On the global asymptotic analysis of a q-discrete Painlevé equation

Pieter Roffelsen

A thesis submitted in partial fulfilment of
requirements for the degree of
Doctor of Philosophy

School of Mathematics and Statistics
The University of Sydney

January 2017

Statement of Originality

This is to certify that to the best of my knowledge, the content of this thesis is my own work. This thesis has not been submitted for any degree or other purposes.
I certify that the intellectual content of this thesis is the product of my own work and that all the assistance received in preparing this thesis and sources have been acknowledged.

Pieter Roffelsen

Publications

Chapter 3 and Appendices B and C of this thesis, are published in
N. Joshi and P. Roffelsen. Analytic solutions of q-$P(A_1)$ near its critical points. *Nonlinearity*, 29(12):3696-742, 2016.

Acknowledgements

The past three and a half years as a PhD student have been a wonderful experience. During this time I had the pleasure to meet many mathematicians from all over the world. I would like to thank all of them, apologising in advance to those who are not explicitly mentioned.

Most importantly of all, I would like to express my sincere gratitude towards Nalini Joshi, for her supervision and constant enthusiasm over the last few years. She has taught me a great deal about mathematics, the mathematical world and her encouragements over the years have been invaluable to me.

Nalini has put together a close group of students and researchers, creating a pleasant and stimulating environment to work in. I would like to thank all its members, in particular James Gregory, Steven Luu, Matthew Nolan, Yang Shi and Michael Touitou, who I have had many interesting exchanges of ideas with and have been responsible for most of the healthy and necessary distractions throughout these years.

I am grateful to Nalini for her support in attending several international conferences, which have been highlights of my time as a PhD student. There are many people with who I had inspiring and enjoyable conversations at such conferences or otherwise. In particular I would like to thank Walter Van Assche, James Atkinson, Pavlos Kassotakis, Oleg Lisovyy, Yousuke Ohyama, Christopher Ormerod, Hidetaka Sakai, Jacques Sauloy, Teruhisa Tsuda and Changgui Zhang.

Let me also thank my undergraduate supervisor Erik Koelink, for first introducing me to Painlevé equations and his continued support and interest over the years. I still vividly remember when he first showed me pictures of the pole distribution of rational Painlevé functions, which initiated my fascination with these equations. This fascination quickly turned into an obsession when I got the opportunity to write my master's thesis under supervision of Peter Clarkson. He also first put in my mind the idea of doing a PhD in Australia with Nalini. I will always be grateful to him. I would also like to express my gratitude to my first collaborator Davide Masoero. I feel fortunate to be working with him exactly on the subject which first triggered my interest in the Painlevé world.

Residing in Australia, has given me the opportunity to explore and experience true nature, to which I am ever grateful. However, living on the other side of the planet, has its drawbacks. I would like to thank my family and friends back home for their unconditional support, even though some wished I would have ventured less far. Finally, I would like to thank my girlfriend Laura, for her love and support, especially in hectic times.

Contents

CHAPTER 1

Introduction

The Painlevé equations are universal integrable systems. Their solutions turn out to have many physically relevant properties, which can be deduced precisely because these equations are integrable. In this thesis we are concerned with a nonlinear q-difference equation, which is a discrete version of a Painlevé equation. Our main objective is to derive global asymptotic properties of its solutions.

1.1 Painlevé Equations

One of the most fundamental properties of linear differential equations, is that we can read of the equation itself, where solutions might have singularities. To put it differently, the equation determines a puncturing of the Riemann sphere, such that any local solution has an unique meromorphic continuation to the universal cover space of the resulting punctured sphere. Around the turn of the nineteenth century, Painlevé and his school wished to find and classify nonlinear differential equations, particularly of second order, which share this remarkable property with the linear ones, appropriately referred to as the *Painlevé property* nowadays.

To be exact, Painlevé [69, 70], Gambier [17], Fuchs [15] and their colleagues classified all second order differential equations, having the Painlevé property, of the form

$$\omega'' = H(\omega, \omega', \zeta),$$

where $' = \frac{d}{d\zeta}$ and H meromorphic in ζ and rational in ω and ω'. They ended up with a list of fifty such equations, of which six are not trivially integrable, the six *Painlevé equations*, given in Appendix A.

Just as many linear equations are used to define classical or linear special functions as their solutions, the Painlevé equations give rise to so called *Painlevé functions* or *transcendents*. Indeed solutions of Painlevé equations are generically higher transcendental, which roughly means they can not be expressed in terms of earlier known functions, and therefore define truly new functions. Nowadays Painlevé transcendents are widely recogized as *nonlinear special functions*, where we particularly mention that they are included in the NIST handbook of Mathematical Functions, see Clarkson [10].

The inspiring *sixth Painlevé equation* is given by

$$P_{\rm VI}\quad \begin{cases} w'' = \frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-\zeta}\right)(w')^2-\left(\frac{1}{\zeta}+\frac{1}{\zeta-1}+\frac{1}{w-\zeta}\right)w' \\ \quad +\frac{w(w-1)(w-\zeta)}{2\zeta^2(\zeta-1)^2}\left((\theta_\infty-1)^2-\frac{\theta_x^2\zeta}{w^2}+\frac{\theta_y^2(\zeta-1)}{(w-1)^2}+\frac{(1-\theta_z^2)\zeta(\zeta-1)}{(w-\zeta)^2}\right), \end{cases} \tag{1.1}$$

where $\theta_{x,y,z,\infty}\in\mathbb{C}$ are complex parameters. The Painlevé property manifests itself as follows, any local solution of the sixth Painlevé equation on $\mathbb{P}^1\setminus\{0,1,\infty\}$, has an unique meromorphic continuation to the corresponding universal covering space, see also Theorem 2.1.1. In particular $P_{\rm VI}$ transcendents can only branch at $\zeta=0$, $\zeta=1$ or $\zeta=\infty$, the *critical points* of the equation. The sixth Painlevé equation is often referred to as the mother equation, as for one, it has the most complex parameters of the six Painlevé equations, and secondly, the other Painlevé equations can be obtained by coalescence limits of it. One can hence think of $P_{\rm VI}$ as the universal differential equation on the nonlinear level, similar to Euler's hypergeometric differential equation on the linear level,

$$\zeta(1-\zeta)\omega''+(c-(a+b+1)\zeta)\omega'-ab\omega=0. \tag{1.2}$$

The sixth Painlevé equation is of great importance to this thesis, as we study a q-analog of it, called the q-$P(A_1)$ equation. To put it broadly, a *q-analog* of an object, is a generalisation involving an extra parameter q in the complex plane, such that in the limit $q\to 1$, called the *continuum limit*, the original object is recovered. One is generally interested in objects characterised by some property, like the Painlevé property, and hence a proper q-analog should somehow share this property or a q-analog of it.

Let us recall that the theory of classical special functions has always co-existed with a q-discrete theory of q-special functions. One of the most prominent examples are the hypergeometric and q-hypergeometric functions. The famous Gauss hypergeometric function, defined by

$${}_2F_1\begin{bmatrix}a,\ b\\ c\end{bmatrix};\zeta\Bigg]=\sum_{n=0}^\infty\frac{(a)_n(b)_n}{(c)_n(1)_n}\zeta^n,\quad (x)_n:=\prod_{0\le i\le n-1}(x+i),\quad (x\in\mathbb{C},n\in\mathbb{N})$$

defines a solution to Euler's hypergeometric differential equation (1.2). In 1846, Heine [33, 34] introduced the q-hypergeometric function,

$${}_2\phi_1\left[\begin{matrix}a,\ b\\ c\end{matrix};q,\zeta\right]=\sum_{n=0}^\infty\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}\zeta^n,\quad (x;q)_n:=\prod_{0\le i\le n-1}(1-q^ix),\quad (x\in\mathbb{C},n\in\mathbb{N}) \tag{1.3}$$

generalising Gauss' hypergeometric function, which appropriately satisfies a q-discrete analog of Euler's hypergeometric differential equation. Given such a longstanding and fruitful tradition within the special function community, it is quite remarkable that a q-Painlevé theory had to wait for almost a century after the pioneering works of Painlevé and his colleagues. Perhaps the deeper reason for this, is the difficulty in defining an appropriate q-analog of the Painlevé property.

1.2 Discrete Painlevé Equations

The early appearances of discrete Painlevé equations in the literature were not in the context of Painlevé equations. Probably the first such example is the following *additive* or *d-discrete equation*

$$x_{n+1} + x_n + x_{n-1} = \frac{t_n}{x_n} + 1, \quad t_n = t_0 + dn, \quad (n \in \mathbb{Z}) \tag{1.4}$$

in a paper by Shohat [81] in 1939, in which unfortunately, no connection with Painlevé equations was made. It had to wait until 1990, when Brézin and Kazakov [7] first calculated the continuum limit $d \to 0$ of this equation, in the context of a field-theoretical model of two-dimensional gravity. The result was the first Painlevé equation $\omega'' = 6\omega^2 + \zeta$, and equation (1.4) soon become known as d-P_{I}. This observation initiated an exciting new research area, extending the Painlevé world into the discrete regime, opening the hunt for new *discrete Painlevé equations.*

A question of much discussion during the nineties, was what the discrete analog of the Painlevé property should be? A first candidate, called the *singularity confinement property*, was proposed by Grammaticos, Ramani and Papageorgiou [22] in 1991. Let us discuss it by example using d-P_{I} (1.4). Reflecting on the Painlevé property for differential equations, we are concerned with the continuation of local solutions of d-P_{I}. Starting with some initial conditions, say $x_m = \mu$ and $x_{m-1} = \nu$, equation (1.4) allows for a straightforward continuation, both in the *forward time direction* $n \mapsto n+1$, and in the *backward time direction* $n \mapsto n-1$, unless at some time n_0 we find $x_{n_0} = 0$. Let us consider the forward time direction, with $x_{n_0} = 0$ and $x_{n_0-1} = \nu_0$. Then equation (1.4) gives $x_{n_0+1} = \infty$, which is not at all problematic as we can easily move to projective space. However when we calculate the further iterates, we find $x_{n_0+2} = \infty$, $x_{n_0+3} = 0$ and finally $x_{n_0+4} = \infty - \infty = ?$, which is a singularity, in the sense that the solution is undefined. Just as the Painlevé property forbids movable essential singularities, the singularity confinement property entails that the singularity in the fourth iterate is in fact an apparent one. Indeed, upon closer inspection, setting $x_{n_0} = \epsilon$, one finds

$$x_{n_0+1} \sim \frac{t_{n_0}}{\epsilon}, \quad x_{n_0+2} \sim -\frac{t_{n_0}}{\epsilon}, \quad x_{n_0+3} \sim -\frac{t_{n_0+3}}{t_{n_0}}\epsilon, \quad x_{n_0+4} \sim \frac{t_{n_0}\nu_0 + 2d}{t_{n_0+3}}, \qquad (\epsilon \to 0)$$

and hence letting $\epsilon \to 0$, gives a regular value for x_{n_0+4}, recovering the initial value $x_{n_0-1} = \nu_0$. By direct calculation x_{n_0+5} is also well-defined and generically nonvanishing in the limit $\epsilon \to 0$, and we say that the singularity is *confined.*

Grammaticos, Ramani and their collaborators [21] used the singularity confinement property with great success to derive many different discrete Painlevé equations. Particularly they were the first to write down the q-difference equation, which is the main subject of this thesis, given by

$$q\text{-}P(A_1) \quad \begin{cases} \dfrac{(fg - t^2)(\overline{f}g - qt^2)}{(fg-1)(\overline{f}g-1)} = \dfrac{(g-b_1t)(g-b_2t)(g-b_3t)(g-b_4t)}{(g-b_5)(g-b_6)(g-b_7)(g-b_8)}, \\ \dfrac{(\overline{f}g - qt^2)(\overline{f}\overline{g} - q^2t^2)}{(\overline{f}g-1)(\overline{f}\overline{g}-1)} = \dfrac{(\overline{f}-b_1^{-1}qt)(\overline{f}-b_2^{-1}qt)(\overline{f}-b_3^{-1}qt)(\overline{f}-b_4^{-1}qt)}{(\overline{f}-b_5^{-1})(\overline{f}-b_6^{-1})(\overline{f}-b_7^{-1})(\overline{f}-b_8^{-1})}, \end{cases}$$

where $f = f(t)$ and $g = g(t)$ are the dependent variables, t is the independent variable, we denote $\overline{f} = f(qt)$ and $\overline{g} = g(qt)$, and $b_1, \ldots, b_8 \in \mathbb{C}^*$ are complex parameters satisfying the single constraint

$$q = \frac{b_1b_2b_3b_4}{b_5b_6b_7b_8}. \tag{1.5}$$

Grammaticos and Ramani [20] originally called this equation asymmetric q-P_{VI}, as it was the first q-difference equation whose symmetric form reduces to the sixth Painlevé equation in the continuum limit.

By the end of the nineties there were many different discrete Painlevé equations, and there was no method of classification known. To make things worse, the singularity confinement turned out to be merely a necessary condition for integrability, Hietarinta and Viallet [35] first constructed a now well known counterexample, i.e. a mapping which satisfies the singularity confinement property but is not integrable.

The time was ripe for a new approach, and it was Sakai [77], who around the turn of the century, inspired by Okamoto's work [66] on the continuous counterparts in the seventies, gave a new characterisation of discrete Painlevé equations and in particular gave a complete classification of them. To give a rough idea of Sakai's theory, note that the Painlevé property implies that the solution space of P_{VI} can be identified with the local solution space on any simply connected open domain in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, which one can choose as small as one pleases. Taking this idea to its extreme, we let the domain shrink to a point, at which stage P_{VI} becomes an equation for germs of meromorphic functions at that point. Okamoto [66] understood that, following such a procedure with P_{VI} appropriately rewritten in system form, the corresponding solution space becomes an algebraic surface, called the *initial value space* or *space of initial conditions*. Given any two distinct times and a path between them, the sixth Painlevé equation induces an isomorphism between the two corresponding initial value spaces, via meromorphic continuation along such path.

Sakai [77] went in the opposite direction, he started with and classified initial value spaces, and ended up with continuous and discrete Painlevé equations as isomorphisms of these spaces. Within this context, the fundamental role played by symmetry groups, underlying the Painlevé equations, becomes particularly visible. We also mention the book by Noumi [63] in this regard, as it nicely illustrates the prominent role of symmetries in Painlevé equations. The equation under consideration in this thesis has an initial value space of $A_1^{(1)}$ surface type in Sakai's classification, hence the name "q-$P(A_1)$ equation".

1.3 Global Asymptotic Analysis

Many of the linear special functions were established because of their applications, particularly in physics. Even though the Painlevé equations were initially derived from purely mathematical considerations, they have also found their way into many different areas of mathematical physics. Examples include random matrix theory, quantum gravity, statistical mechanics and conformal field theory. What essentially makes the Painlevé equations useful as models, is that we have effective methods to study *global* asymptotic properties of its solutions. To illustrate the latter point, let us get back to the Gauss hypergeometric functions, which are basically the *local* asymptotic series corresponding to the critical behaviours of solutions of (1.2) near any of the critical points 0, 1 or ∞. Quite remarkably, it is possible to write down explicit *connection formulae* which relate these critical behaviours near different critical points. We call such connection formulae *global* asymptotic results and we remark that they are generally much harder to establish than their local counterparts. For a long time it was thought that such explicit global results could only be obtained for linear special functions of hypergeometric type.

Fokas et al. [14] define the *global asymptotic analysis* or simply *global analysis*, of for instance the sixth Painlevé equation, as the classification of all possible critical behaviours of solutions near the critical points 0, 1 and ∞, and solving the corresponding *connection problem* between each of these critical points. More explicitly, the critical behaviours should typically be parameterised by two integration constants near each critical point, and the connection problem concerns relating these pairs of integration constants coming from different critical points explicitly. Let us remark that the global analysis of the sixth Painlevé equation has been completed [28]. Jimbo [42] dealt with the generic case and most of the special cases have been dealt with by Guzzetti [24, 25, 26]. From this perspective the sixth Painlevé equation can hence be considered "solved". We refer to Fokas et al. [14] for an overview of the global analysis of continuous Painlevé equations.

The history of the asymptotic analysis of Painlevé equations is vast and goes back a long way, starting with Boutroux [6] more than a hundred years ago. On the other hand, very little asymptotic investigations have been carried out for discrete Painlevé equations, and only one on the global analysis of such equations. We mention two recent works which are the discovery of the nonlinear Stokes phenomenon in d-P_{I} by Joshi and Lustri [47], and in d-P_{II} by Joshi et al. [48]. Mano [61] was the first to study the global analysis of a discrete equation, more precisely q-P_{VI} or q-$P(A_3)$ in Sakai's classification. Mano's work can be considered a q-analog of Jimbo's [42] classical work on the sixth Painlevé equation. In particular he completes the global analysis for q-P_{VI} in the generic case, up to writing down the connection formulae in explicit form. In this thesis we are concerned with the global analysis of the q-$P(A_1)$ equation, and it is hence the second work of this kind. So far we have not explained what methods allow us tackle the global analysis of Painlevé equations, this is where a third characterisation of Painlevé equations comes into play.

1.4 The Isomonodromic Deformation Method

Initially Painlevé's classification was incomplete, as the sixth Painlevé equation was overlooked. Fuchs [15] derived the missing equation, from a very different viewpoint, which lies at the foundation of the isomonodromic deformation method. Fuchs considered the generic linear differential equation of second order, say in z, with four regular singular points $z = 0$, $z = 1$, $z = \infty$ and $z = \zeta$. He was interested in deformations of the coefficients in the equation, by varying the location of the fourth regular singular point $z = \zeta$, which preserves the associated *monodromy*. He observed that in such case, the deformation of the coefficients is characterised by a second order nonlinear differential equation, the sixth Painlevé equation. In particular Fuchs showed that there must exist an accompanying linear equation, involving differentiation with respect to ζ, which is consistent with the second order regular singular equation. Such a pair of equations is called a *Lax pair* nowadays.

Using this Lax pair, one can construct a mapping, from the solution space of the sixth Painlevé equation, to the monodromy space of Fuchsian equations with four regular singular points. That is, the monodromy data are integrals of motion of the sixth Painlevé equation. This mapping, often referred to as the *monodromy mapping*, is part of the celebrated Riemann-Hilbert correspondence. Though this correspondence is a transcendental one, it is possible to evaluate it in the limit where ζ approaches one of the critical points. That is, one can explicitly relate the critical behaviour of solutions near a critical point, with corresponding monodromy of the associated linear equation. It is this property, which allows for the connection problems to be solved explicitly.

Let us remark that q-P_{VI} was first derived by Jimbo and Sakai [43], analogously to P_{VI}, by considering the isomonodromic deformation of a Fuchsian q-difference system. It was Yamada [85] who first derived a Lax pair associated with q-$P(A_1)$, which will the focus of intense study in this thesis.

1.5 Outline of Thesis

We start our journey with a review of some of the fundamental aspects of discrete Painlevé equations in Chapter 2, specialised to q-$P(A_1)$. We discuss the confinement of singularities for solutions of this equation, and delve deeper into some of its algebro-geometric aspects within Sakai's theory. In particular we consider the initial value space of the equation, which allows us to define what we actually mean with a q-$P(A_1)$ transcendent. In fact we consider two viewpoints, that of solutions with discrete time, and that of meromorphic solutions on a connected open domain. We set up the basic analytic theory necessary to discuss the global asymptotic analysis of q-$P(A_1)$, and introduce its symmetric form, which, in the continuum limit, reduces the Painlevé VI.

In Chapter 3 we concern ourselves with the asymptotic analysis of q-$P(A_1)$ transcendents near the critical points $t = 0$ and $t = \infty$. We first follow the method of dominant balance, which gives an autonomous sytem for the leading order behaviour. We identify this autonomous system as a QRT mapping, which allows us to parameterise its solutions completely. The generic two-parameter solution involves complex powers, and there are two one-parameter families which involve logarithms. We show that, associated with the generic solutions of the autonomous system, there exist full asymptotic expansions of solutions of the q-$P(A_1)$ equation. These expansions are convergent and hence define true solutions of our equation of interest. We tabulate all the different critical behaviours obtained, both for q-$P(A_1)$ and its symmetric form. We then calculate the continuum limit of the different critical behaviours on a formal level and show that they coincides with the known ones for the sixth Painlevé equation.

We wish to relate the critical behaviours near $t = 0$ and $t = \infty$, i.e. solve the connection problem, using the isomonodromic deformation approach. For this we first discuss some of the classical concepts involved, first worked out by Birkhoff [5] and his school, such as Fuchsian q-difference equations and monodromy, in Chapter 4. We then interpret Yamada's

Lax pair within this framework, and work out in what sense monodromy is preserved under the q-$P(A_1)$ deformation. This is a nontrivial task, as Yamada derived his Lax pair from considerations different to isomonodromy. Having set up all the analytic aspects of the Lax pair, we are ready to combine it with the tabulated critical behaviours.

The final part of the thesis, Chapter 5, is concerned with explicitly calculating the monodromy of Yamada's Lax pair associated with different critical behaviours around $t = 0$ and $t = \infty$, which is often called the direct monodromy problem. We find that in both asymptotic limits $t \to 0$ and $t \to \infty$, the monodromy problem factorises into two copies of a simpler one, which we call the model equation. The integration constants, parameterising the critical behaviours, enter the two copies of the model equation naturally. The explicit solution of the monodromy problem for the model equation, then yields an explicit parameterisation of the monodromy of Yamada's Lax pair in terms of integration constants characterising the critical behaviour at $t = 0$, and a similar explicit parameterisation in terms of the integration constants characterising the critical behaviour at $t = \infty$. Finally we obtain explicit relations between the critical behaviour at $t = 0$ and $t = \infty$ of q-$P(A_1)$ transcendents.

1.6 Notations and Conventions

We use $\mathbb{N}$ to denote the natural numbers including 0, and we use $\mathbb{P} := \mathbb{P}^1$ to denote the Riemann sphere, as we will only be concerned with complex projective space of dimension one. Recall that $\mathbb{P}$ is obtained by taking the quotient of $\mathbb{C}^2 \setminus \{(0,0)\}$, with respect to the equivalence relation

$$(x_1, x_2) \sim (x_1', x_2') \iff \exists_{\lambda \in \mathbb{C}} \left[(x_1, x_2) = (\lambda x_1', \lambda x_2')\right],$$

and we denote the equivalence class corresponding to (x_1, x_2) by $[x_1, x_2]$. We identify $\mathbb{C} \subseteq \mathbb{P}$ via $x \simeq [x, 1]$ for $x \in \mathbb{C}$ and denote $\infty := [1, 0] \in \mathbb{P}$. For any subset $V \subseteq \mathbb{P}$, we write

$$V^* = V \setminus \{0\}.$$

Throughout this thesis we denote by q a complex number with $0 < |q| < 1$. We write

$$q^{\mathbb{Z}} = \{q^n : n \in \mathbb{Z}\}, \quad q^{\mathbb{Z}} t_0 = \{q^n t_0 : n \in \mathbb{Z}\}, \quad (t_0 \in \mathbb{P})$$

and for any $T_0 \subseteq \mathbb{P}$,

$$q^{\mathbb{Z}} T_0 = \bigcup_{t_0 \in T_0} q^{\mathbb{Z}} t_0.$$

We are concerned with the q-discrete Painlevé equation q-$P(A_1)$ (1.2), whose parameters we compactly denote by $\mathbf{b} = (b_1, \ldots, b_8)$, with corresponding parameter space

$$\mathcal{B}_q = \{\mathbf{b} \in {\mathbb{C}^*}^8 : \text{subject to (1.5)}\}.$$

Sometimes we like q to vary with the parameters $\mathbf{b}$, and we write

$$\mathcal{B} = \{\mathbf{b} \in \mathbb{C}^{*8}\}, \quad q(\mathbf{b}) := \frac{b_1 b_2 b_3 b_4}{b_5 b_6 b_7 b_8}.$$

Given some specific parameter values $\mathbf{b}_* \in \mathcal{B}_q$, we write q-$P(A_1)(\mathbf{b}_*)$, to refer to equation (1.2) with $\mathbf{b} = \mathbf{b}_*$, if we wish to stipulate the particular parameter values.

The natural domain for a q-difference equation, which we appropriately call *q-domain*, is a nonempty set $T \subseteq \mathbb{P}$, invariant under multiplication by q, i.e. $qT = T$. We call T a *discrete q-domain*, if for any choice of $t_0 \in T$, we have $T = q^{\mathbb{Z}} t_0$, in which case it is custom to write $f_s = f(t_s)$ and $g_s = g(t_s)$ where $t_s = q^s t_0$ for $s \in \mathbb{Z}$. The opposite interpretation is to assume T is a nonempty connected open subset of $\mathbb{P}$, which we call a *continuous q-domain*. In this case one is often interested in solutions $(f(t), g(t))$ which depend meromorphically on t. This interpretation is particularly appropriate in view of the so called *continuum limit* of q-$P(A_1)$, discussed in more detail in Section 2.5.1.

1.6.1 Asymptotic Notation

As asymptotics play an important role in this thesis, and their exist various different conventions concerning asymptotic notation in the literature, let us fix ours once and for all. We are only concerned with asymptotics of complex functions on subsets of the Riemann sphere. Let $D \subseteq \mathbb{P}$ and $t_0 \in \mathbb{P}$ be a limit point of D. Let $f(t)$ and $g(t)$ be complex functions which contain D in their domain. We then say that $f(t)$ is *of order* $g(t)$ as $t \to t_0$ in D, if there exists a $c > 0$ and a punctured open environment U of t_0 in $\mathbb{P}$, such that

$$|f(t)| \leq c|g(t)|, \tag{1.6}$$

holds for $t \in U \cap D$. We denote this symbolically using Landau big O notation [2, 57],

$$f(t) = \mathcal{O}(g(t)), \quad (t \to t_0)$$

in D. Similarly we say that $f(t)$ is *of order less than* $g(t)$ as $t \to 0$ in D, if for every $c > 0$, there exists a punctured open environment U of t_0 in $\mathbb{P}$, such that (1.6) holds for $t \in U \cap D$. We denote this symbolically using Landau small o notation,

$$f(t) = o(g(t)), \quad (t \to t_0)$$

in D. Sometimes we find it more natural to use Hardy's notation [30],

$$f(t) \preccurlyeq g(t) \iff f(t) = \mathcal{O}(g(t)), \quad f(t) \prec g(t) \iff f(t) = o(g(t)).$$

Let us also compare this with Vinogradow's asymptotic notation [84], which reads $f(t) \ll g(t)$ iff $f(t) = \mathcal{O}(g(t))$.

We say that $f(t)$ and $g(t)$ have *the same order of magnitude* as $t \to t_0$ in D, if both $f(t) = \mathcal{O}(g(t))$ and $g(t) = \mathcal{O}(f(t))$ as $t \to t_0$ in D, which we denote by

$$f(t) \asymp g(t), \quad (t \to t_0)$$

in D. Finally we say that $f(t)$ is *asymptotic* to $g(t)$ as $t \to t_0$ in D, if

$$\lim_{t \to t_0, t \in D} \frac{f(t)}{g(t)} = 1,$$

which we denote symbolically by

$$f(t) \sim g(t) \quad (t \to t_0),$$

in D. If the set D itself is a (punctured) open environment of t_0, then we do not specify it explicitly. As an example, if $f(t)$ is given by a convergent power series expansion about $t = 0$,

$$f(t) = \sum_{n=0}^{\infty} f_n t^n,$$

then we write for instance

$$f(t) - f_0 - f_1 t = \mathcal{O}(t^2), \quad (t \to 0)$$

and equivalently

$$f(t) = f_0 + f_1 t + \mathcal{O}(t^2). \quad (t \to 0)$$

CHAPTER 2

Analytic and Algebro-Geometric Aspects

As the Painlevé property is intrinsically related to meromorphic continuation of solutions, we start this chapter by considering meromorphic continuation of local solutions of the q-$P(A_1)$ equation on continuous q-domains. We then consider the singularity confinement property for q-$P(A_1)$, which generalises this to continuation of solutions on discrete q-domains. In Section 2.2, we delve into the algebro-geometric side of the story, following Sakai's method [77]. After this, we are in position, to make precise the notion of solutions of q-$P(A_1)$, in Section 2.3. We discuss two different interpretations, one of discrete solutions, and one of meromorphic solutions. We then set up the global asymptotic analysis of the q-$P(A_1)$ equation, considering both interpretations, in Section 2.4. We conclude this chapter with the so called symmetric form of the q-$P(A_1)$ equation, and its continuum limit to P_{VI}, in Section 2.5.

2.1 Meromorphic Continuation

Let us recall the following well-known fundamental result concerning the sixth Painlevé equation.

Theorem 2.1.1. *Any local solution of P_{VI}, i.e. meromorphic solution on a (simply) connected open subset of $\mathbb{P}\setminus\{0,1,\infty\}$, can be meromorphically continued to an unique solution on the universal covering space of $\mathbb{P}\setminus\{0,1,\infty\}$.*

Proof. See for instance Hinkkanen and Laine [37] and Joshi and Kruskal [45]. □

In the above theorem, the Painlevé property manifests itself in the meromorphic continuation of solutions. A naive q-analog for q-$P(A_1)$ would be the following lemma.

Lemma 2.1.2. *Let $\mathbf{b}\in\mathcal{B}_q$, and f and g be meromorphic functions on a connected open set $T_0\subseteq\mathbb{C}^*$, such that $T_0\cap q^{-1}T_0\neq\emptyset$ and (f,g) satisfies q-$P(A_1)$ on this intersection. Then there exists an unique meromorphic continuation of this solution to the continuous q-domain $T=q^{\mathbb{Z}}T_0$.*

Proof. The proof is elementary, we simply use the q-$P(A_1)$ equation to extend the domain of f and g recursively, both in the forward time direction $t \mapsto qt$, and the backward time direction $t \mapsto q^{-1}t$. To make this explicit, let us rewrite q-$P(A_1)$ as

$$\overline{f}g = \frac{qt^2(fg-t^2)\dot{p}_2(g) - qt^4(fg-1)\dot{p}_1(g/t)}{(fg-t^2)\dot{p}_2(g) - qt^4(fg-1)\dot{p}_1(g/t)}, \tag{2.1a}$$

$$\overline{f}\overline{g} = \frac{q^2t^2(\overline{f}g-qt^2)p_2(\overline{f}) - q^3t^4(\overline{f}g-1)p_1(f/(qt))}{t^2(\overline{f}g-qt^2)p_2(\overline{f}) - q^3t^4(\overline{f}g-1)p_1(f/(qt))}, \tag{2.1b}$$

where the polynomials $p_1(x), p_2(x), \dot{p}_1(x), \dot{p}_2(x)$ are defined by

$$p_1(x) = (1-b_1x)(1-b_2x)(1-b_3x)(1-b_4x), \tag{2.2}$$

$$p_2(x) = (1-b_5x)(1-b_6x)(1-b_7x)(1-b_8x), \tag{2.3}$$

$$\dot{p}_1(x) = (1-x/b_1)(1-x/b_2)(1-x/b_3)(1-x/b_4),$$

$$\dot{p}_2(x) = (1-x/b_5)(1-x/b_6)(1-x/b_7)(1-x/b_8).$$

Next we would like to use the first equation (2.1a) to extend the domain of f to $T_0 \cup qT_0$. Note that the numerator of the right-hand side of (2.1a) is divisible by g, hence the only obstacle would be for the denominator of the right-hand side of (2.1a) to be identically zero. If this would be the case, then, using that f and g satisfy q-$P(A_1)$ on $T_0 \cap q^{-1}T_0$ and T_0 is connected, it follows immediately that $(f, g) = p_i$ on T_0 for some $1 \leq i \leq 8$, where the "singular" solutions $p_1, \ldots, p_8$ are defined by

$$p_1 = (\tfrac{1}{b_1}t, b_1t), \qquad p_2 = (\tfrac{1}{b_2}t, b_2t), \qquad p_3 = (\tfrac{1}{b_3}t, b_3t), \qquad p_4 = (\tfrac{1}{b_4}t, b_4t), \tag{2.4a}$$

$$p_5 = (\tfrac{1}{b_5}, b_5), \qquad p_6 = (\tfrac{1}{b_6}, b_6), \qquad p_7 = (\tfrac{1}{b_7}, b_7), \qquad p_8 = (\tfrac{1}{b_8}, b_8). \tag{2.4b}$$

In any case, unique meromorphic continuation to $T_0 \cup qT_0$ is guaranteed. Similarly the second equation (2.1b) can be used to extend the domain of g to $T_0 \cup qT_0$. We proceed inductively to extend the domain of f and g to $q^{\mathbb{N}}T_0$. Obviously a similar approach in the backward time direction allows us to consequently extend the domain of the solution to $q^{\mathbb{Z}}T_0$. □

Let us emphasise that the connectedness of T_0 in the above lemma is crucial. If we drop this assumption, local meromorphic solutions might not have a meromorphic continuation to the whole q-domain. This is why we demand that continuous q-domains are connected.

2.1.1 Singularity Confinement

The result of Lemma 2.1.2 holds for a large class of q-discrete equations, many of which would not be considered integrable. Indeed, what makes the q-$P(A_1)$ equation special is that, quite remarkably, unique continuation of solutions holds even on discrete q-domains. That is to say, take any $t_0 \in \mathbb{C}^*$ with initial values $(f_0, g_0) \in \mathbb{P} \times \mathbb{P}$, not equal to one of the points $p_1 \ldots p_8$ with $t = t_0$ as defined in (2.4), then there exists a unique continuation of this (albeit initially trivial) solution to $q^{\mathbb{Z}}t_0$. Note that it is a priori unclear what is meant with continuation beyond points on $q^{\mathbb{Z}}t_0$ where the q-$P(A_1)$ equation becomes singular. This is where *singularity confinement* comes into play.

Generally speaking, a *singularity* of a discrete mapping is an apparent loss of information where, for instance, a curve of initial values gets mapped to a point under the iteration. The singularity is said to be *confined*, if this information is recovered after taking a sufficient number of further iterates with the use of a certain continuity argument we discuss by example below. A discrete mapping is said to have the *singularity confinement property* if all its singularities are confined. Grammaticos, Ramani and collaborators, see the overview [21] and references therein, have used this property with great success in deriving many interesting integrable mappings and more specifically discrete Painlevé equations.

As an example, let us have a look at the case $(f_0, g_0) = (f_0, b_5)$ with generic $f_0 \in \mathbb{P}$. From (2.1) we obtain $f_1 = b_5^{-1}$, and hence both g_0 and f_1 are independent of the initial value f_0. When trying to calculate g_1, we find that the right-hand side of (2.1b) takes the form $\frac{0}{0}$, at which stage further iteration of the solution seems hopeless. Grammaticos et al [22] saw a resolution to this obstacle. Let us perturb the initial conditions by introducing a small parameter ϵ, setting $(f_0, g_0) = (f_0, b_5 + \epsilon)$. Calculating f_1 and g_1 again, we find

$$f_1 = b_5^{-1} + \mathcal{O}(\epsilon), \quad g_1 = G_1(f_0, t; \mathbf{b}) + \mathcal{O}(\epsilon),$$

as $\epsilon \to 0$, for some rational function $G_1(f_0, t_0; \mathbf{b})$, which is non-constant with respect to f_0. At this point the singularity is said to be *confined*, as not only letting $\epsilon \to 0$, we have a sensible continuation by setting $g_1 = G_1(f_0, t_0; \mathbf{b})$, but also the initial value f_0 is recovered as g_1 depends on it.

We write the corresponding *singularity pattern* symbolically as

$$(f_0, b_5) \overset{R}{\longmapsto} (b_5^{-1}, b_5) \overset{S}{\longmapsto} (b_5^{-1}, g_1), \tag{2.5}$$

where we think of $R = R(t)$ as the mapping sending (f, g) to $(\overline{f}, g)$ and $S = S(t)$ as the mapping sending $(\overline{f}, g)$ to $(\overline{f}, \overline{g})$. The *singularity patterns* of the q-$P(A_1)$ equation can now be written as

$$\begin{aligned}
&(f_0, b_j) \overset{R}{\longmapsto} (\tfrac{1}{b_j}, b_j) \overset{S}{\longmapsto} (\tfrac{1}{b_j}, g_1),\\
&(f_0, b_i t) \overset{R}{\longmapsto} (\tfrac{q}{b_i} t, b_i t) \overset{S}{\longmapsto} (\tfrac{q}{b_i} t, g_1),\\
&(\tfrac{1}{b_j}, g_0) \overset{S}{\longmapsto} (\tfrac{1}{b_j}, b_j) \overset{R}{\longmapsto} (f_2, b_j),\\
&(\tfrac{q}{b_i} t, g_0) \overset{S}{\longmapsto} (\tfrac{q}{b_i} t, b_i t) \overset{R}{\longmapsto} (f_2, b_i t),
\end{aligned}$$

for $i \in \{1, 2, 3, 4\}$ and $j \in \{5, 6, 7, 8\}$, where for instance the last one with $i = 1$ is obtained by considering $f_1 = \frac{q}{b_1} t$ and generic $g_0 \in \mathbb{P}$. Also note that the singularity patterns in the backward time direction $t \mapsto q^{-1} t$ are obtained simply be reversing the arrows in the above equations and replacing R and S by R^{-1} and S^{-1} respectively. In particular all singularities are confined and hence the q-$P(A_1)$ equation satisfies the *singularity confinement property*. We put these observations on a more rigorous mathematical ground by discussing the algebro-geometric aspects of the q-$P(A_1)$ equation in Section 2.2.

2.2 An Algebro-Geometric Interpretation

In this section we work out the initial value space of the q-$P(A_1)$ equation in Sakai's framework.

2.2.1 The q-$P(A_1)$ Mapping

Recall that any polynomial $P(x,y) \in \mathbb{C}[x,y]$ can be *homogenised* by defining

$$P^h(x_1,x_2,y_1,y_2) = x_2^{d_x} y_2^{d_y} P(x_1/x_2, y_1/y_2), \qquad d_x = \deg_x(P), \quad d_y = \deg_y(P),$$

and hence used to define a mapping $P^h : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ which satisfies

$$P^h((\lambda_x x_1, \lambda_x x_2), (\lambda_y y_1, \lambda_y y_2)) = \lambda_x^{d_x} \lambda_y^{d_y} P^h((x_1,x_2),(y_1,y_2)),$$

for $\lambda_x, \lambda_y \in \mathbb{C}^*$. We define the *locus* of P in $\mathbb{P} \times \mathbb{P}$ by

$$l(P) = \{(\mathrm{x},\mathrm{y}) \in \mathbb{P} \times \mathbb{P} : P^h((x_1,x_2),(y_1,y_2)) = 0\},$$

where we used shorthand notation $\mathrm{x} = [x_1, x_2]$ and $\mathrm{y} = [y_1, y_2]$. Next given a rational function $R(x,y) \in \mathbb{C}(x,y)$, let us write $R(x,y) = P(x,y)/Q(x,y)$ where P and Q are polynomials without common divisors. Assuming P and Q have the same degrees in x and y to make the discussion easier, we define

$$R : \mathbb{P} \times \mathbb{P} \dashrightarrow \mathbb{P}, (\mathrm{x},\mathrm{y}) \mapsto [P^h((x_1,x_2),(y_1,y_2)), Q^h((x_1,x_2),(y_1,y_2))],$$

which is well-defined on the complement of the *indeterminacy locus* of R in $\mathbb{P} \times \mathbb{P}$, defined by $l(R) := l(P) \cap l(Q)$, i.e. the intersection of loci of P and Q.

To put the singularity confinement property in an algebro-geometric perspective, recall that for $t \in \mathbb{C}^*$, we denote by $R = R(t)$ the *rational mapping* which sends (f,g) to $(\overline{f}, g)$. More precisely $R = (R_1, R_2)$, where $R_1 = R_1(f,g) \in \mathbb{C}(f,g)$, and $R_2 = R_2(f,g) = g$, with $R_1(f,g)$ given by the right-hand side of (2.1a) divided by g. Note that we can also easily construct an inverse of R by writing f in terms of $\overline{f}$ and g, using the first equation of q-$P(A_1)$. We write $R^{-1} = (R_1^{-1}, R_2^{-1})$ with $R_1^{-1} = R_1^{-1}(\overline{f}, g) \in \mathbb{C}(\overline{f}, g)$ and $R_2^{-1} = R_2^{-1}(f,g) = g$. The rational mapping R is called *birational* as it has an inverse rational mapping. Similarly $S = S(t)$ is the birational mapping which sends $(\overline{f}, g)$ to $(\overline{f}, \overline{g})$. We now think of q-$P(A_1)$ as the birational mapping given by the composition $\mathcal{T}(t) := S(t) \circ R(t)$.

Let us consider $R_1(t)$ as a mapping from $\mathbb{P} \times \mathbb{P}$ to $\mathbb{P}$. An easy calculation shows that, for generic t, its indeterminacy locus is given by

$$l(R_1(t)) = \{p_1(t), \ldots p_4(t), p_5, \ldots p_8\}, \tag{2.6}$$

where the p_i are as defined in (2.4), and we identified $\mathbb{C}^2 \subseteq \mathbb{P} \times \mathbb{P}$ as usual. We restrict our discussion to generic values of the parameters $\mathbf{b}$ and $t \in \mathbb{C}^*$ here. The singularities p_i of the mapping R are often referred to as *base points*. At this point there is an invaluable tool from algebraic geometry which allows us to resolve these base points, called the *blowup*. The idea is to replace the, in this case, algebraic manifold $\mathbb{P} \times \mathbb{P}$ for a larger one, by *blowing up* the base points, and lifting the mapping R to it. Before we enter into the details the blowup, note that the indeterminacy locus of S_2 is given by

$$l(S_2(t)) = \{p'_1(t), \ldots p'_4(t), p'_5, \ldots p'_8\}, \tag{2.7}$$

where $p'_i(t) = (\frac{q}{b_i}t, b_i t)$ for $1 \leq i \leq 4$ and $p'_j = p_j$ for $5 \leq j \leq 8$.

2.2.2 The Blowup Procedure

As we are just interested in the resolution of base points in two-dimensional complex manifolds, we only consider the *blowup procedure* of such spaces. We keep our discussion brief and refer the interested reader to Duistermaat [13], where a lot of standard machinery from algebraic geometry is explained in the context of discrete integrable systems. Let us start with the simplest example of blowing up the origin in $\mathbb{C}^2$. Each point $z \in \mathbb{C}^2$, defines an unique line through the origin and hence a point in projective space $\mathbb{P}$, except when $z = 0$. We define

$$\begin{aligned} B_0\mathbb{C}^2 &= \{(z, l) \in \mathbb{C}^2 \times \mathbb{P} : z \text{ lies on } l\} \\ &= \{((x, y), [l_1, l_2]) \in \mathbb{C}^2 \times \mathbb{P} : xl_2 = yl_1\}, \end{aligned}$$

and define the *projection map* $\pi_0 : B_0\mathbb{C}^2 \to \mathbb{C}^2$ by forgetting the second coordinate. We refer to $B_0\mathbb{C}^2$ as the *blowup* of $\mathbb{C}^2$ at the origin and define the *exceptional divisor* by $E_0 = \pi_0^{-1}(0)$. From a set-theoretical point of view all we did was replace the origin by a copy of $\mathbb{P}$, however $B_0\mathbb{C}^2$ can in fact be made into a complex manifold such that the projection map becomes a holomorphic mapping. We do this by defining an *atlas* consisting of two charts (U_1, φ_1) and (U_2, φ_2), where

$$U_i = \{(z, l) \in B_0\mathbb{C}^2 : l_i \neq 0\}, \qquad (i \in \{1, 2\})$$

with

$$\begin{aligned} \varphi_1 &: U_1 \to \mathbb{C}^2, ((x, y), [l_1, l_2]) \mapsto (x, l_2/l_1), \\ \varphi_2 &: U_2 \to \mathbb{C}^2, ((x, y), [l_1, l_2]) \mapsto (l_1/l_2, y). \end{aligned}$$

We use the constructed atlas to induce a topology on $B_0\mathbb{C}^2$ and turn it into a complex manifold. It is easy to check that this is a valid construction as all the transition maps are holomorphic and that the projection map becomes a holomorphic mapping. It is often useful to work with the *inverse charts*, given by

$$\begin{aligned} x &= x_1, & x &= x_2 y_2, \\ y &= x_1 y_1, & y &= y_2, \\ l &= [1, y_1], & l &= [x_2, 1], \end{aligned}$$

where $\phi_1((x, y), l) = (x_1, y_1)$ and $\phi_2((x, y), l) = (x_2, y_2)$, for $(x_1, y_1), (x_2, y_2) \in \mathbb{C}^2$. Let us denote two special points of the exceptional divisor E_0 by $0_0 = (0, [0, 1])$ and $\infty_0 = (0, [1, 0])$, where the 0 subscripts emphasise that we blew up the origin. Then, in the first chart, $\{x_1 = 0, y_1 \in \mathbb{C}\}$ parameterises $E_0 \setminus \{0_0\}$. Similarly, in the second chart, $\{y_2 = 0, x_2 \in \mathbb{C}\}$ parameterises $E_0 \setminus \{\infty_0\}$.

Given an arbitrary two-dimensional complex manifold M and a point $p \in M$, we can now easily construct the *blowup* of M at p, by working locally in a chart containing p. Firstly we construct at chart $\varphi : U \to D$, where $p \in U \subseteq M$ and $D \subseteq \mathbb{C}^2$ open, such that $\varphi(p) = 0$. Next we define the blowup of D at the origin simply as $B_0 D = \pi_0^{-1}(D)$ and the corresponding projection $\pi_D : B_0 D \to D$ by restriction. Lastly we glue $M^* := M \setminus \{p\}$ and $B_0 D$ together by identifying $U \setminus \{p\}$ and $B_0 D \setminus E_0$ via $\pi^{-1} \circ \varphi$, giving the complex manifold $B_p M$ with an associated projection mapping $\pi : B_p M \to M$ defined in the obvious way and we call $E = \pi^{-1}(p)$ the exceptional divisor. We leave it to interested reader to work out the analytic details and in particular prove that the obtained blowup is independent of the choice of chart φ up to projection preserving isomorphism. Note that the projection π is an isomorphism onto M^*, when restricted to $B_p M \setminus E$, and we therefore identify $M^* \subseteq B_p M$.

To end our discussion, we would like to note that, as the blowup procedure is a local operation, given distinct points $p, q \in M$, the order in which we blow up M at p and q is irrelevant. That is, let $B_p B_q M$ be obtained by first blowing up M at q with corresponding projection π_q and subsequently at p with corresponding projection π_{pq} and let $\pi = \pi_{pq} \circ \pi_q$ be the composed projection on M, which we refer to as *total projection* from here on. Similarly let $B_q B_p M$ be obtained by blowing up at p and q in opposite order with corresponding total projection π' on M. Then there exists an isomorphism Ψ from $B_p B_q M$ to $B_q B_p M$ such that $\pi = \pi' \circ \Psi$. Hence, given a two-dimensional complex manifold M and distinct points $p_1, \ldots, p_n$ on it, we can now safely speak about the manifold and corresponding total projection π obtained by blowing up M at $p_1, \ldots, p_n$, with associated exceptional divisors $E_i = \pi^{-1}(p_i)$ for $1 \leq i \leq n$.

2.2.3 The Initial Value Space

We are now in the position to resolve the base points of the birational mappings $R(t)$ and $S(t)$ and hence of the q-$P(A_1)$ mapping. Let us, for $t \in \mathbb{C}^*$, define $X(t)$ as the complex manifold obtained by blowing up $\mathbb{P} \times \mathbb{P}$ at the eight base points in the indeterminacy locus (2.6) of $R_1(t)$. We denote the corresponding total projection by $\pi_X = \pi_{X(t)}$ and exceptional divisors by $E_i = \pi_X^{-1}(p_i)$ for $1 \leq i \leq 8$. Similarly let $Y(t)$ be the complex manifold obtained by blowing up $\mathbb{P} \times \mathbb{P}$ at the eight base points in the indeterminacy locus (2.7) of $S_1(t)$, with corresponding total projection by $\pi_Y = \pi_{Y(t)}$ and exceptional divisors $E_i' = \pi_Y^{-1}(p_i')$ for $1 \leq i \leq 8$.

Using machinery from birational geometry, we can lift the birational mapping $R(t)$ to an isomorphism $\widehat{R}(t)$ from $X(t)$ to $Y(t)$, and the birational mapping $S(t)$ to an isomorphism $\widehat{S}(t)$ from $Y(t)$ to $X(qt)$, such that the diagram 2.1 commutes [77]. We do not set up this machinery here but instead discuss these lifts more heuristically in particular charts of the domains and co-domains involved.

But before discussing $\widehat{R}(t)$ and $\widehat{S}(t)$ in more detail, note that correspondingly the q-$P(A_1)$ mapping is lifted to the isomorphism $\widehat{\mathcal{T}}(t) := \widehat{S}(t) \circ \widehat{R}(t)$ from $X(t)$ to $X(qt)$ for $t \in \mathbb{C}^*$. In particular the complex manifold $X(t)$ can be considered the *initial value space* of our equation in consideration, and following Sakai [77], this manifold is called an $A_1^{(1)}$*-surface*, hence the

$$\begin{tikzcd}
X(t) \arrow[r, "\widehat{R}(t)"] \arrow[d, "\pi_{X(t)}"] & Y(t) \arrow[r, "\widehat{S}(t)"] \arrow[d, "\pi_{Y(t)}"] & X(qt) \arrow[d, "\pi_{X(qt)}"] \\
\mathbb{P}\times\mathbb{P} \arrow[r, dashed, "R(t)"] & \mathbb{P}\times\mathbb{P} \arrow[r, dashed, "S(t)"] & \mathbb{P}\times\mathbb{P}
\end{tikzcd}$$

Figure 2.1: Commuting Diagram

name "q-$P(A_1)$ equation". We study this surface in more detail in Section 2.2.4.

Recall that the birational mapping $R_1(t)$ is not well-defined at $(f,g)=(b_5^{-1},b_5)$. In $X(t)$ this point has been blown up to the exceptional divisor E_5. Let us consider the following two *inverse charts*, which cover E_5 in $X(t)$,

$$\begin{aligned} f-b_5^{-1}&=f_{51}, & f-b_5^{-1}&=f_{52}g_{52},\\ g-b_5&=f_{51}g_{51}, & g-b_5&=g_{52},\\ l_5&=[1,g_{51}], & l_5&=[f_{52},1],\end{aligned}$$

both defined for (f_{51},g_{51}) and (f_{52},g_{52}) in an open neighbourhood of $(0,0)$ containing $\{0\}\times\mathbb{C}$ and $\mathbb{C}\times\{0\}$ respectively. Writing U_1 and U_2 for the open subsets of $X(t)$ they cover respectively, we find that $U_1\cap E_5$ is parameterised by $\{f_{51}=0\}$ and $U_2\cap E_5$ is parameterised by $\{g_{52}=0\}$. We use 0_5 and ∞_5 to denote the elements of the singletons $E_5\setminus U_1=\{0_5\}$ and $E_5\setminus U_2=\{\infty_5\}$.

So let us explore what the mapping $\widehat{R}(t)$ looks like on U_1. We use the coordinates f_{51} and g_{51} to express $R_2=f_{51}g_{51}+b_5$ and, using (2.1a),

$$\begin{aligned} gR_1 &=\frac{qt^2(fg-t^2)f_{51}g_{51}(g-b_6)(g-b_7)(g-b_8)-f_{51}(f_{51}g_{51}+b_5^{-1}g_{51}+b_5)b_1b_2b_3b_4t^4\dot{p}_1(g/t)}{(fg-t^2)f_{51}g_{51}(g-b_6)(g-b_7)(g-b_8)-f_{51}(f_{51}g_{51}+b_5^{-1}g_{51}+b_5)b_1b_2b_3b_4t^4\dot{p}_1(g/t)}\\ &=\frac{qt^2(fg-t^2)g_{51}(g-b_6)(g-b_7)(g-b_8)-(f_{51}g_{51}+b_5^{-1}g_{51}+b_5)b_{1\text{-}4}t^4\dot{p}_1(g/t)}{(fg-t^2)g_{51}(g-b_6)(g-b_7)(g-b_8)-(f_{51}g_{51}+b_5^{-1}g_{51}+b_5)b_{1\text{-}4}t^4\dot{p}_1(g/t)}, \end{aligned} \tag{2.8}$$

where we temporarily use the notation $b_{1\text{-}4}=b_1b_2b_3b_4$, the second equality follows from dividing out the common factor f_{51} in numerator and denominator, and only the relevant factors $(g-b_5)$ and $(fg-1)$ have been rewritten in terms of f_{51} and g_{51}. Note that the indeterminacy at p_5 has been resolved, indeed, because of the cancellation of the factor f_{51}, the numerator and denominator do not have common zeros in a neighbourhood of $E_5\cap U_1$ in U_1. Specialising to the exceptional divisor by setting $f_{51}=0$, we find

$$b_5R_1=\frac{qt^2(1-t^2)g_{51}(b_5-b_6)(b_5-b_7)(b_5-b_8)-(b_5^{-1}g_{51}+b_5)b_1b_2b_3b_4t^4\dot{p}_1(b_5/t)}{(1-t^2)g_{51}(b_5-b_6)(b_5-b_7)(b_5-b_8)-(b_5^{-1}g_{51}+b_5)b_1b_2b_3b_4t^4\dot{p}_1(b_5/t)},$$

in particular the only point on $E_5\cap U_1$ which gets send on the exceptional divisor E_5' in $Y(t)$, i.e. such that $R_1=b_5^{-1}$, is parameterised by $(f_{51},g_{51})=(0,0)$, which is the point ∞_5. In fact, considering (2.8), we find that all the elements of U_1 which get send on the exceptional divisor E_5' in $Y(t)$, are parameterised exactly by $g_{51}=0$, as the right-hand side of (2.8) equals 1 iff $g_{51}=0$, in U_1. To understand where such elements get mapped to in E_5', we have to work in inverse charts on $Y(t)$ which cover E_5'. We choose charts

$$\begin{aligned} \overline{f}-b_5^{-1}&=\overline{f}_{51}', & \qquad \overline{f}-b_5^{-1}&=\overline{f}_{52}'g_{52}',\\ g-b_5&=\overline{f}_{51}'g_{51}', & g-b_5&=g_{52}',\\ l_5'&=[1,g_{51}'], & l_5'&=[\overline{f}_{52}',1], \end{aligned}$$

covering open subsets U_1' and U_2' of $Y(t)$ respectively and we denote $E_5'\setminus U_1'=\{0_5'\}$ and $E_5'\setminus U_2'=\{\infty_5'\}$ as before. To explore $\widehat{R}$ in the first chart, we set

$$R_1=\overline{f}_{51}'+b_5^{-1},\qquad R_2=\overline{f}_{51}'g_{51}'+b_5,$$

which gives

$$\begin{aligned} \overline{f}_{51}'&=\frac{(qt^2-1)(fg-t^2)g_{51}(g-b_6)(g-b_7)(g-b_8)}{(fg-t^2)g_{51}(g-b_6)(g-b_7)(g-b_8)-(f_{51}g_{51}+b_5^{-1}g_{51}+b_5)b_1b_2b_3b_4t^4\dot{p}_1(g/t)},\\ g_{51}'&=b_5f_{51}\frac{(fg-t^2)g_{51}(g-b_6)(g-b_7)(g-b_8)-(f_{51}g_{51}+b_5^{-1}g_{51}+b_5)b_1b_2b_3b_4t^4\dot{p}_1(g/t)}{(qt^2-1)(fg-t^2)(g-b_6)(g-b_7)(g-b_8)}. \end{aligned}$$

Note that $\overline{f}_{51}'=0$ parameterises $E_5'\cap U_1$, hence it is clear that $g_{51}=0$ parameterises the points in U_1 send to E_5' by $\widehat{R}$. Specialising to $g_{51}=0$ we find $\overline{f}_{51}'=0$ and

$$g_{51}'=-b_5^2f_{51}\frac{(b_5-b_1t)(b_5-b_2t)(b_5-b_3t)(b_5-b_4t)}{(qt^2-1)(b_5f_{51}+1-t^2)(b_5-b_6)(b_5-b_7)(b_5-b_8)}.$$

In particular setting $f_{51}=0$ we find $g_{51}'=0$, that is, the point $\infty_5\in X(t)$ is send to $\infty_5'\in Y(t)$ by $\widehat{R}(t)$. Similarly we find $\widehat{S}(t)(\infty_5')=\overline{\infty}_5\in X(qt)$, hence $\mathcal{T}(t)(\infty_5)=\infty_5'$ and it is easy to see that

$$E_5\cap\mathcal{T}(t)^{-1}(\overline{E}_5)=\{\infty_5\}.$$

To put things in perspective, by exploring the mapping $R(t)$ in different charts of $X(t)$ and $Y(t)$, we find a canonical way to lift it to a mapping $\widehat{R}(t)$ from $X(t)$ and $Y(t)$, as we did above for two specific charts around E_5 and E_5'.

2.2.4 Sakai's Theory

In this section we delve ourselves more deeply into the geometric aspects of the q-$P(A_1)$ mapping, following Sakai [77] closely. While most of the thesis is self-contained, we use the algebro-geometric machinery, which can be found in Sakai's exposition [77] on discrete Painlevé equations, without setting it up ourselves here. Let us also mention Kajiwara et al. [52] for an overview of the subject. We note that Hay et al. [32] work out Sakai's method explicitly for q-$P(A_1)$ as well.

The Induced Picard Group Isomorphism

Recalling that the manifold $X(t)$ was obtained by blowing up $\mathbb{P}\times\mathbb{P}$ at the eight base points $p_1(t),\ldots,p_4(t),p_5,\ldots,p_8$, the corresponding *Picard group* takes the form

$$\mathrm{Pic}(X(t)) = \mathbb{Z}h_f + \mathbb{Z}h_g + \mathbb{Z}e_1 + \ldots + \mathbb{Z}e_8,$$

where e_i corresponds to the linear equivalence class of the exceptional divisor E_i, for $1\le i\le 8$, and h_f and h_g stand for the linear equivalence classes of $\pi_X^{-1}(L_f)$ and $\pi_X^{-1}(L_g)$ respectively with L_f and L_g any lines $f\equiv$ *constant* and $g\equiv$ *constant* in $\mathbb{P}\times\mathbb{P}$ respectively not containing base points. We define the symmetric bilinear *intersection form* $\cdot$ on the Picard group by setting the "*intersection numbers*" of the generators equal to

$$\begin{array}{llll} h_f\cdot h_f=0, & h_g\cdot h_g=0, & h_f\cdot h_g=1, & \\ h_f\cdot e_i=0, & h_g\cdot e_i=0, & e_i\cdot e_i=-1, & e_i\cdot e_j=0, \end{array} \tag{2.9}$$

for $1\le i,j\le 8$ with $i\neq j$. Equations (2.9) remind us of the fact that for instance (different) lines with $f\equiv$ *constant* do not intersect, but any line with $f\equiv$ *constant* intersects exactly ones with a line given by $g\equiv$ *constant*. Similarly the Picard groups of $Y(t)$ and $X(qt)$, equipped with intersection forms, are given by

$$\mathrm{Pic}(Y(t)) = \mathbb{Z}h'_{\overline{f}} + \mathbb{Z}h'_g + \mathbb{Z}e'_1 + \ldots + \mathbb{Z}e'_8,$$
$$\mathrm{Pic}(X(qt)) = \mathbb{Z}h_{\overline{f}} + \mathbb{Z}h_{\overline{g}} + \mathbb{Z}\overline{e}_1 + \ldots + \mathbb{Z}\overline{e}_8.$$

Now the isomorphism $\widehat{R}(t)$ induces an *intersection preserving* isomorphism ϕ_R between the Picard groups of $X(t)$ and $Y(t)$, i.e. satisfying

$$\phi_R(c_1)\cdot\phi_R(c_2) = c_1\cdot c_2,$$

for $c_1,c_2\in\mathrm{Pic}(X(t))$.
It is fairly straightforward to calculate ϕ_R explicitly. First of all, as R leaves the g-coordinate invariant, i.e. $R_2(f,g)=g$, we immediately obtain $\phi_R(h_g)=h'_g$. Next let us consider the line L_g in $\mathbb{P}\times\mathbb{P}$ given by $g\equiv b_5$. The lift of L_5 is parameterised by $g_{51}=0$ in $X(t)$ and the corresponding equivalence class in the Picard group is given by h_g-e_5. Now recall $\widehat{R}(t)$ sends $\{g_{51}=0\}$ to $\{\overline{f}_{51}=0\}$ in $Y(t)$, and we infer $\phi_R(h_g-e_5)=e'_5$. Using the linearity of ϕ_R we find $\phi_R(e_5)=h'_g-e'_5$. The other exceptional divisors are handled essentially the same and we find

$$\phi_R(h_g)=h'_g, \qquad \phi_R(e_i)=h'_g-e'_i. \quad (1\le i\le 8)$$

To calculate $\phi_R(h_f)$, let us write

$$\phi_R(h_f) = uh_{\overline{f}} + vh'_g + w_1e'_1 + \ldots + w_8e'_8.$$

As ϕ_R is intersection preserving, we have

$$u = \phi_R(h_f)\cdot h'_g = \phi_R(h_f)\cdot\phi_R(h_g) = h_f\cdot h_g = 1,$$

and, for $1 \leq i \leq 8$,

$$w_i = \phi_R(h_f) \cdot (-e_i') = \phi_R(h_f) \cdot \phi_R(e_i - h_g) = h_f \cdot (e_i - h_g) = -1.$$

Furthermore

$$0 = h_f \cdot h_f = \phi_R(h_f) \cdot \phi_R(h_f) = 2v - 8,$$

giving $v = 4$ and hence

$$\phi_R(h_f) = h_{\overline{f}} + 4h_g' - (e_1' + \ldots + e_8').$$

In summary ϕ_R acts on the Picard group $\mathrm{Pic}(X(t))$ by

$$\begin{aligned}\phi_R: \quad h_f &\mapsto h_f' + 4h_g' - (e_1' + \ldots e_8'),\\ h_g &\mapsto h_g',\\ e_i &\mapsto h_g' - e_i'. \qquad (1 \leq i \leq 8)\end{aligned}$$

Completely analogous we find that ϕ_S acts on $\mathrm{Pic}(Y(t))$ by

$$\begin{aligned}\phi_S: \quad h_f' &\mapsto h_{\overline{f}},\\ h_g' &\mapsto 4h_{\overline{f}} + h_{\overline{g}} - (\overline{e}_1 + \ldots \overline{e}_8),\\ e_i' &\mapsto h_{\overline{f}} - \overline{e}_i, \qquad (1 \leq i \leq 8)\end{aligned}$$

and hence $\phi_{\mathcal{T}} = \phi_S \circ \phi_R$ acts on the Picard group of $X(t)$ by

$$\begin{aligned}\phi_{\mathcal{T}}: \quad h_f &\mapsto 9h_{\overline{f}} + 4h_{\overline{g}} - 3(\overline{e}_1 + \ldots \overline{e}_8),\\ h_g &\mapsto 4h_{\overline{f}} + h_{\overline{g}} - (\overline{e}_1 + \ldots \overline{e}_8),\\ e_i &\mapsto 3h_{\overline{f}} + h_{\overline{g}} + \overline{e}_i - (\overline{e}_1 + \ldots \overline{e}_8). \qquad (1 \leq i \leq 8)\end{aligned}$$

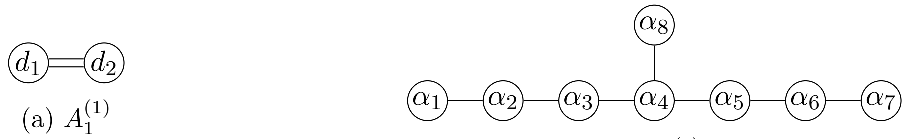
Irreducible Divisors and the $A_1^{(1)}$-Surface

Consider the bidegree $(1,1)$ curves δ_1 and δ_2 in $\mathbb{P}\times\mathbb{P}$, given by $fg = t^2$ and $fg = 1$ respectively, in affine coordinates. So δ_1 meets each of the base points p_1, p_2, p_3, p_4 once, and δ_2 meets each of the base points p_5, p_6, p_7, p_8 once. We let D_1 and D_2 be the corresponding total transforms in $X(t)$ of these curves respectively. The associated classes in the Picard group of $X(t)$, are given respectively by

$$d_1 = h_f + h_g - (e_1 + e_2 + e_3 + e_4), \qquad d_2 = h_f + h_g - (e_5 + e_6 + e_7 + e_8).$$

To ease the notation a bit, we use, for instance, d_1 to denote the total transform of D_1 in $X(t)$, for any $t \in \mathbb{C}^*$. Similarly we no longer distinguish between h_f and $h_{\overline{f}}$ notation wise. Now D_1 and D_2 have self-intersection -2 and they are the *irreducible divisors* of X. The *anti-canonical divisor* of X is given by

$$\delta = -K_X = 2h_f + 2h_g - (e_1 + \ldots + e_8),$$



(a) $A_1^{(1)}$ (b) $E_7^{(1)}$

Figure 2.2: Dynkin Diagrams

and we have the unique decomposition $\delta = d_1 + d_2$ [31]. A small calculation shows

$$\phi_{\mathcal{T}}(d_1) = d_1, \quad \phi_{\mathcal{T}}(d_2) = d_2, \tag{2.10}$$

and we consider the root lattice $Q = \mathbb{Z}d_1 + \mathbb{Z}d_2$ with intersection form inherited from the Picard group. Note that $d_1 \cdot d_2 = 2$ and we readily identify the intersection matrix of $\{d_1, d_2\}$ as the *generalized Cartan matrix* of $A_1^{(1)}$ type multiplied by -1, that is

$$\begin{pmatrix} d_1 \cdot d_1 & d_1 \cdot d_2 \\ d_2 \cdot d_1 & d_2 \cdot d_2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} = -\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

The corresponding Dynkin diagram is the graph 2.2a, with nodes d_1 and d_2 connected by two edges as $d_1 \cdot d_2 = 2$.

The Weyl Group Action

Rational surfaces are intrinsically related to affine Weyl groups, see for instance Looijenga [60]. In this Section we work out the affine Weyl group action corresponding to the $A_1^{(1)}$ surface. We refer the interested reader to Kac [51], for an overview of the theory of affine Weyl groups. The orthogonal complement of $\mathbb{Z}d_1 + \mathbb{Z}d_2$ in $\mathrm{Pic}(X)$,

$$\delta^{\perp} := \{\alpha \in Pic(X_b) \mid \alpha \cdot d_1 = 0, \alpha \cdot d_2 = 0\},$$

admits a $\mathbb{Z}$-basis $\{\alpha_1 \ldots \alpha_8\}$, where

$$\begin{aligned} \alpha_1 &= e_2 - e_1, & \alpha_2 &= e_3 - e_2, & \alpha_3 &= e_4 - e_3, & \alpha_4 &= h_g - e_4 - e_5, \\ \alpha_5 &= e_5 - e_6, & \alpha_6 &= e_6 - e_7, & \alpha_7 &= e_7 - e_8, & \alpha_8 &= h_f - h_g. \end{aligned}$$

Observe that $\delta \in \delta^{\perp}$ and we have

$$\delta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + 2\alpha_8.$$

The basis elements $\alpha_1, \ldots, \alpha_8$ all have self-intersection -2. Furthermore we have the associated Dynkin Diagram 2.2b, of $E_7^{(1)}$ type, with nodes the basis elements, such that different basis elements are connected by n edges, iff their intersection product equals n, for $n \in \mathbb{N}$.

We define *reflections* by

$$w_i : Pic(X) \to Pic(X), \alpha \mapsto \alpha - 2\frac{\alpha \cdot \alpha_i}{\alpha_i \cdot \alpha_i}\alpha_i.$$

for $1 \leq i \leq 8$, and the Dynkin automorphism π on $Pic(X)$ by

$$\pi(h_f) = h_g, \qquad \pi(h_g) = h_f, \qquad \pi(e_i) = e_{9-i}. \quad (1 \leq i \leq 8)$$

By $W = W(E_7^{(1)})$ we denote the affine Weyl group, which is the group generated by the reflections $w_1, \ldots, w_8$. Similarly the extended affine Weyl group $\widetilde{W} = \widetilde{W}(E_7^{(1)})$ is the group generated by the reflections $w_1, \ldots, w_8$ and the Dynkin automorphism π. By definition we have an action of $\widetilde{W}$ on $\mathrm{Pic}(X)$, which of course leaves $\delta^\perp$ invariant. The action of the generators of $\widetilde{W}$ on the basis elements of $\delta^\perp$ is given in Table 2.1, where blank entries represent invariance. Let us note the following fundamental relations

$$\begin{aligned} w_i^2 &= \pi^2 = 1, & \pi w_i &= w_{8-i}\pi, \\ (w_i w_j)^3 &= 1, \quad (|i-j| = 1) & w_i w_j &= w_j w_i, \quad (|i-j| \neq 1) \end{aligned}$$

for $i, j \in \{1, \ldots, 7\}$, and

$$\begin{aligned} w_8^2 &= 1, & \pi w_8 &= w_8 \pi, \\ (w_4 w_8)^3 = (w_8 w_4)^3 &= 1, & w_8 w_j &= w_j w_8. \quad (j \neq 4) \end{aligned}$$

	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	π
α_1	$-\alpha_1$	$\alpha_1+\alpha_2$							α_7
α_2	$\alpha_1+\alpha_2$	$-\alpha_2$	$\alpha_2+\alpha_3$						α_6
α_3		$\alpha_2+\alpha_3$	$-\alpha_3$	$\alpha_3+\alpha_4$					α_5
α_4			$\alpha_3+\alpha_4$	$-\alpha_4$	$\alpha_4+\alpha_5$			$\alpha_4+\alpha_8$	
α_5				$\alpha_4+\alpha_5$	$-\alpha_5$	$\alpha_5+\alpha_6$			α_3
α_6					$\alpha_5+\alpha_6$	$-\alpha_6$	$\alpha_6+\alpha_7$		α_2
α_7						$\alpha_6+\alpha_7$	$-\alpha_7$		α_1
α_8				$\alpha_4+\alpha_8$				$-\alpha_8$	

Table 2.1: Extended Weyl group action on $\delta^\perp$

It is easy to check that the time evolution $\phi_{\mathcal{T}}$ acts on the root lattice $E_7^{(1)}$ as

$$\begin{aligned} \phi_{\mathcal{T}}: \quad & \alpha_i \mapsto \alpha_i, \quad (i \neq 4, 8) \\ & \alpha_4 \mapsto \alpha_4 - \delta, \\ & \alpha_8 \mapsto \alpha_8 + 2\delta, \end{aligned}$$

and hence defines a *translation* in the root lattice by integer multiples of δ. In particular, the equation under consideration is indeed a Painlevé equation as defined in Sakai [77]. Finally we can explicitly write $\phi_{\mathcal{T}}$ as

$$\begin{aligned} \phi_{\mathcal{T}} = & s_4 \circ s_3 \circ s_2 \circ s_5 \circ s_4 \circ s_3 \circ s_6 \circ s_5 \circ s_4 \circ s_7 \circ s_6 \circ s_5 \circ s_8 \circ s_4 \circ s_3 \circ s_2 \circ s_1 \circ \\ & s_2 \circ s_3 \circ s_4 \circ s_5 \circ s_8 \circ s_4 \circ s_3 \circ s_2 \circ s_6 \circ s_5 \circ s_4 \circ s_3 \circ s_7 \circ s_6 \circ s_5 \circ s_4 \circ s_8. \end{aligned}$$

Bäcklund Transformations

The q-$P(A_1)$ equation has various Bäcklund transformations, which relate solutions of the equation for possibly different parameter values. The full affine Weyl group of $E_7^{(1)}$ type acts on q-$P(A_1)$, however we only use four Bäcklund transformations in this study, given by

$$\mathcal{T}_k\left(f, g, \mathbf{b}\right) = \left(f^{(k)}, g^{(k)}, \mathbf{b}^{(k)}\right), \tag{2.11}$$

for $k \in \{1, 2, 3, 4\}$, where

$$\begin{aligned}
f^{(1)}(t) &= \frac{t}{f(t)}, & f^{(2)}(t) &= tg\left(\frac{1}{t}\right), & f^{(3)}(t) &= g\left(q^{-\frac{1}{2}}t\right), & f^{(4)}(t) &= \frac{1}{g(\frac{1}{t})},\\
g^{(1)}(t) &= \frac{t}{g(t)}, & g^{(2)}(t) &= tf\left(\frac{1}{t}\right), & g^{(3)}(t) &= f\left(q^{\frac{1}{2}}t\right), & g^{(4)}(t) &= \frac{1}{f(\frac{1}{t})},
\end{aligned}$$

with for $1 \leq i \leq 4$ and $5 \leq j \leq 8$,

$$\begin{aligned}
b_i^{(1)} &= b_{i+4}^{-1}, & b_i^{(2)} &= b_{i+4}^{-1}, & b_i^{(3)} &= q^{\frac{1}{2}}b_i^{-1}, & b_i^{(4)} &= b_i,\\
b_j^{(1)} &= b_{j-4}^{-1}, & b_j^{(2)} &= b_{j-4}^{-1}, & b_j^{(3)} &= b_j^{-1}, & b_j^{(4)} &= b_j.
\end{aligned}$$

Note that each of these transformations $\mathcal{T}_k$ leaves $q(\mathbf{b})$ invariant and indeed maps solutions of q-$P(A_1)(\mathbf{b})$ to solutions of q-$P(A_1)(\mathbf{b}^{(k)})$ for $k \in \{1, 2, 3, 4\}$. We remark that these transformations are not independent, for instance $\mathcal{T}_1\mathcal{T}_2 = \mathcal{T}_2\mathcal{T}_1 = \mathcal{T}_4$, and that $\mathcal{T}_2$,$\mathcal{T}_3$ and $\mathcal{T}_4$ change the independent variable.

2.2.5 Singularity Confinement Revised

We have seen that the time evolution $\mathcal{T}$ can be lifted to an isomorphism $\widehat{\mathcal{T}}$ between the initial value space X at consecutive times, such that the diagram 2.1 commutes. Singularity confinement is now a consequence of the analyticity of $\widehat{\mathcal{T}}$. Indeed looking back at the example in Section 2.1.1, we have, for $f_0 \neq b_5^{-1}$,

$$\begin{aligned}
\widehat{\mathcal{T}}\left[(f_0, b_5)\right] &= \widehat{\mathcal{T}}\left[\lim_{\epsilon\to 0}(f_0, b_5+\epsilon)\right]\\
&= \lim_{\epsilon\to 0}\widehat{\mathcal{T}}\left[(f_0, b_5+\epsilon)\right]\\
&= \lim_{\epsilon\to 0}\mathcal{T}\left[(f_0, b_5+\epsilon)\right],
\end{aligned}$$

where the last equality follows from the fact that the diagram 2.1 commutes, and for $\epsilon \neq 0$ small enough, $(f_0, b_5 + \epsilon)$ does not equal any base point. Analogously to the singularity pattern (2.5), we have, on the level of Picard groups,

$$h_g - e_5 \overset{\phi_R}{\longmapsto} e_5' \overset{\phi_S}{\longmapsto} h_{\overline{f}} - \overline{e}_5.$$

2.3 Notion of Solutions

For differential equations, the notion of a solution is well defined. However there are two interpretations for difference equations. We describe them here for q-$P(A_1)$.

2.3.1 Discrete Solutions

Firstly there are solutions on discrete q-domains. We call solutions on such domains *discrete solutions*, or *solutions with discrete time.* To be precise, following Section 2.2.3, after fixing a $t_0 \in \mathbb{C}^*$, we call a sequence $(f_s, g_s)_{s\in\mathbb{Z}}$, where $(f_s, g_s) \in X(q^s t_0)$ for $s \in \mathbb{Z}$, a *discrete solution* of q-$P(A_1)$, if and only if

$$\widehat{\mathcal{T}}(f_s, g_s) = (f_{s+1}, g_{s+1}). \qquad (s \in \mathbb{Z})$$

We say that the *domain* of the solution is $q^{\mathbb{Z}} t_0$. We define the *discrete solution space* of q-$P(A_1)$ on $q^{\mathbb{Z}} t_0$ by

$$\mathcal{S}_d(t_0) = \mathcal{S}_d(q^{\mathbb{Z}} t_0) := \{\text{discrete solutions of } q\text{-}P(A_1) \text{ with domain } q^{\mathbb{Z}} t_0\}.$$

As a trivial remark, since $\widehat{\mathcal{T}}$ is an isomorphism, this space can easily be identified with the initial value space $X(t_0)$.

Now we can *project* a discrete solution pointwise to get back to, say the classical notion of a solution in $\mathbb{P} \times \mathbb{P}$, i.e. we define

$$\pi_X(f_s, g_s)_{s\in\mathbb{Z}} = (f_s^c, g_s^c)_{s\in\mathbb{Z}}, \quad (f_s^c, g_s^c) = \pi_{X(t_s)}(f_s, g_s). \qquad (s \in \mathbb{Z})$$

We project the entire discrete solution space $\mathcal{S}_d(t_0)$ to obtain the *classical discrete solution space*

$$\mathcal{S}_c(t_0) := \pi_X\left[\mathcal{S}_d(t_0)\right].$$

Note that π_X now defines a bijection between $\mathcal{S}_d(t_0)$ and $\mathcal{S}_c(t_0)$ and we often treat the two notions of discrete solutions as the same thing under this bijection. In Section 2.2.3 we calculated that $\widehat{\mathcal{T}}(\infty_5) = \infty_5$, and hence $(f_s, g_s) = \infty_5$, for all $s \in \mathbb{Z}$, defines a discrete solution. Its projection is of course given by $(f_s^c, g_s^c) = p_5$ for all $s \in \mathbb{Z}$. In fact, for each of the eight base points, there exists exactly one discrete solution never leaving the corresponding exceptional line, or equivalently one classical discrete solution everywhere equal to it. We call these solutions *base solutions.* Let us mention the following result.

Proposition 2.3.1. *For generic parameter values, any discrete solution of q-$P(A_1)$, whose value, at two consecutive times, lies on the same exceptional line, is a base solution.*

Proof. In Section 2.2.3, we saw that the only element of the exceptional line E_5, whose image under the time evolution $\widehat{\mathcal{T}}$ remains in E_5, is the element $\infty_5 \in X$. Hence, if a discrete solution takes a value on the exceptional line E_5 for two consecutive times, it must equal the base solution which equals ∞_5 everywhere. The other cases are dealt with similarly. □

Singular and Regular Solutions

Remember that the irreducible divisors D_1 and D_2 are invariant under the time evolution, by equations (2.10). Indeed, explicitly, if $(f,g) \in \mathbb{C}^2 \subseteq \mathbb{P} \times \mathbb{P}$, not equal to a base point, satisfies $fg = t^2$, then $\overline{f}g = 1$ and $\overline{f}\overline{g} = q^2t^2$. Hence, if we take any $t_0 \in \mathbb{C}^*$ and $g_0 \in \mathbb{C}^*$, then

$$g_s = q^{s(s+1)} t_0^{2s} g_0, \quad f_s = q^{2s} t_0^2 / g_s, \quad (s \in \mathbb{Z})$$

defines a solution to the equations

$$\begin{cases} \overline{f}g = 1, \\ \overline{f}\overline{g} = q^2 t^2, \end{cases} \tag{2.12}$$

and a discrete solution to the q-$P(A_1)$ equation. Similarly, corresponding to the exceptional divisor D_2, we take any $t_0 \in \mathbb{C}^*$ and $f_0 \in \mathbb{C}^*$, and set

$$f_s = q^{s^2} t_0^{2s} f_0, \quad g_s = 1/f_s, \quad (s \in \mathbb{Z}) \tag{2.13}$$

which defines a solution to the equations

$$\begin{cases} \overline{f}g = qt^2, \\ \overline{f}\overline{g} = 1, \end{cases} \tag{2.14}$$

and a discrete solution to the q-$P(A_1)$ equation. We call these solutions *singular*. For all intent and purposes, they do not form intrinsically interesting solutions of the q-$P(A_1)$ equation. We often disregard them as they cause problems in the analytic analysis, especially when we consider the analytic aspects of Yamada's Lax pair in Chapter 4. Indeed Yamada's Lax pair (2.21) is singular on D_1 and D_2. We will therefore denote, by

$$\mathcal{S}_d^*(t_0) \cong \mathcal{S}_c^*(t_0),$$

the space of discrete solutions living on $q^{\mathbb{Z}} t_0$ excluding base solutions and singular solutions. We call its elements *regular solutions*. Let us write

$$\widetilde{\delta}_1 = \{(f,g) \in \mathbb{C}^2 : fg = t^2\} \setminus \{p_1, p_2, p_3, p_4\}, \tag{2.15a}$$

$$\widetilde{\delta}_2 = \{(f,g) \in \mathbb{C}^2 : fg = 1\} \setminus \{p_5, p_6, p_7, p_8\}. \tag{2.15b}$$

Note that a regular solution can hit a base point, but it can never hit the same base point twice consecutively, by Proposition 2.3.1. On the contrary, if we know that a solution takes a value in $\widetilde{\delta}_1$ or $\widetilde{\delta}_2$ at some time, then it is a singular solution. So regular solutions never hit the sets $\widetilde{\delta}_1$ and $\widetilde{\delta}_2$. Or described differently, Kajiwara et al. [52] call these sets "inaccessible".

2.3.2 Meromorphic Solutions

The interpretation of meromorphic solutions turns a difference equation into a functional equation. Halburg and Korhonen [29] show that the existence of meromorphic solutions defined on the entire complex plane is closely related to the integrability of d-difference equations. Barnett et al. [3] carry out a similar study for q-difference equations. Let us also mentions the work by Zheng and Chen [86] and Qi and Yang [73] on q-difference equations in this regard.

We are interested in a weaker notion of meromorphic solutions, namely those defined on continuous q-domains. To be precise, fix a continuous q-domain $T \subseteq \mathbb{C}^*$, then we call functions $f(t)$ and $g(t)$, or $(f(t), g(t))$, a meromorphic solution of q-$P(A_1)$, if they are meromorphic, not identically zero, and satisfy,

$$\begin{aligned}(gf - t^2)(g\overline{f} - qt^2)(g - b_5)(g - b_6)(g - b_7)(g - b_8) = \\ (gf - 1)(g\overline{f} - 1)(g - b_1t)(g - b_2t)(g - b_3t)(g - b_4t), \quad (2.16a)\end{aligned}$$

$$\begin{aligned}(g\overline{f} - qt^2)(\overline{g}\overline{f} - q^2t^2)(\overline{f} - b_5^{-1})(\overline{f} - b_6^{-1})(\overline{f} - b_7^{-1})(\overline{f} - b_8^{-1}) = \\ (g\overline{f} - 1)(\overline{g}\overline{f} - 1)(\overline{f} - b_1^{-1}qt)(\overline{f} - b_2^{-1}qt)(\overline{f} - b_3^{-1}qt)(\overline{f} - b_4^{-1}qt), \quad (2.16b)\end{aligned}$$

on T, which is just q-$P(A_1)$ rewritten slightly, where $f = f(t)$, $\overline{f} = f(qt)$ and so on as usual. We define the *meromorphic solution space* on T by

$$\mathcal{S}_m(T) = \{\text{meromorphic solutions of } q\text{-}P(A_1) \text{ with domain } T\}.$$

Note that for any $t_0 \in T$, we have a mapping $\pi(t_0)$ from $\mathcal{S}_m(T)$ to $\mathcal{S}_c(t_0)$, defined by restricting the domain. We say that $(f, g) \in \mathcal{S}_m(T)$ *covers*

$$\pi(t_0)\,[(f, g)] := (f(q^s t_0), g(q^s t_0))_{s \in \mathbb{Z}}.$$

This mapping allows us to immediately translate results obtained on the meromorphic level, to results on the discrete level. Which begs the question whether the meromorphic solutions cover all the discrete solutions.

