CM92]. The second part concerns the analytic continuation and uniqueness of the solution for P_{IV} in an extended sector in the complex plane. Our results show that tritronquee solutions exist for P_{IV} and are unique. The third part concerns a related study of solutions of the discrete first Painlevé equation in a limit that is related to our results for P_{IV}.

Theorem 2.2 of Section 2.1 uses tools from functional analysis in order to establish the existence and uniqueness of a one-parameter family of solutions of a special case of the second Painlevé equation. The family is characterised by the asymptotic behaviour of the solutions as the independent variable approaches infinity. We note that this result has been referenced and used numerous times in the literature without a rigorous proof. To the best of our knowledge, our proof is the first one given in the literature for this widely used result.

Theorem 2.3 of Section 2.2 is analogous to Theorem 2.2 of Section 2.1. It provides a functional-analytic proof of the existence and uniqueness of a one parameter family of solutions of a special case of the second Painlevé equation. This family is similarly characterised by the asymptotic behaviour of the solutions as the independent variable approaches infinity. We do not know of any rigorous proof of the existence and uniqueness this family, making our result novel.

Theorem 2.8 of Section 2.2.2 provides the existence and uniqueness of a pair of solutions, $Y_{\pm}$, of a special case of the fourth Painlevé equation, which have no poles of large modulus in the half-plane $\Re(x) > 0$. The pair $Y_{\pm}$ is analogous to similar solutions of other Painlevé equations [JK01, Nov12]. The behaviour of $Y_{\pm}$ over the real line is later studied in Theorem 2.28 of Section 2.2.4, where an estimation of the interval of existence is obtained. The existence and uniqueness of $Y_{\pm}$, combined with the study of their behaviour over the real axis is the key result of this thesis.

Chapter 2

The Painlevé Equations

In this chapter we consider the existence of tritronquée solutions of the fourth Painlevé equation. In particular, we show regularity properties of such solutions on the real line. Similar problems for the other Painlevé equations have been considered in the literature [JK01]. To lay the groundwork for new results, we also provide rigorous proofs here for the existence and uniqueness of analogous solutions of the second Painlevé equation, which include the so-called Hastings-McLeod solution.

The existence of true solutions that are asymptotic to power series expansions relies on Wasow's theorem [Was02]. These expansions are valid in a sector near the limit of interest, which is infinity in this case. To extend the solution into finite regions, we use functional analytic arguments and estimates, which provide us with a lower bound for the interval of existence.

Before we consider the approach of this chapter, we set the scene by transforming $\mathrm{P_{IV}}$ to a form in which the first derivative no longer appears. Both $\mathrm{P_I}$ and $\mathrm{P_{II}}$ have this form, which makes the subsequent arguments simpler. For each given pair of parameters $a \in \mathbb{C}$, $b \in \mathbb{C}$, $\mathrm{P_{IV}}$ is traditionally written as

$$\mathrm{P_{IV}}(a,b): \quad \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \frac{1}{2w}\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2-a)w + \frac{b}{w}. \tag{2.1}$$

Clearly, it involves the first derivative $w'(z)$ on the right-hand side. However, there is a change of dependent variable

$$w(z) = y^2(z),$$

which results in a new equation of the form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} = f(z,y),$$

where f is rational in y and polynomial in x. (See Equation (2.16).)

We will focus on the case $a = 0 = b$ in $\mathrm{P_{IV}}$, because it leads to f being polynomial in y. We choose this case for conciseness and simplicity. We believe the methods provided in this chapter may be extended to the case of nonzero a without major change in the proofs. We also take the parameter in $\mathrm{P_{II}}$ to be zero for similar reasons.

We consider the limit $x \to \pm\infty$ on the real line in both directions. We use standard arguments based on the fixed point theorem on the positive semi-axis while majorizing arguments are used on the negative semi-axis.

We start by considering $\mathrm{P_{II}}$ in Section 2.1. The main results of the thesis lie in Section 2.2 where we consider solutions of $\mathrm{P_{IV}}$. A different perspective is explored in Section 2.3 where we study solution behaviours in the neighbourhood of equilibria of the time-dependent Hamiltonian of $\mathrm{P_{IV}}$. The chapter concludes with Section 2.4, where the initial value problem of $\mathrm{P_{IV}}$ is explored numerically. These numerical results support and illustrate the qualitative results proved in Section 2.2.

2.1 The Second Painlevé Equation

Although referred to in the literature, the one-parameter family of solutions of Equation (1.15) with $\alpha = 2$ does not have a rigorous existence and uniqueness result. We provide such a proof here. Our proof relies on the following formulation of the contraction mapping theorem for integral operators, given in [GLS90].

Theorem 2.1 (Theorem 11.2.1 of [GLS90])**.** *Assume the following:*

(i) *$\mathcal{K}$ is a closed subset of a Banach space $\mathcal{B}$ of functions taking an interval $J \subseteq \mathbb{R}$ into $\mathbb{R}^n$;*

(ii) *F and G are operators mapping $\mathcal{K}$ into $\mathcal{B}$ satisfying*

$$\|F(x) - F(y)\| \leq L_F \|x - y\|, \quad \text{and} \quad \|G(x) - G(y)\| \leq L_G \|x - y\|,$$

for some constants L_F and L_G and for all x and y in $\mathcal{K}$;

(iii) *the mapping*

$$x \mapsto r \star x,$$

defined by

$$(r \star x)(t) = \int_J r(t, s) x(s) \mathrm{d}s,$$

is a linear and continuous operator from $\mathcal{B}$ into itself, with norm ϱ;

(iv) *the function*

$$x \mapsto F(x) + r \star G(x),$$

defined by

$$\left(F(x) + r \star G(x)\right)(t) = F(x)(t) + \int_J r(t, s) G(x)(s) \mathrm{d}s,$$

takes $\mathcal{K}$ into itself;

(v) *the constants L_F, L_G and ϱ satisfy $L_F + L_G \varrho < 1$.*

Then

$$x(t) = F(x)(t) + \int_J r(t, s) G(x)(s) \mathrm{d}s, \quad t \in J,$$

has a unique solution in $\mathcal{K}$.

Using the above, we can prove the following

Theorem 2.2. *Fix a real number k, and a number $M > |k|$. Let*

$$\mathcal{B} = C_w(\mathbb{R}), \tag{2.2}$$

be the weighted space of continuous functions defined over the real line with the weight function

$$w(t) := \frac{H(t - t_0)}{\mathrm{Ai}(t)},$$

and norm

$$\|u\| = \sup_{t\in\mathbb{R}} |w(t)u(t)|. \tag{2.3}$$

Here t_0 *is chosen larger than all zeros of* Ai'*, And* H *denotes the Heaviside step function, defined by*

$$H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}.$$

Also, let

$$\mathcal{K} = \{u \in \mathcal{B} : \|u\| \leq M\}, \tag{2.4}$$

denote the closed ball in $\mathcal{B}$ *of radius* M *centered at zero. The integral equation*

$$y_k(t) = k\,\mathrm{Ai}(t) + 2\pi \int_t^{+\infty} \{\mathrm{Ai}(t)\,\mathrm{Bi}(s) - \mathrm{Bi}(t)\,\mathrm{Ai}(s)\} y_k^3(s)\mathrm{d}s, \tag{2.5}$$

has a unique solution in $\mathcal{K}$*, provided that* t_0 *is chosen large enough so that*

$$\begin{aligned} \rho(t_0) &\leq \frac{M-|k|}{M^3}, \\ \rho(t_0) &< \frac{1}{3M^2}, \end{aligned}$$

where

$$\rho(t_0) := \frac{2\pi}{\mathrm{Ai}(t_0)} \int_{t_0}^{+\infty} \mathrm{Ai}^3(s)\left(\mathrm{Ai}(t_0)\,\mathrm{Bi}(s) - \mathrm{Ai}(s)\,\mathrm{Bi}(t_0)\right)\mathrm{d}s.$$

In particular, Equation (2.5) *with* $k \neq 0$ *has a unique solution in*

$$\mathcal{K} = \left\{u \in \mathcal{B} : \|u\| \leq \frac{3}{2}|k|\right\},$$

provided that

$$\rho(t_0) < \frac{4}{27k^2}.$$

Proof. The proof is an application of Theorem 2.1, with $\mathcal{B}$ and $\mathcal{K}$ as defined by equations (2.2) and (2.4) respectively, $J = \mathbb{R}, n = 1$ and in addition

$$\begin{aligned} F(x)(t) &= k\,\mathrm{Ai}(t), \\ G(x)(s) &= \frac{x^3(s)}{\mathrm{Ai}^2(s)}, \\ r(t,s) &= 2\pi\,\mathrm{Ai}^2(s)H(s-t)\left(\mathrm{Ai}(t)\,\mathrm{Bi}(s) - \mathrm{Ai}(s)\,\mathrm{Bi}(t)\right). \end{aligned}$$

The condition (i) of Theorem 2.1 clearly holds. Let $x \in \mathcal{K}$, then by definition

$$\begin{aligned} \|F(x)\| &= \sup_{t\in\mathbb{R}} \left|\frac{k\,\mathrm{Ai}(t)}{\mathrm{Ai}(t)} H(t-t_0)\right| = |k| < +\infty, \\ \|G(x)\| &= \sup_{t\in\mathbb{R}} \left|\frac{x^3(t)}{\mathrm{Ai}^3(t)} H(t-t_0)\right| = \sup_{t\geq t_0} \left|\frac{x^3(t)}{\mathrm{Ai}^3(t)}\right| = \|x\|^3 = M^3 < +\infty. \end{aligned}$$

Since F is independent of x, we may take $L_F = 0$. Furthermore given $x, y \in \mathcal{K}$

$$\begin{aligned}\|G(x) - G(y)\| &= \sup_{s\in\mathbb{R}} \left| \frac{x^3(s) - y^3(s)}{\mathrm{Ai}^2(s)} \frac{H(s-t_0)}{\mathrm{Ai}(s)} \right| \\ &= \sup_{s\geq t_0} \left| \frac{x^2(s) + x(s)y(s) + y^2(s)}{\mathrm{Ai}^2(s)} \frac{x(s) - y(s)}{\mathrm{Ai}(s)} \right| \\ &\leq 3M^2 \|x - y\|.\end{aligned}$$

Thus we have $L_G \leq 3M^2$, and the validity of the condition (ii) of Theorem 2.1.

Observe that by the definition of the weighted norm (2.3), we have

$$\begin{aligned}\|r \star x\| &= \sup_{t\geq t_0} \left| \frac{2\pi}{\mathrm{Ai}(t)} \int_t^{+\infty} \mathrm{Ai}^2(s) \left(\mathrm{Ai}(t)\,\mathrm{Bi}(s) - \mathrm{Ai}(s)\,\mathrm{Bi}(t)\right) x(s)\mathrm{d}s \right| \\ &= \sup_{t\geq t_0} \frac{2\pi}{\mathrm{Ai}(t)} \left| \int_t^{+\infty} \mathrm{Ai}^2(s) \left(\mathrm{Ai}(t)\,\mathrm{Bi}(s) - \mathrm{Ai}(s)\,\mathrm{Bi}(t)\right) x(s)\mathrm{d}s \right| \\ &= \sup_{t\geq t_0} \frac{2\pi}{\mathrm{Ai}(t)} \left| \int_t^{+\infty} \mathrm{Ai}^3(s) \left(\mathrm{Ai}(t)\,\mathrm{Bi}(s) - \mathrm{Ai}(s)\,\mathrm{Bi}(t)\right) \frac{x(s)}{\mathrm{Ai}(s)}\mathrm{d}s \right| \\ &\leq \sup_{t\geq t_0} \frac{2\pi}{\mathrm{Ai}(t)} \int_t^{+\infty} \left|\mathrm{Ai}^3(s) \left(\mathrm{Ai}(t)\,\mathrm{Bi}(s) - \mathrm{Ai}(s)\,\mathrm{Bi}(t)\right)\right| \left|\frac{x(s)}{\mathrm{Ai}(s)}\right| \mathrm{d}s.\end{aligned}$$

Using Hölder's inequality on the final integral above, we obtain further

$$\begin{aligned}\|r \star x\| &\leq \sup_{t\geq t_0} \frac{2\pi}{\mathrm{Ai}(t)} \left(\sup_{s\geq t} \left| \frac{x(s)}{\mathrm{Ai}(s)} \right| \right) \int_t^{+\infty} \left|\mathrm{Ai}^3(s) \left(\mathrm{Ai}(t)\,\mathrm{Bi}(s) - \mathrm{Ai}(s)\,\mathrm{Bi}(t)\right)\right| \mathrm{d}s \\ &\leq \left(\sup_{t\geq t_0} \frac{2\pi}{\mathrm{Ai}(t)} \int_t^{+\infty} \mathrm{Ai}^3(s) \left|\mathrm{Ai}(t)\,\mathrm{Bi}(s) - \mathrm{Ai}(s)\,\mathrm{Bi}(t)\right| \mathrm{d}s \right) \|x\|.\end{aligned}$$

It follows that the operator

$$x \mapsto r \star x,$$

is continuous, with operator norm

$$\varrho(t_0) \leq \sup_{t\geq t_0} \frac{2\pi}{\mathrm{Ai}(t)} \int_t^{+\infty} \mathrm{Ai}^3(s) \left|\mathrm{Ai}(t)\,\mathrm{Bi}(s) - \mathrm{Ai}(s)\,\mathrm{Bi}(t)\right| \mathrm{d}s =: \rho(t_0). \tag{2.6}$$

In fact, applying Leibniz's rule for differentiation under the integral sign yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{2\pi}{\mathrm{Ai}(t)} \int_t^{+\infty} \mathrm{Ai}^3(s) \left|\mathrm{Ai}(t)\,\mathrm{Bi}(s) - \mathrm{Ai}(s)\,\mathrm{Bi}(t)\right| \mathrm{d}s \right) \\ &= -2\pi\,\mathrm{Ai}^2(t) |\,\mathrm{Ai}(t)\,\mathrm{Bi}(t) - \mathrm{Ai}(t)\,\mathrm{Bi}(t)| \\ &+ 2\pi \int_t^{+\infty} \mathrm{Ai}^3(s) \frac{\left(\mathrm{Ai}'(t)\,\mathrm{Bi}(s) - \mathrm{Ai}(s)\,\mathrm{Bi}'(t)\right)\mathrm{Ai}(t) - \left(\mathrm{Ai}(t)\,\mathrm{Bi}(s) - \mathrm{Ai}(s)\,\mathrm{Bi}(t)\right)\mathrm{Ai}'(t)}{\mathrm{Ai}^2(t)} \mathrm{d}s \\ &= -2 \int_t^{+\infty} \frac{\mathrm{Ai}^4(s)}{\mathrm{Ai}^2(t)} \mathrm{d}s < 0,\end{aligned}$$

where we have used the fact that the Wronskian of the Airy functions Ai and Bi is $1/\pi$. It follows that the supremum in Equation (2.6) is attained at $t = t_0$, giving

$$\rho(t_0) = \frac{2\pi}{\operatorname{Ai}(t_0)} \int_{t_0}^{+\infty} \operatorname{Ai}^3(s) \left|\operatorname{Ai}(t_0)\operatorname{Bi}(s) - \operatorname{Ai}(s)\operatorname{Bi}(t_0)\right| \mathrm{d}s,$$

and proving that the condition (iii) of Theorem 2.1 holds.

Let $x \in \mathcal{K}$. We now estimate the norm of $F(x) + r \star G(x)$:

$$\begin{aligned}
\|F(x) + r \star G(x)\| &= \sup_{t \geq t_0} \left| \left(k \operatorname{Ai}(t) + 2\pi \int_t^{+\infty} (\operatorname{Ai}(t)\operatorname{Bi}(s) - \operatorname{Ai}(s)\operatorname{Bi}(t))\, x^3(s) \mathrm{d}s \right) \frac{1}{\operatorname{Ai}(t)} \right| \\
&= \sup_{t \geq t_0} \left| k + \frac{2\pi}{\operatorname{Ai}(t)} \int_t^{+\infty} (\operatorname{Ai}(t)\operatorname{Bi}(s) - \operatorname{Ai}(s)\operatorname{Bi}(t))\, x^3(s) \mathrm{d}s \right| \\
&\leq \sup_{t \geq t_0} \left(|k| + \frac{2\pi}{\operatorname{Ai}(t)} \int_t^{+\infty} \operatorname{Ai}^3(s) (\operatorname{Ai}(t)\operatorname{Bi}(s) - \operatorname{Ai}(s)\operatorname{Bi}(t)) \frac{|x^3(s)|}{\operatorname{Ai}^3(s)} \mathrm{d}s \right) \\
&\leq \sup_{t \geq t_0} \left(|k| + \frac{2\pi}{\operatorname{Ai}(t)} \int_t^{\infty} \operatorname{Ai}^3(s) (\operatorname{Ai}(t)\operatorname{Bi}(s) - \operatorname{Ai}(s)\operatorname{Bi}(t))\, \mathrm{d}s \|x\|^3 \right) \\
&= |k| + \frac{2\pi}{\operatorname{Ai}(t_0)} \int_{t_0}^{\infty} \operatorname{Ai}^3(s) (\operatorname{Ai}(t_0)\operatorname{Bi}(s) - \operatorname{Ai}(s)\operatorname{Bi}(t_0))\, \mathrm{d}s \|x\|^3 \\
&\leq |k| + \rho(t_0) M^3.
\end{aligned}$$

We require that $\|F(x) + r \star G(x)\| \leq M$, resulting in the condition

$$\rho(t_0) \leq \frac{M - |k|}{M^3}, \tag{2.7}$$

under which the condition (iv) of Theorem 2.1 is valid.

The condition (v) of Theorem 2.1 is satisfied if

$$\rho(t_0) < \frac{1}{3M^2}. \tag{2.8}$$

If $k \neq 0$, the choice

$$M = \frac{3}{2}|k|,$$

results in the equivalence of the simultaneous conditions of Equations (2.7), (2.8) with

$$\rho(t_0) < \frac{4}{27k^2}. \tag{2.9}$$

It is routine to show that

$$\lim_{t_0 \to +\infty} \rho(t_0) = 0,$$

and thus Equation (2.9) holds for sufficiently large $t_0 = t_0(k)$. The case $k = 0$ leads to the zero solution. This completes the proof of the theorem. ☐

2.2 The Fourth Painlevé Equation

In this Section, we explore the fourth Painlevé equation from both the real and the complex perspective. In particular, we first prove an existence and uniqueness result the family of functions of Equation (1.17). Then, we provide estimates on intervals on which solutions are analytic. We also discuss solutions which have no poles of large modulus in certain sectors of the complex plane.

2.2.1 Existence and Uniqueness of a One-Parameter Family

We provide an existence and uniqueness proof for the functions η_k of Equation (1.17). We will restrict ourselves to the case in which $\nu = -\frac{1}{2}$. The proof will make use of validity of the inequality

$$\frac{1}{D_{-\frac{1}{2}}(t)} \geq \frac{1}{(1+t^2)^{\frac{1}{4}}},$$

for sufficiently large t (see Figure 2.1).