Question 2.3.2. *Does there exist, for every $t_0 \in \mathbb{C}^*$, a continuous q-domain $T \subseteq \mathbb{C}^*$, such that $t_0 \in T$ and $\pi(t_0)$ is surjective?*

As in the discrete case, we call a meromorphic solution a *base solution*, if it equals one of the base points (2.4) identically. Quite interestingly, the only meromorphic solutions which cover discrete base solutions, are the meromorphic base solutions. Indeed we have the following result, closely related to Proposition 2.3.1, which, although easily proven, appears to be new.

Proposition 2.3.3. *Assume b_1, b_2, b_3, b_4 are mutually unequal and b_5, b_6, b_7, b_8 are mutually unequal. Let (f, g) be a meromorphic solution of q-$P(A_1)$, with continuous q-domain $T \subseteq \mathbb{C}^*$. If there is a $t_0 \in T$, with $t_0^2 \neq 1, q^{-1}, q^{-2}$, such that, for some $i \in \{1, 2, 3, 4\}$,*

$$(f(t_0), g(t_0)) = (t_0/b_i, b_i t_0), \quad (f(qt_0), g(qt_0)) = (qt_0/b_i, qb_i t_0),$$

then $(f(t), g(t)) \equiv (t/b_i, b_i t)$ on T, i.e. the solution is a base solution.
Similarly, if for some $j \in \{5, 6, 7, 8\}$,

$$(f(t_0), g(t_0)) = (1/b_j, b_j), \quad (f(qt_0), g(qt_0)) = (1/b_j, b_j),$$

then $(f(t), g(t)) \equiv (1/b_j, b_j)$ on T, i.e. the solution is a base solution.

Proof. This can be proven using just a power series method. Let $t_0 \in T$, and let us discuss the case $j = 5$, remarking that the other ones can be dealt with analogously. So we assume

$$(f(t_0), g(t_0)) = (1/b_5, b_5), \quad (f(qt_0), g(qt_0)) = (1/b_5, b_5).$$

As $f(t)$ and $g(t)$ are meromorphic at $t = t_0$ and $t = qt_0$, we have converging power series expansions

$$\begin{aligned} f(t) &= 1/b_5 + f_1(t-t_0) + f_2(t-t_0)^2 + \dots, \\ f(qt) &= 1/b_5 + \overline{f}_1(t-t_0) + \overline{f}_2(t-t_0)^2 + \dots, \\ g(t) &= b_5 + g_1(t-t_0) + g_2(t-t_0)^2 + \dots, \\ g(qt) &= b_5 + \overline{g}_1(t-t_0) + \overline{g}_2(t-t_0)^2 + \dots, \end{aligned}$$

about $t = t_0$. We substitute these expansions into (2.1), and compare coefficients of powers of $(t - t_0)$. The constant terms cancel out, and when we compare the terms of order one, we find

$$\begin{aligned} &(1-t_0^2)(1-qt_0^2)(b_5-b_6)(b_5-b_7)(b_5-b_8)g_1 = 0, \\ &(1-qt_0^2)(1-q^2t_0^2)(b_5^{-1}-b_6^{-1})(b_5^{-1}-b_7^{-1})(b_5^{-1}-b_8^{-1})\overline{f}_1 = 0. \end{aligned}$$

Therefore $g_1 = 0$ and $\overline{f}_1 = 0$. Comparing the terms of order two, we find exactly the same two equations, with g_1 and $\overline{f}_1$ replaced by g_2 and $\overline{f}_2$ respectively. By induction, we easily work out that $g_n = 0$ and $\overline{f}_n = 0$ for all $n \geq 1$. Therefore $g(t) \equiv b_5$ and $f(qt) \equiv 1/b_5$ on T. The theorem follows. □

Singular, Regular and Nowhere Singular Solutions

As in the discrete case, we call a meromorphic solution of q-$P(A_1)$ *singular*, if it satisfies either (2.12) or (2.14). Explicitly, the meromorphic solutions of (2.12) are given by

$$f(t) = c(t)\theta_q(t)^2, \quad g(t) = c(t)^{-1}t^2\theta_q(t)^{-2},$$

where $c(t)$ is any q-periodic and meromorphic function, and $\theta_q(t)$ is a q-*special function* which we introduce properly in Section 4.1, defined by equation (4.7), see also [18]. For now, all we require to know is that $\theta_q(t)$ is holomorphic on $\mathbb{C}^*$ and satisfies $\theta_q(qt) = t^{-1}\theta_q(t)$. Similarly the meromorphic solutions of (2.14), are given by

$$f(t) = c(t)t\theta_q(t)^{-2}, \quad g(t) = c(t)^{-1}t^{-1}\theta_q(t)^2,$$

where again $c(t)$ any meromorphic q-periodic function. These singular solutions often form problems in our analysis and we therefore mostly exclude them. By $\mathcal{S}_m^*(T)$ we denote the space of meromorphic solutions on T which are neither base solutions nor singular solutions. We call its elements *regular solutions*. Unfortunately, in contrast to base solutions, it is possible for a regular meromorphic solution to cover a discrete singular solution. We call a regular meromorphic solution *nowhere singular*, if it does not cover any discrete singular solutions. To be explicit, a regular meromorphic solution (f,g), is nowhere singular, if for any $t_0 \in \mathbb{C}^*$ in its domain, $(f(q^s t_0), g(q^s t_0))_{s\in\mathbb{Z}}$ is not a singular discrete solution. We write

$$\mathcal{S}_m^{**}(T) = \{(f,g) \in \mathcal{S}_m^*(T) : (f,g) \text{ is nowhere singular}\}, \tag{2.17}$$

for the space of nowhere singular meromorphic solutions on T. Let us emphasise that it is very well possible for a (nowhere singular) regular meromorphic solution, to hit a base point at some time $t = t_0 \in T$. However, by Proposition 2.3.3, it cannot hit the same base point at time $t = qt_0$ as well. On the contrary, if a regular meromorphic solution (f,g), assumes a value in $\widetilde{\delta}_1$ or $\widetilde{\delta}_2$, at some time $t = t_0$, then it never escapes D_1 or D_2 respectively on $q^{\mathbb{Z}} t_0$, i.e. it covers a singular discrete solution. Therefore, a nowhere singular meromorphic solution never hits the sets $\widetilde{\delta}_1$ or $\widetilde{\delta}_2$.

Let us note that the growth of the singular solutions is quite wild as $t \to 0$ or $t \to \infty$. Consider for instance the discrete singular solutions defined by (2.13), say with $t_0 = 1$. Then f_s goes to zero like q^{s^2}, whereas f_s grows like q^{-s^2} as $s \to \infty$. Comparing this with $t_s = q^s$, we see that g_s vanishes beyond all orders of t_s, whereas f_s grows beyond all orders of t_s, as $s \to \infty$. In the regular case, we do not expect such asymptotics.

2.4 Global Asymptotic Analysis

Fokas et al. [14] define the *global asymptotic analysis* of a (continuous) Painlevé equation as the study of *critical behaviour* of solutions near *critical points* and corresponding *connection problem* between different critical points. As an example, the critical points of the sixth Painlevé equation are 0, 1 and ∞, as these are the only points where a solution might fail to be meromorphic and hence branching might occur. Furthermore they describe a Painlevé equation as "*solved*", when we have complete knowledge of all critical behaviours near the different critical points, parameterised effectively, and explicit *connection formulae* in terms of the parameters involved, connecting these behaviours between any two critical points. We set out a q-analog for q-$P(A_1)$ of this perspective.

2.4.1 Critical Behaviour near Critical Points

Part of the global asymptotic analysis of the q-$P(A_1)$ equation is the study of critical behaviour near critical points, which, on itself, is essentially a local problem. Typically we would like to obtain a complete tabulation of different critical behaviours near a critical point. The only points in $\mathbb{P}$ which are invariant under the time evolution $t \mapsto qt$, are $t = 0$ and $t = \infty$. These are the only points where branching of solutions can occur and hence the *critical points* of q-$P(A_1)$. Let us first consider discrete solutions, say living on $q^{\mathbb{Z}} t_0$. We would like an explicit parameterisation of all possible critical behaviours near $t = 0$ and $t = \infty$. Following Guzzetti [28], we symbolically denote this, for $u \in \{0, \infty\}$, by

$$(f_s, g_s) = (f^u(t_s; c_1^u, c_2^u), g^u(t_s; c_1^u, c_2^u)), \quad t_s = q^s t_0, \qquad (t_s \to u) \tag{2.18}$$

where c_1^u, c_2^u are complex *integration constants*, and we wrote $t_s \to 0$ for $s \to \infty$ and $t_s \to \infty$ for $s \to -\infty$. We consider our parametrisation, or tabulation, *complete*, if for every solution

$(f_s, g_s)_{s\in\mathbb{Z}} \in \mathcal{S}_d^*(t_0)$, there are unique c_1^u, c_2^u such that (2.18) holds for $u \in \{0, \infty\}$. We remark that this should not be confused with the notion of completeness of the solution space, as studied in for instance [12], for P_I.

Similarly, for meromorphic solutions on some fixed continuous q-domain T, we symbolically write, for $u \in \{0, \infty\}$,

$$(f(t), g(t)) = (f^u(t; c_1^u(t), c_2^u(t)), g^u(t; c_1^u(t), c_2^u(t))), \qquad (t \to u \text{ in } T) \tag{2.19}$$

where $c_1^u(t), c_2^u(t)$ are now *q-integration constants*, i.e. q-periodic functions on T. Again *completeness* entails that our parameterisation covers every element of $\mathcal{S}_m^{**}(T)$. The analysis of critical behaviour near $t = 0$ and $t = \infty$ is the subject of Chapter 3.

2.4.2 The q-$P(A_1)$ Connection Problem

Getting back to the parameterisation (2.18), if it is complete, then for any element $(f_s, g_s)_{s\in\mathbb{Z}} \in \mathcal{S}_d^*(t_0)$, there exist unique c_1^0, c_2^0 and c_1^∞, c_2^∞ such that

$$(f_s, g_s) = \begin{cases} (f^0(t_s; c_1^0, c_2^0), g^0(t_s; c_1^0, c_2^0)), & (t_s \to 0) \\ (f^\infty(t_s; c_1^\infty, c_2^\infty), g^\infty(t_s; c_1^\infty, c_2^\infty)), & (t_s \to \infty) \end{cases}$$

giving rise to the q-$P(A_1)$ *connection problem*, which constitutes determining explicit formulae

$$\begin{cases} c_1^0 = c_1^0(c_1^\infty, c_2^\infty), \\ c_2^0 = c_2^0(c_1^\infty, c_2^\infty), \end{cases} \qquad \begin{cases} c_1^\infty = c_1^\infty(c_1^0, c_2^0), \\ c_2^\infty = c_2^\infty(c_1^0, c_2^0), \end{cases} \tag{2.20}$$

which are called, using the terminology in [28], *connection formulae in closed form.*

Of course there is a natural analog for the meromorphic case. Note that there is no principal objection, in completing the local analysis of behaviour of solutions near critical points, for nonlinear equations. However, generically speaking, there is no hope in solving the connection problem for such equations. Indeed, even for linear equations, the Riemann-Hilbert correspondence is a transcendental one. It is exactly the *integrability* of the q-$P(A_1)$ equation, in this case the existence of a Lax pair, which gives us a technique to solve the q-$P(A_1)$ connection problem.

2.4.3 Yamada's Lax pair

Yamada [85] derived the following *Lax pair* for q-$P(A_1)$,

$$L_1: \quad u(z,t)y(qz,t) + v(z,t)y(z,t) + w(z,t)y(z/q,t) = 0, \tag{2.21a}$$

$$L_2: \quad h_0(z,t)y(z,qt) + h_1(z,t)y(z,t) + h_2(z,t)y(z/q,t) = 0, \tag{2.21b}$$

where the coefficients in L_1, the *spectral equation*, are given by

$$
\begin{aligned}
u(z,t) =& p_2(qz)(1-z/f),\\
w(z,t) =& qp_1(z/t)(1-qz/f),\\
v(z,t) =& \frac{1-z/f}{1-qgz}\left[q^2g^3t^{-2}(t^2-1)\frac{f-qz}{fg-1}z^2p_2(1/g)-(1-qgz/t^2)p_2(qz)\right]+\\
& \frac{1-qz/f}{1-gz/t^2}\left[qg^3t^{-4}(1-t^2)\frac{f-z}{fg-t^2}z^2p_1(t/g)-q(1-gz)p_1(z/t)\right],
\end{aligned}
$$

with the polynomials p_1 and p_2 as defined in equations (2.2) and (2.3) respectively. The coefficients in L_2, called the *deformation equation*, are given by

$$h_0(z,t)=qt^2gz(f-z),\quad h_1(z,t)=(gz-1)t^2,\quad h_2(z,t)=t^2-gz.$$

We call z the *spectral variable* and t the *time* or *Painlevé variable*. The crucial property of the Lax pair (2.21), is that the *compatibility* of the spectral and deformation equation, is equivalent to (f,g) satisfying the q-$P(A_1)$ equation. To illustrate this point, let us recast Yamada's Lax pair into system form by setting

$$Y(z,t)=\begin{pmatrix} y(z,t)\\ y(z/q,t)\end{pmatrix},$$

which gives

$$Y(qz,t)=A(z,t;f,g)Y(z,t), \tag{2.22a}$$
$$Y(z,qt)=H(z,t;f,g)Y(z,t), \tag{2.22b}$$

with

$$A(z,t)=\begin{pmatrix}-\frac{v(z)}{u(z)} & -\frac{w(z)}{u(z)}\\ 1 & 0\end{pmatrix},\qquad H(z,t)=\begin{pmatrix}-\frac{h_1(z)}{h_0(z)} & -\frac{h_2(z)}{h_0(z)}\\ \frac{u(z/q)h_2(z/q)}{w(z/q)h_0(z/q)} & \frac{v(z/q)h_2(z/q)-w(z/q)h_1(z/q)}{w(z/q)h_0(z/q)}\end{pmatrix},$$

where we suppressed the (f,g) dependence throughout.
Now assume we have a fundamental solution $Y(z,t)$ of (2.22), then

$$
\begin{aligned}
Y(qz,qt)&=A(z,qt;\overline{f},\overline{g})Y(z,qt)=A(z,qt;\overline{f},\overline{g})H(z,t;f,g)Y(z,t),\\
Y(qz,qt)&=H(qz,t;f,g)Y(qz,t)=H(qz,t;f,g)A(z,t;f,g)Y(z,t),
\end{aligned}
$$

which yields the compatibility condition

$$A(z,qt;\overline{f},\overline{g})H(z,t;f,g)=H(qz,t;f,g)A(z,t;f,g). \tag{2.23}$$

Theorem 2.4.1. *The q-$P(A_1)$ equation, interpreted as a system of algebraic relations between points (t,f,g) and $(qt,\overline{f},\overline{g})$, is equivalent to the consistency condition of (2.22), given by equation (2.23).*

Proof. See Yamada [85]. □

2.4.4 Isomonodromic Deformation

We give a rough sketch of how an isomonodromic deformation method can be made effective to solve the q-$P(A_1)$ connection problem. We emphasise that this is just a sketch, and many aspects have been oversimplified. Firstly note that the spectral equation $L_1 = L_1(t, f, g)$ constitutes a second order linear q-difference equation in the spectral variable, with the Painlevé variables t, f, g entering the coefficients. We now think of the time evolution of q-$P(A_1)$ as a deformation of the spectral equation. We can associate to the spectral equation L_1, its *monodromy* $m(L_1)$, which is the connection matrix, relating canonical solutions of the spectral equation near $z = 0$ and $z = \infty$, discussed properly in Chapter 4. Now the crucial observation is, that the monodromy of the spectral equation is preserved by the q-$P(A_1)$ deformation, i.e.

$$m(L(qt, \overline{f}, \overline{g})) = m(L(t, f, g)).$$

Hence the q-$P(A_1)$ deformation is called an *isomonodromic deformation*, and we can construct the *monodromy mapping*

$$M : \mathcal{S}_c^*(t_0) \to \mathcal{M}, (f, g) \to m(L(\text{-}, f(\text{-}), g(\text{-}))), \tag{2.24}$$

where $\mathcal{M}$ denotes the *monodromy space*, and "-" indicates one can take any time one pleases. Now let us get back to the q-$P(A_1)$ connection problem, described in Section 2.4.2. Using the notation in Section 2.4.2, a method of attack to solve this problem, is to find explicit formulae

$$M(f, g) = M(c_1^0, c_2^0), \quad M(f, g) = M(c_1^\infty, c_2^\infty), \tag{2.25}$$

which when combined, lead to

$$M(c_1^0, c_2^0) = M(c_1^\infty, c_2^\infty), \tag{2.26}$$

which we call *parametric connection formulae*, following Guzzetti [28]. Determining formulae (2.25), requires analysing Yamada's Lax pair in the limits $t \to 0$ and $t \to \infty$, which is the subject of Chapter 5. Using (2.26), one should be able to derive connection formulae in closed form (2.20).

2.4.5 Further Directions

Finer aspects of the global asymptotic analysis of a discrete Painlevé equation, include studying the distribution of zeros, poles, base points, and other special points of solutions on q-domains. Furthermore special solutions, such as algebraic solutions, classical solutions and rational solutions, should take a special role within this framework. As an example, note that the set $S_m^{**}(\mathbb{P})$ consists exactly of all the rational solutions, for given parameter values. An isomonodromic deformation method might also be an effective tool to classify the algebraic solutions of q-$P(A_1)$. Indeed such an approach has been made successful for the sixth Painlevé equation with $\theta_x = \theta_y = \theta_z = 0$, by Dubrovin and Mazzocco [11], and later on for general parameter values by Lisovyy and Tykhyy [59].

2.5 The Symmetric Form

There are some natural conditions on the parameters $\mathbf{b}$ wich allow a reduction of q-$P(A_1)$ to its symmetric form,

$$\frac{\left(x\hat{x}-\xi t^2\right)\left(x\underset{\smile}{x}-\xi^{-1}t^2\right)}{\left(x\hat{x}-1\right)\left(x\underset{\smile}{x}-1\right)}=\frac{\left(x-at\right)\left(x-a^{-1}t\right)\left(x-bt\right)\left(x-b^{-1}t\right)}{\left(x-c\right)\left(x-c^{-1}\right)\left(x-d\right)\left(x-d^{-1}\right)}, \tag{2.27}$$

where $a,b,c,d\in\mathbb{C}^*$ are complex parameters and $\xi\in\mathbb{C}^*$ defines the time evolution,

$$\hat{t}=\xi t,\qquad x=x(t),\qquad \hat{x}=x(\xi t),\qquad \underset{\smile}{x}=x\left(\xi^{-1}t\right).$$

We write $\mathbf{b}_s=(a,b,c,d)$ and denote the parameter space of symmetric q-$P(A_1)$ by

$$\mathcal{B}_s=\{(a,b,c,d)\in\mathbb{C}^4|a,b,c,d\neq 0\}.$$

Considering q-$P(A_1)$, let $\xi^2=q$ and assume that the parameters $\mathbf{b}$ satisfy

$$b_1b_2=\xi,\qquad b_3b_4=\xi,\qquad b_5b_6=1,\qquad b_7b_8=1. \tag{2.28}$$

Let us write

$$a=b_1\xi^{-\frac{1}{2}},\qquad b=b_3\xi^{-\frac{1}{2}},\qquad c=b_5,\qquad d=b_7,$$

then q-$P(A_1)$ takes the form

$$\begin{aligned}\frac{(fg-t^2)(\overline{f}g-qt^2)}{(fg-1)(\overline{f}g-1)}&=\frac{(g-a\xi^{\frac{1}{2}}t)(g-a^{-1}\xi^{\frac{1}{2}}t)(g-b\xi^{\frac{1}{2}}t)(g-b^{-1}\xi^{\frac{1}{2}}t)}{(g-c)(g-c^{-1})(g-d)(g-d^{-1})},\\ \frac{(\overline{f}g-qt^2)(\overline{f}\overline{g}-q^2t^2)}{(\overline{f}g-1)(\overline{f}\overline{g}-1)}&=\frac{(\overline{f}-a\xi^{\frac{3}{2}}t)(\overline{f}-a^{-1}\xi^{\frac{3}{2}}t)(\overline{f}-b\xi^{\frac{3}{2}}t)(\overline{f}-b^{-1}\xi^{\frac{3}{2}}t)}{(\overline{f}-c)(\overline{f}-c^{-1})(\overline{f}-d)(\overline{f}-d^{-1})},\end{aligned}$$

Hence, for any solution $(f(t),g(t))$ of q-$P(A_1)$ on a discrete q-domain $T=q^{\mathbb{Z}}t_0$,

$$x(\xi^{2n}t_0')=f(q^nt_0),\quad x(\xi^{2n+1}t_0')=g(q^nt_0),\qquad (n\in\mathbb{Z})$$

defines a solution of symmetric q-$P(A_1)$, on the discrete ξ-domain $\widehat{T}:=\xi^{\mathbb{Z}}t_0'$ with $t_0'=\xi^{-\frac{1}{2}}t_0$. On a continuous q-domain things are a bit more delicate. Indeed, suppose $(f(t),g(t))$ is a meromorphic solution of q-$P(A_1)$ on a continuous ξ-domain T. Then

$$x(t)=f(\xi^{\frac{1}{2}}t), \tag{2.29}$$

defines a solution of symmetric q-$P(A_1)$ on $\widehat{T}=\xi^{-\frac{1}{2}}T$, if (f,g) satisfies the *symmetry condition*

$$f\left(t\right)=g\left(\xi^{-1}t\right).\qquad (t\in T) \tag{2.30}$$

Note that Bäcklund transformation $\mathcal{T}_3$ leaves q-$P(A_1)(\mathbf{b})$ invariant for this specific choice of parameters, i.e.

$$f^{(3)}(t) = g(\xi^{-1}t), \qquad g^{(3)}(t) = f(\xi t),$$

also defines a meromorphic solution of q-$P(A_1)(\mathbf{b})$ on the connected ξ-domain T. Condition (2.29) is equivalent to $\mathcal{T}_3$ acting trivial on (f, g), i.e. $(f^{(3)}, g^{(3)}) = (f, g)$. In Section 3.8 we find that the 2.30 appears naturally in the asymptotic description of meromorphic solutions.

2.5.1 A Continuum Limit

Grammaticos and Ramani [20] calculated the following formal *continuum limit* of the symmetric q-$P(A_1)$ equation (2.27) to the sixth Painlevé equation. We set

$$a = -\xi^{\alpha}, \qquad b = \xi^{\beta}, \qquad c = -\xi^{\gamma}, \qquad d = \xi^{\delta}, \tag{2.31}$$

for some fixed $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Then, by letting $\xi \to 1$, symmetric q-$P(A_1)$ becomes

$$\begin{aligned} x_0'' =& \tfrac{1}{2}\left(\frac{1}{x_0+1} + \frac{1}{x_0-1} + \frac{1}{x_0+t} + \frac{1}{x_0-t}\right) x_0'^2 \\ &- \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{t+1} + \frac{1}{x_0-t} - \frac{1}{x_0+t}\right) x_0' \\ &+ \frac{(x_0^2-t^2)(x_0^2-1)}{t^2(t^2-1)} \left(\frac{(\alpha^2-\frac{1}{4})t}{(x_0+t)^2} - \frac{(\beta^2-\frac{1}{4})t}{(x_0-t)^2} - \frac{\gamma^2}{(x_0+1)^2} + \frac{\delta^2}{(x_0-1)^2}\right), \end{aligned} \tag{2.32}$$

which is a non-canonical form of the sixth Painlevé equation, which we refer to as alt-P_{VI}. Indeed, the change of variables

$$t = \frac{1-r}{1+r}, \qquad r^2 = \zeta, \qquad x_0 = \frac{r-w}{r+w}, \tag{2.33}$$

gives P_{VI} in canonical form (1.1), with parameter values

$$\theta_x = \delta, \quad \theta_y = \alpha, \quad \theta_z = \beta, \quad \theta_\infty = \gamma + 1. \tag{2.34}$$

The above continuum limit is the result of a formal calculation, based on presumed expansions

$$x(t;\xi) = x_0(t) + (\xi-1)x_1(t) + (\xi-1)^2 x_2(t) + \mathcal{O}((\xi-1)^3), \tag{2.35}$$

$$\begin{aligned} x(\xi t;\xi) =& x_0(t) + (\xi-1)[x_1(t) + t x_0'(t)] + (\xi-1)^2[x_2(t) + t x_1'(t) + \tfrac{1}{2}t^2 x_0''(t)] \\ &+ \mathcal{O}((\xi-1)^3), \end{aligned} \tag{2.36}$$

$$\begin{aligned} x(\xi^{-1} t;\xi) =& x_0(t) + (\xi-1)[x_1(t) - t x_0'(t)] + (\xi-1)^2[x_2(t) - t x_1'(t) + \tfrac{1}{2}t^2 x_0''(t) \\ &+ t x_0'(t)] + \mathcal{O}((\xi-1)^3), \end{aligned} \tag{2.37}$$

as $\xi \to 1$. Indeed upon substitution in symmetric q-$P(A_1)$, after multiplying out the denominators on both sides, one easily finds that the constant terms, as well as the $(\xi-1)^1$ terms, cancel out. Considering the $(\xi-1)^2$ terms, we find an equation involving only x_0, equivalent to (2.32), after some cancellation. We infer the following result.

Lemma 2.5.1. *Let $x(t;\xi)$ denote a family of meromorphic solutions of symmetric q-$P(A_1)$, with parameter values (2.31), on continuous ξ-domains $T(\xi)$, for $\xi \in Q$, where $Q \subseteq \{z \in \mathbb{C}^* : |z| < 1\}$ such that $1 \in \overline{Q}$. Given a non-empty open set $D \subseteq \mathbb{C}^*$ in the t-plane such that $D \subseteq T(\xi)$ and $x(t;\xi)$ analytic in t on D, for $\xi \in Q$ close to 1, and (2.35) holding uniformly in t on D as $\xi \to 1$ in Q, for some complex functions x_0, x_1, x_2 on D. Then*

$$x_0(t) = \lim_{\xi \to 1, \xi \in Q} x(t;\xi),$$

is an analytic solution of alt-P_{VI} (2.32) *on D.*

Proof. This follows from elementary complex analysis. Indeed note that, as (2.35) holds uniformly in t on D as $\xi \to 1$ in Q, we immediately obtain that x_0, x_1 and x_2 are analytic on D. Next, for any bounded open set $U \subseteq \overline{U} \subseteq D$, we can easily derive that (2.36) and (2.37) hold uniformly in t on U as $\xi \to 1$ in Q. The lemma follows. □

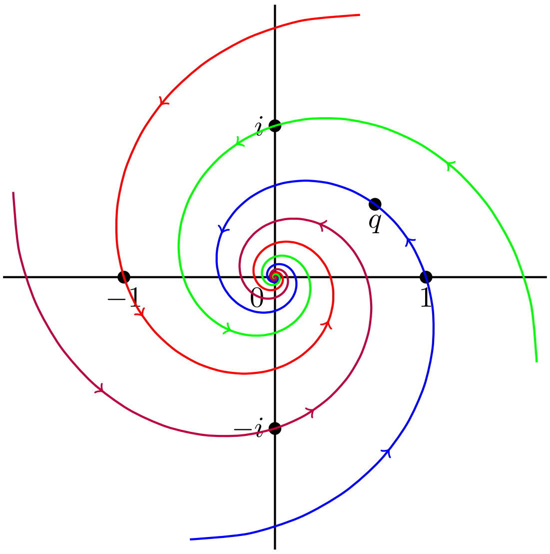
To get some intuition on what is happening in the continuum limit, we keep track of four particular q-spirals in the t-plane, depicted in Figure 2.3a. As we are more used to working with q instead of ξ, we set $q = \xi$ temporarily. Recall that we assume $|q| < 1$ and, defining q^s by the principal branch for $s \in \mathbb{R}$, we consider the q-spirals:

- $s_1^q = q^{\mathbb{R}}$, depicted in **blue**;
- $s_2^q = -q^{\mathbb{R}}$, depicted in **red**;
- $s_3^q = iq^{\mathbb{R}}$, depicted in **green**;
- $s_4^q = -iq^{\mathbb{R}}$, depicted in **purple**;

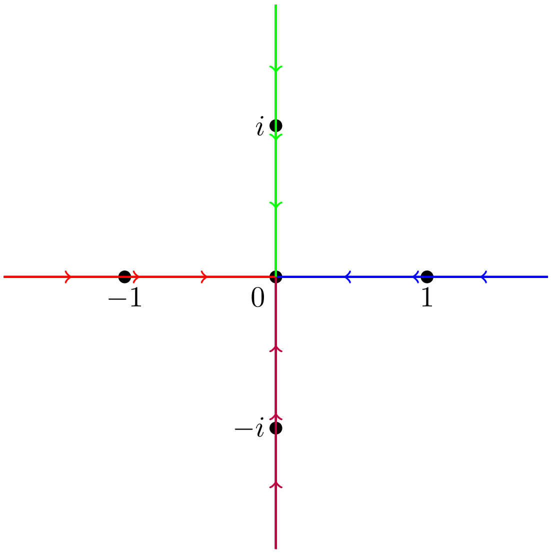
where we think of s_2^q as a hypothetical branch cut, and the arrows on the spirals indicate the direction of the time evolution $t \mapsto qt$, running from $t = \infty$ to $t = 0$.

From Figure 2.3a, it is clear that the hypothetical branch cut, in the continuum limit $q \to 1$, depends very strongly on the angle at which q approaches 1 in the complex plane. Indeed, fix a nonzero q_* in the (open) unit disc and consider the corresponding picture (2.3a). Now say we let q vary along $q_*^{\mathbb{R}}$ on the inside of the unit disc, then the q-spirals in (2.3a) remain completely invariant. In particular letting q approach 1 along $q_*^{\mathbb{R}}$, the resulting t-plane still has the spiral $-q_*^{\mathbb{R}}$ as a hypothetical branch cut. As such branch cuts are highly non-standard in the study of complex differential equations, we only consider continuum limits in which q approaches 1 tangentially to the real axis in the unit disc. We denote such a limit by $q \to 1^-$. However, we would like to note that there is principally nothing wrong with continuum limits where q approaches 1 from a different angle. In this context we would particularly like to mention Sauloy's thesis [78], in which he works out such continuum limits rigorously for Fuchsian linear q-difference equations, including their monodromy.

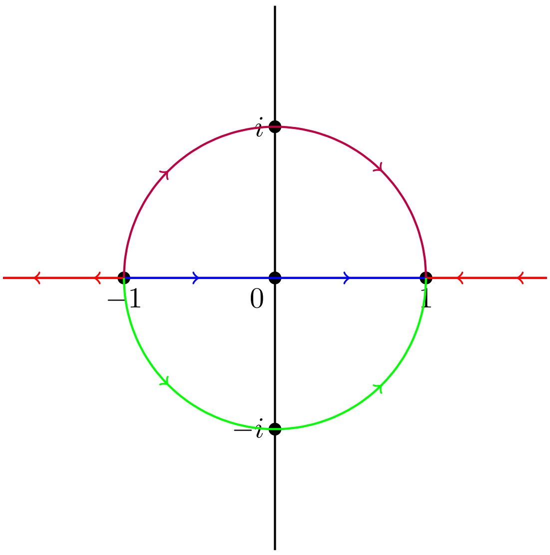
Note that, in Lemma 2.5.1, we did not specify any angle at which q approaches 1 in the continuum limit. However this lemma only deals with the local problem of convergence away from critical points.



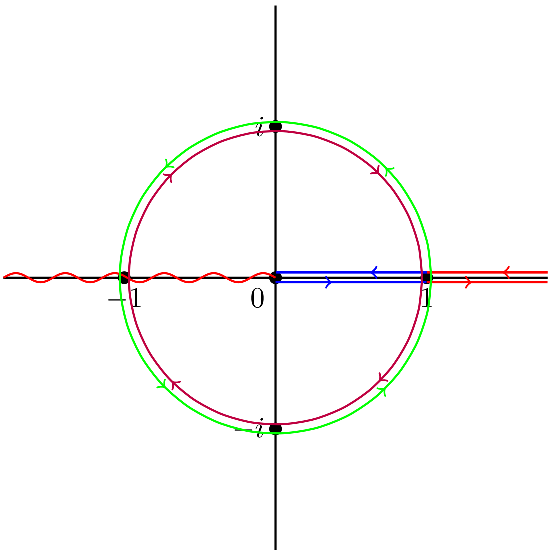
(a) complex t-plane



(b) complex t-plane, after $q \to 1^-$



(c) complex r-plane



(d) complex ζ-plane

Figure 2.3: Continuum Limit in Pictures

From now on we consider the case where $q \to 1^-$, i.e. q approaches 1 tangential to the real axis from within the unit disc. Under such a limit the four q-spirals $s_1^q, s_2^q, s_3^q, s_4^q$, depicted in Figure 2.3a, are stretched out to the four open half-axes

$$s_1^1 = (\infty, 0), \quad s_2^1 = (-\infty, 0), \quad s_3^1 = (i\infty, 0), \quad s_4^1 = (-i\infty, 0),$$

of the t-plane respectively, as depicted in Figure 2.3b, where we kept track of the direction of time. Furthermore, note that under this limit, two base points of symmetric q-$P(A_1)$ merge to the base point $x_0 = 1$ of alt-P_{VI}, and similarly two base points merge to $x_0 = -1$, two base points merge to $x_0 = t$ and the remaining two base points merge to $x_0 = -t$. The critical points $t = 0$ and $t = \infty$ remain critical points of alt-P_{VI}, plus two new critical points $t = 1$ and $t = -1$ are formed as $q \to 1^-$.

Now we consider the change of variables $(x_0, t) \to (\omega, r)$ defined in (2.33). Both the change of the dependent and independent variable are via a Möbius transformation and hence automorphism of $\mathbb{P}$. Firstly, the half-axes $s_1^1, s_2^1, s_3^1, s_4^1$ are send to paths

$$s_1^r = (-1, 1), \quad s_2^r = (-1, -\infty] \cup [\infty, 1), \quad s_3^r = \{-e^{\pi i\theta} : -1 < \theta < 1\}, \quad s_4^r = \{e^{\pi i\theta} : -1 < \theta < 1\},$$

respectively, as shown in Figure 2.3c. Furthermore the base points $x_0 = 1$, $x_0 = -1$, $x_0 = t$ and $x_0 = -t$ of alt-P_{VI}, are send to the well known base points $\omega = 0$, $\omega = \infty$, $\omega = r^2 = \zeta$ and $\omega = 1$ of the sixth Painlevé equation respectively. Similarly the critical points $t = 0$, $t = \infty$, $t = -1$ and $t = 1$ of alt-P_{VI} are send to $r = 1$, $r = -1$, $r = \infty$ and $r = 0$ respectively.

Finally $\zeta = r^2$ gives the well-known critical points $\zeta = 0$, $\zeta = 1$ and $\zeta = \infty$ of P_{VI}. Note however, that $r = 1$ and $r = -1$ correspond to $\zeta = 1$ in different sheets of the universal covering space of $\mathbb{P} \setminus \{0, 1, \infty\}$. Indeed the change of variables $\zeta = r^2$ forces us to choose a branch cut, which we set equal to the negative real axis $(-\infty, 0)$ in the ζ-plane, see the oscillating red line in Figure 2.3d. Also the hypothetical branch cut s_2^r becomes $(1, \infty]$ in the ζ-plane. Finally the paths s_3^r and s_4^r with starting point $r = -1$ and ending point $r = 1$ respectively, both become loops tracing out the unit circle in the ζ-plane, starting and finishing at $\zeta = 1$, going around $\zeta = 0$ once in anti-clockwise and clockwise direction respectively.

Let us reflect on the global asymptotic analysis of q-$P(A_1)$, as set out in Section 2.4, from this perspective. Firstly, as $q \to 1^-$, the local classification of both critical behaviour of solutions near $t = 0$ and $t = \infty$, should coincide with that of the sixth Painlevé equation near $\zeta = 1$, after the relevant change of variables. Furthermore the q-$P(A_1)$ connection problem should reduce to the connection problem of P_{VI}, on relating critical behaviour near $\zeta = 1$ of solutions, in different sheets of the universal covering space of $\mathbb{P} \setminus \{0, 1, \infty\}$, related by simple loops around $\zeta = 0$.

We conclude with a summary of this chapter, in which we have discussed all the basic analytic and algebro-geometric aspects of the q-$P(A_1)$ equation. Using the singularity confinement property and Sakai's theory, we defined the notion of solutions of the q-$P(A_1)$ equation, and we saw that local discrete solutions can be uniquely continued on discrete q-domains, and local meromorphic solutions can be uniquely continued meromorphically on continuous q-domains. We formulated what constitutes the global asymptotic analysis of the q-$P(A_1)$ equation, involving in particular classifying critical behaviours of solutions near the origin and infinity, and the corresponding q-$P(A_1)$ connection problem, relating those critical behaviours. We finished the chapter with the symmetric form of the q-$P(A_1)$ equation, and a heuristic discussion of its continuum limit to P_{VI}.

CHAPTER 3

Local Behaviour of Solutions Near Critical Points

In this chapter we study the *local behaviour* of q-$P(A_1)$ transcendents near the critical points $t=0$ and $t=\infty$. We do this by a typical *local asymptotic analysis* of differential or difference equations. Firstly we study the leading order behaviour of solutions of q-$P(A_1)$, and we find that it is characterised by an autonomous system. We derive the general solution of this system and subsequently formally calculate the full asymptotic expansion of the proposedly corresponding q-$P(A_1)$ transcendent, which contains the freedom of two q-constants. Finally we show that this full expansion is always convergent, for suitably chosen q-constants. But before going down this path, we warm up by considering solutions of q-$P(A_1)$, described by very simple behaviour near critical points. Most of this chapter is published in Joshi and Roffelsen [50].

3.1 Solutions Which Are Meromorphic at a Critical Point

In the language of Section 2.3, we classify the solutions spaces $\mathcal{S}_m^{**}(\mathbb{C})$ and $\mathcal{S}_m^{**}(\mathbb{P}^*)$ in this section. Let us start by studying solutions which are holomorphic at the origin. These solutions play a special role in the more general solution we derive later, as they correspond to constant solutions of the *leading order autonomous system* (3.22). We classify the holomorphic solutions using the power series method in combination with the q-Briot-Bouquet Theorem B.3 to prove convergence.

We note that Ohyama [64, 65] classified the meromorphic solutions of the discrete Painlevé equations q-P_{VI}, q-P_{V} and q-P_{III} around the origin in this fashion. We consider q-$P(A_1)$, rewritten as in equations (2.16). Suppose we have a power series solution around $t=0$, say

$$f(t)=\sum_{n=0}^{\infty} f_n t^n, \qquad g(t)=\sum_{n=0}^{\infty} g_n t^n.$$

Evaluating equation (2.16) at $t = 0$ gives

$$(f_0 g_0 - 1)^2 f_0^4 = f_0^2 g_0^2 (f_0 - b_5^{-1})(f_0 - b_6^{-1})(f_0 - b_7^{-1})(f_0 - b_8^{-1}), \tag{3.1a}$$

$$(f_0 g_0 - 1)^2 g_0^4 = f_0^2 g_0^2 (g_0 - b_5)(g_0 - b_6)(g_0 - b_7)(g_0 - b_8). \tag{3.1b}$$

Equation (3.1) has several trivial solutions, given by $(f_0, g_0) = (0,0)$ and $(f_0, g_0) = (b_i^{-1}, b_i)$ for $i = 5,6,7,8$. Furthermore there are generally three nontrivial solutions, given by

$$\left(f_0^{(0,1)}, g_0^{(0,1)}\right) = \left(\frac{b_5 b_6 - b_7 b_8}{b_5 b_6 (b_7 + b_8) - b_7 b_8 (b_5 + b_6)}, \frac{b_5 b_6 - b_7 b_8}{b_5 + b_6 - (b_7 + b_8)}\right), \tag{3.2a}$$

$$\left(f_0^{(0,2)}, g_0^{(0,2)}\right) = \left(\frac{b_6 b_7 - b_8 b_5}{b_6 b_7 (b_8 + b_5) - b_8 b_5 (b_6 + b_7)}, \frac{b_6 b_7 - b_8 b_5}{b_6 + b_7 - (b_8 + b_5)}\right), \tag{3.2b}$$

$$\left(f_0^{(0,3)}, g_0^{(0,3)}\right) = \left(\frac{b_5 b_7 - b_6 b_8}{b_5 b_7 (b_6 + b_8) - b_6 b_8 (b_5 + b_7)}, \frac{b_5 b_7 - b_6 b_8}{b_5 + b_7 - (b_6 + b_8)}\right). \tag{3.2c}$$

If $(f_0, g_0) = (0,0)$, then there are no terms t^n with $n < 4$ appearing in (2.16), equating the coefficients of t^4 in (2.16) gives

$$(f_1 g_1 - 1)^2 b_1 b_2 b_3 b_4 = (g_1 - b_1)(g_1 - b_2)(g_1 - b_3)(g_1 - b_4), \tag{3.3a}$$

$$(f_1 g_1 - 1)^2 \frac{1}{b_1 b_2 b_3 b_4} = (f_1 - b_1^{-1})(f_1 - b_2^{-1})(f_1 - b_3^{-1})(f_1 - b_4^{-1}). \tag{3.3b}$$

Equation (3.3) has several trivial solutions, given by $(f_1, g_1) = (0,0)$ and $(f_1, g_1) = (b_i^{-1}, b_i)$ for $i = 1,2,3,4$. Furthermore there are generally three nontrivial solutions, given by

$$\left(f_1^{(1,1)}, g_1^{(1,1)}\right) = \left(\frac{b_1 + b_2 - (b_3 + b_4)}{b_1 b_2 - b_3 b_4}, \frac{b_1 b_2 (b_3 + b_4) - b_3 b_4 (b_1 + b_2)}{b_1 b_2 - b_3 b_4}\right), \tag{3.4a}$$

$$\left(f_1^{(1,2)}, g_1^{(1,2)}\right) = \left(\frac{b_2 + b_3 - (b_4 + b_1)}{b_2 b_3 - b_4 b_1}, \frac{b_2 b_3 (b_4 + b_1) - b_4 b_1 (b_2 + b_3)}{b_2 b_3 - b_4 b_1}\right), \tag{3.4b}$$

$$\left(f_1^{(1,3)}, g_1^{(1,3)}\right) = \left(\frac{b_1 + b_3 - (b_2 + b_4)}{b_1 b_3 - b_2 b_4}, \frac{b_1 b_3 (b_2 + b_4) - b_2 b_4 (b_1 + b_3)}{b_1 b_3 - b_2 b_4}\right). \tag{3.4c}$$

Each of the cases in equations (3.2) and (3.4) generically determines an unique converging power series solution.

Proposition 3.1.1. *For $k \in \{1,2,3\}$, the q-$P(A_1)$ equation has an unique power series solution*

$$f^{(0,k)}(t) = \sum_{n=0}^{\infty} f_n^{(0,k)} t^n, \qquad g^{(0,k)}(t) = \sum_{n=0}^{\infty} g_n^{(0,k)} t^n, \tag{3.5}$$

with $f_1^{(0,k)}$ and $g_1^{(0,k)}$ as defined in equation (3.2), given that the following conditions are *satisfied for the case $k=1$, $k=2$ and $k=3$ respectively,*

$$\frac{b_5 b_6}{b_7 b_8} \notin q^{\mathbb{Z}}, \qquad b_5+b_6 \neq b_7+b_8 \qquad \textit{and} \qquad b_5^{-1}+b_6^{-1} \neq b_7^{-1}+b_8^{-1}, \tag{3.6}$$

$$\frac{b_6 b_7}{b_5 b_8} \notin q^{\mathbb{Z}}, \qquad b_6+b_7 \neq b_5+b_8 \qquad \textit{and} \qquad b_6^{-1}+b_7^{-1} \neq b_5^{-1}+b_8^{-1}, \tag{3.7}$$

$$\frac{b_5 b_7}{b_6 b_8} \notin q^{\mathbb{Z}}, \qquad b_5+b_7 \neq b_6+b_8 \qquad \textit{and} \qquad b_5^{-1}+b_7^{-1} \neq b_6^{-1}+b_8^{-1}. \tag{3.8}$$

Furthermore, each of these power series solutions has a positive radius of convergence and an unique meromorphic continuation to $\mathbb{C}$.

Proof. We discuss the case $k=1$. Let us first note that the assumptions made ensure that $f_0^{(1)}$ and $g_0^{(1)}$ are well-defined and non-zero. In accordance with the notation in the q-Briot-Bouquet theorem B.3, we rewrite q-$P(A_1)$ as

$$\overline{f} = H_1(t,f,g), \qquad \overline{g} = H_2(t,f,g), \tag{3.9}$$

for some rational functions H_1 and H_2.
We apply the q-Briot-Bouquet theorem B.3 with $m=1$ and $n=2$ to this system, where $y_1=f$, $y_2=g$ and

$$\mathbf{Y} = (f_0,g_0) = \left(f_0^{(1)}, g_0^{(1)}\right).$$

It is not hard to see that $H(t,f,g)$ is holomorphic at $(t,f,g)=(0,f_0,g_0)$ and $H(0,f_0,g_0) = (f_0,g_0)$, as this is essentially the calculation done to obtain the case (3.2a). To establish this it is helpful to think of H as the composition of $R(t,f,g) := R(t)(f,g)$ and $S(t,\overline{f},g) := S(t)(\overline{f},g)$ as defined in Section 2.2, i.e.

$$H_1(t,f,g) = R_1(t,f,g) = S_1(t,R(t,f,g)), \quad H_2(t,f,g) = S_2(t,R_1(t,f,g),g) = S_2(t,R(t,f,g)).$$

In particular this is helpful to calculate the Jacobian, using the chain rule,

$$\begin{aligned}\begin{pmatrix} \frac{\partial H_1}{\partial f}(0,\mathbf{Y}) & \frac{\partial H_1}{\partial g}(0,\mathbf{Y}) \\ \frac{\partial H_2}{\partial f}(0,\mathbf{Y}) & \frac{\partial H_2}{\partial g}(0,\mathbf{Y}) \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ \frac{\partial S_2}{\partial \overline{f}}(0,\mathbf{Y}) & \frac{\partial S_2}{\partial g}(0,\mathbf{Y}) \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial R_1}{\partial f}(0,\mathbf{Y}) & \frac{\partial R_1}{\partial g}(0,\mathbf{Y}) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -\frac{(b_5+b_6-b_7-b_8)^2(b_5b_6+b_7b_8)}{(b_5b_6(b_7+b_8)-b_7b_8(b_5+b_6))^2} \\ \frac{(b_5b_6+b_7b_8)(b_5b_6(b_7+b_8)-b_7b_8(b_5+b_6))^2}{b_5b_6b_7b_8(b_5+b_6-b_7-b_8)^2} & \frac{(b_5b_6+b_7b_8)^2}{b_5b_6b_7b_8}-1 \end{pmatrix}.\end{aligned}$$

The eigenvalues of this matrix are equal to $\frac{b_5b_6}{b_7b_8}$ and $\frac{b_7b_8}{b_5b_6}$. Since $\frac{b_5b_6}{b_7b_8} \neq q^n$ for any $n \in \mathbb{Z}^*$, we can apply the q-Briot-Bouquet theorem B.3 to obtain the desired results. As to the last line of the proposition, this is a direct consequence of Lemma 2.1.2. □

Proposition 3.1.2. *For $k \in \{1,2,3\}$, the q-$P(A_1)$ equation has an unique power series solution*

$$f^{(1,k)}(t) = \sum_{n=1}^{\infty} f_n^{(1,k)} t^n, \qquad g^{(1,k)}(t) = \sum_{n=1}^{\infty} g_n^{(1,k)} t^n, \tag{3.10}$$

with $f_1^{(1,k)}$ and $g_1^{(1,k)}$ as defined in equation (3.4), given that the following conditions are *satisfied for the case $k=1$, $k=2$ and $k=3$ respectively,*

$$\frac{b_1b_2}{b_3b_4}\notin q^{\mathbb{Z}},\qquad b_1+b_2\neq b_3+b_4\qquad \textit{and}\qquad b_1^{-1}+b_2^{-1}\neq b_3^{-1}+b_4^{-1}, \tag{3.11}$$

$$\frac{b_2b_3}{b_1b_4}\notin q^{\mathbb{Z}},\qquad b_2+b_3\neq b_1+b_4\qquad \textit{and}\qquad b_2^{-1}+b_3^{-1}\neq b_1^{-1}+b_4^{-1}, \tag{3.12}$$

$$\frac{b_1b_3}{b_2b_4}\notin q^{\mathbb{Z}},\qquad b_1+b_3\neq b_2+b_4\qquad \textit{and}\qquad b_1^{-1}+b_3^{-1}\neq b_2^{-1}+b_4^{-1}. \tag{3.13}$$

Furthermore, each of these power series solutions has a positive radius of convergence and an unique meromorphic continuation to $\mathbb{C}$.

Proof. Note we can apply the q-Briot-Bouquet Theorem B.3 as done in the proof of Proposition 3.1.1. However, for a more elegant proof, we make use of one of the many symmetries of q-$P(A_1)$. Indeed applying the Bäcklund transformation $\mathcal{T}_1$, defined in (2.11), to each of the solutions defined in Proposition 3.1.1, gives the desired results directly. □

By Remark B.5, the solutions defined in Propositions 3.1.1 and 3.1.2 are also analytic in the parameters $\mathbf{b}$.

Theorem 3.1.3. *For generic parameter values $\mathbf{b}\in\mathcal{B}_q$, i.e.*

$$\frac{b_{i_1}b_{i_2}}{b_{i_3}b_{i_4}}\notin q^{\mathbb{Z}},\quad b_{i_1}^{\pm}+b_{i_2}^{\pm}\neq b_{i_3}^{\pm}+b_{i_4}^{\pm},\qquad (\{i_1,i_2,i_3,i_4\}=\{1,2,3,4\},\{5,6,7,8\}) \tag{3.14}$$

the solutions of q-$P(A_1)$, defined in Propositions 3.1.1 and 3.1.2, are all solutions meromorphic at the origin, excluding the singular ones (2.4)*.*

Proof. The proof is a bit laborious. We consider a Laurent series solution of (2.16),

$$f(t)=\sum_{n=k}^{\infty}f_nt^n,\qquad g(t)=\sum_{n=l}^{\infty}g_nt^n,$$

where $k,l\in\mathbb{Z}$ and $f_k,g_l\neq 0$. We distinguish 16 different scenarios given by

$$\begin{array}{cccccc} k<0, & k=0, & k=1, & \text{or} & k>1; & \text{and}\\ l<0, & l=0, & l=1, & \text{or} & l>1. & \end{array}$$

We have already studied the cases $k,l=0$ and $k,l=1$, to prove the theorem it remains to discard the other ones. Using the Bäcklund transformations $\mathcal{T}_1$ and $\mathcal{T}_3$, we can reduce the number of cases to be discarded to the following five,

$$k<0,l>1;\quad k=0,l=1;\quad k=0,l>1;\quad k=1,l>1;\quad k,l>1.$$

From (3.1a) it follows immediately that the cases $k=0,l>1$ and $k=0,l>1$ are impossible. Considering the case $k,l>1$, by calculating the coefficients of t^{l+3} and t^{k+3} in (3.1a) and

(3.1b) respectively, we find

$$(b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4)g_l = 0, \quad q^{k+3}\frac{1}{b_1b_2b_3b_4}(b_1 + b_2 + b_3 + b_4)f_k = 0.$$

As $f_k, g_l \neq 0$, the above two equations can easily be used to derive

$$b_1 + b_2 = 0 = b_3 + b_4, \quad b_1 + b_3 = 0 = b_3 + b_4 \quad \text{or} \quad b_1 + b_4 = 0 = b_2 + b_3.$$

In any case our assumption (3.14) is violated. Let us now consider the case $k = 1, l > 1$. Calculating the coefficients of t^{l+3} in (3.1a), and equation (3.3b) give respectively

$$f_1 = \tfrac{1}{2}(b_1^{-1} + b_2^{-1} + b_3^{-1} + b_4^{-1}), \quad \frac{1}{b_1b_2b_3b_4} = (f_1 - b_1^{-1})(f_1 - b_2^{-1})(f_1 - b_3^{-1})(f_1 - b_4^{-1}),$$

which combined again violate (3.14). We are left with the scenario $k < 0, l > 1$, which can be dealt with easily by comparing the lowest powers of t appearing on the left- and right-hand sides of (2.16), distinguishing $k + l < 0, k + l = 0, k + l = 1, k + l = 2$ and $k + l > 2$. □

We can translate all these results to a classification of solutions meromorphic at $t = \infty$, simply by employing Bäcklund transformation $\mathcal{T}_4$. For $k \in \{1, 2, 3\}$, we define solutions

$$(\check{f}^{(0,k)}, \check{g}^{(0,k)}) = \mathcal{T}_4(f^{(0,k)}, g^{(0,k)}), \qquad (\check{f}^{(1,k)}, \check{g}^{(1,k)}) = \mathcal{T}_4(f^{(1,k)}, g^{(1,k)}), \tag{3.15}$$

which are meromorphic at $t = \infty$, with asymptotic characterisations

$$\check{f}^{(0,k)}(t) = \frac{1}{g_0^{(0,k)}} + \mathcal{O}(t^{-1}), \qquad \check{g}^{(0,k)}(t) = \frac{1}{f_0^{(0,k)}} + \mathcal{O}(t^{-1}),$$
$$\check{f}^{(1,k)}(t) = \frac{1}{g_1^{(1,k)}}t + \mathcal{O}(1), \qquad \check{g}^{(1,k)}(t) = \frac{1}{f_1^{(1,k)}}t + \mathcal{O}(1),$$

as $t \to \infty$. Note that, by Lemma 2.1.2, each of these solutions has an unique meromorphic continuation to $\mathbb{P}^*$. Not only are the solutions, meromorphic at a critical point, special because of their local behaviour near that critical point, but also as they are globally uni-valued. We therefore pose the following question.

Question 3.1.4. *Do there exist regular solutions of q-$P(A_1)$, other than the ones meromorphic at the origin or infinity, which are meromorphic on the entire doubly punctured Riemann sphere $\mathbb{P} \setminus \{0, \infty\}$?*

We remark that the existence of meromorphic solutions of a discrete equation has been related to integrability of the equation in question, see for instance Halburd and Korhone [29] and the references therein. Let us discuss an explicit example. We let $q = s^4$ and take parameter values

$$b_1 = sa, \qquad b_3 = sb, \qquad b_5 = s^2, \qquad b_7 = ab,$$
$$b_2 = sa^{-1}, \qquad b_4 = sb^{-1}, \qquad b_6 = s^{-2}, \qquad b_8 = \frac{1}{ab}.$$

for some $a, b \in \mathbb{C}^*$. Then q-$P(A_1)$ admits the *rational solution*

$$f(t) = x(s^{-1}t), \quad g(t) = x(st), \quad x(t) := -\frac{1-ut}{1-ut^{-1}}, \quad u := \frac{1+ab}{a+b},$$

where, not coincidently, $x(t)$ defines a solution of symmetric q-$P(A_1)$ with $c = s^2$ and $d = ab$. By calculating the leading order behaviour near $t = 0$ and $t = \infty$, we identify

$$f = f^{(1,3)} = \check{f}^{(1,3)}, \quad g = g^{(1,3)} = \check{g}^{(1,3)}.$$

This solution was obtained by reverse engineering, using the continuum limit, starting from the following known rational solution of the sixth Painlevé equation,

$$w(\zeta) = \frac{\theta_y}{\theta_y + \theta_z}\zeta + \frac{\theta_z}{\theta_y + \theta_z}, \tag{3.16}$$

with $\theta_x = 1$, $\theta_\infty = 1 - \theta_x - \theta_y$ and $\theta_y, \theta_z \in \mathbb{C}^*$. Indeed the continuum limit of $x(t)$, as defined in the beginning of Section 2.5.1, gives the following solution of alt-P_{VI} (2.32),

$$x_0(t) = -\frac{1-u_0 t}{1-u_0 t^{-1}}, \quad u_0 := \frac{\alpha+\beta}{\alpha-\beta},$$

which is related to (3.16) via the change of variables (2.33) and (2.34).

Remark 3.1.5. Considering the special parameter values (2.28), the solutions $(f^{(0,1)}, g^{(0,1)})$ and $(f^{(0,3)}, g^{(0,3)})$, defined in Proposition 3.1.1, satisfy the symmetry condition (2.30), and hence give rise to solutions of symmetric q-$P(A_1)$. The same holds for the solutions $(f^{(1,1)}, g^{(1,1)})$ and $(f^{(1,3)}, g^{(1,3)})$, defined in Proposition 3.1.2. Upon calculating the continuum limit of the leading order terms, they coincide with those of the four meromorphic solutions of the sixth Painlevé equation at $\zeta = 1$, as classified by Kaneko [53].

3.2 The Leading Order Autonomous System

We study the leading order behaviour of solutions in more detail in this section. For complex functions f and g we write $f(t) \asymp g(t)$ as $t \to t_0$ if and only if $f(t) = \mathcal{O}(g(t))$ and $g(t) = \mathcal{O}(f(t))$ as $t \to t_0$. Note that the solutions (f, g) defined in Propositions 3.1.1 and 3.1.2 satisfy respectively $f, g \asymp 1$ and $f, g \asymp t$ as $t \to 0$. We therefore consider, on a formal level, any of the following 25 combinations of asymptotic relations as $t \to 0$, for a solution (f, g) of q-$P(A_1)$,

$$\begin{array}{llllll} f \prec t, & f \asymp t, & t \prec f \prec 1, & f \asymp 1 & \text{or} & f \succ 1; \quad \textit{and} \\ g \prec t, & g \asymp t, & t \prec g \prec 1, & g \asymp 1 & \text{or} & g \succ 1. \end{array}$$

Using Bäcklund transformations $\mathcal{T}_1$ and $\mathcal{T}_3$ (2.11), we can reduce the number of individual cases to be studied to 9. We assume that there exist $m, n \in \mathbb{N}^*$ such that

$$t^m \prec f, g \prec t^{-n}, \quad (t \to 0)$$

to exclude singular solutions and the zero solution. By some laborious comparison of dominant and subdominant terms in equations (2.16), just like in the proof of Theorem 3.1.3, it is possible to show that, for generic parameter values, the only 3 consistent combinations are

$$f, g \asymp t, \qquad t \prec f, g \prec 1, \qquad f, g \asymp 1. \tag{3.17}$$

Furthermore, there are 6 combinations which are only conditionally consistent, given by

$$f, g \prec t, \qquad f \prec t \text{ and } g \asymp t, \qquad f \asymp t \text{ and } g \prec t, \tag{3.18a}$$
$$f, g \succ 1, \qquad f \succ 1 \text{ and } g \asymp 1, \qquad f \asymp 1 \text{ and } g \succ 1. \tag{3.18b}$$

For example, $f, g \prec t$ is only consistent if

$$b_1 + b_2 + b_3 + b_4 = 0 \qquad \text{and} \qquad b_1^{-1} + b_2^{-1} + b_3^{-1} + b_4^{-1} = 0, \tag{3.19}$$

and $f \prec t$ with $g \asymp t$ is only consistent if

$$b_1 + b_2 = b_3 + b_4, \qquad b_1 + b_3 = b_2 + b_4 \qquad \text{or} \qquad b_1 + b_4 = b_2 + b_3. \tag{3.20}$$

We give explicit examples of such cases in Section 3.5. The interested reader can find the conditions, of the other conditionally consistent combinations, using Bäcklund transformations $\mathcal{T}_1$ and $\mathcal{T}_3$. The remaining combinations are inconsistent for all parameter values $\mathbf{b} \in \mathcal{B}_q$. Let us focus on the case $t \prec f, g \prec 1$ in (3.17). We put $f = tf_1$, and $g = tg_1$, then $1 \prec f_1, g_1 \prec t^{-1}$ as $t \to 0$, and by substitution into equations (2.16), we obtain

$$(f_1 g_1 - 1)(\overline{f}_1 g_1 - 1) \sim \left(b_1^{-1} g_1 - 1\right)\left(b_2^{-1} g_1 - 1\right)\left(b_3^{-1} g_1 - 1\right)\left(b_4^{-1} g_1 - 1\right), \tag{3.21a}$$
$$(\overline{f}_1 g_1 - 1)(\overline{f}_1 \overline{g}_1 - 1) \sim \left(b_1 \overline{f}_1 - 1\right)\left(b_2 \overline{f}_1 - 1\right)\left(b_3 \overline{f}_1 - 1\right)\left(b_4 \overline{f}_1 - 1\right), \tag{3.21b}$$

as $t \to 0$. So asymptotically f_1 and g_1 satisfy an autonomous system. Inspired by these equations, we study the following autonomous system,

$$(FG - 1)(\overline{F}G - 1) = \left(b_1^{-1} G - 1\right)\left(b_2^{-1} G - 1\right)\left(b_3^{-1} G - 1\right)\left(b_4^{-1} G - 1\right), \tag{3.22a}$$
$$(\overline{F}G - 1)(\overline{FG} - 1) = \left(b_1 \overline{F} - 1\right)\left(b_2 \overline{F} - 1\right)\left(b_3 \overline{F} - 1\right)\left(b_4 \overline{F} - 1\right), \tag{3.22b}$$

which we refer to as the *leading order autonomous system*. We identify this system as a *QRT mapping* (C.1) with

$$A_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & S_2^- & -S_1^- \\ S_4^- & -S_3^- & 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where $S_i^\pm$ denotes the ith degree elementary symmetric polynomial in $b_1^{\pm 1}$, $b_2^{\pm 1}$, $b_3^{\pm 1}$ and $b_4^{\pm 1}$, that is,

$$(z - b_1^{\pm 1})(z - b_2^{\pm 1})(z - b_3^{\pm 1})(z - b_4^{\pm 1}) = z^4 - S_1^\pm z^3 + S_2^\pm z^2 - S_3^\pm z + S_4^\pm.$$

We note that this is not a big surprise, as many discrete Painlevé equations were initially discovered by Grammaticos, Ramani and collaborators [21], via deautonomisation of QRT mappings. We refer the reader to the Appendix C for more details on QRT mappings. Note that the QRT mapping under consideration is particularly simple as many entries of the matrices A_1 and A_2 are zero. Indeed, condition (C.8) is satisfied, which means it is linearisable.

We wish to construct full asymptotic expansions of solutions of q-$P(A_1)$, starting from solutions of the leading order autonomous system. However solutions of q-$P(A_1)$ live on q-domains, possibly open. This requires us to adopt a somewhat unusual interpretation of the system (3.22). Even though it is an autonomous system, we think of it as a system of q-difference equations on some q-domain.

3.2.1 Generic Solution of Leading Order System

We apply the method as described in Section C.1, to parameterise the generic solution of the leading order autonomous system (3.22) . First of all, the invariant of (3.22) is given by

$$I(F,G) = \frac{F^2 + S_2^- FG + S_4^- G^2 - S_1^- F - S_3^- G}{FG - 1},$$

and we set $I(F,G) = P$. The linear system (C.9) becomes

$$F + \overline{F} + (S_2^- - P)G = S_1^-, \qquad G + \overline{G} + b_1b_2b_3b_4(S_2^- - P)\overline{F} = S_1^+. \tag{3.23}$$

If $b_1b_2b_3b_4(P - S_2^-)^2 \neq 4$, there exists an equilibrium solution (F_{eq}, G_{eq}) to this system given by

$$F_{eq} = \frac{S_1^+(P - S_2^-) + 2S_1^-}{4 - b_1b_2b_3b_4(P - S_2^-)^2}, \qquad G_{eq} = \frac{S_1^-(P - S_2^-) + 2S_1^+}{4 - b_1b_2b_3b_4(P - S_2^-)^2}. \tag{3.24}$$

The special case $b_1b_2b_3b_4(P - S_2^-)^2 = 4$, requires a separate analysis, which we discuss in Section 3.2.3. The matrix M (C.12) equals

$$M = \begin{pmatrix} -1 & P - S_2^- \\ -b_1b_2b_3b_4(P - S_2^-) & b_1b_2b_3b_4(P - S_2^-)^2 - 1 \end{pmatrix},$$

and its characteristic equation is given by

$$|M - \lambda I| = \lambda^2 + \left(2 - b_1b_2b_3b_4(P - S_2^-)^2\right)\lambda + 1 = 0. \tag{3.25}$$

At this stage, we consider P as a formal variable satisfying $\overline{P} = P$, and as such, the characteristic equation of M does not have a solution $\lambda \in \mathbb{C}(P)$. However we can rewrite (3.25) as

$$b_1b_2b_3b_4(P - S_2^-)^2 = \lambda + 2 + \lambda^{-1} = \left(\lambda^{\frac{1}{2}} + \lambda^{-\frac{1}{2}}\right)^2,$$

which inspires us to reparameterise

$$P = \epsilon_0 + \frac{\Lambda}{b_1 b_2 b_3 b_4} + \Lambda^{-1}, \tag{3.26}$$

where $\overline{\Lambda} = \Lambda$, giving

$$|M - \lambda I| = \left(\lambda - \frac{\Lambda^2}{b_1 b_2 b_3 b_4}\right)\left(\lambda - \frac{b_1 b_2 b_3 b_4}{\Lambda^2}\right).$$

We put $\lambda = \frac{\Lambda^2}{b_1 b_2 b_3 b_4}$, and M can be diagonalised as follows,

$$M = Q \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} Q^{-1}, \qquad Q = \begin{pmatrix} 1 & 1 \\ \Lambda & \frac{b_1 b_2 b_3 b_4}{\Lambda} \end{pmatrix}.$$

We introduce an independent variable ϕ characterised by $\overline{\phi} = \lambda\phi$, which allows us to write the generic solution to the linear system (3.23) as

$$F(\phi) = F_{\text{eq}}(\Lambda, \mathbf{b}) + \phi + \mu\phi^{-1}, \qquad G(\phi) = G_{\text{eq}}(\Lambda, \mathbf{b}) + \Lambda\phi + \frac{b_1 b_2 b_3 b_4}{\Lambda}\mu\phi^{-1}, \tag{3.27}$$

where μ is an arbitrary periodic constant, i.e. $\overline{\mu} = \mu$, and by substituting identity (3.26) into equations (3.24),

$$\begin{aligned} F_{\text{eq}}(\Lambda, \mathbf{b}) &= -\frac{b_1 b_2 b_3 b_4 \Lambda \left(S_1^+ + 2S_1^- \Lambda + S_3^- \Lambda^2\right)}{(b_1 b_2 b_3 b_4 - \Lambda^2)^2}, \\ G_{\text{eq}}(\Lambda, \mathbf{b}) &= -\frac{b_1 b_2 b_3 b_4 \Lambda \left(S_3^+ + 2S_1^+ \Lambda + S_1^- \Lambda^2\right)}{(b_1 b_2 b_3 b_4 - \Lambda^2)^2}. \end{aligned}$$

By direct calculation we find that the identity $I(F, G) = P$, for F and G as defined in equation (3.27), is equivalent to

$$\mu = \mu(\Lambda, \mathbf{b}) := \frac{\Lambda(\Lambda + b_1 b_2)(\Lambda + b_1 b_3)(\Lambda + b_1 b_4)(\Lambda + b_2 b_3)(\Lambda + b_2 b_4)(\Lambda + b_3 b_4)}{(b_1 b_2 b_3 b_4 - \Lambda^2)^4}. \tag{3.28}$$

So F and G as defined in equation (3.27), with $\mu = \mu(\Lambda, \mathbf{b})$ as defined above, satisfy equations (3.23) and (C.4). Hence, by Lemma C.1,

$$F(\phi, \Lambda) = \phi + F_{\text{eq}}(\Lambda, \mathbf{b}) + \mu(\Lambda, \mathbf{b})\phi^{-1}, \qquad G(\phi, \Lambda) = \Lambda\phi + G_{\text{eq}}(\Lambda, \mathbf{b}) + \frac{b_1 b_2 b_3 b_4}{\Lambda}\mu(\Lambda, \mathbf{b})\phi^{-1}, \tag{3.29}$$

defines a formal solution to the QRT mapping (3.22), where Λ and ϕ satisfy

$$\overline{\Lambda} = \Lambda, \qquad \overline{\phi} = \lambda\phi, \qquad \lambda = \frac{\Lambda^2}{b_1 b_2 b_3 b_4}. \tag{3.30}$$

We emphasise that this is a formal solution. However we can easily use it to construct true solutions of the leading order autonomous system. Choose any $\Lambda_0, \phi_0 \in \mathbb{C}^*$, then

$$F_s := F(\phi_s, \Lambda_0), \quad G_s := G(\phi_s, \Lambda_0), \quad \phi_s := \lambda_0^s \phi_0, \quad \lambda_0 := \frac{\Lambda_0^2}{b_1 b_2 b_3 b_4}, \quad (s \in \mathbb{Z}) \tag{3.31}$$

defines a true solution to (3.22).

3.2.2 Six Special Families of Solutions

Note that the autonomous system (3.22) reduces to the system of algebraic equations (3.3) if we assume $\overline{F}_1 = F_1$ and $\overline{G}_1 = G_1$. In particular, equations (3.4) give three constant solutions to system (3.22). In this section we see that any of these constant solutions has two associated 1-parameter families of solutions of (3.22). We denote the roots of $\mu(\Lambda, \mathbf{b})$ by $\Lambda = \Lambda_k^\pm$ $(k = 1, 2, 3)$, where

$$\begin{aligned} \Lambda_1^+ &= -b_1 b_2, & \Lambda_2^+ &= -b_2 b_3, & \Lambda_3^+ &= -b_1 b_3, \\ \Lambda_1^- &= -b_3 b_4, & \Lambda_2^- &= -b_1 b_4, & \Lambda_3^- &= -b_2 b_4. \end{aligned}$$

Let $k \in \{1, 2, 3\}$, then we have

$$F_{\mathrm{eq}}\left(\Lambda_k^\pm, \mathbf{b}\right) = f_1^{(1,k)}, \qquad G_{\mathrm{eq}}\left(\Lambda_k^\pm, \mathbf{b}\right) = g_1^{(1,k)}, \tag{3.32}$$

where the $f_1^{(1,k)}$ and $g_1^{(1,k)}$, as defined in (3.4), denote a constant solutions of (3.22). Associated we find two special 1-parameter families of solutions, by setting $\Lambda = \Lambda_k^\pm$ in (3.29), given by

$$F_k^\pm(\phi) := \phi + f_1^{(1,k)}, \qquad G_k^\pm(\phi) := \Lambda_k^\pm \phi + g_1^{(1,k)}, \qquad \overline{\phi} = \lambda_k^{\pm 1} \phi, \tag{3.33}$$

where

$$\lambda_1 = \frac{b_1 b_2}{b_3 b_4}, \qquad \lambda_2 = \frac{b_2 b_3}{b_1 b_4}, \qquad \lambda_3 = \frac{b_1 b_3}{b_2 b_4}. \tag{3.34}$$

Note that for the particular choice $\phi = 0$, the families $\left(F_k^+(\phi), G_k^+(\phi)\right)$ and $\left(F_k^-(\phi), G_k^-(\phi)\right)$ coincide with the constant solution $\left(f_1^{(1,k)}, g_1^{(1,k)}\right)$.

3.2.3 Logarithmic Type Solutions

We consider the remaining case

$$b_1 b_2 b_3 b_4 (P - S_2^-)^2 = 4,$$

for the linear system (3.23). Note that the equilibrium solution (3.24) no longer exists and we show that this case gives rise to *logarithmic type* solutions. We write $r_\pm = \pm\sqrt{b_1 b_2 b_3 b_4}$ and assume

$$P = S_2^- + \frac{2}{r_\pm}. \tag{3.35}$$

The system of equations (3.23) becomes

$$\overline{F} = -F + \frac{2}{r_\pm} G + S_1^-, \qquad \overline{G} = -2r_\pm F + 3G + 2r_\pm S_1^- + S_1^+. \tag{3.36}$$

We write $V = G - r_\pm F$, then

$$\overline{V} = V + S_1^+ + r_\pm S_1^-,$$

and we therefore introduce a formal variable χ which satisfies

$$\overline{\chi} = \chi + 1, \tag{3.37}$$

and set

$$V = \left(S_1^+ + r_\pm S_-\right)\chi.$$

This allows the first equation in (3.36) to be rewritten as

$$\overline{F} = F + 2\left(\frac{1}{r_\pm} S_1^+ + S_-\right)\chi + S_1^-,$$

which gives

$$F(\chi) = F_0 - \frac{1}{r_\pm} S_1^+ \chi + \left(\frac{1}{r_\pm} S_1^+ + S_1^-\right)\chi^2, \tag{3.38}$$

for some F_0 with $\overline{F}_0 = F_0$.
As $V = G - r_\pm F$, we obtain a corresponding expression for G,

$$G(\chi) = r_\pm F_0 + r_\pm S_1^- \chi + \left(S_1^+ + r_\pm S_1^-\right)\chi^2. \tag{3.39}$$

Upon substitution of (3.38) and (3.39) into the identity $I(F,G) = P$, or equivalently into the leading order autonomous system (3.22), we find

$$F_0 = \frac{2 + r_\pm S_2^-}{S_1^+ + r_\pm S_1^-}.$$

We conclude that the general formal solution of the leading order autonomous system (3.22), subject to (3.35), is given by

$$F_l^\pm(\chi) = \frac{2 + r_\pm S_2^-}{S_1^+ + r_\pm S_1^-} - \frac{1}{r_\pm} S_1^+ \chi + \left(\frac{1}{r_\pm} S_1^+ + S_1^-\right)\chi^2, \tag{3.40a}$$

$$G_l^\pm(\chi) = \frac{2r_\pm + S_2^+}{S_1^+ + r_\pm S_1^-} + r_\pm S_1^- \chi + \left(S_1^+ + r_\pm S_1^-\right)\chi^2, \tag{3.40b}$$

where χ satisfies (3.37).
The subscripts 'l' stand for *logarithmic type*, as the time evolution of χ, equation (3.37), is characteristic for $\log_q(t)$ when interpreted as a q-difference equation in t. Note that we used $S_1^+ + r_\pm S_1^- \neq 0$ in the above derivation, we leave the degenerate case $S_1^+ + r_\pm S_1^- = 0$ to the interested reader.

We again emphasise that the obtained solution is a formal one. However we can easily use it to construct true solutions of the leading order autonomous system. Choose any $\chi_0 \in \mathbb{C}$, then

$$F_s := F_l^{\pm}(\chi_s), \quad G_s := G_l^{\pm}(\chi_s), \quad \chi_s := \chi_0 + s, \quad (s \in \mathbb{Z}) \tag{3.41}$$

defines a true solution to (3.22).

Remark 3.2.1. We would like to note that the classification of solutions of (3.22) is now complete. That is, given any initial data $(F_0, G_0) \in \mathbb{C}^2$, such that $F_0 \cdot G_0 \neq 0, 1$. Let $F_{s+1} = \overline{F}_s$ and $G_{s+1} = \overline{G}_s$ be defined recursively by (3.22) for $s \in \mathbb{Z}$. Then $(F_s, G_s)_{s\in\mathbb{Z}}$ is captured by (3.31) or (3.41). Indeed let $P = I(F_0, G_0)$, and assume $b_1b_2b_3b_4(P - S_2^-)^2 \neq 4$, then (3.26) has two distinct solutions $\Lambda = \Lambda_0, \Lambda_0' \in \mathbb{C}$, which are related by $\Lambda_0\Lambda_0' = b_1b_2b_3b_4$. Hence $\mu(\Lambda_0) = 0$ iff $\mu(\Lambda_0') = 0$. Assume $\mu(\Lambda_0) \neq 0$, then the system

$$F_0 = \phi_0 + F_{\text{eq}}(\Lambda_0, \mathbf{b}) + \mu(\Lambda_0, \mathbf{b})\phi_0^{-1}, \quad G_0 = \Lambda_0\phi_0 + G_{\text{eq}}(\Lambda_0, \mathbf{b}) + \frac{b_1b_2b_3b_4}{\Lambda_0}\mu(\Lambda_0, \mathbf{b})\phi_0^{-1},$$

has an unique solution $\phi_0 \in \mathbb{C}^*$, and $(F_s, G_s)_{s\in\mathbb{Z}}$ is given by (3.31). Of course the choice $\Lambda = \Lambda_0'$ would have led to the same result. This, however, is no longer the case when $\mu(\Lambda_0) = 0$. We leave it to the reader to work through these degenerate cases as well as the logarithmic one, $b_1b_2b_3b_4(P - S_2^-)^2 = 4$.

3.3 A Formal Series Solution

In Section 3.2 we saw that, heuristically speaking, the generic leading order behaviour of q-$P(A_1)$ transcendents is described by the autonomous system (3.22). Furthermore we derived a general formal parameterisation of the solutions of this autonomous system. In this section we study the complete formal expansion of q-$P(A_1)$ transcendents corresponding to the general formal solution of the autonomous system. To this end, we consider the following formal *solution ansatz*,

$$f = \sum_{i=1}^{\infty} F_i t^i, \qquad g = \sum_{i=1}^{\infty} G_i t^i. \tag{3.42}$$

This approach reduces to the power series method if we assume that the coefficients F_i and G_i are plain complex numbers. However for now we work with these coefficients on a formal level, for example,

$$\overline{f} = \sum_{i=1}^{\infty} q^i \overline{F}_i t^i, \qquad \overline{g} = \sum_{i=1}^{\infty} q^i \overline{G}_i t^i.$$

We substitute these formal series into equations (2.16) and compare coefficients of t order by order. First of all, note that no terms t^n with $n < 4$ occur in equations (2.16). By comparing the coefficients of t^4 in equations (2.16) we recover the leading order autonomous system (3.22) with $F = F_1$ and $G = G_1$. As to the higher order coefficients, for $n > 1$, by comparing the coefficients of t^{n+3} in equations (2.16), we obtain

$$\begin{aligned} &G_1(G_1F_1-1)q^n\overline{F}_n+qG_1(G_1\overline{F}_1-1)F_n+q\left(2G_1F_1\overline{F}_1-F_1-\overline{F}_1\right)G_n= \\ &\qquad \frac{q}{b_1b_2b_3b_4}Q^{(1)}(G_1)G_n+R_n^{(1)}\left[(F_i)_{1\leq i<n},(\overline{F}_i)_{1\leq i<n},(G_i)_{1\leq i<n}\right], \qquad (3.43a)\\ &q\overline{F}_1\left(\overline{F}_1\overline{G}_1-1\right)G_n+\overline{F}_1\left(G_1\overline{F}_1-1\right)q^n\overline{G}_n+\left(2F_1G_1\overline{G}_1-G_1-\overline{G}_1\right)q^n\overline{F}_n= \\ &\qquad b_1b_2b_3b_4Q^{(-1)}\left(\overline{F}_1\right)q^n\overline{F}_n+R_n^{(2)}\left[(\overline{F}_i)_{1\leq i<n},(G_i)_{1\leq i<n},(\overline{G}_i)_{1\leq i<n}\right], \qquad (3.43b)\end{aligned}$$

for certain $R_n^{(1)}$ and $R_n^{(2)}$ which are polynomial with respect to their inputs, where the polynomials $Q^{(i)}(z)$ are defined by

$$Q^{(i)}(z)=\frac{d}{dx}\left[\left(x-b_1^i\right)\left(x-b_2^i\right)\left(x-b_3^i\right)\left(x-b_4^i\right)\right],$$

for $i\in\{1,-1\}$.

Note that these equations are linear autonomous equations with respect to F_n and G_n. It is straightforward to obtain explicit expressions for $R_n^{(1)}$ and $R_n^{(2)}$, these are however rather lengthy, which is why we omit them. As an example, $R_2^{(1)}$ and $R_2^{(2)}$ are given by

$$\begin{aligned} R_2^{(1)}\left(F_1,\overline{F}_1,G_1\right)&=\left(b_5^{-1}+b_6^{-1}+b_7^{-1}+b_8^{-1}\right)qG_1(F_1G_1-1)\left(\overline{F}_1G_1-1\right),\\ R_2^{(2)}\left(\overline{F}_1,G_1,\overline{G}_1\right)&=\left(b_5+b_6+b_7+b_8\right)q^2\overline{F}_1\left(\overline{F}_1G_1-1\right)\left(\overline{F}_1\overline{G}_1-1\right).\end{aligned}$$

Furthermore the polynomials $R_n^{(1)}$ and $R_n^{(2)}$ are of degree at most $n+3$ with respect to the weighted gradation $\deg_w$ on $\mathbb{C}\left[\cup_{i=1}^{\infty}\{F_i,\overline{F}_i,G_i,\overline{G}_i\}\right]$, which is uniquely defined by its values on the generators of this polynomial ring, as

$$\deg_w F_i=\deg_w\overline{F}_i=\deg_w G_i=\deg_w\overline{G}_i=i. \qquad (i\in\mathbb{N}^*)$$

The importance of this observation becomes clear when we substitute the generic formal solution (3.29) to equations (3.22) for F_1 and G_1. Indeed, if we set $F_1=F_1(\phi)=F(\phi)$ and $G_1=G_1(\phi)=G(\phi)$ as defined in equations (3.29), then $F_1(\phi)$ and $G_1(\phi)$ are Laurent polynomials in ϕ of degree 1 in both ϕ and ϕ^{-1}. Hence the right-hand sides of equations (3.43) for $n=2$, are Laurent polynomials in ϕ of at most degree $n+3=5$ in both ϕ and ϕ^{-1}, which shows that the system of equations (3.43) for $n=2$ possibly has a solution $(F_2(\phi),G_2(\phi))$, such that $F_2(\phi)$ and $G_2(\phi)$ are Laurent polynomials in ϕ of at most degree 2 in both ϕ and ϕ^{-1}. Indeed a lengthy calculation confirms this. More generally, we conjecture that there is an unique solution $((F_n(\phi))_{n=1}^{\infty},(G_n(\phi))_{n=1}^{\infty})$ to equations (3.43) with $F_1(\phi)=F(\phi)$ and $G_1(\phi)=G(\phi)$ as above, such that $F_n(\phi)$ and $G_n(\phi)$ are Laurent polynomials in ϕ of at most degree n in both ϕ and ϕ^{-1}. An equivalent formulation of this statement is given in Conjecture 3.3.3. This however seems difficult to prove directly and we hence prove a weaker version, which states that there is an unique solution where the coefficients $F_n(\phi)$ and $G_n(\phi)$ are Laurent series in ϕ with highest order term of degree less or equal to n, for $n\in\mathbb{N}^*$.

Theorem 3.3.1. *There exists an unique formal series solution to q-$P(A_1)$ of the form*

$$f^{0,+}(t,\phi;\Lambda,\mathbf{b}) = \sum_{n=1}^{\infty} F_n^{0,+}(\phi;\Lambda,\mathbf{b})t^n, \quad g^{0,+}(t,\phi;\Lambda,\mathbf{b}) = \sum_{n=1}^{\infty} G_n^{0,+}(\phi;\Lambda,\mathbf{b})t^n, \tag{3.44}$$

with, for $n \in \mathbb{N}^*$,

$$F_n^{0,+}(\phi;\Lambda,\mathbf{b}) = \sum_{i=-\infty}^{n} F_{n,i}^{0,+}(\Lambda,\mathbf{b})\phi^i, \quad G_n^{0,+}(\phi;\Lambda,\mathbf{b}) = \sum_{i=-\infty}^{n} G_{n,i}^{0,+}(\Lambda,\mathbf{b})\phi^i, \tag{3.45}$$

where $F_{1,1}^{0,+}(\Lambda,\mathbf{b}) = 1$, $G_{1,1}^{0,+}(\Lambda,\mathbf{b}) = \Lambda$ and Λ and ϕ satisfy equations (3.30), with

$$q = q(\mathbf{b}) = \frac{b_1b_2b_3b_4}{b_5b_6b_7b_8}, \qquad \lambda = \lambda(\Lambda,\mathbf{b}) = \frac{\Lambda^2}{b_1b_2b_3b_4}.$$

For $n \in \mathbb{N}^*$ and $i \in \mathbb{Z}_{\leq n}$, the coefficients $F_{n,i}^{0,+}(\Lambda,\mathbf{b})$ and $G_{n,i}^{0,+}(\Lambda,\mathbf{b})$ are rational functions in their inputs, which are regular at points $(\Lambda,\mathbf{b}) \in \mathbb{C}^* \times \mathcal{B}$ such that

$$1 \notin Q := \{q_1^m q_2^n : (m,n) \in \mathbb{N}^2 \setminus \{(0,0)\}\}, \tag{3.46}$$

where $q_1 = q_1(\mathbf{b},\Lambda) = q\lambda$ and $q_2 = q_2(\mathbf{b},\Lambda) = \lambda^{-1}$.
Furthermore, for fixed $\mathbf{b} \in \mathcal{B}$ with $|q| < 1$, for any $\Lambda \in L_0(\mathbf{b})$, where

$$L_0(\mathbf{b}) := \{x \in \mathbb{C}^* : |b_1b_2b_3b_4| < |x|^2 < |b_5b_6b_7b_8|\}, \tag{3.47}$$

condition (3.46) is satisfied and this formal solution, written in terms of the variables $\zeta_1 = t\phi$ and $\zeta_2 = \phi^{-1}$,

$$f^{0,+}(\zeta_1\zeta_2,\zeta_2^{-1};\Lambda,\mathbf{b}) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} F_{n,n-m}^{0,+}(\Lambda,\mathbf{b})\zeta_1^n\zeta_2^m, \tag{3.48a}$$

$$g^{0,+}(\zeta_1\zeta_2,\zeta_2^{-1};\Lambda,\mathbf{b}) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} G_{n,n-m}^{0,+}(\Lambda,\mathbf{b})\zeta_1^n\zeta_2^m, \tag{3.48b}$$

converges near $(\zeta_1,\zeta_2) = (0,0)$.
In fact, these expansions are also analytic in Λ. That is, for any $L \subseteq L_0(\mathbf{b})$ open with $\overline{L} \subseteq L_0(\mathbf{b})$, there is an open environment $Z \subseteq \mathbb{C}^2$ of $\mathbf{0}$, such that the series (3.48) converge uniformly on $Z \times L$, defining holomorphic functions on this set in $(\boldsymbol{\zeta},\Lambda)$.

Proof. We apply the q-Briot-Bouquet Theorem B.3 with $m = 2$ to q-$P(A_1)$, after a change of dependent and independent variables. More precisely, inspired by equations (3.48), we introduce the following variables,

$$\zeta_1 = t\phi, \qquad \zeta_2 = \phi^{-1}, \qquad y_1 = \frac{f}{\zeta_1} - 1, \qquad y_2 = \frac{g}{\zeta_1} - \Lambda, \tag{3.49}$$

where ζ_1 and ζ_2, in accordance with equations (3.30), satisfy

$$\overline{\zeta}_1 = q_1\zeta_1, \qquad \overline{\zeta}_2 = q_2\zeta_2.$$

As $t = \zeta_1\zeta_2$, we can rewrite q-$P(A_1)$ in terms of these new variables as

$$y_1(q_1\zeta_1, q_2\zeta_2) = H_1\left(\zeta_1, \zeta_2, y_1(\zeta_1,\zeta_2), y_2(\zeta_1,\zeta_2); \Lambda, \mathbf{b}\right), \tag{3.50a}$$
$$y_2(q_1\zeta_1, q_2\zeta_2) = H_2\left(\zeta_1, \zeta_2, y_1(\zeta_1,\zeta_2), y_2(\zeta_1,\zeta_2); \Lambda, \mathbf{b}\right), \tag{3.50b}$$

for certain rational functions $H_1(\zeta_1, \zeta_2, y_1, y_2; \Lambda, \mathbf{b})$ and $H_2(\zeta_1, \zeta_2, y_1, y_2; \Lambda, \mathbf{b})$.
We apply the q-Briot-Bouquet Theorem B.3 to this system of q-difference equations. We denote

$$\boldsymbol{\zeta} = (\zeta_1, \zeta_2), \qquad \mathbf{y} = (y_1, y_2), \qquad \mathbf{q} = (q_1, q_2),$$

and leave it to the interested reader to write down $H_1(\boldsymbol{\zeta}, \mathbf{y}; \Lambda, \mathbf{b})$ and $H_2(\boldsymbol{\zeta}, \mathbf{y}; \Lambda, \mathbf{b})$ explicitly. A rather lengthy calculation shows

$$H_1(\mathbf{0}, \mathbf{y}; \Lambda, \mathbf{b}) = \frac{(y_2+\Lambda)^2}{\Lambda^2(y_1+1)} - 1, \qquad H_2(\mathbf{0}, \mathbf{y}; \Lambda, \mathbf{b}) = \frac{(y_2+\Lambda)^3}{\Lambda^2(y_1+1)^2} - \Lambda,$$

in particular $H(\mathbf{0}, \mathbf{0}; \mathbf{q}, \Lambda) = \mathbf{0}$ and we have

$$D(\Lambda, \mathbf{b}) := \begin{pmatrix} \frac{\partial H_1}{\partial y_1}(\mathbf{0}, \mathbf{0}; \Lambda, \mathbf{b}) & \frac{\partial H_1}{\partial y_2}(\mathbf{0}, \mathbf{0}; \Lambda, \mathbf{b}) \\ \frac{\partial H_2}{\partial y_1}(\mathbf{0}, \mathbf{0}; \Lambda, \mathbf{b}) & \frac{\partial H_2}{\partial y_2}(\mathbf{0}, \mathbf{0}; \Lambda, \mathbf{b}) \end{pmatrix} = \begin{pmatrix} -1 & 2\Lambda^{-1} \\ -2\Lambda & 3 \end{pmatrix}. \tag{3.51}$$

Note that 1 is the only eigenvalue of $D(\Lambda, \mathbf{b})$, with multiplicity 2. Therefore, by the q-Briot-Bouquet Theorem B.3, if conditions (3.46) are satisfied, then the system of q-difference equations (3.50) has an unique power series solution of the form

$$y_i(\zeta_1, \zeta_2; \Lambda, \mathbf{b}) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} y_{n,m}^{(i)}(\Lambda, \mathbf{b})\zeta_1^n\zeta_2^m, \tag{3.52}$$

with $y_{0,0}^{(i)}(\Lambda, \mathbf{b}) = 0$ for $i \in \{1, 2\}$.
Associated via equations (3.49), we have the following expansions for $f = f(\zeta_1, \zeta_2; \Lambda, \mathbf{b})$ and $g = g(\zeta_1, \zeta_2; \Lambda, \mathbf{b})$,

$$f(\zeta_1, \zeta_2; \Lambda, \mathbf{b}) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} f_{n,m}(\Lambda, \mathbf{b})\zeta_1^n\zeta_2^m, \quad g(\zeta_1, \zeta_2; \Lambda, \mathbf{b}) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} g_{n,m}(\Lambda, \mathbf{b})\zeta_1^n\zeta_2^m, \tag{3.53}$$

where the coefficients are defined by

$$f_{n,m}(\Lambda, \mathbf{b}) = y_{n-1,m}^{(1)}(\Lambda, \mathbf{b}), \qquad g_{n,m}(\Lambda, \mathbf{b}) = y_{n-1,m}^{(2)}(\Lambda, \mathbf{b}),$$

for $n \in \mathbb{N}^*$ and $m \in \mathbb{N}$ with $(n, m) \neq (1, 0)$, and

$$f_{1,0}(\Lambda, \mathbf{b}) = 1, \qquad g_{1,0}(\Lambda, \mathbf{b}) = \Lambda.$$

Rewriting these expansions in terms of the original independent variables t and ϕ, the formulas

$$f^{0,+}(t,\phi;\Lambda,\mathbf{b}) = f\left(t\phi,\phi^{-1};\Lambda,\mathbf{b}\right), \qquad g^{0,+}(t,\phi;\Lambda,\mathbf{b}) = g\left(t\phi,\phi^{-1};\Lambda,\mathbf{b}\right),$$

define formal series solutions of q-$P(A_1)$, precisely as described in equations (3.44). Furthermore the q-Briot-Bouquet Theorem B.3 implies that the power series (3.53) converge in an open environment of $(\zeta_1,\zeta_2)=(0,0)$, if 1 is not a limit point of Q. Note that this condition is trivially satisfied if $0<|q_1|,|q_2|<1$, which is equivalent to $\Lambda\in L_0(\mathbf{b})$. We conclude that the series (3.48) indeed converge locally at $(\zeta_1,\zeta_2)=(0,0)$, for $\Lambda\in L_0(\mathbf{b})$. Strictly speaking, this only shows that the series defines a solution in the two variables t and ϕ for a fixed $\Lambda\in\mathbb{C}^*$ such that condition (3.46) holds. Note however, that the proof of Theorem B.3 gives an explicit recursion for the coefficients, which proves that the coefficients $F_{n,i}^{0,+}(\Lambda,\mathbf{b})$ and $G_{n,i}^{0,+}(\Lambda,\mathbf{b})$ are rational functions in their inputs and the formal series solution defines a solution on a formal level. This finishes the proof of the first part of the theorem.

As to the second part, we would like to prove that the solutions (3.53) depend analytically on Λ, which is equivalent to proving that the expansions (3.52) are analytic in Λ. To this end we apply Theorem B.4. As, for any $L\subseteq L_0(\mathbf{b})$ with $\overline{L}\subseteq L_0(\mathbf{b})$, the set $\overline{L}$ is compact, a simple compactness argument shows that it suffices to prove that for any $\Lambda_0\in L_0(\mathbf{b})$, there is an open environment $L\subseteq L_0(\mathbf{b})$ of Λ_0, and an open environment $Z\subseteq\mathbb{C}^2$ of $\mathbf{0}$, such that the series (3.52) converge uniformly on $Z\times L$.

So let us take a $\Lambda_0\in L_0(\mathbf{b})$, we denote $\mathbf{q}_0=(q_1(\Lambda_0,\mathbf{b}),q_2(\Lambda_0,\mathbf{b}))$ and determine an $r>0$ such that

$$B^2_{\max}(\mathbf{q}_0,r)\subseteq\overline{B}^2_{\max}(\mathbf{q}_0,r)\subseteq B^2_{\max}(\mathbf{0},1)\setminus\{\mathbf{q}\in\mathbb{C}^2|q_1q_2=0\}\subseteq\mathbb{C}^2,$$

and set $U=B^2_{\max}(\mathbf{q}_0,r)$.
We have to modify the functions $H_1(\boldsymbol{\zeta},\mathbf{y};\Lambda,\mathbf{b})$ and $H_2(\boldsymbol{\zeta},\mathbf{y};\Lambda,\mathbf{b})$ a bit in order to be able to apply Theorem B.4, as Λ and $\mathbf{b}$ are not independent of $\mathbf{q}=(q_1,q_2)$. Indeed, we have to reparameterise all variables in terms of q_1 and q_2. To this end, we keep the value of b_i fixed for $2\le i\le 8$, but allow b_1 and Λ to vary with $\mathbf{q}$. More explicitly, we define

$$b_1'(\mathbf{q})=\frac{q_1q_2b_5b_6b_7b_8}{b_2b_3b_4}, \qquad \mathbf{b}'(\mathbf{q})=(b_1'(\mathbf{q}),b_2,b_3,b_4,b_5,b_6,b_7,b_8), \qquad \Lambda(\mathbf{q})=(b_5b_6b_7b_8)^{\frac{1}{2}}q_1^{\frac{1}{2}},$$

for $\mathbf{q}\in U$, where we choose the sign of the square root such that $\Lambda(\mathbf{q}_0)=\Lambda_0$.
Note that at $\mathbf{q}=\mathbf{q}_0$, the original values of the parameters are recovered, as

$$\mathbf{b}'(\mathbf{q}_0)=\mathbf{b}, \qquad \Lambda(\mathbf{q}_0)=\Lambda_0,$$

and $\Lambda(\mathbf{q})$ is a univalued holomorphic function on U.
We modify H, by setting

$$\tilde{H}(\boldsymbol{\zeta},\mathbf{y};\mathbf{q})=H(\boldsymbol{\zeta},\mathbf{y};\Lambda(\mathbf{q}),\mathbf{b}'(\mathbf{q})).$$

The function $\tilde{H}(\boldsymbol{\zeta},\mathbf{y};\mathbf{q})$ is holomorphic at $(\boldsymbol{\zeta},\mathbf{y},\mathbf{q})=(\mathbf{0},\mathbf{0},\mathbf{q}')$ with $\tilde{H}(\mathbf{0},\mathbf{0},\mathbf{q}')=\mathbf{0}$, for every

$\mathbf{q}' \in U$. The relevant Jacobian matrix of $\tilde{H}$ is given by

$$\tilde{D}(\mathbf{q}) := \begin{pmatrix} \frac{\partial \tilde{H}_1}{\partial y_1}(\mathbf{0},\mathbf{0};\mathbf{q}) & \frac{\partial \tilde{H}_1}{\partial y_2}(\mathbf{0},\mathbf{0};\mathbf{q}) \\ \frac{\partial \tilde{H}_2}{\partial y_1}(\mathbf{0},\mathbf{0};\mathbf{q}) & \frac{\partial \tilde{H}_2}{\partial y_2}(\mathbf{0},\mathbf{0};\mathbf{q}) \end{pmatrix} = D(\Lambda(\mathbf{q}),\mathbf{b}'(\mathbf{q})) = \begin{pmatrix} -1 & 2\Lambda(\mathbf{q})^{-1} \\ -2\Lambda(\mathbf{q}) & 3 \end{pmatrix},$$

for $\mathbf{q} \in U$, where $D(\Lambda,\mathbf{b})$ is the Jacobian matrix of H, as defined in equation (3.51). Again 1 is the only eigenvalue of $\tilde{D}(\mathbf{q})$, which is not an element of Q_0 as defined in (B.5) with $m = 2$, for $\mathbf{q} \in U$. We can hence apply Theorem B.4, which gives open environments $Z \subseteq \mathbb{C}^2$ and $V \subseteq U$ of $\mathbf{0}$ and $\mathbf{q}_0$ respectively, such that the series $y_i(\boldsymbol{\zeta};\Lambda(\mathbf{q}),\mathbf{b}'(\mathbf{q}))$, with notation as in equation (3.52) for $i = 1,2$, converge uniformly on $Z \times V$, defining holomorphic functions in $(\boldsymbol{\zeta},\mathbf{q})$ on this set. To undo the reparameterisation (3.3), we define

$$\mathbf{q}(s) = \left(\frac{s^2}{b_5b_6b_7b_8}, \frac{b_1b_2b_3b_4}{s^2}\right),$$

and determine an open connected environment $L \subseteq L_0(\mathbf{b})$ of Λ_0, such that

$$\{\mathbf{q}(s) : s \in L\} \subseteq V.$$

Then we know that the series

$$Y_i(\boldsymbol{\zeta},s) := y_i\left(\boldsymbol{\zeta};\Lambda\left(\mathbf{q}(s)\right),\mathbf{b}'\left(\mathbf{q}(s)\right)\right),$$

converge uniformly on $Z \times L$, defining holomorphic functions in $(\boldsymbol{\zeta},s)$ on this set. Note however, that we have, for $s \in L$,

$$\Lambda\left(\mathbf{q}(s)\right) = s, \qquad \mathbf{b}'\left(\mathbf{q}(s)\right) = \mathbf{b},$$

and hence

$$Y_i(\boldsymbol{\zeta},s) = y_i\left(\boldsymbol{\zeta};s,\mathbf{b}\right).$$

The theorem follows. □

Remark 3.3.2. In fact, the expansions (3.48) also depend analytically on the parameters $\mathbf{b}$. That is, given $\mathbf{b}_0 \in \mathcal{B}$ and $\Lambda_0 \in L_0(\mathbf{b}_0)$, there exist open environments $Z \subseteq \mathbb{C}^2$, $L \subseteq \mathbb{C}$ and $B \subseteq \mathcal{B}$ of $\mathbf{0}$, Λ_0 and $\mathbf{b}_0$ respectively, such that for any $(\Lambda,\mathbf{b}) \in L \times B$, we have $\Lambda \in L_0(\mathbf{b})$ and the series (3.48) converge uniformly on $Z \times L \times B$, defining holomorphic functions on this set in $(\boldsymbol{\zeta},\Lambda,\mathbf{b})$. This can be proven easily by incorporating parameters in Theorem B.4, see Remark B.5.

As desired, we have

$$F_1^{0,+}(\phi;\Lambda,\mathbf{b}) = F(\phi), \qquad G_1^{0,+}(\phi;\Lambda,\mathbf{b}) = G(\phi), \tag{3.54}$$

where F and G are defined as in equations (3.29). Furthermore the coefficients $F_n^{0,+}(\phi;\Lambda,\mathbf{b})$ and $G_n^{0,+}(\phi;\Lambda,\mathbf{b})$ indeed satisfy equations (3.43).

In Theorem 3.3.1 the plus superscripts reflect the fact that there are only finitely many positive powers of ϕ occuring in the Laurent series (3.45), we define the dual 'minus' solutions as follows

$$f^{0,-}(t,\phi;\Lambda,\mathbf{b})=f^{0,+}\left(t,\mu(\Lambda,\mathbf{b})\phi^{-1};\frac{b_1b_2b_3b_4}{\Lambda},\mathbf{b}\right),\tag{3.55a}$$

$$g^{0,-}(t,\phi;\Lambda,\mathbf{b})=g^{0,+}\left(t,\mu(\Lambda,\mathbf{b})\phi^{-1};\frac{b_1b_2b_3b_4}{\Lambda},\mathbf{b}\right).\tag{3.55b}$$

Note that indeed, by Theorem 3.3.1, this defines a formal solution to q-$P(A_1)$, as

$$\overline{\mu(\Lambda,\mathbf{b})\phi^{-1}}=\frac{\left(\frac{b_1b_2b_3b_4}{\Lambda}\right)^2}{b_1b_2b_3b_4}\mu(\Lambda,\mathbf{b})\phi^{-1}.$$

Analogously to the expansions (3.44) and (3.45), we have

$$f^{0,-}(t,\phi;\Lambda,\mathbf{b})=\sum_{n=1}^{\infty}F_n^{0,-}(\phi;\Lambda,\mathbf{b})t^n,\quad g^{0,-}(t,\phi;\Lambda,\mathbf{b})=\sum_{n=1}^{\infty}G_n^{0,-}(\phi;\Lambda,\mathbf{b})t^n,\tag{3.56}$$

with, for $n\in\mathbb{N}^*$,

$$F_n^{0,-}(\phi;\Lambda,\mathbf{b})=\sum_{i=-n}^{\infty}F_{n,i}^{0,-}(\Lambda,\mathbf{b})\phi^i,\quad G_n^{0,-}(\phi;\Lambda,\mathbf{b})=\sum_{i=-n}^{\infty}G_{n,i}^{0,-}(\Lambda,\mathbf{b})\phi^i,\tag{3.57}$$

where, for $i\in\mathbb{Z}_{\geq -n}$,

$$F_{n,i}^{0,-}(\Lambda,\mathbf{b})=F_{n,-i}^{0,+}\left(\frac{b_1b_2b_3b_4}{\Lambda},\mathbf{b}\right)\mu(\Lambda,\mathbf{b})^{-i},\quad G_{n,i}^{0,-}(\Lambda,\mathbf{b})=G_{n,-i}^{0,+}\left(\frac{b_1b_2b_3b_4}{\Lambda},\mathbf{b}\right)\mu(\Lambda,\mathbf{b})^{-i}.$$

Using the symmetries

$$\mu\left(\frac{b_1b_2b_3b_4}{\Lambda}\right)=\mu(\Lambda,\mathbf{b}),\quad F_{eq}\left(\frac{b_1b_2b_3b_4}{\Lambda}\right)=F_{\mathrm{eq}}(\Lambda,\mathbf{b}),\quad G_{eq}\left(\frac{b_1b_2b_3b_4}{\Lambda}\right)=G_{\mathrm{eq}}(\Lambda,\mathbf{b}),$$

it is easy to see that

$$F_1^{0,-}(\phi;\Lambda,\mathbf{b})=F(\phi,\Lambda)=F_1^{0,+}(\phi;\Lambda,\mathbf{b}),\quad G_1^{0,-}(\phi;\Lambda,\mathbf{b})=G(\phi,\Lambda)=G_1^{0,+}(\phi;\Lambda,\mathbf{b}),\tag{3.58}$$

where $F(\phi,\Lambda)$ and $G(\phi,\Lambda)$ are as defined in equations (3.29).
Note that this implies that the coefficients of the formal 'plus' and 'minus' series solutions, (3.45) and (3.57), satisfy the same recursive system of difference equations (3.43), with the same initial values (3.58). Motivated by this plausibility argument, we formulate the following conjecture.

Conjecture 3.3.3. The formal series solutions (3.44) and (3.56) are equal, that is,

$$f^{0,+}(t,\phi;\Lambda,\mathbf{b})=f^{0,-}(t,\phi;\Lambda,\mathbf{b}),\qquad g^{0,+}(t,\phi;\Lambda,\mathbf{b})=g^{0,-}(t,\phi;\Lambda,\mathbf{b}),$$

or equivalently, for $n \in \mathbb{N}^*$, the Laurent series (3.45) terminate at $i = -n$, that is,

$$F_n^{0,+}(\phi;\Lambda,\mathbf{b}) = \sum_{i=-n}^{n} F_{n,i}^{0,+}(\Lambda,\mathbf{b})\phi^i, \qquad G_n^{0,+}(\phi;\Lambda,\mathbf{b}) = \sum_{i=-n}^{n} G_{n,i}^{0,+}(\Lambda,\mathbf{b})\phi^i. \tag{3.59}$$

In particular, by equations (3.55), we have, for $n \in \mathbb{N}^*$ and $i \leq n$,

$$\mu(\Lambda,\mathbf{b})^i F_{n,i}^{0,+}\left(\frac{b_1b_2b_3b_4}{\Lambda},\mathbf{b}\right) = F_{n,-i}^{0,+}(\phi;\Lambda,\mathbf{b}), \tag{3.60a}$$

$$\mu(\Lambda,\mathbf{b})^i G_{n,i}^{0,+}\left(\frac{b_1b_2b_3b_4}{\Lambda},\mathbf{b}\right) = G_{n,-i}^{0,+}(\phi;\Lambda,\mathbf{b}). \tag{3.60b}$$

Equations (3.54) show that (3.59) is true when $n = 1$ and we have checked the case $n = 2$ using Mathematica. As an additional check, Proposition 3.5.1 is consistent with equations (3.60).

Remark 3.3.4. By equations (3.54) and (3.29), we see that the coefficients $F_1^{0,+}$ and $G_1^{0,+}$ are only singular when $\Lambda^2 = b_1b_2b_3b_4$. Reflecting on the proofs of Theorem 3.3.1 and B.3, this implies that condition (3.46) in Theorem 3.3.1 can be relaxed to $1 \notin Q_1$, where $Q_1 \subseteq Q$ equals

$$Q_1 = \{q_1^m q_2^n : (m,n) \in \mathbb{N}^2 \setminus \{0,0\} \text{ with } m \geq 1\} \cup \{q_2\}.$$

In particular, if $|q_2| = 1$ with $q_2 \neq 1$ and $|q_1| < 1$, then the convergence of expansions (3.48) still holds. If Conjecture 3.3.3 is true, then condition (3.46) can be relaxed further, to $1 \notin Q_{\text{rel}}$, where $Q_{\text{rel}} \subseteq Q$ is defined as

$$Q_{\text{rel}} = \{q_1^m q_2^n : (m,n) \in \mathbb{N}^2 \setminus \{0,0\} \text{ with } n \leq m+1\}.$$

3.3.1 Formal Series Solution about Infinity

The Bäcklund transformation $\mathcal{T}_4$ defined in (2.11), shows that the critical points 0 and ∞ play an essentially equivalent role in q-$P(A_1)$. Using Bäcklund transformation $\mathcal{T}_2$ and Theorem 3.3.1, it is easy to see that

$$f = tg^{0,+}\left(\frac{1}{t},\widetilde{\phi};\widetilde{\Lambda},\mathbf{b}^{(2)}\right), \qquad g = tf^{0,+}\left(\frac{1}{t},\widetilde{\phi};\widetilde{\Lambda},\mathbf{b}^{(2)}\right), \tag{3.61}$$

defines a formal solution to q-$P(A_1)(\mathbf{b})$, if $\widetilde{\Lambda}$ and $\widetilde{\phi}$ satisfy

$$\overline{\widetilde{\Lambda}} = \widetilde{\Lambda}, \qquad \overline{\widetilde{\phi}} = \widetilde{\lambda}^{-1}\widetilde{\phi}, \qquad \widetilde{\lambda} = \frac{\widetilde{\Lambda}^2}{b_1^{(2)}b_2^{(2)}b_3^{(2)}b_4^{(2)}}. \tag{3.62}$$

We introduce formal variables Λ_∞ and ϕ_∞ satisfying

$$\overline{\Lambda}_\infty = \Lambda_\infty, \qquad \overline{\phi}_\infty = \lambda_\infty\phi_\infty, \qquad \lambda_\infty = \frac{\Lambda_\infty^2}{b_5b_6b_7b_8}, \tag{3.63}$$

and set
$$\widetilde{\Lambda}=\frac{1}{\Lambda_\infty},\qquad \widetilde{\phi}=\Lambda_\infty\phi_\infty.$$
Observe that equations (3.62) are satisfied, and upon substitution into equation (3.61), we find that

$$f^{\infty,+}(t,\phi_\infty;\Lambda_\infty,\mathbf{b})=tg^{0,+}\left(\frac{1}{t},\Lambda_\infty\phi_\infty;\frac{1}{\Lambda_\infty},\mathbf{b}^{(2)}\right), \tag{3.64a}$$
$$g^{\infty,+}(t,\phi_\infty;\Lambda_\infty,\mathbf{b})=tf^{0,+}\left(\frac{1}{t},\Lambda_\infty\phi_\infty;\frac{1}{\Lambda_\infty},\mathbf{b}^{(2)}\right), \tag{3.64b}$$

defines a formal series solution to q-$P(A_1)(\mathbf{b})$ at $t=\infty$.
Indeed, expanding this solution in t and ϕ_∞, we find

$$f^{\infty,+}(t,\phi_\infty;\Lambda_\infty,\mathbf{b})=\sum_{n=0}^{\infty}F_n^{\infty,+}(\phi_\infty;\Lambda_\infty,\mathbf{b})t^{-n},$$
$$g^{\infty,+}(t,\phi_\infty;\Lambda_\infty,\mathbf{b})=\sum_{n=0}^{\infty}G_n^{\infty,+}(\phi_\infty;\Lambda_\infty,\mathbf{b})t^{-n},$$

with, for $n\in\mathbb{N}$,

$$F_n^{\infty,+}(\phi_\infty;\Lambda_\infty,\mathbf{b})=G_{n+1}^{0,+}\left(\Lambda_\infty\phi_\infty;\frac{1}{\Lambda_\infty},\mathbf{b}^{(2)}\right),$$
$$G_n^{\infty,+}(\phi_\infty;\Lambda_\infty,\mathbf{b})=F_{n+1}^{0,+}\left(\Lambda_\infty\phi_\infty;\frac{1}{\Lambda_\infty},\mathbf{b}^{(2)}\right),$$

and hence

$$F_n^{\infty,+}(\phi_\infty;\Lambda_\infty,\mathbf{b})=\sum_{i=-\infty}^{n+1}F_{n,i}^{\infty,+}(\Lambda_\infty,\mathbf{b})\phi_\infty^i,\quad G_n^{\infty,+}(\phi_\infty;\Lambda_\infty,\mathbf{b})=\sum_{i=-\infty}^{n+1}G_{n,i}^{\infty,+}(\Lambda_\infty,\mathbf{b})\phi_\infty^i,$$

where, for $n\in\mathbb{N}$ and $i\in\mathbb{Z}_{\leq n+1}$,

$$F_{n,i}^{\infty,+}(\Lambda_\infty,\mathbf{b})=\Lambda_\infty^iG_{n+1,i}^{0,+}\left(\frac{1}{\Lambda_\infty},\mathbf{b}^{(2)}\right),\quad G_{n,i}^{\infty,+}(\Lambda_\infty,\mathbf{b})=\Lambda_\infty^iF_{n+1,i}^{0,+}\left(\frac{1}{\Lambda_\infty},\mathbf{b}^{(2)}\right).$$

Of course we can formulate analogous convergence results to the ones in Theorem 3.3.1. To obtain the dual 'minus' solutions at infinity, we again take Λ_∞ and ϕ_∞ satisfying equations (3.63), and set
$$\widetilde{\Lambda}=\frac{\Lambda_\infty}{b_5b_6b_7b_8},\qquad \widetilde{\phi}=\mu\left(\frac{\Lambda_\infty}{b_5b_6b_7b_8},\mathbf{b}^{(2)}\right)\frac{1}{\Lambda_\infty\phi_\infty}.$$
in equations (3.61).

3.3.2 Symmetries of Formal Series Solution

In Section 2.2.4 we discussed several Bäcklund transformations of q-$P(A_1)$. Using these we can find symmetries of the formal series solution (3.44). We discuss three such examples. First of all, note that for any permutation

$$\sigma \in Sym(\{1,2,3,4\}) \times Sym(\{5,6,7,8\}), \tag{3.65}$$

q-$P(A_1)$ is invariant under permutation of the parameters $\sigma(b)_i = b_{\sigma(i)}$ correspondingly for $1 \leq i \leq 8$, and using Theorem 3.3.1 we deduce

$$f^{0,+}(t,\phi;\Lambda,\mathbf{b}) = f^{0,+}\left(t,\phi;\Lambda,\sigma(\mathbf{b})\right), \qquad g^{0,+}(t,\phi;\Lambda,\mathbf{b}) = g^{0,+}\left(t,\phi;\Lambda,\sigma(\mathbf{b})\right). \tag{3.66}$$

Next, we would like to derive a symmetry of the formal series solution (3.44) by application of Bäcklund transformation $\mathcal{T}_1$ as defined in (2.11). Consider formal variables ϕ and Λ satisfying (3.30) and put

$$\widetilde{\phi} = \frac{1}{t\phi}, \qquad \widetilde{\Lambda} = \frac{1}{\Lambda}.$$

Then we have

$$\overline{\widetilde{\Lambda}} = \widetilde{\Lambda}, \qquad \overline{\widetilde{\phi}} = \frac{\widetilde{\Lambda}^2}{b_1^{(1)}b_2^{(1)}b_3^{(1)}b_4^{(1)}}\widetilde{\phi},$$

and by Theorem 3.3.1 this implies that

$$f^{0,+}\left(t,\widetilde{\phi};\widetilde{\Lambda},\mathbf{b}^{(1)}\right) = f^{0,+}\left(t,\frac{1}{t\phi};\frac{1}{\Lambda},\mathbf{b}^{(1)}\right), \quad g^{0,+}\left(t,\widetilde{\phi};\widetilde{\Lambda},\mathbf{b}^{(1)}\right) = g^{0,+}\left(t,\frac{1}{t\phi};\frac{1}{\Lambda},\mathbf{b}^{(1)}\right),$$

defines a formal solution to q-$P(A_1)(\mathbf{b}^{(1)})$.
We apply Bäcklund transformation $\mathcal{T}_1$, which shows that

$$f(t,\phi) = \frac{t}{f^{0,+}\left(t,\frac{1}{t\phi};\frac{1}{\Lambda},\mathbf{b}^{(1)}\right)}, \qquad g(t,\phi) = \frac{t}{g^{0,+}\left(t,\frac{1}{t\phi};\frac{1}{\Lambda},\mathbf{b}^{(1)}\right)}, \tag{3.67}$$

defines a formal solution to q-$P(A_1)(\mathbf{b})$.
We expand this solution in powers of t and ϕ and prove that it is exactly the formal series solution (3.44). First of all, for the denominators in (3.67), expanding in t gives

$$f^{0,+}\left(t,\frac{1}{t\phi};\frac{1}{\Lambda},\mathbf{b}^{(1)}\right) = \sum_{m=0}^{\infty}\widetilde{f}_m(\phi;\Lambda,\mathbf{b})t^m,$$
$$g^{0,+}\left(t,\frac{1}{t\phi};\frac{1}{\Lambda},\mathbf{b}^{(1)}\right) = \sum_{m=0}^{\infty}\widetilde{g}_m(\phi;\Lambda,\mathbf{b})t^m,$$

where, for $m \in \mathbb{N}$,

$$\widetilde{f}_m(\phi;\Lambda,\mathbf{b}) = \sum_{i=-\infty}^{m-1} F^{0,+}_{m-i,-i}\left(\frac{1}{\Lambda},\mathbf{b}^{(1)}\right)\phi^i, \tag{3.68a}$$

$$\widetilde{g}_m(\phi;\Lambda,\mathbf{b}) = \sum_{i=-\infty}^{m-1} G^{0,+}_{m-i,-i}\left(\frac{1}{\Lambda},\mathbf{b}^{(1)}\right)\phi^i. \tag{3.68b}$$

We can hence expand equations (3.67) in t, using for instance the *Lagrange inversion formula*, to obtain

$$f(t,\phi) = \frac{1}{\widetilde{f}_0(\phi;\Lambda,\mathbf{b})}t - \frac{\widetilde{f}_1(\phi;\Lambda,\mathbf{b})}{\widetilde{f}_0(\phi;\Lambda,\mathbf{b})^2}t^2 + \dots, \tag{3.69a}$$

$$g(t,\phi) = \frac{1}{\widetilde{g}_0(\phi;\Lambda,\mathbf{b})}t - \frac{\widetilde{g}_1(\phi;\Lambda,\mathbf{b})}{\widetilde{g}_0(\phi;\Lambda,\mathbf{b})^2}t^2 + \dots, \tag{3.69b}$$

and compare the result with the formal series solution (3.44).
Indeed, by expanding the coefficients of the series (3.69) with respect to ϕ, we see that they are of exactly the same form as solutions (3.44), that is, we can find $\widetilde{F}_{n,i}$ and $\widetilde{G}_{n,i}$ for $i \in \mathbb{N}_{\leq n}$ and $n \in \mathbb{N}^*$ such that

$$f(t,\phi) = \sum_{n=1}^{\infty}\sum_{i=-\infty}^{n} \widetilde{F}_{n,i}t^n\phi^i, \qquad g(t,\phi) = \sum_{n=1}^{\infty}\sum_{i=-\infty}^{n} \widetilde{G}_{n,i}t^n\phi^i.$$

In particular, calculating $\widetilde{F}_{1,1}$ and $\widetilde{G}_{1,1}$ gives

$$\widetilde{F}_{1,1} = \frac{1}{F^{0,+}_{1,1}\left(\frac{1}{\Lambda},\mathbf{b}^{(1)}\right)} = 1, \qquad \widetilde{G}_{1,1} = \frac{1}{G^{0,+}_{1,1}\left(\frac{1}{\Lambda},\mathbf{b}^{(1)}\right)} = \Lambda.$$

Therefore, by the uniqueness property of the formal series solution (3.44) in Theorem 3.3.1, we have

$$f(t,\phi) = f^{0,+}(t,\phi;\Lambda,\mathbf{b}), \qquad g(t,\phi) = g^{0,+}(t,\phi;\Lambda,\mathbf{b}),$$

and hence, by the definition of f and g (3.67), we obtain the formal identities

$$f^{0,+}(t,\phi;\Lambda,\mathbf{b})f^{0,+}\left(t,\frac{1}{t\phi};\frac{1}{\Lambda},\mathbf{b}^{(1)}\right) = t, \quad g^{0,+}(t,\phi;\Lambda,\mathbf{b})g^{0,+}\left(t,\frac{1}{t\phi};\frac{1}{\Lambda},\mathbf{b}^{(1)}\right) = t. \tag{3.70}$$

These equations induce a countable number of identities among the coefficients, each one given by comparing the coefficients of a positive power of t. In particular, comparing the coefficients of the lowest order term t, we obtain

$$F^{0,+}_1(\phi;\Lambda,\mathbf{b})\widetilde{f}_0(\phi;\Lambda,\mathbf{b}) = 1, \qquad G^{0,+}_1(\phi;\Lambda,\mathbf{b})\widetilde{g}_0(\phi;\Lambda,\mathbf{b}) = 1. \tag{3.71}$$

Combining this identity with equations (3.54), (3.21) and (3.68), we find generating functions,

$$\frac{x}{1+F_{\text{eq}}(\Lambda,\mathbf{b})x+\mu(\Lambda,\mathbf{b})x^2}=\sum_{i=1}^{\infty}F_{i,i}^{0,+}\left(\frac{1}{\Lambda},\mathbf{b}^{(1)}\right)x^i, \tag{3.72a}$$

$$\frac{x}{\Lambda+G_{\text{eq}}(\Lambda,\mathbf{b})x+\frac{b_1b_2b_3b_4}{\Lambda}\mu(\Lambda,\mathbf{b})x^2}=\sum_{i=1}^{\infty}G_{i,i}^{0,+}\left(\frac{1}{\Lambda},\mathbf{b}^{(1)}\right)x^i. \tag{3.72b}$$

Similarly, using Bäcklund transformation $\mathcal{T}_3$, we find formal identities

$$f^{0,+}(t,\phi;\Lambda,\mathbf{b})=g^{0,+}\left(q^{-\frac{1}{2}}t,q^{-\frac{1}{2}}\frac{b_1b_2b_3b_4}{\Lambda}\phi;\frac{\Lambda}{b_5b_6b_7b_8},\mathbf{b}^{(3)}\right), \tag{3.73}$$

$$g^{0,+}(t,\phi;\Lambda,\mathbf{b})=f^{0,+}\left(q^{-\frac{1}{2}}t,q^{\frac{1}{2}}\Lambda\phi;\frac{\Lambda}{b_5b_6b_7b_8},\mathbf{b}^{(3)}\right). \tag{3.74}$$

These equations plays an important role in Section 3.8, where we consider the formal series solution in the perspective of the reduction to symmetric q-$P(A_1)$, as described in Section 2.5.

3.4 Constructing True Solutions

In this section we use the formal series solution (3.44) to construct true solutions of q-$P(A_1)$. The idea is relatively straightforward, we replace the formal variables Λ and ϕ with actual functions satisfying equations (3.30). We first discuss how to construct discrete solutions.

3.4.1 Discrete Solutions

As usual, we adopt the discrete time interpretation

$$t_s=q^st_0,\quad f_s=f(t_s),\quad g_s=g(t_s).\qquad (s\in\mathbb{Z})$$

In this setting, we interpret equations (3.30) as follows,

$$\Lambda_{s+1}=\Lambda_s,\quad \phi_{s+1}=\lambda_s\phi_s,\quad \lambda_s=\frac{\Lambda_s^2}{b_1b_2b_3b_4}. \tag{3.75}$$

Let us take any $\phi_0\in\mathbb{C}^*$ and $\Lambda_0\in L_0(\mathbf{b})$, as defined in (3.47). In accordance with Theorem 3.3.1 and equations (3.75), we put

$$\lambda_0=\frac{\Lambda_0^2}{b_1b_2b_3b_4},\qquad q_1=q\lambda_0,\qquad q_2=\lambda_0^{-1},$$

and define, for $s\in\mathbb{Z}$,

$$\phi_s=\lambda_0^s\phi_0,\quad (\zeta_1)_s=q_1^s\phi_0t_0,\quad (\zeta_2)_s=q_2^s\phi_0^{-1}.$$

As $\Lambda_0 \in L_0(\mathbf{b})$, Theorem 3.3.1 shows that there is an $r > 0$ such that the expansions (3.48) with $\Lambda = \Lambda_0$ converge for all $(\zeta_1, \zeta_2) \in \mathbb{C}^2$ with $|\zeta_1|, |\zeta_2| < r$. Note that $0 < |q_1|, |q_2| < 1$ and we determine an $S \in \mathbb{Z}$, such that, for all $s \geq S$,

$$|(\zeta_1)_s|, |(\zeta_2)_s| < r.$$

Then we know, that for all $s \geq S$,

$$f_s = f^{0,+}(t_s, \phi_s; \Lambda_0, \mathbf{b}) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} F^{0,+}_{n,n-m}(\Lambda_0, \mathbf{b})(\zeta_1)^n_s (\zeta_2)^m_s, \tag{3.76a}$$

$$g_s = g^{0,+}(t_s, \phi_s; \Lambda_0, \mathbf{b}) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} G^{0,+}_{n,n-m}(\Lambda_0, \mathbf{b})(\zeta_1)^n_s (\zeta_2)^m_s, \tag{3.76b}$$

are well-defined, and converge uniformly in s on $\mathbb{Z}_{\geq S}$, defining a solution of q-$P(A_1)$.
Guaranteed by the singularity confinement property, we have an unique continuation of $(f_s, g_s)_{s \geq S}$ to a full solution $(f_s, g_s)_{s \in \mathbb{Z}}$ in $\mathbb{P} \times \mathbb{P}$. Note that this solution is completely determined by our initial choices for Λ_0 and ϕ_0, that is, writing

$$(f_s, g_s)_{s \in \mathbb{Z}} = (f_s(\Lambda_0, \phi_0), g_s(\Lambda_0, \phi_0))_{s \in \mathbb{Z}},$$

we found a family of solutions of q-$P(A_1)$ on discrete q-domains, with two arbitrary *integration constants* $\Lambda_0 \in L_0(\mathbf{b})$ and $\phi_0 \in \mathbb{C}^*$. Note that the leading order behaviour is given by

$$f_s \sim (q\lambda_0)^s t_0 \phi_0, \quad g_s \sim \Lambda_0 (q\lambda_0)^s t_0 \phi_0. \qquad (s \to \infty) \tag{3.77}$$

3.4.2 Meromorphic Solutions

To construct solutions on a continuous q-domain, we replace the formal variables ϕ and Λ, in the formal series solution (3.44), by analytic functions on this q-domain which satisfy equations (3.30). Before stating the main theorem of this section, let us introduce some notation. For a set $V \subseteq \mathbb{C}^*$, we denote its closure in $\mathbb{C}^*$ by $\overline{V}^*$.

Theorem 3.4.1. *Let $\mathbf{b} \in \mathcal{B}_q$. Suppose we have a continuous q-domain T, a function $\Lambda(t)$ which is analytic on T and q-periodic, i.e. $\Lambda(qt) = \Lambda(t)$, satisfying $\Lambda(t) \in L_0(\mathbf{b})$, for $t \in T$. Let $\phi(t)$ be a nonvanishing analytic function on T, satisfying*

$$\phi(qt) = \lambda(t)\phi(t), \quad \lambda(t) := \frac{\Lambda(t)^2}{b_1 b_2 b_3 b_4}. \quad (t \in T) \tag{3.78}$$

Then there exists an unique (nowhere singular) meromorphic solution $(f(t), g(t))$ of q-$P(A_1)$ on T, characterised by the fact that, for every continuous q-domain $V \subseteq \overline{V}^* \subseteq T$, there is *an $r>0$, such that the series*

$$f^{0,+}(t,\phi(t);\Lambda(t),\mathbf{b}) = \sum_{n=1}^{\infty}\sum_{i=-\infty}^{n} F_{n,i}^{0,+}(\Lambda(t),\mathbf{b})t^n\phi(t)^i, \tag{3.79a}$$

$$g^{0,+}(t,\phi(t);\Lambda(t),\mathbf{b}) = \sum_{n=1}^{\infty}\sum_{i=-\infty}^{n} G_{n,i}^{0,+}(\Lambda(t),\mathbf{b})t^n\phi(t)^i, \tag{3.79b}$$

converge uniformly on

$$V\cap\{t\in\mathbb{C}^*:|t|<r\},$$

and we have $f(t)\equiv f^{0,+}(t,\phi(t);\Lambda(t),\mathbf{b})$ and $g(t)\equiv g^{0,+}(t,\phi(t);\Lambda(t),\mathbf{b})$ on this set. In particular the leading order behaviour of this solution within T, i.e. on V as above, is given by

$$f(t)\sim\phi(t)t,\quad g(t)\sim\Lambda(t)\phi(t)t. \qquad (t\to 0) \tag{3.80}$$

Proof. Let us take any continuous q-domain $V\subseteq T$, such that $\overline{V}^*\subseteq T$. We define

$$V_{\text{ann}} = \overline{V}\cap\{t\in\mathbb{C}:1\leq|t|\leq|q|^{-1}\},$$

then V_{ann} is a compact subset of T, and we set

$$\lambda^+ = \sup_{t\in V_{\text{ann}}}|\lambda(t)|,\quad \lambda^- = \inf_{t\in V_{\text{ann}}}|\lambda(t)|,\quad \phi^+ = \sup_{t\in V_{\text{ann}}}|\phi(t)|,\quad \phi^- = \inf_{t\in V_{\text{ann}}}|\phi(t)|.$$

As $\Lambda(t)\in L_0(\mathbf{b})$ for $t\in T$, we have

$$1<\lambda^-\leq\lambda^+<|q|^{-1}. \tag{3.81}$$

By equation (3.78), we obtain,

$$(\lambda^-)^{\log_{|q|}(|t|)}\phi^-\leq|\phi(t)|\leq(\lambda^+)^{\log_{|q|}(|t|)+1}\phi^+,$$

for all $t\in V$.
Let us introduce the variables

$$\zeta_1(t)=t\phi(t),\qquad \zeta_2(t)=\phi(t)^{-1},$$

then we have inequalities

$$|\zeta_1(t)|\leq\lambda^+\phi^+(|q|\lambda^+)^{\log_{|q|}(|t|)}, \tag{3.82a}$$

$$|\zeta_2(t)|\leq\left(\phi^-\right)^{-1}(\lambda^-)^{-\log_{|q|}(|t|)}, \tag{3.82b}$$

for $t\in V$.
Determine $L\subseteq L_0(\mathbf{b})$ open with $\overline{L}\subseteq L_0(\mathbf{b})$, such that

$$\{\Lambda(t):t\in V\}\subseteq L.$$

By Theorem 3.3.1, there is an open environment $Z \subseteq \mathbb{C}^2$ of $\mathbf{0}$, such that the series (3.48) converge uniformly in $(\boldsymbol{\zeta}, \Lambda)$ on $Z \times L$. By inequalities (3.81) and (3.82), we can determine an $r > 0$ such that $\boldsymbol{\zeta}(t) \in Z$ for $|t| < r$. It follows that the series (3.79) converge uniformly on (3.4.1), defining an analytic solution of q-$P(A_1)$ on this set. We apply Lemma (2.1.2) to meromorphically continue this solution to a solution (f_V, g_V) on V. As we can do so for any continuous q-domain $V \subseteq \overline{V}^* \subseteq T$, we take the union of these solutions $f_V(t)$ and $g_V(t)$, giving an unique meromorphic solution $(f(t), g(t))$ of q-$P(A_1)$ on T. □

Note that we can formulate a real version of Theorem 3.4.1. That is, assume that the parameters $\mathbf{b}$ satisfy

$$\overline{\{b_1, b_2, b_3, b_4\}} = \{b_1, b_2, b_3, b_4\}, \qquad \overline{\{b_5, b_6, b_7, b_8\}} = \{b_5, b_6, b_7, b_8\},$$

with $q \in (0, 1)$.
We let $\phi(t)$ and $\Lambda(t)$ be real valued continuous functions satisfying $\Lambda(t) \in L_0(\mathbf{b})$ and (3.78) on $\mathbb{R}_+$. Then there is an $r > 0$ such that the series expansions (3.79) converge uniformly on $(0, r)$ and $f(t) = f^{0,+}(t, \phi(t); \Lambda(t), \mathbf{b})$ and $g(t) = g^{0,+}(t, \phi(t); \Lambda(t), \mathbf{b})$ define a real-valued continuous solution of q-$P(A_1)$ on $(0, r)$. If $\phi(t)$ and $\Lambda(t)$ are real analytic, then $f(t)$ and $g(t)$ are real analytic on $(0, r)$ and there exists an unique piecewise real analytic continuation to a solution on $\mathbb{R}_+$.

In the coming sections, we discuss special cases of the construction in Theorem 3.4.1, leading to different types of interesting leading order behaviour.

3.4.3 Complex Power Type Critical Behaviour

We consider Theorem 3.4.1, where we choose $\Lambda(t) \equiv \Lambda \in L_0(\mathbf{b})$ constant. Then the associated $\lambda(t) \equiv \lambda$ is also constant and we determine a $\rho \in \mathbb{C}$ such that $\exp[\rho \ln q] = \lambda$. We choose a $\phi_0 \in \mathbb{C}^*$ and set $\phi(t) = \phi_0 t^\rho$. As $\rho \notin \mathbb{Z}$, we have to impose a branchcut on the domain $T \subseteq \mathbb{C}^*$, and in order to meet the requirement $qT = T$, we set this branchcut equal to $-q^{\mathbb{R}}$. That is, we define

$$T = \mathbb{C}^* \setminus \{-q^s : s \in \mathbb{R}\}, \quad q^s := \exp[s \ln q]. \qquad (s \in \mathbb{R}) \tag{3.83}$$

Then we can define the complex exponential t^ρ uni-valued on T, with $\phi(qt) = \lambda\phi(t)$ for all $t \in T$. Explicitly we define t^ρ on T as follows. Let $t \in T$, then there are unique $s \in \mathbb{R}$ and $\theta \in (-\pi, \pi)$ such that $t = q^s e^{i\theta}$, and we set

$$t^\rho = \lambda^s e^{-\Im(\rho)\theta} e^{\Re(\rho)\theta i}. \tag{3.84}$$

Applying Theorem 3.4.1 gives us an unique meromorphic solution $(f(t), g(t))$ of q-$P(A_1)$ on T, such that $f(t)$ and $g(t)$ are described by

$$f(t) = f^{0,+}\left(t, \phi_0 t^\rho; q^\rho, \mathbf{b}\right) = \sum_{n=1}^{\infty} \sum_{i=-\infty}^{n} F_{n,i}^{0,+}\left(q^\rho, \mathbf{b}\right) \phi_0^i t^{\rho i + n}, \tag{3.85a}$$

$$g(t) = g^{0,+}\left(t, \phi_0 t^\rho; q^\rho, \mathbf{b}\right) = \sum_{n=1}^{\infty} \sum_{i=-\infty}^{n} G_{n,i}^{0,+}\left(q^\rho, \mathbf{b}\right) \phi_0^i t^{\rho i + n}, \tag{3.85b}$$

for t close to 0, on every continuous q-domain $V \subseteq \overline{V}^* \subseteq T$.
However, since λ is constant, we can do a bit better. There is an $r > 0$, such that the expansions (3.85) converge uniformly on

$$T \cap \{t \in \mathbb{C}^* : |t| < r\}.$$

An interesting special case occurs when $\lambda \in q^{\mathbb{R}}$, then we can choose $\rho \in \mathbb{R}$, and as $1 < |\lambda| < |q|^{-1}$, we have $-1 < \rho < 0$. In Section 3.8.2 we identify the critical behaviour (3.85) with the complex power type behaviour of solutions of Painlevé VI found by Jimbo [42] near critical points, in the continuum limit $q \to 1$. We remark that Mano [61] found similar complex power type critical behaviour for solutions of q-P_{VI} (4.2), the q-analog of P_{VI} derived by Jimbo and Sakai [43].

3.4.4 Oscillatory Type Critical Behaviour

Another case of special interest is given by setting $\lambda = e^{\theta i}$ in Theorem 3.4.1, where $\theta \in \mathbb{R} \backslash 2\pi\mathbb{Z}$. Indeed, by Remark 3.3.4, the expansions (3.44) are well-defined in this case, and converge. This gives rise to solutions of q-$P(A_1)$ with leading order behaviour of *oscillatory type*. Indeed, given a continuous q-domain T and a nonvanishing function $\phi(t)$ satisfying $\phi(qt) = e^{\theta i}\phi(t)$ on T, setting

$$\Lambda(t) = \Lambda = \pm(b_1 b_2 b_3 b_4)^{\frac{1}{2}} e^{\frac{1}{2}\theta i},$$

we can construct an unique meromorphic solution $(f(t), g(t))$ of q-$P(A_1)$ on T, such that, for every continuous q-domain $V \subseteq \overline{V}^* \subseteq T$, there is an $r > 0$, such that the series (3.79) converge uniformly in t on

$$V \cap \{t \in \mathbb{C}^* : |t| < r\},$$

and we have $f(t) \equiv f^{0,+}(t, \phi(t); \Lambda, \mathbf{b})$ and $g(t) \equiv g^{0,+}(t, \phi(t); \Lambda, \mathbf{b})$ on this set.
The leading order behaviour of this solution, as $t \to 0$ within T, is given by

$$f(t) = t\left(\phi(t) + F_{\mathrm{eq}}(\Lambda, \mathbf{b}) + \mu(\Lambda, \mathbf{b})\phi(t)^{-1}\right) + \mathcal{O}\left(t^2\right), \tag{3.86a}$$

$$g(t) = t(\Lambda\phi(t) + G_{\mathrm{eq}}(\Lambda, \mathbf{b}) + \frac{b_1 b_2 b_3 b_4}{\Lambda}\mu(\Lambda, \mathbf{b})\phi(t)^{-1}) + \mathcal{O}\left(t^2\right). \tag{3.86b}$$

If $\theta \in \pi\mathbb{Q}$, then $\phi(t)$ is periodic, leading to a vast number of possible oscillatory type asymptotics in equations (3.87), for different choices of $\phi(t)$. On the other hand, for any $\phi_0 \in \mathbb{C}^*$, we can set $\phi(t) = \phi_0 t^\rho$, as defined in (3.84), where $\rho = i\theta \ln(q)^{-1}$, on T as defined in (3.83). Then we have $\phi(q^s t) = \phi(t)$, where $s = \frac{2\pi}{\theta} \in \mathbb{R}$, which gives oscillatory type asymptotics in equations (3.86) on q-spirals as well. Heuristically speaking, the latter critical behaviour is related to the oscillatory type critical behaviour of solutions of the sixth Painlevé equation, obtained by Guzzetti [24], via the continuum limit.

Let us get back to the general case. Applying Bäcklund Transformation $\mathcal{T}_1$ to the solution $(f(t), g(t))$, we find that

$$\widetilde{f}(t) = \frac{t}{f(t)}, \qquad \widetilde{g}(t) = \frac{t}{g(t)},$$

defines a meromorphic solution of q-$P(A_1)(\mathbf{b}^{(1)})$ on T, with

$$\widetilde{f}(t) = \frac{1}{\phi(t) + F_{\mathrm{eq}}(\Lambda, \mathbf{b}) + \mu(\Lambda, \mathbf{b})\phi(t)^{-1} + \mathcal{O}(t)}, \tag{3.87a}$$

$$\widetilde{g}(t) = \frac{1}{\Lambda\phi(t) + G_{\mathrm{eq}}(\Lambda, \mathbf{b}) + \frac{b_1 b_2 b_3 b_4}{\Lambda}\mu(\Lambda, \mathbf{b})\phi(t)^{-1} + \mathcal{O}(t)}, \tag{3.87b}$$

as $t \to 0$ in T. We remark that Guzzetti [24] also obtained *inverse oscillatory type* critical behaviour for the sixth Painlevé equation.

Accumulating Poles and Base Points

Note that the leading order terms in equations (3.86) can not vanish identically on a non-empty open subset of T, except for a very special choice of the parameters $\mathbf{b} \in \mathcal{B}_q$ and $\lambda = e^{\theta i} = -1$, which we leave to the interested reader to explore. However, for special choices of $\phi(t)$, poles of $\widetilde{f}(t)$ and $\widetilde{g}(t)$ might accumulate at $t = 0$ in T. For example, let us again take a $\phi_0 \in \mathbb{C}^*$, and set $\phi(t) = \phi_0 t^\rho$, as defined in (3.84), where $\rho = i\theta \ln(q)^{-1}$, on T as defined in (3.83). Then we have $\phi(q^s t) = \phi(t)$, where $s = \frac{2\pi}{\theta} \in \mathbb{R}$. Let x_1 and x_2 denote the zeros of

$$x + F_{\mathrm{eq}}(\Lambda, \mathbf{b}) + \mu(\Lambda, \mathbf{b})x^{-1}.$$

Say $t_i \in T$ satisfies $\phi(t_i) = x_i$ for $i = 1, 2$, and assume $s > 0$. Then the leading order term of $f(t)$ in (3.86) vanishes on the spirals $\{q^{ns}t_i : n \in \mathbb{N}\}$, with $i = 1, 2$, which accumulate at $t = 0$. One can image that asymptotic to these spirals, there exists approximate spirals of true poles of $f(t)$, accumulating at $t = 0$. We do not pursue to make such estimates rigorous here, but note that a similar argument has been employed by Guzzetti [28] to prove existence of critical behaviour of solutions of the sixth Painlevé equation, at for instance the critical point 0, with two rays of poles accumulating at the critical point.

Now let us consider the critical behaviour (3.86) in light of for instance the base point $p_1 = (t/b_1, b_1 t)$. The two polynomial equations

$$\phi + F_{\mathrm{eq}}(\Lambda, \mathbf{b}) + \mu(\Lambda, \mathbf{b})\phi^{-1} = 1/b_1, \tag{3.88}$$

$$\Lambda\phi + G_{\mathrm{eq}}(\Lambda, \mathbf{b}) + \frac{b_1 b_2 b_3 b_4}{\Lambda}\mu(\Lambda, \mathbf{b})\phi^{-1} = b_1, \tag{3.89}$$

have the common solution

$$\phi^* = \frac{b_1(\Lambda + b_2 b_3)(\Lambda + b_2 b_4)(\Lambda + b_3 b_4)}{(b_1 b_2 b_3 b_4 - \Lambda^2)^2},$$

which can be checked by direct computation. Let $\phi(t)$ be as above, and suppose $t_0 \in T$ is such that $\phi(t_0) = \phi^*$. Then we have

$$(f(t), g(t)) = (t/b_1, b_1 t) + \mathcal{O}\left(t^2\right),$$

as $t \to 0$ in $\{q^{ns}t_0 : n \in \mathbb{N}\}$. Again it is not implausible for an approximate spiral asymptotic to $\{q^{ns}t_0 : n \in \mathbb{N}\}$ to exist on which the solution hits base points, accumulating at $t = 0$. Note that we can not realise the case $s = 1$, which would heuristically speaking contradict 2.3.3 asymptotically.

3.4.5 An Asymptotic Formula

Given $\Lambda(t)$ and $\phi(t)$, we are interested in obtaining numerics of the via Theorem 3.4.1 associated solution. From a theoretical point of view, we could obtain arbitrary accurate numerics of the solution $(f(t), g(t))$ in T close to $t = 0$, by calculating sufficiently many coefficients in the series (3.79). However, in practice these coefficients seem to be quite hard to calculate for large n. As an example, writing the coefficients $F_{2,0}^{0,+}(\Lambda, \mathbf{b})$ and $G_{2,0}^{0,+}(\Lambda, \mathbf{b})$ of expansions (3.45), down explicitly as a ratio of polynomials in Λ and $b_1, \ldots, b_8$ already requires a couple of pages. Despite this drawback, note that for continuous q-domains $V \subseteq \overline{V}^* \subseteq T$, we have $1 \prec \phi(t) \prec t^{-1}$ on V as $t \to 0$. Therefore

$$\phi(t)^{-1} \prec t, \qquad \phi(t)^i t^n \prec t,$$

for $i \in \mathbb{Z}_{<n}$ and $n \geq 2$, as $t \to 0$ in V.
Hence, by Theorem 3.4.1 and equations (3.79), we have

$$f(t) = f^{0,+}(t, \phi(t); \Lambda(t), \mathbf{b}) = F_{1,0}^{0,+}(\Lambda(t), \mathbf{b})t + \sum_{n=1}^{\infty} F_{n,n}^{0,+}(\Lambda(t), \mathbf{b})t^n \phi(t)^n + o(t),$$
$$g(t) = g^{0,+}(t, \phi(t); \Lambda(t), \mathbf{b}) = G_{1,0}^{0,+}(\Lambda(t), \mathbf{b})t + \sum_{n=1}^{\infty} G_{n,n}^{0,+}(\Lambda(t), \mathbf{b})t^n \phi(t)^n + o(t),$$

as $t \to 0$ in V.
And therefore, using equations (3.72), we obtain

$$f(t) = F_{1,0}^{0,+}(\Lambda(t), \mathbf{b})t + \frac{1}{F_1^{0,+}\left(t^{-1}\phi(t)^{-1}, \Lambda(t)^{-1}, \mathbf{b}^{(1)}\right)} + o(t),$$
$$g(t) = G_{1,0}^{0,+}(\Lambda(t), \mathbf{b})t + \frac{1}{G_1^{0,+}\left(t^{-1}\phi(t)^{-1}, \Lambda(t)^{-1}, \mathbf{b}^{(1)}\right)} + o(t),$$

as $t \to 0$ in V, and explicit formulas for $F_1^{0,+}$ and $G_1^{0,+}$ are given by equations (3.29) and (3.54).

3.4.6 Critical Behaviour near Infinity

As $t=0$ and $t=\infty$ play an equivalent role in q-$P(A_1)$, we can easily formulate an analog result for Theorem 3.4.1 around $t=\infty$.

Theorem 3.4.2. *Let $\mathbf{b}\in\mathcal{B}_q$, suppose we have a continuous q-domain T, a function $\Lambda_\infty(t)$ which is analytic on T and q-periodic, i.e. $\Lambda_\infty(qt)=\Lambda_\infty(t)$, satisfying $\Lambda_\infty(t)\in L_0(\mathbf{b})$, for $t\in T$. Let $\phi_\infty(t)$ be a nonvanishing analytic function on T, satisfying*

$$\phi_\infty(qt)=\lambda_\infty(t)\phi_\infty(t),\quad \lambda_\infty(t):=\frac{\Lambda_\infty(t)^2}{b_5b_6b_7b_8}.\quad (t\in T) \tag{3.90}$$

Then there exists an unique (nowhere singular) meromorphic solution $(f(t),g(t))$ of q-$P(A_1)$ on T, characteristed by the fact that, for every continuous q-domain $V\subseteq\overline{V}^*\subseteq T$, there is an $r>0$, such that the series

$$f^{\infty,+}(t,\phi_\infty(t);\Lambda_\infty(t),\mathbf{b})=\sum_{n=0}^{\infty}\sum_{i=-\infty}^{n+1}F_{n,i}^{\infty,+}(\Lambda_\infty(t),\mathbf{b})t^{-n}\phi_\infty(t)^i, \tag{3.91a}$$

$$g^{\infty,+}(t,\phi_\infty(t);\Lambda_\infty(t),\mathbf{b})=\sum_{n=0}^{\infty}\sum_{i=-\infty}^{n+1}G_{n,i}^{\infty,+}(\Lambda_\infty(t),\mathbf{b})t^{-n}\phi_\infty(t)^i, \tag{3.91b}$$

converge uniformly on

$$V\cap\{t\in\mathbb{C}^*:|t^{-1}|<r\},$$

and we have $f(t)\equiv f^{\infty,+}(t,\phi_\infty(t);\Lambda_\infty(t),\mathbf{b})$ and $g(t)\equiv g^{\infty,+}(t,\phi_\infty(t);\Lambda_\infty(t),\mathbf{b})$ on this set. In particular the leading order behaviour of this solution within T, i.e. in V as above, is given by

$$f(t)\sim\phi_\infty(t),\quad g(t)\sim\Lambda_\infty(t)\phi_\infty(t).\qquad (t\to\infty) \tag{3.92}$$

Proof. The proof is analogous to the proof of Theorem 3.4.1. □

Similar to the beginning of Section 3.4, we construct solutions on a discrete q-domain $q^{\mathbb{Z}}t_0$, by taking a $\Lambda_\infty\in L_0(\mathbf{b})$ and $\phi_\infty\in\mathbb{C}^*$, setting

$$t_s=q^st_0,\quad \lambda_\infty=\frac{\Lambda_\infty^2}{b_5b_6b_7b_8},$$

and determining an $S\in\mathbb{Z}$, such that

$$f_s=f^{0,+}(t_s,\lambda_\infty^s\phi_\infty;\Lambda_0,\mathbf{b}),\quad g_s=g^{0,+}(t_s,\lambda_\infty^s\phi_\infty;\Lambda_0,\mathbf{b}),$$

converge uniformly in s on $\mathbb{Z}_{\leq S}$, defining a solution of q-$P(A_1)$.
Guaranteed by the the singularity confinement property, we have an unique continuation of $(f_s,g_s)_{s\leq S}$ to a full solution $(f_s,g_s)_{s\in\mathbb{Z}}$ in $\mathbb{P}\times\mathbb{P}$. Note that this solution is completely determined by our initial choices for Λ_∞ and ϕ_∞, that is, writing

$$(f_s,g_s)_{s\in\mathbb{Z}}=(f_s(\Lambda_\infty,\phi_\infty),g_s(\Lambda_\infty,\phi_\infty))_{s\in\mathbb{Z}},$$

we found a family of solutions of q-$P(A_1)$ on discrete q-domains, with two arbitrary *integration constants* $\Lambda_\infty \in L_0(\mathbf{b})$ and $\phi_\infty \in \mathbb{C}^*$. Furthermore we have asymptotic formula

$$f_s \sim \lambda_\infty^s \phi_\infty, \quad g_s \sim \Lambda_\infty \lambda_\infty^s \phi_\infty. \qquad (s \to -\infty)$$

3.5 Six Special One-Parameter Families of Critical Behaviour

As a consequence of Conjecture 3.3.3, we expect the inner summations in (3.45) to terminate at $i = 0$, i.e. all negative powers of ϕ to disappear, when Λ is equal to any of the roots of $\mu(\Lambda, \mathbf{b})$. Indeed we have the following result.

Proposition 3.5.1. *Let $k \in \{1, 2, 3\}$, and $\Lambda_k^\pm$ and λ_k be defined as in Section (3.2.2), where we fix the sign $\pm$ throughout the proposition. Take $\mathbf{b} \in \mathcal{B}$ such that*

$$1 \notin Q_s := \{(\lambda_k^{\pm 1} q)^{m-1} q^n : (m, n) \in \mathbb{N}^2 \setminus \{(1, 0)\}. \tag{3.93}$$

Then, setting $\Lambda = \Lambda_k^\pm$, the formal solution (3.44) of q-$P(A_1)$, defined in Theorem 3.3.1, takes the form

$$f^{0,+}(t, \phi; \Lambda_k^\pm, \mathbf{b}) = \sum_{n=1}^{\infty} \sum_{i=0}^{n} F_{n,i}^{0,+}(\Lambda_k^\pm, \mathbf{b}) \phi^i t^n, \tag{3.94a}$$

$$g^{0,+}(t, \phi; \Lambda_k^\pm, \mathbf{b}) = \sum_{n=1}^{\infty} \sum_{i=0}^{n} G_{n,i}^{0,+}(\Lambda_k^\pm, \mathbf{b}) \phi^i t^n, \tag{3.94b}$$

where ϕ satisfies $\overline{\phi} = \lambda_k^{\pm 1} \phi$.
Assuming $|q| < 1$, $|\lambda_k^{\pm 1}| < |q|^{-1}$ and $\lambda_k^{\pm 1} \notin q^{\mathbb{N}^*}$, condition (3.93) is satisfied and this formal solution, written in terms of the variables t and $\zeta_1 = t\phi$,

$$f^{0,+}(t, \zeta_1/t; \Lambda_k^\pm, \mathbf{b}) = \sum_{m=1}^{\infty} F_{m,0}^{0,+}(\Lambda_k^\pm, \mathbf{b}) t^m + \sum_{i=1}^{\infty} \sum_{m=0}^{\infty} F_{m+i,i}^{0,+}(\Lambda_k^\pm, \mathbf{b}) \zeta_1^i t^m, \tag{3.95a}$$

$$g^{0,+}(t, \zeta_1/t; \Lambda_k^\pm, \mathbf{b}) = \sum_{m=1}^{\infty} G_{m,0}^{0,+}(\Lambda_k^\pm, \mathbf{b}) t^m + \sum_{i=1}^{\infty} \sum_{m=0}^{\infty} G_{m+i,i}^{0,+}(\Lambda_k^\pm, \mathbf{b}) \zeta_1^i t^m, \tag{3.95b}$$

converges near $(t, \zeta_1) = (0, 0)$.
Furthermore, the pair of isolated power series in (3.95), equals the solution $(f^{(1,k)}, g^{(1,k)})$, holomorphic at $t = 0$, defined in Proposition 3.1.2, that is,

$$f^{0,+}(t, 0; \Lambda_k^\pm, \mathbf{b}) = \sum_{m=1}^{\infty} F_{m,0}^{0,+}(\Lambda_k^\pm, \mathbf{b}) t^m = f^{(1,k)}(t), \tag{3.96a}$$

$$g^{0,+}(t, 0; \Lambda_k^\pm, \mathbf{b}) = \sum_{m=1}^{\infty} G_{m,0}^{0,+}(\Lambda_k^\pm, \mathbf{b}) t^m = g^{(1,k)}(t), \tag{3.96b}$$

and in particular these do not depend on the choice of sign $\pm$ in $\Lambda_k^\pm$.

Proof. For notational simplicity, we discuss the particular case $\Lambda = \Lambda_1^+ = -b_1b_2$, noting that the other cases can be dealt with analogously. We assume condition (3.93) with $k = 1$ and $\pm = +$, plus the additional conditions

$$b_1 + b_2 \neq b_3 + b_4, \qquad b_1^{-1} + b_2^{-1} \neq b_3^{-1} + b_4^{-1}, \qquad 1 \notin \lambda_1^2 Q_s. \tag{3.97}$$

Once we have proven the proposition with these additional assumptions, we can easily discard them by analytic continuation using Remark 3.3.2. Indeed, given the proposition, we find, that condition (3.46) in Theorem 3.3.1 can be replaced by $1 \notin Q_s$ when $\Lambda = \Lambda_1^+$, as in Remark 3.3.4. The idea of the proof is to construct a formal solution $(f(t,\phi), g(t,\phi))$ of q-$P(A_1)$, which has an expansion in t and ϕ, exactly of the form (3.94), and subsequently use the uniqueness property in Theorem 3.3.1 to conclude

$$f^{0,+}(t,\phi;\Lambda_1^+,\mathbf{b}) = f(t,\phi), \qquad g^{0,+}(t,\phi;\Lambda_1^+,\mathbf{b}) = g(t,\phi). \tag{3.98}$$

Firstly, by (3.93), we have $\lambda_1 \notin q^{\mathbb{Z}}$, and using the first two conditions in (3.97), we construct the solution $(f^{(1,1)}, g^{(1,1)})$ of q-$P(A_1)$, holomorphic at $t = 0$, defined in Proposition 3.1.2. Next we apply the following change of variables

$$f(t,\phi) = f^{(1,1)}(t) + \zeta_1\left(1 + y_1(t,\zeta_1)\right), \qquad g(t,\phi) = g^{(1,1)}(t) + \zeta_1\left(-b_1b_2 + y_2(t,\zeta_1)\right), \tag{3.99}$$

which allows us to rewrite q-$P(A_1)$ as

$$y_1(qt, q\lambda\zeta_1) = H_1\left(t, \zeta_1, y_1(t,\zeta_1), y_2(t,\zeta_2)\right), \tag{3.100a}$$
$$y_2(qt, q\lambda\zeta_1) = H_2\left(t, \zeta_1, y_1(t,\zeta_1), y_2(t,\zeta_2)\right), \tag{3.100b}$$

for some functions $H_1(t,\zeta_1,y_1,y_2)$ and $H_2(t,\zeta_1,y_1,y_2)$ which are rational in the elements of

$$\left\{t, \zeta_1, y_1, y_2, f^{(1,1)}(t), g^{(1,1)}(t), f^{(1,1)}(qt), g^{(1,1)}(qt)\right\}. \tag{3.101}$$

We wish to apply the q-Briot Bouquet theorem B.3 with $\mathbf{Y} = (0,0)$, therefore the first condition we have to check is that H_1 and H_2 are holomorphic at $(t,\zeta_1,y_1,y_2) = (0,0,0,0)$. As H_1 and H_2 are rational in the elements of (3.101), it is enough to expand H_1 and H_2 as series in t, ζ_1, y_1, y_2 and check that no negative powers appear. Expanding H_1 and H_2 in ζ_1, we find for $i = 1, 2$,

$$H_i(t,\zeta_1,y_1,y_2) = h_{-1}^{(i)}(t)\zeta_1^{-1} + h_0^{(i)}(t,y_1,y_2) + h_1^{(i)}(t,y_1,y_2)\zeta_1 + \ldots.$$

The coefficients $h_{-1}^{(i)}(t)$ are rational in t, $f^{(1,1)}(t)$, $g^{(1,1)}(t)$, $f^{(1,1)}(qt)$ and $g^{(1,1)}(qt)$. Formally speaking $h_{-1}^{(1)}(t) = 0$ and $h_{-1}^{(2)}(t) = 0$ is equivalent to the q-$P(A_1)$ equation with $f = f^{(1,1)}(t)$ and $g = g^{(1,1)}(t)$. That is, $h_{-1}^{(1)}(t)$ and $h_{-1}^{(2)}(t)$ are identically zero, precisely because we are perturbing around a solution of q-$P(A_1)$. We conclude, for $i = 1, 2$,

$$H_i(t,\zeta_1,y_1,y_2) = h_0^{(i)}(t,y_1,y_2) + h_1^{(i)}(t,y_1,y_2)\zeta_1 + h_2^{(i)}(t,y_1,y_2)\zeta_1^2 + \ldots.$$

In a similar fashion one can calculate that $H_i(t,\zeta_1,y_1,y_2)$ enjoys a power series expansion in the other variables y_1, y_2 and t, for $i=1,2$. The y_1 and y_2 cases are rather trivial, but in the t case, we use the fact that we perturb around a solution of q-$P(A_1)$, holomorphic at $t=0$, in an essential way. We conclude that H_1 and H_2 are holomorphic at $(t,\zeta_1,y_1,y_2)=(0,0,0,0)$, and calculate

$$H_1(0,0,y_1,y_2) = -\lambda_1^{-1}y_1 - \frac{1}{b_1b_2}\left(1+\lambda_1^{-1}\right)y_2,$$
$$H_2(0,0,y_1,y_2) = b_3b_4\left(1+\lambda_1^{-1}\right)y_1 + \left(1+\lambda_1^{-1}+\lambda_1^{-2}\right)y_2.$$

So $H_i(0,0,0,0)=0$ for $i=1,2$, and the Jacobi matrix

$$\begin{pmatrix} \frac{\partial H_1}{\partial y_1}(0,0,0,0) & \frac{\partial H_1}{\partial y_2}(0,0,0,0) \\ \frac{\partial H_1}{\partial y_1}(0,0,0,0) & \frac{\partial H_1}{\partial y_2}(0,0,0,0) \end{pmatrix} = \begin{pmatrix} -\lambda_1^{-1} & -\frac{1}{b_1b_2}\left(1+\lambda_1^{-1}\right) \\ b_3b_4\left(1+\lambda_1^{-1}\right) & 1+\lambda_1^{-1}+\lambda_1^{-2} \end{pmatrix},$$

has eigenvalues 1 and λ_1^{-2}.
By (3.93) and the third additional assumption in (3.97), we can apply the q-Briot Bouquet Theorem B.3, to obtain an unique power series solution to (3.100) of the form

$$y_i(t,\zeta_1) = \sum_{m,n=0}^{\infty} y_{m,n}^{(i)} t^m \zeta_1^n,$$

with $y_{0,0}^{(i)}=0$ for $i=1,2$.
Associated via equations (3.99), we have the solution $(f(t,\phi), g(t,\phi))$ of q-$P(A_1)$ with

$$f(t,\phi) = \sum_{n=1}^{\infty}\sum_{i=0}^{n} f_{n,i}\phi^i t^n, \qquad g(t,\phi) = \sum_{n=1}^{\infty}\sum_{i=0}^{n} g_{n,i}\phi^i t^n,$$

where

$$f_{1,1}=1, \qquad f_{1,0}=f_1^{(1,1)}, \qquad f_{n,0}=f_n^{(1,1)}, \qquad f_{n,i}=y_{n-i,i-1}^{(1)}, \tag{3.102}$$
$$g_{1,1}=-b_1b_2, \qquad g_{1,0}=g_1^{(1,1)}, \qquad g_{n,0}=g_n^{(1,1)}, \qquad g_{n,i}=y_{n-i,i-1}^{(2)}, \tag{3.103}$$

for $1\leq i\leq n$ and $n\in\mathbb{N}_{\geq 2}$.
By the uniqueness property in Theorem 3.3.1 we conclude that (3.98) must hold. The remaining convergence result follows from the q-Briot Bouquet Theorem B.3. □

The proof of Proposition 3.5.1 is not particularly elegant. This lies in the fact that we are dealing with a strongly resonant case in light of the general solution of a q-Briot Bouquet type equation. We do not want to delve too far into this issue, but just like to point out that the difficulty comes from the fact that in the case of solutions, holomorphic at $t=0$, the two eigenvalues of the relevant Jacobi matrix are each other's reciprocals, as the proof of Proposition 3.1.1 shows. We avoid this issue by a change of dependent and independent variables, with the cost of dealing with some additional assumptions (3.97).

Remark 3.5.2. Equations (3.96) allow us to analytically continue, for instance the solution

$\left(f^{(1,1)}(t), g^{(1,1)}(t)\right)$ defined in Proposition 3.1.2, to the degenerate parameter cases $b_1 + b_2 = b_3 + b_4$ and $b_1^{-1} + b_2^{-1} = b_3^{-1} + b_4^{-1}$.

Let us discuss the particular case $\Lambda = \Lambda_1^+ = -b_1 b_2$ in Proposition 3.5.1, in more detail. We choose some parameter values $\mathbf{b} \in \mathcal{B}$, such that $|q| < 1$, $|\lambda_1| < |q|^{-1}$ and $\lambda_1 \notin q^{\mathbb{N}^*}$. In particular condition (3.93) is satisfied. Strictly speaking, Theorem 3.4.1 is only applicable if $\Lambda = -b_1 b_2 \in L_0(\mathbf{b})$, or equivalently,

$$1 < |\lambda_1| < |q|^{-1}.$$

However, the convergence result of (3.95) in Proposition 3.5.1, allows us to easily extend the results of Theorem 3.4.1 to the cases $|\lambda_1| < 1$ and $|\lambda_1| = 1$. Indeed, let us consider the case $|\lambda_1| < 1$, and take some analytic function $\phi(t)$ which satisfies $\phi(qt) = \lambda_1 \phi(t)$ on a continuous q-domain $T \subseteq \mathbb{C}^*$. Then there exists an unique meromorphic solution $(f(t), g(t))$ of q-$P(A_1)$ on T, characterised by

$$f(t) = f^{0,+}(t, \phi(t); -b_1 b_2, \mathbf{b}) = \sum_{n=1}^{\infty} \sum_{i=0}^{n} F_{n,i}^{0,+}(-b_1 b_2, \mathbf{b}) \phi(t)^i t^n, \tag{3.104a}$$
$$g(t) = g^{0,+}(t, \phi(t); -b_1 b_2, \mathbf{b}) = \sum_{n=1}^{\infty} \sum_{i=0}^{n} G_{n,i}^{0,+}(-b_1 b_2, \mathbf{b}) \phi(t)^i t^n, \tag{3.104b}$$

for t small in T, as the right-hand sides converge uniformly in t on any continouous q-domain $V \subseteq \overline{V}^* \subseteq T$, intersected with a disk centered at the origin with radius chosen small enough. In particular, by equations (3.96), the leading order behaviour of $f(t)$ and $g(t)$ is given by

$$f(t) = f^{(1,1)}(t) + \phi(t) t + \mathcal{O}\left(\phi(t) t^2\right),$$
$$g(t) = g^{(1,1)}(t) - b_1 b_2 \phi(t) t + \mathcal{O}\left(\phi(t) t^2\right),$$

as $t \to 0$ in V as above.
Of course the choice $\phi(t) \equiv 0$ gives $f(t) = f^{(1,1)}(t)$ and $g(t) = g^{(1,1)}(t)$. Now let us realise the case $f \prec t$ with $g \asymp t$ in (3.18), by assuming the condition

$$b_1 + b_2 = b_3 + b_4, \tag{3.105}$$

given in (3.20). Indeed the leading term of $f^{(1,1)}(t)$ vanishes, as $f_1^{(1,1)} = 0$, and hence we generically have $f \prec t$ and $g \asymp t$ as $t \to 0$ in $V \subseteq T$ as above. If we also set

$$b_1^{-1} + b_2^{-1} = b_3^{-1} + b_4^{-1}, \tag{3.106}$$

then the leading term of $g^{(1,1)}(t)$ also vanishes, as $g_1^{(1,1)} = 0$, and this realises the case $f, g \prec t$ as $t \to 0$ in (3.18). Note that (3.105) and (3.106) imply $b_1 = -b_2$ and $b_3 = -b_4$, so condition (3.19) is trivially satisfied, as expected. To give the reader an appreciation how far the rabbit hole of degenerations goes, let us consider the case

$$b_1 = -b_2, \qquad b_3 = -b_4, \qquad b_5 = -b_6, \qquad b_7 = -b_8, \qquad b_2 = b_6. \tag{3.107}$$

The solution $(f(t), g(t))$ takes the form

$$f(t) = t\phi(t), \quad g(t) = b_2^2 t\phi(t), \qquad \phi(qt) = \lambda_1\phi(t), \quad \lambda_1 = \left(\frac{b_2}{b_4}\right)^2, \quad q = \left(\frac{b_4}{b_8}\right)^2, \tag{3.108}$$

where the parameters b_2, b_4 and b_8 can be chosen at our pleasure. In particular, let us fix some $b_4, b_8 \in \mathbb{C}^*$ with $|b_4| < |b_8|$. Then, for any $m \in \mathbb{N}$, we can choose $b_2 \in \mathbb{C}^*$ small enough, such that $|\lambda_1| < |q|^m$, which gives

$$f(t), g(t) \prec t^{m+1},$$

as $t \to 0$ in any continuous q-domain $V \subseteq \overline{V}^* \subseteq T$.

Let us return to the generic case (3.104), application of Bäcklund transformation $\mathcal{T}_1$ and a permutation $\mathbf{b} \mapsto \mathbf{b}^{(1)}$ of the parameters, gives an associated solution $(\tilde{f}(t), \tilde{g}(t))$ of q-$P(A_1)(\mathbf{b})$, with

$$\tilde{f}(t) = f^{(0,1)}(t)\left[1 + f^{(0,1)}(t)\widetilde{\phi}(t)\right]^{-1} + \mathcal{O}\left(\widetilde{\phi}(t)t\right), \tag{3.109a}$$

$$\tilde{g}(t) = g^{(0,1)}(t)\left[1 - b_5^{-1}b_6^{-1}g^{(0,1)}(t)\widetilde{\phi}(t)\right]^{-1} + \mathcal{O}\left(\widetilde{\phi}(t)t\right), \tag{3.109b}$$

as $t \to 0$ in V as above, where $\widetilde{\phi}(qt) = \lambda\widetilde{\phi}(t)$ with $\lambda = \frac{b_7 b_8}{b_5 b_6}$, subject to conditions $|q| < 1$, $|\lambda| < 1$, $\lambda \notin q^{\mathbb{N}^*}$ and, to ensure the validity of the asymptotics (3.109),

$$b_5 + b_6 \neq b_7 + b_8 \qquad \text{and} \qquad b_5^{-1} + b_6^{-1} \neq b_7^{-1} + b_8^{-1}.$$

Setting $\widetilde{\phi}(t) \equiv 0$, gives the solution $\tilde{f}(t) = f^{(0,1)}(t)$ and $\tilde{g}(t) = g^{(0,1)}(t)$ defined in Proposition 3.1.1.