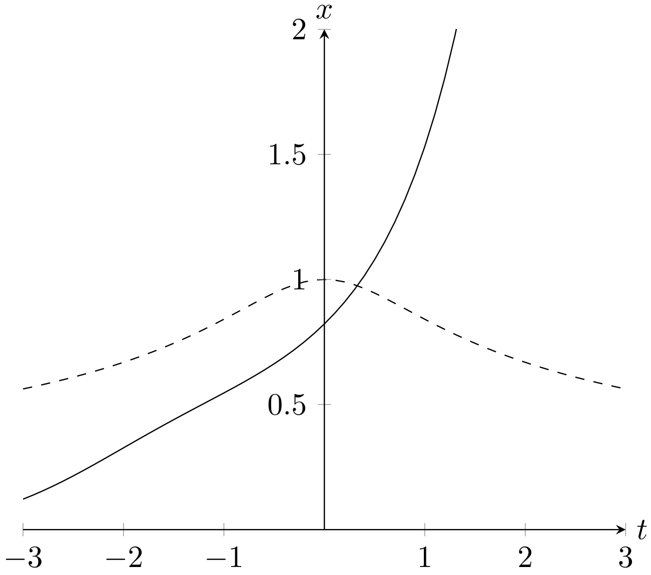

FIGURE 2.1: Plots of the graphs of the functions $1/D_{-1/2}(t)$ (solid), and $1/(1+t^2)^{1/4}$ (dashed).

Theorem 2.3. *Fix real numbers t_0, k and M such that*

$$\frac{1}{D_{-\frac{1}{2}}(t)} \geq \frac{1}{(1+t^2)^{\frac{1}{4}}},$$

for all $t \geq t_0$, and $M > |k|$. Let

$$\mathcal{B} = C_w(\mathbb{R}), \tag{2.10}$$

be the weighted space of continuous functions defined over the real line with the weight function

$$w(t) := \frac{H(t-t_0)}{D_{-\frac{1}{2}}(t)},$$

and norm

$$\|u\| = \sup_{t\in\mathbb{R}} |w(t)u(t)|. \tag{2.11}$$

Also, let

$$\mathcal{K} = \{u \in \mathcal{B} : \|u\| \leq M\}, \tag{2.12}$$

denote the closed ball in $\mathcal{B}$ of radius M centered at zero. The integral equation

$$x(t) = kD_{-\frac{1}{2}}(t) - \sqrt{2}\int_t^{+\infty}\left(D_{-\frac{1}{2}}(t)D_{-\frac{1}{2}}(-s) - D_{-\frac{1}{2}}(s)D_{-\frac{1}{2}}(-t)\right)\left(3x^5(s) + 2sx^3(s)\right)\mathrm{d}s,$$

whose solutions satisfy the differential equation (1.16a) with $\nu = -1/2$, has a unique solution in $\mathcal{K}$, provided that t_0 is chosen large enough so that

$$\rho(t_0) \leq \frac{M - |k|}{3M^5 + 2M^3}, \tag{2.13a}$$

$$\rho(t_0) < \frac{1}{15M^4 + 6M^2}, \tag{2.13b}$$

where

$$\rho(t_0) := \frac{\sqrt{2}}{D_{-\frac{1}{2}}(t_0)}\int_{t_0}^{+\infty}(1+s^2)^{\frac{1}{2}}D^3_{-\frac{1}{2}}(s)\left(D_{-\frac{1}{2}}(t_0)D_{-\frac{1}{2}}(-s) - D_{-\frac{1}{2}}(s)D_{-\frac{1}{2}}(-t_0)\right)\mathrm{d}s.$$

Proof. The proof is an application of Theorem 2.1, with $\mathcal{B}$ and $\mathcal{K}$ as defined by equations (2.10) and (2.12) respectively, $J = \mathbb{R}, n = 1$ and in addition

$$\begin{aligned}
F(x)(t) &= kD_{-\frac{1}{2}}(t), \\
G(x)(s) &= \frac{1}{(1+s^2)^{\frac{1}{2}}D_{-\frac{1}{2}}(s)^2}\left(3x^5(s) + 2sx^3(s)\right), \\
r(t,s) &= -\sqrt{2}(1+s^2)^{\frac{1}{2}}D_{-\frac{1}{2}}(s)^2\left(D_{-\frac{1}{2}}(t)D_{-\frac{1}{2}}(-s) - D_{-\frac{1}{2}}(s)D_{-\frac{1}{2}}(-t)\right)H(t-s).
\end{aligned}$$

The condition (i) of Theorem 2.1 clearly holds.

Let $x \in \mathcal{K}$, then by definition

$$\|F(x)\| = \sup_{t\in\mathbb{R}}\left|kD_{-\frac{1}{2}}(t)\frac{H(t-t_0)}{D_{-\frac{1}{2}}(t)}\right| = |k| < +\infty,$$

$$\begin{aligned}
\|G(x)\| &= \sup_{t\in\mathbb{R}}\left|\frac{1}{(1+t^2)^{\frac{1}{2}}D_{-\frac{1}{2}}(t)^3}\left(3x^5(t) + 2tx^3(t)\right)H(t-t_0)\right| \\
&\leq \sup_{t\geq t_0}\left|\frac{1}{(1+t^2)^{\frac{1}{2}}D_{-\frac{1}{2}}(t)^3}\left(3M^2(1+t^2)^{\frac{1}{2}}|x^3(t)| + 2(1+t^2)^{\frac{1}{2}}|x^3(t)|\right)\right| \\
&\leq M^3\left(3M^2 + 2\right) = 3M^5 + 2M^3 < +\infty.
\end{aligned}$$

Since F is independent of x, we may take $L_F = 0$. Furthermore given $x, y \in \mathcal{K}$

$$\begin{aligned}
\|G(x) - G(y)\| &= \sup_{s\in\mathbb{R}} \left| \frac{1}{(1+s^2)^{\frac{1}{2}} D_{-\frac{1}{2}}(s)^3} \left(3x^5(s) - 3y^5(s) + 2sx^3(s) - 2sy^3(s)\right) H(s-t_0) \right| \\
&= \sup_{s\geq t_0} \left| \frac{1}{(1+s^2)^{\frac{1}{2}} D_{-\frac{1}{2}}(s)^2} \right. \\
&\quad \left[3\left(x^4(s) + x^3(s)y(s) + x^2(s)y^2(s) + x(s)y^3(s) + y^4(s)\right)\right. \\
&\quad \left.\left. + 2s\left(x^2(s) + x(s)y(s) + y(s)^2\right)\right] \left[\frac{1}{D_{-\frac{1}{2}}(s)}\right] (x(s) - y(s)) \right| \\
&\leq \sup_{s\geq t_0} \left\{ \frac{1}{(1+s^2)^{\frac{1}{2}} D_{-\frac{1}{2}}(s)^2} \right. \\
&\quad \left[3M^2(1+s^2)^{\frac{1}{2}}\left(x^2(s) + 3|x(s)y(s)| + y^2(s)\right)\right. \\
&\quad \left.\left. +2(1+s^2)^{\frac{1}{2}}\left(x^2(s) + |x(s)y(s)| + y^2(s)\right)\right] \left[\frac{1}{D_{-\frac{1}{2}}(s)}\right] |x(s) - y(s)| \right\} \\
&\leq \left[3M^2\left(5M^2\right) + 6M^2\right] \|x - y\| = \left(15M^4 + 6M^2\right) \|x - y\|.
\end{aligned}$$

Thus we have $L_G \leq 15M^4 + 6M^2$, and the validity of the condition (ii) of Theorem 2.1.

Observe that by the definition of the weighed norm (2.11), we have

$$\begin{aligned}
\|r \star x\| &= \sup_{t\geq t_0} \left| -\sqrt{2}w(t) \int_t^{+\infty} \frac{(1+s^2)^{\frac{1}{2}}}{w^2(s)} \left(D_{-\frac{1}{2}}(t)D_{-\frac{1}{2}}(-s) - D_{-\frac{1}{2}}(s)D_{-\frac{1}{2}}(-t)\right) x(s)\mathrm{d}s \right| \\
&= \sup_{t\geq t_0} \sqrt{2}w(t) \left| \int_t^{+\infty} \frac{(1+s^2)^{\frac{1}{2}}}{w^3(s)} \left(D_{-\frac{1}{2}}(t)D_{-\frac{1}{2}}(-s) - D_{-\frac{1}{2}}(s)D_{-\frac{1}{2}}(-t)\right) x(s)w(s)\mathrm{d}s \right| \\
&\leq \sup_{t\geq t_0} \sqrt{2}w(t) \int_t^{+\infty} \frac{(1+s^2)^{\frac{1}{2}}}{w^3(s)} \left(D_{-\frac{1}{2}}(t)D_{-\frac{1}{2}}(-s) - D_{-\frac{1}{2}}(s)D_{-\frac{1}{2}}(-t)\right) |x(s)|w(s)\mathrm{d}s.
\end{aligned}$$

Using Hölder's inequality on the final integral above, we obtain further

$$\begin{aligned}
\|r \star x\| &\leq \sup_{t\geq t_0} \sqrt{2}w(t) \left(\sup_{s\geq t} |x(s)|w(s)\right) \int_t^{+\infty} \frac{(1+s^2)^{\frac{1}{2}}}{w^3(s)} \left(D_{-\frac{1}{2}}(t)D_{-\frac{1}{2}}(-s) - D_{-\frac{1}{2}}(s)D_{-\frac{1}{2}}(-t)\right) \mathrm{d}s \\
&\leq \left[\sup_{t\geq t_0} \sqrt{2}w(t) \int_t^{+\infty} \frac{(1+s^2)^{\frac{1}{2}}}{w^3(s)} \left(D_{-\frac{1}{2}}(t)D_{-\frac{1}{2}}(-s) - D_{-\frac{1}{2}}(s)D_{-\frac{1}{2}}(-t)\right) \mathrm{d}s\right] \|x\|.
\end{aligned}$$

It follows that the operator

$$x \mapsto r \star x,$$

is continuous, with operator norm

$$\varrho(t_0) \leq \sup_{t\geq t_0} \sqrt{2}w(t) \int_t^{+\infty} \frac{(1+s^2)^{\frac{1}{2}}}{w^3(s)} \left(D_{-\frac{1}{2}}(t)D_{-\frac{1}{2}}(-s) - D_{-\frac{1}{2}}(s)D_{-\frac{1}{2}}(-t)\right) \mathrm{d}s =: \rho(t_0). \tag{2.14}$$

Differentiation of Equation (2.14) shows that the function inside the supremum is monotonically decreasing. Hence the supremum is attained at the point $t = t_0$

$$\rho(t_0) = \sqrt{2}w(t_0) \int_{t_0}^{+\infty} \frac{(1+s^2)^{\frac{1}{2}}}{w^3(s)} \left(D_{-\frac{1}{2}}(t_0) D_{-\frac{1}{2}}(-s) - D_{-\frac{1}{2}}(s) D_{-\frac{1}{2}}(-t_0) \right) \mathrm{d}s.$$

This proves that the condition (iii) of Theorem 2.1 is valid.

Let $x \in \mathcal{K}$. We now estimate the norm of $F(x) + r \star G(x)$.

$$\begin{aligned}
&\|F(x) + r \star G(x)\| \\
&= \sup_{t \geq t_0} \left| \left[k D_{-\frac{1}{2}}(t) - \sqrt{2} \int_t^{+\infty} \left(D_{-\frac{1}{2}}(t) D_{-\frac{1}{2}}(-s) - D_{-\frac{1}{2}}(s) D_{-\frac{1}{2}}(-t) \right) \left(3x^5(s) + 2sx^3(s) \right) \mathrm{d}s \right] w(t) \right| \\
&\leq \sup_{t \geq t_0} |k| D_{-\frac{1}{2}}(t) w(t) + \sqrt{2} w(t) \int_t^{+\infty} \left(D_{-\frac{1}{2}}(t) D_{-\frac{1}{2}}(-s) - D_{-\frac{1}{2}}(s) D_{-\frac{1}{2}}(-t) \right) \left| 3x^5(s) + 2sx^3(s) \right| \mathrm{d}s \\
&\leq \sup_{t \geq t_0} |k| D_{-\frac{1}{2}}(t) w(t) \\
&+ \sqrt{2} w(t) \int_t^{+\infty} \left(D_{-\frac{1}{2}}(t) D_{-\frac{1}{2}}(-s) - D_{-\frac{1}{2}}(s) D_{-\frac{1}{2}}(-t) \right) \left(3M^2 + 2 \right) (1+s^2)^{\frac{1}{2}} |x^3(s)| \mathrm{d}s \\
&\leq \sup_{t \geq t_0} |k| D_{-\frac{1}{2}}(t) w(t) + (3M^2+2) \sqrt{2} w(t) \int_t^{+\infty} \frac{D_{-\frac{1}{2}}(t) D_{-\frac{1}{2}}(-s) - D_{-\frac{1}{2}}(s) D_{-\frac{1}{2}}(-t)}{w^3(s)} (1+s^2)^{\frac{1}{2}} \mathrm{d}s \|x\|^3 \\
&\leq \sup_{t \geq t_0} |k| D_{-\frac{1}{2}}(t) w(t) + (3M^2+2) \rho(t_0) M^3 = |k| + (3M^5 + 2M^3) \rho(t_0).
\end{aligned}$$

We require that $\|F(x) + r \star G(x)\| \leq M$, resulting in the condition

$$\rho(t_0) \leq \frac{M - |k|}{3M^5 + 2M^3},$$

under which the condition (iv) of Theorem 2.1 holds.

The condition (v) of Theorem 2.1 is satisfied if

$$\rho(t_0) < \frac{1}{15M^4 + 6M^2},$$

which completes the proof of the theorem.

$\square$

Remark 2.4. *The choice*

$$\begin{aligned}
M = \frac{1}{108} \Bigg(& 9 \sqrt[3]{125|k|^3 + 4\sqrt{6750k^4 + 4884k^2 + 256} + 312|k|} \\
& + \frac{9\left(25k^2 - 16\right)}{\sqrt[3]{125|k|^3 + 4\sqrt{6750k^4 + 4884k^2 + 256} + 312|k|}} + 45k \Bigg),
\end{aligned}$$

results in the equality of the right-hand sides of equations (2.13). In particular, if $k = 1/(2\pi)$, which is the value corresponding to the absence of the leading order oscillatory term in Equation (1.20), based on the connection formulae (1.21) we get the condition

$$\rho(t_0) \lessapprox 2.69913.$$

Taking t_0 to be the abscissa of the intersection of the two curves of Figure 2.1 (which is roughly 0.32797*), and evaluating the integral numerically results in the relation*

$$0.896619 \lessapprox 2.69913,$$

which holds. Hence, we have numerical evidence that the solution $\eta_{1/(2\pi)}$ is continuous on the interval $(0.328, +\infty)$*.*

2.2.2 $\mathrm{P_{IV}}$ in the Complex Plane

In this section, we will discuss special asymptotic behaviours of solutions of the fourth Painlevé equation in the complex plane, based on Wasow's theorem [Was02]. In particular, we will prove the existence of tronquée and tritronquée type solutions, similarly to the case of the first- and second Painlevé equations.

We continue studying $\mathrm{P_{IV}}$ using the change of variables

$$w(z) = 2\sqrt{2}y^2(x), \quad x = -\sqrt{2}z, \tag{2.15}$$

which transforms Equation (2.1) into

$$\mathrm{P_{IV}^{1/2}}(a,b): \quad y'' = 3y^5 - 2xy^3 + \frac{1}{4}x^2y - \frac{1}{2}ay + \frac{b}{32y^3}. \tag{2.16}$$

Furthermore, we let the constants a, b both be zero, and focus on

$$\mathrm{S_{IV}}: \quad y'' = 3y^5 - 2xy^3 + \frac{1}{4}x^2y = \frac{1}{4}y(6y^2-x)(2y^2-x) =: f(x,y), \tag{2.17}$$

which is equivalent to Equation (1.16a) under the reflection $\xi \mapsto -\xi$. The zero set of the right-hand side of Equation (2.17) will be used to study the convexity of its solutions. See Figure 2.2.

Figure 2.3 shows a solution of Equation (2.17) obtained via numerical integration. It suggests the existence of solutions of Equation (2.17) with the asymptotic behaviour

$$y(x) \sim \pm\sqrt{\frac{x}{6}}, \quad \text{as } x \to +\infty. \tag{2.18}$$

There is also numerical evidence of solutions with the asymptotic behaviour

$$y \sim \sqrt{\frac{x}{2}}, \quad \text{as } x \to +\infty,$$

provided that one works with high precision, and furthermore, the convexity plot in Figure 2.2 suggests the existence of solutions $y(x)$ of $\mathrm{P_{IV}}$, which approach 0 as $x \to \infty$. While we do not consider this special case here, we rediscover it through the analysis presented in Section 3.3 in Chapter 3.

To get an analytical foothold of the situation, we try substituting the fractional power series ansatz

$$y_f(x) = x^{1/2}\sum_{n=0}^{\infty}\frac{a_n}{x^{4n}}, \tag{2.19}$$

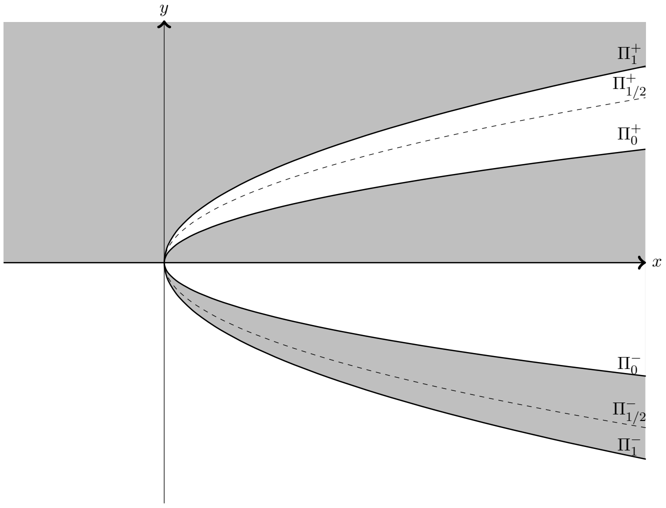

FIGURE 2.2: The zero set of $f(x, y) = \frac{1}{4}y(2y^2 - x)(6y^2 - x)$ (solid), as well as two curves on which $f(x, y)$ has extrema with respect to y (dashed). Solutions of Equation (2.17) are convex in the shaded set $\{x \leq 0, y > 0\} \cup \{x > 0, y > \Pi_1^+(x)\} \cup \{x > 0, 0 < y < \Pi_0^+(x)\} \cup \{x > 0, \Pi_1^-(x) < y < \Pi_0^-(x)\}$, and concave in the white set $\{x \leq 0, y < 0\} \cup \{x > 0, \Pi_0^+(x) < y < \Pi_1^+(x)\} \cup \{x > 0, \Pi_0^-(x) < y < 0\} \cup \{x > 0, y < \Pi_1^-(x)\}$.