3.6 The Logarithmic Case

In Sections 3.4 and 3.5 we have been able to find complete expansions of the formal solutions of q-$P(A_1)$ associated with the generic and 6 special families of formal solutions of the leading order autonomous system (3.22). Subsequently we have been able to turn such formal solutions into true solutions by appropriate substitutions for the formal variables entering the formal solutions. The only case left the discuss is the *logarithmic type* solutions of the leading order system, discussed in Section 3.2.3. However this case does not seem very straightforward, even on a formal level. We expect the complete expansion of the formal solution of q-$P(A_1)$, associated with (3.40), to take the form

$$f(t,\chi) = \sum_{n=1}^{\infty} F_n^{\pm}(\chi)t^n, \quad g(t,\chi) = \sum_{n=1}^{\infty} G_n^{\pm}(\chi)t^n, \tag{3.110}$$

where of course $F_1^{\pm}(\chi) = F_l^{\pm}(\chi)$ and $G_1^{\pm}(\chi) = G_l^{\pm}(\chi)$ as defined in (3.40), and more generally the coefficients $F_n^{\pm}(\chi)$ and $G_n^{\pm}(\chi)$ are polynomials of degree $2n$ in χ,

$$F_n^{\pm}(\chi) = \sum_{i=0}^{2n} F_{n,i}^{\pm}\chi^i, \quad G_n^{\pm}(\chi) = \sum_{i=0}^{2n} G_{n,i}^{\pm}\chi^i.$$

By direct calculation using Mathematica, we found that there indeed exist unique degree 4 polynomials in χ for $F_2^{\pm}(\chi)$ and $G_2^{\pm}(\chi)$, solving (3.43) with $n = 2$. However carrying out such a computation becomes very unattractive already for $n = 3$. A theoretical understanding is required, but the author does not know a method of attack, at the time of writing this thesis. Logarithmic type critical behaviour has been obtained for solutions of the sixth Painlevé equation, see for instance Guzzetti [25], however convergence of the corresponding complete expansions is still an open problem [28] at the time of writing.

Let us assume that indeed the complete formal expansion (3.110) is valid. Say we wish to construct true solutions of q-$P(A_1)$ on some discrete q-domain $q^{\mathbb{Z}}t_0$, typically we would set

$$t_s = q^s t_0, \quad \chi_s = \chi_0 + s, \quad (s \in \mathbb{Z})$$

where we can choose $\chi_0 \in \mathbb{C}$ at pleasure. Note that, for $n \in \mathbb{N}^*$,

$$F_n^{\pm}(\chi_s)t_s^n \sim F_{n,2n}^{\pm} t_0^n q^{sn} s^{2n}, \quad G_n^{\pm}(\chi_s)t_s^n \sim G_{n,2n}^{\pm} t_0^n q^{sn} s^{2n}. \quad (s \to \infty)$$

Since $q^{sn}s^{2n} \to 0$ as $s \to \infty$, we would hope that

$$f_s := f(t_s, \chi_s) = \sum_{n=1}^{\infty} F_n^{\pm}(\chi_s)t_s^n, \quad g_s := g(t_s, \chi_s) = \sum_{n=1}^{\infty} G_n^{\pm}(\chi_s)t_s^n,$$

converge uniformly on $\{s \in \mathbb{Z} : s \geq S\}$, for some $S \in \mathbb{Z}$ large enough, defining a true solution of q-$P(A_1)$. Note that this solution has one free parameter $\chi_0 \in \mathbb{C}$.

As to solutions on open q-domains, we would typically consider

$$\chi(t) = \log_q t + c(t),$$

where $c(t)$ any q-periodic function.

3.7 Summary and Outlook

Let us summarise what we have done so far. We started with a somewhat heuristic comparison of possible asymptotic growths as $t \to 0$ of solutions of q-$P(A_1)$ in Section 3.2. This lead to three different cases, which we write down again for convenience of the reader,

$$f, g \asymp t, \qquad t \prec f, g \prec 1, \qquad f, g \asymp 1. \qquad (t \to 0)$$

Assuming $t \preceq f, g \prec 1$, we showed that the leading order behaviour satisfies an autonomous system (3.22). We derived the general solution of this autonomous system, and we were able to construct, for any solution of the autonomous system, an associated solution of q-$P(A_1)$ with that particular critical behaviour, except for the logarithmic type solutions. By applying Bäcklund Transformation $\mathcal{T}_1$ we found additional critical behaviours, which correspond formally to the case $f, g \asymp 1$. Indeed, if we assume $f, g \asymp 1$, and consider the following formal ansatz, analogously to (3.42),

$$f = \sum_{i=0}^{\infty} F_i t^i, \qquad g = \sum_{i=0}^{\infty} G_i t^i.$$

Then we find that the first formal terms satisfy

$$G_0^2(F_0 G_) - 1)(\overline{F}_0 G_0 - 1) = F_0 \overline{F}_0 (G_0 - b_5)(G_0 - b_6)(G_0 - b_7)(G_0 - b_8), \tag{3.111a}$$

$$\overline{F}_0^2(\overline{F}_0 G_0 - 1)(\overline{F}_0 \overline{G}_0 - 1) = G_0 \overline{G}_0 (\overline{F}_0 - b_5^{-1})(\overline{F}_0 - b_6^{-1})(\overline{F}_0 - b_7^{-1})(\overline{F}_0 - b_8^{-1}). \tag{3.111b}$$

Now the formal leading order behaviour of the inverse complex power type, power series II, inverse oscillatory type and inverse logarithmic type solutions, as depicted in Table 3.1, all define solutions of system (3.111). In Table 3.1 the different critical behaviours near $t = 0$ are summarised. We discuss each case on the discrete level, so $(f_s, g_s)_{s \in \mathbb{Z}} \in \mathcal{S}_c^*(t_0)$, where $t_s = q^s t_0$ for $s \in \mathbb{Z}$ and $t_0 \in \mathbb{C}^*$. We emphasise that the logarithmic type and inverse logarithmic type critical behaviours are conjectural. In the table we have written down only the formal leading order behaviour. As to the inverse oscillatory and the inverse logarithmic type behaviour, one should apply the permutation $\mathbf{b} \mapsto \mathbf{b}^{(1)}$ to obtain the correct formulas for critical behaviour of solutions for q-$P(A_1)(\mathbf{b})$. Furthermore we refer to the q-$P(A_1)$ transcendents, meromorphic at the origin, defined in Propositions 3.1.2 and 3.1.1 as "Power Series I" and "Power Series II" respectively in Table 3.1. We remark that one can obtain a completely similar table for the critical behaviour about $t = \infty$.

A number of fundamental questions now arise. Firstly, we only proved that there exists a solution with given leading order behaviour, but we did not prove that they are uniquely characterised by this leading order behaviour. As an example, consider the discrete solution constructed in Section 3.4.1, whose leading order behaviour is given by equations (3.77). Is it true that there is only one discrete solution with leading order behaviour given by (3.77)? We pose the following "uniqueness" problem.

Problem 3.7.1 (Uniqueness)**.** *Show that the leading order critical behaviours of the solutions in Table 3.1 determine the corresponding solution of q-$P(A_1)$ uniquely.*

There are several to approach this problem. One way would be via a fixed point argument, and another would be to show that the monodromy mapping (2.24) is injective, and that there is only one monodromy datum corresponding to a given leading order behaviour, considering the isomonodromic deformation framework in Section 2.4.4.

Another important question is whether our table of critical behaviours 3.1 is *complete*. We pose the following "completeness" problem.

Problem 3.7.2 (Completeness)**.** *Show that Table 3.1 lists all critical behaviours of solutions of q-$P(A_1)$ near $t = 0$, for generic parameter values.*

A method of attack to solve this problem would be to show that the monodromy data, corresponding to all the different critical behaviours in our table, exhaust the monodromy space $\mathcal{M}$. Finally let us recall the q-$P(A_1)$ connection problem, described in Section 2.4.2, for which an isomonodromic deformation method can again be made effective to solve it, as sketched in Section 2.4.4.

We remark that the critical behaviours in Table 3.1 all correspond to relatively moderate growth, in the sense that there is no exponential growth or similar among them as $t \to 0$. We expect that there are no solutions, other than the singular ones, which have such wild behaviour near critical points. Establishing such a result rigorously, just by a local analysis such as the method of dominant balance to arrive at a contradiction, seems difficult, especially since the singular solutions show that exponential growth or decay of solutions is not inherently inconsistent. This is of course closely related to the completeness problem 3.7.2.

Let us also note that some of the continuous Painlevé equations exhibit the nonlinear Stokes phenomenon, typically coming from an divergent asymptotic expansion near a critical point. See for instance Joshi and Kruskal [44] and Kapaev [54] for P_{I}, Joshi and Kruskal [44] and Its and Kapaev [40] for P_{II}, and Kitaev [55] for P_{IV}. Recently the nonlinear Stokes phenomenon has also been observed in additive discrete Painlevé I by Joshi and Lustri [47] and additive discrete Painlevé II by Luu et al. [48]. As to q-$P(A_1)$, we found that the generic critical behaviour is given by convergent asymptotic expansions, both near $t = 0$ and $t = \infty$. The same holds true for P_{VI}, near its critical points. However, we also came across logarithmic type formal leading order behaviour, and a conjectural corresponding full expansion. Even in the Painlevé six case, it is not known whether the complete logarithmic type expansions are convergent or divergent.

Remark 3.7.3. Another possible approach to solving Problems 3.7.1 and 3.7.2, than the ones already mentioned, would be to do an asymptotic analysis in the initial value space $\widehat{X}(t)$ of q-$P(A_1)$, in the large and small t limit. In this regard we mention the works of Joshi and collaborators [12, 38, 49, 46].

3.8 Reduction to Symmetric Form and Continuum Limit

In Section 2.5, we discussed the natural reduction of q-$P(A_1)$ to its symmetric form. This reduction allows us to easily translate much of the work done in the previous sections, to symmetric q-$P(A_1)$. So let us consider the formal series solution (3.44) and assume (2.28), then we have

$$\hat{\hat{\Lambda}} = \Lambda, \qquad \hat{\hat{\phi}} = \lambda\phi, \qquad \lambda = \left(\frac{\Lambda}{\xi}\right)^2. \tag{3.112}$$

In order to make sense of condition (2.30), we have to define the time evolution $\hat{\ }$ on Λ and ϕ. Inspired by equations (3.112), we set

$$\hat{\Lambda} = \Lambda, \qquad \hat{\phi} = \frac{\Lambda}{\xi}\phi. \tag{3.113}$$

Note that this is indeed consistent with (3.112) and condition (2.30) becomes

$$f^{(0,+)}\left(t,\phi;\Lambda,\mathbf{b}(\mathbf{b}_s,\xi)\right)=g^{(0,+)}\left(\xi^{-1}t,\underset{\smile}{\phi};\Lambda,\mathbf{b}(\mathbf{b}_s,\xi)\right), \tag{3.114}$$

where we denoted

$$\mathbf{b}(\mathbf{b}_s,\xi)=\left(a\xi^{\frac{1}{2}},a^{-1}\xi^{\frac{1}{2}},b\xi^{\frac{1}{2}},b^{-1}\xi^{\frac{1}{2}},c,c^{-1},d,d^{-1}\right). \tag{3.115}$$

We prove that this condition indeed holds, which implies that the formal series solution (3.44) reduces 'naturally' to a solution of symmetric q-$P(A_1)$. By equation (3.73), we have

$$\begin{aligned}f^{0,+}(t,\phi;\Lambda,\mathbf{b}(\mathbf{b}_s,\xi))&=g^{0,+}\left(\xi^{-1}t,\xi^{-1}\frac{b_1b_2b_3b_4}{\Lambda}\phi;\frac{\Lambda}{b_5b_6b_7b_8},\mathbf{b}^{(3)}(\mathbf{b}_s,\xi)\right)\\&=g^{0,+}\left(\xi^{-1}t,\underset{\smile}{\phi};\Lambda,\mathbf{b}^{(3)}(\mathbf{b}_s,\xi)\right),\end{aligned}$$

where

$$\mathbf{b}^{(3)}(\mathbf{b}_s,\xi)=\left(a^{-1}\xi^{\frac{1}{2}},a\xi^{\frac{1}{2}},b^{-1}\xi^{\frac{1}{2}},b\xi^{\frac{1}{2}},c^{-1},c,d^{-1},d\right). \tag{3.116}$$

so it remains to prove

$$g^{(0,+)}\left(\xi^{-1}t,\underset{\smile}{\phi};\Lambda,\mathbf{b}(\mathbf{b}_s,\xi)\right)=g^{0,+}\left(\xi^{-1}t,\underset{\smile}{\phi};\Lambda,\mathbf{b}^{(3)}(\mathbf{b}_s,\xi)\right).$$

This identity, however, follows directly from equation (3.66), by comparing (3.115) and (3.116), where the permutation $\sigma\in Sym(\{1,2,3,4\})\times Sym(\{5,6,7,8\})$ equals

$$\sigma=(1\ 2)(3\ 4)(5\ 6)(7\ 8).$$

We conclude that, assuming equations (3.113), the symmetry condition (3.114) always holds. To put it differently, consider solutions defined by Theorem 3.4.1 on some q-domain T which also happens to be a ξ-domain. Then the symmetry condition (2.30) holds, if and only if $\Lambda(t)$ and $\phi(t)$ satisfy equations (3.113), which is only the case for very special choices of $\Lambda(t)$ and $\phi(t)$.

Formula (2.29), however, does not allow for any straightforward interpretation on a formal level. Luckily we are working with formal variables, so let us for a moment, denote the time evolution $t\mapsto\xi^{\frac{1}{2}}t$ by $\tilde{t}$, so $\tilde{t}=\xi^{\frac{1}{2}}t$ and in general $\tilde{\tilde{\cdot}}=\hat{\cdot}$. We simply introduce new formal variables Λ' and ϕ' which are forced to satisfy

$$\widetilde{\Lambda}=\xi\Lambda',\qquad\widetilde{\phi}=\xi^{-\frac{1}{2}}\phi',$$

and hence, by equations (3.113), satisfy

$$\hat{\Lambda}'=\Lambda',\qquad\hat{\phi}'=\Lambda'\phi'. \tag{3.117}$$

We conclude, using equation (2.29), that

$$x^{0,+}\left(t,\phi';\Lambda',\xi,\mathbf{b}_s\right)=f^{0,+}\left(\xi^{\frac{1}{2}}t,\xi^{-\frac{1}{2}}\phi';\xi\Lambda',\mathbf{b}(\mathbf{b}_s,\xi)\right),$$

defines a solution of symmetric q-$P(A_1)$.
Despite the appearance of square roots of ξ in the above expression, the coefficients in the expansion are rational in ξ and we have the following result.

Theorem 3.8.1. *There exists an unique formal series solution of the symmetric q-$P(A_1)$ equation (2.27), of the form*

$$x^{0,+}(t,\phi;\Lambda,\xi,\mathbf{b}_s)=\sum_{n=1}^{\infty}\sum_{i=-\infty}^{n}x^{0,+}_{n,i}(\Lambda,\xi,\mathbf{b}_s)\,t^n\phi^i, \tag{3.118}$$

where $x^{0,+}_{1,1}(\Lambda,\xi,\mathbf{b}_s)=1$ and Λ and ϕ satisfy (3.117).
For $n\in\mathbb{N}^*$ and $i\in\mathbb{Z}_{\leq n}$, the coefficient $x^{0,+}_{n,i}(\Lambda,\xi,\mathbf{b}_s)$ is a rational function in its inputs which is regular at points $(\Lambda,\xi,\mathbf{b}_s)\in\mathbb{C}^*\times\mathbb{C}^*\times\mathcal{B}_s$ satisfying

$$1\notin Q:=\{q_1^m q_2^n:(m,n)\in\mathbb{N}^2\setminus\{(0,0)\}\}, \tag{3.119}$$

where $q_1=q_1(\xi,\Lambda_s)=\xi\Lambda_s$ and $q_2=q_2(\Lambda_s)=\Lambda_s^{-1}$.
Furthermore, let $|\xi|<1$ and $\Lambda\in L_0^s:=\{z\in\mathbb{C}:1<|z|<|\xi|^{-1}\}$, then condition (3.119) is satisfied and this formal solution, written in terms of the variables $\zeta_1=t\phi$ and $\zeta_2=\phi^{-1}$,

$$x^{0,+}(\zeta_1\zeta_2,\zeta_2^{-1};\Lambda,\xi,\mathbf{b}_s)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}x^{0,+}_{n,n-m}(\Lambda,\xi,\mathbf{b}_s)\zeta_1^n\zeta_2^m, \tag{3.120}$$

converges near $(\zeta_1,\zeta_2)=(0,0)$.
In fact, these expansions also depend analytically on Λ_s. That is, for any $L\subseteq L_0^s$ open with $\overline{L}\subseteq L_0^s$, there is an open environment $Z\subseteq\mathbb{C}^2$ of $\mathbf{0}$, such that the series (3.120) converges uniformly on $Z\times L$, defining holomorphic functions on this set in $(\boldsymbol{\zeta},\Lambda)$.

Proof. We prove this analogously to Theorem 3.3.1. □

For the formal series solution (3.118), conjecture 3.3.3 implies that the coefficients $x^{0,+}_{n,i}(\Lambda,\mathbf{b}_s,\xi)$ vanish for $i<-n$ and $n\in\mathbb{N}^*$. Indeed, by direct computation we checked this assertion for $n=1,2,3$. As to the case $n=1$, it is easy to see that

$$x_1^{0,+}=x_1^{0,+}(\phi;\Lambda,\xi,\mathbf{b}_s)=\sum_{i=-\infty}^{1}x^{0,+}_{1,i}(\Lambda,\xi,\mathbf{b}_s)\,\phi^i,$$

equals

$$x_1^{0,+}=\phi-\Lambda\frac{a+a^{-1}+b+b^{-1}}{(\Lambda-1)^2}+\frac{\Lambda(\Lambda+ab)(\Lambda+\frac{a}{b})(\Lambda+\frac{b}{a})(\Lambda+\frac{1}{ab})}{(\Lambda-1)^4(\Lambda+1)^2}\phi^{-1}, \tag{3.121}$$

which defines a solution to the autonomous QRT mapping

$$(x_1\overline{x}_1-1)\,(x_1\underline{x}_1-1)=(x_1-a)\left(x_1-a^{-1}\right)(x_1-b)\left(x_1-b^{-1}\right).$$

3.8.1 Constructing True Solutions

Similar to Theorem 3.4.1, we can use the formal series solution (3.118) to construct true solutions to symmetric q-$P(A_1)$. As an example we construct complex power type series solutions, let $\phi_0, \rho \in \mathbb{C}^*$, set $\Lambda = \xi^\rho$, say with respect to the principal branch, and assume

$$1 < |\Lambda| < |\xi|^{-1}. \tag{3.122}$$

Define t^ρ analogously to (3.84) on a domain T defined by equation (3.83), with q replaced by ξ. Then there is an unique meromorphic solution $x(t)$ of symmetric q-$P(A_1)$ on T such that

$$x(t) = x^{0,+}\left(t, \phi_0 t^\rho; \xi^\rho, \xi, \mathbf{b}_s\right) = \sum_{n=1}^{\infty} \sum_{i=-\infty}^{n} x_{n,i}^{0,+}\left(\xi^\rho, \xi, \mathbf{b}_s\right) \phi_0^i t^{\rho i+n}, \tag{3.123}$$

for t close to 0. More precisely, there is an $r > 0$, such that the expansion on the right-hand side of equation (3.123) converges uniformly in t on

$$T \cap \{t \in \mathbb{C}^* : |t| < r\},$$

and the equation holds on this set.

We will not work out all the different critical behaviours of solutions of symmetric q-$P(A_1)$, as we did for q-$P(A_1)$ to finally obtain Table 3.1. In stead we simply give the corresponding Table 3.2 for symmetric q-$P(A_1)$. In Table 3.2 we adopted the discrete time interpretation $(x_s)_{s\in\mathbb{Z}}$, with $t_s = \xi^s t_0$ and $t_0 \in \mathbb{C}^*$. We have only written down the formal leading order behaviour, but we remark that the corresponding complete expansions are in form identical to those for q-$P(A_1)$.

Note that symmetric q-$P(A_1)$ has four solutions which are meromorphic at $t = 0$, depicted by “power series I” and “power series II” in the table. Corresponding to each of the two power series of type I, there are two special complex power type critical behaviours. Similarly, corresponding to each of the two power series of type II, there are two inverse special complex power type critical behaviours.

Just as in the q-$P(A_1)$ case, there are solutions of its symmetric form with oscillatory type and inverse oscillatory type critical behaviour. Furthermore, we have logarithmic and inverse logarithmic type critical behaviour, where $x_{1,0}^l(a,b)$, in the formula of the formal leading order behaviour, is given by

$$x_{1,0}^l(a,b) = \frac{ab(a + a^{-1} - (b + b^{-1}) + 4)(b + b^{-1} - (a + a^{-1}) + 4)}{8(a + b)(1 + ab)}).$$

One might wonder why there is no term “$x_{1,1}^l(a,b)\chi_s$” in the formula. This is because we have the freedom of substitution $\chi_s \mapsto \chi_s + r$, for any $r \in \mathbb{C}$, which allows us to scale such a term away. The same of course applies to equations (3.40).

3.8.2 Continuum Limit of Complex Power Type Critical Behaviour

We wish to calculate the continuum limit $\xi \to 1$ as described in Section 2.5.1, of solutions which are described by some critical behaviour. Doing such rigorous does not seem easy, even on a formal level. Nonetheless let us consider the solutions defined by (3.123), where we restrict ourselves to $\xi \in (0,1)$, and define $\mathbf{b}_s = \mathbf{b}_s(\xi)$ by equations (2.31), for some fixed $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. Considering equation (3.122), we fix a $\rho \in \mathbb{C}$ with $-1 < \Re(\rho) < 0$. We define, as in equation (3.123),

$$x(t;\phi_0,\rho,\xi) = x^{0,+}\left(t,\phi_0 t^\rho;\xi^\rho,\xi,\mathbf{b}_s(\xi)\right) = \sum_{n=1}^{\infty}\sum_{i=-\infty}^{n} x_{n,i}^{0,+}\left(\xi^\rho,\xi,\mathbf{b}_s(\xi)\right)\phi_0^i t^{\rho i+n}, \tag{3.124}$$

with the principal branch cut, which converges uniformly in t on an open disc punctured at the origin. Via meromorphic continuation we extend the domain of $x(t;\phi_0,\rho,\xi)$ to $\mathbb{C}\setminus(-\infty,0)$. Using Theorem B.4, it is not hard to see that this solution also depends analytically on ξ for $\xi \in (0,1)$. However, we are interested in the limit $\xi \uparrow 1$ of solution (3.124), but condition (3.119) for the existence of the formal series solution (3.118), is not satisfied at $\xi = 1$. Despite the fact that Theorem 3.8.1 becomes inapplicable in this limit, we do expect the solution (3.124) converges to a true solution of the differential equation (2.32) as $\xi \uparrow 1$. Proving this rigorously, probably requires an extension of Theorem B.4 which incorporates limits of the variable $\mathbf{q}$ as it approaches the boundary of $B^m_{\max}(\mathbf{0},1)$ under some specific assumptions, an interesting direction for future research. Instead, we proceed by heuristically calculating the continuum limit on a formal level. By equation (3.121), we have,

$$\lim_{\xi\to 1} x_{1,0}^{0,+}\left(\xi^\rho,\xi,\mathbf{b}_s(\xi)\right) = \lim_{\xi\to 1} -\xi^\rho \frac{\xi^\beta+\xi^{-\beta}-\xi^\alpha-\xi^{-\alpha}}{(\xi^\rho-1)^2} = \frac{\alpha^2-\beta^2}{\rho^2},$$

where the second equality is obtained by applying L'Hôpital's rule twice.
Similarly, we find

$$\lim_{\xi\to 1} x_{1,-1}^{0,+}\left(\xi^\rho,\xi,\mathbf{b}_s(\xi)\right) = \frac{\rho(\rho-\alpha-\beta)(\rho-\alpha-\beta)(\rho+\alpha-\beta)(\rho-\alpha+\beta))(\rho+\alpha+\beta)}{4\rho^4},$$

which motivates us to make the following bold move.
We assume that the limit,

$$x_{n,i}^{c}\left(\rho,\mathbf{b}_c\right) = \lim_{\xi\to 1} x_{n,i}^{0,+}\left(\xi^\rho,\xi,\mathbf{b}_s(\xi)\right),$$

exists on a formal level, for all $i \in \mathbb{Z}_{\leq n}$ and $n \in \mathbb{N}^*$, where $\mathbf{b}_c = (\alpha,\beta,\gamma,\delta)$.
We hence obtain the following formal solution to the differential equation (2.32),

$$x^c\left(t;\phi_0,\rho,\mathbf{b}_c\right) = \sum_{n=1}^{\infty}\sum_{i=-\infty}^{n} x_{n,i}^{c}\left(\rho,\mathbf{b}_c\right)\phi_0^i t^{\rho i+n},$$

and assuming convergence, its leading order behaviour is given by

$$x^c(t;\phi_0,\rho,\mathbf{b}_c)=\phi_0 t^{1+\rho}+\frac{\alpha^2-\beta^2}{\rho^2}t+\\ \frac{(\rho-\alpha-\beta)(\rho+\alpha-\beta)(\rho-\alpha+\beta)(\rho+\alpha+\beta)}{4\rho^4}\phi_0^{-1}t^{1-\rho}+\mathcal{O}\left(t^{2-2|\Re(\rho)|}\right),$$

as $t\to 0$.
Applying the change of variables (2.33) to this solution, we find an associated solution $w^0(\zeta;\rho,s,\mathbf{b}_c)$ of P_{VI} with parameter values parameters (2.34), whose leading order behaviour is given by

$$w^0(\zeta;s,\rho,\mathbf{b}_c)=1-s(1-\zeta)^{1+\rho}-\frac{1}{2}\left(1+\frac{\alpha^2-\beta^2}{\rho^2}\right)(1-\zeta)\\ -\frac{(\rho-\alpha-\beta)(\rho+\alpha-\beta)(\rho-\alpha+\beta)(\rho+\alpha+\beta)}{16\rho^4}s^{-1}(1-\zeta)^{1-\rho}+\mathcal{O}\left((1-\zeta)^{2-2|\Re(\rho)|}\right),$$

as $\zeta\to 1$, where $s=2^{-1-2\rho}\phi_0$.
This is exactly the critical behaviour around $\zeta=1$ which characterises the solutions obtained by Jimbo [42] for P_{VI}. Guzzetti [26] states that the full expansion of the solution $\omega(t)$ of P_{VI}, is given by

$$\omega(\zeta)=1+\sum_{n=1}^{\infty}\sum_{i=-n}^{n}\omega_{n,i}(\rho,\mathbf{b}_c)\,s^i(1-\zeta)^{\rho i+n}, \tag{3.125}$$

where $\omega_{1,1}=1$ and the remaining coefficients can be determined uniquely via substitution into P_{VI} and comparing coefficients. So the continuous counterpart of Conjecture 3.3.3 is true. That is, there are no terms $t^{\rho i+n}$ in expansion (3.125), with $i<-n$ and $n\in\mathbb{N}^*$. However, also in the continuous case, this is not easy to derive from the equation itself. Indeed Guzzetti [26] shows how to determine the coefficients $\omega_{n,i}$ recursively and observes that for at least $n\le 3$, no terms $t^{\rho i+n}$ with $i<-n$ have to be introduced, but he does not give a proof of this fact for general n. The form of the complete expansion (3.125) is easier to understand via the linear problem of P_{VI}. Quite remarkably, recent work by Lisovyy and collaborators [4, 16] gives explicit formulae for all the coefficients in the asymptotic expansion of the τ-function associated with $\omega(\zeta)$, in terms of conformal blocks. Similarly, it might also be possible to find explicit expressions for the coefficients $F^{0,+}_{n,i}$ and $G^{0,+}_{n,i}$ in the formal series solution (3.44).

3.8.3 Comparison with Painlevé Six

Guzzetti [27] gives a tabulation of critical behaviours of solutions of P_{VI}. We have found reflections of all these different critical behaviours in the q-$P(A_1)$ case. Indeed, considering Table 3.1, using the terminology in Guzzetti [27], we have encountered complex power behaviour, oscillatory behaviour, logarithmic behaviour, Taylor expansions, inverse oscillatory behaviour and inverse logarithmic behaviour. Furthermore each of these reduce to critical behaviour of solutions of symmetric q-$P(A_1)$, depicted in Table 3.2. In the previous section we showed how, at least the formal leading order behaviour of complex power type of solutions, converges to the corresponding complex power behaviour of solutions for the sixth Painlevé equation, in the continuum limit. In fact this can be made to work for each of the formal leading order behaviours, resulting in a similar table for the sixth Painlevé equation, as given by Guzzetti [27]. We say made to work, as one of course has to make sure that terms such as χ in the logarithmic case, have to depend appropriately on ξ for the continuum limit $\xi \to 1$ to be sensible. Indeed, if we set

$$\chi = \frac{1}{\xi - 1}\left(\frac{\log_\xi(t)}{\xi - 1} + r\right),$$

with $r \in \mathbb{C}$, and the parameter values $\mathbf{b}_s$ as in (2.31), then the logarithmic leading order behaviour, given in Table 3.2, satisfies

$$t\left(\tfrac{1}{2}(a + a^{-1} + b + b^{-1})\chi^2 + x^l_{1,0}(a,b)\right) \to t\left(\tfrac{1}{2}(\beta^2 - \alpha^2)(\log(t) + r)^2 + \frac{\alpha^2 + \beta^2}{\alpha^2 - \beta^2}\right),$$

as $\xi \to 1$. We invite the interested reader to compare this with the Painlevé six case under the change of variables (2.33), and confirm that they indeed coincide.

3.9 Tables of Critical Behaviours

Formal Leading Order Behaviour	Terms Involved	Int. Const.	Disc. Par.	Ref.
Complex Power Type				
$f_s \sim t_s\left(\phi_s + F_{eq}(\Lambda) + \mu(\Lambda)\phi_s^{-1}\right)$ $g_s \sim t_s\left(\Lambda\phi_s + G_{eq}(\Lambda) + \frac{\Lambda}{\lambda}\mu(\Lambda)\phi_s^{-1}\right)$	$\phi_s = \lambda^s\phi_0$ $\lambda = \frac{\Lambda^2}{b_1b_2b_3b_4}$	$\phi_0 \in \mathbb{C}^*$ $\Lambda \in L_0(\mathbf{b})$		(3.76)
Special Complex Power Type				
$f_s \sim t_s(\phi_s + f_1^{(1,k)})$ $g_s \sim t_s(\Lambda_k^{\pm}\phi_s + g_1^{(1,k)})$	$\phi_s = \lambda_k^{\pm s}\phi_0$	$\phi_0 \in \mathbb{C}^*$	$k \in \{1,2,3\}$ $\pm \in \{+,-\}$	(3.104)
Power Series I				
$f_s \sim t_s f_1^{(1,k)}$ $g_s \sim t_s g_1^{(1,k)}$			$k \in \{1,2,3\}$	(3.10)
Oscillatory Type				
$f_s \sim t_s\left(\phi_s + F_{eq}(\Lambda) + \mu(\Lambda)\phi_s^{-1}\right)$ $g_s \sim t_s\left(\Lambda\phi_s + G_{eq}(\Lambda) + \frac{\Lambda}{\lambda}\mu(\Lambda)\phi_s^{-1}\right)$	$\phi_s = e^{\theta i s}\phi_0$ $\Lambda = \pm(b_1b_2b_3b_4)^{\frac{1}{2}}e^{\frac{1}{2}\theta i}$	$\phi_0 \in \mathbb{C}^*$ $\theta \in \mathbb{R}\setminus 2\pi\mathbb{Z}$	$\pm \in \{+,-\}$	(3.86)
Logarithmic Type				
$f_s \sim t_s\left(F_{1,0}^{\pm} + F_{1,1}^{\pm}\chi_s + F_{1,2}^{\pm}\chi_s^2\right)$ $g_s \sim t_s\left(G_{1,0}^{\pm} + G_{1,1}^{\pm}\chi_s + G_{1,2}^{\pm}\chi_s^2\right)$	$\chi_s = \chi_0 + s$	$\chi_0 \in \mathbb{C}$	$\pm \in \{+,-\}$	(3.110)
Inverse Special Complex Power Type				
$f_s \sim f_0^{(0,k)}[1 + f_0^{(0,k)}\phi_s]^{-1}$ $g_s \sim g_0^{(0,k)}[1 + \Lambda_k^{\pm}\vert_{\mathbf{b}\mapsto\mathbf{b}^{(1)}} g_0^{(0,k)}\phi_s]^{-1}$	$\phi_s = \lambda_k^{\pm s}\vert_{\mathbf{b}\mapsto\mathbf{b}^{(1)}}\phi_0$	$\phi_0 \in \mathbb{C}^*$	$k \in \{1,2,3\}$ $\pm \in \{+,-\}$	(3.109)
Power Series II				
$f_s \sim f_0^{(0,k)}$ $g_s \sim g_0^{(0,k)}$			$k \in \{1,2,3\}$	(3.5)
Inverse Oscillatory Type (for parameter values $\mathbf{b}^{(1)}$)				
$f_s \sim \left[\phi_s + F_{eq}(\Lambda) + \mu(\Lambda)\phi_s^{-1}\right]^{-1}$ $g_s \sim \left[\Lambda\phi_s + G_{eq}(\Lambda) + \frac{\Lambda}{\lambda}\mu(\Lambda)\phi_s^{-1}\right]^{-1}$	$\phi_s = e^{\theta i s}\phi_0$ $\Lambda = \pm(b_1b_2b_3b_4)^{\frac{1}{2}}e^{\frac{1}{2}\theta i}$	$\phi_0 \in \mathbb{C}^*$ $\theta \in \mathbb{R}\setminus 2\pi\mathbb{Z}$	$\pm \in \{+,-\}$	(3.87)
Inverse Logarithmic Type (for parameter values $\mathbf{b}^{(1)}$)				
$f_s \sim [F_{1,0}^{\pm} + F_{1,1}^{\pm}\chi_s + F_{1,2}^{\pm}\chi_s^2]^{-1}$ $g_s \sim [G_{1,0}^{\pm} + G_{1,1}^{\pm}\chi_s + G_{1,2}^{\pm}\chi_s^2]^{-1}$	$\chi_s = \chi_0 + s$	$\chi_0 \in \mathbb{C}$	$\pm \in \{+,-\}$	

Table 3.1: Critical Behaviours of solutions of q-$P(A_1)$ near $t_s := q^s t_0 = 0$, or $s = \infty$, where "disc. par." stands for "discrete parameters", and "ref." stands for "reference".

Formal Leading Order Behaviour	Terms Involved	Int. Const.
Complex Power Type		
$x_s \sim t_s(\phi_s + x_{1,0}(\Lambda) + x_{1,-1}(\Lambda)\phi_s^{-1})$	$\phi_s = \lambda^s \phi_0$	$\phi_0 \in \mathbb{C}^*$ $\Lambda \in L_0^s$
Special Complex Power Type		
$x_s \sim t_s(\phi_s + \frac{a+b}{1+ab})$	$\phi_s = (-ab)^s \phi_0$	$\phi_0 \in \mathbb{C}^*$
$x_s \sim t_s(\phi_s + \frac{a+b}{1+ab})$	$\phi_s = (-ab)^{-s} \phi_0$	$\phi_0 \in \mathbb{C}^*$
$x_s \sim t_s(\phi_s + \frac{1+ab}{a+b})$	$\phi_s = (-a/b)^s \phi_0$	$\phi_0 \in \mathbb{C}^*$
$x_s \sim t_s(\phi_s + \frac{1+ab}{a+b})$	$\phi_s = (-b/a)^s \phi_0$	$\phi_0 \in \mathbb{C}^*$
Power Series I		
$x_s \sim t_s \frac{a+b}{1+ab}$		
$x_s \sim t_s \frac{1+ab}{a+b}$		
Oscillatory Type		
$x_s \sim t_s(\phi_s + x_{1,0}(e^{\theta i}) + x_{1,-1}(e^{\theta i})\phi_s^{-1})$	$\phi_s = e^{\theta i s} \phi_0$	$\phi_0 \in \mathbb{C}^*$ $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$
Logarithmic Type		
$x_s \sim t_s(\frac{1}{2}(a + a^{-1} + b + b^{-1})\chi_s^2 + x_{1,0}^l(a,b))$	$\chi_s = \chi_0 + s$	$\chi_0 \in \mathbb{C}$
Inverse Special Complex Power Type		
$x_s \sim [\phi_s + \frac{c+d}{1+cd}]^{-1}$	$\phi_s = (-cd)^s \phi_0$	$\phi_0 \in \mathbb{C}^*$
$x_s \sim [\phi_s + \frac{c+d}{1+cd}]^{-1}$	$\phi_s = (-cd)^{-s} \phi_0$	$\phi_0 \in \mathbb{C}^*$
$x_s \sim [\phi_s + \frac{1+cd}{c+d}]^{-1}$	$\phi_s = (-c/d)^s \phi_0$	$\phi_0 \in \mathbb{C}^*$
$x_s \sim [\phi_s + \frac{1+cd}{c+d}]^{-1}$	$\phi_s = (-d/c)^s \phi_0$	$\phi_0 \in \mathbb{C}^*$
Power Series II		
$x_s \sim \frac{1+cd}{c+d}$		
$x_s \sim \frac{c+d}{1+cd}$		
Inverse Oscillatory Type (with parameter values $a \leftrightarrow c$,$b \leftrightarrow d$)		
$x_s \sim \left[\phi_s + x_{1,0}(e^{\theta i}) + x_{1,-1}(e^{\theta i})\phi_s^{-1}\right]^{-1}$	$\phi_s = e^{\theta i s} \phi_0$	$\phi_0 \in \mathbb{C}^*$ $\theta \in \mathbb{R} \setminus 2\pi\mathbb{Z}$
Inverse Logarithmic Type		
$x_s \sim \left[\frac{1}{2}(c + c^{-1} + d + d^{-1})\chi_s^2 + x_{1,0}^l(c,d)\right]^{-1}$	$\chi_s = \chi_0 + s$	$\chi_0 \in \mathbb{C}$

Table 3.2: Critical behaviours of solutions of symmetric q-$P(A_1)$ near $t_s := \xi^s t_0 = 0$, or $s = \infty$, where "int. const." stands for "integration constants".

CHAPTER 4

Linear q-Difference Equations and Isomonodromy

The theory of linear q-difference equations goes back a long way. A classical approach to the global asymptotic analysis of such equations was completed by Birkhoff [5], in which he treats the *Riemann-Hilbert problem* for regular singular q-difference systems without resonance. More specifically, after appropriate normalisation, he studies the system

$$Y(qz) = A(z)Y(z), \tag{4.1}$$

where $A(z)$ is a complex $m \times m$ matrix polynomial of degree n,

$$A(z) = A_0 + zA_1 + \ldots + z^n A_n.$$

Under some generic assumptions on the eigenvalues of the matrices A_0 and A_n, Birkhoff's student Carmichael [9] constructed canonical fundamental solutions $Y^0(z)$ and $Y^\infty(z)$ about $z = 0$ and $z = \infty$ respectively. These fundamental solutions are related by

$$Y^\infty(z) = Y^0(z)P(z),$$

for some matrix $P(z)$, called the *connection matrix*, which obviously satisfies $P(qz) = P(z)$. For q-difference equations, it is essentially this connection matrix which constitutes the monodromy of the equation. Roughly speaking Birkhoff [5] worked out an exact correspondence between linear first order q-difference systems (4.1), up to $GL_n(\mathbb{C})$ conjugation, and their connection matrices, again up to some action, which we call the *Riemann-Hilbert-Birkhoff correspondence*. Our main interest lies in scalar q-difference equations of the form

$$u_0(z)y(z) + u_1(z)y(qz) + \ldots u_{m-1}(z)y(q^{m-1}z) = 0,$$

where $u_0(z), \ldots u_{m-1}(z)$ are some polynomials, as the spectral equation in Yamada's Lax pair (2.21) is of this type with $m = 3$. We therefore also give a treatment of the subject customised to such second order scalar q-difference equations. We note that a modern treatment of the subject in total generality, including irregular cases, has been carried out by Ramis, Sauloy and Zhang [76]. See also Sauloy [78, 80] for an analytic approach.

We start our discussion with first order scalar q-difference equations and q-elliptic functions, as they form the building blocks of the theory of linear q-difference equations. We then discuss Birkhoff's theory in more detail, and consider the analog for scalar q-difference equations. We discuss a particular case, which we call the model equation, as it plays a vital role in the analysis of Yamada's Lax pair in Chapter 5.

We then turn our attention to Yamada's Lax pair (2.21). We define fundamental solutions near $z = 0$ and $z = \infty$, discuss how the q-$P(A_1)$ time evolution deforms them, and to what extend it leaves the monodromy, i.e. the connection matrix, invariant, constituting an *isomonodromic deformation.*

4.1 First Order q-Difference Equations

Not only are first order q-difference equations a great introduction to *classical q-theory*, they also play a fundamental role as *scaling factors* in the higher order cases. Much of the material in this section goes back a long way and we refer to Gasper and Rahman [18], for an exposition of classical q-theory.

In this section we consider q-difference equations of the form

$$y(qz) = a(z)y(z), \tag{4.2}$$

with $a(z) \not\equiv 0$ meromorphic on $\mathbb{C}^*$, where we are interested in solutions which are meromorphic on the same doubly punctured Riemann sphere. The simplest case of (4.2) is of course given by $a(z) \equiv 1$, whose solutions we call *q-elliptic functions*, which we discuss in detail in Section 4.1.1. Let us make the following trivial but important remark, given nonzero solutions y and $y*$ of (4.2), their quotient $p = y/y^*$ is a q-elliptic function. Or, to put it differently, once one solution is found, the equation is essentially solved. We call $z = 0$ or $z = \infty$, an *ordinary (critical) point* of (4.2), iff respectively $a(z)$ is holomorphic at $z = 0$ with $a(0) = 1$ or $a(z)$ is holomorphic at $z = \infty$ with $a(\infty) = 1$.

Lemma 4.1.1. *If $z = 0$ is an ordinary point of (4.2), then (4.2) admits an unique holomorphic solution $y(z)$ with $y(0) = 1$ around $z = 0$. If $a(z)$ is meromorphic on $\mathbb{C}$, then $y(z)$ has an unique meromorphic continuation to $\mathbb{C}$. Furthermore, if $1/a(z)$ is entire, then $y(z)$ is entire. Similarly, if $z = \infty$ is an ordinary point of (4.2), then (4.2) admits an unique holomorphic solution $y(z)$ with $y(\infty) = 1$ around $z = \infty$. If $a(z)$ is meromorphic on $\mathbb{P}^*$, then $y(z)$ has an unique meromorphic continuation to $\mathbb{P}^*$. Furthermore, if $a(1/z)$ is entire, then $y(1/z)$ is entire.*

Proof. Suppose $z = 0$ is an ordinary point of (4.2), and temporarily assume we found a solution $y(z)$, holomorphic at $z = 0$, with $y(0) = 1$. From (4.2) we immediately obtain

$$\frac{y(z)}{y(q^{n+1}z)} = \prod_{k=0}^{n} \frac{1}{a(q^k z)},$$

for $n \in \mathbb{N}$. Letting $n \to \infty$, we find the *infinite product representation*

$$y(z) = \prod_{k=0}^{\infty} \frac{1}{a(q^k z)}. \tag{4.3}$$

Hence, to obtain the first part of the lemma, we can use the infinite product (4.3) to define a solution of (4.2). Indeed, let R be the radius of convergence of $1/a(z)$ about $z = 0$, then it is an elementary exercise in complex analysis, to show that the infinite product (4.3) converges uniformly in z on $\{z \in \mathbb{C} : |z| \leq r\}$, for any $0 < r < R$. Hence (4.3) defines a solution $y(z)$ which is holomorphic on $\{z \in \mathbb{C} : |z| < R\}$, and it is the same infinite product representation which allows us to meromorphically continue it on $\mathbb{C}$. Also note that indeed $y(0) = 1$ and obviously, if $1/a(z)$ is entire, then $y(z)$ is entire. The case $z = \infty$ is dealt with similarly. $\square$

We are now able to define one of the main building blocks of classical q-theory, the *infinite q-Pochhammer symbol*, which we obtain by taking $a(z) = 1/(1 - z)$ in Lemma 4.1.1, giving an entire function $(\text{-}; z)_\infty$, with infinite product representation

$$(z; q)_\infty = \prod_{k=0}^{\infty} (1 - q^k z), \tag{4.4}$$

satisfying

$$(qz; q)_\infty = \frac{1}{1 - z}(z; q)_\infty. \tag{4.5}$$

Its finite counterparts, simply called *q-Pochammer symbols*, are given by

$$(z; q)_n = \prod_{k=0}^{n-1} (1 - q^k z),$$

for $n \in \mathbb{N}$, where the empty product, corresponding to $n = 0$, is set equal to 1 as usual. From the infinite product representation (4.4) we derive that the zeros of the infinite q-Pochhammer symbol $(z; q)_\infty$ are all simple and given by $z = q^{-n}$, for $n \in \mathbb{N}$. Similarly we apply Lemma 4.1.1 with $\widetilde{a}(z) = 1 - z^{-1}$, to obtain a function $y(z)$, which is analytic on $\mathbb{P}^*$, satisfying $y(\infty) = 1$, and

$$y(qz) = \frac{z - 1}{z} y(z). \tag{4.6}$$

We easily identify $y(z) = (q/z; q)_\infty$ and consider the product

$$\theta_q(z) = \theta(z; q) := (z; q)_\infty \cdot (q/z; q)_\infty. \tag{4.7}$$

Our interest in this product, comes from the observation, that it satisfies the following *shift* and *reflection relation*,

$$\theta_q(qz) = -z^{-1}\theta_q(z), \quad \theta_q(z) = \theta_q(q/z),$$

respectively, where the first is a direct consequence of (4.5) and (4.6), and the latter follows from the definition. We refer to (4.7) as the *q-theta function*. It bears its name from being a

q-analog of usual *theta functions* in elliptic function theory, which we make explicit in Section 4.1.1. Note that the q-theta function is holomorphic on $\mathbb{C}^*$, with only simple zeros, located on $q^{\mathbb{Z}}$. It is easy to see that $(z;q)_\infty$ and hence $(q/z;q)_\infty$ are not rational functions, hence $\theta_q(z)$ has an essential singularity at $z=0$ and $z=\infty$.

We are now able to construct a meromorphic solution of (4.2) with $a(z)\equiv\alpha\in\mathbb{C}^*$, by setting

$$y(z)=e_q(z;\alpha):=\frac{\theta_q(z/\alpha)}{\theta_q(z)},$$

which indeed satisfies $y(qz)=\alpha y(z)$. Now consider (4.2) for any meromorphic function $a(z)$ on $\mathbb{C}$. Determine $\alpha\in\mathbb{C}^*$ and $n\in\mathbb{Z}$ such that

$$a(z)=\alpha z^n\widetilde{a}(z),$$

where $\widetilde{a}(z)$ holomorphic at $z=0$ with $\widetilde{a}(0)=1$. Then we can scale (4.2) by setting

$$y(z)=\theta_q(-z)^{-n}e_q(z;\alpha)\widetilde{y}(z),$$

which gives

$$\widetilde{y}(qz)=\widetilde{a}(z)\widetilde{y}(z).$$

Now $z=0$ is an ordinary point of this equation, and we can apply Lemma 4.1.1 to obtain a nonzero solution $\widetilde{y}(z)$, holomorphic at $z=0$. Similarly, we can construct solutions of (4.2) about $z=\infty$, for any meromorphic function $a(z)$ on $\mathbb{C}^*$.

Let us end the discussion with an important example, the case where $a(z)=r(z)$ is a rational function, say with $r(0)=r(\infty)=1$, which means that both $z=0$ and $z=\infty$ are ordinary points of (4.2). We write

$$r(z)=\frac{(1-z/p_1)\cdot\ldots\cdot(1-z/p_n)}{(1-z/q_1)\cdot\ldots\cdot(1-z/q_n)},$$

for some $n\in\mathbb{N}$, where we assume that the numerator and denominator do not have common divisors, and

$$\prod_{k=1}^n\frac{q_k}{p_k}=1.$$

We define a *canonical solution* at $z=0$ by

$$y_0(z)=\frac{(z/q_1,\ldots,z/q_n;q)_\infty}{(z/p_1,\ldots,z/p_n;q)_\infty},$$

where we used the shorthand notation

$$(z_1,\ldots,z_s;q)_\infty=(z_1;q)_\infty\cdot\ldots\cdot(z_n;q)_\infty.\qquad(z_1,\ldots,z_n\in\mathbb{C}^*)$$

Similarly, we define a *canonical solution* at $z=\infty$ by

$$y_\infty(z)=\frac{(qp_1/z,\ldots,qp_n/z;q)_\infty}{(qq_1/z,\ldots,qq_n/z;q)_\infty}.$$

The solutions $y_0(z)$ and $y_\infty(z)$, are related by the q-elliptic function,

$$p(z) := \frac{y_\infty(z)}{y_0(z)} = \frac{\theta_q(z/q_1, \ldots, z/q_n)}{\theta_q(z/p_1, \ldots, z/p_n)},$$

which is the connection "matrix", where we used shorthand notation

$$\theta_q(z_1, \ldots, z_n) = \theta(z_1, \ldots, z_s; q) = \theta_q(z_1) \cdot \ldots \cdot \theta_q(z_n). \qquad (z_1, \ldots, z_n \in \mathbb{C}^*)$$

Note that the q-elliptic function is written as a quotient of q-theta functions. In the coming section we see that any q-elliptic function has such a representation. Albeit a bit trivial, the Riemann-Hilbert-Birkhoff correspondence, specialised to this case, is the correspondence between rank 1 first order equations with $z = 0$ and $z = \infty$ being ordinary points, and corresponding connection "matrices" up to multiplication by nonzero complex numbers,

$$r(z) \leftrightarrow [p(z)]_{\mathbb{C}^*}.$$

4.1.1 *q*-Elliptic Functions

In this section we set up the basics of a q-analog of classical elliptic function theory. Recall that a *q-periodic function* or *q-constant* is defined as any complex function $p(z)$ satisfying $p(qz) = p(z)$ on its q-domain. Now a *q-elliptic function* is a q-periodic function which is meromorphic on the entire doubly punctured Riemann sphere $\mathbb{C}^*$.

All the results in this section can be obtained by direct translation of the classical results on elliptic functions. However, it seems more appropriate to develop the fundamentals directly from a q-discrete perspective. We therefore first clarify the relation between q-elliptic and classical elliptic functions, and then rederive the basic results from scratch for q-elliptic functions.

Let us take a q-elliptic function $p(z)$ and set

$$E(\zeta) = p(\exp[\log(q)\zeta]), \tag{4.8}$$

for any choice of branch for $\log(q)$. Then $E(\zeta)$ is a meromorphic function on $\mathbb{C}$ which satisfies

$$E(\zeta + 1) = E(\zeta), \quad E(\zeta + 2\pi i/\log(q)) = E(\zeta),$$

and hence $E(\zeta)$ is an elliptic function with *fundamental periods* $\omega_1 = 1$ and $\omega_2 = 2\pi i/\log(q)$. Conversely, given an elliptic function $E(\zeta)$ with fundamental periods ω_1 and ω_2, setting

$$p(z) = E\left(\frac{\omega_2}{2\pi i}\log z\right), \quad q = \exp\left[2\pi i \frac{\omega_1}{\omega_2}\right],$$

where the choice of branch of $\log(z)$ is irrelevant, defines a q-elliptic function $p(z)$. Note that in this correspondence, the condition $\omega_1/\omega_2 \notin \mathbb{R}$ is equivalent to $|q| \neq 1$. To extend the correspondence, following Rains [75], for $\alpha \in \mathbb{C}^*$ and $n \in \mathbb{Z}$, we call a meromorphic function

$\theta(z)$ on $\mathbb{C}^*$, a *q-theta function* of *multiplier* αz^n, if it satisfies

$$\theta(qz) = \alpha z^n \theta(z).$$

To justify this definition, analogously to (4.8) we set

$$\Theta(\zeta) = \theta(\exp[\log(q)\zeta]), \tag{4.9}$$

then $\Theta(\zeta)$ is periodic with respect to the fundamental period $\omega_2 = 2\pi i/\log(q)$, and *quasi-periodic* with respect to the fundamental period $\omega_1 = 1$,

$$\Theta(\zeta + 1) = \alpha q^{n\zeta}\Theta(\zeta),$$

and hence a *theta function* in the usual sense. Note that the q-theta function (4.7) is a q-theta function of multiplier $-z^{-1}$. Similarly q-elliptic functions are q-theta functions of multiplier 1.

Getting back to q-elliptic functions, we define a *fundamental domain* or *annulus* of a q-elliptic function, as any subset of $\mathbb{C}^*$, of the form

$$\mathrm{An}(r) := \{z \in \mathbb{C} : |q|r \leq |z| < r\},$$

for some $r > 0$. Considering the group action of $q^{\mathbb{Z}}$ on $\mathbb{C}^*$ by multiplication, note that a fundamental annulus indeed contains exactly one element of each orbit of this action. Geometrically we can think of such a fundamental annulus as a torus by glueing together the inner and outer boundary in $\mathbb{C}^*$ consistent with the $q^{\mathbb{Z}}$ action.

Let us prove some basic results we are familiar with from the usual elliptic function theory.

Lemma 4.1.2. *Analytic q-elliptic functions are constant.*

Proof. Let $p(z)$ be a q-elliptic function which is analytic. Then it is bounded on the compact set $\overline{\mathrm{An}(1)}$, hence the via (4.8) corresponding elliptic function $E(\zeta)$ is bounded and entire. By Liouville's Theorem, $E(\zeta)$ and hence $p(z)$ is constant. □

Corollary 4.1.3. *q-elliptic functions without zeros are constant.*

Proof. As for any nonzero q-elliptic function $p(z)$, the function $1/p(z)$ is also a q-elliptic function, this follows directly from Lemma 4.1.2. □

Lemma 4.1.4. *Suppose $\theta(z)$ is a q-theta function of multiplier αz^n, and $\theta(z)$ has neither zeros nor poles in $\mathbb{C}^*$, then $n = 0$ and $\theta(z) = cz^k$ for some $c \in \mathbb{C}$ and $k \in \mathbb{Z}$, in particular $\alpha = q^k \in q^{\mathbb{Z}}$.*

Proof. Differentiating $\theta(qz) = \alpha z^n \theta(z)$, we find

$$q\theta'(qz) = \lambda z^n(nz^{-1}\theta(z) + \theta'(z)),$$

and hence

$$f(z) = \frac{z\theta'(z)}{\theta(z)},$$

defines an analytic function on $\mathbb{C}^*$, satisfying

$$f(qz) = n + f(z). \tag{4.10}$$

In particular $g(z) := f'(z)/z$ is an analytic q-elliptic function. By Lemma 4.1.2, we know that $g(z)$ is constant, say $g(z) \equiv g_0 \in \mathbb{C}$. We immediately obtain

$$f(z) = \tfrac{1}{2} g_0 z^2 + f_0,$$

for some $f_0 \in \mathbb{C}$. This can only be consistent with (4.10), if $n = 0$ and $g_0 = 0$, so $f(z) \equiv f_0$ is constant. Hence $\theta(z)$ is an analytic function on $\mathbb{C}^*$ without zeros, satisfying

$$\frac{\theta'(z)}{\theta(z)} = \frac{f_0}{z}.$$

We easily derive that $f_0 = q^k$ for some $k \in \mathbb{Z}$ and $\theta(z) = cz^k$ for some $c \in \mathbb{C}^*$. □

We now have all the tools to classify q-elliptic functions.

Theorem 4.1.5. *Let $p(z)$ be a nonzero q-elliptic function, say with n zeros and m poles, counting multiplicities, within any fundamental annulus. We fix a particular fundamental annulus and denote the zeros and poles of $p(z)$ in it respectively by $a_1, \dots a_n$ and $b_1, \dots, b_m$, with repetition according to multiplicity. Then $m = n$ and there exist unique $c \in \mathbb{C}^*$ and $k \in \mathbb{Z}$ such that*

$$p(z) = cz^k \prod_{i=1}^{n} \frac{\theta_q(z/a_i)}{\theta_q(z/b_i)}, \quad q^k = \prod_{i=1}^{n} \frac{b_i}{a_i}. \tag{4.11}$$

Proof. We simply set

$$\theta(z) = p(z) \prod_{i=1}^{m} \theta_q(z/b_i) \cdot \prod_{i=1}^{n} \theta_q(z/a_i)^{-1},$$

then $\theta(z)$ is a q-theta function of multiplier αz^{n-m}, with

$$\alpha := \prod_{i=1}^{n} b_i \cdot \prod_{i=1}^{m} a_i^{-1}.$$

Furthermore, $\theta(z)$ has neither zeros nor poles on $\mathbb{C}^*$, and we obtain the theorem by application of Lemma 4.1.4. □

The second equation in (4.11) should remind us of the fact that the sum of poles minus the sum of zeros of an elliptic function within a fundamental domain, taking into account their multiplicities, is an element of the period lattice. We define the *degree* of a nonzero q-elliptic function $p(z)$, to be number of zeros or equivalently number of poles, counting multiplicity, within a fundamental annulus.

Corollary 4.1.6. *The degree of a non-constant q-elliptic function is at least* 2.

Proof. Consider (4.11) with $n = 1$. Then $b_1/a_1 \in q^{\mathbb{Z}}$, so $p(z)$ has a both a zero and pole at $z = b_1$, which is nonsense. Hence the degree of a q-elliptic function cannot be 1. □

Corollary 4.1.7. *Let $p(z)$ and $q(z)$ be nonzero q-elliptic functions with identical zeros, poles and their multiplicities, within some fundamental annulus, then there is a $c \in \mathbb{C}^*$ such that $p(z) = cq(z)$.*

We often require only a weaker version of Theorem 3.1.3, given by the following

Corollary 4.1.8. *Let $p(z)$ be a nonzero q-elliptic function of degree $n \in \mathbb{N}$, then there are $a_1, \ldots a_n \in \mathbb{C}^*$, $b_1, \ldots b_n \in \mathbb{C}^*$ and $c \in \mathbb{C}^*$, such that*

$$p(z) = c \prod_{i=1}^{n} \frac{\theta_q(z/a_i)}{\theta_q(z/b_i)}, \quad 1 = \prod_{i=1}^{n} \frac{b_i}{a_i}. \tag{4.12}$$

The above corollary tells us that any q-elliptic function is basically the quotient of two analytic q-theta functions of common multiplier. Indeed, considering equation (4.12), writing

$$\theta^a(z) = c \prod_{i=1}^{n} \theta_q(z/a_i), \quad \theta^b(z) = \prod_{i=1}^{n} \theta_q(z/b_i), \quad \alpha = (-1)^n \prod_{i=1}^{n} a_i = (-1)^n \prod_{i=1}^{n} b_i,$$

we have $p(z) = \theta^a(z)/\theta^b(z)$, where $\theta^a(z)$ and $\theta^b(z)$ are both analytic q-theta functions of multiplier αz^{-n}. Motivated by this observation, let us define, for $n \in \mathbb{N}^*$ and $\alpha \in \mathbb{C}^*$,

$$V_q^n(\alpha) = \{\text{analytic } q\text{-theta functions of multiplier } \alpha z^{-n}\},$$

then $V_q^n(\alpha)$ denotes a complex vector space under the usual function addition and scalar multiplication. If $\theta \in V_q^n(\alpha)$, and $a \in \mathbb{C}^*$ is such that $\theta(a) = 0$, then $\widetilde{\theta}(z) = \theta(z)/\theta_q(z/a)$ is an element of $V_q^{n-1}(-\alpha/a)$. Using this we can easily derive that any nonzero element $\theta \in V_q^n(\alpha)$, is of the form

$$\theta(z) = c \prod_{i=1}^{n} \theta_q(z/a_i), \quad (-1)^n \prod_{i=1}^{n} a_i = \alpha, \tag{4.13}$$

for some $a_1, \ldots, a_n \in \mathbb{C}^*$ and $c \in \mathbb{C}^*$. To put it differently, in (4.13) we can take $c \in \mathbb{C}^*$ and $a_1, \ldots a_{n-1}$ at pleasure to define an element of $V_q^n(\alpha)$. Counting the number of freedoms, this fits in neatly with the following

Theorem 4.1.9. *Let $n \in \mathbb{N}^*$ and $\alpha \in \mathbb{C}^*$, then $V_q^n(\alpha)$ is a complex vector space of dimension n.*

Proof. We proceed by induction. Note the case $n = 1$ is trivial, indeed

$$V_q^1(\alpha) = \{c\theta_q(-z/\alpha) : c \in \mathbb{C}\},$$

and hence $V_q^1(\alpha)$ is one-dimensional for all $\alpha \in \mathbb{C}^*$. Now suppose the statement of the theorem holds for some $n \in \mathbb{N}^*$, and take any $\alpha \in \mathbb{C}^*$. We fix an $a \in \mathbb{C}^*$ and construct an element $\theta^* \in V_q^{n+1}(\alpha)$ which satisfies $\theta^*(a) = 1$. Then any element $\theta \in V_q^{n+1}(\alpha)$ can be written uniquely as

$$\theta(z) = \theta_q(z/a)\widetilde{\theta}(z) + \theta(a)\theta^*(z),$$

where $\widetilde{\theta} \in V_q^n(-\alpha/a)$. It is easy to see that this gives us a decomposition

$$V_q^{n+1}(\alpha) \cong V_q^n(-\alpha/a) \oplus \mathbb{C},$$

and the theorem follows by induction. $\square$

We finish our discussion with a useful *addition formula* for q-theta functions. From Ormerod and Rains [67], we take the following identity among analytic q-theta functions of multiplier z^{-2},

$$\theta_q(bc, c/b)\theta_q(az, z/a) - \theta_q(ac, c/a)\theta_q(bz, z/b) = c/a\theta_q(ab, a/b)\theta_q(cz, z/c). \tag{4.14}$$

To validate it, all we have to do is observe that the left and right-hand side have common roots $z = q^{\mathbb{Z}}c^{\pm}$, hence they differ by at most a multiplicative constant, and subsequently we check that they agree at $z = b$. In light of Theorem 4.1.9, equation (4.14) should be read as follows. Given any three elements of $V_q^2(1)$, we know that they must be linearly dependent, and the addition formula gives us the explicit linear dependence. We require a slightly different parameterisation of the addition formula, given by

$$\begin{aligned}\theta_q(\alpha^{-1}bc, c/b)\theta_q(\alpha^{-1}az, z/a) - \theta_q(\alpha^{-1}ac, c/a)\theta_q(\alpha^{-1}bz, z/b) = \\ c/a\theta_q(\alpha^{-1}ab, a/b)\theta_q(\alpha^{-1}cz, z/c),\end{aligned} \tag{4.15}$$

which gives the explicit linear dependence between any three elements of $V_q^2(\alpha)$.

4.2 Birkhoff's Theory

We discuss Birkhoff's aproach [5] to the global asymptotic analysis of linear q-difference systems, more or less following Mano's summary [61] of it. We keep our discussion brief, and give more details in Section 4.3, adapted to the case we are interested in. Analogously to the first order equation (4.2), our starting point is the following first order matrix equation

$$Y(qz) = A(z)Y(z), \tag{4.16}$$

where we limit our discussion to the rank two case, with

$$A(z) = A_0 + zA_1 + z^2A_2 + \ldots + z^nA_n, \tag{4.17}$$

where $n \in \mathbb{N}$ is called the degree of the equation and $A_0, A_n \in GL_2(\mathbb{C})$, are assumed diagonalisable. We diagonalise A_0 and A_n by

$$A_0 = M_0 \begin{pmatrix}\theta_1 & 0 \\ 0 & \theta_2\end{pmatrix} M_0^{-1}, \quad A_n = M_\infty \begin{pmatrix}\kappa_1 & 0 \\ 0 & \kappa_2\end{pmatrix} M_\infty^{-1}, \tag{4.18}$$

where we think of the eigenvalues $\{\theta_1, \theta_2\}$ and $\{\kappa_1, \kappa_2\}$ as prescribed. Furthermore we prescribe the roots of the determinant of $A(z)$,

$$|A(z)| = c \prod_{k=1}^{2n} (z - x_k). \quad (c \in \mathbb{C}^*) \tag{4.19}$$

Note that we can only do this subject to

$$\kappa_1 \kappa_2 \prod_{k=1}^{2n} x_k = \theta_1 \theta_2, \tag{4.20}$$

which is the q-analog of *Fuchs' relation* for linear differential equations. We note that gauging equation (4.16) by an element of $GL_2(\mathbb{C})$, leaves all the analytic data invariant, and we are to some extend only interested in the equation up to such overall gauging. By prescribing the eigenvalues of A_0 and A_n, we essentially have $4n$ free parameters, and by also taking into account (4.19), we are left with $2n+1$ free parameters. Then, if we consider $A(z)$ up to overall gauging by $GL_2(\mathbb{C})$, these numbers become $4n-3$ and $2n-2$ respectively.

We assume $\theta_1/\theta_2, \kappa_1/\kappa_2 \notin q^{\mathbb{Z}}$, which is a q-analog of the *non-resonant condition* for differential equations. Carmichael [9] shows that, for fixed M_0 and M_∞ satisfying (4.18), there exist unique fundamental solutions of (4.16) of the form,

$$Y_0(z) = M_0 \Phi_0(z) \begin{pmatrix} e_q(z;\theta_1) & 0 \\ 0 & e_q(z;\theta_2) \end{pmatrix}, \tag{4.21a}$$

$$Y_\infty(z) = \theta_q(-z)^{-n} M_\infty \Phi_\infty(z) \begin{pmatrix} e_q(z;\kappa_1) & 0 \\ 0 & e_q(z;\kappa_2) \end{pmatrix}, \tag{4.21b}$$

about $z=0$ and $z=\infty$ respectively, where $\Phi_0(z)$ analytic at $z=0$, with $\Phi_0(0)=I$, and $\Phi_\infty(z)$ analytic at $z=\infty$, with $\Phi_\infty(\infty)=I$. Furthermore $\Phi_\infty(z)$ and $\Phi_0(z)^{-1}$ can be analytically continued, i.e. without poles, on $\mathbb{C}^*$.

We now define the *connection matrix* $P(z)$ by

$$Y_\infty(z) = Y_0(z) P(z),$$

which is meromorphic on $\mathbb{C}^*$ and satisfies $P(qz) = P(z)$, so its entries are q-elliptic functions. Considering (4.21), we have

$$P(z) = \theta_q(-z)^{-n} \begin{pmatrix} e_q(z;\theta_1)^{-1} & 0 \\ 0 & e_q(z;\theta_2)^{-1} \end{pmatrix} Q(z) \begin{pmatrix} e_q(z;\kappa_1) & 0 \\ 0 & e_q(z;\kappa_2) \end{pmatrix},$$

where the *associated connection matrix* $Q(z)$ is defined by

$$Q(z) = \Phi_0(z)^{-1} M_0^{-1} M_\infty \Phi_\infty(z).$$

Note that $Q(z)$ is analytic and satisfies

$$Q(qz) = z^{-n}\begin{pmatrix}\theta_1 & 0\\ 0 & \theta_2\end{pmatrix} Q(z)\begin{pmatrix}\kappa_1^{-1} & 0\\ 0 & \kappa_2^{-1}\end{pmatrix},$$

that is, for $i,j\in\{1,2\}$, the entry $Q_{ij}(z)$ is an element of $V_q^n\left(\theta_i\kappa_j^{-1}\right)$. By Theorem 4.1.9, we see that $Q(z)$ lives in a $4n$ dimensional space. However, note that the diagonalisations in (4.18), are only uniquely defined up to right-multiplication of M_0 and M_∞ by diagonal matrices. Say $M_0' = M_0F_0$ and $M_\infty' = M_\infty F_\infty$, for diagonal matrices $F_0, F_\infty \in GL_2(\mathbb{C})$, then the corresponding fundamental solutions (4.21) become

$$\Phi_0'(z) = F_0^{-1}\Phi_0(z)F_0, \quad \Phi_\infty'(z) = F_\infty^{-1}\Phi_\infty(z)F_\infty,$$

and

$$Q'(z) = F_0^{-1}Q(z)F_\infty.$$

To rigidify the situation, we consider $Q(z)$ only up to multiplication from the left and right by invertible diagonal matrices, and denote

$$[Q(z)]\,, \tag{4.22}$$

for the corresponding equivalence class. Note that $[Q(z)]$ lives in a $4n-3$ dimensional space. We now have a well-defined mapping

$$A(z)\mapsto [Q(z)]\,,$$

which is easily seen to be constant on $GL_2(\mathbb{C})$-conjugation classes of $A(z)$. The *Riemann-Hilbert-Birkhoff correspondence* is the bijective correspondence between matrix polynomials $A(z)$ (4.17), with prescribed exponents at zero and infinity, up to conjugation by $GL_2(\mathbb{C})$, and corresponding $[Q(z)]$, living in the orbit space of $V_q^n\left(\theta_i\kappa_j^{-1}\right)_{1\leq i,j\leq 2}$, with respect to multiplication of invertible diagonal matrices from the left and right, which we symbolically write as

$$[A(z)]_{GL_2(\mathbb{C})} \leftrightarrow [Q(z)]\,. \tag{4.23}$$

We can specialise this correspondence, by taking into account the prescribed zeros $x_1,\ldots,x_{2n}$ of $|A(z)|$, which gives another $2n-1$ constraints on $[Q(z)]$, as one can show that

$$|Q(z)| = \mathit{constant}\times\theta_q(z/x_1,\ldots,z/x_{2n}).$$

A few remarks are in order here. Firstly, let us note that (4.16) is strictly speaking irregular singular at $z=\infty$, indeed $z=\infty$ is only regular singular after scaling by $\theta_q(-z)^{-n}$, see (4.21b). As set out by Sauloy [79], a more natural starting point would be to assume $A(z)$ is a matrix with rational entries, such that $A(0), A(\infty)\in GL_2(\mathbb{C})$. Then (4.16) is really Fuchsian.

Secondly, a major difficulty in the theory of linear q-difference equations, is that the field of constants, i.e. q-elliptic functions, is in some sense too large. Indeed, we wish to consider our equation (4.16), up to gauging by only constant complex matrices, whereas, for instance the scaling entries $e_q(z;\kappa_1)$ and $e_q(z;\kappa_2)$ on the right-hand side of (4.21b), are strictly speaking only characterised up to q-elliptic functions. There are several ways to work around this problem. Our approach lies close to van der Put and Singer [83], we consider the scalings involved only symbolically. As an example, considering (4.21b), we put more importance in $\Phi_\infty(z)$, then $Y_\infty(z)$. Similarly we consider the associated connection matrix $Q(z)$ as being more fundamental than the connection matrix $P(z)$.

Le Caine [58] considered equation (4.17) with $n=1$. She showed that the fundamental solutions near $z=0$ and $z=\infty$ can be described in terms of ${}_2\phi_1$ hypergeometric functions (1.3) and determined the corresponding connection matrix explicitly. Jimbo and Sakai [43] derived a q-analog of P_{VI} within Birkhoff's framework. They consider the $n=2$ case of (4.17), with eigenvalues parameterised by

$$\theta_1=\frac{a_1a_2}{b_1}t,\quad \theta_2=\frac{a_1a_2}{b_2}t,\quad \kappa_1=\frac{1}{qb_3},\quad \kappa_2=\frac{1}{qb_4},$$

and zeros of the determinant (4.19), parameterised by

$$x_1=a_1t,\quad x_2=a_2t,\quad x_3=a_3,\quad x_4=a_4,$$

where $t,a_1,\dots a_4,b_1,\dots,b_4\in\mathbb{C}^*$, and Fuchs' equation (4.20) translates to

$$\frac{b_1b_2a_3a_4}{a_1a_2b_3b_4}=q.$$

Now note that for fixed $t\in\mathbb{C}^*$, the matrix $A=A(z,t)$ has essentially $2n+1=5$ parameters, and hence two parameters when considered up to conjugation by $GL_2(\mathbb{C})$. Next we consider a deformation $t\mapsto qt$, such the connection matrix $P=P(z,t)$ is preserved, i.e. $P(z,qt)=P(z,t)$. Jimbo and Sakai [43] show that, by eliminating the gauge freedom in $A(z,t)$ appropriately, setting in particular $A_2=\mathrm{diag}(\kappa_1,\kappa_2)$, the time-evolution of the entries of $A_0=A_0(t)$ and $A_1=A_1(t)$, after some appropriate parameterisation, is equivalent to

$$q\text{-}P_{\mathrm{VI}}\quad\begin{cases}\dfrac{f\overline{f}}{a_3a_4} &= \dfrac{(\overline{g}-tb_1)(\overline{g}-tb_2)}{(\overline{g}-b_3)(\overline{g}-b_4)},\\ \dfrac{g\overline{g}}{b_3b_4} &= \dfrac{(f-ta_1)(f-ta_2)}{(f-a_3)(f-a_4)},\end{cases}$$

which is a q-analog of the sixth Painlevé equation.

4.3 Second Order q-Difference Equations

In this section we study the global asymptotic analysis of second order q-difference equations of the form,

$$u(z)y(qz)+v(z)y(z)+w(z)y(z/q)=0, \tag{4.24}$$

where $u(z),v(z)$ and $w(z)$ are polynomials, without common divisors. We start with a brief discussion on the classification of critical points and corresponding local solutions, for a more complete treatment of the subject we refer to Adams [1].

4.3.1 Classification of Critical Points

Analogously to the continuous theory, to construct solutions about $z = 0$, we consider the *indicial equation*

$$u(0)\lambda + v(0) + w(0)\lambda^{-1} = 0,$$

and we sometimes refer to its roots as *exponents*. In case $u(0) = 0$ or $w(0) = 0$, the indicial equation has solutions $\lambda = 0$ or, formally speaking, $\lambda^{-1} = 0$, and we say that $z = 0$ is an *irregular singular point* of (4.24), or we call the equation *unbalanced* at $z = 0$. Otherwise we say that $z = 0$ is a *regular singular point* or *Fuchsian singularity* of (4.24). Let us focus on the regular singular case and let $\lambda_1, \lambda_2 \in \mathbb{C}^*$ denote corresponding exponents. We rescale (4.24) by setting

$$y(z) = e_q(z; \lambda_1)\psi(z),$$

which gives

$$\lambda_1 u(z)\psi(qz) + v(z)\psi(z) + \lambda_1^{-1} w(z)\psi(z/q) = 0. \tag{4.25}$$

At this point we employ the power series method, we set

$$\psi(z) = \sum_{m=0}^{\infty} a_m z^m,$$

and substitution into (4.25) shows that we can choose a_0 at our pleasure, and consequently the values of all a_m are determined uniquely by comparing coefficients of z^m in (4.25), unless $\lambda_1 q^m = \lambda_2$ for some $m \in \mathbb{N}$, at which stage the power series method potentially fails.

Let us assume $\lambda_1/\lambda_2 \notin q^Z$ for now, we choose $a_0 = 1$ and determine the unique power series solution $\psi_1(z)$ of (4.25) with $\psi_1(0) = 1$. It is easy to show that such a power series solution is always convergent. Now $y_1(z) = e_q(z; \lambda_1)\psi_1(z)$, defines a solution of (4.24), having the q-analog of a Frobenius expansion at $z = 0$. Similarly we define an unique solution $y_2(z) = e_q(z; \lambda_2)\psi_2(z)$ of (4.24), where $\psi_2(z)$ a convergent power series with $\psi_2(0) = 1$. We can now define a *fundamental solution* of (4.24) at $z = 0$,

$$y(z) = \begin{pmatrix} y_1(z) & y_2(z) \end{pmatrix} = \begin{pmatrix} \psi_1(z) & \psi_2(z) \end{pmatrix} \begin{pmatrix} e_q(z; \lambda_1) & 0 \\ 0 & e_q(z; \lambda_2) \end{pmatrix}.$$

We assumed $\lambda_1/\lambda_2 \notin q^{\mathbb{Z}}$, which is called the *non-resonance condition*. In case it is violated we say that $z = 0$ is a regular singular point with *resonance* of (4.24). Let us consider the resonant case where $\lambda_2 = q\lambda_1$, the only case of interest for our purposes. To simplify the discussion, let us assume

$$\begin{aligned} u(z) &= u_0 + u_1 z + u_2 z^2 + \ldots, & u_0 &= 1, \\ v(z) &= v_0 + v_1 z + v_2 z^2 + \ldots, & v_0 &= -1 - q, \\ w(z) &= w_0 + w_1 z + w_2 z^2 + \ldots, & w_0 &= q, \end{aligned}$$

so $\lambda_1 = 1$ and $\lambda_2 = q$. Note that (4.24) has an unique convergent power series solution

$$y_2(z) = 0 + z + a_2 z^2 + \dots,$$

where all the coefficients can be determined by direct substitution into (4.24) as before. However considering $\lambda_1 = 1$, when we substitute a power series

$$y_1(z) = 1 + a_1 z + a_2 z^2 + \dots,$$

into (4.24), and compare coefficients of z, we find

$$u_1 + v_1 + w_1 = 0. \tag{4.26}$$

So generically the power series method fails and, just like in the continuous case, we have to consider series expansions involving logarithms, or more specifically, a function χ which satisfies $\chi(qz) = \chi(z) + 1$, for instance $\chi(z) = \log_q(z)$ or[1]

$$\chi(z) = z\frac{\theta_q'(-z)}{\theta_q(-z)} + \tfrac{1}{2}. \tag{4.27}$$

However, in case (4.26) holds, we say that $z = 0$ is an *ordinary point* of (4.24). More generally, in case $\lambda_1/\lambda_2 \in q^{\mathbb{Z}}$, but the Frobenius method above still allows us to find two linearly independent solutions, we say that $z = 0$ is an *apparent singularity* of (4.24). We classify the critical point $z = \infty$ of (4.24) in a completely analogous fashion, by dividing (4.24) by the highest power of z occuring in the coefficients $u(z)$, $v(z)$ and $w(z)$, so that the resulting equation has coefficients which are polynomial in z^{-1}.

4.3.2 A Standard Form

From Section 4.1 we know that for any rational function $r(z)$, we can find a meromorphic function $S(z)$ on $\mathbb{C}^*$, such that $S(qz) = r(z)S(z)$. This allows us to *scale* or *gauge* equation (4.24) by setting $y(z) = S(z)\widetilde{y}(z)$, which gives

$$r(z)u(z)\widetilde{y}(qz) + v(z)\widetilde{y}(z) + r(z/q)^{-1}w(z)\widetilde{y}(z/q) = 0. \tag{4.28}$$

In particular the choice $r(z) = u(z)^{-1}$, leads to

$$\widetilde{y}(qz) + v(z)\widetilde{y}(z) + u(z/q)w(z)\widetilde{y}(z/q) = 0. \tag{4.29}$$

So we can always bring our q-difference equation in the following form,

$$y(qz) + v(z)y(z) + w(z)y(z/q) = 0, \tag{4.30}$$

with $v(z)$ and $w(z)$ polynomials such that $v(z), w(z)$ and $w(z/q)$ do not have a nonzero common root, and if $v(0) = 0$, then $z = 0$ is not a root of $w(z)$ with multiplicity more than

[1]The choice (4.27) seems particularly appropriate as the corresponding $\chi(z)$ is meromorphic on $\mathbb{C}^*$, with simple poles on $q^{-\mathbb{Z}}$, satisfying $\chi(q^z) = n = \log_q(q^n)$ for $n \in \mathbb{Z}$.

one. Indeed if the latter condition is violated for some root of $v(z)$, then we can easily gauge it away, replacing $v(z)$ and $w(z)$ in (4.30) by lower degree polynomials. This form is essentially the same as the one presumed by Birkhoff 4.16. Note that (4.30) is singular at $z=\infty$, hence, to study the analytic characterisation at $z=\infty$, it is helpful to rescale

$$y(qz)=\theta_q(-z)^{-n}\widetilde{y}(z),\quad \widetilde{v}(z)=z^{-n}v(z),\quad \widetilde{w}(z)=q^nz^{-2n}w(z), \tag{4.31}$$

where $n\in\mathbb{N}$ chosen minimal such that $\widetilde{v}(z)$ and $\widetilde{w}(z)$ are polynomials in z^{-1}. The indicial equation at $z=\infty$ takes the form

$$\kappa+\widetilde{v}(\infty)+\widetilde{w}(\infty)\kappa^{-1}=0. \tag{4.32}$$

It follows immediately that $z=\infty$ is an essential singularity if the degree of $w(z)$ does not equal $2n$. Hence, $z=\infty$ is a regular singular point if and only if the degree of $w(z)$ is $2n$ and the degree of $v(z)$ is less or equal to n. The same thing is true for Birkhoff's form (4.16), which is strictly speaking irregular singular at $z=\infty$. Indeed only after scaling by $\theta_q(-z)^{-n}$, see (4.21b), $z=\infty$ is a regular singular point.

4.4 The Model Equation

Our main interest lies in the global asymptotic analysis of second order linear equations, where $z=0$ is an ordinary point and $z=\infty$ is a regular singular point.

4.4.1 General Set Up

We consider the second order homogeneous q-difference equation

$$y(qz)+v(z)y(z)+w(z)y(z/q)=0, \tag{4.33}$$

where

$$\begin{aligned} v(z)&=v_0+v_1z+\ldots+v_nz^n, & v_0&=-1-q,\\ w(z)&=w_0+w_1z+\ldots+w_{2n}z^{2n}, & w_0&=q, \end{aligned}$$

for some $n\in\mathbb{N}$, which we refer to as the *degree* of (4.33). Furthermore we think of the zeros of $w(z)$ as prescribed,

$$w(z)=q(1-z/x_1)\cdot\ldots\cdot(1-z/x_{2n}). \tag{4.34}$$

and to assure an apparent singularity at $z=0$, the condition (4.26) takes the form $v_1+w_1=0$, so we require

$$v_1=q(x_1^{-1}+\ldots+x_{2n}^{-1}).$$

Lastly, after the scaling (4.31), we prescribe exponents $\kappa_1,\kappa_2\in\mathbb{C}^*$ at $z=\infty$, and hence, by (4.32),

$$v_n=-(\kappa_1+\kappa_2),\qquad w_{2n}=q^{-n}\kappa_1\kappa_2. \tag{4.35}$$

Note that we can not freely prescribe both the zeros $x_1, \ldots, x_{2n} \in \mathbb{C}^*$ and exponents $\kappa_1, \kappa_2 \in \mathbb{C}^*$, as combining (4.34) and (4.35) gives

$$\kappa_1 \kappa_2 = \frac{q^{n+1}}{x_1 \cdot \ldots \cdot x_{2n}}, \tag{4.36}$$

which is the q-analog of Fuchs' relation for linear differential equations. Now, say we prescribe the zeros and exponents subject to (4.36), then (4.33) still has $n-2$ free parameters, coming from the freedom in choosing the coefficients $v_2, \ldots, v_{n-1}$. The degree $n = 2$ case plays a crucial role in the asymptotic analysis of Yamada's Lax pair in Chapter 5, and in Section 4.5 we discuss it in detail.