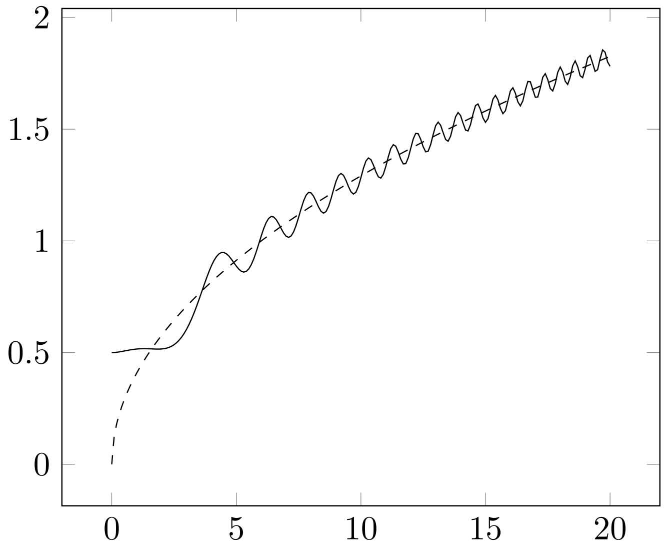

FIGURE 2.3: A numerical solution of Equation (2.17) (solid), which is asymptotic to the curve $y = \sqrt{\frac{x}{6}}$ (dashed) as $x \to +\infty$.

in Equation (2.17). Doing so yields the leading term coefficient $a_0 \in \left\{0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{6}}\right\}$ and the recurrence relation

$$3a_n^{*5} - 2a_n^{*3} + \frac{1}{4}a_n = \frac{1}{4}(8n-7)(8n-9)a_{n-1},$$

where

$$a_n^{*m} = \sum_{n_{m-1}=0}^{n} \cdots \sum_{n_2=0}^{n_3} \sum_{n_1=0}^{n_2} a_{n_1} a_{n_2 - n_1} \cdots a_{n - n_{m-1}},$$

denotes the mth convolution power of the sequence $\{a_n\}$.

The series obtained are not necessarily convergent. Nevertheless, they are of interest, thanks to the following theorem (also used in [JK01] for studying Equation (1.3)) which turns a formal power series solution to an asymptotic series of a genuine solution.

Theorem 2.5 ([Was02], Theorem 12.1)**.** *Let S be an open sector of the complex x-plane with vertex at the origin and a positive central angle not exceeding*

$$\frac{\pi}{q+1},$$

(where q is a non-negative integer). Let $\mathbf{f}(x,\mathbf{w})$ be an r-dimensional vector function of x and the r-dimensional vector $\mathbf{w}$ with the following properties:

1. $\mathbf{f}(x,\mathbf{w})$ is a polynomial in the components w_j of $\mathbf{w}$, $j=1,\dots,r$, with coefficients that are holomorphic in x in the region

$$\{x \in S : 0 < x_0 \le |x| < \infty\},$$

for some x_0.

2. The coefficients of the polynomial $\mathbf{f}(x,\mathbf{w})$ have asymptotic series in powers of x^{-1} as $S \ni x \to \infty$.

3. If $\{f_j(x,\mathbf{w})\}$ denote the components of $\mathbf{f}(x,\mathbf{w})$, then all eigenvalues λ_j, $j=1,\dots,r$ of the limiting Jacobian matrix

$$\lim_{S\ni x\to\infty} \frac{\partial \mathbf{f}}{\partial \mathbf{w}}\Big|_{\mathbf{w}=0},$$

differ from zero.

4. The differential equation $x^{-q}\mathbf{w}' = \mathbf{f}(x,\mathbf{w})$ is formally satisfied by a power series of the form $\sum_{k=1}^{\infty} \mathbf{b}_k x^{-k}$.

If all these conditions hold, there exists, for sufficiently large $x \in S$, a solution $\mathbf{w} = \boldsymbol{\phi}(x)$ of $x^{-q}\mathbf{w}' = \mathbf{f}(x,\mathbf{w})$, such that, in every proper subsector of S

$$\boldsymbol{\phi}(x) \sim \sum_{k=1}^{\infty} \mathbf{b}_k x^{-k}.$$

We are now in position to prove

Theorem 2.6 (Tronquée Solutions)**.** *For any sector S of angle $< \pi/2$ with vertex at the origin there exists a solution of Equation (2.17) with the asymptotic expansion*

$$y(x) \sim \frac{1}{\sqrt{6}}x^{1/2} - \frac{\sqrt{6}}{8}x^{-7/2} + \mathcal{O}\left(x^{-4}\right), \quad \text{as } S \ni x \to \infty, \tag{2.20}$$

with $x^{1/2}$ denoting either branch of the square root.

Proof. The change of variables

$$u(\zeta) = \frac{y(x)}{x^{1/2}} - \frac{1}{\sqrt{6}}, \quad \zeta = \frac{1}{2}x^2,$$

transforms Equation (2.17) to

$$u'' = \frac{1}{16\sqrt{6}\zeta^2} - \frac{u'}{\zeta} + \frac{u}{16\zeta^2} + 3u^5 + 5\sqrt{\frac{3}{2}}u^4 + 3u^3 - \frac{u^2}{\sqrt{6}} - \frac{u}{3},$$

which is equivalent to the system

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= \frac{1}{16\sqrt{6}\zeta^2} - \frac{u_2}{\zeta} + \frac{u_1}{16\zeta^2} + 3u_1^5 + 5\sqrt{\frac{3}{2}}u_1^4 + 3u_1^3 - \frac{u_1^2}{\sqrt{6}} - \frac{u_1}{3}. \end{aligned}$$

The asymptotic series in Equation (2.19) transforms accordingly to

$$u_f(\zeta) = \sum_{n=1}^{\infty} \frac{a_n}{4^n}\zeta^{-2n},$$

and Wasow's theorem is applicable with $q = 0, r = 2$ and $\lambda_1\lambda_2 = \frac{1}{3} \neq 0$. Note that the change of independent variable $\zeta \to x$ decreases the angle of the sector by a factor of 2. $\square$

Remark 2.7. *It is possible to describe solutions $y(x)$ of Equation (2.17) with leading asymptotic behaviour*

$$y(x) \sim \frac{1}{\sqrt{2}}x^{1/2}, \quad x \to \infty$$

using Wasow's theorem as well. We choose to focus on the behaviour given in Equation (2.20), for its analogy with the tritronquée solution of P_I.

One can take two tronquée solutions as defined above, and "sew" them together to create a tritronquée solution in the following way:

Let $\epsilon > 0$ be given and let $y_1(x), y_2(x)$ be tronquée solutions defined in the sectors

$$S_1 : -\epsilon/2 < \arg(x) < \pi/2 - \epsilon, \quad S_2 : -\pi/2 + \epsilon < \arg(x) < \epsilon/2,$$

See Figure 2.4. In the intersection $S_1 \cap S_2$, y_1, y_2 have the same asymptotic expansion,

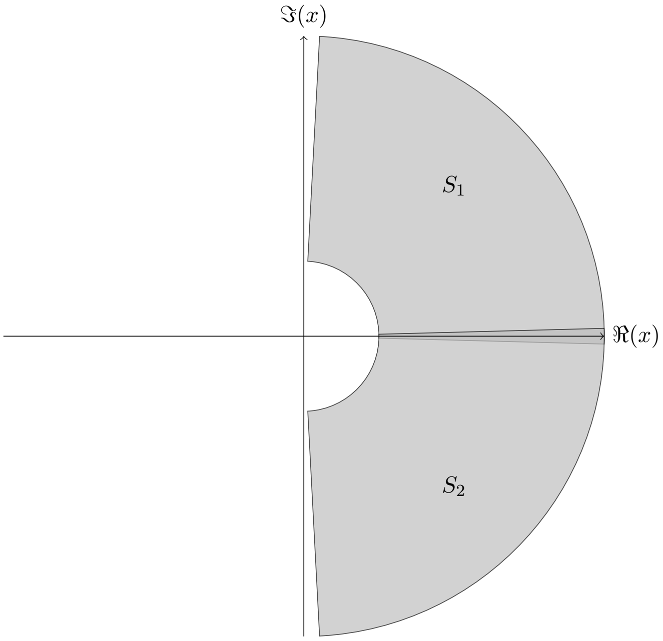

FIGURE 2.4: The sectors S_1 and S_2 used in the construction of the tritronquée solution.

hence

$$v(x) := y_1(x) - y_2(x) = o(x^{-n}), \tag{2.21}$$

for all $n \in \mathbb{N}$. At the same time v satisfies the ODE

$$\begin{aligned} v'' =& [1/4x^2 - 2x(y_1^2 + y_1 y_2 + y_2^2) + 3(y_1^4 + y_1^3 y_2 + y_1^2 y_2^2 + y_1 y_2^3 + y_2^4)]v \\ =&: -f(x)v \sim -\frac{1}{3}x^2 v, \end{aligned}$$

for large real x. Applying Theorem 2.2 from Olver's "Asymptotics and Special Functions" monograph [Olv97] gives the representation

$$\begin{aligned} v(x) =& \frac{c_1}{f^{1/4}} \exp\left(i \int_x^\infty f^{1/2} \mathrm{d}x\right) [1 + o(1)] \\ &+ \frac{c_2}{f^{1/4}} \exp\left(-i \int_x^\infty f^{1/2} \mathrm{d}x\right) [1 + o(1)], \end{aligned}$$

and Equation (2.21) implies $c_1 = c_2 = 0$, so that y_1, y_2 coincide for large x. The identity theorem for analytic functions, combined with letting $\epsilon \to 0$ gives

Theorem 2.8 (Tritronquée Solutions). *There exists a unique pair, $Y_\pm$, of solutions of Equation (2.17) that have no poles of large modulus in the half plane*

$$\{x \in \mathbb{C} : \Re(x) > 0\}, \tag{2.22}$$

with the asymptotic expansion (2.19) with $a_0 = 1/\sqrt{6}$. The pair of solutions is in correspondence with the pair of choices of a branch of $x^{1/2}$ in the half plane (2.22). Correspondingly, the change of variables of Equations (2.15) implies that there is a unique solution of $(\mathrm{P}_{\mathrm{IV}}(0,0))$ with no poles of large modulus, in the half plane

$$\{z \in \mathbb{C} : \Re(z) < 0\},$$

with the asymptotic expansion that is given by applying the change of variables of Equations (2.15) to the asymptotic expansion (2.19) with $a_0 = 1/\sqrt{6}$ for any choice of a branch of $x^{1/2}$.

Remark 2.9. *In order to be able to discuss the solutions $Y_\pm$ of Theorem 2.8 individually, we will distinguish them based on their leading asymptotic behaviour. That is, we will label the pair in such a way that*

$$Y_\pm(x) \sim \pm\sqrt{\frac{x}{6}}, \quad \text{as } x \to +\infty.$$

Remark 2.10. *In [RF13] Equation (2.17) is studied from a numerical perspective. Among other things, the authors describe a solution which appears to have no poles in a half-plane. This solution corresponds to one of the tritronquée solutions defined in Theorem 2.8.*

Our analysis of P_{IV} on the real line will make extensive use of the following functions (see Figure 2.2):

$$\Pi_0^{\pm} : y = \pm\sqrt{\frac{x}{6}},$$
$$\Pi_{1/2}^{\pm} : y = \pm\sqrt{\frac{6+\sqrt{21}}{30}x},$$
$$\Pi_1^{\pm} : y = \pm\sqrt{\frac{x}{2}}.$$

for convenience we will refer to $\Pi_i^{\pm}$, with $i \in \{0, 1/2, 1\}$ both as a function, as well as its graph over $[0, \infty)$. Our goal is to take solutions of Equation (2.17), which are defined by their asymptotic behaviour, and establish their behaviour in the finite domain as well.

2.2.3 A Boundary Value Problem on $\mathbb{R}_{\leq 0}$

In this section, we will demonstrate the continuity and monotonicity of a one-parameter family of solutions of Equation (2.17). This family is, up to the corresponding change of variables, the subset of the family $\{\eta_k\}_{k\in\mathbb{R}}$ from the introduction, with the property that η_k is continuous on $[0, \infty)$. More specifically, we will prove that for any real y_0 there is a unique solution y of Equation (2.17) with $y(0) = y_0$ and

$$\lim_{x\to-\infty} y(x) = 0.$$

We make the dependence on the initial conditions explicit by writing y_{y_0,m_0} for the solution of Equation (2.17) with the initial data $y(0) = y_0, y'(0) = m_0$. We then have the following results.

Lemma 2.11. *Let $x_0 \leq 0$ and let φ be a solution of Equation (2.17) with the initial data*

$$\varphi(x_0) \geq 0, \tag{2.23a}$$
$$\varphi'(x_0) < 0. \tag{2.23b}$$

Then $\varphi(x)$ is positive for any $x < x_0$ in its interval of existence, and this interval of existence is bounded from below.

Proof. Let I be the largest interval of the form (α, x_0), on which φ is defined and $\varphi' < 0$. If $\lim_{x\to\alpha^+} \varphi(x) = +\infty$, there is nothing to prove. If α is a critical point of φ, then it follows from the convexity of φ that $\varphi'(\alpha + \epsilon)$ is positive for sufficiently small $\epsilon > 0$. However, by assumption, $\varphi'(\alpha + \epsilon) < 0$, making this case impossible.

We are left with the case in which $\alpha = -\infty$. Suppose that $\inf I = -\infty$, and observe that in that case the differential inequality

$$\varphi''(x) \geq 3\varphi^5(x),$$

holds for all $x \leq x_0$. Multiplying by the negative quantity $\varphi'(x)$ gives

$$\varphi''(x)\varphi'(x) \leq 3\varphi^5(x)\varphi'(x),$$

and integrating this inequality over the interval $[x, x_0]$ results in

$$\varphi'(x)^2 \geq \varphi^6(x) - (\varphi^6(x_0) - \varphi'(x_0)^2). \tag{2.24}$$

The positivity and convexity of φ over $(-\infty, x_0)$ allows us to find a point $x^* \leq x_0$, such that

$$\frac{3}{4}\varphi^6(x) - (\varphi^6(x_0) - \varphi'(x_0)^2) \geq 0,$$

for all $x \leq x^*$. Thus the inequality (2.24) implies

$$\varphi'(x)^2 \geq \frac{1}{4}\varphi^6(x),$$

for $x \leq x^*$. Taking the square root and integrating yields

$$\varphi^{-2}(x) - \varphi^{-2}(x^*) \leq x - x^*, \tag{2.25}$$

for all $x \leq x^*$. Letting $x \to -\infty$ leads us to a contradiction, as the right-hand side gets arbitrarily negatively large - while the left-hand side remains bounded. It follows that the interval (α, x_0) is in fact bounded from below. ☐

Corollary 2.12. *Let y_0 be a non-negative real number. There exists a solution $y(x)$ of Equation* (2.17) *with $y(0) = y_0$, which is continuous and positive over a bounded interval of the form $(\alpha, 0)$, and $\lim_{x\to\alpha^+} y(x) = +\infty$.*

Proof. Lemma 2.11 implies that all solutions y of Equation (2.17) with $y(0) = y_0 \geq 0$ and $y'(0) < 0$ are positive in the negative part of their interval of existence, $(\alpha, 0)$, and $\lim_{x\to\alpha^+} y(x) = +\infty$. ☐

Lemma 2.13. *Let $y_0 \geq 0$. There exists a solution of Equation* (2.17) *with $y(0) = y_0$, defined over an interval of the form $[\alpha, 0]$, and satisfying $y(\alpha) < 0$.*

Proof. If $y_0 = 0$, the result follows trivially. Otherwise, fix $y_0 > 0$ and $m > 1$, and let f be a solution of Equation (2.17) with the initial conditions

$$y(0) = y_0,$$
$$y'(0) = m.$$

Let ℓ be the line $y - x = y_0$. If f does not have a zero, then its graph and ℓ will have another intersection point, with x-coordinate ξ_0, between 0 and $-y_0$. Rolle's theorem yields a point $\xi_1 \in (\xi_0, 0)$ such that

$$f'(\xi_1) = \frac{f(\xi_0) - f(0)}{\xi_0 - 0} = 1.$$

Applying Rolle's theorem a second time, there is a point $\xi_2 \in (\xi_1, 0)$ such that

$$f''(\xi_2) = \frac{f'(\xi_1) - f'(0)}{\xi_1 - 0} = \frac{1 - m}{\xi_1} > \frac{m - 1}{y_0}. \tag{2.27}$$

On the other hand

$$f''(\xi_2) = \frac{1}{4}f(\xi_2)\left(2f^2(\xi_2) - \xi_2\right)\left(6f^2(\xi_2) - \xi_2\right) < \frac{1}{4}y_0(2y_0^2 + y_0)(6y_0^2 + y_0). \tag{2.28}$$

Equations (2.27),(2.28) show that m is bounded from above. Taking m larger than this value will result in a solution which becomes negative at some point on its interval of existence. ☐

We proceed by making the following definitions.

Definition 2.14. *For a fixed $y_0 \geq 0$, let $A(y_0)$ denote the set of all real numbers m such that $y_{y_0,m}$ is positive on an interval of the form $(\alpha, 0)$, and $\lim_{x\to\alpha^+} y_{y_0,m}(x) = +\infty$. Let $B(y_0)$ denote the set of all real numbers m such that the solution $y_{y_0,m}$ is defined on an interval of the form $[\alpha, 0)$, and $y(\alpha) < 0$ for some real α.*

Lemma 2.15. *Fix $y_0 \geq 0$. The sets $A(y_0), B(y_0)$ are open intervals of the form $(-\infty, a(y_0))$ and $(b(y_0), +\infty)$ respectively.*

Proof. This is a consequence of the continuous dependence of the solution of an ordinary differential equation on its initial conditions [Arn92]. Although its application to proving the openness of $B(y_0)$ is evident, it is worthwhile to discuss that of $A(y_0)$: Let $m \in A(y_0)$, if m is negative, then there exists a small neighbourhood of m, all of its elements are negative, and Lemma 2.11 shows that all such elements are in $A(y_0)$. If m is non-negative, then $y_{y_0,m}$ attains a positive minimum at some point $x_m \leq 0$, and consequently, according to Lemma 2.11, approaches infinity in finite time. Here the continuous dependence on the initial conditions shows that a slight perturbation of m will only slightly affect the location and value of this minimum and thus the solution will remain of the same type. The fact that the sets are intervals follows from the fact that $f(x, y)$ in Equation (2.17) is monotonically increasing with respect to y, when $x \leq 0$. As it implies that if $m_1 < m_2$ then $y_{y_0,m_1}(x) \geq y_{y_0,m_2}(x)$ in their common interval of existence. The fact that $a(y_0) \leq b(y_0)$ is a consequence of the fact that the sets $A(y_0)$ and $B(y_0)$ are disjoint. ☐

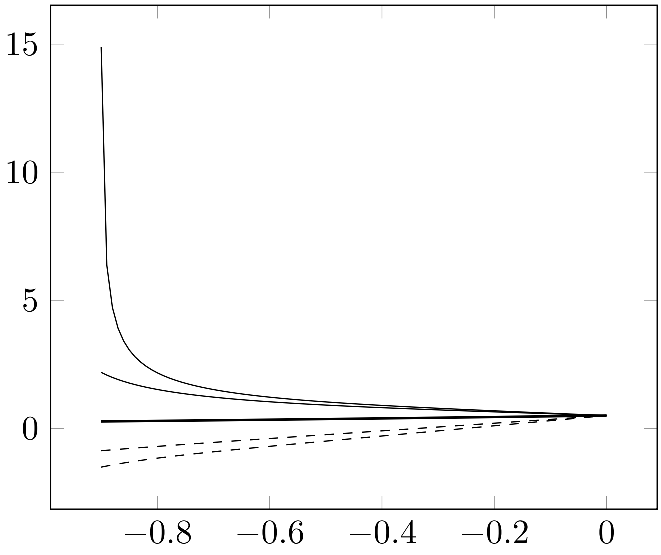

FIGURE 2.5: Solutions $y_{y_0,m}$ of Equation (2.17) with $m \in A(y_0)$ (solid), $m \in B(y_0)$ (dashed), and $m = m^-(y_0)$ (thick).

We will now show that $a(y_0) = b(y_0)$.

Lemma 2.16. *The boundary value problem*

$$u''(x) = q(x)u(x) \tag{2.29a}$$

$$u(0) = \lim_{x\to\infty} u(x) = 0 \tag{2.29b}$$

with continuous $q : [0, \infty) \to [0, \infty)$ has the unique solution $u(x) \equiv 0$.

Proof. Multiply Equation (2.29a) by $u(x)$ and integrate to get

$$\int_0^x u(t)u''(t)\mathrm{d}t = \int_0^x q(t)u(t)^2\mathrm{d}t \geq 0.$$

Applying integration by parts on the left hand side then gives

$$u(x)u'(x) - \int_0^x u'^2(t)\mathrm{d}t \geq 0,$$

which implies

$$u(x)u'(x) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(u^2(x)\right) \geq 0.$$

Since u^2 is monotonically nondecreasing, we must have $u'(0) = 0$ in order to have $\lim_{x\to\infty} u(x) = 0$, and the uniqueness follows. $\square$

Lemma 2.17. *There is a single solution in the family $\{y_{y_0,m}\}_{m\in\mathbb{R}}$ whose initial slope, $y'_{y_0,m}(0)$, is neither in $A(y_0)$, nor $B(y_0)$.*

Proof. Let $m_1, m_2 \in [a(y_0), b(y_0)]$, and set $\phi_1(x) := y_{y_0,m_1}(-x)$, $\phi_2(x) := y_{y_0,m_2}(-x)$ and $u := \phi_1 - \phi_2$. The difference u satisfies the ODE (2.29) with

$$q(x) = 3(\phi_1^4 + \phi_1^3\phi_2 + \phi_1^2\phi_2^2 + \phi_1\phi_2^3 + \phi_2^4) + 2x(\phi_1^2 + \phi_1\phi_2 + \phi_2^2) + \frac{1}{4}x^2 \geq 0,$$

and we find that $\phi_1 \equiv \phi_2$. $\square$

We proceed by making the following definition.

Definition 2.18. *For any $y_0 \geq 0$, define*

$$m^-(y_0) = \sup A(y_0) = \inf B(y_0).$$

Some of the values of m^- are plotted in Section 2.4. Since $m^-(y_0) \notin A(y_0) \cup B(y_0)$, the solutions $y_{y_0,m^-(y_0)}$ are continuous on the nonpositive real semi-axis, and remain positive there (c.f. Figure 2.6). Moreover $y_{y_0,m^-(y_0)}(x) \to 0$ monotonically as $x \to -\infty$, as can be seen due to the nature of the right hand side $f(x,y)$ of Equation (2.17).

Lemma 2.19. *For $\epsilon > 0$ fixed, the restricted function $f|_{\mathbb{R}_{\leq 0}\times[\epsilon,\infty)}$, with f as in Equation (2.17) approaches infinity as $x \to -\infty$, uniformly in y. Consequently, the solutions $y_{y_0,m^-(y_0)}(x)$ approach zero as $x \to -\infty$.*

Proof. We have

$$f(x,y) = 3y^5 + 2|x|y^3 + \frac{1}{4}|x|^2 y \geq 3\epsilon^5 + 2\epsilon^3|x| + \frac{1}{4}|x|^2\epsilon.$$

Let φ denote a solution of the form $y_{y_0,m^-(y_0)}$. Since $\varphi'(0) \notin B(y_0)$, φ is bounded from below on $\mathbb{R} \leq 0$. Furthermore, $\varphi'(x) \leq 0$ for all $x \leq 0$, as can be seen from Lemma 2.11. Thus the limit $l = \lim_{x\to-\infty}\varphi(x)$ exists, and $l \geq 0$. It follows that $\lim_{x\to-\infty}\varphi'(x) = 0$. If $l > 0$, the second derivative $\varphi''(x)$ is unbounded as $x \to -\infty$, which is in contradiction with $\lim_{x\to-\infty}\varphi'(x) = 0$. We conclude that $l = 0$. $\square$

Lemma 2.20. *The function m^- is a monotonically increasing, continuous, odd function defined on the entire real line, satisfying $\lim_{x\to\pm\infty} m^-(x) = \pm\infty$.*

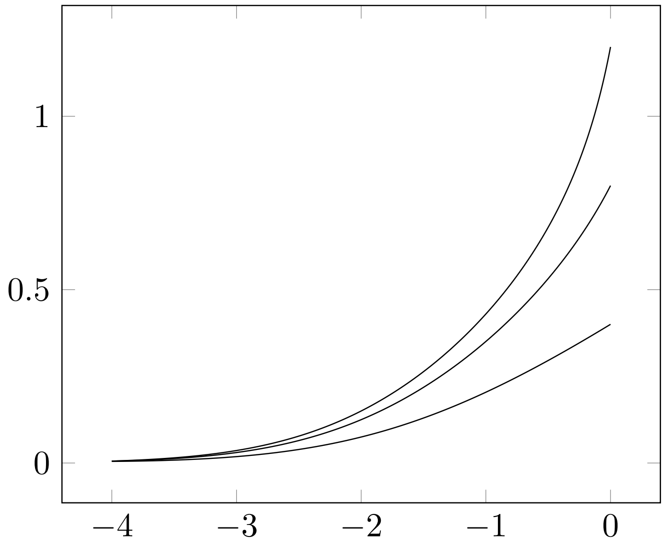

FIGURE 2.6: Graphs of solutions $y_{y_0,m^-(y_0)}$ of Equation (2.17) with $y_0 \in \{0.4, 0.8, 1.2\}$.

Proof. The monotonicity of m^- follows from the monotonicity of $f(x,y)$ with respect to its second argument: Let φ_1, φ_2 denote the solutions $y_{y_1,m^-(y_1)}, y_{y_2,m^-(y_2)}$ with $y_1 > y_2$. Suppose that $m^-(y_1) < m^-(y_2)$, the mean value theorem gives

$$(\varphi_1 - \varphi_2)''(x) = f(x,\varphi_1(x)) - f(x,\varphi_2(x)) = \frac{\partial f}{\partial y}(x,\varphi^*(x))(\varphi_1(x) - \varphi_2(x)),$$

for some $\varphi_2(x) \leq \varphi^*(x) \leq \varphi_1(x)$. This shows that $u = \varphi_1 - \varphi_2$ is a convex function on $\mathbb{R}_{\leq 0}$ with $u(0) > 0, u'(0) < 0$. This is in contradiction with $\lim_{x\to-\infty} u(x) = 0$. The continuity follows, once again from the continuous dependence of $y_{y_0,m}$ on its initial conditions. The symmetry $y(x) \mapsto -y(x)$ of Equation (2.17) shows that we can repeat our analysis above for $y_0 < 0$, and get an odd extension $m^- : \mathbb{R} \to \mathbb{R}$. As for the unboundedness of m^-, suppose by way of contradiction the existence of a number $M > 0$ such that $m^-(y_0) \leq M$ for all $y_0 \geq 0$. Taylor's theorem combined with the convexity of $y_{y_0,m^-(y_0)}$ gives

$$y'(x) = m^-(y_0) + \int_0^x y''(t)\mathrm{d}t \leq m^-(y_0) + \int_0^x f(t, y_0 + m^-(y_0)t)\mathrm{d}t, \tag{2.30}$$

where we have used the monotonic nature of f in the domain $x \leq 0$. For $x_1 := -\frac{y_0}{m^-(y_0)} < 0$ the inequality (2.30) reads

$$\begin{aligned} y'(x_1) &\leq -\frac{y_0^6}{2m^-(y_0)} - \frac{y_0^5}{10(m^-(y_0))^2} - \frac{y_0^4}{48(m^-(y_0))^3} + m^-(y_0) \\ &\leq -\frac{y_0^6}{2M} - \frac{y_0^5}{10M^2} - \frac{y_0^4}{48M^3} + M. \end{aligned}$$

This shows that $y'(x_1)$ is negative, provided that y_0 is taken large enough, which is in contradiction with the fact that y is monotonically increasing on $\mathbb{R}_{\leq 0}$. We conclude that $m^- : \mathbb{R} \to \mathbb{R}$ is an odd, monotonically increasing bijection of the real line. □

2.2.4 The Tritronquée Solution on the Positive Real Semi-Axis

The main result of this section is proving that the interval of existence of the solutions $Y_{\pm}(x)$ of Equation (2.17) mentioned in Remark 2.9 contains $(\alpha, +\infty)$ for a particular α. Without loss of generality, we will state the results for $Y_{-}(x)$ alone. First we state the following result:

Theorem 2.21. *Let $y(x)$ be a solution of Equation (2.17) and let $y(x) = -\sqrt{x/6} - w(x)$. Then $w(x)$ satisfies*

$$w'' = 3w^5 + 5\sqrt{\frac{3}{2}}\sqrt{x}w^4 + 3xw^3 - \frac{x^{3/2}}{\sqrt{6}}w^2 - \frac{x^2}{3}w + \frac{1}{4\sqrt{6}x^{3/2}}. \tag{2.31}$$

If $w(x_0) = 0, w'(x_0) \geq 0$ and $w(x)$ has a maximum M at some $x_m > x_0$, then $w(x_f) = 0$ again, for some $x_f > x_0$.

In order to prove this result we prove the following two lemmas:

Lemma 2.22. *The function*

$$g(x, w) := 3w^5 + 5\sqrt{\frac{3}{2}}\sqrt{x}w^4 + 3xw^3 - \frac{x^{3/2}}{\sqrt{6}}w^2 - \frac{x^2}{3}w + \frac{1}{4\sqrt{6}x^{3/2}}$$

with $w > 0$ fixed is monotonically decreasing with respect to x for $x > q_0 w^2$, where $q_0 \approx 4.65$ is the largest real root of the cubic

$$32q^3 - 315q^2 + 918q - 675.$$

Proof. Write $x = qw^2$, so that

$$g(x, w) = g(qw^2, w) = \frac{1}{4\sqrt{6}q^{3/2}w^3} + \frac{1}{24}\left(-4\sqrt{6}q^{3/2} - 8q^2 + 72q + 60\sqrt{6}\sqrt{q} + 72\right)w^5.$$

The first term is monotonically decreasing with respect to q, and the second term is too, provided that the derivative of the expression inside the parenthesis is negative, i.e. when

$$-16q - 6\sqrt{6}\sqrt{q} + \frac{30\sqrt{6}}{\sqrt{q}} + 72 < 0.$$

This happens when q is greater than the largest real root of the cubic

$$32q^3 - 315q^2 + 918q - 675,$$

which is 4.65 approximately (see Figure 2.7).

$\square$

Lemma 2.23. *Suppose $w(x)$ solves Equation (2.31) with the initial data $w(x_0) = 0, w'(x_0) \geq 0$ for some $x_0 > 0$, and that $w(x) < \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}}\right)\sqrt{x}$. Suppose also that w has a maximum M at $x_m > x_0$. Let x_1, x_2 be points satisfying $x_0 < x_1 < x_m < x_2$ and $w(x_1) = w(x_2)$. We then have $w''(x_1) > w''(x_2)$.*

Proof. Using the previous lemma it suffices to show that $x_1 > q_0 w(x_1)^2$. The hypothesis of this lemma gives

$$x_1 > \frac{w^2(x_1)}{\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}}\right)^2},$$

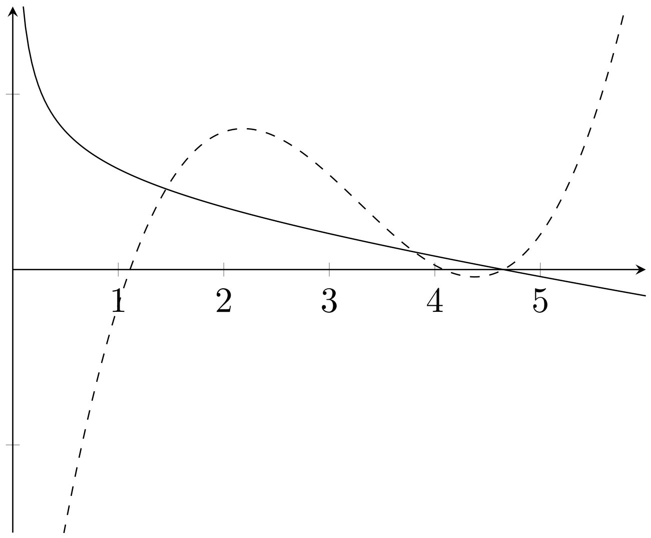

FIGURE 2.7: The graphs of $-16q-6\sqrt{6}\sqrt{q}+\frac{30\sqrt{6}}{\sqrt{q}}+72$ (solid) and $32q^3-315q^2+918q-675$ (dashed).

so that the proof boils down to showing

$$\frac{1}{\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{6}}\right)^2}>q_0,$$

which indeed holds (the left hand side is approximately 11.196 while the right hand side is about 4.65). ☐

We are now in position to prove Theorem 2.21.

Proof of Theorem 2.21. One needs to utilise the differential inequality $w''(x_1)>w''(x_2)$ in an identical way to that used in the proofs of lemmas 1 and 2 in [JK01]. ☐

We will now address the solutions $Y_\pm$ defined in Remark 2.9.

Theorem 2.24. *The solution $Y_-(x)$ does not intersect the parabolic arc Π_0^- on its maximal interval of existence I.*

Proof. Suppose $x_0\in I$ is a point such that $Y_-(x)$ intersects $-\sqrt{x/6}$ from above at x_0, that is the function w from Theorem 2.21 satisfies $w(x_0),w'(x_0)\geq 0$. The asymptotic nature of Equation (2.20) of $Y_-(x)$ implies that w must attain a maximum, so that the conditions of Theorem 2.21 hold and there exists a succeeding intersection.

Suppose now that $x_1\in I$ is a point such $Y_-(x)$ intersects $-\sqrt{x/6}$ from below at x_0. In this case, the asymptotic nature in Equation (2.20) implies that there exists a succeeding intersection.

We conclude that if $Y_-(x)$ intersects Π_0^- once, it must do so infinitely many times, which leaves us with two cases: either the intersection points grow arbitrarily large, or they must accumulate at a finite point. The former is impossible, since such behaviour is incompatible with the asymptotic series of Equation (2.20). The latter is also impossible, since the identity theorem for complex analytic function would then imply that $Y_-(x)\equiv-\sqrt{x/6}$, which is clearly not a solution of Equation (2.17). ☐

We turn to prove some additional lemmas, including a finite-time blowup result.

Lemma 2.25. *Let $x_0 > 0$. If $y(x)$ satisfies Equation (2.17) with the initial data*

$$y(x_0) = -\sqrt{x_0/2},$$
$$y'(x_0) \leq -\frac{1}{2\sqrt{2x_0}},$$

(i.e. the intersection is from above), then the solution develops a singularity $x_p > x_0$, where $\lim_{x\to x_p^-} |y(x)| = \infty$ in finite time.

Proof. As the solution decreases at a negative rate, there exists a point $x_1 > x_0$ such that $y(x) < -\sqrt{x}$ for all $x \geq x_1$. We thus have for all $x > x_1$

$$y'' = 3y^5 - 2xy^3 + \frac{1}{4}x^2y \leq 3y^5 - 2xy^3 \leq y^5.$$

Multiplying by $y'(x) < 0$ and integrating gives

$$\frac{1}{2}y'(x)^2 \geq \frac{1}{6}y(x)^6 + \frac{1}{2}y'(x_1)^2 - \frac{1}{6}y(x_1)^6,$$

or

$$y'(x)^2 \geq \frac{1}{3}y(x)^6 + y'(x_1)^2 - \frac{1}{3}y(x_1)^6 \geq \frac{1}{6}y(x)^6,$$

for $x \geq x_2$ for some x_2. This means that

$$|y'(x)| \geq \frac{1}{\sqrt{6}}|y(x)|^3,$$

or

$$\frac{y'(x)}{y(x)^3} \geq \frac{1}{\sqrt{6}}.$$

Integration of both sides from x_2 to x gives

$$-\frac{1}{2}y(x)^{-2} + \frac{1}{2}y(x_2)^{-2} \geq \frac{1}{\sqrt{6}}(x - x_2),$$

which is a contradiction as $x \to \infty$. □

Lemma 2.26. *Let $x_0 > 0$, and let $y(x)$ be a solution of Equation (2.17) with the initial data*

$$-\sqrt{\frac{x_0}{2}} < y(x_0) < -\sqrt{\frac{x_0}{6}},$$
$$y'(x_0) = 0.$$

Then there exists a point $x_1 > x_0$ such that

$$y(x_1) = -\sqrt{\frac{x_1}{6}}.$$

Proof. The critical point of y at x_0 is evidently a minimum, since

$$y''(x_0) = \frac{1}{4}y(x_0)(6y(x_0)^2 - x_0)(2y(x_0)^2 - x_0) > 0.$$

The slope y' thus remains positive in the interval $[x_0, x_1]$ where x_1 is the supremum of the set

$$S := \left\{ x \geq x_0 : \frac{1}{4} y(t)(6y(t)^2 - t)(2y(t)^2 - t) > 0 \text{ for } x_0 \leq t \leq x \right\}.$$

We consider several cases

- $x_1 = \infty$. In this case $y(x)$ is monotonically increasing, while $-\sqrt{x/6} \to -\infty$ as $x \to -\infty$. It is clear that an intersection must occur.
- $x_1 < \infty$. In this case one of the factors $y(t), 6y(t)^2 - t, 2y(t)^2 - t$ must vanish at x_1. If $y(x_1) = 0$, then the intermediate value theorem furnishes an earlier point in which y intersects Π_0^-. A similar use of the intermediate value theorem shows that $y(x_1)$ cannot equal $\sqrt{x_1/6}$ nor $\sqrt{x_1/2}$. The only case left to eliminate is that in which $y(x_1) = -\sqrt{x_1/2}$. However, this is also impossible, since that would imply the existence of a maximum in (x_0, x_1), which is incomatible with

$$-\sqrt{\frac{x}{2}} < y(x) < -\sqrt{\frac{x}{6}}.$$

□

Lemma 2.27. *Let*

$$x_0 \geq \frac{3^{3/8} 5^{3/4}}{\sqrt[4]{34\sqrt{3} + 7\sqrt{7}}} =: \alpha \approx 1.70196.$$

Suppose $y(x)$ satisfies Equation (2.17) and

$$y(x_0) = -\sqrt{\frac{x_0}{2}}.$$

If in addition $y(x)$ is defined for all $x \geq x_0$ and approaches $-\sqrt{x/6}$, then $y(x)$ must intersect Π_0^- at some point $x_1 > x_0$.