It is sometimes helpful to write (4.33) in system form. Writing

$$Y(z) = \begin{pmatrix} y(z) \\ y(z/q) \end{pmatrix},$$

we obtain

$$Y(qz) = A(z)Y(z), \quad A(z) = \begin{pmatrix} -v(z) & -w(z) \\ 1 & 0 \end{pmatrix}. \tag{4.37}$$

The equivalence between (4.33) and Birkhoff's form (4.17) can now be made explicit, as choosing any factorisation $w(z) = w_1(z)w_2(z)$, with $w_1(z)$ and $w_2(z)$ polynomials of degree n, the rational gauge

$$Y(z) = R(z)\widetilde{Y}(z), \quad R(z) = \begin{pmatrix} 1 & 0 \\ 0 & w_2(z)^{-1} \end{pmatrix},$$

gives

$$\widetilde{Y}(qz) = \widetilde{A}(z)\widetilde{Y}(z), \quad \widetilde{A}(z) = R(qz)^{-1}A(z)R(z) = \begin{pmatrix} -v(z) & -w_1(z) \\ w_2(qz) & 0 \end{pmatrix},$$

which is of the form (4.17).

4.4.2 Fundamental Solution at Origin

As $z = 0$ is an ordinary point of the equation under consideration (4.33), the power series method gives a convergent power series solution

$$y(z) = c_0 + c_1 z + c_2 z^2 + \ldots,$$

for any choice of $c_0, c_1 \in \mathbb{C}$, after which all higher order coefficients are fixed. Let us specify two solutions

$$y_1^0(z) = 1 + 0z + \mathcal{O}\left(z^2\right), \tag{4.38a}$$
$$y_2^0(z) = 0 + 1z + \mathcal{O}\left(z^2\right), \tag{4.38b}$$

and write the corresponding fundamental solution by

$$y^0(z) := \left(y_1^0(z) \quad y_2^0(z)\right) = \left(1 \quad z\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \mathcal{O}\left(z^2\right). \qquad (z \to 0)$$

This is initially only a local solution, but we can use the equation it satisfies (4.33), to meromorphically continue it to the finite complex plane $\mathbb{C}$. Note however, that the zeros of $w(z)$ can cause poles to arise. Let us define the corresponding fundamental solution of the system form (4.37) by

$$Y^0(z) = \begin{pmatrix} y_1^0(z) & y_2^0(z)\\ y_1^0(z/q) & y_2^0(z/q)\end{pmatrix},$$

then the *Wronskian* $\mathcal{W}(y^0)(z) := |Y^0(z)|$ satisfies

$$\mathcal{W}(y^0)(qz) = w(z)\mathcal{W}(y^0),$$

as $|A(z)| = w(z)$. Furthermore we have the asymptotic characterisation

$$\mathcal{W}(y^0)(z) = (q^{-1} - 1)z + \mathcal{O}\left(z^2\right), \qquad (z \to 0)$$

which gives

$$\mathcal{W}(y^0)(z) = (q^{-1} - 1)z(z/x_1, \dots z/x_{2n}; q)_\infty^{-1}. \tag{4.39}$$

We end our analysis of equation (4.33) near $z = 0$ with the following observation.

Lemma 4.4.1. *The matrix function $Y^0(z)^{-1}$ is analytic on $\mathbb{C}^*$.*

Proof. As $Y^0(z)$ is a fundamental solution of (4.37), by (4.39) clearly invertible, its inverse satisfies

$$Y^0(z)^{-1} = Y^0(qz)^{-1}A(z). \tag{4.40}$$

Note that $Y^0(z)^{-1}$ is a meromorphic matrix, which has a convergent Laurent expansion at $z = 0$, with leading term given by

$$Y^0(z)^{-1} = z^{-1}\begin{pmatrix} 0 & 0\\ \frac{q}{q-1} & -\frac{q}{q-1}\end{pmatrix} + \mathcal{O}(1). \qquad (z \to 0)$$

Therefore $Y^0(z)^{-1}$ is analytic in a punctured disc about the origin, and (4.40) guarantees unique analytic continuation to the entire punctured plane $\mathbb{C}^*$. □

4.4.3 Fundamental Solution at Infinity

To construct a fundamental solution at $z = \infty$ of (4.33), we rescale, in accordance with (4.31),

$$y(z) = S_i^\infty(z)\psi_i(z), \quad S_i^\infty(qz) = \kappa_i z^n S_i^\infty(z), \quad S_i(z) = \theta_q(-z)^{-n}e_q(z;\kappa_i), \tag{4.41}$$

which gives, for $i = 1, 2$,

$$\kappa_i\psi_i(qz) + z^{-n}v(z)\psi_i(z) + \kappa_i^{-1}q^n z^{-2n}w(z)\psi_i(z/q) = 0. \tag{4.42}$$

In analogy with Lemma 4.4.1, we have the following result.

Lemma 4.4.2. *If $\kappa_1/\kappa_2 \notin q^{\mathbb{Z}}$, then, for $i = 1, 2$, the q-difference equation (4.42) has an unique solution $\psi_i^\infty(z)$ which is analytic at $z = \infty$ with $\psi_i^\infty(\infty) = 1$. Furthermore this solution is analytic on the entire punctured Riemann sphere $\mathbb{P}^*$.*

Proof. The assumption assures that $z = \infty$ is non-resonant, hence the power series method allows us to find solutions around $z = \infty$ as prescribed. Equation (4.42) now guarantees unique analytic continuation to $\mathbb{P}^*$. □

For now we assume $\kappa_1/\kappa_2 \notin q^{\mathbb{Z}}$, and define a fundamental solution of (4.33) near $z = \infty$, by

$$y^\infty(z) = \begin{pmatrix} y_1^\infty(z) & y_2^\infty(z) \end{pmatrix} := \psi^\infty(z) \begin{pmatrix} S_1^\infty(z) & 0 \\ 0 & S_2^\infty(z) \end{pmatrix}, \quad \psi^\infty(z) := \begin{pmatrix} \psi_1^\infty(z) & \psi_2^\infty(z) \end{pmatrix}.$$

Associated we have a fundamental solution of the system form (4.37),

$$\begin{aligned} Y^\infty(z) &= \begin{pmatrix} y_1^\infty(z) & y_2^\infty(z) \\ y_1^\infty(z/q) & y_2^\infty(z/q) \end{pmatrix} \\ &= \begin{pmatrix} S_1^\infty(z)\psi_1^\infty(z) & S_2^\infty(z)\psi_2^\infty(z) \\ S_1^\infty(z/q)\psi_1^\infty(z/q) & S_2^\infty(z/q)\psi_2^\infty(z/q) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & q^n z^{-n} \end{pmatrix} \begin{pmatrix} \psi_1^\infty(z) & \psi_2^\infty(z) \\ \kappa_1^{-1}\psi_1^\infty(z/q) & \kappa_2^{-1}\psi_2^\infty(z/q) \end{pmatrix} \begin{pmatrix} S_1^\infty(z) & 0 \\ 0 & S_2^\infty(z) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & q^n z^{-n} \end{pmatrix} \Psi^\infty(z) \begin{pmatrix} S_1^\infty(z) & 0 \\ 0 & S_2^\infty(z) \end{pmatrix}, \end{aligned}$$

where we denoted

$$\Psi^\infty(z) = \begin{pmatrix} \psi_1^\infty(z) & \psi_2^\infty(z) \\ \kappa_1^{-1}\psi_1^\infty(z/q) & \kappa_2^{-1}\psi_2^\infty(z/q) \end{pmatrix}.$$

Again we are interested in the Wronskians $\mathcal{W}(y^\infty)(z) = |Y^\infty(z)|$ and $\mathcal{W}(\psi^\infty)(z) = |\Psi^\infty(z)|$, which, by the above calculation, are related by

$$\mathcal{W}(y^\infty)(z) = q^n z^{-n} S_1^\infty(z) S_2^\infty(z) \mathcal{W}(\psi^\infty)(z).$$

From this relation we easily derive

$$\begin{aligned} \mathcal{W}(\psi^\infty)(qz) &= \frac{q^n}{\kappa_1\kappa_2} z^{-2n} w(z) \mathcal{W}(\psi^\infty)(z) \\ &= q^{-1} x_1 \cdot \ldots \cdot x_{2n} z^{-2n} w(z) \mathcal{W}(\psi^\infty)(z) \\ &= (1 - x_1/z) \cdot \ldots \cdot (1 - x_{2n}/z) \mathcal{W}(\psi^\infty)(z), \end{aligned}$$

where in the second equality we used Fuchs' relation (4.36).
Combining this first order equation with the asymptotic characterisation

$$\mathcal{W}(\psi^\infty)(z) = \kappa_2^{-1} - \kappa_1^{-1} + \mathcal{O}\left(z^{-1}\right), \qquad (z \to \infty)$$

we directly obtain an explicit equation for the Wronskian

$$\mathcal{W}(\psi^\infty)(z) = (\kappa_2^{-1} - \kappa_1^{-1})\,(qx_1/z, \ldots, qx_{2n}/z; q)_\infty\,. \tag{4.43}$$

4.4.4 The Connection Matrix

We are now ready to construct a fundamental object in our study, the *connection matrix*,

$$P(z) = Y^0(z)^{-1}Y^\infty(z),$$

which is meromorphic on $\mathbb{C}^*$, and satisfies

$$y^\infty(z) = y^0(z)P(z).$$

From the definition we immediately find $P(qz) = P(z)$, and hence the entries of the connection matrix are q-elliptic functions. Note however, that we made choices on the way, on which $P(z)$ depends. For example, the choice of functions S_1^∞ and $S_2^\infty(z)$ in (4.41), was quite arbitrary. A bit more delicate, the choice of initial conditions in (4.38), contains some freedom as $z = 0$ is an ordinary point. Note however, that

$$\begin{aligned} P(z) &= Y^0(z)^{-1}Y^\infty(z)\\ &= Y^0(z)^{-1}\begin{pmatrix}1 & 0\\ 0 & q^n z^{-n}\end{pmatrix}\Psi^\infty(z)\begin{pmatrix}S_1^\infty(z) & 0\\ 0 & S_2^\infty(z)\end{pmatrix}\\ &= Q(z)\begin{pmatrix}S_1^\infty(z) & 0\\ 0 & S_2^\infty(z)\end{pmatrix}, \end{aligned}$$

where the *associated connection matrix* $Q(z)$ is defined by

$$Q(z) = \begin{pmatrix}Q_{11}(z) & Q_{12}(z)\\ Q_{21}(z) & Q_{22}(z)\end{pmatrix} := Y^0(z)^{-1}\begin{pmatrix}1 & 0\\ 0 & q^n z^{-n}\end{pmatrix}\Psi^\infty(z), \tag{4.44}$$

independent of the particular choice of scalings. Note that $Q(z)$ satisfies

$$y^0(z)Q(z) = \psi^\infty(z), \tag{4.45}$$

and by Lemma 4.40 and 4.4.2, the matrix function $Q(z)$ is analytic on the entire punctured plane $\mathbb{C}^*$. Of course $Q(z)$ fails to be q-periodic, however it does satisfy

$$Q(qz) = Q(z)\begin{pmatrix}\kappa_1^{-1}z^{-n} & 0\\ 0 & \kappa_2^{-1}z^{-n}\end{pmatrix}. \tag{4.46}$$

Hence the entries $Q_{11}(z)$ and $Q_{21}(z)$ are analytic q-theta functions of multiplier $\kappa_1^{-1}z^{-n}$, and the entries $Q_{12}(z)$ and $Q_{22}(z)$ are analytic q-theta functions of multiplier $\kappa_2^{-1}z^{-n}$. To put it differently, writing

$$(Q_{11}, Q_{21}, Q_{12}, Q_{22}) \in M := V_q^n(\kappa_1^{-1}) \oplus V_q^n(\kappa_1^{-1}) \oplus V_q^n(\kappa_2^{-1}) \oplus V_q^n(\kappa_2^{-1}), \tag{4.47}$$

the matrix $Q(z)$ essentially lives in the $4n$ dimensional vector space M.

At this point we have not used the additional information coming from the explicit expressions (4.39) and (4.43), for the Wronskians involved in

$$|Q(z)| = q^n z^{-n} \mathcal{W}(y^0)(z)^{-1} \mathcal{W}(\psi^\infty)(z),$$

which gives

$$|Q(z)| = \frac{q(\kappa_2^{-1} - \kappa_1^{-1})}{1-q} q^n z^{-n-1} \theta_q(z/x_1, \ldots, z/x_{2n}). \tag{4.48}$$

We conclude that the matrix $Q(z)$ in fact lives in M', the "closed" subspace of M defined by the cut (4.48). Note that (4.48) gives $2n$ constraints, $2n-1$ coming from the locations of, for instance, the zeros $x_1, \ldots x_{2n-1}$, after which x_{2n} is automatically a zero because of the second equation in (4.13). The remaining constraint comes from the overall scalar factor in front of the product of q-theta functions. So M' is essentially a $2n$ dimensional space. However, following Birkhoff's approach, we instead consider the bigger space M'' obtained by cutting M by

$$|Q(z)| = \mathit{constant} \times z^{-n-1} \theta_q(z/x_1, \ldots, z/x_{2n}).$$

The Riemann-Hilbert-Birkhoff correspondence gives an injective mapping

$$(v_2, v_3, \ldots, v_{n-1}) \mapsto [Q(z)],$$

where $[Q(z)]$ denotes the orbit of $Q(z)$ in M'' under multiplication by diagonal matrices by the right and lower triangular matrices by the left, as $z = 0$ is an ordinary point. Now M'' is a $2n+1$ dimensional space and hence the corresponding orbit space is $2n-3$ dimensional, whereas the domain of the injective mapping is $n-2$ dimensional. That is, on the scalar level, the monodromy mapping is generically not surjective, and we do not have a Riemann-Hilbert-Birkhoff correspondence.

4.4.5 The Degree Zero and One Cases

Albeit a bit trivial, the degree zero case of (4.33) is given by

$$y(qz) - (1+q)y(z) + qy(z/q) = 0,$$

and hence $\{\kappa_1, \kappa_2\} = \{1, q\}$. Two linear independent solutions are given by $y(z) = 1$ and $y(z) = z$ and the connection problem is trivial.
The degree one case, is given by

$$y(qz) + \left[-(1+q) + q(x_1^{-1} + x_2^{-1})z\right] y(z) + q(1 - z/x_1)(1 - z/x_2)y(z/q) = 0, \tag{4.49}$$

where the exponents κ_1 and κ_2 can hence be set equal to $\kappa_1 = -q/x_1, \kappa_2 = -q/x_2$. From equations (4.47) and (4.13), we immediately obtain

$$Q(z) = \begin{pmatrix} q_{11}\theta_q(qz/x_1) & q_{12}\theta_q(qz/x_2) \\ q_{21}\theta_q(qz/x_1) & q_{22}\theta_q(qz/x_2) \end{pmatrix},$$

where $q_{ij} \in \mathbb{C}$ for $i, j \in \{1, 2\}$. Furthermore (4.48) gives us

$$q_{11}q_{22} - q_{12}q_{21} = \frac{q}{1-q}(x_2^{-1} - x_1^{-1}).$$

Equation 4.49 is in fact reducible to first order equations, indeed one can easily check that

$$\widetilde{y}^0(z) = \left((qz/x_1;q)_\infty^{-1} \quad (qz/x_2;q)_\infty^{-1}\right),$$

defines a fundamental solution to (4.49). We calculate

$$\psi^\infty(z) = \left((x_1/z;q)_\infty \quad (x_2/z;q)_\infty\right),$$

and

$$\widetilde{y}^0(z) = y^0(z)C, \quad C = \begin{pmatrix} 1 & 1 \\ \frac{q}{1-q}x_1^{-1} & \frac{q}{1-q}x_2^{-1} \end{pmatrix}.$$

We have the following connection result

$$\widetilde{y}^0(z)\widetilde{Q}(z) = \psi^\infty(z), \quad \widetilde{Q}(z) = \begin{pmatrix} \theta_q(qz/x_1) & 0 \\ 0 & \theta_q(qz/x_2) \end{pmatrix}, \tag{4.50}$$

and hence, by (4.45), we find

$$Q(z) = C\begin{pmatrix} \theta_q(qz/x_1) & 0 \\ 0 & \theta_q(qz/x_2) \end{pmatrix} = \begin{pmatrix} \theta_q(qz/x_1) & \theta_q(qz/x_2) \\ \frac{q}{1-q}x_1^{-1}\theta_q(qz/x_1) & \frac{q}{1-q}x_2^{-1}\theta_q(qz/x_2) \end{pmatrix}.$$

Note that in this setting, it is more natural to work with $\widetilde{y}^0(z)$ than $y^0(z)$, indeed the connection formula (4.50) is much neater.

4.5 The Degree Two Model Equation

In Section (4.4.5) we saw that the degree zero and one cases of (4.33) are essentially trivial. In this section we analyse the degree two case, the lowest degree non-trivial one. It plays a crucial role in the analysis of the *direct monodromy problem* of Yamada's Lax pair (2.21), as the spectral part reduces to the degree two model equation when the time variable t approaches zero or infinity. The equation under consideration is

$$\begin{aligned} y(qz) + \left[-(1+q) + q(x_1^{-1} + x_2^{-1} + x_3^{-1} + x_4^{-1})z - (\kappa_1+\kappa_2)z^2\right] y(z) \\ + q(1 - z/x_1)(1 - z/x_2)(1 - z/x_3)(1 - z/x_4)y(z/q) = 0, \end{aligned} \tag{4.51}$$

supplemented with Fuchs' equation

$$\kappa_1\kappa_2 = \frac{q^3}{x_1x_2x_3x_4}. \tag{4.52}$$

We use the following short-hand notation for the parameters involved

$$\sigma = (x_1, x_2, x_3, x_4; \kappa_1, \kappa_2),$$

and only consider the generic parameter case, to be precise,

$$\frac{x_i}{x_j} \notin q^{\mathbb{Z}}, \quad \kappa_1 x_i x_j \notin q^{\mathbb{Z}}, \quad \frac{\kappa_1}{\kappa_2} \notin q^{\mathbb{Z}}. \qquad (i, j \in \{1, 2, 3, 4\}) \tag{4.53}$$

4.5.1 Relation with Associated Continuous Dual q-Hahn Polynomials

After some rescaling, and specialising to $z \in q^{\mathbb{Z}}$, equation (4.51) coincides with the recurrence satisfied by the *associated continuous dual q-Hahn polynomials*. Indeed, following Gupta et al. [23], these polynomials are defined by the three-term recurrence

$$p_{n+1}(\mu) - (\mu - a_n)p_n(\mu) + b_n^2 p_{n-1}(\mu) = 0, \tag{4.54}$$

for $n \in \mathbb{N}$, with $p_0(\mu) \equiv 1$ and $p_{-1}(\mu) \equiv 0$, where

$$\begin{aligned} a_n &= (a^{-1} + b^{-1} + c^{-1} + d^{-1})q^n - (1+q)q^{2n-1}, \\ b_n &= \frac{q}{abcd}(1 - aq^{n-1})(1 - bq^{n-1})(1 - cq^{n-1})(1 - dq^{n-1}). \end{aligned}$$

These polynomials generalise the *continuous dual q-Hahn polynomials*, see Koekoek et al. [56, Section 14.3], for a comparison. To relate equation (4.51) to the recurrence (4.54), we rescale

$$y(z) = S(z)\psi(1/z), \quad S(qz) = q^{-2}z^{-2}(1 - qz/x_1)(1 - qz/x_2)(1 - qz/x_3)(1 - qz/x_4)S(z),$$

which gives

$$\begin{aligned} &\psi(qz) + \left[-q^{-1}(\kappa_1 + \kappa_2) + (x_1^{-1} + x_2^{-1} + x_3^{-1} + x_4^{-1})z - (1+q)z^2/q\right]\psi(z) \\ &\qquad + \frac{q}{x_1x_2x_3x_4}(1 - x_1z/q)(1 - x_2z/q)(1 - x_3z/q)(1 - x_4z/q)\psi(z/q) = 0. \end{aligned} \tag{4.55}$$

Now, specialising

$$\rho_n := \psi(q^n), \quad \mu = q^{-1}(\kappa_1 + \kappa_2), \quad (x_1, x_2, x_3, x_4) = (a, b, c, d), \tag{4.56}$$

we find that ρ_n satisfies (4.54) for all $n \in \mathbb{Z}$.
To be explicit, let $S_n = S(q^{-n})$, or more precisely

$$S_{n+1} = \frac{q^{2n}}{(q^n - \frac{1}{x_1})(q^n - \frac{1}{x_2})(q^n - \frac{1}{x_3})(q^n - \frac{1}{x_4})} S_n, \quad (n \in \mathbb{Z})$$

with $S_0 = 1$, then $\rho_n = S_n y(q^{-n})$ satisfies (4.54), for any solution $y(z)$. One can of course obtain exactly the polynomials $p_n(\mu)$, by choosing appropriate initial conditions $y(1) = 1$ and $y(q^{-1}) = 0$, uniform in μ.

Gupta et al. [23] studied the recurrence (4.54) in the large positive n limit, which corresponds to the limit $z \to \infty$ in $q^{\mathbb{Z}}$ for (4.51). Their approach can easily be adopted to the more general case (4.51) with z not confined to $q^{\mathbb{Z}}$. They observed that the large n asymptotics can be expressed in terms of ${}_3\phi_2$ *basic hypergeometric functions.*

4.5.2 Intermezzo on Basic Hypergeometric Functions

Generalising Heine's ${}_2\phi_1$ basic hypergeometric series (1.3), for $r, s \in \mathbb{N}$, the ${}_r\phi_s$ *basic hypergeometric series* are defined by

$$ {}_r\phi_s \left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, \ldots b_s \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} [(-1)^n q^{\binom{n}{2}}]^{1+s-r} z^n, \tag{4.57} $$

where $a_i, b_j \in \mathbb{C}$ for $1 \le i \le r$ and $1 \le j \le s$. Gasper and Rahman [18] wrote an extensive overview on these series. We are mostly concerned with the ${}_3\phi_2$ basic hypergeometric series

$$ {}_3\phi_2 \left[\begin{matrix} a, \ b, \ c \\ d, \ e \end{matrix}; q, z\right] = \sum_{n=0}^{\infty} \frac{(a, b, c; q)_n}{(q, d, e; q)_n} z^n, \tag{4.58} $$

which converges for $|z| < 1$ and enjoys meromorphic continuation to the entire complex plane. Gasper and Rahman [18] give the following two-term tranformations

$$ {}_3\phi_2 \left[\begin{matrix} a, \ b, \ c \\ d, \ e \end{matrix}; q, \frac{de}{abc}\right] = \frac{(e/a, de/(bc); q)_\infty}{(e, de/(abc); q)_\infty} \, {}_3\phi_2 \left[\begin{matrix} a, \ d/b, \ d/c \\ d, \ de/(bc) \end{matrix}; q, \frac{e}{a}\right], \tag{4.59a} $$

$$ = \frac{(b, de/(ab), de/(bc); q)_\infty}{(d, e, de/(abc); q)_\infty} \, {}_3\phi_2 \left[\begin{matrix} d/b, \ e/b, \ de/(abc) \\ de/(ab), \ de/(bc) \end{matrix}; q, b\right], \tag{4.59b} $$

and the three-term transformation

$$ \begin{aligned} {}_3\phi_2 \left[\begin{matrix} a, \ b, \ c \\ d, \ e \end{matrix}; q, \frac{de}{abc}\right] &= \frac{(d/b, d/c, cq/a, q/e; q)_\infty}{(d, cq/e, q/a, d/(bc); q)_\infty} \, {}_3\phi_2 \left[\begin{matrix} c, \ e/a, \ cq/d \\ cq/a, \ bcq/d \end{matrix}; q, \frac{bq}{e}\right] \\ &- \frac{\big(q/e, dq/e, b, c, e/a, de/(bcq), bcq^2/(de); q\big)_\infty}{(e/q, d, bq/e, cq/e, q/a, d/(bc), bcq/d; q)_\infty} \, {}_3\phi_2 \left[\begin{matrix} aq/e, \ bq/e, \ cq/e \\ q^2/e, \ dq/e \end{matrix}; q, \frac{de}{abc}\right]. \end{aligned} \tag{4.60} $$

4.5.3 Outline of Global Asymptotic Analysis

We wish to explicitly determine the associated connection matrix $Q(z)$ (4.44), for the degree two case (4.51). Firstly, we derive explicit formula for the solutions $\psi_i^\infty(z)$ of (4.42) at $z = \infty$ for $i \in \{1, 2\}$, by adopting the approach of Gupta et al. [23]. Writing

$$ \phi(a, b, c, d, e) = {}_3\phi_2 \left[\begin{matrix} a, \ b, \ c \\ d, \ e \end{matrix}; q, \frac{de}{abc}\right] \tag{4.61} $$

Gupta et al. [23] derive the following contiguous relation

$$\begin{aligned}&\frac{de}{bcq}\frac{(1-b)(1-c)(1-d/a)(1-e/a)}{(1-d)(1-e)}\phi(a,qb,qc,qd,qe)\\ &\quad-\left((1-a)\left(1-\frac{de}{abcq}\right)+a\left(1-\frac{d}{aq}\right)\left(1-\frac{e}{aq}\right)+\frac{de}{abcq}(1-b)(1-c)\right)\phi(a,b,c,d,e)\\ &\qquad+(1-d/q)(1-e/q)\phi(a,b/q,c/q,d/q,e/q)=0, \quad (4.62)\end{aligned}$$

and, by rescaling and specialisation of the parameters involved, they use it to construct solutions of the recurrence (4.54) about $n=\infty$. Similarly we use it to construct explicit solutions of (4.42), analytic at $z=\infty$.

We then use the transformations (4.59) and (4.60) to simultaneously find explicit solutions of (4.51), analytic at $z=0$, and corresponding connection matrix. We remark that these results seem new.

4.5.4 Explicit Solutions near Infinity

Recall that the solutions $\psi_i^\infty(z)$ satisfy (4.42), which we rewrite as

$$\begin{aligned}&z^4\psi_i(qz)+z^2/\kappa_i\left[-(1+q)+q(x_1^{-1}+x_2^{-1}+x_3^{-1}+x_4^{-1})z-(\kappa_1+\kappa_2)z^2\right]\psi_i(z)\\ &\quad+q^3/\kappa_i^2(1-z/x_1)(1-z/x_2)(1-z/x_3)(1-z/x_4)\psi_i(z/q)=0, \quad (4.63)\end{aligned}$$

and are characterised by $\psi_i^\infty(\infty)=1$, for $i\in\{1,2\}$. We set

$$\phi(z)=\phi(a,Bz,Cz,aDz,aAz),$$

then, by (4.62), we have

$$\begin{aligned}&\frac{a^2AD}{qBC}\frac{(1-Az)(1-Bz)(1-Cz)(1-Dz)}{(1-aDz)(1-aAz)}\phi(qz)\\ &\quad-\left(1+\frac{a^2AD}{qBC}-\frac{aAD}{q}\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}+\frac{1}{D}\right)z+\frac{aAD}{q}\tfrac{1+q}{q}z^2\right)\phi(z)\\ &\qquad+(1-aDz/q)(1-aAz/q)\phi(z/q)=0.\end{aligned}$$

We subsequently rescale

$$\phi(z)=(aDz,aAz;q)_\infty^{-1}\psi(1/z),$$

then $\psi(z)$ satisfies

$$\begin{aligned}&z^4\psi(qz)+z^2\left[-\frac{aAD}{q^2}(1+q)+\frac{aAD}{q}(A^{-1}+B^{-1}+C^{-1}+D^{-1})z-\left(1+\frac{a^2AD}{qBC}\right)z^2\right]\psi(z)\\ &\quad+\frac{a^2A^2D^2}{q}(1-z/A)(1-z/B)(1-z/C)(1-z/D)\psi(z/q)=0. \quad (4.64)\end{aligned}$$

Let $i \in \{1,2\}$, then this equation coincides with (4.63) exactly for the choice

$$a = \frac{q^2}{x_1 x_4 \kappa_i}, \quad (A,B,C,D) = (x_1,x_2,x_3,x_4),$$

and we obtain the following solution to (4.63),

$$\phi_i(z) = \left(\frac{q^2}{x_1\kappa_i}z^{-1}, \frac{q^2}{x_4\kappa_i}z^{-1}; q\right)_\infty {}_3\phi_2\left[\begin{matrix}\frac{q^2}{x_1x_4\kappa_i}, \ x_2z^{-1}, \ x_3z^{-1} \\ \frac{q^2}{x_1\kappa_i}z^{-1}, \ \frac{q^2}{x_4\kappa_i}z^{-1}\end{matrix}; q, \frac{q^2}{x_2x_3\kappa_i}\right].$$

This form of the solution does not allow easy evaluation of $\phi_i(\infty)$, we therefore apply transformation (4.59a), and discard of constant factors, to finally obtain

$$\psi_1^\infty(z;\sigma) = \left(\frac{q^2}{x_1\kappa_1}z^{-1}, x_1z^{-1}; q\right)_\infty {}_3\phi_2\left[\begin{matrix}\frac{x_2x_3}{q}\kappa_2, \ \frac{x_3x_4}{q}\kappa_2, \ \frac{x_2x_4}{q}\kappa_2 \\ \frac{q^2}{x_1\kappa_1}z^{-1}, \ q\frac{\kappa_2}{\kappa_1}\end{matrix}; q, x_1z^{-1}\right], \tag{4.65a}$$

$$\psi_2^\infty(z;\sigma) = \left(\frac{q^2}{x_1\kappa_2}z^{-1}, x_1z^{-1}; q\right)_\infty {}_3\phi_2\left[\begin{matrix}\frac{x_2x_3}{q}\kappa_1, \ \frac{x_3x_4}{q}\kappa_1, \ \frac{x_2x_4}{q}\kappa_1 \\ \frac{q^2}{x_1\kappa_2}z^{-1}, \ q\frac{\kappa_1}{\kappa_2}\end{matrix}; q, x_1z^{-1}\right], \tag{4.65b}$$

noting that both solutions are analytic at $z=\infty$ with $\psi_1^\infty(\infty;\sigma)=1$ and $\psi_2^\infty(\infty;\sigma)=1$, and hence coinciding with the solutions defined in Lemma 4.4.2. We denote the corresponding fundamental solution by

$$\psi^\infty(z;\sigma) = \begin{pmatrix}\psi_1^\infty(z;\sigma) & \psi_2^\infty(z;\sigma)\end{pmatrix}.$$

Remark 4.5.1. The model equation (4.51) is invariant under permutations of $\{x_1,x_2,x_3,x_4\}$ and switching of κ_1 and κ_2. From the asymptotic characterisation of the solutions (4.65), we immediately obtain that both solutions are invariant under permutations of $\{x_1,x_2,x_3,x_4\}$ and

$$\psi_1^\infty(z;\sigma|_{\kappa_1\leftrightarrow\kappa_2}) = \psi_2^\infty(z;\sigma), \quad \psi_2^\infty(z;\sigma|_{\kappa_1\leftrightarrow\kappa_2}) = \psi_1^\infty(z;\sigma).$$

One can also check this algebraically, using transformations (4.59).

4.5.5 Explicit Solutions at Origin

Using the three-term transformation (4.60), we are able to find solutions which are analytic around the origin. Note that, without relying on meromorphic continuation, a necessary requirement, for this transformation to be sensible, is $|\frac{bq}{e}|<1$. In the following theorem, we apply the three-term transformation (4.60), to the solution (4.65b). For validity of this step we have to assume (4.66). However, in Section 4.5.7, we show that we can always apply a permutation of the parameters such that this condition is satisfied.

Theorem 4.5.2. *Considering the degree two model equation (4.51), if the parameters satisfy*

$$|x_3x_4\kappa_2| < |q|, \tag{4.66}$$

then we have a fundamental solution $\widetilde{y}^0(z;\sigma)$, analytic at $z=0$, given explicitly by

$$\widetilde{y}_1^0(z;\sigma) = \frac{(x_4\kappa_1 z, q)_\infty}{(qx_1^{-1}z, qx_2^{-1}z, qx_3^{-1}z; q)_\infty}\, {}_3\phi_2\left[\begin{matrix}\frac{x_2x_4}{q}\kappa_1,\ \frac{x_1x_4}{q}\kappa_1,\ qx_3^{-1}z \\ q\frac{x_4}{x_3},\ x_4\kappa_1 z\end{matrix};q,\frac{x_3x_4}{q}\kappa_2\right], \tag{4.67a}$$

$$\widetilde{y}_2^0(z;\sigma) = \frac{(x_3\kappa_1 z, q)_\infty}{(qx_1^{-1}z, qx_2^{-1}z, qx_4^{-1}z; q)_\infty}\, {}_3\phi_2\left[\begin{matrix}\frac{x_2x_3}{q}\kappa_1,\ \frac{x_1x_3}{q}\kappa_1,\ qx_4^{-1}z \\ q\frac{x_3}{x_4},\ x_3\kappa_1 z\end{matrix};q,\frac{x_3x_4}{q}\kappa_2\right]. \tag{4.67b}$$

These solutions are related to the solutions at infinity (4.65), by the connection formula

$$\widetilde{y}^0(z;\sigma) = \psi^\infty(z;\sigma)\widetilde{R}(z;\sigma), \tag{4.68}$$

with the connection matrix $\widetilde{R}(z;\sigma)$ given by

$$\widetilde{R}(z;\sigma) = \frac{1}{\theta_q(qz/x_1, qz/x_2)}\begin{pmatrix} r_{11}\theta_q(x_4\kappa_2 z) & r_{12}\theta_q(x_3\kappa_2 z) \\ r_{21}\theta_q(x_4\kappa_1 z) & r_{22}\theta_q(x_3\kappa_1 z)\end{pmatrix}\begin{pmatrix}\frac{1}{\theta_q(qz/x_3)} & 0 \\ 0 & \frac{1}{\theta_q(qz/x_4)}\end{pmatrix}, \tag{4.69}$$

where

$$r_{11} = \frac{\left(\frac{x_3x_4}{q}\kappa_1, \frac{x_2x_4}{q}\kappa_1, \frac{x_1x_4}{q}\kappa_1; q\right)_\infty}{\left(\frac{x_3x_4}{q}\kappa_2, q\frac{x_4}{x_3}, \frac{\kappa_1}{\kappa_2}; q\right)_\infty}, \qquad r_{12} = \frac{\left(\frac{x_3x_4}{q}\kappa_1, \frac{x_2x_3}{q}\kappa_1, \frac{x_1x_3}{q}\kappa_1; q\right)_\infty}{\left(\frac{x_3x_4}{q}\kappa_2, q\frac{x_3}{x_4}, \frac{\kappa_1}{\kappa_2}; q\right)_\infty}, \tag{4.70a}$$

$$r_{21} = \frac{\left(\frac{x_2x_4}{q}\kappa_2, \frac{x_1x_4}{q}\kappa_2; q\right)_\infty}{\left(q\frac{x_4}{x_3}, \frac{\kappa_2}{\kappa_1}; q\right)_\infty}, \qquad r_{22} = \frac{\left(\frac{x_2x_3}{q}\kappa_2, \frac{x_1x_3}{q}\kappa_2; q\right)_\infty}{\left(q\frac{x_3}{x_4}, \frac{\kappa_2}{\kappa_1}; q\right)_\infty}. \tag{4.70b}$$

Proof. Firstly, let us specialise (4.68) to the first column, reading

$$\widetilde{y}_1^0(z;\sigma) = \widetilde{R}_{11}(z)\psi_1^\infty(z;\sigma) + \widetilde{R}_{21}(z)\psi_2^\infty(z;\sigma), \tag{4.71}$$

where the entries $\widetilde{R}_{11}(z)$ and $\widetilde{R}_{21}(z)$ equal

$$\widetilde{R}_{11}(z) = r_{11}\frac{\theta_q(x_4\kappa_2 z)}{\theta_q\left(qz/x_1, qz/x_2, qz/x_3\right)}, \quad \widetilde{R}_{21}(z) = r_{21}\frac{\theta_q(x_4\kappa_1 z)}{\theta_q\left(qz/x_1, qz/x_2, qz/x_3\right)}.$$

Now, we take the expression (4.65b) for $\psi_2^\infty(z;\sigma)$, and apply the three-term transformation (4.60), then, by carefully working out the different factors, we obtain

$$\psi_2^\infty(z;\sigma) = \widetilde{R}_{21}(z)^{-1}\widetilde{y}_1^0(z;\sigma) - \widetilde{R}_{21}(z)^{-1}\widetilde{R}_{11}(z)\psi_1^\infty(z;\sigma),$$

that is, equation (4.71) is indeed correct. Furthermore, note that

$$\widetilde{R}_{11}(qz) = \kappa_1 z^2\widetilde{R}_{11}(z), \quad \widetilde{R}_{21}(qz) = \kappa_2 z^2\widetilde{R}_{21}(z),$$

from which we conclude, using (4.41), that $\widetilde{y}_1^0(z;\sigma)$ satisfies our model equation (4.51). Or to put it differently, the right-hand side of (4.71) defines a solution of the model equation (4.51), regardless of the values of r_{11} and r_{21}. By switching $x_3 \leftrightarrow x_4$ in (4.71), using Remark

(4.5.1), we immediately obtain the second column of (4.68) and the theorem follows. □

Note that the solutions (4.67) are related to $y^0(z)$ by

$$\widetilde{y}^0(z;\sigma) = y^0(z)C(\sigma), \quad C(\sigma) = \begin{pmatrix} \widetilde{y}_1^0(0;\sigma) & \widetilde{y}_2^0(0;\sigma) \\ \frac{d\widetilde{y}_1^0}{dz}(0;\sigma) & \frac{d\widetilde{y}_2^0}{dz}(0;\sigma) \end{pmatrix}.$$

One can of course write $C(\sigma)$ explicitly in terms of ${}_2\phi_1$ hypergeometric functions. By comparison with (4.45), we obtain

$$Q(z) = Q(z;\sigma) = C(\sigma)\widetilde{Q}(z;\sigma), \quad \widetilde{Q}(z;\sigma) = \widetilde{R}(z;\sigma)^{-1}. \tag{4.72}$$

4.5.6 An Explicit Connection Matrix

Considering equations (4.72), we wish to explicitly calculate the inverse of $\widetilde{R}(z;\sigma)$. To do this we first calculate its determinant explicitly. Of course we already have a formula (4.48) for the determinant of $Q(z;\sigma)$, however, calculating $|C(\sigma)|$ takes some effort. In stead let us use some of the q-elliptic theory developed in Section 4.1.1. Firstly, considering (4.69), we evaluate

$$\left|\begin{pmatrix} r_{11}\theta_q(x_4\kappa_2 z) & r_{12}\theta_q(x_3\kappa_2 z) \\ r_{21}\theta_q(x_4\kappa_1 z) & r_{22}\theta_q(x_3\kappa_1 z) \end{pmatrix}\right| = \frac{(\frac{x_3x_4}{q}\kappa_1;q)_\infty}{(q\frac{x_4}{x_3}, q\frac{x_3}{x_4}, \frac{\kappa_1}{\kappa_2}, \frac{\kappa_2}{\kappa_1}, \frac{x_3x_4}{q}\kappa_2;q)_\infty} r(z),$$

with

$$\begin{aligned} r(z) &= \theta_q\left(\frac{x_2x_4}{q}\kappa_1, \frac{x_1x_4}{q}\kappa_1\right)\theta_q(x_3\kappa_1 z, x_4\kappa_2 z) - \theta_q\left(\frac{x_2x_3}{q}\kappa_1, \frac{x_1x_3}{q}\kappa_1\right)\theta_q(x_4\kappa_1 z, x_3\kappa_2 z) \\ &= \frac{x_2x_3}{q}\kappa_1\theta_q\left(\frac{x_4}{x_3}, \frac{\kappa_2}{\kappa_1}\right)\theta_q\left(q^2 z/x_1, qz/x_2\right), \end{aligned}$$

where the second equality follows from the addition formula (4.15) with

$$\alpha = \frac{1}{x_3x_4\kappa_1\kappa_2}, \quad a = \frac{1}{x_3\kappa_1}, \quad b = \frac{1}{x_4\kappa_1}, \quad c = x_2/q.$$

Therefore a little calculation gives us an explicit formula for the determinant

$$\left|\widetilde{R}(z;\sigma)\right| = qz^{-1}\frac{x_3^{-1} - x_4^{-1}}{\kappa_2 - \kappa_1}\frac{(\frac{x_3x_4}{q}\kappa_1;q)_\infty}{(\frac{x_3x_4}{q}\kappa_2;q)_\infty}\theta_q(qz/x_1, qz/x_2, qz/x_3, qz/x_4)^{-1}. \tag{4.73}$$

We can now calculate the inverse of $\widetilde{R}(z;\sigma)$, giving

$$\widetilde{Q}(z;\sigma) = q^{-1}\frac{\kappa_2 - \kappa_1}{x_4^{-1} - x_3^{-1}}\begin{pmatrix} x_3\theta_q(z/x_3) & 0 \\ 0 & x_4\theta_q(z/x_4) \end{pmatrix}\begin{pmatrix} q_{11}\theta_q(x_3\kappa_1 z) & q_{12}\theta_q(x_3\kappa_2 z) \\ q_{21}\theta_q(x_4\kappa_1 z) & q_{22}\theta_q(x_4\kappa_2 z) \end{pmatrix},$$

where

$$q_{11} = \frac{\left(\frac{x_3x_4}{q}\kappa_2, \frac{x_2x_3}{q}\kappa_2, \frac{x_1x_3}{q}\kappa_2; q\right)_\infty}{\left(\frac{x_3x_4}{q}\kappa_1, q\frac{x_3}{x_4}, \frac{\kappa_2}{\kappa_1}; q\right)_\infty}, \qquad q_{12} = -\frac{\left(\frac{x_2x_3}{q}\kappa_1, \frac{x_1x_3}{q}\kappa_1; q\right)_\infty}{\left(q\frac{x_3}{x_4}, \frac{\kappa_1}{\kappa_2}; q\right)_\infty}, \tag{4.74a}$$

$$q_{21} = -\frac{\left(\frac{x_3x_4}{q}\kappa_2, \frac{x_2x_4}{q}\kappa_2, \frac{x_1x_4}{q}\kappa_2; q\right)_\infty}{\left(\frac{x_3x_4}{q}\kappa_1, q\frac{x_4}{x_3}, \frac{\kappa_2}{\kappa_1}; q\right)_\infty}, \qquad q_{22} = \frac{\left(\frac{x_2x_4}{q}\kappa_1, \frac{x_1x_4}{q}\kappa_1; q\right)_\infty}{\left(q\frac{x_4}{x_3}, \frac{\kappa_1}{\kappa_2}; q\right)_\infty}. \tag{4.74b}$$

Of course $\widetilde{Q}(z;\sigma)$ is characterised by

$$\widetilde{y}^0(z;\sigma)\widetilde{Q}(z;\sigma) = \psi^\infty(z;\sigma). \tag{4.75}$$

Comparison of (4.73) and (4.48) gives

$$|C(\sigma)| = \frac{q-1}{q}\frac{1}{x_3^{-1}-x_4^{-1}}\frac{(\frac{x_3x_4}{q}\kappa_2;q)_\infty}{(\frac{x_3x_4}{q}\kappa_1;q)_\infty}.$$

Remark 4.5.3. We emphasise that the explicit formulas in this section, break down when one of the assumptions on the parameters in (4.53) is broken. As an example, suppose $\kappa_1x_1x_2 = q^2$, then (4.65a) becomes

$$\psi_1^\infty(z;\sigma) = \left(x_1z^{-1}, x_2z^{-1}; q\right)_\infty,$$

and

$$\widetilde{y}_1^0(z) = (qz/x_1, qz/x_2; q)_\infty^{-1},$$

defines a solution of (4.51), analytic at $z=0$. Taking any other solution $\widetilde{y}_2^0(z)$, analytic at $z=0$, linearly independent of $\widetilde{y}_1^0(z)$, the connection matrix, relating $\{\psi_1^\infty(z;\sigma), \psi_2^\infty(z;\sigma)\}$ and $\{\widetilde{y}_1^0(z), \widetilde{y}_2^0(z)\}$, is triangular. Note that the explicit connection results in this section break down, as for instance r_{11} and r_{12}, defined in (4.70), are singular. We do not wish to discuss the various degenerations of the model equation (4.51) here.

4.5.7 Symmetries

The calculations in the previous two sections are only valid if condition (4.66) holds. Now suppose that instead $|x_3x_4\kappa_2| \geq |q|$, then, by Fuchs' equation (4.52), we have

$$|x_1x_2\kappa_1| = \frac{|q|^3}{|x_3x_4\kappa_2|} \leq |q|^2 < |q|.$$

Hence we can simply apply a permutation $\kappa_1 \leftrightarrow \kappa_2$, $x_1 \leftrightarrow x_3$ and $x_2 \leftrightarrow x_4$, to all the results, interchanging the rows and columns of $\widetilde{R}(z)$ and $\widetilde{Q}(z)$ respectively in accordance with Remark 4.5.1.

4.6 Yamada's Lax Pair

In Section 2.4.3, we introduced Yamada's Lax pair (2.21). Recall that the compatibility condition (2.23) of the Lax pair, is equivalent to the q-$P(A_1)$ equation, as formulated in Theorem 2.4.1. In Section 4.7, we momentarily forget about the deformation equation and set up the global analysis of the spectral equation, similar to that of the model equation 4.4. In Section 4.8, we study how q-$P(A_1)$, or equivalently the deformation equation, deforms the analytic data of the spectral equation. Finally, in Sections 4.9 and 4.10, we make the heuristic discussion on the isomonodromic deformation method in Section 2.4.4, rigorous.

4.7 Analytic Theory of Spectral Equation

In this section we consider t, f, g as mere constants entering the spectral equation L_1, and study the analytic structure of the spectral equation. We often suppress t, f or g dependence in this section. Let us first remark that L_1 is in polynomial form, i.e. $u(z), v(z), w(z)$ are all polynomials in z of degree five. This is of course trivial for $u(z)$ and $w(z)$, but less so for $v(z)$. We invite the interested reader to check it themself. Let us write

$$\begin{aligned}
u(z,t) &= u_0(t) + u_1(t)z + \ldots + u_5(t)z^5,\\
v(z,t) &= v_0(t) + v_1(t)z + \ldots + v_5(t)z^5,\\
w(z,t) &= w_0(t) + w_1(t)z + \ldots + w_5(t)z^5.
\end{aligned}$$

The Lax pair is singular on the complement of

$$R_p = \{(t,f,g) \in \mathbb{C}^* \times \mathbb{C}^2 : f, g, fg-1, fg-t^2 \neq 0\},$$

and we say that $(t,f,g) \in \mathbb{C}^* \times \mathbb{C}^2$ is in *regular position*, if it is an element of this set.

4.7.1 Fundamental Solution at Origin

It is easy to calculate

$$u_0 = 1, \quad v_0 = -(1+q), \quad w_0 = q,$$

hence the exponents at $z=0$ are $\{1, q\}$, which means $z=0$ is a regular singular point with resonance. Furthermore, a less easy calculation shows

$$u_1 + v_1 + w_1 = 0,$$

so $z=0$ is in fact an ordinary point of L_1. We define two linearly independent solutions at $z=0$ by

$$y_1^0(z;t,f,g) = 1 + 0z + \mathcal{O}\left(z^2\right), \tag{4.76a}$$
$$y_2^0(z;t,f,g) = 0 + 1z + \mathcal{O}\left(z^2\right), \tag{4.76b}$$

and write the corresponding fundamental solution by

$$y^0(z;t,f,g) := \begin{pmatrix} y_1^0(z;t,f,g) & y_2^0(z;t,f,g) \end{pmatrix} = \begin{pmatrix} 1 & z \end{pmatrix} + \mathcal{O}\left(z^2\right). \qquad (z \to 0) \tag{4.77}$$

The technical characterisation is given in the following lemma.

Lemma 4.7.1. *For fixed $(t,f,g) \in R_p$, there exists an unique fundamental formal power series solution $y^0(z;t,f,g)$ of L_1 about $z=0$, characterised asymptotically by (4.77). For any $(t^*,f^*,g^*) \in R_p$, this power series solution converges, locally uniformly in $(z,t,f,g) \in \mathbb{C}\times R_p$, at $(z,t,f,g) = (0,t^*,f^*,g^*)$. The local solution $y^0(z;t,f,g)$ has an unique meromorphic continuation to $\mathbb{C}$ in z, remaining analytic in (t,f,g) on R_p.*

Proof. This can be proven by elementary means, or by for instance using Theorem B.3 and Remark B.5. □

It is helpful to rescale

$$y(z,t) = S^0(z,t)\psi(z,t), \quad S^0(qz,t) = \frac{1}{u(z,t)} S^0(z,t),$$

where we specify

$$S^0(z,t) := (qb_5 z, qb_6 z, qb_7 z, qb_8 z, z/f; q)_\infty,$$

and $\psi(z,t)$ satisfies

$$\psi(qz,t) + v(z,t)\psi(z,t) + u(z/q,t)w(z,t)\psi(z/q,t) = 0.$$

Note that this equation is in standard form (4.33), and we specify an unique fundamental solution by

$$\psi^0(z,t) = \begin{pmatrix} \psi_1^0(z,t) & \psi_2^0(z,t) \end{pmatrix} = \begin{pmatrix} 1 & z \end{pmatrix} + \mathcal{O}\left(z^2\right). \qquad (z \to 0) \tag{4.78}$$

We can easily calculate

$$S^0(0,t) = 1, \qquad \frac{dS^0}{dz}(0,t) = \frac{1}{q-1}(q(b_5+b_6+b_7+b_8)+f^{-1}),$$

and we immediately obtain

$$y^0(z,t) = S^0(z,t)\psi^0(z,t)\begin{pmatrix} 1 & 0 \\ -\frac{1}{q-1}(q(b_5+b_6+b_7+b_8)+f^{-1}) & 1 \end{pmatrix}.$$

Correspondingly we write

$$Y^0(z,t) = \begin{pmatrix} y_1^0(z,t) & y_2^0(z,t) \\ y_1^0(z/q,t) & y_2^0(z/q,t) \end{pmatrix}, \quad \Psi^0(z,t) = \begin{pmatrix} \psi_1^0(z,t) & \psi_2^0(z,t) \\ \psi_1^0(z/q,t) & \psi_2^0(z/q,t) \end{pmatrix},$$

which are related by

$$Y^0(z,t) = \begin{pmatrix} S^0(z,t) & 0 \\ 0 & S^0(z/q,t) \end{pmatrix} \Psi^0(z,t) \cdot \begin{pmatrix} 1 & 0 \\ -\frac{1}{q-1}(q(b_5+b_6+b_7+b_8)+f^{-1}) & 1 \end{pmatrix}. \tag{4.79}$$

By equation (4.39), we have

$$|\Psi^0(z,t)| = (q^{-1}-1)z(z/(qf), qz/f, b_1z/t, b_2z/t, b_3z/t, b_4z/t, b_5z, b_6z, b_7z, b_8z; q)_\infty^{-1},$$

and hence, by (4.79),

$$|Y^0(z,t)| = (q^{-1}-1)z\frac{(z/f, qb_5z, qb_6z, qb_7z, qb_8z; q)_\infty}{(qz/f, b_1z/t, b_2z/t, b_3z/t, b_4z/t; q)_\infty} \tag{4.80a}$$

$$= (q^{-1}-1)z(1-z/f)\frac{(qb_5z, qb_6z, qb_7z, qb_8z; q)_\infty}{(b_1z/t, b_2z/t, b_3z/t, b_4z/t; q)_\infty}. \tag{4.80b}$$

Analogously to Lemma 4.4.1, we have the following result.

Lemma 4.7.2. *The matrix function $\Psi^0(z,t)^{-1} = \Psi^0(z;t,f,g)^{-1}$ is analytic on $\mathbb{C}^* \times R_p$.*

4.7.2 Fundamental Solution at Infinity

We easily calculate

$$u_5 = -\frac{b_1b_2b_3b_4}{f}q^3, \quad v_5 = \frac{b_1b_2b_3b_4}{f}q^3(1+q^{-1})t^{-2}, \quad w_5 = -\frac{b_1b_2b_3b_4}{f}q^3q^{-1}t^{-4},$$

hence the exponents $\{\kappa_1,\kappa_2\}$ at $z=\infty$ are given by $\kappa_1 = t^{-2}$ and $\kappa_2 = q^{-1}t^{-2}$, which means $z=\infty$ is a regular singular point with resonance. Furthermore, a less easy calculation shows

$$t^{-2}u_1 + v_1 + t^2w_1 = 0,$$

so $z=\infty$ is in fact an apparent singularity of L_1. We rescale Yamada's Lax pair a bit, such that $z=\infty$ becomes an ordinary point. We set

$$y(z,t) = S(z,t)\widetilde{y}(z,t), \quad S(qz,t) = t^{-2}S(z,t) \quad S(z,qt) = z^{-2}S(z,t), \tag{4.81}$$

which leads to

$$\widetilde{L}_1: \quad \widetilde{u}(z,t)\widetilde{y}(qz,t) + \widetilde{v}(z,t)\widetilde{y}(z,t) + \widetilde{w}(z,t)\widetilde{y}(z/q,t) = 0, \tag{4.82a}$$

$$\widetilde{L}_2: \quad z^{-2}h_0(z,t)\widetilde{y}(z,qt) + h_1(z,t)\widetilde{y}(z,t) + t^2h_2(z,t)\widetilde{y}(z/q,t) = 0, \tag{4.82b}$$

where $\widetilde{u},\widetilde{v},\widetilde{w}$ are polynomials in z^{-1}, normalised such that $\widetilde{u}(\infty,t)=1$, given by

$$\begin{aligned}
\widetilde{u}(z,t)&=-\frac{f}{b_1b_2b_3b_4}q^{-3}z^{-5}u(z,t),\\
\widetilde{v}(z,t)&=-\frac{f}{b_1b_2b_3b_4}q^{-3}z^{-5}t^2v(z,t),\\
\widetilde{w}(z,t)&=-\frac{f}{b_1b_2b_3b_4}q^{-3}z^{-5}t^4w(z,t).
\end{aligned}$$

We only use the scaling $S(z,t)$ formally, one could for instance take

$$S(z,t)=\frac{\theta_q(zt)^2}{\theta_q(z)^2\theta_q(t)^2},$$

but really the actual candidate is irrelevant. Explicitly, $\widetilde{u}(z,t)$ is for instance given by

$$\widetilde{u}(z,t)=(1-1/(b_5qz))(1-1/(b_6qz))(1-1/(b_7qz))(1-1/(b_8qz))(1-f/z).$$

The corresponding system form is obtained by setting

$$\widetilde{Y}(z,t)=\begin{pmatrix}\widetilde{y}(z,t)\\ \widetilde{y}(z/q,t)\end{pmatrix},$$

which gives

$$\widetilde{Y}(qz,t)=\widetilde{A}(z,t)\widetilde{Y}(z,t), \tag{4.83a}$$
$$\widetilde{Y}(z,qt)=\widetilde{H}(z,t)\widetilde{Y}(z,t), \tag{4.83b}$$

where

$$\begin{aligned}
\widetilde{A}(z,t)&=\begin{pmatrix}-t^2\frac{v(z,t)}{u(z,t)} & -t^4\frac{w(z,t)}{u(z,t)}\\ 1 & 0\end{pmatrix}=\begin{pmatrix}t^2 & 0\\ 0 & 1\end{pmatrix}A(z,t)\begin{pmatrix}1 & 0\\ 0 & t^2\end{pmatrix},\\
\widetilde{H}(z,t)&=z^2\begin{pmatrix}1 & 0\\ 0 & q^{-2}t^{-2}\end{pmatrix}H(z,t)\begin{pmatrix}1 & 0\\ 0 & t^2\end{pmatrix}.
\end{aligned}$$

Returning to the scalar equation $\widetilde{L}_1$, we have

$$\widetilde{u}(\infty,t)=1,\quad \widetilde{v}(\infty,t)=-(1+q^{-1}),\quad \widetilde{w}(\infty,t)=q^{-1}.$$

and of course the non-resonance condition is satisfied and $z=\infty$ is an ordinary point of $\widetilde{L}_1$. We define two linearly independent solutions at $z=\infty$ by

$$y_1^\infty(z;t,f,g)=1+0z^{-1}+\mathcal{O}\left(z^{-2}\right), \tag{4.84a}$$
$$y_2^\infty(z;t,f,g)=0+1z^{-1}+\mathcal{O}\left(z^{-2}\right), \tag{4.84b}$$

and write the corresponding fundamental solution by

$$y^{\infty}(z;t,f,g) := \begin{pmatrix} y_1^{\infty}(z;t,f,g) & y_2^{\infty}(z;t,f,g) \end{pmatrix} = \begin{pmatrix} 1 & z^{-1} \end{pmatrix} + \mathcal{O}\left(z^{-2}\right). \qquad (z \to \infty) \tag{4.85}$$

The technical characterisation is given in the following lemma.

Lemma 4.7.3. *For fixed $(t,f,g) \in R_p$, there exists an unique fundamental formal power series solution $y^{\infty}(z;t,f,g)$ of $\widetilde{L}_1$ about $z=\infty$, characterised asymptotically by (4.85). For any $(t^*,f^*,g^*) \in R_p$, this power series solution converges, locally uniformly in $(z,t,f,g) \in \mathbb{P}^* \times R_p$, at $(z,t,f,g) = (\infty, t^*, f^*, g^*)$. The local solution $y^{\infty}(z;t,f,g)$ has an unique meromorphic continuation to $\mathbb{P}^*$ in z, remaining analytic in (t,f,g) on R_p.*

Proof. This can be proven by elementary means, or by for instance using Theorem B.3 and Remark B.5. $\square$

It is helpful to rescale

$$\widetilde{y}(z,t) = S^{\infty}(z,t)\widetilde{\psi}(z,t), \quad S^{\infty}(qz,t) = \frac{1}{\widetilde{u}(z,t)}S^{\infty}(z,t),$$

where we specify

$$S^{\infty}(z,t) := (1/(b_5 z), 1/(b_6 z), 1/(b_7 z), 1/(b_8 z), qf/z; q)_{\infty}^{-1},$$

and $\widetilde{\psi}(z,t)$ satisfies

$$\widetilde{\psi}(qz,t) + \widetilde{v}(z,t)\widetilde{\psi}(z,t) + \widetilde{u}(z/q,t)\widetilde{w}(z,t)\widetilde{\psi}(z/q,t) = 0.$$

Note that this equation is in the form of (4.42), and we specify an unique fundamental solution by

$$\psi^{\infty}(z,t) = \begin{pmatrix} \psi_1^{\infty}(z,t) & \psi_2^{\infty}(z,t) \end{pmatrix} = \begin{pmatrix} 1 & z^{-1} \end{pmatrix} + \mathcal{O}\left(z^{-2}\right). \qquad (z \to \infty) \tag{4.86}$$

We can easily calculate

$$S^{\infty}(z,t) = 1 - \frac{1}{q-1}(b_5^{-1} + b_6^{-1} + b_7^{-1} + b_8^{-1} + qf)z^{-1} + \mathcal{O}\left(z^{-2}\right), \qquad (z \to \infty)$$

and we immediately obtain

$$y^{\infty}(z,t) = S^{\infty}(z,t)\psi^{\infty}(z,t)\begin{pmatrix} 1 & 0 \\ \frac{1}{q-1}(b_5^{-1} + b_6^{-1} + b_7^{-1} + b_8^{-1} + qf) & 1 \end{pmatrix}.$$

Correspondingly we write

$$Y^{\infty}(z,t) = \begin{pmatrix} y_1^{\infty}(z,t) & y_2^{\infty}(z,t) \\ y_1^{\infty}(z/q,t) & y_2^{\infty}(z/q,t) \end{pmatrix}, \quad \Psi^{\infty}(z,t) = \begin{pmatrix} \psi_1^{\infty}(z,t) & \psi_2^{\infty}(z,t) \\ \psi_1^{\infty}(z/q,t) & \psi_2^{\infty}(z/q,t) \end{pmatrix},$$

which are related by

$$Y^\infty(z,t) = \begin{pmatrix} S^\infty(z,t) & 0 \\ 0 & S^\infty(z/q,t) \end{pmatrix} \Psi^\infty(z,t) \\ \cdot \begin{pmatrix} 1 & 0 \\ \frac{1}{q-1}(b_5^{-1}+b_6^{-1}+b_7^{-1}+b_8^{-1}+qf) & 1 \end{pmatrix}. \tag{4.87}$$

Note that

$$|\Psi^\infty(qz,t)| = \widetilde{u}(z/q,t)\widetilde{w}(z,t)|\Psi^\infty(z,t)|,$$

and hence, using (4.86), we find

$$\begin{aligned}|\Psi^\infty(z,t)| =&(q-1)z^{-1}(q^2f/z, f/z, qt/(b_1z), qt/(b_2z), qt/(b_3z), qt/(b_4z); q)_\infty \times \\ &(q/(b_5z), q/(b_6z), q/(b_7z), q/(b_8z); q)_\infty,\end{aligned}$$

and hence, by (4.87),

$$|Y^\infty(z,t)| = (q-1)z^{-1}(1-f/z)\frac{(qt/(b_1z), qt/(b_2z), qt/(b_3z), qt/(b_4z); q)_\infty}{(1/(b_5z), 1/(b_6z), 1/(b_7z), 1/(b_8z); q)_\infty}. \tag{4.88}$$

Analogously to Lemma 4.4.2 we have the following result.

Lemma 4.7.4. *The matrix function $\Psi^\infty(z,t) = \Psi^\infty(z;t,f,g)$ is analytic on $\mathbb{P}^* \times R_p$.*

4.7.3 The Connection Matrix

Recall that, in Section 4.7.1, we defined a fundamental solution $Y^0(z,t)$ of (2.22a). Similarly, in Section 4.7.2, we defined a fundamental solution $Y^\infty(z,t)$ of (4.83a). We wish to define the *connection matrix* such that

$$y^\infty(z,t) = y^0(z,t)P(z,t), \tag{4.89}$$

and hence we see that $P(z,t)$ should be defined by

$$P(z,t) = Y^0(z;t)^{-1}\begin{pmatrix} 1 & 0 \\ 0 & t^2 \end{pmatrix} Y^\infty(z,t).$$

Indeed equation (4.89) is satisfied and

$$\begin{aligned} P(qz,t) &= Y^0(qz;t)^{-1}\begin{pmatrix} 1 & 0 \\ 0 & t^2 \end{pmatrix} Y^\infty(qz,t) \\ &= Y^0(z;t)^{-1}A(z,t)^{-1}\begin{pmatrix} 1 & 0 \\ 0 & t^2 \end{pmatrix}\begin{pmatrix} t^2 & 0 \\ 0 & 1 \end{pmatrix} A(z,t)\begin{pmatrix} 1 & 0 \\ 0 & t^2 \end{pmatrix} Y^\infty(z,t) \\ &= t^2 Y^0(z;t)^{-1}\begin{pmatrix} 1 & 0 \\ 0 & t^2 \end{pmatrix} Y^\infty(z,t), \end{aligned}$$

that is,

$$P(qz,t) = t^2 P(z,t). \tag{4.90}$$

Note that $P(z,t) = P(z;t,f,g)$ is meromorphic on $\mathbb{C}^* \times R_p$, and to understand its analytic properties better, we consider the *associated connection matrix*

$$\begin{aligned} Q(z,t) &= \frac{S^0(z,t)}{S^\infty(z,t)} P(z,t), \\ &= \theta_q(z/f)\theta_q(qb_5z, qb_6z, qb_7z, qb_8z)P(z,t). \end{aligned}$$

Indeed, from Lemma's 4.7.2 and 4.7.2, we easily derive that $Q(z,t) = Q(z;t,f,g)$ is analytic on $\mathbb{C}^* \times R_p$. Furthermore, by (4.90), $Q(z,t)$ satisfies

$$Q(qz,t) = -\frac{ft^2}{b_1b_2b_3b_4q^3} z^{-5} Q(z,t). \tag{4.91}$$

By Theorem 4.1.9, we see that, for fixed t, $Q(z,t)$ lives in a $4 \times 5 = 20$ dimensional space. However, we can explicitly determine its determinant. Indeed, using equations (4.80a) and (4.88) we derive

$$|P(z,t)| = qft^2 z^{-3} \frac{\theta_q(b_1z/t, b_2z/t, b_3z/t, b_4z/t)}{\theta_q(qb_5z, qb_6z, qb_7z, qb_8z)}, \tag{4.92}$$

and hence

$$|Q(z,t)| = qft^2 z^{-3} \theta_q(z/f)^2 \theta_q(b_1z/t, b_2z/t, b_3z/t, b_4z/t, qb_5z, qb_6z, qb_7z, qb_8z). \tag{4.93}$$

After this cut $Q(z,t)$ still contains essentially $20 - 10 = 10$ free parameters, for fixed t. So far we have used the analytic characterisation of L_1 at $z = 0$ and $z = \infty$ and, say the location of the zeros of $u(z,t)$ and $w(z,t)$. There is one intrinsic property of Yamada's Lax pair we have not used yet, which basically boils down to the following identity

$$v(f,t)v(f/q,t) = u(f/q,t)w(f,t). \tag{4.94}$$

One can verify it by direct calculation, indeed

$$\begin{aligned} v(f,t) &= -q(1-q)\frac{1-fg}{1-fg/t^2} p_1(f/t), \\ v(f/q,t) &= -(1-q^{-1})\frac{1-fg/t^2}{1-fg} p_2(f), \end{aligned}$$

from which the identity can easily be confirmed. At this point (4.94) might look a bit mysterious, however, one can think of it as the condition for $z = f$ to be an apparent singularity, as the following lemma shows.

Lemma 4.7.5. *For (t, f, g) in regular position, we have*

$$\Psi^0(q^n f, t)^{-1} = 0, \qquad (n \in \mathbb{Z}_{\leq -1}) \tag{4.95}$$
$$\Psi^\infty(q^n f, t) = 0, \qquad (n \in \mathbb{Z}_{\geq 2}) \tag{4.96}$$
$$Q(q^n f, t) = 0. \qquad (n \in \mathbb{Z}) \tag{4.97}$$

Proof. Let us start with verifying (4.95). Note that

$$\Psi^0(z,t)^{-1} = \Psi^0(qz,t)^{-1}A^0(z,t), \quad A^0(z,t) := \begin{pmatrix} -v(z,t) & -u(z/q,t)w(z,t) \\ 1 & 0 \end{pmatrix},$$

and as we know, by Lemma 4.7.2, that $\Psi^0(z,t)^{-1}$ is analytic in z on $\mathbb{C}^*$, we only have to check that $\Psi^0(f/q,t)^{-1} = 0$. Now we simply use the above recursion three times to obtain

$$\begin{aligned}
\Psi^0(f/q,t)^{-1} &= \Psi^0(q^2 f,t)^{-1}A^0(qf,t)A^0(f,t)A^0(f/q,t), \\
&= \Psi^0(q^2 f,t)^{-1}\begin{pmatrix} -v(qf,t) & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -v(f,t) & -u(f/q,t)w(f,t) \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -v(f/q,t) & 0 \\ 1 & 0 \end{pmatrix} \\
&= \Psi^0(q^2 f,t)^{-1}\begin{pmatrix} -v(qf,t) & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v(f,t)v(f/q,t) - u(f/q,t)w(f,t) & 0 \\ -v(f/q,t) & 0 \end{pmatrix} \\
&= \Psi^0(q^2 f,t)^{-1}\begin{pmatrix} -v(qf,t) & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -v(f/q,t) & 0 \end{pmatrix} \\
&= \Psi^0(q^2 f,t)^{-1}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&= 0,
\end{aligned}$$

where in the fourth equality we used identity (4.94). Of course the evaluation (4.96) for $\Psi^\infty(z,t)$ is done similarly. As to (4.97), firstly note that it is enough to show that $Q(f/q,t) = 0$, by recursion (4.91). The result then follows by observing that $Q(z,t)$ can be written as a product of analytic matrices in z on $\mathbb{C}^*$, one of which is $\Psi^0(z,t)^{-1}$. Explicitly, by equations (4.79) and (4.87), we have

$$Q(z,t) = M_0\Psi^0(z,t)^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -\frac{q^2 f t^2}{b_1 b_2 b_3 b_4} z^{-5} \end{pmatrix}\Psi^\infty(z,t)M_\infty,$$

where

$$M_0 = \begin{pmatrix} 1 & 0 \\ \frac{1}{q-1}(q(b_5+b_6+b_7+b_8)+f^{-1}) & 1 \end{pmatrix}, \quad M_\infty = \begin{pmatrix} 1 & 0 \\ \frac{1}{q-1}(b_5^{-1}+b_6^{-1}+b_7^{-1}+b_8^{-1}+qf) & 1 \end{pmatrix},$$

and the lemma follows. □

Theorem 4.7.6. *The matrix function*

$$R(z;t,f,g) := \frac{1}{\theta_q(z/f)}Q(z,t) = \theta_q(qb_5 z, qb_6 z, qb_7 z, qb_8 z)P(z,t),$$

is analytic on $\mathbb{C}^* \times R_p$*, satisfying*

$$R(qz,t) = \frac{t^2}{q^4 b_5 b_6 b_7 b_8} z^{-4} R(z,t),$$

and determinant equal to

$$|R(z,t)| = qft^2 z^{-3} \theta_q(b_1 z/t, b_2 z/t, b_3 z/t, b_4 z/t, qb_5 z, qb_6 z, qb_7 z, qb_8 z).$$

Proof. This is a direct consequence of Lemma 4.7.5 and equations (4.91) and (4.93). □

4.8 The Deformation Equation

In Section 4.7 we considered the global asymptotic analysis of the spectral equation L_1. For fixed t, we derived fundamental solutions at $z = 0$ and $z = \infty$ and studied the corresponding connection matrix. Now we wish to let t vary and understand how these, or related objects, behave under the time evolution.

4.8.1 Solutions at the “Spectral” Origin

Perhaps the easiest way to initiate the discussion, is by the use of the power series method to explore what happens when we also take the deformation equation in consideration. Say we set

$$y(z,t) = c_0(t) + c_1(t)z + c_2(t)z^2 + \dots,$$

then, considering only the spectral equation, $c_0(t)$ and $c_1(t)$ can be chosen at pleasure after which all higher order coefficients are fixed by L_1. Now let us consider what happens when we substitute the power series into L_2. All constant terms cancel and when we compare the coefficients of z in L_2, we find

$$c_1(t) = \frac{q}{q-1}(1 - t^{-2}) g(t) c_0(t) + \frac{q^2}{q-1} f(t) g(t) c_0(qt). \tag{4.98}$$

Similarly, comparing the coefficients of z^2, we obtain

$$c_2(t) = \frac{q^2}{q^2-1} g(t) c_0(qt) + \frac{q}{q^2-1}(q - t^{-2}) g(t) c_1(t) + \frac{q^3}{q^2-1} f(t) g(t) c_1(qt).$$

Upon substitution of (4.98) into the above equation, we find an expression for $c_2(t)$ in terms of $\{c_0(t), c_0(qt), c_0(q^2t)\}$. Of course, by comparing higher order coefficients, we find inductively, that for any $n \in \mathbb{N}$, the coefficient $c_n(t)$ can be expressed in terms of $\{c_0(t), c_0(qt), \dots, c_0(q^n t)\}$. Now, if we wish for $y(z,t)$ to simultaneously solve L_1 and L_2, then this leads to $c_0(t)$ to satisfy countably many q-difference equations, a priori. As an example, if we take the expression for $c_2(t)$ we find by substitution into L_1, and compare it with (4.98), then we find, after substitution of (4.98) and a long calculation, that $c_0(t)$ should satisfy

$$\gamma_0(t) c_0(t) + \gamma_1(t) c_0(qt) + \gamma_2(t) c_0(q^2 t) = 0, \tag{4.99}$$

where

$$\begin{aligned}\gamma_0(t) =&(t^2-1)g^2\left[qt^2(fg-t^2)p_2(1/g)-(fg-1)p_1(t/g)\right],\\ \gamma_1(t) =&t^2(fg-1)(fg-t^2)[(b_1+b_2+b_3+b_4)qt-(b_5+b_6+b_7+b_8)q^2t^2+q(qt^2-1)g\\ &+(q^2t^2-1)\overline{g}],\\ \gamma_2(t) =&q^3t^4\overline{f}\overline{g}(fg-1)(fg-t^2),\end{aligned}$$

and we suppressed t dependence of f and g throughout. It turns out that this condition is in fact sufficient. That is, to put it sloppy, let $c_0(t)$ be a solution of (4.99), and define $c_1(t)$ by (4.98), then the unique corresponding power series solution of L_1, also satisfies L_2. There are several ways to make this precise. Let us first discuss the discrete case. For $(t,f,g)\in R_p$, we denote the space of meromorphic solutions of $L_1=L_1(t,f,g)$ in z, by $\mathrm{SOL}_1(t,f,g)$. Note that $\mathrm{SOL}_1(t,f,g)$ is a 2-dimensional vector space over the field of q-elliptic functions. Then we use the deformation equation, to define an operator $\mathcal{L}_2$ as follows. Assume (t,f,g) and $(qt,\overline{f},\overline{g})$ are both in regular position, related by q-$P(A_1)$, then

$$\begin{aligned}\mathcal{L}_2 &: \mathrm{SOL}_1(t,f,g)\rightarrow \mathrm{SOL}_1(qt,\overline{f},\overline{g}), y\mapsto \mathcal{L}_2[y],\\ \mathcal{L}_2[y](z) &:= -\frac{h_1(z,t)}{h_0(z,t)}y(z)-\frac{h_2(z,t)}{h_0(z,t)}y(z/q),\end{aligned}$$

is a well-defined linear operator by Theorem 2.4.1. Recall that $\{y_1^0(z;t,f,g),y_2^0(z;t,f,g)\}$ and $\{y_1^0(z;qt,\overline{f},\overline{g}),y_2^0(z;qt,\overline{f},\overline{g})\}$ are bases of $\mathrm{SOL}_1(t,f,g)$ and $\mathrm{SOL}_1(qt,\overline{f},\overline{g})$ respectively. The question now is, what does $\mathcal{L}_2$ look like with respect to these bases? Well, let us write

$$\begin{aligned}y(z) &= c_0y_1^0(z;t,f,g)+c_1y_2^0(z;t,f,g),\\ \mathcal{L}_2[y](z) &= \overline{c}_0y_1^0(z;t,f,g)+\overline{c}_1y_2^0(z;t,f,g),\end{aligned}$$

then the constants $\{c_0,c_1\}$ and $\{\overline{c}_0,\overline{c}_1\}$, are related exactly by equations (4.98) and (4.99), with the obvious identifications, like $c_0(qt)=\overline{c}_0$. We refer to equation (4.99) as the *auxiliary equation* at $z=0$. As to the continuous time interpretation, we have the following result.

Theorem 4.8.1. *Let* $\mathbf{b}\in B_q$ *be generic*[2] *and* $T\subseteq\mathbb{C}^*$ *be a continuous q-domain. Let* (f,g) *be a nowhere singular meromorphic solution of q-$P(A_1)$ on T. Furthermore suppose $c_0(t)$ denotes a meromorphic solution of the auxiliary equation* (4.99) *on T. Define $c_1(t)$ by* (4.98), *then*

$$y(z,t)=c_0(t)y_1^0(z;t,f(t),g(t))+c_1(t)y_2^0(z;t,f(t),g(t)), \tag{4.100}$$

defines a solution of Yamada's Lax pair, both the spectral and deformation equation, which is meromorphic on $\mathbb{C}\times T$ in (z,t).

Proof. Firstly, let $i\in\{1,2\}$, then $y_i^0(z;t,f,g)$ is meromorphic on $\mathbb{C}\times R_p$, by Lemma 4.7.1. Let us define

$$T_s=\{t\in T:(t,f(t),g(t))\notin R_p\}\cup\{t\in T:f(t)=\infty \text{ or } g(t)=\infty\}.$$

[2]See Remark 4.8.2 for a precise definition

As (f,g) is nowhere singular, we can write this set as

$$\begin{aligned}T_s =&\{t\in T: f(t)=0 \text{ or } g(t)=0\}\cup\{t\in T: f(t)=\infty \text{ or } g(t)=\infty\}\\ &\cup\{t\in T:(f(t),g(t))=p_i \text{ for some } 1\leq i\leq 8\},\end{aligned}$$

and hence it is easy to see that the points in T_s are isolated and do not accumulate in T.

Let T_r be the complement of T_s in T, then (4.100) defines a function $y(z,t)$ which is meromorphic on $\mathbb{C}\times T_r$. Furthermore, it is obviously a solution of the spectral equation L_1. Now we apply the operator $\mathcal{L}_2$, that is, we set

$$\widetilde{y}(z,t):=-\frac{h_1(z,t)}{h_0(z,t)}y(z,t)-\frac{h_2(z,t)}{h_0(z,t)}y(z/q,t),$$

and we wish to show $y(z,qt)=\widetilde{y}(z,t)$, as this implies that $y(z,t)$ satisfies the deformation equation L_2. Well, we can characterise $y(z,qt)$ as the unique solution of $L_1(qt)$, i.e. L_1 with $t\mapsto qt$, meromorphic on $\mathbb{C}\times q^{-1}T_r$, satisfying

$$y(z,qt)=c_0(qt)+c_1(qt)z+\mathcal{O}\left(z^2\right), \tag{4.101}$$

as $z\to 0$, for any fixed $t\in q^{-1}T_r$, such that $c_0(qt)$ and $c_1(qt)$ are finite. By Theorem 2.4.1, we know that $\widetilde{y}(z,t)$ also satisfies $L_1(qt)$, and is meromorphic on $\mathbb{C}\times T_r$. Furthermore, precisely because $c_0(t)$ and $c_1(t)$ satisfy (4.99) and (4.98), we know that $\widetilde{y}(z,t)$ enjoys exactly the same expansion (4.101). The conclusion is that $y(z,t)=\widetilde{y}(z,t)$ holds, as an equality of meromorphic functions on $\mathbb{C}\times(T_r\cap q^{-1}T_r)$. In particular $y(z,t)$ indeed satisfies the deformation equation L_2.

Finally, we wish to show that the singularities of $y(z,t)$, at times in T_s, are at worst poles, i.e. $y(z,t)$ is meromorphic on $\mathbb{C}\times T$. Here we of course use that $y(z,t)$ satisfies L_2, a priori on $\mathbb{C}\times(T_r\cap q^{-1}T_r)$, and is meromorphic on $\mathbb{C}\times T_r$. Indeed, to establish the final piece of the theorem, all we have to show is that $q^{\mathbb{Z}}T_r=T$, or to put it differently,

$$\bigcap_{n\in\mathbb{Z}}q^nT_s=\emptyset. \tag{4.102}$$

For generic parameter values $\mathbf{b}\in B_q$, this is guaranteed. $\square$

Remark 4.8.2. Considering the proof of Theorem 4.8.1, we wish to exclude parameter values $\mathbf{b}\in\mathcal{B}_q$, for which their exist discrete solutions which only take values in the exceptional divisors $E_1,\ldots,E_8$, and the lines $\{f\equiv 0\}$, $\{f\equiv\infty\}$, $\{g\equiv 0\}$ and $\{g\equiv\infty\}$ in X. Note that such discrete solutions only exist in very degenerate parameter cases, and we call the parameters $\mathbf{b}$ generic if such solutions do not exist. An explicit example is given by the very degenerate case (3.108), where $\phi(t)$ can be chosen to have zeros on some q-spiral, and hence the intersection on the left-hand side of (4.102) is non-empty. Working out the exact conditions for which such discrete solutions exist is a laborious combinatorial problem, which we do not wish to work out in detail here.

4.8.2 Solutions at the "Spectral" Infinity

The story around $z=\infty$ is completely similar to that around $z=0$, which we discussed in the previous section. We consider an expansion

$$\widetilde{y}(z,t)=\widetilde{c}_0(t)+\widetilde{c}_1(t)z^{-1}+\widetilde{c}_2(t)z^{-2}+\dots, \tag{4.103}$$

about $z=\infty$, for the rescaled Lax pair $\{\widetilde{L}_1,\widetilde{L}_2\}$. Considering only the spectral equation $\widetilde{L}_1$, the coefficients $\widetilde{c}_0(t)$ and $\widetilde{c}_1(t)$ can be chosen at pleasure after which all higher order coefficients are fixed. Now let us consider what happens when we substitute the power series into $\widetilde{L}_2$. All singular terms cancel and when we compare the constant terms in $\widetilde{L}_2$, we find

$$\widetilde{c}_1(t)=\frac{t^2-1}{(q-1)g(t)}\widetilde{c}_0(t)-\frac{q}{q-1}\widetilde{c}_0(qt). \tag{4.104}$$

Similarly, comparing the coefficients of z^{-1}, we obtain

$$\widetilde{c}_2(t)=\frac{q}{q^2-1}f(t)\widetilde{c}_0(qt)+\frac{qt^2-1}{(q^2-1)g(t)}\widetilde{c}_1(t)+\frac{q}{q^2-1}\widetilde{c}_1(qt).$$

Upon substitution of (4.104) into the above equation, we find an expression for $\widetilde{c}_2(t)$ in terms of $\{\widetilde{c}_0(t),\widetilde{c}_0(qt),\widetilde{c}_0(q^2t)\}$. Comparing this expression for $\widetilde{c}_2(t)$, with the one we obtain from $\widetilde{L}_1$, we find, after substitution of (4.104) and a long calculation, that $\widetilde{c}_0(t)$ should satisfy

$$\widetilde{\gamma}_0(t)\widetilde{c}_0(t)+\widetilde{\gamma}_1(t)\widetilde{c}_0(qt)+\widetilde{\gamma}_2(t)\widetilde{c}_0(q^2t)=0, \tag{4.105}$$

where

$$\begin{aligned}
\widetilde{\gamma}_0(t)=&(t^2-1)g^3\overline{g}\left[(fg-1)p_1(t/g)-(fg-t^2)p_2(1/g)\right],\\
\widetilde{\gamma}_1(t)=&b_1b_2b_3b_4(fg-1)(fg-t^2)\big[(b_5^{-1}+b_6^{-1}+b_7^{-1}+b_8^{-1}-q(b_1^{-1}+b_2^{-1}+b_3^{-1}+b_4^{-1})t)g\overline{g}+\\
&(q^2t^2-1)g+(qt^2-1)\overline{g}\big],\\
\widetilde{\gamma}_2(t)=&-qb_1b_2b_3b_4g\overline{g}(fg-1)(fg-t^2),
\end{aligned}$$

and we suppressed t dependence of f and g throughout. It turns out that this condition is in fact sufficient. That is, to put it sloppy, let $\widetilde{c}_0(t)$ be a solution of (4.105), and define $\widetilde{c}_1(t)$ by (4.104), then the unique corresponding power series solution of $\widetilde{L}_1$, also satisfies $\widetilde{L}_2$. We refer to equation (4.105) as the *auxiliary equation* at $z=\infty$.

Theorem 4.8.3. *Let $\mathbf{b}\in B_q$ be generic*[3] *and $T\subseteq\mathbb{C}^*$ be a continuous q-domain. Let (f,g) be a nowhere singular meromorphic solution of q-$P(A_1)$ on T. Furthermore suppose $\widetilde{c}_0(t)$ denotes a meromorphic solution of the auxiliary equation (4.105) on T. Define $\widetilde{c}_1(t)$ by (4.104), then*

$$\widetilde{y}(z,t)=\widetilde{c}_0(t)y_1^\infty(z;t,f(t),g(t))+\widetilde{c}_1(t)y_2^\infty(z;t,f(t),g(t)), \tag{4.106}$$

defines a solution of the rescaled Lax pair $\{\widetilde{L}_1,\widetilde{L}_2\}$, which is meromorphic on $\mathbb{P}^*\times T$ in (z,t).