Proof. Using Lemma 2.25 above, we may assume

$$y'(x_0) > -\frac{1}{2\sqrt{2x_0}},$$

and

$$y(x) \geq -\sqrt{x/2},$$

for all $x \geq x_0$. If $y'(x_0) \geq 0$, Lemma 2.26 yields an intersection with Π_-. Thus we assume

$$-\frac{1}{2\sqrt{2x_0}} < y'(x_0) < 0,$$

from this point onwards.

According to the intermediate value theorem there exists a point $x_{1/2}$ where

$$y(x_{1/2}) = -\frac{\sqrt{\sqrt{21} + 6}}{\sqrt{30}} \sqrt{x_{1/2}}$$

and this intersection is from below (this is one of the curves where $\frac{\partial f}{\partial y} = 0$. It is in fact a minimum with respect to y). If the slope $y'(x_{1/2}) \geq 0$ an intersection with $\{y = -\sqrt{x/6}\}$

is guaranteed. Also, if y intersects the line $\{y = y(x_{1/2})\}$ at some point $> x_{1/2}$, Rolle's theorem furnishes an intermediate point where $y' = 0$, and once again an intersection with $\{y = -\sqrt{x/6}\}$ is bound to happen.

Thus we may assume that $y(x) \leq y(x_{1/2})$ for all $x \geq x_{1/2}$. Using Taylor's Theorem we have

$$\begin{aligned} y(x) &= y(x_{1/2}) + y'(x_{1/2})(x - x_{1/2}) + \int_{x_{1/2}}^{x} (x-t) f(t, y(t)) dt \\ &\geq y(x_{1/2}) + m_{1/2}(x - x_{1/2}) + \int_{x_{1/2}}^{x} (x-t) f(t, y(t)) dt \\ &\geq y(x_{1/2}) + m_{1/2}(x - x_{1/2}) + \int_{x_{1/2}}^{x} (x-t) f(t, y(x_{1/2})) dt := y_L(x), \end{aligned}$$

where $m_{1/2}$ is the slope of the curve

$$y = -\frac{\sqrt{\sqrt{21}x + 6x}}{\sqrt{30}},$$

at $x = x_{1/2}$, and the last inequality is valid for

$$x_{1/2} \leq x \leq \frac{3^{3/8} 5^{3/4}}{\sqrt[4]{34\sqrt{3} + 7\sqrt{7}}} x_{1/2}.$$

Evaluating the inequality $y(x) \geq y_L(x)$ at the upper bound

$$x_1 = \frac{3^{3/8} 5^{3/4}}{\sqrt[4]{34\sqrt{3} + 7\sqrt{7}}} x_{1/2},$$

gives $y(x_1) \geq -\sqrt{x_1/6}$ as long as

$$x_{1/2}^4 \geq \frac{375\sqrt{3}}{34\sqrt{3} + 7\sqrt{7}}.$$

□

The number α found in the previous lemma can now be used in the following existence theorem.

Theorem 2.28. *The solutions $Y_{\pm}(x)$ is continuous on the interval (α, ∞).*

Proof. We have already established that Y_- does not intersect Π_0^- for $x \geq 0$. The preceding lemma implies that if it intersects Π_1^- at any point $> \alpha$, then it must also intersect Π_0^-. Thus our solution is bounded

$$-\sqrt{x/2} \leq Y_-(x) \leq -\sqrt{x/6},$$

for $x \in (\alpha, \infty)$. The reflection symmetry $y \mapsto -y$ of Equation (2.17) implies that Y_+ has an identical interval of existence. □

2.3 Behaviour near the Equilibria Points

The purpose of this section is studying the fourth Painlevé equation via the Hamiltonian formalism, similarly to what was done for the first Painlevé equation in [DJ11]. Joshi and Radnović [JR16] have used a Hamiltonian formulation for the fourth Painlevé equation (2.1), which can be derived from the symmetric form of Equation (2.1) as in [NY98]:

$$\begin{aligned}\frac{\mathrm{d}y_1}{\mathrm{d}x} &= -y_1(y_1+2y_2+2x)-2\alpha_1,\\ \frac{\mathrm{d}y_2}{\mathrm{d}x} &= y_2(2y_1+y_2+2x)-2\alpha_2,\end{aligned}$$

with the time-dependent Hamiltonian function

$$H(x,y_1,y_2) = -y_1y_2(y_1+y_2+2x)+2\alpha_2y_1-2\alpha_1y_2,$$

where $y_1 = y$ solves Equation (2.1), with $a = 1-\alpha_1-2\alpha_2, b = -2\alpha_1^2$.

Using the change of variables

$$y_1(x) = xu(z), \quad y_2(x) = xv(z), \quad z = x^2/2,$$

we get the system

$$u' = -u(u+2v+2)-\frac{\alpha_1}{z}-\frac{u}{2z}, \tag{2.32a}$$

$$v' = v(2u+v+2)-\frac{\alpha_2}{z}-\frac{v}{2z}. \tag{2.32b}$$

The limiting autonomous system is obtained by neglecting the $\mathcal{O}(1/z)$ terms

$$u' = -u(u+2v+2), \tag{2.33a}$$

$$v' = v(2u+v+2), \tag{2.33b}$$

whose fixed points are

$$\begin{aligned}&(0,0),\\ &(-2,0),\\ &(0,-2),\\ &\left(-\frac{2}{3},-\frac{2}{3}\right).\end{aligned}$$

The corresponding Jacobian matrices are

$$\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix},$$
$$\begin{pmatrix} 2 & 4 \\ 0 & -2 \end{pmatrix},$$
$$\begin{pmatrix} 2 & 0 \\ -4 & -2 \end{pmatrix},$$
$$\begin{pmatrix} \frac{2}{3} & \frac{4}{3} \\ -\frac{4}{3} & -\frac{2}{3} \end{pmatrix},$$

with respective eigenvalues

$$\{-2, 2\},$$
$$\{-2, 2\},$$
$$\{-2, 2\},$$
$$\left\{-\frac{2i}{\sqrt{3}}, \frac{2i}{\sqrt{3}}\right\}.$$

Now, in the neighbourhood of each of the fixed points, we will bring the system (2.32) into the form

$$\frac{\mathrm{d}p}{\mathrm{d}t} = v(t^{-1}, p), \tag{2.34}$$

with $p = (p^+, p^-)$, and v satisfying the conditions of the following Lemma:

Lemma 2.29 (Lemma 5.1 of [DJ11]). *Assume that $v = (v^+, v^-)$ is a $\mathbb{C}^2$-valued complex analytic function on an open neighbourhood D of the origin in $\mathbb{C}^3$ such that $v^\pm(u, p) = \pm p^\pm + w(u, p)$, $w(0, 0) = 0$, and*

$$\frac{\partial w(0, p)}{\partial p}\Big|_{p=0} = 0.$$

Here $p = (p^+, p^-) \in \mathbb{C}^2$ and $\|p\| := \max\{|p^+|, |p^-|\}$. Then there exist strictly positive real numbers $\delta_0, \epsilon_0, C_1, C_2, C_3$, and C_4 such that $\|w(u, p)\| \leq C_1\|p\|^2 + C_2|u|$, and $\|\partial w(u, p)/\partial p\| \leq C_3\|p\| + C_4|u|$ if $\|p\| < \delta_0$ and $|u| < \epsilon_0$. Here the last condition implies the preceding one for $C_1 = C_3/2$ and a suitable C_2. In the sequel we will take D equal to the set of all $(u, p) \in \mathbb{C} \times \mathbb{C}^2$ such that $|u| < \epsilon_0$ and $\|p\| < \delta_0$. For solutions p of (2.34) we will always require that $|t| > 1/\epsilon_0$ and $\|p(t)\| < \delta_0$ for all t in the domain of definition of p.

The proof of this lemma is identical to that given for the corresponding result in [DJ11] and we refer to that paper for details.

At each fixed point, the sequence of lemmas of [DJ11] produces three types of solutions: $p_{t_0, a^-}, p_\uparrow$ and $p_\downarrow$. We describe their asymptotic behaviours in the particular case of Equation (2.32). For a given fixed point, let us follow the convention in [DJ11] and define

$$
\begin{aligned}
c_1 &= -\left(\frac{\partial v(0,p)}{\partial p}\Big|_{p=0} \right)^{-1} \frac{\partial v(u,0)}{\partial u}\Big|_{u=0}, \\
L_1 &= \frac{\partial^2 v(u,p)}{\partial u \partial p}\Big|_{u=0,p=0} + \frac{\partial^2 v(0,p)}{\partial p^2}_{p=0} c_1, \\
\alpha &= (0\ 1) L_1 (0\ 1)^T, \\
R_{\eta,r} &= \left\{ t \in \mathbb{C} : |t| \geq r,\ \left| \mathrm{e}^{-t} t^{\alpha} \right| \leq \eta \right\}.
\end{aligned}
$$

Focusing on the first fixed point (the origin), the simple change of variables $t = 2z, u = p^-, v = p^+$.

$$
\begin{aligned}
{p^+}' &= p^+ + p^- p^+ + \frac{{p^+}^2}{2} - \frac{p^+}{2t} - \frac{\alpha_2}{t} := v^+, \\
{p^-}' &= -p^- - p^- p^+ - \frac{1}{2} {p^-}^2 - \frac{p^-}{2t} - \frac{\alpha_1}{t} =: v^-.
\end{aligned}
$$

The truncated solutions p_{t_0,a^-} from [DJ11] correspond to solutions of Equation (2.1) which approach zero on the domain which consists of applying the transformation $x \mapsto \sqrt{2} x^{1/2}$ to $R_{\eta,r}$ for some $\eta, r > 0$ (which we will call $\hat{R}_{\eta,r}$), and $p_\uparrow, p_\downarrow$ correspond to solutions which approach zero as their argument approaches infinity in the sectors

$$
-\pi/4 + \epsilon < \arg(x) < 3\pi/4 - \epsilon,
$$

and

$$
-3\pi/4 + \epsilon < \arg(x) < \pi/4 - \epsilon,
$$

respectively.

Another fixed point is $(-2, 0)$. The affine map is

$$
\begin{aligned}
u &= -p^- + p^+ - 2, \\
v &= p^-,
\end{aligned}
$$

which results, along with the transformation $t = 2z$, in the following system

$$
\begin{aligned}
{p^+}' &= p^+ - \frac{p^+}{2t} - \frac{1}{2} \left(p^+\right)^2 + p^- p^+ - \frac{\alpha_1}{t} - \frac{\alpha_2}{t} + \frac{1}{t}, \\
{p^-}' &= -p^- - \frac{p^-}{2t} - \frac{1}{2} \left(p^-\right)^2 + p^+ p^- - \frac{\alpha_2}{t}.
\end{aligned}
$$

This time the truncated solutions correspond to solutions of Equation (2.1) which are asymptotic to $-2x$ as $|x| \to \infty$ in $\hat{R}_{\eta,r}$, while $p_\uparrow, p_\downarrow$ have the same behaviour in their corresponding sectors above.

Another fixed point is $(0, -2)$. The affine map is

$$
\begin{aligned}
u &= -p^+, \\
p^- &= p^- + p^+ - 2,
\end{aligned}
$$

and the resulting system is

$$\begin{aligned} {p^{+}}' &= p^{+} - \frac{p^{+}}{2t} - \frac{\left(p^{+}\right)^{2}}{2} - p^{-}p^{+} + \frac{\alpha_1}{t}, \\ {p^{-}}' &= -p^{-} - \frac{p^{-}}{2t} + \frac{\left(p^{-}\right)^{2}}{2} + p^{+}p^{-} - \frac{\alpha_1}{t} - \frac{\alpha_2}{t} + \frac{1}{t}. \end{aligned}$$

This point is symmetric with the point $(-2, 0)$.

Lastly, there is the point $(-2/3, -2/3)$. The affine map is

$$\begin{aligned} u &= \frac{1}{2}\left(\sqrt{3} + i\right) ip^{-} - \frac{1}{2} i \left(\sqrt{3} - i\right) p^{+} - \frac{2}{3}, \\ v &= p^{-} + p^{+} - \frac{2}{3}. \end{aligned}$$

The resulting system is

$$\begin{aligned} {p^{+}}' &= p^{+} + \frac{3\left(p^{-}\right)^{2}}{2} - \frac{p^{+}}{2t} + -\frac{\alpha_2}{2t} - \frac{i\alpha_1}{\sqrt{3}t} - \frac{i\alpha_2}{2\sqrt{3}t} + \frac{1}{6t} + \frac{i}{2\sqrt{3}t}, \\ {p^{-}}' &= -p^{-} - \frac{p^{-}}{2t} - \frac{1}{2} 3\left(p^{+}\right)^{2} + \frac{i\alpha_1}{\sqrt{3}t} + \frac{i\alpha_2}{2\sqrt{3}t} - \frac{\alpha_2}{2t} + \frac{1}{6t} - \frac{i}{2\sqrt{3}t}. \end{aligned}$$

Here the truncated solutions p_{t_0,a^-} and the triply truncated solutions $p_\uparrow, p_\downarrow$ correspond to solutions of Equation (2.1) which are asymptotic to $-2/3x$ in the domains $\hat{R}_{\eta,r}$,

$$-\pi/4 + \epsilon < \arg(x) < 3\pi/4 - \epsilon,$$

and

$$-3\pi/4 + \epsilon < \arg(x) < \pi/4 - \epsilon,$$

respectively.

The above results show that the system (2.33) has four fixed points. We showed in this section that in the neighbourhood of each such point, there exist three types of solutions, which correspond to tronquée or tritronquée solutions, with specific asymptotic behaviours near infinity. In the case of the final fixed point studied above, the solutions $p_\uparrow$ and $p_\downarrow$ match up with the results we found in Section 2.2.2 eariler.

2.4 Numerical Analysis

The Mathematica code below, allows us to approximate the function m^- at various points $y_0 > 0$, with the parameter “eps” being the precision.

```
mMinus[y0, inf, wp, step, eps] = (p0 = 0;
  sol = ParametricNDSolveValue[{y'[x] == a [x] p[x],
    p'[x] == a [x]*y[x]/4*(6 y[x]^2 + x)*(2 y[x]^2 + x)
    ,y[0] == y0,p[0] == m0, a[0] == 1,
    WhenEvent[y[x] < 0, {a[x] -> 0, y[x] -> -1}],
    WhenEvent[y[x] > y0 + 1, {a[x] -> 0, y[x] -> 1}]},
   y[inf], {x, 0, inf}, m0, MaxSteps -> Infinity,
   WorkingPrecision -> wp, Method -> "StiffnessSwitching",
   DiscreteVariables -> a];
  s = step;
  -NestWhile[If[sol[# - s] > 0, # - s, s /= 2; #] &, p0,
    Abs[s] >= eps &] // Quiet)
```

Chapter 3

Discrete Painlevé Equations

In this chapter we will explore the relationship between the fourth Painlevé equation P_{IV} and the discrete first Painlevé equation d-P_I. Although studies of d-P_I concerning the behaviour of its solutions in the asymptotic limit where the independent variable becomes large are known [JL15], asymptotic results for a large parameter are lacking. In particular, we will use the existence of a one-parameter family of solutions of P_{IV} with its asymptotic behaviour, to establish asymptotic results for solutions of d-P_I.

The discrete first Painlevé equation d-P_I plays a role in the study of orthogonal polynomials, and is referred to as the Freud equation there [DK06, Mag95, Mag96].

3.1 The Connection Between P_{IV} and d-P_I

In [HJN16] the authors make use of the connection between the discrete Painlevé equation d-P_I and the continuous Painlevé equation P_{IV} described in [FIK91]. Their starting point is the fourth Painlevé equation, in the form

$$w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4tw^2 + 2(t^2 - \alpha)w + \frac{\beta}{w}.$$

The starting point for the derivation is a collection of transformations for solutions of P_{IV} called Bäcklund transformations (see [HJN16] for further information about such transformations). The Bäcklund transformations (BTs) we consider are given by

$$\widetilde{w}^{\pm}(t) := \frac{w' - w^2 - 2tw \mp \gamma}{2w},$$

where $\widetilde{w}^{\pm}$ with parameters $\widetilde{\alpha}^{\pm}, \widetilde{\beta}^{\pm}$ given by the formulas

$$\begin{aligned} \widetilde{\alpha}^{\pm} &= \frac{1}{4}\left(2 - 2\alpha \pm 3\gamma\right), \\ \widetilde{\beta}^{\pm} &= -\frac{1}{2}\left(1 + \alpha \pm \frac{1}{2}\gamma\right)^2, \end{aligned}$$

and

$$\widehat{w}^{\pm}(t) := -\frac{w' + w^2 + 2tw \mp \gamma}{2w},$$

with

$$\begin{aligned} \widehat{\alpha}^{\pm} &= -\frac{1}{4}(2 + 2\alpha \pm 3\gamma), \\ \widehat{\beta}^{\pm} &= -\frac{1}{2}\left(1 - \alpha \pm \frac{1}{2}\gamma\right)^2. \end{aligned}$$

It is interesting to note that the mappings of the parameters above can be expressed in terms of an integer n, as

$$\alpha_n = -\frac{n}{2} + c_0 + c_1(-1)^n, \tag{3.1a}$$

$$\gamma_n = n - 2c_0 + \frac{2c_1}{3}(-1)^n. \tag{3.1b}$$

Denoting each solution of $\mathrm{P_{IV}}$ with parameters α_n, γ_n by $w_n = w(t; \alpha_n, \gamma_n)$, we obtain $\widehat{w}^+ = w_{n+1}$ and $\widetilde{w}^- = w_{n-1}$, from the corresponding mappings of parameters given above. Starting from any seed solution w_0, one finds a sequence $\{w_n\}_{n=-\infty}^{\infty}$ of solutions by forward and backward iteration of the above Bäcklund transformations.

Actually, we can eliminate the derivative w_n' from the pair of Bäcklund transformations that give this sequence. Doing so, we arrive at the so-called discrete first Painlevé equation

$$\text{d-P}_{\mathrm{I}}: \quad w_{n+1} + w_n + w_{n-1} = -2t + \frac{\gamma_n}{w_n}. \tag{3.2}$$

Notice that the independent variable of this difference equation is n, while c_0 and c_1 in the equation (3.1b) for γ_n and t play the role of parameters. In other words, the roles of variable and parameter are interchanged from those of the differential equation $\mathrm{P_{IV}}$.

3.2 General asymptotic behaviours as $t \to \infty$

In this section, we discuss the asymptotic analysis of solutions of d-$\mathrm{P_I}$ as $t \to \infty$. Although this may appear to be similar to the asymptotic analysis of $\mathrm{P_{IV}}$ in Chapter 2, there are some important differences that make the analysis of this section worthwhile as a separate study.

Our motivation is analogous to the study of special functions, such as Bessel functions $J_\nu(x)$ in the limit as $x \to \infty$ versus the limit $\nu \to \infty$. The results of Chapter 2 relied on a given fixed pair of values of parameters α and β in $\mathrm{P_{IV}}$. In contrast, in this section, these parameters will be varying, whilst t although taken to be arbitrarily large, will stay fixed.