[3] See Remark 4.8.2 for a precise definition

Proof. This is proven analogously to Theorem 4.8.1. □

4.9 The Connection Matrix and Isomonodromy

We are now in position to combine the analytic theory of the spectral equation with the time deformation. We assume $\mathbf{b} \in B_q$ is generic in the sense of Remark 4.8.2. We fix a continuous q-domain $T \subseteq \mathbb{C}^*$ and suppose (f, g) is a nowhere singular meromorphic solution (f, g) of q-$P(A_1)$ on T. Let us take two linearly independent meromorphic solutions $c_0^1(t)$ and $c_0^2(t)$ of the auxiliary equation (4.99) at $z = 0$ on T, whose existence is guaranteed by the work of Praagman [72]. We define $c_1^1(t)$ and $c_1^2(t)$ by equation (4.98), with $c_0 = c_0^1$ and $c_0 = c_0^2$ respectively. We denote

$$C(t) = \begin{pmatrix} c_0^1(t) & c_0^2(t) \\ c_1^1(t) & c_1^2(t) \end{pmatrix},$$

then Theorem 4.8.1 shows that

$$\mathcal{Y}^0(z,t) := Y^0(z; t, f(t), g(t)) \cdot C(t),$$

defines a fundamental solution of the Lax pair in system form (2.22), both of the spectral and the deformation equation. Furthermore $\mathcal{Y}^0(z,t)$ is meromorphic on $\mathbb{C} \times T$.

Similarly, we take two linearly independent meromorphic solutions $\widetilde{c}_0^1(t)$ and $\widetilde{c}_0^2(t)$ of the auxiliary equation (4.105) at $z = \infty$ on T. We define $\widetilde{c}_1^1(t)$ and $\widetilde{c}_1^2(t)$ by equation (4.104), with $\widetilde{c}_0 = \widetilde{c}_0^1$ and $\widetilde{c}_0 = \widetilde{c}_0^2$ respectively. We denote

$$\widetilde{C}(t) = \begin{pmatrix} \widetilde{c}_0^1(t) & \widetilde{c}_0^2(t) \\ \widetilde{c}_1^1(t) & \widetilde{c}_0^2(t) \end{pmatrix},$$

then Theorem 4.8.3 shows that

$$\mathcal{Y}^\infty(z,t) = Y^\infty(z; t, f(t), g(t)) \cdot \widetilde{C}(t),$$

defines a fundamental solution of the Lax pair in system form (4.83), both of the spectral and the deformation equation. Furthermore $\mathcal{Y}^\infty(z,t)$ is meromorphic on $\mathbb{C}\times T$. The corresponding *connection matrix* of Yamada's Lax pair is defined by

$$\mathcal{P}(z,t) = \mathcal{Y}^0(z,t)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & t^2 \end{pmatrix} \mathcal{Y}^\infty(z,t) = C(t)^{-1} P(z; t, f(t), g(t)) \widetilde{C}(t), \tag{4.107}$$

which is meromorphic on $\mathbb{C}^* \times T$. Quite fundamentally, the spectral and time evolution of the connection matrix are given by

$$\mathcal{P}(qz,t) = t^2 \mathcal{P}(z,t), \tag{4.108a}$$
$$\mathcal{P}(z,qt) = z^2 \mathcal{P}(z,t). \tag{4.108b}$$

The first one is just the analog of equation (4.90), and the second one follows from the following calculation,

$$\begin{aligned}\mathcal{P}(z,qt) &= \mathcal{Y}^0(z,qt)^{-1}\begin{pmatrix}1 & 0\\ 0 & q^2t^2\end{pmatrix}\mathcal{Y}^\infty(z,qt)\\ &= \mathcal{Y}^0(z,t)^{-1}H(z,t)^{-1}\begin{pmatrix}1 & 0\\ 0 & q^2t^2\end{pmatrix}\widetilde{H}(z,t)\mathcal{Y}^\infty(z,t)\\ &= \mathcal{Y}^0(z,t)^{-1}H(z,t)^{-1}\begin{pmatrix}1 & 0\\ 0 & q^2t^2\end{pmatrix}\left[z^2\begin{pmatrix}1 & 0\\ 0 & q^{-2}t^{-2}\end{pmatrix}H(z,t)\begin{pmatrix}1 & 0\\ 0 & t^2\end{pmatrix}\right]\mathcal{Y}^\infty(z,t)\\ &= z^2\mathcal{P}(z,t).\end{aligned}$$

To all intent and purposes, it is (4.108b), which is the manifestation of *isomonodromy* of Yamada's Lax pair. Indeed the time evolution of the q-$P(A_1)$ equation is nonlinear and "complex", whereas the time evolution of the associated connection matrix is trivial. If one wishes to be strict, one could take any scalar function $S(z,t)$, meromorphic on $\mathbb{C}^*\times T$, enjoying the same spectral and time evolution (4.108), as the connection matrix, i.e. (4.81), and set

$$\widetilde{\mathcal{P}}(z,t) = S(z,t)^{-1}\mathcal{P}(z,t).$$

Then $\widetilde{\mathcal{P}}(z,t)$ is q-periodic, both in the spectral and time variable, and we have isomonodromy in the strict sense of the word. This approach, however, involves introducing arbitrariness, by the freedom of choosing $S(z,t)$ in the above.

4.9.1 Analytic Characteristics of Connection Matrix

By application of Theorem 4.7.6, we find that

$$\mathcal{R}(z,t) := \theta_q(qb_5z, qb_6z, qb_7z, qb_8z)\mathcal{P}(z,t), \tag{4.109}$$

is meromorphic on $\mathbb{C}^*\times T$, being analytic[4] in z, satisfying

$$\mathcal{R}(qz,t) = \frac{t^2}{q^4b_5b_6b_7b_8}z^{-4}\mathcal{R}(z,t), \tag{4.110}$$

and determinant equal to

$$|\mathcal{R}(z,t)| = qf(t)t^2\frac{|\widetilde{C}(t)|}{|C(t)|}\theta_q(b_1z/t, b_2z/t, b_3z/t, b_4z/t, qb_5z, qb_6z, qb_7z, qb_8z). \tag{4.111}$$

Furthermore the time evolution is given by

$$\mathcal{R}(z,qt) = z^2\mathcal{R}(z,t). \tag{4.112}$$

[4] We say that a meromorphic function $f(z,t)$, is analytic in z, if every point in its domain has an open environment on which $f(z,t)$ can be written as $f(z,t)=g(z,t)h(t)$, with $g(z,t)$ holomorphic and $h(t)$ meromorphic.

4.9.2 The Monodromy Mapping

Before resuming our discussion, it is convenient, using equations (4.98) and (4.104), to express $C(t)$ and $\widetilde{C}(t)$ as follows,

$$C(t) = N(t)C_0(t), \qquad \widetilde{C}(t) = \widetilde{N}(t)\widetilde{C}_0(t), \tag{4.113}$$

where

$$N(t) = \begin{pmatrix} 1 & 0 \\ \frac{q}{q-1}(1-t^{-2})g(t) & \frac{q^2}{q-1}f(t)g(t) \end{pmatrix}, \qquad C_0(t) = \begin{pmatrix} c_0^1(t) & c_0^2(t) \\ c_0^1(qt) & c_0^2(qt) \end{pmatrix},$$
$$\widetilde{N}(t) = \begin{pmatrix} 1 & 0 \\ \frac{t^2-1}{(q-1)g(t)} & -\frac{q}{q-1} \end{pmatrix}, \qquad \widetilde{C}_0(t) = \begin{pmatrix} \tilde{c}_0^1(t) & \tilde{c}_0^2(t) \\ \tilde{c}_0^1(qt) & \tilde{c}_0^2(qt) \end{pmatrix}.$$

To summarise, note that we have assigned, to the meromorphic solution (f,g) of q-$P(A_1)$, and corresponding solutions $\{c_0^1, c_0^2\}$ and $\{\tilde{c}_0^1, \tilde{c}_0^2\}$ of (4.99) and (4.105) respectively, the matrix $\mathcal{R}(z,t)$, symbolically

$$\left(f, g, \{c_0^1, c_0^2\}, \{\tilde{c}_0^1, \tilde{c}_0^2\}\right) \mapsto \mathcal{R}(z,t). \tag{4.114}$$

Now let us do some counting. The left-hand side has principally 10 q-periodic freedoms in t, 2 coming from q-$P(A_1)$, and 4 coming both from choosing a fundamental solution of the auxiliary equation (4.99) and (4.105). The right-hand side, i.e. $\mathcal{R}(z,t)$, basically carries 9 q-periodic freedoms in t. Indeed $\mathcal{R}(z,t)$ lives in a 16 dimensional space, characterised by (4.110), cut by

$$|\mathcal{R}(z,t)| = constant(t) \times \theta_q(b_1z/t, b_2z/t, b_3z/t, b_4z/t, qb_5z, qb_6z, qb_7z, qb_8z), \tag{4.115}$$

which leaves $16-7=9$ freedoms. Indeed, as the overall scaling, the factor before the q-theta functions, in the determinant formula (4.111), depends on the particular choices on the left-hand side of the “correspondence” (4.114), we can only take the locations of the 8 (spirals of) zeros of the determinant in consideration for the cut. The product of the 8 zeros in the determinant (4.115) is prescribed by (4.110), hence we essentially eliminate 7 freedoms in t by the cut (4.115).

So the left-hand side of the correspondence has 10 q-periodic freedoms, whereas the right-hand side has 9. This imbalance is easily understood from equations (4.107) and (4.113). Indeed, take any meromorphic function $\lambda(t)$ which is q-periodic in t, then

$$\left(f, g, \{\lambda c_0^1, \lambda c_0^2\}, \{\lambda\tilde{c}_0^1, \lambda\tilde{c}_0^2\}\right), \tag{4.116}$$

gets send to exactly the same matrix in the correspondence (4.114).

We can eliminate the fundamental solutions of the auxiliary equations from the correspondence (4.114), by sending

$$M_T : (f,g) \mapsto [\mathcal{R}(z,t)], \tag{4.117}$$

where [-] denotes say the orbit under the action of arbitrary left and inverse right multiplication by meromorphic invertible matrices in t which are q-periodic. To be exact, take the space S_T of matrices $\mathcal{Q}(z,t)$, meromorphic on $\mathbb{C}^* \times T$, being analytic in z, satisfying (4.110), (4.112) and (4.115). Loosely speaking this is a 9-dimensional space in the q-periodic sense. Then we consider the action on S_T defined by multiplication from the left and inversely from the right by meromorphic invertible matrices in t on T, which are q-periodic. We then divide out this action, which eliminates $8-1=7$ q-periodic freedoms, and the resulting orbit space $\mathcal{M}_T$ has $9-7=2$ q-periodic "dimensions". We hence obtain a mapping

$$M_T : \mathcal{S}_m^{**}(T) \to \mathcal{M}_T, \tag{4.118}$$

where $\mathcal{S}_m^{**}(T)$ denotes the space of nowhere singular meromorphic solutions on T, as defined in (2.17). We refer to M_T as the *monodromy mapping* and call $M_T(f,g)$ the monodromy corresponding to (f,g). We think of (4.118), or (4.114) after dividing out the freedom in multiplication by $\lambda(t)$ as described in (4.116), of really being a correspondence. However we will not investigate this further here.

Let us remark that the mapping M_T does not coincide, say pointwise, with the Riemann-Hilbert-Birkhoff correspondence (4.23). Of course we have only defined the Riemann-Hilbert-Birkhoff correspondence for non-resonant Fuchsian equations. However, in the case of a Fuchsian system with resonance and trivial monodromy, both at the origin and infinity, one typically sets the Riemann-Hilbert-Birkhoff correspondence to take the form

$$[A(z)]_{GL_2(\mathbb{C})} \leftrightarrow [Q(z)]_*\,,$$

where $[Q(z)]_*$ denotes the orbit of $Q(z)$ under arbitrary multiplication by lower triangular invertible matrices from the left and right. That is to say, in constructing $\mathcal{M}_T$ we divided out more than one would typically do from the Riemann-Hilbert-Birkhoff perspective.

4.10 Time Deformation of Rigid Objects

We can exploit the simple time evolution of for instance $\mathcal{Y}^0(z,t)$, to calculate the more complex time evolution of the rigid object $Y^0(z,t)$. Indeed by equations (4.113), (4.99) and (4.105), we have

$$C(qt) = E(t)C(t), \qquad E(t) := N(qt)\begin{pmatrix} 0 & 1 \\ -\frac{\gamma_0(t)}{\gamma_2(t)} & -\frac{\gamma_1(t)}{\gamma_2(t)} \end{pmatrix} N(t)^{-1},$$
$$\widetilde{C}(qt) = \widetilde{E}(t)\widetilde{C}(t), \qquad \widetilde{E}(t) := \widetilde{N}(qt)\begin{pmatrix} 0 & 1 \\ -\frac{\widetilde{\gamma}_0(t)}{\widetilde{\gamma}_2(t)} & -\frac{\widetilde{\gamma}_1(t)}{\widetilde{\gamma}_2(t)} \end{pmatrix} \widetilde{N}(t)^{-1}.$$

Hence, as $\mathcal{Y}^0(z,t)$ and $\mathcal{Y}^\infty(z,t)$ satisfy the deformation equations (2.22b) and (4.83b) respectively, we have

$$Y^0(z,qt) = H(z,t)Y^0(z,t)E(t)^{-1},$$
$$Y^\infty(z,qt) = \widetilde{H}(z,t)Y^\infty(z,t)\widetilde{E}(t)^{-1}.$$

In fact, let us write $E(t) = E(t, f, g, \overline{f}, \overline{g})$ and $\widetilde{E}(t) = \widetilde{E}(t, f, g, \overline{f}, \overline{g})$, then one can show that for (t, f, g) and $(qt, \overline{f}, \overline{g})$ in regular position, related by q-$P(A_1)$, we have

$$Y^0(z; qt, \overline{f}, \overline{g}) = H(z; t, f, g)Y^0(z; t, f, g)E(t, f, g, \overline{f}, \overline{g})^{-1},$$
$$Y^\infty(z; qt, \overline{f}, \overline{g}) = \widetilde{H}(z; t, f, g)Y^\infty(z; t, f, g)\widetilde{E}(t, f, g, \overline{f}, \overline{g})^{-1}.$$

Similarly we have

$$P(z; qt, \overline{f}, \overline{g}) = z^2 E(t, f, g, \overline{f}, \overline{g})P(z; t, f, g)\widetilde{E}(t, f, g, \overline{f}, \overline{g})^{-1},$$

and also

$$R(z; qt, \overline{f}, \overline{g}) = z^2 E(t, f, g, \overline{f}, \overline{g})R(z; t, f, g)\widetilde{E}(t, f, g, \overline{f}, \overline{g})^{-1}.$$

This time evolution is rather complex, however by taking on the relative position as set out in Section 4.9.2, it takes a much simpler form. That is, following the ideas in Section 4.9.2, we fix a $t_0 \in \mathbb{C}^*$, and write $S(t_0)$ for the space of analytic 2×2 matrix functions $R(z)$, satisfying

$$R(qz) = \frac{t_0^2}{q^4 b_5 b_6 b_7 b_8} z^{-4} R(z), \tag{4.119}$$

and, analogously to (4.115),

$$|R(z)| = constant \times \theta_q(b_1 z/t_0, b_2 z/t_0, b_3 z/t_0, b_4 z/t_0, qb_5 z, qb_6 z, qb_7 z, qb_8 z). \tag{4.120}$$

Then we consider the action φ of $GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$ on $S(t_0)$, which acts by left and inverse right multiplication,

$$\varphi(E, \widetilde{E})(R(z)) = ER(z)\widetilde{E}^{-1}. \qquad (R(z) \in M(t_0)) \tag{4.121}$$

We denote by $\mathcal{M}(t_0) = S(t_0)/\varphi$, the orbit space of $S(t_0)$ under φ. Let us write

$$R_p(t_0) = \{(t, f, g) \in R_p : t \in q^{\mathbb{Z}} t_0\},$$

then we define the monodromy mapping by

$$m : R_p(t_0) \to \mathcal{M}(t_0), (q^n t_0, f, g) \mapsto [z^{-2n} R(z; q^n t_0, f, g)].$$

Now we know that if $(q^n t_0, f, g)$ and $(q^{n+1} t_0, \overline{f}, \overline{g})$ are in regular position, related by q-$P(A_1)$, then $m(q^n t_0, f, g) = m(q^{n+1} t_0, \overline{f}, \overline{g})$. However we get into difficulties when a discrete solution hits a base point. We pose the following conjecture.

Conjecture 4.10.1. Let $(f_s, g_s)_{s\in\mathbb{Z}} \in S_c^*(t_0)$, then, for any $r, s \in \mathbb{Z}$, if $(q^s t_0, f_s, g_s)$ and $(q^r t_0, f_r, g_r)$ are in regular position, then

$$m(q^s t_0, f_s, g_s) = m(q^r t_0, f_r, g_r).$$

Note that if the answer to Question 2.3.2 is affirmative, then this conjecture holds true.

Let us assume momentarily that the above conjecture is correct. Then we obtain a mapping

$$M(t_0) : S_c^*(t_0) \to \mathcal{M}(t_0).$$

We can do the same counting as done in Section 4.9.2. The space $M(t_0)$ is $16 - 7 = 9$ dimensional, and hence $\mathcal{M}(t_0)$ is $9 - 7 = 2$ dimensional, which adds up with $S_c^*(t_0)$ being two dimensional.

CHAPTER 5

The Direct Monodromy Problem

The *direct monodromy problem* entails calculating the monodromy datum of the associated linear problem corresponding to a given Painlevé transcendent explicitly. In our case the transcendents are characterised by some asymptotics at $t = 0$ or $t = \infty$. It turns out that we can completely determine the monodromy exploiting just such asymptotic characterisations. This is of course not unusual in the Painlevé world, Jimbo [42] and Mano [61] show that the same is true in the P_{VI} and q-P_{VI} case.

We start with an overview of our approach to solve the direct monodromy problem for a transcendent, characterised by some given critical behaviour. In Section 5.2, we discuss a particular example in detail, the transcendent $(f, g) = (f^{(1,1)}, g^{(1,1)})$, meromorphic at $t = 0$, defined in Proposition 3.1.2. We then give the monodromy corresponding to the other transcendents, which are meromorphic at a critical point, in Section 5.3. The technical proofs of those results can be found in Appendix D.

We then consider the direct monodromy problem for the generic case near $t = 0$, i.e. a solution with critical behaviour as specified in Theorem 3.4.1. In Sections 5.4 and 5.5 we construct fundamental solutions of Yamada's Lax pair near $z = \infty$ and $z = 0$ respectively, which we relate via an explicit connection matrix in Section 5.6. This leads to an explicit parameterisation of the monodromy of Yamada's Lax pair in terms of the integration constants $\{\phi(t), \Lambda(t)\}$ of the generic solution near $t = 0$, given in Section 5.7.

A similar analysis in Sections 5.8, 5.9 and 5.10, leads to an explicit parameterisation of the monodromy of Yamada's Lax pair in terms of the integration constants $\{\phi_\infty(t), \Lambda_\infty(t)\}$ of the generic solution near $t = \infty$, given in Section 5.11.

Finally we combine the results in Section 5.12, to arrive at parametric connection formulae, relating the critical behaviour of transcendents near $t = 0$ and $t = \infty$, indirectly.

5.1 Overview of Approach

Our starting point is a meromorphic solution of q-$P(A_1)$, given by some critical behaviour near $t = 0$ or $t = \infty$. Let us focus on the case $t = 0$, with some critical behaviour given in Table 3.1. We only wish to give a schematic overview here, hence we warn the reader some parts should be taken with a grain of salt.

We consider Yamada's Lax pair and we wish to construct solutions of it near the "spectral"

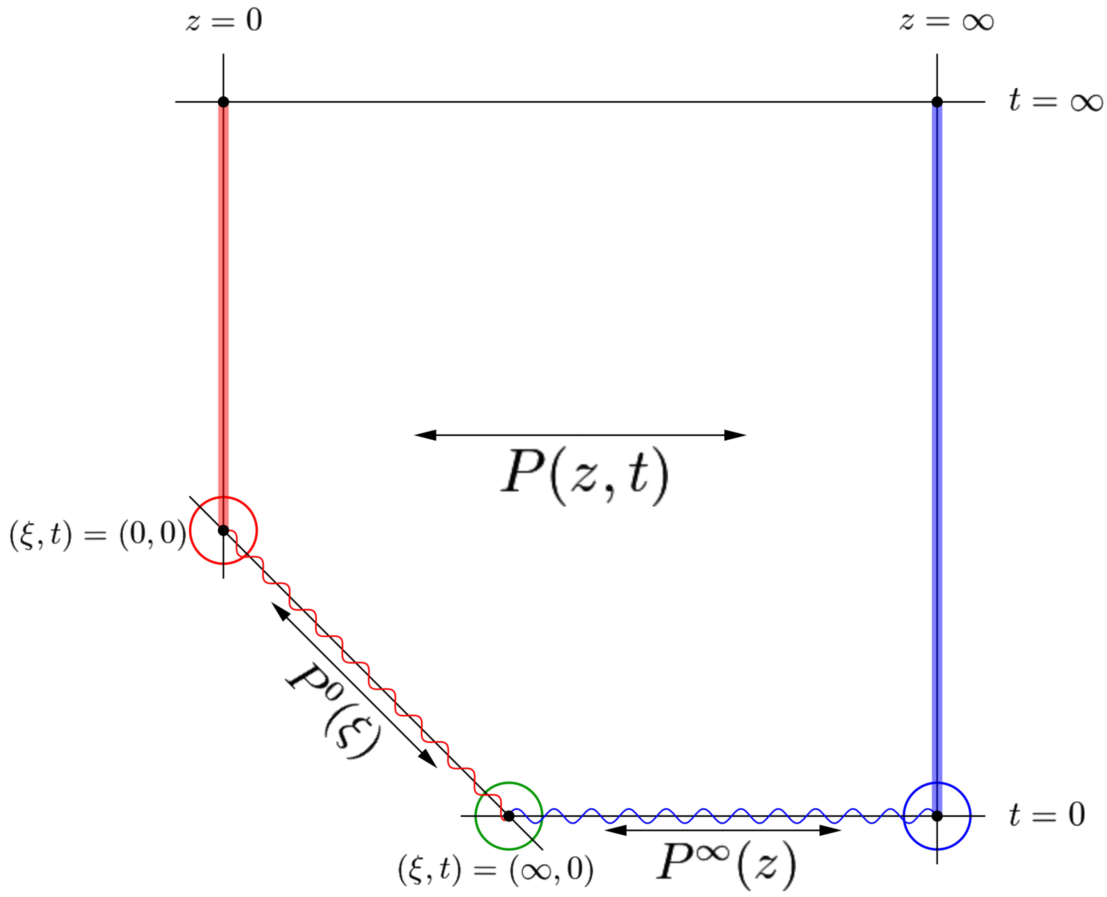


Figure 5.1: Graphical illustration of factorisation of connection matrix in the $t \to 0$ limit.

zero 4.8.1 and the "spectral" infinity 4.8.2, depicted in red and blue respectively in Figure 5.1. As we have an asymptotic characterisation of our meromorphic solution near $t = 0$, we construct a meromorphic fundamental solution $\widetilde{c}_0(t)$ of (4.105), such that, after some specific scaling $\widetilde{c}_0(t) = s^\infty(t)c_0^\infty(t)$, the resulting $c_0^\infty(t)$ has tempered behaviour as $t \to 0$. Correspondingly, by Theorem 4.8.3, we have a fundamental solution $\widetilde{Y}^\infty(z,t) = s^\infty(t)\Psi^\infty(z,t)$ of $\widetilde{L}$, such that the limit

$$\lim_{t \to 0} \Psi^\infty(z,t) \to D_0^\infty(z),$$

exists. We call such a fundamental solution a fundamental solution at $(z,t) = (\infty, 0)$. In Figure 5.1, the point $(z,t) = (\infty, 0)$ is encircled in blue.

The function $\Psi^\infty(z,t)$ satisfies a rescaled form of Yamada's Lax pair, which we denote by L^∞. When we let $t \to 0$ in L^∞, we find that $D_0^\infty(z)$ satisfies a second order linear equation, which is a rescaled version of the degree two model equation (4.51). We have an explicit solution to the connection problem of the limiting equation, given by a connection matrix $P^\infty(z)$. This allows us to "transition" from $z = \infty$ to $z = 0$ along the line $t = 0$ in the (z,t)

plane, depicted by the wiggling blue line in Figure 5.1, by setting

$$\Psi^{\infty,0}(z,t) = \Psi^{\infty}(z,t)P^{\infty}(z)^{-1}.$$

Indeed $\Psi^{\infty,0}(z,t)$ satisfies a rescaled version $L^{\infty,0}$ of Yamada's Lax pair, the limit

$$\lim_{t\to 0}\Psi^{\infty,0}(z,t) =: D_0^{\infty,0}(z),$$

exists, and $D_0^{\infty,0}(z)$ is holomorphic at $z=0$.

We wish to follow a similar approach to construct a solution of Yamada's Lax pair (2.21) near $(z,t)=(0,0)$. However it turns out that the Lax pair is very singular at $(z,t)=(0,0)$ in the (z,t) plane. The same holds true in the case of P_{VI} and q-P_{VI}, as shown by Jimbo [42] and Mano [61] respectively. Further consideration shows that we should instead consider Yamada's Lax pair in the (ξ,t) plane, where $\xi=\frac{z}{t}$. We find that there exists a fundamental solution near $(\xi,t)=(0,0)$, in the (ξ,t) plane,

$$Y^0(z,t) = s^0(t)\Psi^0(\xi,t),$$

where $s^0(t)$ some scaling, such that $\Psi^0(\xi,t)$ is holomorphic at $\xi=0$ and the limit

$$\lim_{t\to 0}\Psi^0(\xi,t) =: D_0^0(\xi),$$

exists. Now $\Psi^0(\xi,t)$ satisfies a rescaled version L^0 of Yamada's Lax pair, and letting $t\to 0$, we find that $D_0^0(\xi)$ satisfies a linear second order equation, which is again a rescaled version of our model equation (4.51). We denote the corresponding connection matrix by $P^0(\xi)$, and we transition from $\xi=0$ to $\xi=\infty$ along $t=0$ in the (ξ,t) plane, depicted by the wiggling red line in Figure 5.1, by setting

$$\Psi^{0,\infty}(\xi,t) = \Psi^0(\xi,t)P^0(\xi).$$

Indeed $\Psi^{0,\infty}(\xi,t)$ satisfies a rescaled version $L^{0,\infty}$ of Yamada's Lax pair, the limit

$$\lim_{t\to 0}\Psi^{0,\infty}(\xi,t) =: D_0^{0,\infty}(\xi),$$

exists, and $D_0^{0,\infty}(\xi)$ is holomorphic at $\xi=\infty$.

Interestingly enough, the Lax pairs $L^{\infty,0}$ and $L^{0,\infty}$ are identical, and hence $\Psi^{0,\infty}(\xi,t) = \Psi^{0,\infty}(\frac{z}{t},t)$ and $\Psi^{\infty,0}(z,t)$ satisfy the same Lax pair, though they are characterised by certain asymptotics approaching $(\xi,t)=(\infty,0)$ via disjoint regimes. To *match* the two fundamental solutions, we introduce a third fundamental solution $\Psi^{\mathrm{tr}}(z,t)$ near $(\xi,t)=(\infty,0)$, depicted by the green circle, which is characterised asymptotically in a larger regime, having non-empty intersection with the regimes corresponding to $\Psi^{0,\infty}(\xi,t)$ and $\Psi^{\infty,0}(z,t)$. It turns out all three solutions are identical (possibly up to some scalar factors), and we find

$$
\begin{aligned}
Y^0(z,t)P(z,t) &= \widetilde{Y}^{\infty}(z,t)\\
&= s^{\infty}(t)\Psi^{\infty}(z,t)\\
&= s^{\infty}(t)\Psi^{\infty,0}(z,t)P^{\infty}(z)\\
&= s^{\infty}(t)\Psi^{\mathrm{tr}}(z,t)P^{\infty}(z)\\
&= s^{\infty}(t)\Psi^{0,\infty}(\xi,t)P^{\infty}(z)\\
&= s^{\infty}(t)\Psi^{0}(\xi,t)P^{0}(\xi)P^{\infty}(z)\\
&= \frac{s^{\infty}(t)}{s^{0}(t)}Y^0(z,t)P^0(\xi)P^{\infty}(z),
\end{aligned}
$$

and we conclude

$$
P(z,t) = \frac{s^{\infty}(t)}{s^{0}(t)}P^0\left(\frac{z}{t}\right)P^{\infty}(z). \tag{5.1}
$$

So we have an explicit factorisation of the Yamada Lax pair connection problem into two copies of the connection problem of our model equation (4.51) as $t \to 0$. We note that a similar factorisation takes place in the case of P_{VI} and q-P_{VI}, as shown by Jimbo [42] and Mano [61] respectively. The way we have represented it here, the factorisation seems quite miraculous. Let us therefore emphasise that such factorisations are in fact common in isomonodromic deformation theory, where we particularly want to mention Gavrylenko and Lisovyy [19] who discuss the generic (continuous) Fuchsian case with arbitrary rank and number of regular singular points.

5.1.1 Overview of Approach near $t=\infty$

Our approach to solving the direct monodromy problem for meromorphic q-$P(A_1)$ transcendents, characterised by some critical behaviour near $t=\infty$, is essentially the same. We consider Yamada's Lax pair and we wish to construct solutions of it near the "spectral" zero 4.8.1 and the "spectral" infinity 4.8.2, depicted in red and blue respectively in Figure 5.2.

We find a scaling $\widehat{s}^0(t)$, such that there exists a fundamental solution $\widehat{Y}^0(z,t)=\widehat{s}^0(t)\widehat{\Psi}^0(z,t)$ of L (2.21) at $(z,t)=(0,\infty)$, such that the limit

$$
\lim_{t\to\infty}\widehat{\Psi}^0(z,t)\to\widehat{D}_0^0(z),
$$

exists. In Figure 5.2, the point $(z,t)=(0,\infty)$ is encircled in red.

The function $\widehat{\Psi}^0(z,t)$ satisfies a rescaled form of Yamada's Lax pair, which we denote by $\widehat{L}^0$. When we let $t\to\infty$ in $\widehat{L}^0$, we find that $\widehat{D}_0^0(z)$ satisfies a second order linear equation, which is a rescaled version of the degree two model equation (4.51). We have an explicit solution to the connection problem of this equation, given by a connection matrix $\widehat{P}^0(z)$. This allows us to "transition" from $z=0$ to $z=\infty$ along the line $t=\infty$ in the (z,t) plane, depicted by the wiggling red line in Figure 5.2, by setting

$$
\widehat{\Psi}^{0,\infty}(z,t)=\widehat{\Psi}^0(z,t)\widehat{P}^0(z).
$$

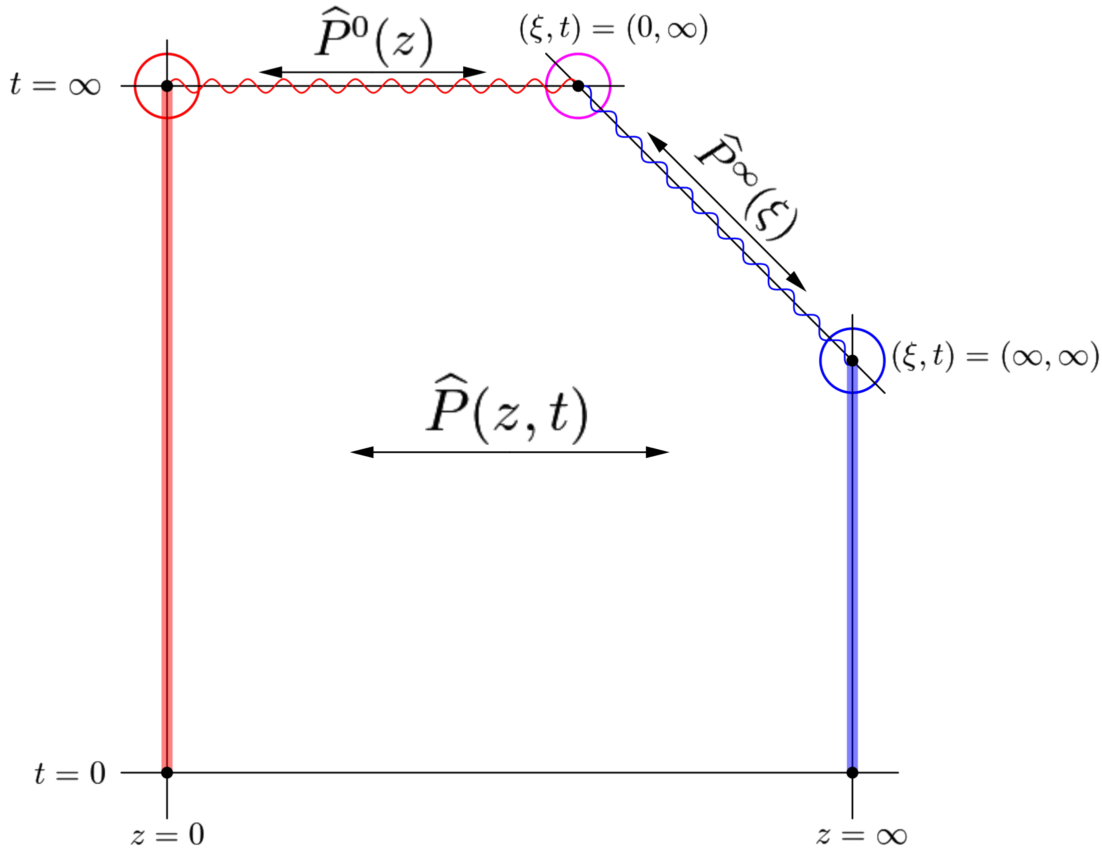


Figure 5.2: Graphical illustration of factorisation of connection matrix in the $t \to \infty$ limit.

We wish to follow a similar approach to construct a solution of Yamada's Lax pair (4.82) near $(z, t) = (\infty, \infty)$. However it turns out that the Lax pair is rather singular at $(z, t) = (\infty, \infty)$ in the (z, t) plane. We resolve this singularity by considering Yamada's Lax pair in the (ξ, t) plane, where $\xi = \frac{z}{t}$. We find that there exists a fundamental solution at $(\xi, t) = (\infty, \infty)$, in the (ξ, t) plane,

$$\widehat{Y}^\infty(z, t) = \widehat{s}^\infty(t)\widehat{\Psi}^\infty(\xi, t),$$

where $\widehat{s}^\infty(t)$ some scaling, such that $\widehat{\Psi}^\infty(\xi, t)$ is holomorphic at $\xi = \infty$, and the limit

$$\lim_{t\to\infty} \widehat{\Psi}^\infty(\xi, t) =: \widehat{D}_0^\infty(\xi),$$

exists. Now $\widehat{\Psi}^\infty(\xi, t)$ satisfies a rescaled version $\widehat{L}^\infty$ of Yamada's Lax pair, and letting $t \to 0$, we find that $\widehat{D}_0^\infty(\xi)$ satisfies a linear second order equation, which is again a rescaled version of our model equation (4.51). We denote the corresponding connection matrix by $\widehat{P}^\infty(\xi)$, and we transition from $\xi = \infty$ to $\xi = 0$ along $t = \infty$ in the (ξ, t) plane, depicted by the wiggling blue line in Figure 5.2, by setting

$$\widehat{\Psi}^{\infty,0}(\xi, t) = \widehat{\Psi}^\infty(\xi, t)\widehat{P}^\infty(\xi).$$

It turns out that the Lax pairs $\widehat{L}^{0,\infty}$ and $\widehat{L}^{\infty,0}$ are identical, and we find $\widehat{\Psi}^{0,\infty}(z,t) = \widehat{\Psi}^{\infty,0}(\xi,t)$, from which we derive the connection result

$$\widehat{Y}^0(z,t)\widehat{P}(z,t) = \widehat{Y}^\infty(z,t),$$

where

$$\widehat{P}(z,t) = \frac{\widehat{s}^\infty(t)}{\widehat{s}^0(t)}\widehat{P}^0\left(z\right)\widehat{P}^\infty\left(\frac{z}{t}\right).$$

5.2 A Special Case

Before tackling the generic case, we first solve the direct monodromy problem for solutions with the simplest critical behaviour near $t=0$ in our Table 3.1. We start with the solution $(f,g) = (f^{(1,1)}, g^{(1,1)})$ meromorphic at $t=0$, defined in Proposition 3.1.2, where we assume the corresponding conditions (3.11) on the parameters. We write

$$f(t) = f_1 t + f_2 t^2 + f_3 t^3 + \ldots, \qquad f_1 = \frac{b_1+b_2-(b_3+b_4)}{b_1b_2-b_3b_4},$$
$$g(t) = g_1 t + g_2 t^2 + g_3 t^3 + \ldots, \qquad g_1 = \frac{b_1b_2(b_3+b_4)-b_3b_4(b_1+b_2)}{b_1b_2-b_3b_4}.$$

Let us recall that this solution is obtained from the generic solution (3.44) by setting $\Lambda = \Lambda_1^\pm$ and $\phi = 0$, as shown in Proposition 3.5.1. It is helpful the keep this in mind as we we will see that the values $\Lambda_1^+ = -b_1b_2$ and $\Lambda_1^- = -b_3b_4$ enter the connection matrices of the limiting second order equations naturally. Later on we find that this also holds in the generic case.

We mention that Kaneko [53] calculates the monodromy corresponding to solutions of P_{VI} which are meromorphic at a critical point. Also Ohyama [64, 65] calculates the connection matrices corresponding to the limiting equations of linear problems associated with q-P_{III}, q-P_{V} and q-P_{VI}, for transcendents meromorphic at the origin, though he does not follow up with a matching procedure or equivalent to establish a factorisation as in (5.1).

5.2.1 Fundamental Solution at $(z,t) = (\infty, 0)$

Let us first recall that Theorem 4.8.3 tells us that we should study equation (4.105) to construct a solution at $(z,t) = (\infty,0)$. The coefficients in (4.105) satisfy

$$\begin{aligned}
\widetilde{\gamma}_0(t) &= b_1b_2b_3b_4(f_1g_1-1)t^2 + \mathcal{O}\left(t^3\right),\\
\widetilde{\gamma}_1(t) &= b_1b_2b_3b_4(f_1g_1-1)(1+q)g_1t^3 + \mathcal{O}\left(t^4\right),\\
\widetilde{\gamma}_2(t) &= b_1b_2b_3b_4(f_1g_1-1)q^2g_1^2t^4 + \mathcal{O}\left(t^5\right),
\end{aligned}$$

as $t \to 0$. Hence this equation is unbalanced, so we apply a scaling

$$\widetilde{c}_0(t) = s^\infty(t)c_0^\infty(t), \quad s^\infty(qt) = \beta t^{-1}s^\infty(t), \quad \beta := -q^{-1}g_1^{-1}, \tag{5.2}$$

where we invite the reader to choose $s^\infty(t)$, meromorphic on $\mathbb{C}^*$, at their pleasure. The resulting equation for $c_0^\infty(t)$ takes the form

$$\gamma_0^\infty(t)c_0^\infty(t) + \gamma_1^\infty(t)c_0^\infty(qt) + \gamma_2^\infty(t)c_0^\infty(q^2t) = 0, \tag{5.3}$$

where

$$(\gamma_0^\infty(t), \gamma_1^\infty(t), \gamma_2^\infty(t)) = \frac{1}{b_1b_2b_3b_4(f_1g_1 - 1)t^2}(\widetilde{\gamma}_0(t), \beta t^{-1}\widetilde{\gamma}_1(t), q^{-1}\beta^2 t^{-2}\widetilde{\gamma}_2(t)).$$

Now $t = 0$ is a regular singular point of (5.3), with exponents $1, q$ and hence resonance, as

$$\gamma_0^\infty(0) = 1, \quad \gamma_1^\infty(0) = -(1 + q^{-1}), \quad \gamma_2^\infty(0) = q^{-1}.$$

However, the no-logarithms condition is satisfied, indeed a little calculation shows

$$\gamma_0^{\infty\prime}(0) + \gamma_1^{\infty\prime}(0) + \gamma_2^{\infty\prime}(0) = 0,$$

and hence $t = 0$ is in fact an ordinary point of (5.3). In particular, for any choice of $c_{0,0}, c_{0,1} \in \mathbb{C}$, there exists an unique meromorphic solution $c_0^\infty(t)$ of (5.3), characterised by

$$c_0^\infty(t) = c_{0,0} + c_{0,1}t + \mathcal{O}\left(t^2\right). \quad (t \to 0) \tag{5.4}$$

Now we rescale $\widetilde{y}(z,t)$ along with $\widetilde{c}_0$ in (5.2), i.e.

$$\widetilde{y}(z,t) = s^\infty(t)\psi^\infty(z,t), \tag{5.5}$$

then $\psi^\infty(z,t)$ satisfies the Lax pair L^∞ given by

$$L_1^\infty: \quad \widetilde{u}(z,t)\psi^\infty(qz,t) + \widetilde{v}(z,t)\psi^\infty(z,t) + \widetilde{w}(z,t)\psi^\infty(z/q,t) = 0, \tag{5.6a}$$

$$L_2^\infty: \quad \beta t^{-1}z^{-2}h_0(z,t)\psi^\infty(z,qt) + h_1(z,t)\psi^\infty(z,t) + t^2h_2(z,t)\psi^\infty(z/q,t) = 0. \tag{5.6b}$$

By Theorem 4.8.3, we know that this system of equations is balanced at $z = \infty$, and we know that (5.3), which could be considered as L^∞ on the line $z = \infty$, is balanced at $t = 0$. It turns out L^∞ is balanced at $(z,t) = (\infty, 0)$, as the following Theorem shows.

Proposition 5.2.1. *For any choice of $c_{0,0}, c_{0,1} \in \mathbb{C}$, there exists an unique solution $\psi^\infty(z,t)$ of the Lax pair (5.6), which is analytic at $(z,t) = (\infty, 0)$, such that $c_0^\infty(t) := \psi^\infty(\infty, t)$ is characterised by the expansion (5.4). Furthermore $\psi^\infty(z,t)$ enjoys an unique meromorphic continuation to $\mathbb{P}^* \times \mathbb{C}$.*

Proof. Let us consider an expansion

$$\psi^\infty(z,t) = c_0^\infty(t) + c_1^\infty(t)z^{-1} + c_2^\infty(t)z^{-2} + \ldots,$$

for a solution of (5.6). Comparing the scaling (5.5) with (4.103), we have

$$\widetilde{c}_0(t) = s^\infty(t)c_0^\infty(t), \quad \widetilde{c}_1(t) = s^\infty(t)c_1^\infty(t), \quad \widetilde{c}_2(t) = s^\infty(t)c_2^\infty(t), \ldots,$$

where the first equation should remind us of (5.2). In particular, analogously to (4.104), we find the following equation from L_2^∞,

$$c_1^\infty(t) = \frac{t^2-1}{(q-1)g(t)}c_0^\infty(t) - \frac{q}{q-1}\beta t^{-1}c_0^\infty(qt). \tag{5.7}$$

Now let us take some $c_{0,0}, c_{0,1} \in \mathbb{C}$, then we know that there exists an unique solution $c_0^\infty(t)$ of (5.3), which is analytic at $t=0$, characterised by the expansion (5.4). Next we define $c_1^\infty(t)$ by (5.7), and note that $c_1^\infty(t)$ is analytic at $t=0$, precisely because $\beta = -q^{-1}g_1^{-1}$.

Claim 5.2.2. *There exists a small open disc D about $t=0$, such that there is an unique solution $\psi^\infty(z,t)$ of L_1^∞, which is meromorphic on $\mathbb{P}^* \times D$, characterised by*

$$\psi^\infty(z,t) = c_0^\infty(t) + c_1^\infty(t)z^{-1} + \mathcal{O}\left(z^{-2}\right), \tag{5.8}$$

holding locally uniformly in t on D as $z \to \infty$.

The proof of this claim follows a typical procedure. Let us first remind ourselves of Remark B.5, which basically tells us that for any well-posed (q-discrete) Cauchy problem, with analytic dependence on some parameters, the corresponding solution also depends analytically on those parameters. To establish the claim, we first observe that the coefficients $\widetilde{u}(z,t), \widetilde{v}(z,t), \widetilde{w}(z,t)$ are holomorphic in (z,t) at $(z,t) = (\infty, 0)$, and hence holomorphic on some open polydisc $D_z \times D_t$ centered at $(z,t) = (\infty,0)$. In fact we have

$$\widetilde{u}(z,t) = q^{-1}\left(1-\frac{1}{b_5qz}\right)\left(1-\frac{1}{b_6qz}\right)\left(1-\frac{1}{b_7qz}\right)\left(1-\frac{1}{b_8qz}\right) + \mathcal{O}(t), \tag{5.9a}$$

$$\widetilde{v}(z,t) = q^{-1}\left[-(1+q) + (b_5^{-1}+b_6^{-1}+b_7^{-1}+b_8^{-1})z^{-1} - \left(\frac{1}{b_1b_2}+\frac{1}{b_3b_4}\right)z^{-2}\right] + \mathcal{O}(t), \tag{5.9b}$$

$$\widetilde{w}(z,t) = q^{-1} + \mathcal{O}(t), \tag{5.9c}$$

locally uniformly in z on $\mathbb{P}^*$ as $t \to 0$. Now let $D \subseteq D_t$ be an open disc centered at $t=0$, such that $c_0^\infty(t)$ and $c_1^\infty(t)$ are holomorphic on D. Then, for any fixed $t \in D$, there exists an unique solution $\psi^\infty(z,t)$ of L_1^∞, which is meromorphic on $\mathbb{P}^*$ in z, characterised by (5.8) as $z \to \infty$. By Remark B.5, equation (5.8) holds locally uniformly in t on D as $z \to \infty$, and the claim follows.

It remains to show that $\psi^\infty(z,t)$ also satisfies L_2^∞, which basically follows from Theorem 4.8.1 after some rescaling. For convenience of the reader we repeat this line of proof once more. We use L_2^∞ as an operator, defining

$$\widetilde{\psi}^\infty(z,t) := -z^2t\frac{h_1(z,t)}{\beta h_0(z,t)}\psi^\infty(z,t) - z^2t^3\frac{h_2(z,t)}{\beta h_0(z,t)}\psi^\infty(z/q,t).$$

Indeed $\widetilde{\psi}^\infty(z,t/q)$ is a solution of L_1^∞, meromorphic on $\mathbb{P}^* \times qD$ and it is easy to see that $\widetilde{\psi}^\infty(z,t/q)$ has the same asymptotic characterisation (5.8), holding locally uniformly in t on qD as $z \to \infty$. As $D \cap qD = qD$, we immediately obtain $\widetilde{\psi}^\infty(z,t/q) = \psi^\infty(z,t)$ on qD, that is, $\psi^\infty(z,t)$ satisfies L_2^∞ on $\mathbb{P}^*\times qD$. The last line of the theorem, on unique meromorphic continuation, is now obvious. $\square$

5.2.2 Transition from $(z,t)=(\infty,0)$ to $(z,t)=(0,0)$

Let us take some $c_{0,0},c_{0,1}\in\mathbb{C}$ and corresponding solution $\psi_0(z,t)$ of L^∞, as defined in Proposition 5.2.1. We write its power series expansion about $(z,t)=(\infty,0)$ as

$$\begin{aligned}\psi^\infty(z,t)&=\sum_{m=0}^{\infty}c_m^\infty(t)z^{-m}=\sum_{m,n=0}^{\infty}c_{m,n}^\infty z^{-m}t^n\\&=\sum_{m=0}^{\infty}d_m^\infty(z)t^m=\sum_{m,n=0}^{\infty}d_{m,n}^\infty z^{-n}t^m,\end{aligned}$$

in particular $c_{m,n}^\infty=d_{n,m}^\infty$ for $m,n\in\mathbb{N}$. Now, we let $t\to 0$ in L_1^∞, which gives, by equations (5.9),

$$\begin{aligned}d_0^\infty(z/q)+\left[-(1+q)+(b_5^{-1}+b_6^{-1}+b_7^{-1}+b_8^{-1})z^{-1}-\left(\frac{1}{b_1b_2}+\frac{1}{b_3b_4}\right)z^{-2}\right]d_m^\infty(z)\\+\left(1-\frac{1}{b_5qz}\right)\left(1-\frac{1}{b_6qz}\right)\left(1-\frac{1}{b_7qz}\right)\left(1-\frac{1}{b_8qz}\right)d_m^\infty(qz)=0.\end{aligned}\tag{5.10}$$

This is exactly the degree two model equation (4.51) under the identification $y(z)=d_m^\infty(1/z)$, with parameter values $\sigma=\sigma_\infty^{\mathrm{I}}$, defined in (5.26), where we note that Fuchs' equation (4.52) is indeed satisfied. By letting $t\to 0$ in L_2^∞, we find, in a similar fashion,

$$d_1^\infty(z)=-\frac{g_1}{q-1}z(d_0^\infty(z)-d_0^\infty(z/q))-\frac{1}{q-1}\left(\frac{g_2}{g_1}-f_1z^{-1}\right)d_0^\infty(z),$$

and hence

$$d_{0,0}=c_{0,0},\quad d_{0,1}=-\frac{g_2}{(q-1)g_1}c_{0,0}+g_1^{-1}c_{0,1}.$$

In particular, we can just as well prescribe $d_{0,0}$ and $d_{0,1}$ to define $\psi^\infty(z,t)$ uniquely. Using the notation in Section 4.5, we define a fundamental solution of (5.10) by

$$D_0^\infty(z)=y^0(z^{-1};\sigma_\infty^{\mathrm{I}}),$$

and denote by $\Psi^\infty(z,t)$, the fundamental solution of L^∞, meromorphic on $\mathbb{P}^*\times\mathbb{C}$, associated to it by $\Psi^\infty(z,0)=D_0^\infty(z)$. Similarly we denote

$$D_0^{\infty,0}(z)=\psi^\infty(z^{-1};\sigma_\infty^{\mathrm{I}}),$$

and we have the connection result

$$D_0^\infty(z)=D_0^{\infty,0}(z)P^\infty(z),\quad P^\infty(z):=Q(z^{-1};\sigma_\infty^{\mathrm{I}})^{-1}.$$

Analogously, we define

$$\Psi^{\infty}(z,t) = \Psi^{\infty,0}(z,t)P^{\infty}(z). \tag{5.11}$$

From (4.46), we obtain

$$P^{\infty}(qz) = \begin{pmatrix} b_1b_2q^2z^2 & 0 \\ 0 & b_3b_4q^2z^2 \end{pmatrix} \cdot P^{\infty}(z),$$

and hence, for $i = 1, 2$, the component $\Psi_i^{\infty,0}(z,t)$ defines a solution of the following Lax pair $L^{\infty,0,i}$,

$$L_1^{\infty,0,i}: \quad q^2\delta_i z^2\widetilde{u}(z,t)\psi_i^{\infty,0}(qz,t) + \widetilde{v}(z,t)\psi_i^{\infty,0}(z,t) + \frac{1}{\delta_i z^2}\widetilde{w}(z,t)\psi_i^{\infty,0}(z/q,t) = 0, \tag{5.12a}$$

$$L_2^{\infty,0,i}: \quad \beta\frac{1}{z^2t}h_0(z,t)\psi_i^{\infty,0}(z,qt) + h_1(z,t)\psi_i^{\infty,0}(z,t) + \frac{t^2}{\delta_i z^2}h_2(z,t)\psi_i^{\infty,0}(z/q,t) = 0. \tag{5.12b}$$

where $\delta_1 = b_1b_2$ and $\delta_2 = b_3b_4$. We note that $\Psi_i^{\infty,0}(z,t)$ is meromorphic on $\mathbb{C}^* \times \mathbb{C}$, and $\Psi_i^{\infty,0}(z,0) = D_0^{\infty,0,i}(z)$ can be characterised as the unique solution of (4.63) with $z \mapsto z^{-1}$ and parameter values $\sigma = \sigma_\infty^{\mathrm{I}}$, holomorphic at $z = 0$, with $D_0^{\infty,0,i}(0) = 1$.

5.2.3 Fundamental Solution at $(\xi, t) = (0, 0)$

Let us first consider equation (4.99), its coefficients $\gamma_0(t)$, $\gamma_1(t)$ and $\gamma_2(t)$ are of order of magnitude t^2, t^5 and t^8 respectively as $t \to 0$. After some scaling

$$c_0(t) = s^0(t)c_0^0(t), \quad s^0(qt) = t^{-3}s^0(t),$$

where we again invite the reader to choose $s^0(t)$ at their pleasure, the resulting equation for $c_0^0(t)$ is balanced at $t = 0$, and some calculation, best done using equations (3.3), gives the exponents

$$\alpha_1 = \beta\delta_1 = -b_1b_2q^{-1}g_1^{-1}, \quad \alpha_2 = \beta\delta_2 = -b_1b_2q^{-1}g_1^{-1}.$$

Now, if we scale Yamada's Lax pair 2.21 in t as above, it is still unbalanced, as $u(z,t)$, $v(z,t)$ and $w(z,t)$ are of order of magnitude t^{-1}, t^{-3} and t^{-5} respectively. Indeed to overcome the imbalance we should also apply a change of independent variables. For $i = 1, 2$, we set

$$y(z,t) = s_i^0(t)\psi_i^0(\xi,t), \quad s_i^0(qt) = \alpha_i t^{-3}s^0(t), \quad \xi = \frac{z}{t}, \tag{5.13}$$

which gives the Lax pair $L^{0,i}$ given by

$$L_1^{0,i}: \quad u(\xi t,t)\psi_i^0(q\xi,t) + v(\xi t,t)\psi_i^0(\xi,t) + w(\xi t,t)\psi_i^0(\xi/q,t) = 0, \tag{5.14}$$

$$L_2^{0,i}: \quad \alpha_i t^{-3}h_0(\xi t,t)\psi_i^0(\xi/q,qt) + h_1(\xi t,t)\psi_i^0(\xi,t) + h_2(\xi t,t)\psi_i^0(\xi/q,t) = 0. \tag{5.15}$$

One can check that the coefficients in $L_1^{0,i}$, are all analytic at $t = 0$, and have expansions similar to (5.9).

Proposition 5.2.3. *For $i = 1, 2$, there exists an unique solution $\Psi_i^0(\xi,t)$ of $L^{0,i}$, which is* *holomorphic at $(\xi,t)=(0,0)$, with $\Psi_i^0(0,0)=1$. Furthermore $\Psi_i^0(\xi,t)$ enjoys an unique meromorphic continuation to $\mathbb{C}\times\mathbb{C}$.*

Proof. This is proven analogously to Proposition 5.2.1. $\square$

5.2.4 Transition from $(\xi,t)=(0,0)$ to $(\xi,t)=(\infty,0)$

For $i=1,2$, we denote the power series expansion of $\Psi_i^0(\xi,t)$, defined in Proposition 5.2.3, about $(\xi,t)=(0,0)$ by

$$\begin{aligned}\Psi_i^0(\xi,t) &= \sum_{m=0}^{\infty} C_m^{0,i}(t)\xi^m = \sum_{m,n=0}^{\infty} C_{m,n}^{0,i}\xi^m t^n\\ &= \sum_{m=0}^{\infty} D_m^{0,i}(\xi)t^m = \sum_{m,n=0}^{\infty} D_{m,n}^{0,i}\xi^n t^m.\end{aligned}$$

When we let $t\to 0$ in $L_1^{0,i}$, we obtain a second order q-difference equation for $D_0^{0,i}(\xi)$, of which the coefficient $\lim_{t\to 0} v(\xi t,t)$, of $D_0^{0,i}(q\xi)$, is a bit complicated. But if we let $t\to 0$ in $L_1^{0,i}$, and consider the order t^2 terms, we obtain the very simple first order equation

$$D_0^{0,i}(\xi) = \left(\delta_i\xi^2-(\delta_i f_1+g_1)\xi+1\right) D_0^{0,i}(\xi/q).$$

Let's consider $i=1$, then $\delta_1 f_1+g_1=b_1+b_2$, and hence

$$D_0^{0,1}(\xi) = (1-b_1\xi)(1-b_2\xi)D_0^{0,i}(\xi/q). \tag{5.16}$$

As $D_0^{0,i}(0)=1$, we immediately obtain

$$D_0^{0,1}(\xi) = (qb_1\xi, qb_2\xi;q)_\infty^{-1}.$$

Similarly, setting $i=2$, we have

$$D_0^{0,2}(\xi) = (1-b_3\xi)(1-b_4\xi)D_0^{0,2}(\xi/q), \tag{5.17}$$

and as $D_0^{0,2}(0)=1$, we obtain

$$D_0^{0,2}(\xi) = (qb_3\xi, qb_4\xi;q)_\infty^{-1}.$$

The connection problem for equations (5.16) and (5.17), are of course trivial, we set

$$D_0^{0,\infty}(\xi) = (D_0^{0,\infty,1}(\xi), D_0^{0,\infty,2}(\xi)) = \left(\left(\frac{1}{b_1\xi},\frac{1}{b_2\xi};q\right)_\infty, \left(\frac{1}{b_3\xi},\frac{1}{b_4\xi};q\right)_\infty\right),$$

then the connection result reads

$$D_0^{0,\infty}(\xi) = D_0^0(\xi)P^0(\xi), \quad P^0(\xi) = \begin{pmatrix}\theta_q(qb_1\xi,qb_2\xi) & 0\\ 0 & \theta_q(qb_3\xi,qb_4\xi)\end{pmatrix},$$

and we define

$$\Psi^{0,\infty}(\xi,t)=\Psi^0(\xi,t)P^0(\xi). \tag{5.18}$$

Now, for $i=1,2$, the component $\Psi_i^{0,\infty}(\xi,t)$ satisfies the Lax pair $L^{0,\infty,i}$, given by

$$\begin{aligned}
L_1^{0,\infty,i}:&\quad q^2\delta_i\xi^2 u(\xi t,t)\psi_i^0(q\xi,t)+v(\xi t,t)\psi_i^0(\xi,t)+\frac{1}{\delta_i\xi^2}w(\xi t,t)\psi_i^0(\xi/q,t)=0,\\
L_2^{0,\infty,i}:&\quad \frac{\alpha_i}{t^3}h_0(\xi t,t)\psi_i^0(\xi/q,qt)+\delta_i\xi^2h_1(\xi t,t)\psi_i^0(\xi,t)+h_2(\xi t,t)\psi_i^0(\xi/q,t)=0.
\end{aligned}$$

Let us remark that we found that the connection problem on the $t=0$ line in the (ξ,t) plane, is trivial. This is precisely the case because we are considering a very special solution of q-$P(A_1)$. Generically we find the full degree two model equation (4.51), on the $t=0$ line in the (ξ,t) plane, just as we found on the $t=0$ line in the (z,t) plane in Section 5.2.2.

5.2.5 The Matching Procedure

Let us first make the crucial observation that, for $i=1,2$, the Lax pairs $L^{\infty,0,i}$ and $L^{0,\infty,i}$ essentially coincide, under the identification $\xi=\frac{z}{t}$. To be precise, the equations $L_2^{\infty,0,i}$ and $L_2^{0,\infty,i}$ are identical, and the equations $L_1^{\infty,0,i}$ and $L_1^{0,\infty,i}$ are a multiple of each other. Let us focus on the case $i=1$. So both $\Psi_1^{\infty,0}(z,t)$ and $\Psi_1^{0,\infty}(\xi,t)$ are solutions of the Lax pair $L^{0,\infty,1}$. We now wish to match these two solutions at $(\xi,t)=(\infty,0)$. Note however, that both solutions are characterised asymptotically near $(\xi,t)=(\infty,0)$, only on different complex lines with empty intersection. Indeed we only have an asymptotic characterisation of $\Psi_1^{\infty,0}(z,0)$ near $z=0$, and of $\Psi_1^{0,\infty}(\xi,0)$ near $\xi=\infty$. As the connection problem on the $t=0$ line in the (ξ,t) plane was trivial, the matching procedure can be done much simpler than in the generic case. We perform the following trick, we consider

$$\Psi_1^{\mathrm{tr}}(z,t):=\left(qb_1\frac{z}{t},qb_2\frac{z}{t};q\right)_\infty\Psi_1^0\left(\frac{z}{t},t\right), \tag{5.19}$$

which is a solution of the Lax pair L^{tr}, given by

$$\begin{aligned}
L_1^{\mathrm{tr}}:&\quad u^{\mathrm{tr}}(z,t)\psi_1^{\mathrm{tr}}(qz,t)+v^{\mathrm{tr}}(z,t)\psi_1^{\mathrm{tr}}(z,t)+w^{\mathrm{tr}}(z,t)\psi_1^{\mathrm{tr}}(z/q,t)=0,\\
L_2^{\mathrm{tr}}:&\quad \frac{\alpha_1}{t^3}h_0(z,t)\psi_1^{\mathrm{tr}}(z,qt)+(1-b_1\xi)(1-b_2\xi)h_1(z,t)\psi_1^{\mathrm{tr}}(z,t)+h_2(z,t)\psi_1^{\mathrm{tr}}(z/q,t)=0,
\end{aligned}$$

where

$$\begin{aligned}
u^{\mathrm{tr}}(z,t)&=(t-qb_1z)(t-qb_2z)(t-b_1z)(t-b_2z)\,u(z,t),\\
v^{\mathrm{tr}}(z,t)&=t^2(t-b_1z)(t-b_2z)\,v(z,t),\\
w^{\mathrm{tr}}(z,t)&=t^4w(z,t).
\end{aligned}$$

Proposition 5.2.4. *There exists an unique solution $\Psi_1^{tr\prime}(z,t)$ of L^{tr}, which is holomorphic at $(z,t)=(0,0)$, with $\Psi_1^{tr\prime}(0,0)=1$. Furthermore $\Psi_1^{tr\prime}(z,t)$ enjoys an unique meromorphic continuation to $\mathbb{C}\times\mathbb{C}$.*

Proof. The proof is completely analogous to the proof of Proposition 5.2.3. □

We now simply compare $\Psi_1^0(\xi,t)$ and $\Psi_1^{0\prime}(\xi,t)$, where, following (5.19), the latter is defined by

$$\Psi_1^{\mathrm{tr}\prime}(\xi t,t)=(qb_1\xi,qb_2\xi;q)_\infty\,\Psi_1^{0\prime}(\xi,t)\,.$$

Evidently both $\Psi_1^0(\xi,t)$ and $\Psi_1^{0\prime}(\xi,t)$ satisfy the Lax pair $L^{0,1}$, both are meromorphic on $\mathbb{C}\times\mathbb{C}$, and

$$\Psi_1^{0\prime}(\xi,0)=(qb_1\xi,qb_2\xi;q)_\infty^{-1}\,\Psi_1^{\mathrm{tr}\prime}(0,0)=(qb_1\xi,qb_2\xi;q)_\infty^{-1}=\Psi_1^0(\xi,0).$$

In particular $\Psi_1^{0\prime}(0,0)=1$ and, by Proposition 5.2.3,

$$\Psi_1^0(\xi,t)=\Psi_1^{0\prime}(\xi,t)=(qb_1\xi,qb_2\xi;q)_\infty^{-1}\,\Psi_1^{\mathrm{tr}\prime}(\xi t,t). \tag{5.20}$$

Similarly we wish to relate $\Psi_1^{\mathrm{tr}\prime}(z,t)$ to $\Psi_1^{\infty,0}(z,t)$. Well, we simply scale

$$\Psi_1^{\mathrm{tr}*}(z,t)=\left(\frac{t}{b_1z},\frac{t}{b_2z};q\right)_\infty^{-1}\Psi_1^{\infty,0}(z,t), \tag{5.21}$$

then $\Psi_1^{\mathrm{tr}*}(z,t)$ is a solution of L^{tr}, meromorphic on $\mathbb{C}^*\times\mathbb{C}$, and we have

$$\Psi_1^{\mathrm{tr}*}(z,0)=\Psi_1^{\infty,0}(z,0)=D_1^{\infty,0}(z)=\psi_1^\infty(z^{-1};\sigma_\infty^{\mathrm{I}}). \tag{5.22}$$

By comparing order t^{-1} terms in L_1^{tr} as $t\to 0$, we find that $D_0^{\mathrm{tr}\prime}(z):=\Psi_1^{\mathrm{tr}\prime}(z,0)$, satisfies the same second order equation as $\psi_1^\infty(z^{-1};\sigma_\infty^{\mathrm{I}})$, i.e. (4.63) with $z\mapsto z^{-1}$ and parameter values $\sigma=\sigma_\infty^{\mathrm{I}}$. Furthermore $D_0^{\mathrm{tr}\prime}(z)$ is holomorphic at $z=0$ and $D_0^{\mathrm{tr}\prime}(0)=1$, from which we conclude, using (5.22),

$$D_0^{\mathrm{tr}\prime}(z)=\psi_1^\infty(z^{-1};\sigma_\infty^{\mathrm{I}})=\Psi_1^{\mathrm{tr}*}(z,0).$$

So both $\Psi_1^{\mathrm{tr}\prime}(z,t)$ and $\Psi_1^{\mathrm{tr}*}(z,t)$ are solutions of L^{tr}, meromorphic on $\mathbb{C}^*\times\mathbb{C}$, such that

$$\Psi_1^{\mathrm{tr}\prime}(z,0)=\Psi_1^{\mathrm{tr}*}(z,0).$$

By considering L_2^{tr} for different powers of t, we easily find, by a typical induction argument, that all coefficients of powers in t of the two solutions agree, and hence

$$\Psi_1^{\mathrm{tr}\prime}(z,t)=\Psi_1^{\mathrm{tr}*}(z,t). \tag{5.23}$$

Putting everything together, we find

$$\begin{aligned}
\Psi_1^{0,\infty}(\xi,t)&=\Psi_1^0(\xi,t)\theta_q(qb_1\xi,qb_2\xi)\\
&=(qb_1\xi,qb_2\xi;q)_\infty^{-1}\,\Psi_1^{\mathrm{tr}\prime}(\xi t,t)\theta_q(qb_1\xi,qb_2\xi)\\
&=\left(\frac{1}{b_1\xi},\frac{1}{b_2\xi};q\right)_\infty\Psi_1^{\mathrm{tr}\prime}(\xi t,t)\\
&=\left(\frac{1}{b_1\xi},\frac{1}{b_2\xi};q\right)_\infty\Psi_1^{\mathrm{tr}*}(z,t)\\
&=\Psi_1^{\infty,0}(z,t),
\end{aligned}$$

where the second equality follows from (5.20), the fourth equality follows from (5.23), and the last equality follows from (5.21). In a completely analogous fashion we find $\Psi_2^{0,\infty}(\xi,t)=\Psi_2^{\infty,0}(z,t)$ and hence

$$\Psi^{0,\infty}(\xi,t)=\Psi^{\infty,0}(z,t). \tag{5.24}$$

5.2.6 Monodromy Corresponding to Transcendent

In the context of Section 4.9, we can now explicitly write down the monodromy corresponding to the q-$P(A_1)$ transcendent under consideration. From equations (5.18), (5.11) and (5.24) we conclude the following connection result,

$$\begin{aligned}\Psi^\infty(z,t)&=\Psi^0(\xi,t)P^0(\xi,t)P^\infty(z)\\&=\Psi^0(\xi,t)\begin{pmatrix}\theta_q(qb_1\xi,qb_2\xi) & 0\\ 0 & \theta_q(qb_3\xi,qb_4\xi)\end{pmatrix}Q(z^{-1};\sigma_\infty^{\mathrm{I}})^{-1}.\end{aligned}$$

Following (5.5) and (5.13), we define

$$Y^\infty(z,t)=s^\infty(t)\Psi^\infty(z,t),\quad Y^0(z,t)=\Psi^0\left(\frac{z}{t},t\right)\cdot\begin{pmatrix}s_1^0(t) & 0\\ 0 & s_2^0(t)\end{pmatrix},$$

then $Y^\infty(z,t)$ and $Y^0(z,t)$ denote fundamental solutions of respectively (4.82) and (2.21). We conclude

$$Y^\infty(z,t)=Y^0(z,t)\mathcal{P}(z,t),$$

where

$$\mathcal{P}(z,t)=s^\infty(t)\begin{pmatrix}s_1^0(t)^{-1}\theta_q(qb_1\frac{z}{t},qb_2\frac{z}{t}) & 0\\ 0 & s_2^0(t)^{-1}\theta_q(qb_3\frac{z}{t},qb_4\frac{z}{t})\end{pmatrix}Q(z^{-1};\sigma_\infty)^{-1}. \tag{5.25}$$

This is consistent with the notation in Section 4.9, where

$$\begin{aligned}c_0^1(t)&:=s_1^0(t)\Psi_1^0(0,t), & c_0^2(t)&:=s_2^0(t)\Psi_2^0(0,t),\\ \tilde{c}_0^1(t)&:=s^\infty(t)\Psi_1^\infty(\infty,t), & \tilde{c}_0^2(t)&:=s^\infty(t)\Psi_2^\infty(\infty,t).\end{aligned}$$

Proposition 5.2.5. *Consider the solution $(f,g)=(f^{(1,1)},g^{(1,1)})$ of q-$P(A_1)$, meromorphic at $t=0$, defined in Proposition 3.1.2, where we assume the corresponding conditions (3.11) on the parameters. Then the monodromy of Yamada's Lax pair corresponding to this solution, is given by*

$$M_{\mathbb{C}^*}(f,g)=[\mathcal{R}(z,t)],$$

with

$$\mathcal{R}(z,t)=\theta_q(qb_5z,qb_6z,qb_7z,qb_8z)\begin{pmatrix}s_1(t)\theta_q(qb_1\frac{z}{t},qb_2\frac{z}{t}) & 0\\ 0 & s_2(t)\theta_q(qb_3\frac{z}{t},qb_4\frac{z}{t})\end{pmatrix}Q(z^{-1};\sigma_\infty^I)^{-1},$$

where $s_1(t)$ and $s_2(t)$ any nonzero meromorphic functions satisfying $s_1(qt)=\frac{t^2}{b_1b_2}s_1(t)$ and

$s_2(qt)=\frac{t^2}{b_3b_4}s_2(t)$, on $\mathbb{C}^*$, and the parameter set σ_∞^I equal to

$$\sigma_\infty^I=\left(qb_5,qb_6,qb_7,qb_8;\frac{1}{b_1b_2},\frac{1}{b_3b_4}\right). \tag{5.26}$$

Proof. This follows directly from equations (5.25) and (4.109). □

Expanding on the above Proposition, we consider Theorem 4.5.2 with $\sigma=\sigma_\infty^{\mathrm{I}}$, where we assume (4.53) holds. Assumption 4.66, is equivalent to

$$\left|\frac{b_1b_2}{b_5b_6}\right|<1.$$

Let us assume this inequality is indeed valid, then

$$\begin{aligned}Q(z^{-1};\sigma_\infty^{\mathrm{I}})^{-1}&=\widetilde{R}(z^{-1};\sigma_\infty^{\mathrm{I}})C(\sigma_\infty^{\mathrm{I}})^{-1}\\&=\frac{1}{\theta_q(qb_5z,qb_6z)}\begin{pmatrix}r_{11}\theta_q(q\frac{b_8}{b_3b_4}z^{-1}) & r_{12}\theta_q(q\frac{b_7}{b_3b_4}z^{-1})\\ r_{21}\theta_q(q\frac{b_8}{b_1b_2}z^{-1}) & r_{22}\theta_q(q\frac{b_7}{b_1b_2}z^{-1})\end{pmatrix}\begin{pmatrix}\frac{1}{\theta_q(qb_7z)} & 0\\ 0 & \frac{1}{\theta_q(qb_8z)}\end{pmatrix}C(\sigma_\infty^{\mathrm{I}})^{-1},\end{aligned}$$

where $r_{11},r_{12},r_{21},r_{22}$ are defined by (4.70) with $\sigma=\sigma_\infty^{\mathrm{I}}$. Hence, we find that $\mathcal{R}(z,t)$, as defined in Proposition 5.2.5, equals

$$\begin{aligned}\mathcal{R}(z,t)=&\begin{pmatrix}s_1(t)\theta_q(qb_1\frac{z}{t},qb_2\frac{z}{t}) & 0\\ 0 & s_2(t)\theta_q(qb_3\frac{z}{t},qb_4\frac{z}{t})\end{pmatrix}\\&\cdot\begin{pmatrix}r_{11}\theta_q(\frac{b_3b_4}{b_8}z) & r_{12}\theta_q(\frac{b_3b_4}{b_7}z)\\ r_{21}\theta_q(\frac{b_1b_2}{b_8}z) & r_{22}\theta_q(\frac{b_1b_2}{b_7}z)\end{pmatrix}\begin{pmatrix}\theta_q(qb_8z) & 0\\ 0 & \theta_q(qb_7z)\end{pmatrix}C(\sigma_\infty^{\mathrm{I}})^{-1}.\end{aligned}$$

We hence obtain

$$\begin{aligned}[\mathcal{R}(z,t)]=\Bigg[&\begin{pmatrix}s_1(t)\theta_q(qb_1\frac{z}{t},qb_2\frac{z}{t}) & 0\\ 0 & s_2(t)\theta_q(qb_3\frac{z}{t},qb_4\frac{z}{t})\end{pmatrix}\\&\cdot\begin{pmatrix}\widetilde{r}_{11}\theta_q(\frac{b_3b_4}{b_8}z) & \widetilde{r}_{12}\theta_q(\frac{b_3b_4}{b_7}z)\\ \widetilde{r}_{21}\theta_q(\frac{b_1b_2}{b_8}z) & \widetilde{r}_{22}\theta_q(\frac{b_1b_2}{b_7}z)\end{pmatrix}\begin{pmatrix}\theta_q(qb_8z) & 0\\ 0 & \theta_q(qb_7z)\end{pmatrix}\Bigg],\end{aligned}$$

where

$$\begin{aligned}\widetilde{r}_{11}&=\left(\frac{b_3b_4}{b_5b_7},\frac{b_3b_4}{b_6b_7};q\right)_\infty, & \widetilde{r}_{12}&=\left(\frac{b_3b_4}{b_5b_8},\frac{b_3b_4}{b_6b_8};q\right)_\infty,\\ \widetilde{r}_{21}&=\left(\frac{b_1b_2}{b_5b_7},\frac{b_1b_2}{b_6b_7};q\right)_\infty, & \widetilde{r}_{22}&=\left(\frac{b_1b_2}{b_5b_8},\frac{b_1b_2}{b_6b_8};q\right)_\infty.\end{aligned}$$

Let us remark that, by permuting $\{b_1,b_2,b_3,b_4\}$, we easily translate all the results in this section, to the other two solutions, meromorphic at the origin, defined in Proposition 3.1.2.

5.3 Transcendents Meromorphic at a Critical Point

In this section we give the monodromy corresponding to q-$P(A_1)$ transcendents, which are meromorphic at the origin or infinity. The relevant proofs can be found in Appendix D. As in Table 3.1, we distinguish between the solutions defined in Propositions 3.1.2 and 3.1.1, by calling them of type I and type II respectively. In particular we have discussed the type I case in Section 5.2.

Considering the transcendents meromorphic at infinity, listed in equation (3.15), we distinguish between the ones of order of magnitude t^0 and those of order of magnitude t^1 as $t \to \infty$, which we respectively call of type I and of type II.

5.3.1 Transcendents Meromorphic at the Origin of Type II

Proposition 5.3.1. *Consider the solution $(f,g) = (f^{(0,1)}, g^{(0,1)})$ of q-$P(A_1)$, meromorphic at $t=0$, defined in Proposition 3.1.1, where we assume the corresponding conditions (3.6) on the parameters. Then the monodromy of Yamada's Lax pair corresponding to this solution, is given by*

$$M_{\mathbb{C}^*}(f,g) = [\mathcal{R}(z,t)]\,,$$

with

$$\mathcal{R}(z,t) = Q\left(\frac{z}{t};\sigma_0^{II}\right)\begin{pmatrix}\theta_q(qb_7z,qb_8z) & 0\\ 0 & \theta_q(qb_5z,qb_6z)\end{pmatrix}\begin{pmatrix}s_1(t) & 0\\ 0 & s_2(t)\end{pmatrix},$$

where $s_1(t)$ and $s_2(t)$ any nonzero meromorphic functions satisfying $s_1(qt) = \frac{t^2}{b_5b_6}s_1(t)$ and $s_2(qt) = \frac{t^2}{b_7b_8}s_2(t)$, on $\mathbb{C}^*$, and the parameter set σ_0^{II} equals

$$\sigma_0^{II} = \left(b_1^{-1}, b_2^{-1}, b_3^{-1}, b_4^{-1}; q^2b_5b_6, q^2b_7b_8\right). \tag{5.27}$$

Proof. This follows directly from equations (D.2) and (4.109). □

Expanding on the above Proposition, we consider Theorem 4.5.2 with $\sigma = \sigma_0^{\mathrm{II}}$, where we assume (4.53) holds. Assumption 4.66, is equivalent to

$$\left|\frac{b_1b_2}{b_5b_6}\right| < 1.$$

Let us assume this inequality is indeed valid, then we obtain

$$[\mathcal{R}(z,t)] = \Bigg[\begin{pmatrix}\theta_q(b_3\frac{z}{t}) & 0\\ 0 & \theta_q(b_4\frac{z}{t})\end{pmatrix}\begin{pmatrix}\widetilde{q}_{11}\theta_q(q^2\frac{b_5b_6}{b_3}\frac{z}{t}) & \widetilde{q}_{12}\theta_q(q^2\frac{b_7b_8}{b_3}\frac{z}{t})\\ \widetilde{q}_{21}\theta_q(q^2\frac{b_5b_6}{b_4}\frac{z}{t}) & \widetilde{q}_{22}\theta_q(q^2\frac{b_7b_8}{b_4}\frac{z}{t})\end{pmatrix}$$
$$\cdot\begin{pmatrix}s_1(t)\theta_q(qb_7z,qb_8z) & 0\\ 0 & s_2(t)\theta_q(qb_5z,qb_6z)\end{pmatrix}\Bigg],$$

where

$$\widetilde{q}_{11} = \left(\frac{b_1b_4}{b_5b_6}, \frac{b_2b_4}{b_5b_6}; q\right)_\infty, \qquad \widetilde{q}_{12} = -\left(\frac{b_1b_4}{b_7b_8}, \frac{b_2b_4}{b_7b_8}; q\right)_\infty,$$
$$\widetilde{q}_{21} = -\left(\frac{b_1b_3}{b_5b_6}, \frac{b_2b_3}{b_5b_6}; q\right)_\infty, \qquad \widetilde{q}_{22} = \left(\frac{b_1b_3}{b_7b_8}, \frac{b_2b_3}{b_7b_8}; q\right)_\infty.$$

Let us remark that, by permuting $\{b_5, b_6, b_7, b_8\}$, we easily translate all the results in this section, to the other two solutions, meromorphic at the origin, defined in Proposition 3.1.1.

5.3.2 Transcendents Meromorphic at Infinity of Type I

Proposition 5.3.2. *Consider the solution $(f, g) = (\check{f}^{(0,1)}, \check{g}^{(0,1)})$, meromorphic at $t = \infty$, defined in Equation (3.15), where we assume the corresponding conditions (3.6) on the parameters. Then the monodromy of Yamada's Lax pair corresponding to this solution, is given by*

$$M_{\mathbb{C}^*}(f, g) = \left[\widehat{\mathcal{R}}(z, t)\right],$$

with

$$\widehat{\mathcal{R}}(z,t) = \theta_q\left(qb_1\frac{z}{t}, qb_2\frac{z}{t}, qb_3\frac{z}{t}, qb_4\frac{z}{t}\right)\begin{pmatrix}\widehat{s}_1(t)\theta_q(qb_7z, qb_8z) & 0\\ 0 & \widehat{s}_2(t)\theta_q(qb_5z, qb_6z)\end{pmatrix} Q\left(\frac{t}{z}; \widehat{\sigma}_\infty^I\right)^{-1},$$

where $\widehat{s}_1(t)$ and $\widehat{s}_2(t)$ any nonzero meromorphic functions satisfying $\widehat{s}_1(qt) = \frac{t^2}{b_5b_6}\widehat{s}_1(t)$ and $\widehat{s}_2(qt) = \frac{t^2}{b_7b_8}\widehat{s}_2(t)$, on $\mathbb{C}^*$, and the parameter set $\widehat{\sigma}_\infty^I$ equals

$$\widehat{\sigma}_\infty^I = \left(qb_3, qb_4, qb_1, qb_2; \frac{1}{qb_7b_8}, \frac{1}{qb_5b_6}\right). \tag{5.28}$$

Proof. This follows directly from equations (5.25) and (4.109). □

Expanding on the above Proposition, we consider Theorem 4.5.2 with $\sigma = \widehat{\sigma}_\infty^{\mathrm{I}}$, where we assume (4.53) holds. Assumption 4.66, is equivalent to

$$\left|\frac{b_1b_2}{b_5b_6}\right| < 1.$$

Let us assume this inequality is indeed valid, then we obtain

$$\left[\widehat{\mathcal{R}}(z,t)\right] = \left[\begin{pmatrix}\widehat{s}_1(t)\theta_q(qb_7z, qb_8z) & 0\\ 0 & \widehat{s}_2(t)\theta_q(qb_5z, qb_6z)\end{pmatrix}\begin{pmatrix}\widetilde{r}_{11}\theta_q(q\frac{b_5b_6}{b_2}\frac{z}{t}) & \widetilde{r}_{12}\theta_q(q\frac{b_5b_6}{b_1}\frac{z}{t})\\ \widetilde{r}_{21}\theta_q(q\frac{b_7b_8}{b_2}\frac{z}{t}) & \widetilde{r}_{22}\theta_q(q\frac{b_7b_8}{b_1}\frac{z}{t})\end{pmatrix}\right.$$
$$\left.\cdot\begin{pmatrix}\theta_q\left(b_2\frac{z}{t}\right) & 0\\ 0 & \theta_q\left(b_1\frac{z}{t}\right)\end{pmatrix}\right],$$

where

$$\widetilde{r}_{11} = \left(\frac{b_2b_4}{b_7b_8}, \frac{b_2b_3}{b_7b_8}; q\right)_\infty, \qquad \widetilde{r}_{12} = \left(\frac{b_1b_4}{b_7b_8}, \frac{b_1b_3}{b_7b_8}; q\right)_\infty,$$
$$\widetilde{r}_{21} = \left(\frac{b_2b_4}{b_5b_6}, \frac{b_2b_3}{b_5b_6}; q\right)_\infty, \qquad \widetilde{r}_{22} = \left(\frac{b_1b_4}{b_5b_6}, \frac{b_1b_3}{b_5b_6}; q\right)_\infty.$$

Let us remark that, by permuting $\{b_5, b_6, b_7, b_8\}$, we easily translate all the results in this section, to the other two solutions of type I, defined in (3.15).

5.3.3 Transcendents Meromorphic at Infinity of Type II

Proposition 5.3.3. *Consider the solution* $(f,g) = (\check{f}^{(1,1)}, \check{g}^{(1,1)})$, *meromorphic at* $t = \infty$, *defined in Equation (3.15), where we assume the corresponding conditions (3.11) on the parameters. Then the monodromy of Yamada's Lax pair corresponding to this solution, is given by*

$$M_{\mathbb{C}^*}(f,g) = \left[\widehat{\mathcal{R}}(z,t)\right],$$

with

$$\widehat{\mathcal{R}}(z,t) = Q\left(z; \widehat{\sigma}_0^{II}\right) \begin{pmatrix} \widehat{s}_1(t)\theta_q\left(qb_1\frac{z}{t}, qb_2\frac{z}{t}\right) & 0 \\ 0 & \widehat{s}_2(t)\theta_q\left(qb_3\frac{z}{t}, qb_4\frac{z}{t}\right) \end{pmatrix},$$

where $\widehat{s}_1(t)$ *and* $\widehat{s}_2(t)$ *any nonzero meromorphic functions satisfying* $\widehat{s}_1(qt) = \frac{t^2}{b_1b_2}\widehat{s}_1(t)$ *and* $\widehat{s}_2(qt) = \frac{t^2}{b_3b_4}\widehat{s}_2(t)$, *on* $\mathbb{C}^*$, *and the parameter set* $\widehat{\sigma}_0^{II}$ *equals*

$$\widehat{\sigma}_0^{II} = \left(b_7^{-1}, b_8^{-1}, b_5^{-1}, b_6^{-1}; qb_3b_4, qb_1b_2\right). \tag{5.29}$$

Proof. This follows directly from equations (5.25) and (4.109). □

Expanding on the above Proposition we consider Theorem 4.5.2 with $\sigma = \widehat{\sigma}_0^{\mathrm{II}}$, where we assume (4.53) holds. Assumption 4.66, is equivalent to

$$\left|\frac{b_1b_2}{b_5b_6}\right| < 1.$$

Let us assume this inequality is indeed valid, then we obtain

$$\left[\widehat{\mathcal{R}}(z,t)\right] = \Bigg[\begin{pmatrix} \theta_q(b_5z) & 0 \\ 0 & \theta_q(b_6z) \end{pmatrix} \begin{pmatrix} \widetilde{q}_{11}\theta_q(q\frac{b_3b_4}{b_5}z) & \widetilde{q}_{12}\theta_q(q\frac{b_1b_2}{b_5}z) \\ \widetilde{q}_{21}\theta_q(q\frac{b_3b_4}{b_6}z) & \widetilde{q}_{22}\theta_q(q\frac{b_1b_2}{b_6}z) \end{pmatrix}$$
$$\cdot \begin{pmatrix} \widehat{s}_1(t)\theta_q\left(qb_1\frac{z}{t}, qb_2\frac{z}{t}\right) & 0 \\ 0 & \widehat{s}_2(t)\theta_q\left(qb_3\frac{z}{t}, qb_4\frac{z}{t}\right) \end{pmatrix} \Bigg],$$

where

$$\widetilde{q}_{11}=\left(\frac{b_1b_2}{b_5b_8},\frac{b_1b_2}{b_5b_7};q\right)_\infty, \qquad \widetilde{q}_{12}=-\left(\frac{b_3b_4}{b_5b_8},\frac{b_3b_4}{b_5b_7};q\right)_\infty,$$
$$\widetilde{q}_{21}=-\left(\frac{b_1b_2}{b_6b_8},\frac{b_1b_2}{b_6b_7};q\right)_\infty, \qquad \widetilde{q}_{22}=\left(\frac{b_3b_4}{b_6b_8},\frac{b_3b_4}{b_6b_7};q\right)_\infty.$$

Let us remark that, by permuting $\{b_1,b_2,b_3,b_4\}$, we easily translate all the results in this section, to the other two solutions of type II, defined in (3.15).

5.4 Generic Case: Analysis near $(z,t)=(\infty,0)$

We wish to calculate the monodromy corresponding to solutions of q-$P(A_1)$ with critical behaviour near $t=0$ as described in Theorem 3.4.1. In fact it is often easier to work with the formal expansion in Theorem 3.3.1 and we hence do most of the analysis on a formal level. We lighten the notation of the formal series solution a bit by writing the formal solution (3.44) as $f=f(t,\phi;\Lambda)$ and $g=g(t,\phi;\Lambda)$, with

$$f=\sum_{n=1}^{\infty}F_nt^n, \quad g=\sum_{n=1}^{\infty}G_nt^n,$$

where for $n\in\mathbb{N}^*$, the coefficients $F_n=F_n(\phi)=F_n(\phi;\Lambda)$ and $G_n=G_n(\phi)=G_n(\phi;\Lambda)$ are defined by

$$F_n=\sum_{i=-\infty}^{n}F_{n,i}\phi^i, \quad G_n=\sum_{i=-\infty}^{n}G_{n,i}\phi^i,$$

with for $i\leq n$, the coefficients $F_{n,i}=F_{n,i}(\Lambda)$ and $G_{n,i}=G_{n,i}(\Lambda)$ equal to

$$F_{n,i}(\Lambda)=F_{n,i}^{0,+}(\Lambda,\mathbf{b}), \quad G_{n,i}(\Lambda)=G_{n,i}^{0,+}(\Lambda,\mathbf{b}).$$

Analogously to (5.2) and (5.5), we rescale the Lax pair $\widetilde{L}$ (4.82), by setting

$$\widetilde{y}(z,t,\phi;\Lambda)=s^\infty(t,\phi,\Lambda)\psi^\infty(z,t,\phi;\Lambda), \quad s^\infty(qt,\lambda\phi,\Lambda)=\beta t^{-1}s^\infty(t,\phi,\Lambda), \tag{5.30}$$

with

$$\beta:=-q^{-1}G_1(\phi;\Lambda)^{-1}, \tag{5.31}$$

where we invite the reader to choose $s^\infty(t,\phi,\Lambda)$, meromorphic on $\mathbb{C}^*\times\mathbb{C}^*\times\mathbb{C}^*$, at their pleasure. The rescaled Lax pair for $\psi^\infty(z,t,\phi)$ takes the form (5.6), which we again denote by L^∞. We sometimes suppress the explicit Λ dependence of formulas, i.e. $\widetilde{y}(z,t,\phi)=\widetilde{y}(z,t,\phi;\Lambda)$, to ease the notation.

5.4.1 Expanding about $z=\infty$

Expanding $\psi^\infty(z,t,\phi)$ in z,

$$\psi^\infty(z,t,\phi;\Lambda)=c_0^\infty(t,\phi;\Lambda)+c_1^\infty(t,\phi;\Lambda)z^{-1}+c_2^\infty(t,\phi;\Lambda)z^{-2}+\dots, \tag{5.32}$$

we have $\widetilde{c}_0 = s^\infty c_0^\infty$, and hence, by equation (4.105), we have

$$\gamma_0^\infty(t,\phi;\Lambda)c_0^\infty(t,\phi;\Lambda)+\gamma_1^\infty(t,\phi;\Lambda)c_0^\infty(qt,\lambda\phi;\Lambda)+\gamma_2^\infty(t,\phi;\Lambda)c_0^\infty(q^2t,\lambda^2\phi;\Lambda)=0, \tag{5.33}$$

where

$$(\gamma_0^\infty,\gamma_1^\infty,\gamma_2^\infty)=\frac{1}{b_1b_2b_3b_4(F_1G_1-1)t^2}(\widetilde{\gamma}_0,\beta t^{-1}\widetilde{\gamma}_1,q^{-1}\beta\overline{\beta}t^{-2}\widetilde{\gamma}_2).$$

Expanding $c_0^\infty(t,\phi)$ in t as

$$c_0^\infty(t,\phi;\Lambda)=c_{0,0}^\infty(\phi;\Lambda)+c_{0,1}^\infty(\phi;\Lambda)t+c_{0,2}^\infty(\phi;\Lambda)t^2+\dots, \tag{5.34}$$

and substituting into (5.33) gives, by comparing coefficients of t^0, the following linear second order difference equation for $c_{0,0}^\infty(\phi)$,

$$q\overline{G}_1c_{0,0}^\infty(\phi)-(G_1+q\overline{G}_1)c_{0,0}^\infty(\lambda\phi)+G_1c_{0,0}^\infty(\lambda^2\phi)=0, \tag{5.35}$$

where we suppressed the ϕ dependence of G_1 and $\overline{G}_1$. Analogously to equation (5.7), we have

$$c_1^\infty(t,\phi)=\frac{t^2-1}{(q-1)g(t,\phi)}c_0^\infty(t,\phi)+\frac{1}{q-1}G_1(\phi)^{-1}t^{-1}c_0^\infty(qt,\lambda\phi). \tag{5.36}$$

We wish $c_1^\infty(t,\phi)$ to have a power series expansion in t about $t=0$, which, considering (5.36), requires $c_{0,0}^\infty(\lambda\phi)=c_{0,0}^\infty(\phi)$. Note that the latter is compatible with (5.35), indeed

$$c_{0,0}^\infty(\phi)=c_{0,0}^\infty(\phi;\Lambda)=k_1(\Lambda), \tag{5.37}$$

satisfies (5.35), for any $k_1(\Lambda)$. Substituting expansion (5.34) into (5.33) gives, by comparing coefficients of t, the following linear second order difference equation for $c_{0,1}^\infty(\phi)$,

$$\overline{G}_1c_{0,1}^\infty(\phi)-(G_1+q\overline{G}_1)c_{0,1}^\infty(\lambda\phi)+qG_1c_{0,1}^\infty(\lambda^2\phi)=k_1(\Lambda)\left[\frac{G_2\overline{G}_1}{G_1}-\frac{\overline{G}_2G_1}{\overline{G}_1}\right]. \tag{5.38}$$

Note that the ϕ^2 terms on the right-hand side of this equation cancel and indeed, there exists an unique formal series $\nu(\phi;\Lambda)$, of the form

$$\nu(\phi;\Lambda)=\sum_{n=-\infty}^{0}\nu_n(\Lambda)\phi^n,$$

such that $c_{0,1}^\infty(\phi;\Lambda)=\nu(\phi;\Lambda)$ defines a solution of (5.38) with $k_1(\Lambda)$ replaced by 1. Furthermore,

$$\nu_h(\phi;\Lambda)=\phi+\frac{q\lambda-1}{\Lambda(q-1)}G_{1,0}(\Lambda)+\frac{\lambda(q\lambda-1)}{\Lambda(q-\lambda)}G_{1,-1}(\Lambda)\phi^{-1},$$

defines a solution of the homogeneous part of (5.38), so

$$c_{0,1}^\infty(\phi;\Lambda):=k_1(\Lambda)\nu(\phi;\Lambda)+k_2(\Lambda)\nu_h(\phi;\Lambda), \tag{5.39}$$

defines a solution to (5.38) for any $k_2(\Lambda)$. The following proposition tells us that, after fixing particular values for $k_1(\Lambda)$ and $k_2(\Lambda)$, there exists an unique corresponding formal series solution of (5.33).

Proposition 5.4.1. *Consider equation (5.33) with $f=f^{0,+}(t,\phi;\Lambda,\mathbf{b})$ and $g=g^{0,+}(t,\phi;\Lambda,\mathbf{b})$ as defined in Theorem 3.3.1. Then, for $i\in\{1,2\}$, there exists an unique formal solution of equation (5.33), of the form*

$$c_0^{\infty,i}(t,\phi;\Lambda)=\sum_{m=0}^{\infty}c_{0,m}^{\infty,i}(\phi;\Lambda)t^m, \tag{5.40}$$

with, for $m\in\mathbb{N}$,

$$c_{0,m}^{\infty,i}(\phi;\Lambda)=\sum_{n=-\infty}^{m}c_{0,m,n}^{\infty,i}(\Lambda)\phi^n, \tag{5.41}$$

where

$$\begin{pmatrix}c_{0,0,0}^{\infty,1}(\Lambda)\\ c_{0,1,1}^{\infty,1}(\Lambda)\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix},\quad \begin{pmatrix}c_{0,0,0}^{\infty,2}(\Lambda)\\ c_{0,1,1}^{\infty,2}(\Lambda)\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}. \tag{5.42}$$

For $m\in\mathbb{N}$ and $n\in\mathbb{Z}_{\leq m}$ the coefficients $c_{0,m,n}^{\infty,i}(\Lambda)$ are rational functions in Λ and the parameters $b_1,\ldots,b_8$, in particular these rational functions are regular at points $(\Lambda,\mathbf{b})\in\mathbb{C}^*\times\mathcal{B}$, satisfying (3.46). Furthermore, for fixed $\mathbf{b}\in\mathcal{B}$ with $|q|<1$, for any $\Lambda\in L_0(\mathbf{b})$, condition (3.46) is satisfied and this formal solution, written in terms of the variables $\zeta_1=t\phi$ and $\zeta_2=\phi^{-1}$,

$$c_0^{\infty,i}\left(\zeta_1\zeta_2,\zeta_2^{-1};\Lambda\right)=\sum_{m,n=0}^{\infty}c_{0,m,m-n}^{\infty,i}(\Lambda)\zeta_1^m\zeta_2^n, \tag{5.43}$$

converges near $(\zeta_1,\zeta_2)=(0,0)$.
In fact, these expansions are also analytic in Λ. That is, for any $L\subseteq L_0(\mathbf{b})$ open with $\overline{L}\subseteq L_0(\mathbf{b})$, there is an open environment $Z\subseteq\mathbb{C}^2$ of $\mathbf{0}$, such that the series (5.43) converge uniformly on $Z\times L$, defining holomorphic functions on this set in $(\boldsymbol{\zeta},\Lambda)$.

Proof. Note that within the context of equations (5.37) and (5.39), the initial conditions (5.42) correspond to the choices $(k_1(\Lambda),k_2(\Lambda))=(1,0)$ and $(k_1(\Lambda),k_2(\Lambda))=(0,1)$ respectively. For any of the two choices, we can prove the proposition similar to Theorem 3.3.1. We rewrite equation (5.33) in appropriate system form and apply the q-Briot Bouquet Theorems B.3 and B.4. $\square$

Remark 5.4.2. Recall that we used the formal series solution in Theorem 3.3.1, to construct true solutions of q-$P(A_1)$ in Theorem 3.4.1, by replacing the formal variables by actual analytic functions. Doing so we can use the formal series solutions in Proposition 5.4.1 to construct corresponding true solutions of (5.33).

5.4.2 Expanding about $t=0$

We expand the solution $\psi^{\infty}(z,t,\phi;\Lambda)$ in t about $t=0$,

$$\psi^{\infty}(z,t,\phi;\Lambda)=d_0^{\infty}(z,\phi;\Lambda)+d_1^{\infty}(z,\phi;\Lambda)t+d_2^{\infty}(z,\phi;\Lambda)t^2+\ldots. \tag{5.44}$$

Substitution in L_2^∞ and comparing the coefficients of terms t^2 gives

$$d_0^\infty(z,\lambda\phi;\Lambda)=d_0^\infty(z,\phi;\Lambda), \tag{5.45}$$

and we hence set

$$d_0^\infty(z,\phi;\Lambda)=d_{0,0}^\infty(z;\Lambda).$$

Similarly substitution of (5.44) in L_1^∞ gives

$$\begin{aligned} d_{0,0}^\infty\left(z/q;\Lambda\right)+\left[-(1+q)+\left(b_5^{-1}+b_6^{-1}+b_7^{-1}+b_8^{-1}\right)z^{-1}+\left(\Lambda^{-1}+\frac{\lambda}{\Lambda}\right)z^{-2}\right]d_{0,0}^\infty(z;\Lambda)\\ +q\left(1-\frac{1}{qb_5z}\right)\left(1-\frac{1}{qb_6z}\right)\left(1-\frac{1}{qb_7z}\right)\left(1-\frac{1}{qb_8z}\right)d_{0,0}^\infty(qz;\Lambda)=0. \end{aligned} \tag{5.46}$$

This is exactly the degree two model equation (4.51) under the identification $y(z;\Lambda)=d_{0,0}^\infty\left(1/z;\Lambda\right)$, with parameter values $\sigma=\sigma_\infty(\Lambda)$, as defined in (5.108), where we note that Fuchs' equation (4.52) is indeed satisfied. We remark that this is consistent with and generalises equation (5.10), as for the choices $\Lambda=\Lambda_1^\pm$, equation (5.46) reduces to (5.10), as expected from Proposition 3.5.1.

For any $l_1(\Lambda),l_2(\Lambda)$, there exists an unique formal power series solution $d_{0,0}^\infty(z;\Lambda)$ in z about $z=\infty$, of equation (5.46) with

$$d_{0,0}^\infty(z;\Lambda)=l_1(\Lambda)+l_2(\Lambda)z^{-1}+\dots. \tag{5.47}$$

Now, by comparing coefficients of z^0t^3 in L_2^∞, we find

$$qc_{0,1}^\infty(\lambda\phi;\Lambda)-c_{0,1}^\infty(\phi;\Lambda)=l_2(\Lambda)(q-1)G_1-k_1(\Lambda)\frac{G_2}{G_1},$$

which, one can easily check, is indeed consistent with (5.38).
By comparing the coefficients of ϕ^1 of both sides of this equation and using (5.39), we find

$$\begin{aligned}(q\lambda-1)k_2(\Lambda)&=(q-1)\Lambda l_2(\Lambda)-k_1(\Lambda)\frac{G_{2,2}(\Lambda)}{\Lambda}\\ &=(q-1)\Lambda l_2(\Lambda)+k_1(\Lambda)G_{eq}\left(\Lambda^{-1},\mathbf{b}^{(1)}\right),\end{aligned}$$

where in the second equality we used

$$G_{2,2}(\Lambda)=-\Lambda G_{eq}\left(\Lambda^{-1},\mathbf{b}^{(1)}\right),$$

which follows directly from (3.72). We conclude

$$k_1(\Lambda)=l_1(\Lambda), \tag{5.48a}$$

$$k_2(\Lambda)=\frac{1}{q\lambda-1}G_{eq}\left(\Lambda^{-1},\mathbf{b}^{(1)}\right)l_1(\Lambda)+\frac{q-1}{q\lambda-1}\Lambda l_2(\Lambda). \tag{5.48b}$$

5.4.3 Main Existence Theorem near $(z,t)=(\infty,0)$

Theorem 5.4.3. *Consider the Lax pair L^∞ (5.6) with $f=f^{0,+}(t,\phi;\Lambda,\mathbf{b})$ and $g=g^{0,+}(t,\phi;\Lambda,\mathbf{b})$ as described in Theorem 3.3.1, and β defined as in (5.31). Then, for $i\in\{1,2\}$, there exists an unique formal series solution of the Lax pair L^∞, of the form*

$$\psi_i^\infty(z,t,\phi;\Lambda)=\sum_{k=0}^{\infty}c_k^{\infty,i}(t,\phi;\Lambda)z^{-k},$$

where, for $k\in\mathbb{N}$,

$$c_k^{\infty,i}(t,\phi;\Lambda)=\sum_{m=0}^{\infty}c_{k,m}^{\infty,i}(\phi;\Lambda)t^m,$$

with, for $m\in\mathbb{N}$,

$$c_{k,m}^{\infty,i}(\phi;\Lambda)=\sum_{n=-\infty}^{m}c_{k,m,n}^{\infty,i}(\Lambda)\phi^n,$$

and initial conditions (5.42).
We note that the notation here coincides with that in Proposition 5.4.1. For $k,m\in\mathbb{N}$ and $n\in\mathbb{Z}_{\leq m}$, the coefficients $c_{k,m,n}^{\infty,i}(\Lambda)$ are rational functions in Λ and the parameters $b_1,\dots,b_8$, in particular these rational functions are regular at points $(\Lambda,\mathbf{b})\in\mathbb{C}^*\times\mathcal{B}$, satisfying (3.46). Furthermore, for fixed $\mathbf{b}\in\mathcal{B}$ with $|q|<1$, for any $\Lambda\in L_0(\mathbf{b})$, condition (3.46) is satisfied and this formal solution, written in terms of the variables z, $\zeta_1=t\phi$ and $\zeta_2=\phi^{-1}$,

$$\psi_i^\infty\left(z,\zeta_1\zeta_2,\zeta_2^{-1};\Lambda\right)=\sum_{k,m,n=0}^{\infty}c_{k,m,m-n}^{\infty,i}(\Lambda)z^{-k}\zeta_1^m\zeta_2^n, \tag{5.49}$$

converges near $(z,\zeta_1,\zeta_2)=(\infty,0,0)$.
In fact, this expansion also depends holomorphically on Λ. That is, for any $L\subseteq L_0(\mathbf{b})$ open with $\overline{L}\subseteq L_0(\mathbf{b})$, there is an open environment $Z\subseteq\mathbb{P}^*\times\mathbb{C}^2$ of $(\infty,0,0)$, such that the series (5.49) converge uniformly on $Z\times L$, defining holomorphic functions on this set in $(z,\zeta_1,\zeta_2,\Lambda)$.

Proof. The proof follows the same lines as the proof of Proposition 5.2.1. We only give a sketch, not to bore the reader with all the analytic details. Considering the case $i=1$, we set $c_0^\infty(t,\phi;\Lambda):=c_0^{\infty,i}(t,\phi;\Lambda)$, as defined in Proposition 5.4.1, in the expansion (5.32). We define $c_1^\infty(t,\phi;\Lambda)$ by equation (5.36). Then, analogously to Claim 5.2.2, we prove, using the q-Briot Bouquet Theorem B.3 and Remark B.5, that there exists an unique formal solution $\psi_1^\infty(z,t,\phi;\Lambda)$ of L_1^∞, as described in the Theorem. It remains to prove that $\psi_1^\infty(z,t,\phi;\Lambda)$ also satisfies L_2^∞, which we establish similarly to the final part of the proof of Proposition 5.2.1. □

Remark 5.4.4. Recall that we used the formal series solution in Theorem 3.3.1, to construct true solutions of q-$P(A_1)$ in Theorem 3.4.1, by replacing the formal variables by actual analytic functions. Doing so we can use the formal series solutions in Theorem 5.4.3, to construct corresponding true solutions of the Lax pair L^∞.

5.4.4 Transition from $(z, t) = (\infty, 0)$ to $(z, t) = (0, 0)$

Note that the two formal series solutions $\psi_1^\infty(z, t, \phi; \Lambda)$ and $\psi_2^\infty(z, t, \phi; \Lambda)$, define a basis of solutions of L^∞. It is more convenient for us to work with a different basis of solutions, given by

$$\Psi_1^\infty(z, t, \phi; \Lambda) := 1 \cdot \psi_1^\infty(z, t, \phi; \Lambda) + \frac{1}{q\lambda - 1} G_{eq}\left(\Lambda^{-1}, \mathbf{b}^{(1)}\right) \psi_2^\infty(z, t, \phi; \Lambda), \tag{5.50a}$$

$$\Psi_2^\infty(z, t, \phi; \Lambda) := 0 \cdot \psi_1^\infty(z, t, \phi; \Lambda) + \frac{q-1}{q\lambda - 1} \Lambda \psi_2^\infty(z, t, \phi; \Lambda). \tag{5.50b}$$

By equations (5.48), these two solutions correspond respectively to $(l_1(\Lambda), l_2(\Lambda)) = (1, 0)$ and $(l_1(\Lambda), l_2(\Lambda)) = (0, 1)$ in (5.47). We write, for $i = 1, 2$,

$$\Psi_i^\infty(z, t, \phi; \Lambda) = \sum_{k=0}^{\infty} D_k^{\infty,i}(z, \phi; \Lambda) t^k, \tag{5.51}$$

where for $k \in \mathbb{N}$,

$$D_k^{\infty,i}(z, \phi; \Lambda) = \sum_{m=-\infty}^{k} D_{k,m}^{\infty,i}(z; \Lambda) \phi^m,$$

with for $m \in \mathbb{Z}_{\leq k}$,

$$D_{k,m}^{\infty,i}(z; \Lambda) = \sum_{n=0}^{\infty} D_{k,m,n}^{\infty,i}(\Lambda) z^{-n}.$$

Note that, by equation (5.45),

$$D_0^\infty(z, \phi; \Lambda) = D_{0,0}^\infty(z; \Lambda),$$

and $D_{0,0}^\infty(z; \Lambda)$ denotes the fundamental solution at $z = \infty$ of equation (5.46), given by

$$D_{0,0}^\infty(z; \Lambda) = y^0(z^{-1}; \sigma_\infty(\Lambda)),$$

where we used the notation in Section 4.5.

As in Section 5.2.2, we denote

$$D_{0,0}^{\infty,0}(z; \Lambda) = \psi^\infty(z^{-1}; \sigma_\infty(\Lambda)),$$

and we have the connection result

$$D_{0,0}^\infty(z; \Lambda) = D_{0,0}^{\infty,0}(z; \Lambda) P^\infty(z; \Lambda), \quad P^\infty(z; \Lambda) := Q(z^{-1}; \sigma_\infty(\Lambda))^{-1}. \tag{5.52}$$

Analogously to (5.11), we symbolically define

$$\Psi^{\infty,0}(z, t, \phi; \Lambda) := \Psi^\infty(z, t, \phi; \Lambda) \cdot P^\infty(z; \Lambda)^{-1}. \tag{5.53}$$

From (4.46), we obtain

$$P^\infty(qz;\Lambda) = \begin{pmatrix} -q^2\Lambda z^2 & 0 \\ 0 & -q^2\Lambda/\lambda z^2 \end{pmatrix} \cdot P^\infty(z;\Lambda),$$

and hence, for $i = 1, 2$, a symbolic computation, shows that the component $\Psi_i^{\infty,0}(z, t, \phi; \Lambda)$ defines a solution of the Lax pair $L^{\infty,0,i}$ (5.12), with $\delta_1 = -\Lambda$ and $\delta_2 = -\Lambda/\lambda$, and β as defined in (5.31).

5.5 Generic Case: Analysis near $(\xi, t) = (0, 0)$

Analogously to (5.13), we rescale the Lax pair L (2.21), by

$$y(z, t, \phi; \Lambda) = s^0(t, \phi; \Lambda)\psi^0(\xi, t, \phi; \Lambda), \quad s^0(qt, \lambda\phi; \Lambda) = \alpha t^{-3} s^0(t, \phi; \Lambda), \quad \xi = \frac{z}{t}, \tag{5.54}$$

where

$$\alpha = \Lambda q^{-1} G_1(\phi; \Lambda)^{-1},$$

which leads to the Lax pair L^0, given by

$$L_1^0: \quad u(\xi t, t)\psi^0(q\xi, t) + v(\xi t, t)\psi^0(\xi, t) + w(\xi t, t)\psi^0(\xi/q, t) = 0, \tag{5.55a}$$

$$L_2^0: \quad \alpha t^{-3} h_0(\xi t, t)\psi^0(\xi/q, qt) + h_1(\xi t, t)\psi^0(\xi, t) + h_2(\xi t, t)\psi^0(\xi/q, t) = 0, \tag{5.55b}$$

where we suppressed ϕ and Λ dependence throughout.
We invite the reader to choose an appropriate $s^0(t, \phi; \Lambda)$, meromorphic on $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$, at their pleasure.

5.5.1 Expanding about $\xi = 0$

We expand $\psi^0(\xi, t, \phi; \Lambda)$ about $\xi = 0$,

$$\psi^0(\xi, t, \phi; \Lambda) = c_0^0(t, \phi; \Lambda) + c_1^0(t, \phi; \Lambda)\xi + c_2^0(t, \phi; \Lambda)\xi^2 + \ldots.$$

We have $c_0 = s^0 c_0^0$, and hence, by (4.99),

$$\gamma_0^0(t, \phi; \Lambda)c_0^0(t, \phi; \Lambda) + \gamma_1^0(t, \phi; \Lambda)c_0^0(qt, \lambda\phi; \Lambda) + \gamma_2^0(t, \phi; \Lambda)c_0^0(q^2 t, \lambda^2\phi; \Lambda) = 0, \tag{5.56}$$

where

$$(\gamma_0^0, \gamma_1^0, \gamma_2^0) = t^{-2}(\gamma_0, \alpha t^{-3}\gamma_1, \alpha\overline{\alpha} q^{-3} t^{-6}\gamma_2).$$

Let us, for $i = 0, 1, 2$, write

$$\gamma_i^0(t, \phi; \Lambda) = \gamma_{i,0}^0(\phi; \Lambda) + \gamma_{i,1}^0(\phi; \Lambda)t + \gamma_{i,2}^0(\phi; \Lambda)t^2 + \ldots. \tag{5.57}$$

Expanding $c_0^0(t, \phi; \Lambda)$ about $t = 0$,

$$c_0^0(t, \phi; \Lambda) = c_{0,0}^0(\phi; \Lambda) + c_{0,1}^0(\phi; \Lambda)t + c_{0,2}^0(\phi; \Lambda)t^2 + \ldots, \tag{5.58}$$

and substitution into (5.56) gives, by comparing coefficients of t^0, the following linear second order difference equation for $c^0_{0,0}(\phi;\Lambda)$,

$$\gamma^0_{0,0}(\phi;\Lambda)c^0_{0,0}(\phi;\Lambda)+\gamma^0_{1,0}(\phi;\Lambda)c^0_{0,0}(\lambda\phi;\Lambda)+\gamma^0_{2,0}(\phi;\Lambda)c^0_{0,0}(\lambda^2\phi;\Lambda)=0, \tag{5.59}$$

where the $\gamma^0_{i,0}(\phi;\Lambda)$ equal

$$\begin{aligned}\gamma^0_{0,0}(\phi;\Lambda)=&G_1[b_1b_2b_3b_4F_1+(b_1b_2+b_1b_3+b_1b_4+b_2b_3+b_2b_4+b_3b_4)G_1-(b_1+b_2+b_3+b_4)G_1^2\\&-b_1b_2b_3-b_1b_2b_4-b_1b_3b_4-b_2b_3b_4+G_1^3],\\ \gamma^0_{1,0}(\phi;\Lambda)=&\Lambda G_1(F_1G_1-1)(b_1+b_2+b_3+b_4-G_1-\overline{G}_1),\\ \gamma^0_{2,0}(\phi;\Lambda)=&\Lambda^2G_1\overline{F}_1(F_1G_1-1).\end{aligned}$$

Lemma 5.5.1. *We have the following identities*

$$\begin{aligned}\gamma^0_{0,0}(\phi;\Lambda)+\gamma^0_{1,0}(\phi;\Lambda)+\gamma^0_{2,0}(\phi;\Lambda)&=0,\\ \gamma^0_{0,0}(\phi;\Lambda)+\gamma^0_{1,0}(\phi;\Lambda)\lambda^{-1}+\gamma^0_{2,0}(\phi;\Lambda)\lambda^{-2}&=0.\end{aligned}$$

Proof. One can either check these identities by direct calculation, or use equations (3.22) with $F=F_1$ and $G=G_1$, to establish them. □

By the above Lemma, we see that for any $k_1(\Lambda),k_2(\Lambda)$,

$$c^0_{0,0}(\phi;\Lambda)=k_1(\Lambda)+k_2(\Lambda)\phi^{-1}, \tag{5.60}$$

defines a solution of equation (5.59).

Proposition 5.5.2. *Consider equation (5.56) with $f=f^{0,+}(t,\phi;\Lambda,\mathbf{b})$ and $g=g^{0,+}(t,\phi;\Lambda,\mathbf{b})$ as defined in Theorem 3.3.1. Then there exists, for $i=1,2$, an unique formal solution of (5.56), of the form*

$$c^{0,i}_0(t,\phi;\Lambda)=\sum_{m=0}^{\infty}c^{0,i}_{0,m}(\phi;\Lambda)t^m \tag{5.61}$$

with for $m\in\mathbb{N}$,

$$c^{0,i}_{0,m}(\phi;\Lambda)=\sum_{n=-\infty}^{m}c^{0,i}_{0,m,n}(\Lambda)\phi^n, \tag{5.62}$$

where

$$\begin{pmatrix}c^{0,1}_{0,0,0}(\Lambda)\\ c^{0,1}_{0,0,-1}(\Lambda)\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix},\qquad \begin{pmatrix}c^{0,2}_{0,0,0}(\Lambda)\\ c^{0,2}_{0,0,-1}(\Lambda)\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}. \tag{5.63}$$

For $m\in\mathbb{N}$ and $n\in\mathbb{Z}_{\leq m}$ the coefficients $c^{0,i}_{0,m,n}(\Lambda)$ are rational functions in Λ and the parameters $b_1,\ldots,b_8$, in particular these rational functions are regular at points $(\Lambda,\mathbf{b})\in\mathbb{C}^*\times\mathcal{B}$, satisfying (3.46). Furthermore, for fixed $\mathbf{b}\in\mathcal{B}$ with $|q|<1$, for any $\Lambda\in L_0(\mathbf{b})$, condition (3.46) is satisfied and this formal solution, written in terms of the variables $\zeta_1=t\phi$

and $\zeta_2 = \phi^{-1}$,

$$c_0^{0,i}\left(\zeta_1\zeta_2, \zeta_2^{-1}; \Lambda\right) = \sum_{m,n=0}^{\infty} c_{0,m,m-n}^{0,i}(\Lambda)\zeta_1^m\zeta_2^n, \tag{5.64}$$

converges near $(\zeta_1, \zeta_2) = (0, 0)$.
In fact, these expansions are also analytic in Λ. *That is, for any* $L \subseteq L_0(\mathbf{b})$ *open with* $\overline{L} \subseteq L_0(\mathbf{b})$, *there is an open environment* $Z \subseteq \mathbb{C}^2$ *of* $\mathbf{0}$, *such that the series* (5.43) *converge uniformly on* $Z \times L$, *defining holomorphic functions on this set in* $(\boldsymbol{\zeta}, \Lambda)$.

Proof. Note that within the context of equation (5.60), the initial conditions (5.63) correspond to the choices $(k_1(\Lambda), k_2(\Lambda)) = (1, 0)$ and $(k_1(\Lambda), k_2(\Lambda)) = (0, 1)$ respectively. For any of the two choices, we can prove the proposition similar to Theorem 3.3.1. We rewrite equation (5.33) in appropriate system form and apply the q-Briot Bouquet Theorems B.3 and B.4. □

Remark 5.5.3. Recall that we used the formal series solution in Theorem 3.3.1, to construct true solutions of q-$P(A_1)$ in Theorem 3.4.1, by replacing the formal variables by actual analytic functions. Doing so we can use the formal series solutions in Proposition 5.5.2 to construct corresponding true solutions of (5.56).

5.5.2 Expanding about $t = 0$

We consider a formal expansion of $\psi^0(\xi, t, \psi; \Lambda)$ in t,

$$\psi^0(\xi, t, \phi; \Lambda) = d_0^0(\xi, \phi; \Lambda) + d_1^0(\xi, \phi; \Lambda)t + d_2^0(\xi, \phi; \Lambda)t^2 + \ldots.$$

By substitution in L_2^0 (5.55b), and comparing the coefficients of order t^2, we obtain

$$(1 - q\xi G_1)d_0^0(\xi, \phi; \Lambda) - d_0^0(q\xi, \phi; \Lambda) + \Lambda q\xi(F_1 - q\xi)d_0^0(\xi, \lambda\phi; \Lambda) = 0. \tag{5.65}$$

Similarly, substitution into L_1^0 gives

$$\delta_0(\xi, \phi; \Lambda)d_0^0(\xi/q, \phi; \Lambda) + \delta_1(\xi, \phi; \Lambda)d_0^0(\xi, \phi; \Lambda) + \delta_2(\xi, \phi; \Lambda)d_0^0(q\xi, \phi; \Lambda) = 0, \tag{5.66}$$

where the coefficients $\delta_i(\xi, \phi; \Lambda)$ are given by

$$\begin{aligned}
\delta_0(\xi, \phi; \Lambda) =& q\left(F_1(\phi)G_1(\phi) - 1\right)(b_1\xi - 1)(b_2\xi - 1)(b_3\xi - 1)(b_4\xi - 1)(q\xi - F_1(\phi)),\\
\delta_1(\xi, \phi; \Lambda) =& \left(F_1(\phi)G_1(\phi) - 1\right)\left[\delta_{1,0}(\phi; \Lambda) + \delta_{1,1}(\phi; \Lambda)\xi + \delta_{1,3}(\phi; \Lambda)\xi^3\right] + \delta_{1,2}(\phi; \Lambda)\xi^2,\\
\delta_2(\xi, \phi; \Lambda) =& \left(F_1(\phi)G_1(\phi) - 1\right)(\xi - F_1(\phi)),
\end{aligned}$$

with

$$\begin{aligned}
\delta_{1,0}(\phi;\Lambda) =&(q+1)F_1,\\
\delta_{1,1}(\phi;\Lambda) =&-q(b_1+b_2+b_3+b_4)F_1-q^2-1,\\
\delta_{1,2}(\phi;\Lambda) =&-q^2(b_1+b_2+b_3+b_4)-q(b_1b_2+b_1b_3+b_1b_4+b_2b_3+b_2b_4+b_3b_4)F_1\\
&+q(b_1b_2b_3+b_1b_2b_4+b_1b_3b_4+b_2b_3b_4)F_1^2-qb_1b_2b_3b_4F_1^3+q(q-1)G_1\\
&+q(q+1)(b_1+b_2+b_3+b_4)F_1G_1-q^2F_1G_1^2,\\
\delta_{1,3}(\phi;\Lambda) =&q^2\Lambda(1+\lambda^{-1}).
\end{aligned}$$

Let us expand $d_0^0(\xi,\phi;\Lambda)$ in ϕ as

$$d_0^0(\xi,\phi;\Lambda)=d_{0,0}^0(\xi;\Lambda)+d_{0,-1}^0(\xi;\Lambda)\phi^{-1}+d_{0,-2}^0(\xi;\Lambda)\phi^{-2}+\dots, \tag{5.67}$$

then, substitution into equation (5.66) gives, by comparing coefficients of ϕ^3,

$$\begin{aligned}
d_{0,0}^0(q\xi;\Lambda)+\left[-(1+q)+q(b_1+b_2+b_3+b_4)\xi+q\left(q\Lambda+\frac{\Lambda}{\lambda}\right)\xi^2\right]d_{0,0}^0(\xi;\Lambda)&\\
+q(1-b_1\xi)(1-b_2\xi)(1-b_3\xi)(1-b_4\xi)d_{0,0}^0(\xi/q;\Lambda)=0.&
\end{aligned} \tag{5.68}$$

Furthermore, substitution into equation (5.65) gives, by comparing coefficients of ϕ^0,

$$\begin{aligned}
d_{0,-1}^0(\xi;\Lambda)=\Lambda\frac{d_{0,0}^0(q\xi;\Lambda)-d_{0,0}^0(\xi;\Lambda)}{q(b_1b_2b_3b_4-\Lambda^2)\xi}+\frac{\Lambda^2}{(b_1b_2b_3b_4-\Lambda^2)^2}\Big[q(b_1b_2b_3b_4-\Lambda^2)\xi&\\
-(b_1b_2b_3+b_1b_2b_4+b_1b_3b_4+b_2b_3b_4+(b_1+b_2+b_3+b_4)\Lambda)\Big]d_{0,0}^0(\xi;\Lambda).&
\end{aligned} \tag{5.69}$$

Lemma 5.5.4. *Given any solution $d_{0,0}^0(\xi;\Lambda)$ of equation (5.68), and defining $d_{0,-1}^0(\xi;\Lambda)$ by equation (5.69), the function*

$$d_0^0(\xi,\phi;\Lambda):=d_{0,0}^0(\xi;\Lambda)+d_{0,-1}^0(\xi;\Lambda)\phi^{-1}, \tag{5.70}$$

solves (5.65) and (5.66) simultaneously.

Proof. We checked this by direct calculation, using Mathematica. □

We identify equation (5.68) as the degree two model equation (4.51), with parameter values $\sigma=\sigma_0(\Lambda)$, as defined in (5.107), where we note that Fuchs' equation (4.52) is indeed satisfied. Considering Proposition 3.5.1, for instance the choice $\Lambda=\Lambda_1^+$, leads to a violation of the condition 4.53, as described in Remark 4.5.3. This is why the connection problem on the line $t=0$ in Section 5.2.4 is trivial.

Now, for any $l_1(\Lambda)$ and $l_2(\Lambda)$, there exists an unique formal power series solution $d_{0,0}^0(\xi;\Lambda)$ in ξ of equation (5.68) with

$$d_{0,0}^0(\xi;\Lambda)=l_1(\Lambda)+l_2(\Lambda)\xi+\dots$$

Comparing expansions (5.58),(5.60) and (5.70), we find

$$k_1(\Lambda) = l_1(\Lambda), \tag{5.71a}$$

$$k_2(\Lambda) = \frac{(q-1)}{q\Lambda(1/\lambda - 1))} l_2(\Lambda) - \frac{1}{\Lambda^2(1/\lambda - 1)^2}\Big[b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4 \tag{5.71b}$$

$$+ (b_1 + b_2 + b_3 + b_4)\Lambda\Big] l_1(\Lambda). \tag{5.71c}$$

5.5.3 Main Existence Theorem near $(\xi, t) = (0, 0)$

Theorem 5.5.5. *Consider the Lax pair L^0 (5.55) with $f = f^{0,+}(t, \phi; \Lambda, \mathbf{b})$ and $g = g^{0,+}(t, \phi; \Lambda, \mathbf{b})$ as defined in Theorem 3.3.1. Then, for $i \in \{1, 2\}$, there exists an unique formal series solution of the Lax pair L^0, of the form*

$$\psi_i^0(\xi, t, \phi; \Lambda) = \sum_{k=0}^{\infty} c_k^{0,i}(t, \phi; \Lambda)\xi^k,$$

where for $k \in \mathbb{N}$,

$$c_k^{0,i}(t, \phi; \Lambda) = \sum_{m=0}^{\infty} c_{k,m}^{0,i}(\phi; \Lambda)t^m,$$

with for $m \in \mathbb{N}$,

$$c_{k,m}^{0,i}(\phi; \Lambda) = \sum_{n=-\infty}^{m} c_{k,m,n}^{0,i}(\Lambda)\phi^n,$$

and initial conditions (5.63).
We note that the notation here coincides with that in Proposition 5.5.2. For $k, m \in \mathbb{N}$ and $n \in \mathbb{Z}_{\leq m}$, the coefficients $c_{k,m,n}^{\infty,i}(\Lambda)$ are rational functions in Λ and the parameters $b_1, \ldots, b_8$, in particular these rational functions are regular at points $(\Lambda, \mathbf{b}) \in \mathbb{C}^* \times \mathcal{B}$, satisfying (3.46). Furthermore, for fixed $\mathbf{b} \in \mathcal{B}$ with $|q| < 1$, for any $\Lambda \in L_0(\mathbf{b})$, condition (3.46) is satisfied and this formal solution, written in terms of the variables ξ, $\zeta_1 = t\phi$ and $\zeta_2 = \phi^{-1}$,

$$\psi_i^0\left(\xi, \zeta_1\zeta_2, \zeta_2^{-1}; \Lambda\right) = \sum_{k,m,n=0}^{\infty} c_{k,m,m-n}^{0,i}(\Lambda)\xi^k\zeta_1^m\zeta_2^n, \tag{5.72}$$

converges near $(\xi, \zeta_1, \zeta_2) = (0, 0, 0)$.
In fact, this expansion also depends holomorphically on Λ. That is, for any $L \subseteq L_0(\mathbf{b})$ open with $\overline{L} \subseteq L_0(\mathbf{b})$, there is an open environment $Z \subseteq \mathbb{C} \times \mathbb{C}^2$ of $(0, 0, 0)$, such that the series (5.72) converge uniformly on $Z \times L$, defining holomorphic functions on this set in $(z, \zeta_1, \zeta_2, \Lambda)$.

Proof. We prove this analogous to Theorem 5.5.5. $\square$

Remark 5.5.6. Recall that we used the formal series solution in Theorem 3.3.1, to construct true solutions of q-$P(A_1)$ in Theorem 3.4.1, by replacing the formal variables by actual analytic functions. Doing so we can use the formal series solutions in Theorem 5.5.5 to construct corresponding true solutions of the Lax pair L^0 (5.55).

5.5.4 Transition from $(\xi, t) = (0, 0)$ to $(\xi, t) = (\infty, 0)$

Note that the two formal series solutions $\psi_1^0(\xi, t, \phi; \Lambda)$ and $\psi_2^0(\xi, t, \phi; \Lambda)$, define a basis of solutions of L^0. It is more convenient for us to work with a different basis of solutions, given by

$$\Psi_1^0(\xi, t, \phi; \Lambda) := 1 \cdot \psi_1^0(\xi, t, \phi; \Lambda) + k_2^*(\Lambda)\psi_2^0(\xi, t, \phi; \Lambda), \tag{5.73a}$$

$$\Psi_2^0(\xi, t, \phi; \Lambda) := 0 \cdot \psi_1^0(\xi, t, \phi; \Lambda) + \frac{(q-1)}{q\Lambda(1/\lambda - 1))}\psi_2^0(\xi, t, \phi; \Lambda). \tag{5.73b}$$

where $k_2^*(\Lambda)$ is defined by

$$k_2^*(\Lambda) = -\frac{1}{\Lambda^2(1/\lambda - 1)^2}\Big[b_1b_2b_3 + b_1b_2b_4 + b_1b_3b_4 + b_2b_3b_4 + (b_1 + b_2 + b_3 + b_4)\Lambda\Big].$$

By equations(5.71), these two solutions correspond respectively to $(l_1(\Lambda), l_2(\Lambda)) = (1, 0)$ and $(l_1(\Lambda), l_2(\Lambda)) = (0, 1)$ in (5.47). Let us write, for $i = 1, 2$,

$$\Psi_i^0(\xi, t, \phi; \Lambda) = \sum_{k=0}^{\infty} D_k^{0,i}(\xi, \phi; \Lambda)t^k, \tag{5.74}$$

where for $k \in \mathbb{N}$,

$$D_k^{0,i}(\xi, \phi; \Lambda) = \sum_{m=-\infty}^{k} D_{k,m}^{0,i}(\xi; \Lambda)\phi^m,$$

with for $m \in \mathbb{Z}_{\leq k}$,

$$D_{k,m}^{0,i}(\xi; \Lambda) = \sum_{n=0}^{\infty} D_{k,m,n}^{0,i}(\Lambda)\xi^n.$$

Then we have, by Lemma 5.5.4,

$$D_0^0(\xi, \phi; \Lambda) := D_{0,0}^0(\xi; \Lambda) + D_{0,-1}^0(\xi; \Lambda)\phi^{-1},$$

where

$$D_{0,0}^0(\xi; \Lambda) = y^0(\xi; \sigma_0(\Lambda)).$$

We denote

$$D_{0,0}^{0,\infty}(\xi; \Lambda) = \psi^\infty(\xi; \sigma_0(\Lambda)),$$

and we have the connection result

$$D_{0,0}^{0,\infty}(\xi; \Lambda) = D_{0,0}^0(\xi; \Lambda)P^0(\xi; \Lambda), \quad P^0(\xi; \Lambda) := Q(\xi; \sigma_0(\Lambda)). \tag{5.75}$$

Analogously to (5.11), we symbolically define

$$\Psi^{0,\infty}(\xi, t, \phi; \Lambda) := \Psi^0(\xi, t, \phi; \Lambda) \cdot P^0(\xi; \Lambda). \tag{5.76}$$

From (4.46), we obtain

$$P^0(q\xi;\Lambda) = P^0(\xi;\Lambda)\cdot\begin{pmatrix}-\frac{1}{q^2\Lambda}\xi^{-2} & 0\\ 0 & -\frac{\lambda}{q\Lambda}\xi^{-2}\end{pmatrix},$$

and hence, for $i = 1, 2$, a symbolic computation, shows that the component $\Psi_i^{0,\infty}(\xi, t, \phi; \Lambda)$, defines a solution of the Lax pair $L^{0,\infty,i}$, given by

$$L_1^{0,\infty,i}:\quad q^2\widetilde{\delta}_i\xi^2 u(\xi t,t)\psi_i^{0,\infty}(q\xi,t) + v(\xi t,t)\psi_i^{0,\infty}(\xi,t) + \frac{1}{\widetilde{\delta}_i\xi^2}w(\xi t,t)\psi_i^{0,\infty}(\xi/q,t) = 0, \tag{5.77a}$$

$$L_2^{0,\infty,i}:\quad \frac{\alpha}{t^3}h_0(\xi t,t)\psi_i^{0,\infty}(\xi/q,qt) + \widetilde{\delta}_i\xi^2 h_1(\xi t,t)\psi_i^{0,\infty}(\xi,t) + h_2(\xi t,t)\psi_i^{0,\infty}(\xi/q,t) = 0, \tag{5.77b}$$

where $\widetilde{\delta}_1 = -\Lambda$ and $\widetilde{\delta}_2 = -\Lambda/(q\lambda)$, and we suppressed ϕ and Λ dependence throughout.

5.6 Generic Case: Matching near $t = 0$

In Section 5.4 we constructed a fundamental formal solution $\Psi^\infty(z, t, \phi; \Lambda)$ of the Lax pair L^∞ (5.6), where $\beta := -q^{-1}G_1(\phi; \Lambda)^{-1}$. This formal solution converges for appropriate values of Λ, as described in Theorem 5.4.3. Similarly we constructed a fundamental formal solution $\Psi^0(z, t, \phi; \Lambda)$ of the Lax pair L^0 (5.55), in Section 5.5. This formal solution converges for appropriate values of Λ, as described in Theorem 5.5.5.

We wish to relate the fundamental solution $\Psi^\infty(z, t, \phi; \Lambda)$ of L^∞, with the fundamental solution $\Psi^0(\xi, t, \phi; \Lambda)$ of the Lax pair L^0. However, this is not sensible on the formal level. Indeed we first have to substitute actual analytic functions for Λ and ϕ, as done in Theorem 3.4.1, after which connecting the fundamental solutions becomes possible. To ease the notation and technical details, we restrict ourselves to $\Lambda(t) \equiv \Lambda$ constant.

5.6.1 True Solutions of Lax pairs

Recalling the definition (3.47) of $L_0(\mathbf{b})$, we consider we fix some $\Lambda \in L_0(\mathbf{b})$, take a continuous q-domain T, and fix a function $\phi(t)$ which is analytic and nonvanishing on T, satisfying

$$\phi(qt) = \lambda\phi(t), \quad \lambda := \frac{\Lambda^2}{b_1b_2b_3b_4}. \quad (t \in T)$$

Let $(f, g) = (f(t), g(t))$ be the meromorphic solution of q-$P(A_1)$, as defined in Theorem 3.4.1. We fix a continuous q-domain $V \subseteq \overline{V}^* \subseteq T$ and consider the Lax pair L^∞, with $\beta = \beta(t) := -q^{-1}G_1(\phi(t); \Lambda)^{-1}$. Theorem 5.4.3 shows that, analogous to Theorem 3.4.1,

$$\psi^\infty(z,t) := (\psi_1^\infty(z,t,\phi(t);\Lambda), \psi_2^\infty(z,t,\phi(t);\Lambda)),$$

defines a fundamental solution of L^∞ for (z, t) close to $(\infty, 0)$ in $\mathbb{P}^* \times V$, which has an unique meromorphic continuation to $\mathbb{P}^* \times V$. We use the change of basis (5.50), to define the corresponding fundamental solution

$$\Psi^\infty(z,t) := (\Psi_1^\infty(z,t,\phi(t);\Lambda), \Psi_2^\infty(z,t,\phi(t);\Lambda)),$$

which satisfies

$$\Psi^\infty(z,0) = y^0(z^{-1};\sigma_\infty(\Lambda)). \tag{5.78}$$

Following (5.53), we define

$$\Psi^{\infty,0}(z,t) := \Psi^\infty(z,t)\cdot P^\infty(z;\Lambda)^{-1}, \tag{5.79}$$

then

$$\Psi^{\infty,0}(z,0) = \psi^\infty(z^{-1};\sigma_\infty(\Lambda)), \tag{5.80}$$

and for $i=1,2$, the component $\Psi_i^{\infty,0}(z,t)$ defines a solution of the Lax pair $L^{\infty,0,i}$ (5.12), with $\delta_1 = -\Lambda$ and $\delta_2 = -\Lambda/\lambda$.

Next we consider the Lax pair L^0 (5.55), with $\alpha = \Lambda q^{-1}G_1(\phi;\Lambda)^{-1}$. Using Theorem 5.5.5 and the change of basis (5.73), we define a fundamental solution of L^0,

$$\Psi_1^0(\xi,t) = (\Psi_1^0(\xi,t,\phi(t);\Lambda), \Psi_2^0(\xi,t,\phi(t);\Lambda)),$$

meromorphic on $\mathbb{C}\times V$ in (ξ,t), which satisfies

$$\Psi_1^0(\xi,0) = y^0(\xi;\sigma_0(\Lambda)).$$

Following (5.76), we define

$$\Psi^{0,\infty}(\xi,t) := \Psi^0(\xi,t)\cdot P^0(\xi;\Lambda), \tag{5.81}$$

then

$$\Psi^{0,\infty}(\xi,0) = \psi^\infty(\xi;\sigma_0(\Lambda)), \tag{5.82}$$

and for $i=1,2$, the component $\Psi_i^{0,\infty}(\xi,t)$ defines a solution of the Lax pair $L^{0,\infty,i}$ (5.77), with $\widetilde{\delta}_1 = -\Lambda$ and $\widetilde{\delta}_2 = -\Lambda/(q\lambda)$.

We now wish to relate $\Psi_1^{\infty,0}(z,t)$ and $\Psi_1^{0,\infty}(\xi,t)$, and we wish to relate $\Psi_2^{\infty,0}(z,t)$ and $\Psi_2^{0,\infty}(\xi,t)$. Let us make the crucial observation that $\Psi_1^{\infty,0}(z,t)$ and $\Psi_1^{0,\infty}(\xi,t)$ satisfy the same Lax pair. That is, the Lax pairs $L^{\infty,0,1}$ and $L^{0,\infty,1}$ are identical under the formal identification

$$\psi_1^{\infty,0}(z,t,\phi;\Lambda) = \psi_1^{0,\infty}(\xi,t,\phi;\Lambda), \quad \xi = \frac{z}{t}. \tag{5.83}$$

Similarly, the Lax pairs $L^{\infty,0,2}$ and $L^{0,\infty,2}$ are identical under the formal identification

$$\psi_2^{\infty,0}(z,t,\phi;\Lambda) = \frac{t\phi}{z}\psi_2^{0,\infty}(\xi,t,\phi;\Lambda), \quad \xi = \frac{z}{t}.$$

We will establish the following proposition.

Proposition 5.6.1. *We have the following identities,*

$$\Psi_1^{\infty,0}(z,t) = \Psi_1^{0,\infty}(\xi,t), \quad \Psi_2^{\infty,0}(z,t) = \frac{ct\phi(t)}{z}\Psi_2^{0,\infty}(\xi,t),$$

where the constant c is given by

$$c = -\frac{q\lambda - 1}{\lambda - 1}. \tag{5.84}$$

The proof of this proposition is not easy. Even though we know that in both identities, the left- and right-hand sides satisfy the same Lax pair, it is not possible to match them directly. Instead, we will introduce an additional solutions of the Lax pair in both cases, which allows us to make all the matches. To this end, we temporarily go back to the formal setting, i.e. with ϕ and Λ formal variables.

5.6.2 Formal Transition

Recall the symbolic transition (5.53), which by equation (5.51) and a formal computation, reads

$$\Psi^{\infty,0}(z,t,\phi;\Lambda) = \sum_{k=0}^{\infty}\sum_{m=-\infty}^{k} D_{k,m}^{\infty,0}(z;\Lambda)t^k\phi^m,$$

where for $k \in \mathbb{N}$ and $m \in \mathbb{Z}_{\leq k}$,

$$D_{k,m}^{\infty,0}(z;\Lambda) = D_{k,m}^{\infty,i}(z;\Lambda) \cdot P^{\infty}(z;\Lambda)^{-1},$$

with, by equation (5.52),

$$D_{0,0}^{\infty,0}(z;\Lambda) = \psi^{\infty}(z^{-1};\sigma_{\infty}(\Lambda)).$$

We emphasise that these identities are merely symbolic. Nonetheless, motivated by this symbolic calculation, we consider, for $i = 1, 2$, the Lax pair $L^{\infty,0,i}$ (5.12), where $\delta_1 = -\Lambda$ and $\delta_2 = -\Lambda/\lambda$, and β as defined in (5.31). And we consider a formal solution $\psi_i^{\mathrm{tr}}(z,t,\phi;\Lambda)$ of this Lax pair of the form

$$\psi_i^{\mathrm{tr}}(z,t,\phi;\Lambda) = \sum_{k=0}^{\infty}\sum_{m=-\infty}^{k} d_{k,m}^{\mathrm{tr},i}(z;\Lambda)t^k\phi^m,$$

where

$$d_{0,0}^{\mathrm{tr},i}(z;\Lambda) = \sum_{n=0}^{\infty} d_{0,0,n}^{\mathrm{tr},i}(\Lambda)z^n,$$

with $d_{0,0,0}^{\mathrm{tr},i}(\Lambda) = 1$.

Via substitution in $L_1^{\infty,0,i}$, we quickly recover $d_{0,0}^{\mathrm{tr},i}(z;\Lambda) = \psi^{\infty}(z^{-1};\sigma_{\infty}(\Lambda))$. Furthermore, by considering $L_2^{\infty,0,i}$, we find by induction, that for $k \in \mathbb{N}$ and $m \in \mathbb{Z}_{\leq k}$, the coefficient $d_{k,m}^{\mathrm{tr},i}(z;\Lambda)$ enjoys a Laurent expansion in z about $z = 0$ with lowest order term at least z^{-k}. To be precise, we have the following result.

Lemma 5.6.2. *Let $i \in \{1,2\}$ and consider the Lax pair $L^{\infty,0,i}$ (5.12) with $f = f^{0,+}(t,\phi;\Lambda,\mathbf{b})$ and $g = g^{0,+}(t,\phi;\Lambda,\mathbf{b})$ as described in Theorem 3.3.1, where $\delta_1 = -\Lambda$ and $\delta_2 = -\Lambda/\lambda$, and*

β as defined in (5.31). Then there exists an unique formal series solution of the Lax pair $L^{\infty,0,i}$, of the form

$$\Psi_i^{tr}(z,t,\phi;\Lambda)=\sum_{k=0}^{\infty}\sum_{m=-\infty}^{k}D_{k,m}^{tr,i}(z;\Lambda)t^k\phi^m, \tag{5.85}$$

where for $k\in\mathbb{N}$ and $m\in\mathbb{Z}_{\leq k}$,

$$D_{k,m}^{tr,i}(z;\Lambda)=\sum_{n=-k}^{\infty}D_{k,m,n}^{tr,i}(\Lambda)z^n,$$

with $D_{0,0,0}^{tr,i}(\Lambda)=1$.
For k and $m,n\in\mathbb{Z}_{\leq k}$, the coefficients $D_{k,m,n}^{tr,i}(\Lambda)$ are rational functions in Λ and the parameters $b_1,\ldots,b_8$, in particular these rational functions are regular at points $(\Lambda,\mathbf{b})\in\mathbb{C}^*\times\mathcal{B}$, satisfying (3.46). Furthermore, for fixed $\mathbf{b}\in\mathcal{B}$ with $|q|<1$, for any $\Lambda\in L_0(\mathbf{b})$, condition (3.46) is satisfied and this formal solution, written in terms of the variables z, $\mu=t\phi/z$ and $\zeta_2=\phi^{-1}$,

$$\Psi_i^{tr}(z,z\mu\zeta_2,\zeta_2^{-1};\Lambda)=\sum_{k,m,n=0}^{\infty}D_{k,k-m,n-k}^{tr,i}(\Lambda)\mu^k\zeta_2^m z^n, \tag{5.86}$$

converges near $(z,\mu,\zeta_2)=(0,0,0)$.
In fact, this expansion also depends holomorphically on Λ. That is, for any $L\subseteq L_0(\mathbf{b})$ open with $\overline{L}\subseteq L_0(\mathbf{b})$, there is an open environment $Z\subseteq\mathbb{C}^3$ of $(0,0,0)$, such that the series (5.86) converge uniformly on $Z\times L$, defining holomorphic functions on this set in (z,μ,ζ_2,Λ).

Proof. We prove this analogous to Theorem 5.4.3. □

Recall that $\Psi^{0,\infty}(\xi,t,\phi;\Lambda)$, defined symbolically in equation (5.76), is expressed in terms of the independent variables ξ,t,ϕ,Λ. We hence rewrite, for $i=1,2$, the formal series $\Psi_i^{\mathrm{tr}}(z,t,\phi;\Lambda)$, defined in Lemma 5.6.2, in terms of these variables,

$$\widetilde{\Psi}_i^{\mathrm{tr}}(\xi,t,\phi;\Lambda):=\Psi_i^{\mathrm{tr}}(\xi t,t,\phi;\Lambda). \tag{5.87}$$

Using (5.85), we have

$$\begin{aligned}\widetilde{\Psi}_i^{\mathrm{tr}}(\xi,t,\phi;\Lambda)&=\sum_{k=0}^{\infty}\sum_{m=-\infty}^{k}\sum_{n=-k}^{\infty}D_{k,m,n}^{\mathrm{tr},i}(\Lambda)t^{k+n}\phi^m\xi^n\\&=\sum_{k'=0}^{\infty}\sum_{n=-\infty}^{k'}\sum_{m=-\infty}^{k'-n}D_{k'-n,m,n}^{\mathrm{tr},i}(\Lambda)t^{k'}\phi^m\xi^n,\end{aligned} \tag{5.88}$$

where the last equality is the result of the change of summation $k'=k+n$. Note that

$\widetilde{\Psi}_i^{\mathrm{tr}}(\xi,t,\phi;\Lambda)$ defines a formal solution of the Lax pair $\widetilde{L}^{\mathrm{tr},i}$, given by

$$\widetilde{L}_1^{\mathrm{tr},i}: \quad q^2\delta_i\xi^2 u(\xi t,t)\widetilde{\psi}_i^{\mathrm{tr}}(q\xi,t)+v(\xi t,t)\widetilde{\psi}_i^{\mathrm{tr}}(\xi,t)+\frac{1}{\delta_i\xi^2}w(\xi t,t)\widetilde{\psi}_i^{\mathrm{tr}}(\xi/q,t)=0, \tag{5.89a}$$

$$\widetilde{L}_2^{\mathrm{tr},i}: \quad \beta t^{-3}h_0(\xi t,t)\widetilde{\psi}_i^{\mathrm{tr}}(\xi/q,qt)+\xi^2 h_1(\xi t,t)\widetilde{\psi}_i^{\mathrm{tr}}(\xi,t)+\frac{1}{\delta_i}h_2(\xi t,t)\widetilde{\psi}_i^{\mathrm{tr}}(\xi/q,t)=0, \tag{5.89b}$$

where again $\delta_1=-\Lambda$ and $\delta_2=-\Lambda/\lambda$, and β as defined in (5.31), and we suppressed the Λ and ϕ dependence throughout.

Lemma 5.6.3. *For $i=1$ and $i=2$, the inner m summation in the formal series (5.88), can also be bounded from above by k' and $k'+1$ respectively, that is*

$$\widetilde{\Psi}_i^{tr}(\xi,t,\phi;\Lambda)=\sum_{k'=0}^{\infty}\sum_{n=-\infty}^{k'}\sum_{m=-\infty}^{\min(k'+i-1,k'-n)} D_{k'-n,m,n}^{tr,i}(\Lambda)t^{k'}\phi^m\xi^n.$$

Proof. We use $l=k'-n$ to rewrite (5.88) as

$$\widetilde{\Psi}_i^{\mathrm{tr}}(\xi,t,\phi;\Lambda)=\sum_{l=0}^{\infty}\sum_{k'=0}^{\infty}\sum_{m=-\infty}^{l} D_{l,m,k'-l}^{\mathrm{tr},i}(\Lambda)t^{k'}\phi^m\xi^{k'-l}.$$

Our goal is to show

$$\widetilde{\Psi}_i^{\mathrm{tr}}(\xi,t,\phi;\Lambda)=\sum_{l=0}^{\infty}\sum_{k'=0}^{\infty}\sum_{m=-\infty}^{\min(k'+i-1,l)} D_{l,m,k'-l}^{\mathrm{tr},i}(\Lambda)t^{k'}\phi^m\xi^{k'-l}.$$

In both the case $i=1$ and $i=2$, this is easily proven by double induction, the outer one with respect to l and the inner one with respect to k', using the fact that $\widetilde{\Psi}_i^{\mathrm{tr}}(\xi,t,\phi;\Lambda)$ satisfies $\widetilde{L}_2^{\mathrm{tr},i}$. □

The following result gives us an analytic characterisation of the formal series (5.87), which allows us to complete the matching procedure.

Lemma 5.6.4. *Let $i\in\{1,2\}$ and consider the Lax pair $\widetilde{L}^{tr,i}$ (5.89) with $f=f^{0,+}(t,\phi;\Lambda,\mathbf{b})$ and $g=g^{0,+}(t,\phi;\Lambda,\mathbf{b})$ as described in Theorem 3.3.1. Then $\widetilde{\Psi}_i^{tr}(\xi,t,\phi;\Lambda)$ (5.87) can be characterised as the unique formal series solution of the Lax pair $\widetilde{L}^{tr,i}$, of the form*

$$\widetilde{\Psi}_i^{tr}(\xi,t,\phi;\Lambda)=\sum_{k=0}^{\infty}\sum_{m=-\infty}^{k+i-1}\widetilde{D}_{k,m}^{tr,i}(\xi;\Lambda)t^k\phi^m,$$

where for $k\in\mathbb{N}$ and $m\in\mathbb{Z}_{\leq k+i-1}$,

$$\widetilde{D}_{k,m}^{tr,i}(\xi;\Lambda)=\sum_{n=-\infty}^{k}\widetilde{D}_{k,m,n}^{tr,i}(\Lambda)\xi^n,$$

with $\widetilde{D}_{0,0,0}^{tr,i}(\Lambda)=1$.

For k, $m \in \mathbb{Z}_{\leq k+i-1}$ and $n \in \mathbb{Z}_{\leq k}$, the coefficients $\widetilde{D}^{tr,i}_{k,m,n}(\Lambda)$ are rational functions in Λ and the parameters $b_1, \ldots, b_8$, in particular these rational functions are regular at points $(\Lambda, \mathbf{b}) \in \mathbb{C}^* \times \mathcal{B}$, satisfying (3.46). Furthermore, for fixed $\mathbf{b} \in \mathcal{B}$ with $|q| < 1$, for any $\Lambda \in L_0(\mathbf{b})$, condition (3.46) is satisfied and the following formal series, written in terms of the variables ξ, $\tau = \xi t \phi$ and $\zeta_2 = \phi^{-1}$,

$$\widetilde{\Psi}_1^{tr}(\xi, \xi^{-1}\tau\zeta_2, \zeta_2^{-1}; \Lambda) = \sum_{k,m,n=0}^{\infty} \widetilde{D}^{tr,1}_{k,k-m,k-n}(\Lambda)\tau^k\zeta_2^m\xi^{-n}, \tag{5.90}$$

$$\phi^{-1}\widetilde{\Psi}_2^{tr}(\xi, \xi^{-1}\tau\zeta_2, \zeta_2^{-1}; \Lambda) = \sum_{k,m,n=0}^{\infty} \widetilde{D}^{tr,2}_{k,k-m+1,k-n}(\Lambda)\tau^k\zeta_2^m\xi^{-n}, \tag{5.91}$$

converge near $(\xi, \tau, \zeta_2) = (\infty, 0, 0)$.
In fact, these expansions also depends holomorphically on Λ. That is, for any $L \subseteq L_0(\mathbf{b})$ open with $\overline{L} \subseteq L_0(\mathbf{b})$, there is an open environment $Z \subseteq \mathbb{P}^* \times \mathbb{C}^2$ of $(\infty, 0, 0)$, such that the series (5.90) converge uniformly on $Z \times L$, defining holomorphic functions on this set in $(\xi, \tau, \zeta_2, \Lambda)$.

Proof. In both the case $i = 1$ and $i = 2$, we first prove the existence of such a formal solution analogous to Theorem 5.4.3. We then conclude that this formal series solution must equal $\widetilde{\Psi}_i^{\mathrm{tr}}(\xi, t, \phi; \Lambda)$, because of Lemma 5.6.3. □

By direct computation, we can calculate

$$\widetilde{D}^{\mathrm{tr},2}_{0,1,0}(\Lambda) = 0, \quad \widetilde{D}^{\mathrm{tr},2}_{0,1,-1}(\Lambda) = c, \tag{5.92}$$

where c as defined in (5.84), and we note that the first equality can also be deduced from Lemma 5.6.3.

5.6.3 Matching of True Solutions

We now return to the set up in Section 5.6.1 and prove Proposition 5.6.1. We use Lemma 5.6.2, to construct, for $i = 1, 2$, a true solution

$$\Psi_i^{\mathrm{tr}}(z, t) = \Psi_i^{\mathrm{tr}}(z, t, \phi(t); \Lambda),$$

of the Lax pair $L^{\infty,0,i}$, meromorphic on $\mathbb{C}^* \times V$. Note that by the second part of Lemma 5.6.2, the limit

$$\lim_{t \to 0, t \in V} \Psi_i^{\mathrm{tr}}(z, t) = D^{\mathrm{tr},i}_{0,0}(z; \Lambda), \tag{5.93}$$

exists, for z close but not equal to 0.

Lemma 5.6.5. *We have the identities*

$$\Psi_1^{\infty,0}(z, t) = \Psi_1^{tr}(z, t), \quad \Psi_2^{\infty,0}(z, t) = \Psi_2^{tr}(z, t). \tag{5.94}$$

Proof. We define

$$s(z) = \frac{\theta_q(z/\lambda)}{\theta_q(z)},$$

then $s(qz) = \lambda s(z)$ and it is easy to see that

$$s(z)\Psi_2^{\mathrm{tr}}(z,t),$$

defines a solution of the Lax pair $L^{\infty,0,1}$, as $\Psi_2^{\mathrm{tr}}(z,t)$ defines a solution of $L^{\infty,0,2}$. So both $\Psi_1^{\mathrm{tr}}(z,t)$ and $s(z)\Psi_2^{\mathrm{tr}}(z,t)$ define a solution of the Lax pair $L^{\infty,0,1}$, and it is easy to see that they are linearly independent. As $\Psi_1^{\infty,0}(z,t)$ also satisfies the Lax pair $L^{\infty,0,1}$, there must exist meromorphic functions $p_1(z,t)$ and $p_2(z,t)$ on $\mathbb{C}^* \times V$, which are q-periodic with respect to z and t, such that

$$\Psi_1^{\infty,0}(z,t) = p_1(z,t)\Psi_1^{\mathrm{tr}}(z,t) + p_2(z,t)s(z)\Psi_2^{\mathrm{tr}}(z,t). \tag{5.95}$$

Now we take any $t_0 \in V$, such that $p_1(z,t_0)$ and $p_2(z,t_0)$ are not identically singular. We set $t = q^n t_0$ in (5.95), and let $n \to \infty$, giving, by equation (5.93),

$$\Psi_1^{\infty,0}(z,0) = p_1(z,t_0)D_{0,0}^{\mathrm{tr},1}(z;\Lambda) + p_2(z,t_0)s(z)D_{0,0}^{\mathrm{tr},2}(z;\Lambda), \tag{5.96}$$

for z close but not equal to 0. By equation (5.80), we know that $\Psi_1^{\infty,0}(z,0)$ is holomorphic at $z = 0$ with $\Psi_1^{\infty,0}(0,0) = 1$. By Lemma 5.6.2, we know that both $D_{0,0}^{\mathrm{tr},1}(z;\Lambda)$ and $D_{0,0}^{\mathrm{tr},2}(z;\Lambda)$ are convergent power series with constant term equal to 1. Note that $p_2(z,t_0)s(z)$ cannot be regular at $z = 0$, unless it is identically zero, and hence we must have $p_1(z,t_0) \equiv 1$ and $p_2(z,t_0) \equiv 0$, by equation (5.96). As this holds for any $t_0 \in V$, such that $p_1(z,t_0)$ and $p_2(z,t_0)$ are not identically singular, we conclude that the first identity in (5.94) holds. We establish the other one analogously. □

Let $i \in \{1,2\}$, then, following (5.87),

$$\widetilde{\Psi}_i^{\mathrm{tr}}(\xi,t) := \Psi_i^{\mathrm{tr}}(\xi t,t), \tag{5.97}$$

defines a solution of the Lax pair $\widetilde{L}^{\mathrm{tr},i}$ (5.89), meromorphic on $\mathbb{C}^* \times V$ in (ξ,t). Furthermore, by the second part of Lemma 5.6.4, we know that the limits

$$\lim_{t\to 0, t\in V} \widetilde{\Psi}_1^{\mathrm{tr}}(\xi,t) = \widetilde{D}_{0,0}^{\mathrm{tr},1}(\xi;\Lambda), \tag{5.98a}$$

$$\lim_{t\to 0, t\in V} \phi(t)^{-1}\widetilde{\Psi}_2^{\mathrm{tr}}(\xi,t) = \widetilde{D}_{0,1}^{\mathrm{tr},2}(\xi;\Lambda), \tag{5.98b}$$

exists, for ξ close but not equal to ∞.

Lemma 5.6.6. *We have the identities,*

$$\Psi_1^{0,\infty}(\xi,t) = \widetilde{\Psi}_1^{tr}(\xi,t), \qquad \frac{c\phi(t)}{\xi}\Psi_2^{0,\infty}(\xi,t) = \widetilde{\Psi}_2^{tr}(\xi,t), \tag{5.99}$$

where the constant c is defined in (5.84).

Proof. Let us focus on the first equality. We note that the Lax pairs $\widetilde{L}^{\mathrm{tr},1}$ (5.89) and $L^{0,\infty,1}$ (5.77) are equivalent, i.e. $\widetilde{L}_1^{\mathrm{tr},1}$ and $L_1^{0,\infty,1}$ are identical, and $\widetilde{L}_2^{\mathrm{tr},1}$ and $L_2^{0,\infty,1}$ are a multiple of each other. Let us define

$$s(\xi)=\xi\frac{\theta_q(\xi/\lambda)}{\theta_q(\xi)},$$

then $s(q\xi)=q\lambda s(\xi)$ and we note that $s(\xi)\Psi_2^{0,\infty}(\xi,t)$ is a solution of $L^{0,\infty,1}$. So both $\Psi_1^{0,\infty}(\xi,t)$ and $s(\xi)\Psi_2^{0,\infty}(\xi,t)$ are solutions of $L^{0,\infty,1}$, and it is easy to see that they are linearly independent. As $\widetilde{\Psi}_1^{\text{tr}}(\xi,t)$ also satisfies $L^{0,\infty,1}$, there must exist meromorphic functions $p_1(\xi,t)$ and $p_2(\xi,t)$ on $\mathbb{C}^*\times V$, which are q-periodic with respect to ξ and t, such that

$$\widetilde{\Psi}_1^{\text{tr}}(\xi,t)=p_1(\xi,t)\Psi_1^{0,\infty}(\xi,t)+p_2(\xi,t)s(\xi)\Psi_2^{0,\infty}(\xi,t). \tag{5.100}$$

Take any $t_0\in V$ such that $p_1(\xi,t)$ and $p_2(\xi,t)$ are not identically singular. We set $t=q^nt_0$ and let $n\to\infty$ in (5.100), which gives, by equations (5.98a) and (5.82),

$$\widetilde{D}_{0,0}^{\text{tr},1}(\xi;\Lambda)=p_1(\xi,t_0)\psi_1^{\infty}(\xi;\sigma_0(\Lambda))+p_2(\xi,t_0)s(\xi)\psi_2^{\infty}(\xi;\sigma_0(\Lambda)), \tag{5.101}$$

for ξ close but not equal to ∞. Now we recall that, by Lemma 5.6.4, we know that $\widetilde{D}_{0,0}^{\text{tr},1}(\xi;\Lambda)$ has a convergent power series expansion about $\xi=\infty$. Note also that $\psi_1^{\infty}(\xi;\sigma_0(\Lambda))$ and $\psi_2^{\infty}(\xi;\sigma_0(\Lambda))$ are holomorphic and equal to 1 at $\xi=\infty$. However $p_2(\xi,t)s(\xi)$ is not regular at $\xi=\infty$ or identically zero. Combined we find, by equation (5.101), that we must have $p_1(\xi,t_0)\equiv 1$ and $p_2(\xi,t_0)\equiv 0$. We conclude that the first identity in (5.99) holds. As to the second one, let us write

$$\Psi(\xi,t)=\frac{\xi}{c\phi(t)}\widetilde{\Psi}_2^{\text{tr}}(\xi,t).$$

We wish to prove

$$\Psi(\xi,t)=\Psi_2^{0,\infty}(\xi,t). \tag{5.102}$$

Firstly, we note that both $\Psi_2^{0,\infty}(\xi,t)$ and $\Psi(\xi,t)$ define a solution of the Lax pair $L^{0,\infty,2}$ (5.77). Furthermore, we know that, by equation (5.98b), the limit

$$\lim_{t\to 0,t\in V}\Psi(\xi,t)=\frac{\xi}{c}\widetilde{D}_{0,1}^{\text{tr},2}(\xi;\Lambda),$$

exists for ξ close but not equal to ∞. Similarly we have $\Psi_2^{0,\infty}(\xi,0)=\psi_2^{\infty}(\xi;\sigma_0(\Lambda))$, by (5.82). Finally, by Lemma 5.6.4 and equations (5.92), we know that $\frac{\xi}{c}\widetilde{D}_{0,1}^{\text{tr},2}(\xi;\Lambda)$ has a convergent power series expansion in ξ about $\xi=\infty$, with

$$\frac{\xi}{c}\widetilde{D}_{0,1}^{\text{tr},2}(\xi;\Lambda)=1+\mathcal{O}(\xi^{-1}).\quad(\xi\to\infty)$$

A similar argument as above, where we write $\Psi(\xi,t)$ as a linear combination of $\Psi_2^{0,\infty}(\xi,t)$ and $s(\xi)^{-1}\Psi_1^{0,\infty}(\xi,t)$, finally gives (5.102). □

Proof of Proposition 5.6.1. The Proposition is now a consequence of equation (5.97) and Lemmas 5.6.5 and 5.6.6. □

By putting together Proposition 5.6.1 and equations (5.81) and (5.79), we find

$$\Psi^\infty(z,t) = \Psi^0\left(\frac{z}{t},t\right) Q\left(\frac{z}{t},\sigma_0(\Lambda(t))\right)\begin{pmatrix}1 & 0\\ 0 & \frac{ct\phi(t)}{z}\end{pmatrix} Q\left(z^{-1};\sigma_\infty(\Lambda(t))\right)^{-1}. \tag{5.103}$$

By equation (5.30), we know that

$$\widetilde{Y}^\infty(z,t) := s^\infty(t)\Psi^\infty(z,t), \quad s^\infty(t) := s^\infty(t,\phi(t);\Lambda), \tag{5.104}$$

defines a fundamental solution of Yamada's Lax pair $\widetilde{L}$ (5.6), which is meromorphic on $\mathbb{P}^* \times V$. We note that $s^\infty(t)$ satisfies

$$s^\infty(qt) = -q^{-1}G_1(\phi(t);\Lambda)^{-1}t^{-1}s^\infty(t).$$

Similarly, by (5.54), we know that

$$Y^0(z,t) = s^0(t)\Psi^0\left(\frac{z}{t},t\right), \quad s^0(t) := s^0(t,\phi(t);\Lambda), \tag{5.105}$$

defines a fundamental solution of Yamada's Lax pair L (2.21), which is meromorphic on $\mathbb{C}\times V$. We note that $s^0(t)$ satisfies

$$s^0(qt) = \Lambda q^{-1}G_1(\phi(t);\Lambda)^{-1}t^{-3}s^0(t).$$

Combining equations (5.103), (5.104) and (5.105), we obtain

$$Y^\infty(z,t) = Y^0(z,t)\mathcal{P}(z,t), \tag{5.106}$$

with

$$\mathcal{P}(z,t) = s(t)Q\left(\frac{z}{t},\sigma_0(\Lambda)\right)\begin{pmatrix}1 & 0\\ 0 & \frac{ct\phi(t)}{z}\end{pmatrix} Q\left(z^{-1};\sigma_\infty(\Lambda)\right)^{-1},$$

where $s(t)$ is the meromorphic function on T, satisfying $s(qt) = -t^2/\Lambda s(t)$, defined by

$$s(t) = \frac{s^\infty(t)}{s^0(t)}.$$

This is consistent with the notation in Section 4.9, where

$$c_0^1(t) := s^0(t)\Psi_1^0(0,t), \qquad c_0^2(t) := s^0(t)\Psi_2^0(0,t),$$
$$\widetilde{c}_0^1(t) := s^\infty(t)\Psi_1^\infty(\infty,t), \qquad \widetilde{c}_0^2(t) := s^\infty(t)\Psi_2^\infty(\infty,t).$$

Recall that the functions $Y^\infty(z,t)$ and $Y^0(z,t)$ are defined on $\mathbb{P}^*\times V$ and $\mathbb{C}\times V$ respectively, and the connection result (5.106) is valid on $\mathbb{C}^*\times V$. Now recall that we fixed V to be any continuous q-domain with $V \subseteq \overline{V}^* \subseteq T$, at the beginning of Section 5.6.1. By doing the same analysis on another continuous q-domain W with $W \subseteq \overline{W}^* \subseteq T$, we obtain identical results on the intersection $V\cap W$. We conclude that we can extend the domains of $Y^\infty(z,t)$ and $Y^0(z,t)$ to $\mathbb{P}^*\times T$ and $\mathbb{C}\times T$ respectively, and the connection result (5.106) is valid on $\mathbb{C}^*\times T$.