We find two families of asymptotic behaviours. In this section, we show that there exist 2-parameter solutions with oscillatory behaviour (for real n). In the next section, we consider behaviours that do not appear to be covered by this result.

The asymptotic behaviour of the sequence of solutions of Equations (3.2) can also be arrived at by a scaling approach: we start by the change of dependent variable

$$w_n = t v_n,$$

which results in the equation

$$v_{n+1} + v_n + v_{n-1} = -2 + \frac{\gamma_n}{t^2 v_n}. \tag{3.3}$$

As $t \to +\infty$, Equation (3.3) has the maximal dominant balance

$$v_{n+1}^{(0)} + v_n^{(0)} + v_{n-1}^{(0)} = -2.$$

It is elementary to obtain the solution

$$v_n^{(0)} = -\frac{2}{3} + C_1 \mathrm{e}^{\frac{2\pi i n}{3}} + C_2 \mathrm{e}^{-\frac{2\pi i n}{3}},$$

where C_1 and C_2 depend on t only. These provide the dominant behaviour of Equation (3.3).

To prove that $v_n^{(0)}$ indeed provide leading order behaviours of the solution of Equation (3.3), we convert this difference equation to a summation equation (i.e., discrete analogue of an integral equation). First, transform the dependent variable in Equation (3.3) according to

$$v_n = -\frac{2}{3} + u_n,$$

which results in the equation

$$u_{n+1} + u_n + u_{n-1} = \frac{\gamma_n}{t^2\left(-\frac{2}{3} + u_n\right)}. \tag{3.4}$$

The solution of the associated homogeneous equation of Equation (3.4) is

$$u_n^h = c_1 \mathrm{e}^{\frac{2\pi i n}{3}} + c_2 \mathrm{e}^{-\frac{2\pi i n}{3}}.$$

We make use of the associated homogeneous solution in order to convert Equation (3.4) into a summation equation.

Theorem 3.1. *The solutions u_n of Equation (3.4) satisfy the summation equation*

$$u_n = p_0 \mathrm{e}^{\frac{2\pi i n}{3}} + q_0 \mathrm{e}^{\frac{-2\pi i n}{3}} + \sum_{l=n_0}^{n-1} \sum_{k=n_0}^{l} \frac{\mathrm{e}^{\frac{2\pi i (n+k-2l-1)}{3}} \gamma_k}{t^2\left(-\frac{2}{3} + u_k\right)}, \tag{3.5}$$

where p_0 and q_0 are constants.

Proof. First we multiply Equation (3.4) by $\mathrm{e}^{\frac{2\pi i n}{3}}$ and rearrange the terms, to get

$$\mathrm{e}^{-\frac{2\pi i}{3}} \left(\mathrm{e}^{\frac{2\pi i (n+1)}{3}} u_{n+1} - \mathrm{e}^{\frac{2\pi i n}{3}} u_n\right) + \left(1 + \mathrm{e}^{-\frac{2\pi i}{3}}\right) \left(\mathrm{e}^{\frac{2\pi i n}{3}} u_n - \mathrm{e}^{\frac{2\pi i (n-1)}{3}} u_{n-1}\right) = \frac{\mathrm{e}^{\frac{2\pi i n}{3}} \gamma_n}{t^2\left(-\frac{2}{3} + u_n\right)}.$$

Taking the indefinite sum yields

$$\mathrm{e}^{\frac{2\pi i n}{3}} u_{n+1} + \left(1 + \mathrm{e}^{-\frac{2\pi i}{3}}\right) \mathrm{e}^{\frac{2\pi i n}{3}} u_n = a_0 + \sum_{k=n_0}^{n} \frac{\mathrm{e}^{\frac{2\pi i k}{3}} \gamma_k}{t^2\left(-\frac{2}{3} + u_k\right)},$$

where a_0 depends on t and the lower summation limit n_0. Rearranging gives

$$u_{n+1} + \left(1 + \mathrm{e}^{-\frac{2\pi i}{3}}\right) u_n = a_0 \mathrm{e}^{-\frac{2\pi i n}{3}} + \sum_{k=n_0}^{n} \frac{\mathrm{e}^{\frac{2\pi i (k-n)}{3}} \gamma_k}{t^2\left(-\frac{2}{3} + u_k\right)},$$

and multiplying by $\mathrm{e}^{-\frac{2\pi i n}{3}}$ results in the equation

$$\mathrm{e}^{\frac{2\pi i}{3}} \left(\mathrm{e}^{-\frac{2\pi i (n+1)}{3}} u_{n+1} - \mathrm{e}^{-\frac{2\pi i n}{3}} u_n\right) = a_0 \mathrm{e}^{-\frac{4\pi i n}{3}} + \sum_{k=n_0}^{n} \frac{\mathrm{e}^{\frac{2\pi i (k-2n)}{3}} \gamma_k}{t^2\left(-\frac{2}{3} + u_k\right)}.$$

Rearranging gives

$$\mathrm{e}^{-\frac{2\pi i(n+1)}{3}}u_{n+1}-\mathrm{e}^{-\frac{2\pi i n}{3}}u_n=a_0\mathrm{e}^{-\frac{4\pi i n}{3}}\mathrm{e}^{-\frac{2\pi i}{3}}+\sum_{k=n_0}^{n}\frac{\mathrm{e}^{\frac{2\pi i(k-2n-1)}{3}}\gamma_k}{t^2\left(-\frac{2}{3}+u_k\right)},$$

and taking a second indefinite summation gives

$$\mathrm{e}^{-\frac{2\pi i n}{3}}u_n=b_0+\sum_{l=n_0}^{n-1}\left[a_0\mathrm{e}^{-\frac{4\pi i l}{3}}\mathrm{e}^{-\frac{2\pi i}{3}}+\sum_{k=n_0}^{l}\frac{\mathrm{e}^{\frac{2\pi i(k-2l-1)}{3}}\gamma_k}{t^2\left(-\frac{2}{3}+u_k\right)}\right],$$

where b_0 is a constant depending on t and n_0. Simplifying gives

$$u_n=b_0\mathrm{e}^{\frac{2\pi i n}{3}}+a_0\mathrm{e}^{-\frac{\pi i}{2}}\frac{e^{\frac{2\pi i n-4\pi i n_0}{3}}-e^{-\frac{2\pi i n}{3}}}{\sqrt{3}}+\sum_{l=n_0}^{n-1}\sum_{k=n_0}^{l}\frac{\mathrm{e}^{\frac{2\pi i(n+k-2l-1)}{3}}\gamma_k}{t^2\left(-\frac{2}{3}+u_k\right)}.$$

We obtain the desired summation equation, where

$$\begin{aligned} p_0 &= b_0-\frac{i\mathrm{e}^{\frac{-4\pi i n_0}{3}}}{\sqrt{3}}a_0, \\ q_0 &= \frac{i}{\sqrt{3}}a_0. \end{aligned}$$

□

We now show that the fixed point theorem can be applied to the summation equation (3.5) to provide a two-parameter family of solutions with the leading order behaviour described above. The argument is analogous to those given in Sections 2.1 and 2.2 for the case of the second and fourth Painlevé equations in Chapter 2 and for reasons of simplicity, we do not give the technical details of the proof here.

Theorem 3.2. *For each given $\epsilon>0$, p_0, q_0, n_0, there exists N such that a unique solution of the summation equation (3.5) exists for $n_0\le n\le N$ and $1/\epsilon<|t|$. Moreover, this solution has the leading order asymptotic behaviour*

$$u_n=p_0\mathrm{e}^{\frac{2\pi i n}{3}}+q_0\mathrm{e}^{\frac{-2\pi i n}{3}}+\mathcal{O}\left(\frac{1}{t^2}\right), \tag{3.6}$$

for $t\to\infty$.

Since $w_n(t)$ is a solution of $\mathrm{P_{IV}}$, it is natural to ask whether the asymptotic behaviour (3.6) found above is related to the ones we deduced in Chapter 2. The answer to this question is affirmative, as described in the following Remark.

Remark 3.3. *The behaviour given by Equation (3.6) is the same as the asymptotic behaviour (2.18), in the limit $t\to-\infty$.*

Proof. Since

$$w_n=t\left(-\frac{2}{3}+u_n\right),$$

and using (2.15), keeping in mind that $t = z$, we have

$$y_0(x) \sim \pm\sqrt{\frac{x}{6}},$$

where $y_n^2(x) = w_n(-x/\sqrt{2})/(2\sqrt{2})$ and $t = -x/\sqrt{2}$. We have the desired result, in the special case where $\alpha_n = \gamma_n = 0$ in Equation (3.1). Furthermore, one could relate the values of p_0 and q_0 with that of a of Equation (1.20) by expanding the oscillating term and comparing the coefficients. □

3.3 Special asymptotic behaviours as $t \to \infty$

We show in this section that there also exists a 0-parameter solution of Equation (3.2), which has a power series expansion in powers of t. Estimates based on the recursion relation indicate that this is a divergent asymptotic series as $t \to \infty$. This divergence property suggests that there may be exponentially small terms that are hidden by this series, giving rise to Stokes phenomena in the t-plane. (See [JL15] for such a study in the case where both $n \gg 1$ and $t \gg 1$ and Equation (3.2) is studied in a continuum-limit-like approach.)

In investigating Equation (3.3) to obtain the 2-parameter asymptotic behaviour in the previous section, we assumed that v_n cannot be vanishingly small. Now we consider the possibility $v_n \ll 1$.

Multiplying Equation (3.3) by v_n we get

$$(\overline{v} + v + \underline{v})v = -2v + \frac{\gamma_n}{t^2}. \tag{3.7}$$

where $v = v_n, \overline{v} = v_{n+1}$ and $\underline{v} = v_{n-1}$. The left-hand side of Equation (3.7) is of order $\mathcal{O}\left(v^2\right)$ - while the two terms on the right-hand side are of order $\mathcal{O}(v)$ and $\mathcal{O}(1)$ respectively. The method of dominant balances thus suggests

$$v \sim \frac{\gamma_n}{2t^2},$$

and one can see that this behaviour is consistent with Equation (3.7) by recursive substitution.

Starting with this behaviour, we find a formal series solution of the form

$$v_n = \sum_{k=1}^{\infty} \frac{a_k(n)}{t^{2k}}, \tag{3.8}$$

for $t \to \infty$. To find the coefficients $a_k(n)$, we substitute this series into Equation (3.7).

This substitution results in the equation

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{\left(\overline{a}_k + a_k + \underline{a}_k\right) a_l}{t^{2k+2l}} = -2\sum_{k=2}^{\infty} \frac{a_k}{t^{2k}},$$

where we have used the value $a_1(n) = \gamma_n/2$. Letting $m = k + l$, and changing the order of summation, we find

$$\sum_{m=2}^{\infty} \sum_{k=1}^{m-1} \frac{\left(\overline{a}_k + a_k + \underline{a}_k\right) a_{m-k}}{t^{2m}} = -2 \sum_{k=2}^{\infty} \frac{a_k}{t^{2k}}.$$

Thus, one gets the recurrence relation

$$-2a_m = \sum_{k=1}^{m-1} a_{m-k} \left(\overline{a}_k + a_k + \underline{a}_k\right). \tag{3.9}$$

Taking $\gamma_n = \gamma_0 + n$, one can explicitly calculate the first few coefficients, as given in Table 3.1

k	$a_k(n)$
1	$\frac{1}{2}\left(\gamma_0 + n\right)$
2	$-\frac{3}{8}\left(\gamma_0 + n\right)^2$
3	$\frac{9}{16}\left(\gamma_0 + n\right)^3 + \frac{3}{16}\left(\gamma_0 + n\right)$
4	$-\frac{135}{128}\left(\gamma_0 + n\right)^4 - \frac{81}{64}\left(\gamma_0 + n\right)^2$
5	$\frac{567}{256}\left(\gamma_0 + n\right)^5 + \frac{783}{128}\left(\gamma_0 + n\right)^3 + \frac{315}{256}\left(\gamma_0 + n\right)$
6	$-\frac{5103(\gamma_0+n)^6}{1024} - \frac{26325(\gamma_0+n)^4}{1024} - \frac{18711(\gamma_0+n)^2}{1024}$

TABLE 3.1: The coefficients $a_k(n)$ for $\gamma_n = \gamma_0 + n$ and $1 \leq k \leq 6$.

One can see that each of the coefficients a_k listed in Table 3.1 is a polynomial of degree k in the variable $\gamma_0 + n$. This is shown to be true in the following result:

Proposition 3.4. *For every positive integer k there exist coefficients $d_{k,1}, \ldots, d_{k,k}$ such that*

$$a_k(n) = \sum_{l=1}^{k} d_{k,l}(\gamma_0 + n)^l.$$

Proof. Table 3.1 shows that the base case for a proof by induction is valid. Assuming the proposition holds for all positive integers $k < m$, the recurrence relation (3.9) gives

$$a_m = -\frac{1}{2} \sum_{k=1}^{m-1} \sum_{l=1}^{m-k} d_{m-k,l}(\gamma_0 + n)^l \sum_{p=1}^{k} d_{k,p} \left[(\gamma_0 + n + 1)^p + (\gamma_0 + n)^p + (\gamma_0 + n - 1)^p\right],$$

which is a polynomial of degree m in $\gamma_0 + n$. □

Letting $\gamma_0 = 0$, and plotting the ratios

$$\frac{a_{m+1}(n)}{a_m(n)} \tag{3.10}$$

for various values of n, we observe the fact that they appear to behave like $-(m+n-1)$ (see Figure 3.1). Furthermore, in the case where $\gamma_0 = 1/2$, the ratios (3.10) seem to behave like $-(m + n + 1/2 - 1)$, as presented in Table 3.2. These cases lead to the asymptotic recurrence relation

$$a_{m+1}(n) \sim -(m + \gamma_0 + n - 1)a_m(n)$$

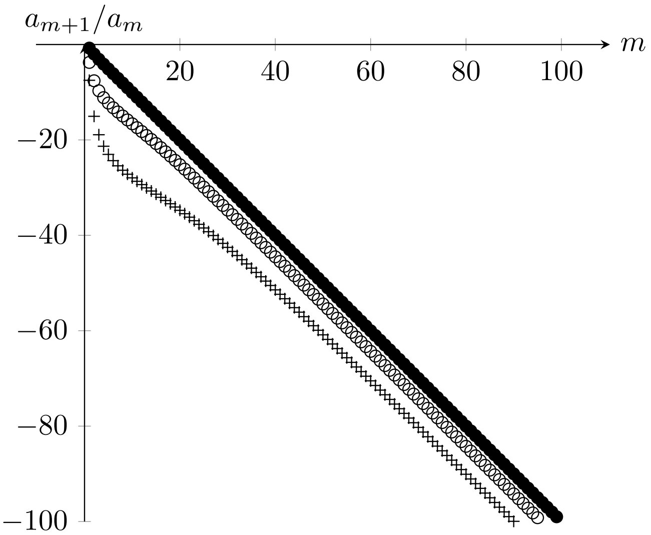

FIGURE 3.1: The ratios $a_{m+1}(n)/a_m(n)$ for $\gamma_0 = 0$ and $n = 1$ (solid), 5 (hollow) or 10 (plus).

which can be solved exactly, suggesting Conjecture 3.5:

m	$\frac{a_{m+1}(1)}{a_m(1)}$	$-(m+1+1/2-1)$
1	-1.125	-1.5
50	-50.522	-50.5
100	-100.511	-100.5
150	-150.507	-150.5
200	-200.505	-200.5
250	-250.504	-250.5
300	-300.503	-300.5

TABLE 3.2: Ratios of consecutive coefficients $a_m(1)$ and their linear growth for $\gamma_0 = 1/2$.

Conjecture 3.5. *We have for any fixed n*

$$a_m(n) \sim C_n(-1)^{m+1}\frac{\Gamma(m+\gamma_0+n-1))}{\Gamma(\gamma_0+n)},$$

where C_n is some real coefficient. Thus, the asymptotic series (3.8) diverges.

3.4 Convexity-Like Approach

The discrete first Painlevé equation can also be studied in a similar way to second order ODEs where the first derivative of the dependent variable does not appear explicitly. This is done by rewriting it in terms of a difference operator as

$$\Delta^2 v_n = -2 - 3v_n + \frac{\gamma_n}{t^2 v_n},$$

where

$$\Delta^2 v_n = v_{n+1} + v_{n-1} - 2v_n.$$

Analogously with the continuous Painlevé equations, we examine the zero set of the right-hand side

$$\frac{-2t^2 v_n - 3t^2 v_n^2 + \gamma_n}{t^2 v_n} = 0.$$

In terms of v_n the solutions are

$$v_n = -\frac{1}{3} \pm \frac{\sqrt{t^2 + 3\gamma_n}}{3|t|},$$

which are real, provided that

$$\gamma_n \geq -\frac{t^2}{3}.$$

Figures 3.2,3.3, and 3.4 show some values of the solutions v_n of Equation (3.3) with the initial values $v_0 = 1, v_1 = 2$ and $t = 1$, $t = 5$ and $t = 10$. One can see that as t increases v_n appears better and better approximated by a sinusoidal wave of period 3, as predicted by Theorem 3.2. The special asymptotic behaviour of Equation (3.8) appears to be unstable, hence a plot of the corresponding solution is not presented here.

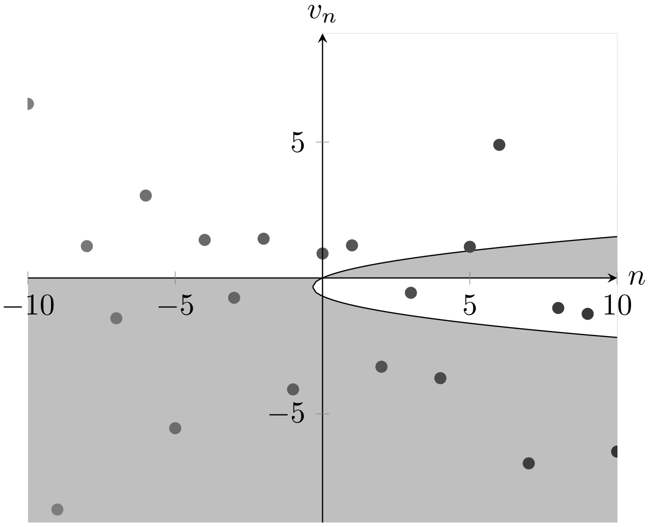

FIGURE 3.2: Starting with initial values $v_0 = 0.9$, $v_1 = 1.2$, the values of iterates v_n, with $t = 1$, are plotted as dots for each n in the interval $-10 \leq n \leq 10$. Each dot is assigned a grayscale intensity that varies with n so that v_n with the largest n appears as black while those with the smallest n appear as light gray. These dots are overlaid on the convexity-like plot for Equation (3.3), in which shaded regions indicate the positivity of $\Delta^2 v_n$.

3.5 Summary

In this Chapter, our main focus was the discrete first Painlevé equation d-P$_{\text{I}}$ in the limit as its parameter $t \to \infty$. We have established its general asymptotic behaviour as

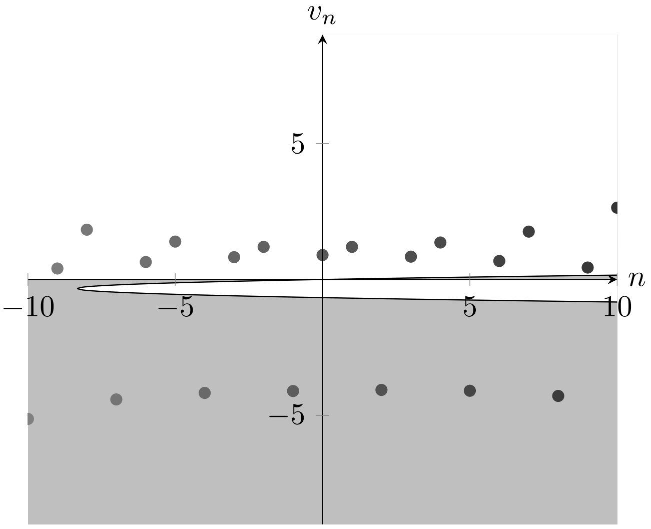

FIGURE 3.3: Starting with initial values $v_0 = 0.9$, $v_1 = 1.2$, the values of iterates v_n, with $t = 5$, are plotted as dots for each n in the interval $-10 \leq n \leq 10$. Each dot is assigned a grayscale intensity that varies with n so that v_n with the largest n appears as black while those with the smallest n appear as light gray. These dots are overlaid on the convexity-like plot for Equation (3.3), in which shaded regions indicate the positivity of $\Delta^2 v_n$.

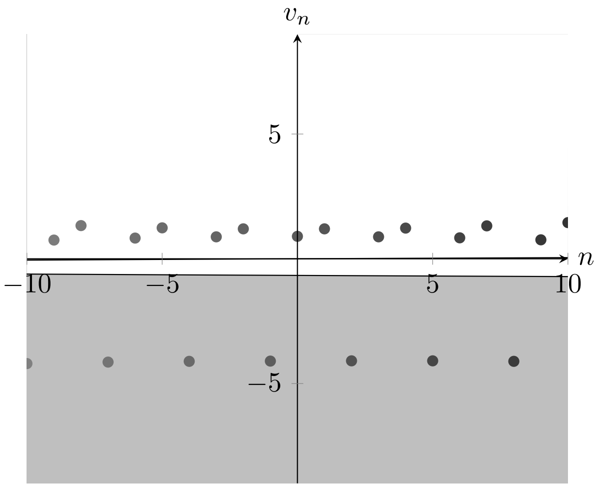

FIGURE 3.4: Starting with initial values $v_0 = 0.9$, $v_1 = 1.2$, the values of iterates v_n, with $t = 1$, are plotted as dots for each n in the interval $-10 \leq n \leq 10$. Each dot is assigned a grayscale intensity that varies with n so that v_n with the largest n appears as black while those with the smallest n appear as light gray. These dots are overlaid on the convexity-like plot for Equation (3.3), in which shaded regions indicate the positivity of $\Delta^2 v_n$.

$t \to \infty$ for the first time and related this to the analogous asymptotic behaviours we found in $\mathrm{P_{IV}}$. However, we also discovered an additional behaviour, which could be interpreted as a formal asymptotic series expansion of a discrete tronquée solution of d-$\mathrm{P_I}$. We provided evidence that indicates that this expansion is a divergent and therefore hides a free parameter. Finally, we have also produced some numerical evidence, involving plots of solutions of d-$\mathrm{P_I}$, demonstrating the preceding analysis.

Chapter 4

Conclusion

In this thesis, we studied asymptotic behaviours of continuous and discrete Painlevé equations. In particular, we focused on the second and fourth Painlevé transcendents and studying behaviours that arise in the limit as their independent variable $x \to \infty$. Despite widespread interest and long history of such investigations, there are some solutions, whose properties remain undiscovered. These are the solutions that Boutroux called *tronquée* or *tritronquée* solutions. We establish concrete bounds on their domains of existence and asymptotic validity.

One question that remains unanswered in the literature concerns the finite properties of solutions defined in such asymptotic limits. We have deduced some new results in the finite domain for $\mathrm{P_{II}}$ and for $\mathrm{P_{IV}}$.

For $\mathrm{P_{II}}$, perhaps the most famous solution in the literature is the so-called Hastings-McLeod solution, which is a special case of a family of solutions that decay as $x \to +\infty$ on the real line [CM88]. We extend the study of this family of solutions to an interval with a finite lower bound, which is given explicitly as a function of the free parameter in the asymptotic behaviour.

For $\mathrm{P_{IV}}$, there are also families of solutions studied at infinity. However, the main interest has been in connection formulas between these behaviours. In this thesis, our focus has been on the extension of these solutions to the finite domain. We start with an analysis similar to [JK01], by studying regions of convexity of the solutions on the real line. We identify a solution that lies between two parabolæ, which turns out to be a tritronquée solution, in the sense defined by Boutroux. We provide numerical information about this solution. In particular, we define and investigate a one-parameter family of *tangent solutions*, and demonstrate evidence for its convergence to the tritronquée solution.

The above results were obtained in the limit $x \to \infty$, but the role of Painlevé transcendents as nonlinear special functions also suggests questions in other directions. For example, recurrence relations for the solutions of $\mathrm{P_{IV}}$ obtained from their Bäcklund transformations, lead immediately to a discrete Painlevé equation known as d-$\mathrm{P_{I}}$. In the latter equation, the parameters of $\mathrm{P_{IV}}$ become the independent variable n, while the independent variable is fixed as a parameter. We consider d-$\mathrm{P_{I}}$ in the limit that the latter parameter $t \to \infty$ and find two types of asymptotic behaviours.

We have also managed to use asymptotic data of continuous equations ($\mathrm{P_{IV}}$) to deduce asymptotic behaviour of discrete equations (d-$\mathrm{P_{I}}$). Although it is not widely known in the literature, $\mathrm{P_{IV}}$ also has solutions that have power-like growth in the asymptotic limit. It is likely that such solutions are related to special families of solutions of d-$\mathrm{P_{I}}$. Moreover, relations of the same kind between other continuous and discrete Painlevé equations may lead to similar results.

One tool that has been used repeatedly is the contraction mapping theorem in the context of an integral equation , arising from ODEs of the type

$$x''(t) - q(t)x(t) = R(x(t), t). \tag{4.1}$$

Thinking of the right-hand side as a function of t alone, that is, as a forcing term, allowed us to formulate equivalent integral equations . If x_1, x_2 are linearly independent solutions of Equation (4.1) with Wronskian

$$W \equiv W(t) = x_1(t)x_2'(t) - x_1'(t)x_2(t)$$

then solutions of

$$x(t) = c_1x_1(t) + c_2x_2(t) - \int_a^t \frac{x_1(t)x_2(s) - x_1(s)x_2(t)}{W(s)} R(x(s), s)\mathrm{d}s$$

with fixed a, are all solutions of Equation (4.1). Furthermore, if the basis $\{x_1, x_2\}$ is chosen with particular behaviour at $+\infty$, we may let a tend to $+\infty$ and obtain an integral equation which encapsulates one-parameter families of solutions of Equation (4.1) having similar behaviour as $t \to +\infty$. This has been discussed here for the second- third- and fourth- Painlevé equations, and the same reasoning remains valid when applied to other equations.

Appendix A

Numerics

A.1 The Painlevé Equations on the Real Line

A.1.1 One Parameter Family of Solutions of the Second Painlevé Equation

In this section, we will present a method that allows one to approximate the initial values $y(0)$ and $y'(0)$ of solutions of $\mathrm{P_{II}}$, satisfying $y(x) \to 0$ as $x \to \infty$ (such as those in Figure 1.7. Fixing the initial value

$$y(0) = y_0 \neq 0,$$

the convexity plot 1.7 suggests the following trichotomy:

- $y(x)$ exists over an interval of the form $[0, \alpha)$, and has a constant sign there. In addition $\lim_{x \to \alpha^-} y(x) = \mathrm{sgn}(y_0)\infty$.
- $y(x) = 0$ for some positive value of x.
- $y(x) \to 0$ as $x \to \infty$.

Thus by modifying the initial slope $y'(0)$ one can "close in" on the one that makes the solution decay to zero.

A.1.2 Tangent Solutions for the Fourth Painlevé Equation

In this section we shall define and study numerically a 1-paramteric family of solutions of (2.17).

Definition A.1. *Let $x_0 > 0$, the solution y_{x_0} of (2.17) is defined as the one satisfying the initial conditions*

$$y_{x_0}(x_0) = \sqrt{\frac{x_0}{6}}$$
$$y'_{x_0}(x_0) = \frac{1}{2\sqrt{6x_0}}$$

We shall refer to y_{x_0} as the "tangent solution" of (2.17) based at x_0, since its initial conditions precisely describe tangency to the parabolic branch

$$y = \sqrt{\frac{x}{6}}$$

at the point $x = x_0$.

x_0 wp	1	10	50	100	150	200	250	300
10	-3.322158202	-6.314226432	-5.837486777	-5.439292980	-5.726193681	-5.710288161	-5.889044098	-5.547082398
20	-3.322154914	-6.443308764	-7.474563554	-8.051176608	-8.418359479	-8.491892696	-8.580908904	-8.579835265
100	-3.322154914	-6.443308651	-7.474217811	-8.051715964	-8.434998404	-8.611994854	-8.795069863	-8.875642979

FIGURE A.1: The approximate position of the earliest pole smaller than x_0

In order to obtain high-precision results, we use the "StiffnessSwitching" method in Mathematica's NDSolve. Tables A.1 display the position of the earliest encountered poles of the tangent solutions y_{x_0} for various values of x_0. One can also see how the value of the parameter "WorkingPrecision" (wp) used affects the results.

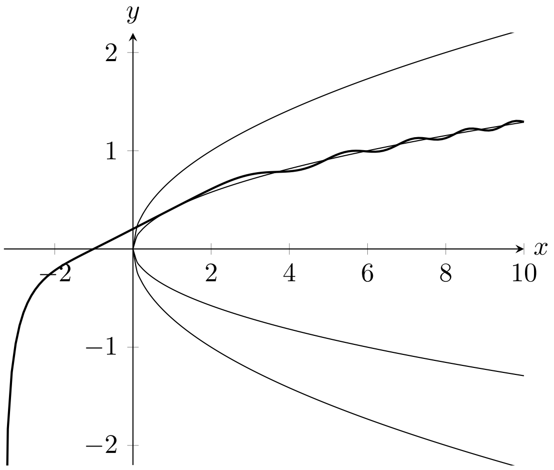

FIGURE A.2: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 1$ and WorkingPrecision=10

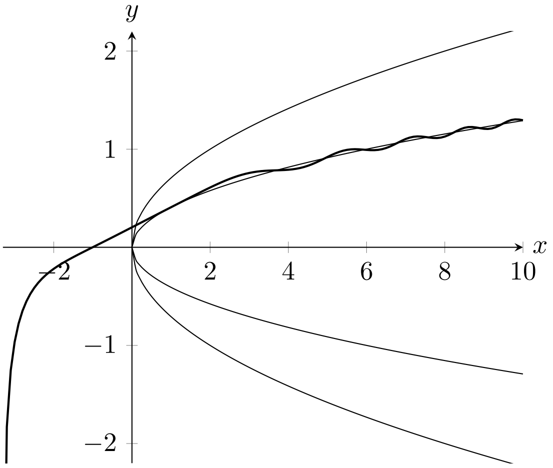

FIGURE A.3: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 1$ and WorkingPrecision=20

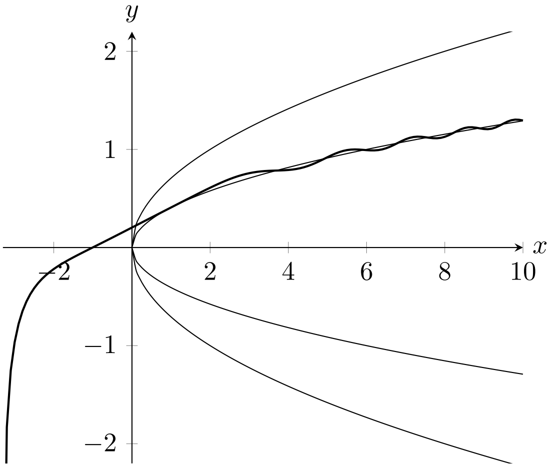

FIGURE A.4: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 1$ and WorkingPrecision=100

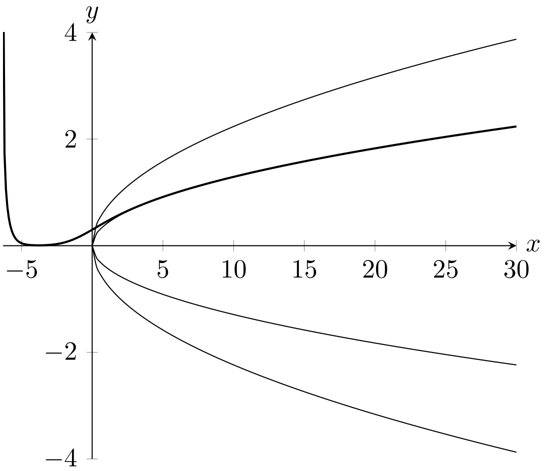

FIGURE A.5: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 10$ and WorkingPrecision=10

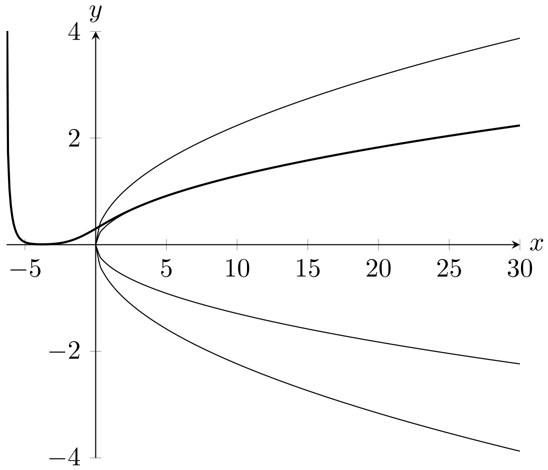

FIGURE A.6: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 10$ and WorkingPrecision=20

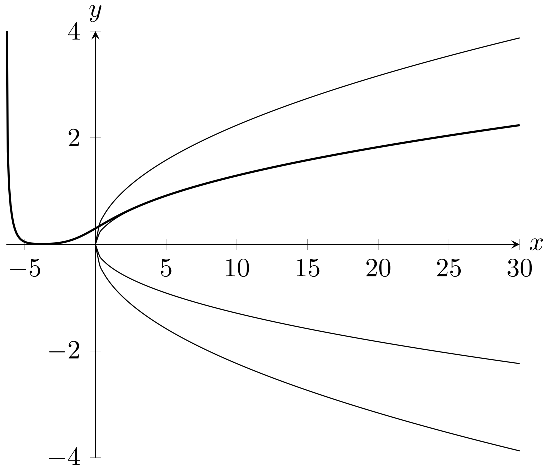

FIGURE A.7: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 10$ and WorkingPrecision=100

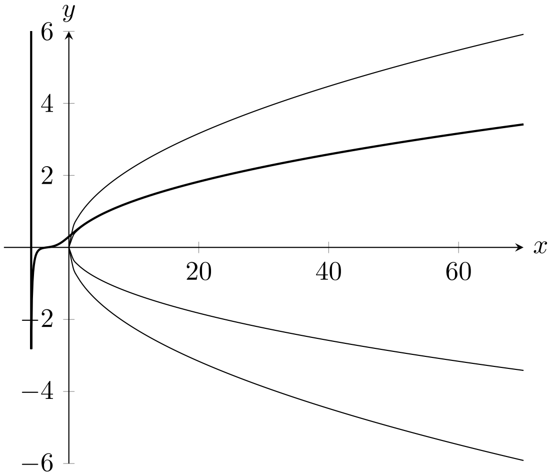

FIGURE A.8: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 50$ and WorkingPrecision=10

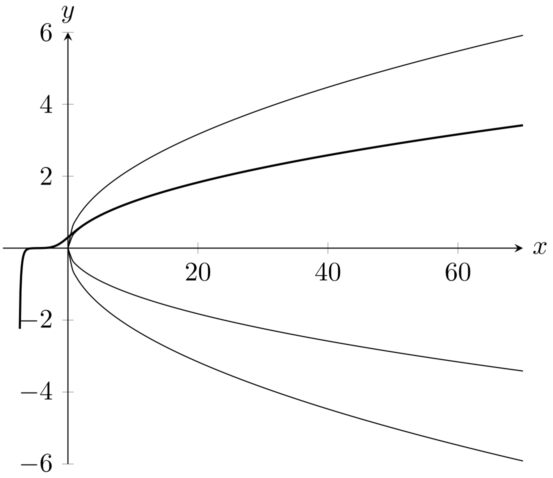

FIGURE A.9: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 50$ and WorkingPrecision=20

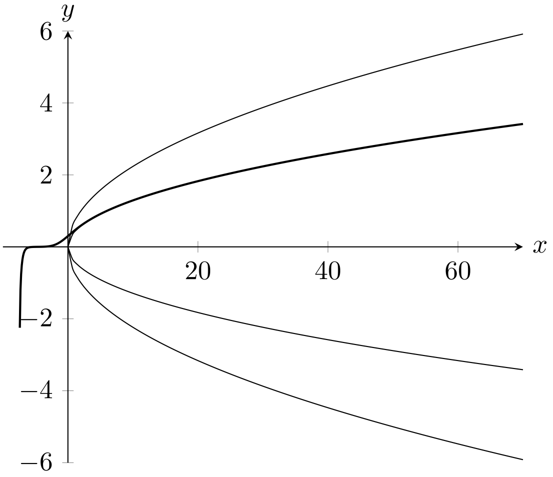

FIGURE A.10: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 50$ and WorkingPrecision=100

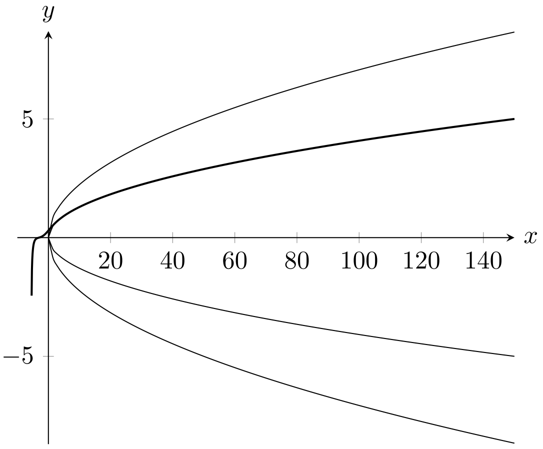

FIGURE A.11: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 100$ and WorkingPrecision=10

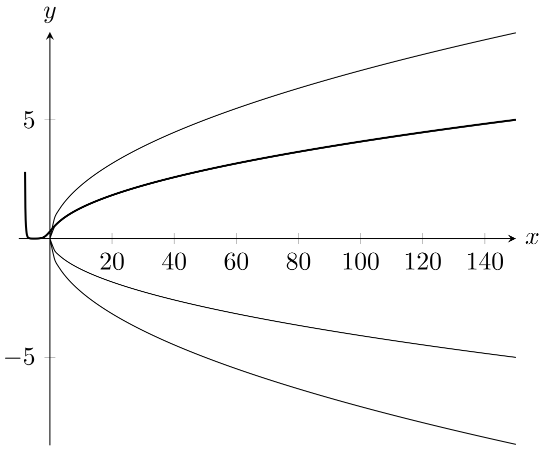

FIGURE A.12: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 100$ and WorkingPrecision=20

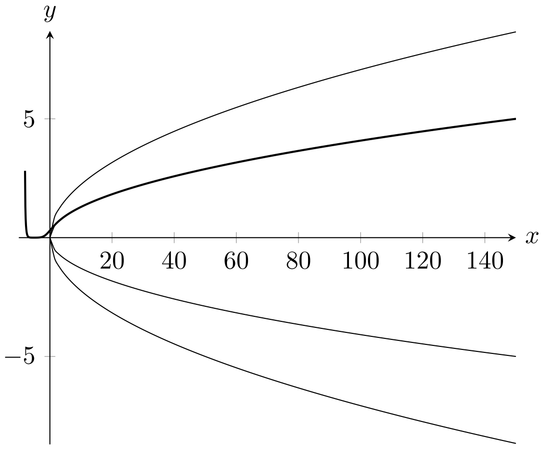

FIGURE A.13: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 100$ and WorkingPrecision=100

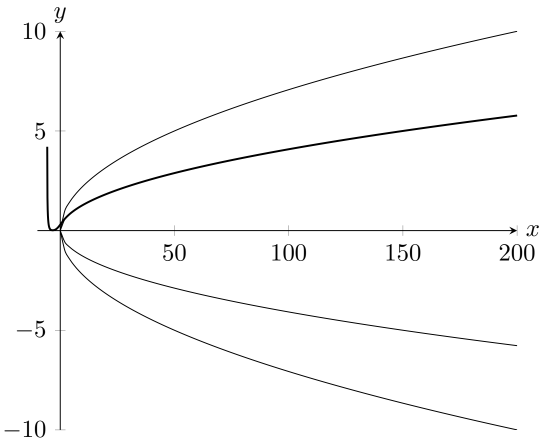

FIGURE A.14: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 150$ and WorkingPrecision=10

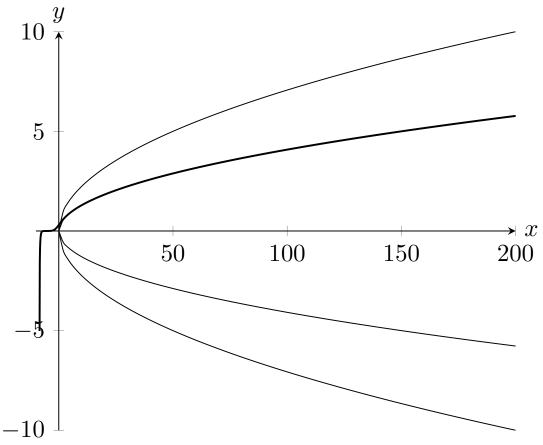

FIGURE A.15: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 150$ and WorkingPrecision=20

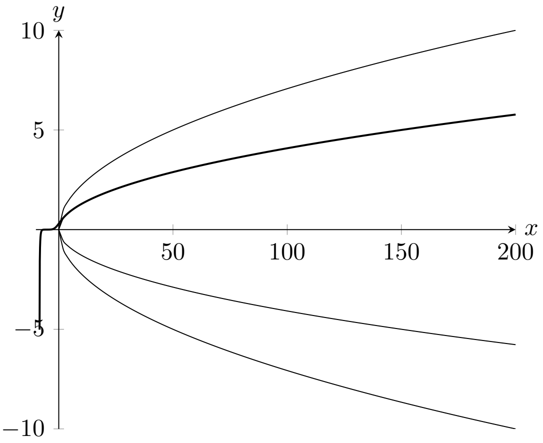

FIGURE A.16: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 150$ and WorkingPrecision=100

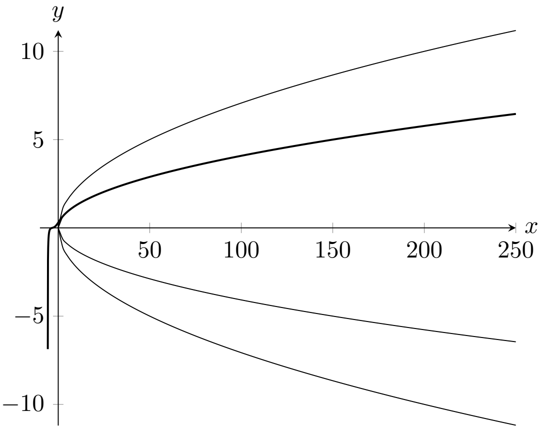

FIGURE A.17: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 200$ and WorkingPrecision=10

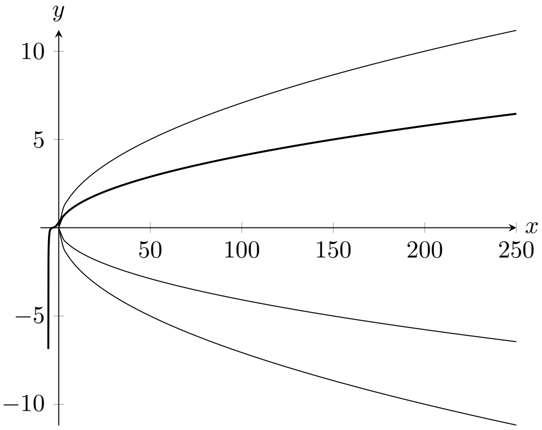

FIGURE A.18: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 200$ and WorkingPrecision=20

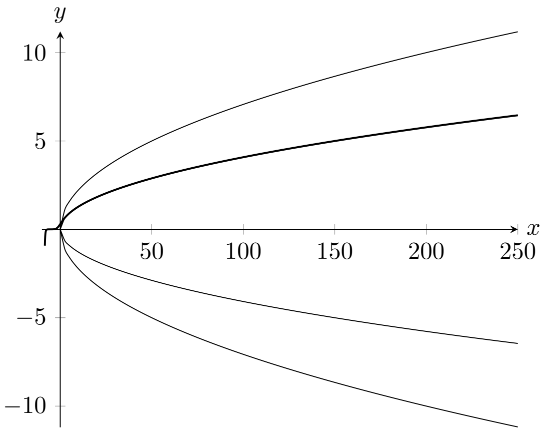

FIGURE A.19: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 200$ and WorkingPrecision=100

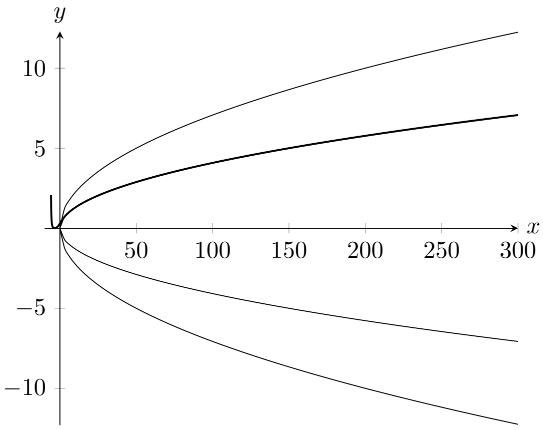

FIGURE A.20: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 250$ and WorkingPrecision=10

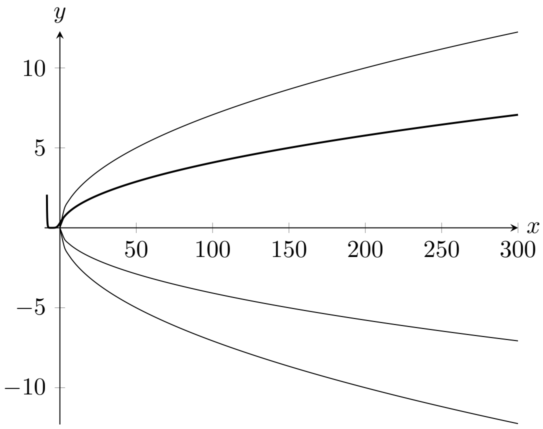

FIGURE A.21: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 250$ and WorkingPrecision=20

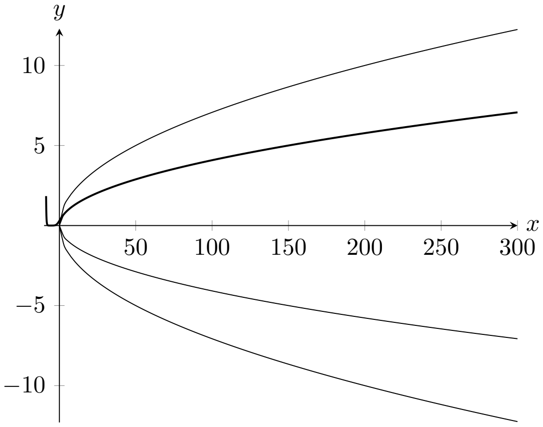

FIGURE A.22: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 250$ and WorkingPrecision=100

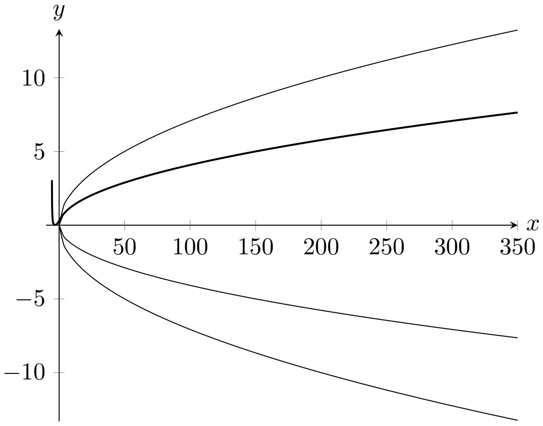

FIGURE A.23: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 300$ and WorkingPrecision=10

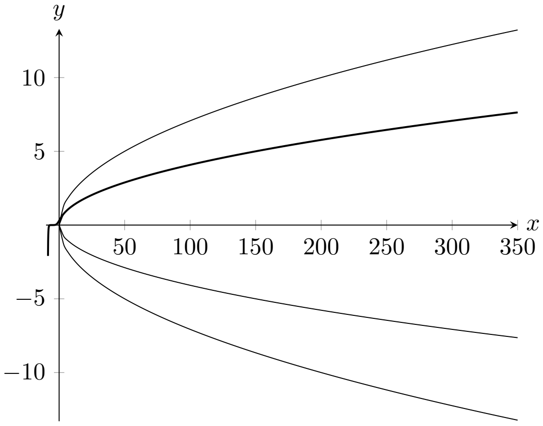

FIGURE A.24: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 300$ and WorkingPrecision=20

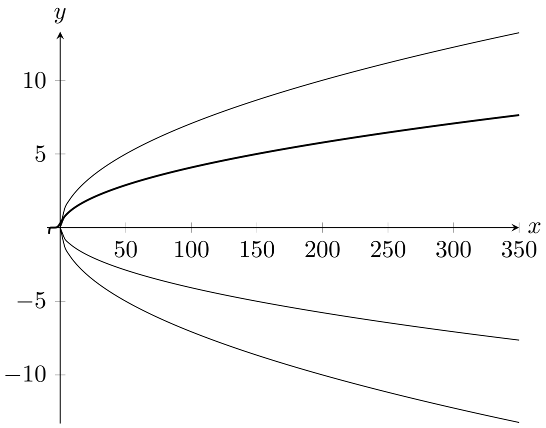

FIGURE A.25: Graph of the tangent solution y_{x_0} of (2.17) with $x_0 = 300$ and WorkingPrecision=100

A.2 The Painlevé Equations in the Complex Plane

In Chapter 1, we have presented figures displaying the pole distribution of solutions of various Painlevé equations. The numerical method behind these, is based on the Padé approximant . The Padé approximant of order $[m/n]$ of a function $f(x)$ is the rational function

$$R_{m,n}(x) = \frac{\sum_{j=0}^{m} a_j x^j}{1 + \sum_{k=1}^{n} b_j x^j},$$

satisfying

$$\begin{aligned} f(0) &= R_{m,n}(0), \\ f'(0) &= R'_{m,n}(0), \\ f''(0) &= R''_{m,n}(0), \\ &\vdots \\ f^{(m+n)}(0) &= R^{(m+n)}_{m,n}(0). \end{aligned}$$

A.2.1 The First Painlevé Equation

The Maclaurin series of degree n of a solution of $\mathrm{P_I}$ can be found by recursively defining the derivatives

```
d[n_] := d[n] = Expand[D[d[n - 1], x] /. y''[x] -> 6 y[x]^2 - x]
```

with the initial conditions

```
d[0] = y[x]; d[1] = y'[x];
```

The command

```
ListPlot[{Re[#], Im[#]} & /@ (x /.
     NSolve[Denominator[
         PadeApproximant[
          Sum[(d[k] /. {y[x] -> y0, y'[x] -> y1, x -> 0})/k! x^k
          , {k, 0, m+n}], {x, 0, {m, n}}]] == 0]),
  PlotRange -> {{-10, 10}, {-10, 10}}, AspectRatio -> 1]
```

then produces that Padé approximant of order $[m/n]$ of the solution of $\mathrm{P_I}$ with the initial conditions $y(0) = y0, y'(0) = y1$, and plots its poles.

In particular, in Figure 1.1 we have used the Padé approximant of order $[80/80]$ with initial conditions $y(0) = y'(0) = 0$, in Figure 1.2 we have used the Padé approximant of order $[100/100]$ with initial conditions $[100/100]$ with initial conditions $y(0) = 0.16, y'(0) = 0.2193934573994778462$, and in Figure 1.3 we have used the Padé approximant of order $[100/100]$ with initial conditions $y(0) = -0.1875543083404949, y'(0) = -0.3049055602612289$.

A.2.2 The Second Painlevé Equation

The high-order derivatives of a solution of $\mathrm{P_{II}}$ can be defined recursively, by

```
d[n_] := d[n]
= Expand[D[d[n - 1], x] /. y''[x] -> 2 y[x]^3 +x y[x]]
```

with the same initial conditions as in the case of P_I

```
d[0] = y[x]; d[1] = y'[x];
```

Producing the figures for the second Painlevé equation follows the same lines as in the case of the first Painlevé equation. In Figure 1.8 we have used the Padé approximant of order $[60/60]$ with the initial conditions $y(0) = 0.1, y'(0) = -0.07348321929513198102$, and in Figure 1.9 we have used the Padé approximant of order $[60/60]$ with the initial conditions $y(0) = 0.3670615515480784, y'(0) = -0.2953721054475501$.

Appendix B

Asymptotic Expansions

In this appendix, we define asymptotic notations, which enable us to compare functions to other known functions.

Definition B.1. *Let* $F : D \to \mathbb{C}$ *and* $G : D \to \mathbb{C}$ *be two functions defined over a domain* $D \subseteq \mathbb{C}$*. Suppose that* x_0 *is a limit point of* D*, and let* ρ *be a path lying in* D *with* x_0 *as an end-point. Assume* $|G(x)|$ *is bounded below, i.e., non-zero, along* ρ*.*

1. F *is said to be* much much less than G *as* $x \underset{\rho}{\to} x_0$*, or*

$$F \ll G, \quad \text{as } x \underset{\rho}{\to} x_0,$$

if and only if

$$\lim_{\substack{x \to x_0 \\ \rho}} \frac{F(x)}{G(x)} = 0.$$

2. F *is said to be* asymptotic to G *as* $x \underset{\rho}{\to} x_0$*, or*

$$F \sim G, \quad \text{as } x \underset{\rho}{\to} x_0,$$

if and only if

$$\lim_{\substack{x \to x_0 \\ \rho}} \frac{F(x) - G(x)}{G(x)} = 0,$$

$$\Leftrightarrow$$

$$\lim_{\substack{x \to x_0 \\ \rho}} \frac{F(x)}{G(x)} = 1.$$

3. F *is said to be of the* order of G *as* $x \underset{\rho}{\to} x_0$*, or*

$$F = \mathcal{O}(g), \quad \text{as } x \underset{\rho}{\to} x_0,$$

if and only if

$$\text{there exists a constant } K \text{ such that } \lim_{\substack{x \to x_0 \\ \rho}} \left| \frac{F(x)}{G(x)} \right| \leq K.$$

Remark B.2. *Note that there are some alternative notations for the above concepts.*

$$F \ll G \quad \text{as } x \underset{\rho}{\to} x_0 \;\Leftrightarrow\; G \gg F \quad \text{as } x \underset{\rho}{\to} x_0,$$
$$F \ll G \quad \text{as } x \underset{\rho}{\to} x_0 \;\Leftrightarrow\; F = o(G) \quad \text{as } x \underset{\rho}{\to} x_0.$$

Where the path ρ is understood from the context, we omit it for simplicity.

Let ρ be the positive real-axis $\mathbb{R}^+$ and x_0 be infinity. In this case, we write $x \to +\infty$ instead of $x \underset{\mathbb{R}^+}{\to} \infty$.

In Definition B.1, we saw how to approximate F by G in a limit, which leaves a correction term: $F(x) - G(x)$. In general, taking more and more corrections leads to an infinite series of approximations given in terms of "reference functions[1]"The simplest reference functions are powers of $(x - x_0)$, for the limit $x \underset{\rho}{\to} x_0$. Given a function $F(x)$ and a limit $x \to x_0$, Poincaré defined asymptotic series in the following way.

Definition B.3. *F is said to be* asymptotic to the series $\sum_{k=0}^{\infty} a_k (x - x_0)^k$ *as $x \underset{\rho}{\to} x_0$, if and only if for each integer $N \geq 0$, we have*

$$\lim_{\substack{x \to x_0 \\ \rho}} \frac{F(x) - \sum_{k=0}^{N} a_k \big(x - x_0\big)^k}{\big(x - x_0\big)^N} = 0. \tag{B.1}$$

In that case, we write

$$F(x) \sim \sum_{k=0}^{\infty} a_k \big(x - x_0\big)^k, \quad x \underset{\rho}{\to} x_0.$$

In the case when $x_0 = \infty$, the asymptotic series becomes $\sum_{k=0}^{\infty} a_k / x^k$.

Taylor series expansions of analytic functions[2] provide simple examples of asymptotic series.

Example B.4. *The function given by*

$$\exp(x) + \exp(-1/x)$$

shares the same asymptotic series as $\exp(x)$, *as* $x \to 0+$. *Note that the term* $\exp(-1/x)$ *lies "beyond all orders" of the power series expansion of* $\exp(x)$ *as* $x \to 0+$, *since Poincaré's definition does not reveal this term in the limit.*

The study of terms that are undetectable by the reference functions leads to what are known as *trans-series*.

---

[1] See Olver [Olv97] for generalizations of reference functions.

[2] Recall that a function $F(x)$ is *analytic* at a point $x_0 \in \mathbb{C}$ if $\mathrm{d}F/\mathrm{d}x$ exists in a neighbourhood of a. In that case, we can expand the function in a Taylor series: $F(x) = \sum_{k=0}^{\infty} F_k (x - x_0)^k$ and the result converges.

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