5.7 Monodromy Corresponding to Critical Behaviour at $t = 0$

Theorem 5.7.1. *Let (f, g) be a meromorphic solution of q-$P(A_1)$ on a continuous q-domain T, characterised by critical behaviour near $t = 0$ as in Theorem 3.4.1, by analytic functions $\Lambda(t)$ and $\phi(t)$. Then the corresponding monodromy of Yamada's Lax pair is given by*

$$M_T(f, g) = [\mathcal{R}(z, t)],$$

with

$$\mathcal{R}(z,t) = s(t)\theta_q(qb_5z, qb_6z, qb_7z, qb_8z)Q\left(\frac{z}{t}, \sigma_0(\Lambda(t))\right)\begin{pmatrix}1 & 0\\ 0 & \frac{c(t)t\phi(t)}{z}\end{pmatrix}Q\left(z^{-1}; \sigma_\infty(\Lambda(t))\right)^{-1},$$

where $s(t)$ is any nonzero meromorphic function satisfying $s(qt) = -t^2/\Lambda(t)s(t)$ on $\mathbb{C}^*$, the two sets of parameters $\sigma_0(\lambda)$ and $\sigma_\infty(\Lambda)$ are defined by

$$\sigma_0(\Lambda) = \left(b_1^{-1}, b_2^{-1}, b_3^{-1}, b_4^{-1}; -q^2\Lambda, -q\frac{\Lambda}{\lambda}\right), \tag{5.107}$$

$$\sigma_\infty(\Lambda) = \left(qb_5, qb_6, qb_7, qb_8; -\Lambda^{-1}, -\frac{\lambda}{\Lambda}\right), \tag{5.108}$$

where we suppressed the t dependence, and

$$c(t) = -\frac{q\lambda(t) - 1}{\lambda(t) - 1}, \qquad \lambda(t) = \frac{\Lambda(t)^2}{b_1b_2b_3b_4}.$$

Proof. Note that equation (5.106), gives the theorem for the case where $\Lambda(t) \equiv \Lambda$ is constant. To obtain the generic case, i.e. $\Lambda(t)$ not necessarily constant, we can simply follow the proof of (5.106), with Λ replaced by $\Lambda(t)$ everywhere. One has to be a bit more careful with limits such as (5.93) and (5.98), which now have to be taken on q-spirals, i.e. setting $t = q^n t_0$ and letting $n \to \infty$. Similarly an equation like (5.78), becomes

$$\lim_{n\to\infty} \Psi^\infty(z, q^n t_0) = y^0(z^{-1}; \sigma_\infty(\Lambda(t_0))).$$

The theorem now follows from (4.109). □

Let us consider the setting in Theorem 5.7.1, where, for the sake of simplicity, we assume that $\Lambda(t) \equiv \Lambda$ is constant, with of course $\Lambda \in L_0()$. We wish to use the explicit results in Section 4.5. Considering the condition (4.53) for $\sigma_0(\lambda)$ and $\sigma_\infty(\Lambda)$, we assume

$$\frac{b_i}{b_j} \notin q^{\mathbb{Z}}, \tag{5.109}$$

for $i, j \in \{1, 2, 3, 4\}$ and $i, j \in \{5, 6, 7, 8\}$. Furthermore it is required that $\Lambda \in L_0^*(\mathbf{b})$, where

$$L_0^*(\mathbf{b}) := L_0(\mathbf{b}) \setminus q^{\mathbb{Z}}\left(\{-b_ib_j : 1 \le i < j \le 4\} \cup \{-b_ib_j : 5 \le i < j \le 8\}\right). \tag{5.110}$$

Now to apply the results in Section 4.5 directly, i.e. without having to permute the parameters, assumption 4.66 has to be satisfied, which translates to

$$|b_1b_2|<|\Lambda|,\quad |\Lambda|<|b_5b_6|, \tag{5.111}$$

for the parameter values $\sigma_0(\lambda)$ and $\sigma_\infty(\Lambda)$ respectively. Without loss of generality, we may assume that the parameters $\mathbf{b}$ are chosen such that

$$|b_1|\leq|b_2|\leq|b_3|\leq|b_4|,\quad |b_5|\geq|b_6|\geq|b_7|\geq b_8|. \tag{5.112}$$

In this case, equation (5.111) is trivially satisfied, as we have

$$|b_1b_2|^2\leq|b_1b_2b_3b_4|\leq|\Lambda|^2\leq|b_5b_6b_7b_8|\leq|b_5b_6|^2,$$

where in the second and third inequality we used $\Lambda\in L_0()$.
Therefore, assuming (5.109) and (5.112) hold, and $\Lambda\in L_0^*(\mathbf{b})$, we find

$$\begin{aligned}[\mathcal{R}(z,t)]=\Bigg[s(t)&\begin{pmatrix}\theta_q\left(b_3\frac{z}{t}\right) & 0\\ 0 & \theta_q\left(b_4\frac{z}{t}\right)\end{pmatrix}\begin{pmatrix}q_{11}\theta_q\left(-q^2\frac{\Lambda}{b_3}\frac{z}{t}\right) & q_{12}\theta_q\left(-q\frac{\Lambda}{b_3\lambda}\frac{z}{t}\right)\\ q_{21}\theta_q\left(-q^2\frac{\Lambda}{b_4}\frac{z}{t}\right) & q_{22}\theta_q\left(-q\frac{\Lambda}{b_4\lambda}\frac{z}{t}\right)\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & \frac{c(t)t\phi(t)}{z}\end{pmatrix}\\ &\begin{pmatrix}r_{11}\theta_q\left(-\frac{\Lambda}{b_8\lambda}z\right) & r_{12}\theta_q\left(-\frac{\Lambda}{b_7\lambda}z\right)\\ r_{21}\theta_q\left(-\frac{\Lambda}{b_8}z\right) & r_{22}\theta_q\left(-\frac{\Lambda}{b_7}z\right)\end{pmatrix}\begin{pmatrix}\theta_q(qb_8z) & 0\\ 0 & \theta_q(qb_7z)\end{pmatrix}\Bigg],\end{aligned}$$

where the q_{ij} are defined in (4.74) with $\sigma=\sigma_0(\lambda)$, and the r_{ij} are defined by (4.70) with $\sigma=\sigma_\infty(\Lambda)$, for $i,j\in\{1,2\}$. As an additional check, one can now verify the equations (4.110) and (4.112) directly.

5.8 Generic Case: Analysis near $(z,t)=(0,\infty)$

We wish to calculate the monodromy corresponding to solutions of q-$P(A_1)$ with critical behaviour near $t=\infty$ as described in Theorem 3.4.2. In fact it is often easier to work with the formal expansion in equation (3.64) and we hence do most of the analysis on a formal level. As the analysis is very similar to the one near $t=0$, we skip over some details. We lighten the notation of the formal series solution a bit by writing the formal solution (3.64) as $f^\infty=f^\infty(t,\phi_\infty;\Lambda_\infty)$ and $g^\infty=g(t,\phi_\infty;\Lambda_\infty)$, with

$$f^\infty=\sum_{n=0}^{\infty}F_n^\infty t^{-n},\quad g^\infty=\sum_{n=0}^{\infty}G_n^\infty t^{-n},$$

where for $n\in\mathbb{N}$, the coefficients $F_n^\infty=F_n^\infty(\phi_\infty)=F_n^\infty(\phi_\infty;\Lambda_\infty)$ and $G_n^\infty=G_n(\phi)=G_n(\phi;\Lambda)$ are defined by

$$F_n^\infty=\sum_{i=-\infty}^{n+1}F_{n,i}^\infty\phi_\infty^i,\quad G_n^\infty=\sum_{i=-\infty}^{n+1}G_{n,i}^\infty\phi_\infty^i,$$

with for $i \leq n+1$, the coefficients $F^{\infty}_{n,i} = F^{\infty}_{n,i}(\Lambda_\infty)$ and $G^{\infty}_{n,i} = G^{\infty}_{n,i}(\Lambda_\infty)$ equal to

$$F^{\infty}_{n,i}(\Lambda_\infty) = F^{\infty,+}_{n,i}(\Lambda_\infty, \mathbf{b}), \quad G^{\infty}_{n,i}(\Lambda_\infty) = G^{\infty,+}_{n,i}(\Lambda_\infty, \mathbf{b}).$$

Furthermore we recall that ϕ_∞ satisfies $\overline{\phi}_\infty = \lambda_\infty \phi_\infty$, where λ_∞ as defined in (3.63).

We rescale the Yamada Lax pair L (2.21), by setting

$$y(z,t,\phi_\infty;\Lambda_\infty) = \widehat{s}^0(t,\phi_\infty;\Lambda_\infty)\widehat{\psi}^0(z,t,\phi_\infty;\Lambda_\infty), \tag{5.113}$$

where

$$\widehat{s}^0(qt,\lambda_\infty\phi_\infty;\Lambda_\infty) = \widehat{\alpha}\widehat{s}^0(t,\phi_\infty;\Lambda_\infty),$$

with

$$\widehat{\alpha} = -q^{-1}\Lambda_\infty G^{\infty}_0(\phi_\infty;\Lambda_\infty)^{-1}, \tag{5.114}$$

and we denote the corresponding rescaled Lax pair for $\widehat{\psi}^0(z,t,\phi_\infty;\Lambda_\infty)$ by $\widehat{L}^0$.

5.8.1 Expanding about $z = 0$

Expanding $\widehat{\psi}^0(z,t,\phi_\infty;\Lambda_\infty)$ in z around $z=0$,

$$\widehat{\psi}^0(z,t,\phi_\infty;\Lambda_\infty) = \widehat{c}^0_0(t,\phi_\infty;\Lambda_\infty) + \widehat{c}^0_1(t,\phi_\infty;\Lambda_\infty)z + \widehat{c}^0_2(t,\phi_\infty;\Lambda_\infty)z^2 + \ldots, \tag{5.115}$$

where $\widehat{c}^0_0 = \widehat{s}^0 c_0$, by equation (4.99), satisfies

$$\widehat{\gamma}^0_0(t,\phi_\infty)\widehat{c}^0_0(t,\phi_\infty) + \widehat{\gamma}^0_1(t,\phi_\infty)\widehat{c}^0_0(qt,\lambda_\infty\phi_\infty) + \widehat{\gamma}^0_2(t,\phi_\infty)\widehat{c}^0_0(q^2t,\lambda_\infty^2\phi_\infty), \tag{5.116}$$

where

$$(\widehat{\gamma}^0_0(t,\phi_\infty), \widehat{\gamma}^0_1(t,\phi_\infty), \widehat{\gamma}^0_2(t,\phi_\infty)) = (\gamma_0(t,\phi_\infty), \widehat{\alpha}\gamma_1(t,\phi_\infty), \widehat{\alpha}\overline{\widehat{\alpha}}\gamma_2(t,\phi_\infty)),$$

and we suppressed the Λ_∞ dependence throughout.

Proposition 5.8.1. *Consider equation (5.116) with $f = f^{\infty,+}(t,\phi_\infty;\Lambda_\infty)$ and $g = g^{\infty,+}(t,\phi_\infty;\Lambda_\infty)$ as defined in equation (3.64). Then there exists, for $i=1,2$, an unique formal solution of (5.116), of the form*

$$\widehat{c}^{0,i}_0(t,\phi_\infty;\Lambda_\infty) = \sum_{m=0}^{\infty} \widehat{c}^{0,i}_{0,m}(\phi_\infty;\Lambda_\infty)t^{-m},$$

with for $m \in \mathbb{N}$,

$$\widehat{c}^{0,i}_{0,m}(\phi_\infty;\Lambda_\infty) = \sum_{n=-\infty}^{m} \widehat{c}^{0,i}_{0,m,n}(\Lambda_\infty)\phi_\infty^n,$$

where

$$\begin{pmatrix} \widehat{c}^{0,1}_{0,0,0}(\Lambda_\infty) \\ \widehat{c}^{0,1}_{0,0,-1}(\Lambda_\infty) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \widehat{c}^{0,2}_{0,0,0}(\Lambda_\infty) \\ \widehat{c}^{0,2}_{0,0,-1}(\Lambda_\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5.117}$$

For $m \in \mathbb{N}$ and $n \in \mathbb{Z}_{\leq m}$ the coefficients $\widehat{c}^{0,i}_{0,m,n}(\Lambda_\infty)$ are rational functions in Λ_∞ and the parameters $b_1,\ldots,b_8$, in particular these rational functions are regular at points $(\Lambda_\infty,\mathbf{b}) \in$

$\mathbb{C}^* \times \mathcal{B}$, satisfying

$$1 \notin \{q_1^k q_2^l : (k,l) \in \mathbb{N}^2 \setminus \{(0,0)\}\}, \tag{5.118}$$

where $q_1 = q^{-1}\lambda_\infty$ and $q_2 = \lambda_\infty^{-1}$.
Furthermore, for fixed $\mathbf{b} \in \mathcal{B}$ with $|q| < 1$, for any $\Lambda_\infty \in L_0(\mathbf{b})$, condition (5.118) is satisfied and this formal solution, written in terms of the variables $\zeta_1 = t^{-1}\phi_\infty$ and $\zeta_2 = \phi_\infty^{-1}$,

$$\widehat{c}_0^{0,i}\left(\zeta_1^{-1}\zeta_2^{-1}, \zeta_2^{-1}; \Lambda_\infty\right) = \sum_{m,n=0}^{\infty} \widehat{c}_{0,m,m-n}^{0,i}(\Lambda_\infty)\zeta_1^m \zeta_2^n, \tag{5.119}$$

converges near $(\zeta_1, \zeta_2) = (0,0)$.
In fact, these expansions are also analytic in Λ. That is, for any $L \subseteq L_0(\mathbf{b})$ open with $\overline{L} \subseteq L_0(\mathbf{b})$, there is an open environment $Z \subseteq \mathbb{C}^2$ of $\mathbf{0}$, such that the series (5.119) converge uniformly on $Z \times L$, defining holomorphic functions on this set in $(\boldsymbol{\zeta}, \Lambda)$.

Proof. This is proven analogous to Proposition 5.4.1. $\square$

Remark 5.8.2. Recall that we used the formal series solution in equation (3.64), to construct true solutions of q-$P(A_1)$ in Theorem 3.4.2, by replacing the formal variables by actual analytic functions. Doing so we can use the formal series solutions in Proposition 5.8.1 to construct corresponding true solutions of (5.116).

5.8.2 Expanding about $t = \infty$

We consider the following formal expansion of $\widehat{\psi}^0(z,t,\phi_\infty;\Lambda_\infty)$ in t at $t = \infty$,

$$\widehat{\psi}^0(z,t,\phi_\infty;\Lambda_\infty) = \widehat{d}_0^0(z,\phi_\infty;\Lambda_\infty) + \widehat{d}_1^0(z,\phi_\infty;\Lambda_\infty)t^{-1} + \widehat{d}_2^0(z,\phi_\infty;\Lambda_\infty)t^{-2} + \dots,$$

which, upon substitution in $\widehat{L}_2^0$ and comparing leading order terms, gives

$$\widehat{d}_0^0(z,\phi_\infty;\Lambda_\infty) + (qzG_0^\infty - 1)\,\widehat{d}_0^0(qz,\phi_\infty;\Lambda_\infty) - q\Lambda_\infty z(F_0^\infty - qz)\widehat{d}_0^0(qz,\lambda_\infty\phi_\infty;\Lambda_\infty) = 0. \tag{5.120}$$

Considering a solution of equation (5.120), which takes the form

$$\widehat{d}_0^0(z,\phi_\infty;\Lambda_\infty) = \widehat{d}_{0,0}^0(z;\Lambda_\infty) + \widehat{d}_{0,-1}^0(z;\Lambda_\infty)\phi_\infty^{-1},$$

a calculation shows that, equation (5.120) is equivalent to

$$q\widehat{d}_{0,0}^0(z/q) + \left[-(1+q) + q(b_5+b_6+b_7+b_8)z + \left(q\Lambda_\infty + q^2\frac{\Lambda_\infty}{\lambda_\infty}\right)z^2\right]\widehat{d}_{0,0}^0(z) + (1-b_5qz)(1-b_6qz)(1-b_7qz)(1-b_8qz)\widehat{d}_{0,0}^0(qz) = 0, \tag{5.121}$$

together with

$$\widehat{d}_{0,-1}^0(z) = -\frac{\lambda_\infty}{\Lambda_\infty(\lambda_\infty - 1)z}\left[\widehat{d}_{0,0}^0(q^{-1}z) + \left(\Lambda_\infty z^2 + (G_{0,0}^\infty - \Lambda_\infty F_{0,0}^\infty)z - 1\right)\widehat{d}_{0,0}^0(z)\right],$$

where we suppressed Λ_∞ dependence of $\widehat{d}^0_{0,0}(z;\Lambda_\infty)$ and $\widehat{d}^0_{0,-1}(z;\Lambda_\infty)$. We identify (5.121) as the degree two model equation (4.51), via the scaling

$$\widehat{d}^0_{0,0}(z;\Lambda_\infty) = s_*(z)y(z;\Lambda_\infty), \quad s_*(z) = (b_5qz, b_6qz, b_7qz, b_8qz; q)_\infty\,, \tag{5.122}$$

with parameter values $\sigma = \widehat{\sigma}_0(\Lambda_\infty)$, defined in (5.153), noting that Fuchs' equation (4.52) is satisfied. For any $l_1(\Lambda_\infty)$ and $l_2(\Lambda_\infty)$, there exists an unique formal power series solution $\widehat{d}^0_{0,0}(z;\Lambda_\infty)$ of (5.121) with

$$\widehat{d}^0_{0,0}(z;\Lambda_\infty) = l_1(\Lambda_\infty) + l_1(\Lambda_\infty)z + \dots. \tag{5.123}$$

Now, by Proposition 5.8.1, the general formal solution of (5.116), takes the form

$$\widehat{c}^0_0(t,\phi_\infty;\Lambda_\infty) = k_1(\Lambda_\infty)\widehat{c}^{0,1}_0(t,\phi_\infty;\Lambda_\infty) + k_2(\Lambda_\infty)\widehat{c}^{0,2}_0(t,\phi_\infty;\Lambda_\infty), \tag{5.124}$$

with $k_1(\Lambda_\infty)$ and $k_2(\Lambda_\infty)$ free.
Analogously to equations (5.71), the free constants in (5.123) and (5.124), are related by

$$k_1(\Lambda_\infty) = l_1(\Lambda_\infty), \tag{5.125a}$$

$$k_2(\Lambda_\infty) = \frac{\lambda_\infty}{q\Lambda_\infty(\lambda_\infty - 1)}\left(q\left(\Lambda_\infty F^\infty_{0,0}(\Lambda_\infty) - G^\infty_{0,0}(\Lambda_\infty)\right)l_1(\Lambda_\infty) + (q-1)l_2(\Lambda_\infty)\right). \tag{5.125b}$$

5.8.3 Main Existence Theorem near $(z,t) = (0,\infty)$

Theorem 5.8.3. *Consider the Lax pair $\widehat{L}^0$, obtained by scaling L (2.21) by (5.113), with $f = f^{\infty,+}(t,\phi_\infty;\Lambda_\infty)$ and $g = g^{\infty,+}(t,\phi_\infty;\Lambda_\infty)$ as defined in equation (3.64). Then, for $i \in \{1,2\}$, there exists an unique formal series solution of the Lax pair $\widehat{L}^0$, of the form*

$$\widehat{\psi}^0_i(z,t,\phi_\infty;\Lambda_\infty) = \sum_{k=0}^{\infty}\widehat{c}^{0,i}_k(t,\phi_\infty;\Lambda_\infty)z^k,$$

where, for $k \in \mathbb{N}$,

$$\widehat{c}^{0,i}_k(t,\phi_\infty;\Lambda_\infty) = \sum_{m=0}^{\infty}\widehat{c}^{0,i}_{k,m}(\phi_\infty;\Lambda_\infty)t^{-m},$$

with, for $m \in \mathbb{N}$,

$$\widehat{c}^{0,i}_{k,m}(\phi_\infty;\Lambda_\infty) = \sum_{n=-\infty}^{m}\widehat{c}^{0,i}_{k,m,n}(\Lambda_\infty)\phi_\infty^n,$$

and initial conditions (5.117).
We note that the notation here coincides with that in Proposition 5.8.1. For $k,m \in \mathbb{N}$ and $n \in \mathbb{Z}_{\leq m}$, the coefficients $\widehat{c}^{0,i}_{k,m,n}(\Lambda_\infty)$ are rational functions in Λ_∞ and the parameters $b_1,\dots,b_8$, in particular these rational functions are regular at points $(\Lambda_\infty,\mathbf{b}) \in \mathbb{C}^*\times\mathcal{B}$, satisfying (5.118). Furthermore, for fixed $\mathbf{b} \in \mathcal{B}$ with $|q|<1$, for any $\Lambda_\infty \in L_0(\mathbf{b})$, condition (3.46) is satisfied

and this formal solution, written in terms of the variables z, $\zeta_1=t^{-1}\phi_\infty$ and $\zeta_2=\phi_\infty^{-1}$,

$$\widehat{\psi}_i^0\left(z,\zeta_1^{-1}\zeta_2^{-1},\zeta_2^{-1};\Lambda\right)=\sum_{k,m,n=0}^{\infty}\widehat{c}_{k,m,m-n}^{\,0,i}(\Lambda_\infty)z^k\zeta_1^m\zeta_2^n, \tag{5.126}$$

converges near $(z,\zeta_1,\zeta_2)=(0,0,0)$.
In fact, this expansion also depends holomorphically on Λ_∞. That is, for any $L\subseteq L_0(\mathbf{b})$ open with $\overline{L}\subseteq L_0(\mathbf{b})$, there is an open environment $Z\subseteq\mathbb{C}^3$ of $(0,0,0)$, such that the series (5.126) converge uniformly on $Z\times L$, defining holomorphic functions on this set in $(z,\zeta_1,\zeta_2,\Lambda_\infty)$.

Proof. We prove this analogous to Theorem 5.4.3. □

Remark 5.8.4. Recall that we used the formal series solution in equation (3.64), to construct true solutions of q-$P(A_1)$ in Theorem 3.4.2, by replacing the formal variables by actual analytic functions. Doing so we can use the formal series solutions in Theorem 5.8.3 to construct corresponding true solutions of the Lax pair $\widehat{L}^0$.

5.8.4 Transition from $(z,t)=(0,\infty)$ to $(z,t)=(\infty,\infty)$

Note that the two formal series solutions $\widehat{\psi}_1^0(z,t,\phi_\infty;\Lambda_\infty)$ and $\widehat{\psi}_2^0(z,t,\phi_\infty;\Lambda_\infty)$, defined in Theorem 5.8.3, form a basis of formal solutions of $\widehat{L}^0$. It is more convenient for us to work with a different basis of solutions, given by

$$\widehat{\Psi}_1^0(z,t,\phi_\infty;\Lambda_\infty):=1\cdot\widehat{\psi}_1^0(z,t,\phi_\infty;\Lambda_\infty)+k_2^*(\Lambda_\infty)\widehat{\psi}_2^0(z,t,\phi_\infty;\Lambda_\infty), \tag{5.127a}$$

$$\widehat{\Psi}_2^0(z,t,\phi_\infty;\Lambda_\infty):=0\cdot\widehat{\psi}_1^0(z,t,\phi_\infty;\Lambda_\infty)+\frac{(q-1)}{q\Lambda_\infty(1-1/\lambda_\infty)}\psi_2^0(z,t,\phi_\infty;\Lambda_\infty). \tag{5.127b}$$

where $k_2^*(\Lambda_\infty)$ is defined by

$$k_2^*(\Lambda_\infty)=\frac{\lambda_\infty}{q\Lambda_\infty(\lambda_\infty-1)}\left(q\left(\Lambda_\infty F_{0,0}^\infty(\Lambda_\infty)-G_{0,0}^\infty(\Lambda_\infty)\right)+(q-1)u\right),$$

where, recalling the definition of $s_*(z)$ in (5.122), the constant $u\in\mathbb{C}$ is defined uniquely by

$$s_*(z)=1+uz+\mathcal{O}\left(z^2\right),\quad(z\to0)$$

or explicitly

$$u=\frac{q}{q-1}\left(b_5+b_6+b_7+b_8\right).$$

By equations (5.125), these two solutions correspond respectively to $(l_1(\Lambda_\infty),l_2(\Lambda_\infty))=(1,u)$ and $(l_1(\Lambda_\infty),l_2(\Lambda_\infty))=(0,1)$ in (5.47). The reason for this choice of basis, is that, writing for $i=1,2$,

$$\widehat{\Psi}_i^0(z,t,\phi_\infty;\Lambda_\infty)=\sum_{k=0}^{\infty}\sum_{m=-\infty}^{k}\widehat{D}_{k,m}^{0,i}(z;\Lambda_\infty)t^k\phi_\infty^m,$$

we have

$$\widehat{D}_{0,0}^0(z;\Lambda_\infty):=(\widehat{D}_{0,0}^{0,1}(z;\Lambda_\infty),\widehat{D}_{0,0}^{0,2}(z;\Lambda_\infty))=s_*(z)y^0(z;\widehat{\sigma}_0(\Lambda_\infty)).$$

We define

$$\widehat{D}_{0,0}^{0,\infty}(z;\Lambda_\infty)=\left(\frac{1}{b_5z},\frac{1}{b_6z},\frac{1}{b_7z},\frac{1}{b_8z};q\right)^{-1}\psi^\infty(z;\widehat{\sigma}_0(\Lambda_\infty)),$$

which leads to the connection result

$$\widehat{D}_{0,0}^{0,\infty}(z;\Lambda_\infty)=\widehat{D}_{0,0}^{0}(z;\Lambda_\infty)\widehat{P}^0(z;\Lambda_\infty), \tag{5.128}$$

where

$$\widehat{P}^0(z;\Lambda_\infty):=\theta_q(qb_5z,qb_6z,qb_7z,qb_8z)^{-1}Q(z;\widehat{\sigma}_0(\Lambda_\infty)).$$

We symbolically define

$$\widehat{\Psi}^{0,\infty}(z,t,\phi_\infty;\Lambda_\infty):=\widehat{\Psi}^0(z,t,\phi_\infty;\Lambda_\infty)\cdot\widehat{P}^0(z;\Lambda_\infty). \tag{5.129}$$

From (4.46), we obtain

$$\widehat{P}^0(qz;\Lambda_\infty)=\widehat{P}^0(z;\Lambda_\infty)\cdot\begin{pmatrix}-q^3\frac{\Lambda_\infty}{\lambda_\infty}z^2 & 0\\ 0 & -q^2\Lambda_\infty z^2\end{pmatrix},$$

and hence, for $i=1,2$, a symbolic computation, shows that the component $\widehat{\Psi}_i^{0,\infty}(z,t,\phi_\infty;\Lambda_\infty)$, defines a solution of the Lax pair $\widehat{L}^{0,\infty,i}$, given by

$$\widehat{L}_1^{0,\infty,i}:\quad q^2\epsilon_iz^2u(z,t)\widehat{\psi}_i^{0,\infty}(qz,t)+v(z,t)\widehat{\psi}_i^{0,\infty}(z,t)+\frac{1}{\epsilon_iz^2}w(z,t)\widehat{\Psi}_i^{0,\infty}(z/q,t)=0, \tag{5.130a}$$

$$\widehat{L}_2^{0,\infty,i}:\quad \widehat{\alpha}h_0(z,t)\widehat{\Psi}_i^{0,\infty}(z,qt)+h_1(z,t)\widehat{\Psi}_i^{0,\infty}(z,t)+\frac{1}{\epsilon_iz^2}h_2(z,t)\widehat{\Psi}_i^{0,\infty}(z/q,t)=0, \tag{5.130b}$$

where $\epsilon_1=-q\Lambda_\infty/\lambda_\infty$, $\epsilon_2=-\Lambda_\infty$ and $\widehat{\alpha}$ as defined in (5.114).

5.9 Generic Case: Analysis near $(\xi,t)=(\infty,\infty)$

we rescale the Lax pair $\widetilde{L}$ (4.82), by setting

$$\widetilde{y}(z,t,\phi_\infty;\Lambda_\infty)=\widehat{s}^\infty(t,\phi_\infty,\Lambda_\infty)\widehat{\psi}^\infty(\xi,t,\phi_\infty;\Lambda_\infty),\quad \xi=\frac{z}{t}, \tag{5.131}$$

where

$$\widehat{s}^\infty(qt,\lambda_\infty\phi_\infty,\Lambda_\infty)=\widehat{\beta}t^2\widehat{s}^\infty(t,\phi_\infty,\Lambda_\infty),$$

with

$$\beta:=q^{-1}G_0^\infty(\phi_\infty;\Lambda_\infty)^{-1}, \tag{5.132}$$

where we invite the reader to choose $\widehat{s}^\infty(t,\phi_\infty,\Lambda_\infty)$, meromorphic on $\mathbb{C}^*\times\mathbb{C}^*\times\mathbb{C}^*$, at their pleasure. We denote the rescaled Lax pair, which $\psi^\infty(z,t,\phi)$ satisfies, by $\widehat{L}^\infty$.

5.9.1 Expanding about $\xi = \infty$

We formally expand $\widehat{\psi}^\infty(\xi, t, \phi_\infty; \Lambda_\infty)$ in ξ about $\xi = \infty$,

$$\widehat{\psi}^\infty(\xi, t, \phi_\infty; \Lambda_\infty) = \widehat{c}_0^\infty(t, \phi_\infty; \Lambda_\infty) + \widehat{c}_1^\infty(t, \phi_\infty; \Lambda_\infty)\xi^{-1} + \widehat{c}_2^\infty(t, \phi_\infty; \Lambda_\infty)\xi^{-2} + \ldots,$$

which, using equation (4.105), leads to

$$\widehat{\gamma}_0^\infty(t, \phi_\infty)\widehat{c}_0^\infty(t, \phi_\infty) + \widehat{\gamma}_1^\infty(t, \phi_\infty)\widehat{c}_0^\infty(qt, \lambda_\infty\phi_\infty) + \widehat{\gamma}_2^\infty(t, \phi_\infty)\widehat{c}_0^\infty(q^2t, \lambda_\infty^2\phi_\infty), \tag{5.133}$$

where

$$(\widehat{\gamma}_0^\infty(t, \phi_\infty), \widehat{\gamma}_1^\infty(t, \phi_\infty), \widehat{\gamma}_2^\infty(t, \phi_\infty)) = (\widetilde{\gamma}_0(t, \phi_\infty), \widehat{\beta}t^2\widetilde{\gamma}_1(t, \phi_\infty), q^2\widehat{\beta}\overline{\widehat{\beta}}t^4\widetilde{\gamma}_2(t, \phi_\infty)),$$

and we suppressed the Λ_∞ dependence throughout.

Proposition 5.9.1. *Consider equation (5.133) with $f = f^{\infty,+}(t, \phi_\infty; \Lambda_\infty)$ and $g = g^{\infty,+}(t, \phi_\infty; \Lambda_\infty)$ as defined in equation (3.64). Then there exists, for $i = 1, 2$, an unique formal solution of (5.133), of the form*

$$\widehat{c}_0^{\infty,i}(t, \phi_\infty; \Lambda_\infty) = \sum_{m=0}^{\infty} \widehat{c}_{\infty,m}^{0,i}(\phi_\infty; \Lambda_\infty)t^{-m},$$

with for $m \in \mathbb{N}$,

$$\widehat{c}_{\infty,m}^{0,i}(\phi_\infty; \Lambda_\infty) = \sum_{n=-\infty}^{m} \widehat{c}_{\infty,m,n}^{0,i}(\Lambda_\infty)\phi_\infty^n,$$

where

$$\begin{pmatrix} \widehat{c}_{0,0,0}^{\infty,1}(\Lambda_\infty) \\ \widehat{c}_{0,1,1}^{\infty,1}(\Lambda_\infty) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \widehat{c}_{0,0,0}^{\infty,2}(\Lambda_\infty) \\ \widehat{c}_{0,1,1}^{\infty,2}(\Lambda_\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{5.134}$$

For $m \in \mathbb{N}$ and $n \in \mathbb{Z}_{\leq m}$ the coefficients $\widehat{c}_{\infty,m,n}^{0,i}(\Lambda_\infty)$ are rational functions in Λ_∞ and the parameters $b_1, \ldots, b_8$, in particular these rational functions are regular at points $(\Lambda_\infty, \mathbf{b}) \in \mathbb{C}^* \times \mathcal{B}$, satisfying (5.118).
Furthermore, for fixed $\mathbf{b} \in \mathcal{B}$ with $|q| < 1$, for any $\Lambda_\infty \in L_0(\mathbf{b})$, condition (5.118) is satisfied and this formal solution, written in terms of the variables $\zeta_1 = t^{-1}\phi_\infty$ and $\zeta_2 = \phi_\infty^{-1}$,

$$\widehat{c}_0^{\infty,i}\left(\zeta_1^{-1}\zeta_2^{-1}, \zeta_2^{-1}; \Lambda_\infty\right) = \sum_{m,n=0}^{\infty} \widehat{c}_{\infty,m,m-n}^{0,i}(\Lambda_\infty)\zeta_1^m\zeta_2^n, \tag{5.135}$$

converges near $(\zeta_1, \zeta_2) = (0, 0)$.
In fact, these expansions are also analytic in Λ. That is, for any $L \subseteq L_0(\mathbf{b})$ open with $\overline{L} \subseteq L_0(\mathbf{b})$, there is an open environment $Z \subseteq \mathbb{C}^2$ of $\mathbf{0}$, such that the series (5.135) converge uniformly on $Z \times L$, defining holomorphic functions on this set in $(\boldsymbol{\zeta}, \Lambda)$.

Proof. We prove this analogous to Proposition 5.4.1. □

Remark 5.9.2. Recall that we used the formal series solution in equation (3.64), to construct true solutions of q-$P(A_1)$ in Theorem 3.4.2, by replacing the formal variables by actual analytic functions. Doing so we can use the formal series solutions in Proposition 5.9.1 to construct corresponding true solutions of (5.133).

5.9.2 Expanding about $t = \infty$

We consider the following formal expansion of $\widehat{\psi}^{\infty}(\xi, t, \phi_\infty; \Lambda_\infty)$ in t at $t = \infty$,

$$\widehat{\psi}^{\infty}(\xi, t, \phi_\infty; \Lambda_\infty) = \widehat{d}_0^{\infty}(\xi, \phi_\infty; \Lambda_\infty) + \widehat{d}_1^{\infty}(\xi, \phi_\infty; \Lambda_\infty)t^{-1} + \widehat{d}_2^{\infty}(\xi, \phi_\infty; \Lambda_\infty)t^{-2} + \dots,$$

which, upon substitution in $\widehat{L}_2^{\infty}$ and comparing leading order terms, gives

$$\widehat{d}_0^{\infty}(\xi, \lambda_\infty\phi_\infty; \Lambda_\infty) = \widehat{d}_0^{\infty}(\xi, \phi_\infty; \Lambda_\infty). \tag{5.136}$$

We hence set $\widehat{d}_0^{\infty}(\xi, \phi_\infty; \Lambda_\infty) = \widehat{d}_{0,0}^{\infty}(\xi; \Lambda_\infty)$, which, by considering $\widehat{L}_1^{\infty}$, leads to

$$\begin{aligned} q\widehat{d}_{0,0}^{\infty}(q\xi; \Lambda_\infty) + \left[-(1+q) + (b_1^{-1} + b_2^{-1} + b_3^{-1} + b_4^{-1})\xi^{-1} + \left(\frac{1}{q\Lambda_\infty} + \frac{\lambda_\infty}{q\Lambda_\infty}\right)\xi^{-2}\right]\widehat{d}_{0,0}^{\infty}(\xi; \Lambda_\infty) \\ + \left(1 - \frac{1}{b_1\xi}\right)\left(1 - \frac{1}{b_2\xi}\right)\left(1 - \frac{1}{b_3\xi}\right)\left(1 - \frac{1}{b_4\xi}\right)\widehat{d}_{0,0}^{\infty}(q^{-1}\xi; \Lambda_\infty) = 0, \end{aligned} \tag{5.137}$$

which we identify with the degree two model equation (4.51), via the rescaling

$$\widehat{d}_{0,0}^{\infty}(\xi; \Lambda_\infty) = s_\diamond(\xi)y(\xi^{-1}; \Lambda_\infty), \quad s_\diamond(\xi) = \left(b_1^{-1}\xi^{-1}, b_2^{-1}\xi^{-1}, b_3^{-1}\xi^{-1}, b_4^{-1}\xi^{-1}; q\right), \tag{5.138}$$

with the parameter values $\sigma = \widehat{\sigma}_\infty(\Lambda_\infty)$, defined in (5.154), where we note that Fuchs' equation (4.52) is indeed satisfied.
For any $l_1(\Lambda_\infty)$ and $l_2(\Lambda_\infty)$, there exists an unique formal power series solution $\widehat{d}_{0,0}^{\infty}(\xi; \Lambda_\infty)$ of (5.137) with

$$\widehat{d}_{0,0}^{\infty}(\xi; \Lambda_\infty) = l_1(\Lambda_\infty) + l_2(\Lambda_\infty)\xi^{-1} + \dots. \tag{5.139}$$

Now, by Proposition 5.9.1, the general formal solution of (5.133), takes the form

$$\widehat{c}_0^{\infty}(t, \phi_\infty; \Lambda_\infty) = k_1(\Lambda_\infty)\widehat{c}_0^{\infty,1}(t, \phi_\infty; \Lambda_\infty) + k_2(\Lambda_\infty)\widehat{c}_0^{\infty,2}(t, \phi_\infty; \Lambda_\infty), \tag{5.140}$$

with $k_1(\Lambda_\infty)$ and $k_2(\Lambda_\infty)$ free.
Analogously to equations (5.71), the free constants in (5.139) and (5.140), are related by

$$k_1(\Lambda_\infty) = l_1(\Lambda_\infty), \tag{5.141a}$$

$$k_2(\Lambda_\infty) = \frac{q}{q - \lambda_\infty}\left(\frac{\Lambda_\infty}{G_{0,1}^{\infty}(\Lambda_\infty)}l_1(\Lambda_\infty) + (q-1)G_{0,1}^{\infty}(\Lambda_\infty)l_2(\Lambda_\infty)\right). \tag{5.141b}$$

5.9.3 Main Existence Theorem near $(z, t) = (\infty, \infty)$

Theorem 5.9.3. *Consider the Lax pair $\widehat{L}^{\infty}$, obtained by scaling $\widetilde{L}$ (4.82) by (5.131), with $f = f^{\infty,+}(t, \phi_\infty; \Lambda_\infty)$ and $g = g^{\infty,+}(t, \phi_\infty; \Lambda_\infty)$ as defined in equation (3.64). Then, for*

$i \in \{1, 2\}$, there exists an unique formal series solution of the Lax pair $\widehat{L}^{\infty}$, of the form

$$\widehat{\psi}_i^{\infty}(\xi, t, \phi_{\infty}; \Lambda_{\infty}) = \sum_{k=0}^{\infty} \widehat{c}_k^{\infty,i}(t, \phi_{\infty}; \Lambda_{\infty})\xi^{-k},$$

where, for $k \in \mathbb{N}$,

$$\widehat{c}_k^{\infty,i}(t, \phi_{\infty}; \Lambda_{\infty}) = \sum_{m=0}^{\infty} \widehat{c}_{k,m}^{\infty,i}(\phi_{\infty}; \Lambda_{\infty})t^{-m},$$

with, for $m \in \mathbb{N}$,

$$\widehat{c}_{k,m}^{\infty,i}(\phi_{\infty}; \Lambda_{\infty}) = \sum_{n=-\infty}^{m} \widehat{c}_{k,m,n}^{\infty,i}(\Lambda_{\infty})\phi_{\infty}^n,$$

and initial conditions (5.134).
We note that the notation here coincides with that in Proposition 5.9.1. For $k, m \in \mathbb{N}$ and $n \in \mathbb{Z}_{\leq m}$, the coefficients $\widehat{c}_{k,m,n}^{\infty,i}(\Lambda_{\infty})$ are rational functions in Λ_{∞} and the parameters $b_1, \ldots, b_8$, in particular these rational functions are regular at points $(\Lambda_{\infty}, \mathbf{b}) \in \mathbb{C}^* \times \mathcal{B}$, satisfying (5.118). Furthermore, for fixed $\mathbf{b} \in \mathcal{B}$ with $|q| < 1$, for any $\Lambda_{\infty} \in L_0(\mathbf{b})$, condition (3.46) is satisfied and this formal solution, written in terms of the variables ξ, $\zeta_1 = t^{-1}\phi_{\infty}$ and $\zeta_2 = \phi_{\infty}^{-1}$,

$$\widehat{\psi}_i^{\infty,i}\left(\xi, \zeta_1^{-1}\zeta_2^{-1}, \zeta_2^{-1}; \Lambda\right) = \sum_{k,m,n=0}^{\infty} \widehat{c}_{k,m,m-n}^{\infty,i}(\Lambda_{\infty})\xi^{-k}\zeta_1^m\zeta_2^n, \tag{5.142}$$

converges near $(\xi, \zeta_1, \zeta_2) = (\infty, 0, 0)$.
In fact, this expansion also depends holomorphically on Λ_{∞}. That is, for any $L \subseteq L_0(\mathbf{b})$ open with $\overline{L} \subseteq L_0(\mathbf{b})$, there is an open environment $Z \subseteq \mathbb{P}^* \times \mathbb{C}^2$ of $(\infty, 0, 0)$, such that the series (5.142) converge uniformly on $Z \times L$, defining holomorphic functions on this set in $(\xi, \zeta_1, \zeta_2, \Lambda_{\infty})$.

Proof. We prove this analogous to Theorem 5.4.3. □

Remark 5.9.4. Recall that we used the formal series solution in equation (3.64), to construct true solutions of q-$P(A_1)$ in Theorem 3.4.2, by replacing the formal variables by actual analytic functions. Doing so we can use the formal series solutions in Theorem 5.9.3 to construct corresponding true solutions of the Lax pair $\widehat{L}^{\infty}$.

5.9.4 Transition from $(\xi, t) = (\infty, \infty)$ to $(\xi, t) = (0, \infty)$

Note that the two formal series solutions $\widehat{\psi}_1^{\infty}(\xi, t, \phi_{\infty}; \Lambda_{\infty})$ and $\widehat{\psi}_2^{\infty}(\xi, t, \phi_{\infty}; \Lambda_{\infty})$, defined in Theorem 5.9.3, form a basis of formal solutions of $\widehat{L}^{\infty}$. It is more convenient for us to work with a different basis of solutions, given by

$$\widehat{\Psi}_1^{\infty}(\xi, t, \phi_{\infty}; \Lambda_{\infty}) := 1 \cdot \widehat{\psi}_1^{\infty}(\xi, t, \phi_{\infty}; \Lambda_{\infty}) + k_2^*(\Lambda_{\infty})\widehat{\psi}_2^{\infty}(\xi, t, \phi_{\infty}; \Lambda_{\infty}), \tag{5.143a}$$

$$\widehat{\Psi}_2^{\infty}(\xi, t, \phi_{\infty}; \Lambda_{\infty}) := 0 \cdot \widehat{\psi}_1^{\infty}(\xi, t, \phi_{\infty}; \Lambda_{\infty}) + \frac{q(q-1)}{q-\lambda_{\infty}}G_{0,1}^{\infty}(\Lambda_{\infty})\psi_2^{\infty}(\xi, t, \phi_{\infty}; \Lambda_{\infty}). \tag{5.143b}$$

where $k_2^*(\Lambda_\infty)$ is defined by

$$k_2^*(\Lambda_\infty) = \frac{q}{q-\lambda_\infty}\left(\frac{\Lambda_\infty}{G_{0,1}^\infty(\Lambda_\infty)} + (q-1)G_{0,1}^\infty(\Lambda_\infty)v\right),$$

where, recalling the definition in (5.138) of $s_\diamond(\xi)$, the constant $v \in \mathbb{C}$ is defined uniquely by

$$s_\diamond(\xi) = 1 + v\xi^{-1} + \mathcal{O}\left(\xi^{-2}\right), \quad (\xi \to \infty)$$

and explicitly given by

$$v = \frac{1}{q-1}\left(b_1^{-1} + b_2^{-1} + b_3^{-1} + b_4^{-1}\right).$$

By equations (5.141), these two solutions correspond respectively to $(l_1(\Lambda_\infty), l_2(\Lambda_\infty)) = (1, v)$ and $(l_1(\Lambda_\infty), l_2(\Lambda_\infty)) = (0, 1)$ in (5.139). The reason for this choice of basis, is that, writing for $i = 1, 2$,

$$\widehat{\Psi}_i^\infty(\xi, t, \phi; \Lambda) = \sum_{k=0}^\infty \sum_{m=-\infty}^k \widehat{D}_{k,m}^{\infty,i}(\xi; \Lambda_\infty) t^k \phi_\infty^m,$$

we have

$$\widehat{D}_{0,0}^\infty(\xi; \Lambda_\infty) := (\widehat{D}_{0,0}^{\infty,1}(\xi; \Lambda_\infty), \widehat{D}_{0,0}^{\infty,2}(\xi; \Lambda_\infty)) = s_\diamond(\xi) y^0(\xi^{-1}; \widehat{\sigma}_\infty(\Lambda_\infty)).$$

We define

$$\widehat{D}_{0,0}^{\infty,0}(\xi; \Lambda_\infty) = (qb_1\xi, qb_2\xi, qb_3\xi, qb_4\xi; q)^{-1}\,\psi^\infty(\xi^{-1}; \widehat{\sigma}_\infty(\Lambda_\infty)),$$

which leads to the connection result

$$\widehat{D}_{0,0}^{\infty,0}(\xi; \Lambda_\infty) = \widehat{D}_{0,0}^\infty(\xi; \Lambda_\infty) \widehat{P}^\infty(\xi; \Lambda_\infty)^{-1}, \tag{5.144}$$

where

$$\widehat{P}^\infty(\xi; \Lambda_\infty) := \theta_q(qb_1\xi, qb_2\xi, qb_3\xi, qb_4\xi) Q(\xi^{-1}; \widehat{\sigma}_\infty(\Lambda_\infty))^{-1}.$$

We symbolically define

$$\widehat{\Psi}^{\infty,0}(\xi, t, \phi_\infty; \Lambda_\infty) := \widehat{\Psi}^\infty(\xi, t, \phi_\infty; \Lambda_\infty) \cdot \widehat{P}^\infty(\xi; \Lambda_\infty)^{-1}. \tag{5.145}$$

From (4.46), we obtain

$$\widehat{P}^\infty(q\xi; \Lambda_\infty) = \begin{pmatrix} -\frac{\lambda_\infty}{q^2\Lambda_\infty}\xi^{-2} & 0 \\ 0 & -\frac{1}{q^2\Lambda_\infty}\xi^{-2} \end{pmatrix} \cdot \widehat{P}^\infty(\xi; \Lambda_\infty),$$

and hence, for $i = 1, 2$, a symbolic computation, shows that the component $\widehat{\Psi}_i^{\infty,0}(\xi, t, \phi_\infty; \Lambda_\infty)$, defines a solution of the Lax pair $\widehat{L}^{\infty,0,i}$, given by

$$\widehat{L}_1^{\infty,0,i}: \quad \widetilde{\epsilon}_1 z^{-2} u(\xi t, t)\psi_i^0(q\xi, t) + v(\xi t, t)\psi_i^0(\xi, t) + \frac{1}{q^2\widetilde{\epsilon}_i} z^2 w(\xi t, t)\psi_i^0(\xi/q, t) = 0, \tag{5.146a}$$

$$\widehat{L}_2^{\infty,0,i}: \quad \frac{\widehat{\beta}}{q^2\widetilde{\epsilon}_i} h_0(\xi t, t)\psi_i^0(\xi/q, qt) + h_1(\xi t, t)\psi_i^0(\xi, t) + \frac{1}{q^2\widetilde{\epsilon}_i} z^2 h_2(\xi t, t)\psi_i^0(\xi/q, t) = 0. \tag{5.146b}$$

where $\widetilde{\epsilon}_1 = -\frac{\lambda_\infty}{q^2\Lambda_\infty}$, $\widetilde{\epsilon}_2 = -\frac{1}{q^2\Lambda_\infty}$ and $\widehat{\beta}$ as defined in (5.132).

5.10 Generic Case: Matching near $t = \infty$

As explained in Section 5.6, the matching procedure is only sensible on the level of true solutions, i.e. with $\Lambda_\infty = \Lambda_\infty(t)$ and $\phi_\infty = \phi_\infty(t)$ analytic functions. So let (f, g) be a meromorphic solution of q-$P(A_1)$ on a continuous q-domain T, characterised by critical behaviour near $t = \infty$ as in Theorem 3.4.2, by analytic functions $\Lambda_\infty(t)$ and $\phi_\infty(t)$.

We fix a continuous q-domain $V \subseteq \overline{V}^* \subseteq T$ and consider the Lax pair $\widehat{L}^0$, obtained by scaling L (2.21) by (5.113). Then Theorem 5.8.3, shows us that

$$\widehat{\psi}^0(z,t) := \left(\widehat{\psi}_1^0(z, t, \phi_\infty(t); \Lambda_\infty(t)), \widehat{\psi}_2^0(z, t, \phi_\infty(t); \Lambda_\infty(t))\right),$$

defines a fundamental solution of $\widehat{L}^0$, for (z, t) close to $(0, \infty)$ in $\mathbb{C} \times V$, which has an unique meromorphic continuation to $\mathbb{C} \times V$. We use the change of basis (5.127), to define the corresponding fundamental solution of $\widehat{L}^0$,

$$\widehat{\Psi}^0(z,t) := (\widehat{\Psi}_1^0(z, t, \phi_\infty(t); \Lambda_\infty(t)), \widehat{\Psi}_2^0(z, t, \phi_\infty(t); \Lambda_\infty(t))).$$

Following (5.129), we define

$$\widehat{\Psi}^{0,\infty}(z,t) := \widehat{\Psi}^0(z,t) \cdot \widehat{P}^0(z; \Lambda_\infty(t)). \tag{5.147}$$

Then we know that the component $\widehat{\Psi}_i^{0,\infty}(z,t)$, defines a solution of the Lax pair $\widehat{L}^{0,\infty,i}$ (5.130), where $\epsilon_1 = -q\Lambda_\infty(t)/\lambda_\infty(t)$, $\epsilon_2 = -\Lambda_\infty(t)$ and $\widehat{\alpha}$ as defined in (5.114).

Next we consider the Lax pair $\widehat{L}^\infty$, obtained by scaling $\widetilde{L}$ (4.82) by (5.131). Then Theorem 5.9.3, shows us that

$$\widehat{\psi}^\infty(\xi,t) := \left(\widehat{\psi}_1^\infty(\xi, t, \phi_\infty(t); \Lambda_\infty(t)), \widehat{\psi}_2^\infty(\xi, t, \phi_\infty(t); \Lambda_\infty(t))\right),$$

defines a fundamental solution of $\widehat{L}^\infty$, for (ξ, t) close to (∞, ∞) in $\mathbb{P}^* \times V$, which has an unique meromorphic continuation to $\mathbb{P}^* \times V$. We use the change of basis (5.143), to define the corresponding fundamental solution of $\widehat{L}^\infty$,

$$\widehat{\Psi}^\infty(\xi,t) := \left(\widehat{\Psi}_1^\infty(\xi, t, \phi_\infty(t); \Lambda_\infty(t)), \widehat{\Psi}_2^\infty(\xi, t, \phi_\infty(t); \Lambda_\infty(t))\right).$$

Following (5.145), we define

$$\widehat{\Psi}^{\infty,0}(\xi,t) := \widehat{\Psi}^{\infty}(\xi,t)\cdot \widehat{P}^{\infty}(\xi;\Lambda_\infty(t))^{-1}. \tag{5.148}$$

Then we know that the component $\widehat{\Psi}_i^{\infty,0}(\xi,t)$, defines a solution of the Lax pair $\widehat{L}^{\infty,0,i}$ (5.146), where $\widetilde{\epsilon}_1 = -\frac{\lambda_\infty(t)}{q^2\Lambda_\infty(t)}$, $\widetilde{\epsilon}_2 = -\frac{1}{q^2\Lambda_\infty(t)}$ and $\widehat{\beta}$ as defined in (5.132).

We now wish to relate $\widehat{\Psi}_1^{0,\infty}(z,t)$ and $\widehat{\Psi}_1^{\infty,0}(\xi,t)$, and we wish to relate $\widehat{\Psi}_2^{0,\infty}(z,t)$ and $\widehat{\Psi}_2^{\infty,0}(\xi,t)$. Let us make the crucial observation that $\widehat{\Psi}_2^{0,\infty}(z,t)$ and $\widehat{\Psi}_2^{\infty,0}(\xi,t)$ satisfy the same Lax pair. That is, the Lax pairs $\widehat{L}^{0,\infty,2}$ and $\widehat{L}^{\infty,0,2}$ are identical under the formal identification

$$\widehat{\psi}_2^{0,\infty}(z,t) = \widehat{\psi}_2^{\infty,0}(\xi,t), \quad \xi = \frac{z}{t}.$$

Similarly, the Lax pairs $\widehat{L}^{0,\infty,1}$ and $\widehat{L}^{\infty,0,1}$ are identical under the formal identification

$$\widehat{\psi}_1^{0,\infty}(z,t) = \frac{z}{\phi_\infty}\widehat{\psi}_1^{\infty,0}(\xi,t), \quad \xi = \frac{z}{t}.$$

Analogous to Proposition 5.6.1, we have the following result.

Proposition 5.10.1. *The following identities hold true,*

$$\widehat{\Psi}_1^{0,\infty}(z,t) = \frac{z}{\widehat{c}(t)\phi_\infty(t)}\widehat{\Psi}_1^{\infty,0}(\xi,t), \quad \widehat{\Psi}_2^{0,\infty}(z,t) = \widehat{\Psi}_2^{\infty,0}(\xi,t),$$

where the q-constant $\widehat{c}(t)$ is defined in (5.155).

Proof. We proof this analogous to Propostion 5.6.1. □

By putting together Proposition 5.10.1 and equations (5.147) and (5.148), we find

$$\widehat{\Psi}^{\infty}\left(\frac{z}{t},t\right) = \widehat{\Psi}^{0}(z,t)\,\frac{\theta_q\left(qb_1\frac{z}{t},qb_2\frac{z}{t},qb_3\frac{z}{t},qb_4\frac{z}{t}\right)}{\theta_q(qb_5z,qb_6z,qb_7z,qb_8z)}Q\left(z,\widehat{\sigma}_0(\Lambda_\infty(t))\right) \cdot \begin{pmatrix}\frac{\widehat{c}_0(t)\phi_\infty(t)}{z} & 0\\ 0 & 1\end{pmatrix} Q\left(\frac{t}{z};\widehat{\sigma}_\infty(\Lambda_\infty(t))\right)^{-1}. \tag{5.149}$$

By (5.113), we know that

$$Y^0(z,t) = \widehat{s}^0(t)\widehat{\Psi}^0(z,t), \quad s^0(t) := s^0(t,\phi_\infty(t);\Lambda_\infty(t)), \tag{5.150}$$

defines a fundamental solution of Yamada's Lax pair L (2.21), which is meromorphic on $\mathbb{C}\times V$. We note that $\widehat{s}^0(t)$ satisfies

$$\widehat{s}^0(qt) = -q^{-1}\Lambda_\infty G_0^\infty(\phi_\infty(t);\Lambda_\infty(t))^{-1}\widehat{s}^0(t).$$

Similarly, by equation (5.131), we know that

$$\widehat{Y}^{\infty}(z,t) := \widehat{s}^{\infty}(t)\Psi^{\infty}(z,t), \quad \widehat{s}^{\infty}(t) := \widehat{s}^{\infty}(t,\phi_\infty(t);\Lambda_\infty(t)), \tag{5.151}$$

defines a fundamental solution of Yamada's Lax pair $\widetilde{L}$ (5.6), which is meromorphic on $\mathbb{P}^* \times V$. We note that $\widehat{s}^\infty(t)$ satisfies

$$\widehat{s}^\infty(qt) = q^{-1} G_0^\infty(\phi_\infty(t); \Lambda_\infty(t))^{-1} t^2 \widehat{s}^\infty(t).$$

Combining equations (5.149), (5.150) and (5.151), we obtain

$$\widehat{Y}^\infty(z,t) = \widehat{Y}^0(z,t)\widehat{\mathcal{P}}(z,t), \tag{5.152}$$

with

$$\widehat{\mathcal{P}}(z,t) = \widehat{s}(t)\frac{\theta_q\left(qb_1\frac{z}{t}, qb_2\frac{z}{t}, qb_3\frac{z}{t}, qb_4\frac{z}{t}\right)}{\theta_q(qb_5 z, qb_6 z, qb_7 z, qb_8 z)} Q\left(z, \widehat{\sigma}_0(\Lambda_\infty(t))\right) \begin{pmatrix} \frac{\widehat{c}_0(t)\phi_\infty(t)}{z} & 0 \\ 0 & 1 \end{pmatrix} Q\left(\frac{t}{z}; \widehat{\sigma}_\infty(\Lambda_\infty(t))\right)^{-1},$$

where $\widehat{s}(t)$ is the meromorphic function on T, satisfying $\widehat{s}(qt) = -t^2/\Lambda_\infty(t)\widehat{s}(t)$, defined by

$$\widehat{s}(t) = \frac{\widehat{s}^\infty(t)}{\widehat{s}^0(t)}.$$

This is consistent with the notation in Section 4.9, where

$$c_0^1(t) := \widehat{s}^0(t)\widehat{\Psi}_1^0(0,t), \qquad c_0^2(t) := \widehat{s}^0(t)\widehat{\Psi}_2^0(0,t),$$
$$\widetilde{c}_0^1(t) := \widehat{s}^\infty(t)\widehat{\Psi}_1^\infty(\infty,t), \qquad \widetilde{c}_0^2(t) := \widehat{s}^\infty(t)\widehat{\Psi}_2^\infty(\infty,t).$$

Recall that the functions $\widehat{Y}^0(z,t)$ and $\widehat{Y}^\infty(z,t)$ are defined on $\mathbb{C} \times V$ and $\mathbb{P}^* \times V$ respectively, and the connection result (5.152) is valid on $\mathbb{C}^* \times V$. Now recall that we fixed V to be any continuous q-domain with $V \subseteq \overline{V}^* \subseteq T$, at the beginning this section. By doing the same analysis on another continuous q-domain W with $W \subseteq \overline{W}^* \subseteq T$, we obtain identical results on the intersection $V \cap W$. We conclude that we can extend the domains of $\widehat{Y}^0(z,t)$ and $\widehat{Y}^\infty(z,t)$ to $\mathbb{C} \times T$ and $\mathbb{P}^* \times T$ respectively, and the connection result (5.152) is valid on $\mathbb{C}^* \times T$.

5.11 Monodromy Corresponding to Critical Behaviour at $t = \infty$

Theorem 5.11.1. *Let (f,g) be a meromorphic solution of q-$P(A_1)$ on a continuous q-domain T, characterised by critical behaviour near $t = \infty$ as in Theorem 3.4.2, by analytic functions $\Lambda_\infty(t)$ and $\phi_\infty(t)$. Then the corresponding monodromy of Yamada's Lax pair is given by*

$$M_T(f,g) = [\widehat{\mathcal{R}}(z,t)],$$

with

$$\widehat{\mathcal{R}}(z,t) = \widehat{s}(t)\theta_q\left(qb_1\frac{z}{t}, qb_2\frac{z}{t}, qb_3\frac{z}{t}, qb_4\frac{z}{t}\right) Q\left(z, \widehat{\sigma}_0(\Lambda_\infty(t))\right) \begin{pmatrix} \frac{\widehat{c}(t)\phi_\infty(t)}{z} & 0 \\ 0 & 1 \end{pmatrix} Q\left(\frac{t}{z}; \widehat{\sigma}_\infty(\Lambda_\infty(t))\right)^{-1},$$

where $\widehat{s}(t)$ any nonzero meromorphic function satisfying $\widehat{s}(qt) = -t^2/\Lambda_\infty(t)\widehat{s}(t)$ on $\mathbb{C}^*$, the two sets of parameters $\widehat{\sigma}_0(\lambda)$ and $\widehat{\sigma}_\infty(\Lambda)$ are defined by

$$\widehat{\sigma}_0(\Lambda_\infty) = \left(b_7^{-1}, b_8^{-1}, b_5^{-1}, b_6^{-1}; -q\Lambda_\infty, -q^2\frac{\Lambda_\infty}{\lambda_\infty}\right), \tag{5.153}$$

$$\widehat{\sigma}_\infty(\Lambda_\infty) = \left(qb_3, qb_4, qb_1, qb_2; -\frac{1}{q\Lambda_\infty}, -\frac{\lambda_\infty}{q\Lambda_\infty}\right), \tag{5.154}$$

and

$$\widehat{c}(t) = -\frac{\lambda_\infty(t)-1}{\lambda_\infty(t)-q}, \quad \lambda_\infty(t) = \frac{\Lambda_\infty(t)^2}{b_5b_6b_7b_8}. \tag{5.155}$$

Proof. This follows directly from the connection result 5.152. □

Let us consider the setting in Theorem 5.11.1, where, for the sake of simplicity, we assume that $\Lambda_\infty(t) \equiv \Lambda_\infty$ is constant, with of course $\Lambda_\infty \in L_0(\mathbf{b})$. As in Section 5.7, we use the explicit results in Section 4.5. Considering the condition (4.53) for $\widehat{\sigma}_0(\lambda)$ and $\widehat{\sigma}_\infty(\Lambda)$, we assume (5.109) and $\Lambda_\infty \in L_0^*(\mathbf{b})$, where we recall the definition of $L_0^*(\mathbf{b})$ in (5.110). Now to apply the results in Section 4.5 directly, i.e. without having to permute the parameters, assumption 4.66 has to be satisfied, which translates to

$$|qb_7b_8| < |\Lambda_\infty|, \quad |\Lambda_\infty| < |q^{-1}b_3b_4|, \tag{5.156}$$

for the parameter values $\widehat{\sigma}_0(\Lambda_\infty)$ and $\widehat{\sigma}_\infty(\Lambda_\infty)$ respectively. Without loss of generality, we again assume (5.112), as equation (5.156) is now trivially satisfied, indeed

$$|qb_7b_8|^2 < |qb_5b_6b_7b_8| = |b_1b_2b_3b_4| < |\Lambda_\infty|^2 < |b_5b_6b_7b_8| = |q^{-1}b_1b_2b_3b_4| < |q^{-1}b_3b_4|^2.$$

Therefore, assuming (5.109) and (5.112) hold, and $\Lambda_\infty \in L_0^*(\mathbf{b})$, we find

$$\begin{aligned}[\widehat{\mathcal{R}}(z,t)] = \Bigg[\widehat{s}(t)&\begin{pmatrix}\theta_q(b_5z) & 0\\ 0 & \theta_q(b_6z)\end{pmatrix}\begin{pmatrix}q_{11}\theta_q\left(-q\frac{\Lambda_\infty}{b_5}z\right) & q_{12}\theta_q\left(-q^2\frac{\Lambda_\infty}{b_5\lambda_\infty}z\right)\\ q_{21}\theta_q\left(-q\frac{\Lambda_\infty}{b_6}z\right) & q_{22}\theta_q\left(-q^2\frac{\Lambda_\infty}{b_6\lambda_\infty}z\right)\end{pmatrix}\begin{pmatrix}\frac{\widehat{c}(t)\phi_\infty(t)}{z} & 0\\ 0 & 1\end{pmatrix}\\ &\begin{pmatrix}r_{11}\theta_q\left(-q\frac{\Lambda_\infty}{b_2\lambda_\infty}\frac{z}{t}\right) & r_{12}\theta_q\left(-q\frac{\Lambda_\infty}{b_1\lambda_\infty}\frac{z}{t}\right)\\ r_{21}\theta_q\left(-q\frac{\Lambda_\infty}{b_2}\frac{z}{t}\right) & r_{22}\theta_q\left(-q\frac{\Lambda_\infty}{b_1}\frac{z}{t}\right)\end{pmatrix}\begin{pmatrix}\theta_q\left(qb_2\frac{z}{t}\right) & 0\\ 0 & \theta_q\left(qb_1\frac{z}{t}\right)\end{pmatrix}\Bigg],\end{aligned}$$

where the q_{ij} are defined in (4.74) with $\sigma = \widehat{\sigma}_0(\lambda)$, and the r_{ij} are defined by (4.70) with $\sigma = \widehat{\sigma}_\infty(\Lambda)$, for $i,j \in \{1,2\}$. As an additional check, one can now verify the equations (4.110) and (4.112) directly.

5.12 Parametric Connection Formulae

We have determined the monodromy of Yamada's Lax pair corresponding to several of the critical behaviours near $t=0$ and $t=\infty$, including the generic ones. Now let us combine these results. Take some meromorphic q-$P(A_1)$ transcendent (f,g), on a continuous q-domain T, and assume its critical behaviour near $t=0$ is as specified in Theorem 3.4.1, for some particular integration constants

$$\{\phi(t), \Lambda(t)\}. \tag{5.157}$$

Similarly let us assume its critical behaviour near $t = \infty$ is as specified in Theorem 3.4.2, for some particular integration constants

$$\{\phi_\infty(t), \Lambda_\infty(t)\}. \tag{5.158}$$

Then we conclude from Theorems 5.7.1 and 5.11.1, that there exist invertible meromorphic q-periodic matrices $E(t)$ and $F(t)$ on T, and a nonzero meromorphic function $s_*(t)$ on T, such that the sets of integration constants (5.157) and (5.158) are related by

$$s_*(t)\theta_q(qb_5z, qb_6z, qb_7z, qb_8z)E(t)Q\left(\frac{z}{t}, \sigma_0(\Lambda(t))\right)\begin{pmatrix}1 & 0\\ 0 & \frac{c(t)t\phi(t)}{z}\end{pmatrix}Q\left(z^{-1}; \sigma_\infty(\Lambda(t))\right)^{-1} =$$

$$\theta_q\left(qb_1\frac{z}{t}, qb_2\frac{z}{t}, qb_3\frac{z}{t}, qb_4\frac{z}{t}\right)Q\left(z, \widehat{\sigma}_0(\Lambda_\infty(t))\right)\begin{pmatrix}\frac{\widehat{c}(t)\phi_\infty(t)}{z} & 0\\ 0 & 1\end{pmatrix}Q\left(\frac{t}{z}; \widehat{\sigma}_\infty(\Lambda_\infty(t))\right)^{-1}F(t),$$

where $s_*(t)$ satisfies $s_*(qt) = \Lambda_\infty(t)/\Lambda(t)s_*(t)$.
The main objective now is to use this equality to derive closed connection formulae 2.20. However this does not seem to be an easy task. At the heart of the problem lies that the space $\mathcal{M}_T$ is seemingly complicated. For any $t_0 \in T$, we can easily justify localising the relation at $t = t_0$, giving

$$E_0R_0(z) = \widehat{R}_0(z)F_0, \tag{5.159}$$

where

$$R_0(z) = \theta_q(qb_5z, qb_6z, qb_7z, qb_8z)Q\left(\frac{z}{t_0}, \sigma_0(\Lambda_0^0)\right)\begin{pmatrix}1 & 0\\ 0 & \frac{c^0t_0\phi_0^0}{z}\end{pmatrix}Q\left(z^{-1}; \sigma_\infty(\Lambda_0^0)\right)^{-1},$$

$$\widehat{R}_0(z) = \theta_q\left(qb_1\frac{z}{t_0}, qb_2\frac{z}{t_0}, qb_3\frac{z}{t_0}, qb_4\frac{z}{t_0}\right)Q\left(z, \widehat{\sigma}_0(\Lambda_\infty^0)\right)\begin{pmatrix}\frac{\widehat{c}(t_0)\phi_\infty^0}{z} & 0\\ 0 & 1\end{pmatrix}Q\left(\frac{t_0}{z}; \widehat{\sigma}_\infty(\Lambda_\infty^0)\right)^{-1},$$

with $E_0, F_0 \in GL_2(\mathbb{C})$, and we denoted

$$\phi_0^0 = \phi(t_0), \qquad \Lambda_0^0 = \Lambda(t_0), \qquad c^0 = c(t_0),$$
$$\phi_\infty^0 = \phi_\infty(t_0), \qquad \Lambda_\infty^0 = \Lambda_\infty(t_0), \qquad \widehat{c}^0 = \widehat{c}(t_0).$$

Note that equation (5.159) is now an equality between elements of $S(t_0)$, using the notation in (4.10). By heuristic counting of dimensions in Section 4.10, we know that $\mathcal{M}(t_0)$ should be a two-dimensional space. However finding appropriate coordinates on $\mathcal{M}(t_0)$ seems quite nontrivial. It is natural to associate to any $R(z) \in S(t_0)$, the object

$$I_R(w_1, w_2) := \mathrm{Tr}\left[R(w_1)R(w_2)^{-1}\right],$$

as this association is constant on orbits $[R(z)] \in \mathcal{M}(t_0)$. In particular, considering (5.159), we have

$$I_{R_0}(w_1, w_2) = I_{\widehat{R}_0}(w_1, w_2).$$

By evaluating both sides at particular points we might be able to find useful relations between the integration constants. We leave this issue open for future research.

CHAPTER 6

Conclusion

In this thesis we made effective the global asymptotic analysis of the q-difference Painlevé equation q-$P(A_1)$, by combining a local asymptotic analysis of its solutions, with an isomonodromic deformation method applied to a Lax pair derived by Yamada [85]. The final result is a conjecturally complete description of critical behaviours of solutions near the critical point $t = 0$ and the critical point $t = \infty$, supplemented with explicit parametric connection formulae, relating the critical behaviours near the two different critical points indirectly.

The local asymptotic analysis consisted roughly of three steps. Firstly we studied the leading order behaviour of q-$P(A_1)$ transcendents near $t = 0$, by the method of dominant balance, and found that it is characterised by an autonomous system. We identified this autonomous system as a QRT mapping, which allowed us to parametrise its generic solution in terms of two integration q-constants. The second step involved finding the full formal asymptotic expansion of the solution of q-$P(A_1)$, corresponding to the generic solution of the autonomous system. The third step consisted of proving that the formal asymptotic expansion always converges for appropriate choices of the two integration q-constants. We then used a Bäcklund transformation to translate all the results to similar ones around $t = \infty$.

We note that, besides the generic solution of the autonomous system, there also exist two one-parameter families of solutions, expressed in terms of logarithms. We did not complete steps two and three above for these families of solutions, and this will be an interesting direction for future research. We compared the results heuristically in the continuum limit with the known results for the sixth Painlevé equation, and observed that they essentially coincide.

Rigorously proving that our description of different critical behaviours of q-$P(A_1)$ transcendents is complete, the completeness problem 3.7.2, is a fundamental but seemingly difficult one. The q-$P(A_1)$ connection problem entails relating the critical behaviours near the two different critical points explicitly. To solve this problem, we employed the isomonodromic deformation method.

Yamada [85] constructed a Lax pair, consisting of a second order scalar q-difference equation, the spectral equation, together with a deformation equation, whose compatibility is equivalent to q-$P(A_1)$. However Yamada derived his Lax pair within a geometric framework of algebraic curves, different from the isomonodromic deformation point of view. We therefore studied the analytic properties of Yamada's Lax pair, and showed that under specific scalings, involving solutions of two auxiliary second order linear q-difference equations, the monodromy, i.e. connection matrix, is preserved by the q-$P(A_1)$ deformation. We hence have a well defined mapping, the monodromy mapping, which sends a q-$P(A_1)$ transcendent to corresponding monodromy of Yamada's Lax pair.

We then turned our attention to the direct monodromy problem, which entails explicitly calculating the monodromy corresponding to a given q-$P(A_1)$ transcendent. We succeeded in explicitly solving the direct monodromy problem, for the solutions characterised by the generic critical behaviour involving two q-constants, both near $t = 0$ and near $t = \infty$. Equating the results, yields explicit parametric connection formulae, relating the critical behaviours near the two different critical points indirectly. Deducing direct connection formulae, from these parametric ones, is a problem which is currently being explored. We note that in both the $t \to 0$ and the $t \to \infty$ limit, the connection matrix of Yamada's Lax pair factorises in two copies of a connection matrix associated with a simpler linear q-difference equation, which we called the degree two model equation. This degree two model equation is related to the *associated continuous dual q-Hahn polynomials*, and we showed that the solutions of the degree two model equation can be expressed in terms of ${}_3\phi_2$ hypergeometric functions, inspired by the work of Gupta et al. [23]. Furthermore we explicitly determined the corresponding connection matrix.

There are several ways to approach the completeness problem. One is by doing an asymptotic analysis within the initial value space of the q-$P(A_1)$ equation, as $t \to 0$ or $t \to \infty$. This would bring together the local asymptotic analysis of solutions and the algebro-geometric side of the q-$P(A_1)$ equation. Let us note that such asymptotic studies have been carried out for several of the continuous Painlevé equations, see Joshi and collaborators [12, 38, 49]. To our knowledge, only one such study has been carried out for a discrete Painlevé equation, namely a q-discrete version of P_{I}, see Joshi and Lobb [46]. Another way to solve the completeness problem, would be to prove that the monodromy mapping is bijective, and show that each monodromy datum of Yamada's Lax pair corresponds to some unique critical behaviour in our list, both near $t = 0$ and $t = \infty$. We were not able to establish such a result, particularly as the spectral equation is scalar with resonance and trivial monodromy, both near the spectral origin and infinity. It recently came to our attention that Rains and Ormerod [68] constructed a different Lax pair for the q-$P(A_1)$ equation, which is in system form, with the origin and infinity of the spectral equation being regular singular without resonance. It would be of interest to perform a similar study of this Lax pair, and particularly use it to solve the completeness problem.

Appendix A

The Painlevé Equations

The six Painlevé equations are given by the following nonlinear differential equations,

$$
\begin{aligned}
P_{\mathrm{I}}: \quad \omega'' =& 6\omega^2 + \zeta, \\
P_{\mathrm{II}}: \quad \omega'' =& 2\omega^3 + \zeta\omega + \alpha, \\
P_{\mathrm{III}}: \quad \omega'' =& \frac{1}{\omega}\left(\omega'\right)^2 - \frac{1}{\zeta}\omega' + \frac{\alpha}{\zeta}\omega^2 + \frac{\beta}{\zeta} + \gamma\omega^3 + \frac{\delta}{\omega}, \\
P_{\mathrm{IV}}: \quad \omega'' =& \frac{1}{2\omega}\left(\omega'\right)^2 + \tfrac{3}{2}\omega^3 + 4\zeta\omega^2 + 2(\zeta^2 - \alpha)\omega + \frac{\beta}{\omega}, \\
P_{\mathrm{V}}: \quad \omega'' =& \left(\frac{1}{2\omega} + \frac{1}{\omega - 1}\right)\left(\omega'\right)^2 - \frac{1}{\zeta}\omega' + \frac{(\omega-1)^2}{\zeta^2}\left(\alpha\omega + \frac{\beta}{\omega}\right) + \gamma\frac{\omega}{\zeta} + \delta\frac{\omega(\omega+1)}{\omega - 1}, \\
P_{\mathrm{VI}}: \quad \omega'' =& \tfrac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-\zeta}\right)\left(w'\right)^2 - \left(\frac{1}{\zeta} + \frac{1}{\zeta - 1} + \frac{1}{w - \zeta}\right)w' \\
& + \frac{w(w-1)(w-\zeta)}{2\zeta^2(\zeta-1)^2}\left(\gamma^2 - \frac{\delta^2\zeta}{w^2} + \frac{\alpha^2(\zeta-1)}{(w-1)^2} + \frac{(1-\beta^2)\zeta(\zeta-1)}{(w-\zeta)^2}\right),
\end{aligned}
$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ are complex parameters.

Appendix B

q-Briot-Bouquet theorem

In 1856, Briot and Bouquet [8] analysed the existence and uniqueness of ordinary differential equations of a specific type, which are appropriately called Briot-Bouquet equations nowadays. Let us formulate their classical result.

Theorem B.1 (Briot-Bouquet theorem (dimension one case))**.** *Let $H(z, y)$ be an analytic function at $(z, y) = (0, 0)$ with $H(0, 0) = 0$, and denote $\lambda = \frac{\partial H}{\partial y}(0, 0)$. If $\lambda \notin \mathbb{N}^*$, then the differential equation*

$$y'(z) = H(z, y(z)),$$

has an unique power series solution with zero constant term. Furthermore this power series converges, defining a holomorphic solution of the differential equation near $z = 0$.

We refer the interested reader to the book by Hille [36] for more on the continuous side of the subject. In 1890 Poincaré [71] analysed q-analogs of the Briot-Bouquet equations and proved the so called q-Briot-Bouquet theorem, which is Theorem B.3, with $m = 1$, $|q| > 1$ and $\mathbf{Y} = 0$. Let us formulate the dimension one case of Poincaré's result separately.

Theorem B.2 (**q**-Briot-Bouquet theorem (dimension one case))**.** *Let $|q| > 1$ and $H(z, y)$ be an analytic function at $(z, y) = (0, 0)$ with $H(0, 0) = 0$, and denote $\lambda = \frac{\partial H}{\partial y}(0, 0)$. If $\lambda \notin q^{\mathbb{N}^*}$, then the q-difference equation*

$$y(qz) = H(z, y(z)),$$

has an unique power series solution with zero constant term. Furthermore this power series converges, defining a holomorphic solution of the q-difference equation near $z = 0$.

Unfortunately Poincaré [71] only discusses the case $|q| > 1$, whereas in the cases $|q| < 1$ and $|q| = 1$ an extra subtlety arises, which one does not see in the dimension one continuous case B.1. Indeed, let us consider the following example,

$$y(qz) = z + \lambda y(z) + zy(z),$$

so $H(z, y) = z + \lambda u + zy$ in Theorem B.2. We can immediately write down the full formal power series solution, which is given by

$$y(z)=\sum_{n=1}^{\infty} b_n z^n, \qquad b_n=\prod_{k=1}^{n}(q^k-\lambda)^{-1}. \quad (n\in\mathbb{N}^*) \tag{B.1}$$

Not surprisingly, the formal power series solution exist, iff $\lambda \notin q^{\mathbb{N}^*}$. Let us assume this is indeed satisfied, then the power series indeed always converges if $|q|>1$. There are three cases to consider, $\lambda=0$, $|\lambda|\neq 0,1$ and $|\lambda|=1$. If $\lambda=0$, then

$$y(z)=\sum_{n=1}^{\infty} q^{-\frac{1}{2}n(n+1)} z^n,$$

which does not converge when $|q|<1$. The hidden reason for this, in the perspective of Theorem B.2, is that although $\lambda\notin q^{\mathbb{N}^*}$, the set $q^{\mathbb{N}^*}$ does have $\lambda=0$ as a limit point. In case $|\lambda|\neq 0,1$, it is easy to see that the power series solution (B.1) converges, regardless of the modulus of q. In the final case $|\lambda|=1$, the power series solution converges when $|q|\neq 1$. However if $|q|=1$, things are much more complicated and in the literature this case is often referred as the *resonant case*. It turns out we can generalise Theorem B.2 to the regime $|q|\leq 1$, where, for the power series to converge, we require $\lambda\notin\overline{q^{\mathbb{N}^*}}$.

In this section we discuss an extension of the classical q-Briot-Bouquet Theorem to several independent variables and, more importantly, we prove that the constructed solutions depend analytically on various parameters involved. This is a crucial ingredient in the proof of Theorem 3.4.1, where we use the formal series solution defined in Theorem 3.3.1, to construct true solutions of q-$P(A_1)$. We use standard multi-index notation, for $n\in\mathbb{N}^*$, for $\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^n$, we set

$$|\alpha|=\alpha_1+\ldots+\alpha_n.$$

For $\alpha,\beta\in\mathbb{N}^n$, we write $\alpha\leq\beta$ if and only if for all $1\leq i\leq n$ we have $\alpha_i\leq\beta_i$. This defines a partial order on $\mathbb{N}^n$, and we say $\alpha<\beta$ if and only of $\alpha\leq\beta$ and $\alpha\neq\beta$.
If $\mathbf{y}\in\mathbb{C}^n$, we define

$$\mathbf{y}^\alpha=y_1^{\alpha_1}\cdot\ldots\cdot y_n^{\alpha_n}.$$

The following Theorem is an extension of the q-Briot-Bouquet Theorem to several independent variables $t_1,\ldots t_m$, with a coupled time evolution, $\overline{t}_i=q_it_i$, where $q_i\in\mathbb{C}^*$ for $1\leq i\leq m$.

Theorem B.3 (***q*-Briot-Bouquet theorem (several independent variables)**). *Let $m,n\in\mathbb{N}^*$ and let us denote*

$$\mathbf{t}=(t_1,\ldots,t_m),\quad \mathbf{q}=(q_1,\ldots,q_m)\quad \overline{\mathbf{t}}=(q_1t_1,\ldots,q_mt_m),\quad \mathbf{y}=(y_1,\ldots,y_n). \tag{B.2}$$

Let $H(\mathbf{t},\mathbf{y};\mathbf{q})=(H_1(\mathbf{t},\mathbf{y};\mathbf{q}),\ldots,H_n(\mathbf{t},\mathbf{y};\mathbf{q}))$ be a vector valued function. Assume there is a $\mathbf{Y}\in\mathbb{C}^n$, such that $H(\mathbf{t},\mathbf{y})$ is holomorphic at $(\mathbf{t},\mathbf{y})=(\mathbf{0},\mathbf{Y})$ with $H(\mathbf{0},\mathbf{Y})=\mathbf{Y}$. Suppose the eigenvalues of the Jacobi matrix

$$D=\left(\frac{\partial H_j}{\partial y_k}(\mathbf{0},\mathbf{Y})\right)_{1\leq j,k\leq n},$$

are not elements of the set

$$Q := \{\mathbf{q}^\alpha \mid \alpha \in \mathbb{N}^m \setminus \{0\}\}.$$

Then the system of q-difference equations

$$y_j\left(\bar{\mathbf{t}};\mathbf{q}\right) = H_j(\mathbf{t},\mathbf{y}(\mathbf{t});\mathbf{q}) \qquad (1 \le j \le n), \tag{B.3}$$

has an unique power series solution of the form,

$$y_j(\mathbf{t};\mathbf{q}) = Y_j + \sum_{\alpha\in\mathbb{N}^m\setminus\{0\}} b_\alpha^{(j)}(\mathbf{q})\mathbf{t}^\alpha \qquad (1 \le j \le n).$$

Furthermore, if the eigenvalues of the matrix D are not limit points of the set Q, then these power series convergence in an open environment of $\mathbf{t} = \mathbf{0}$.

Proof. Several variables can easily be incorporated in the proof by Poincaré [71]. □

Note that the time evolutions of the independent variables in the above theorem are coupled, i.e. although we have several independent variables, it is not a partial difference system. Opposite to the continuous case, not much work has been done in the direction of partial difference systems of Briot-Bouquet type. Let us mention the work of Tahara and Yamazawa [82] which seems a first in this direction.

We would like to improve Theorem B.3, by showing that the solution $\mathbf{y}$ depends analytically on $\mathbf{q}$, as formulated in Theorem B.4. For notational simplicity, we restrict ourselves to the case $\mathbf{Y} = 0$. Iwasaki et al. [41][Prop. 1.1.1] give an elegant proof of the classical Briot-Bouquet Theorem with several dependent variables. The proof of Theorem B.4 is basically an adaptation of their proof for the q-case, where every estimate is done uniformly in $\mathbf{q}$. Before we formulate the theorem, let us introduce some notation. We define the *max norm* $||\cdot||_{\max}$ on $\mathbb{C}^n$ by

$$||\mathbf{v}||_{\max} = \max_{1\le i\le n} |v_i|,$$

for $\mathbf{v} \in \mathbb{C}^n$, and for matrices $A \in \mathbb{C}^{n\times n}$, we set

$$||A||_{\max} = \max_{1\le i,j\le n} |A_{ij}|.$$

We have the following inequality

$$||Av||_{\max} \le n||A||_{\max}||v||_{\max}, \tag{B.4}$$

for $A \in \mathbb{C}^{n\times n}$ and $\mathbf{v} \in \mathbb{C}^n$.
For $\mathbf{v} \in \mathbb{C}^n$ and $R > 0$, we define $B^n_{\max}(\mathbf{v}, R)$ and $\overline{B}^n_{\max}(\mathbf{v}, R)$ to be respectively the open and closed ball of radius R centered at $\mathbf{v}$ in $\mathbb{C}^n$ with respect to the $||\cdot||_{\max}$ norm.

Theorem B.4 (**q**-Briot-Bouquet theorem (several independent variables, uniform in $\mathbf{q}$))**.** *Let $m, n \in \mathbb{N}^*$ and denote $\mathbf{t}, \mathbf{q}, \bar{\mathbf{t}}$ and $\mathbf{y}$ as in (B.2). Let $H(\mathbf{t},\mathbf{y};\mathbf{q}) = (H_1(\mathbf{t},\mathbf{y};\mathbf{q}),\ldots,H_n(\mathbf{t},\mathbf{y};\mathbf{q}))$ be a vector valued function. Assume there is an open set $U \subseteq B^m_{max}(\mathbf{0}, 1) \subseteq \mathbb{C}^m$ such that, for every $\mathbf{q}_0 \in U$, the function $H(\mathbf{t},\mathbf{y};\mathbf{q})$ is holomorphic at $(\mathbf{t},\mathbf{y};\mathbf{q}) = (\mathbf{0},\mathbf{0};\mathbf{q}_0)$ with $H(\mathbf{0},\mathbf{0};\mathbf{q}_0) =$*

0. For $\mathbf{q} \in U$, let us denote the Jacobian matrix of H with respect to $\mathbf{y}$ at $(\mathbf{t}, \mathbf{y}) = (\mathbf{0}, \mathbf{0})$ by

$$D(\mathbf{q}) = \left(\frac{\partial H_j}{\partial y_k}(\mathbf{0}, \mathbf{0}; \mathbf{q})\right)_{1 \le j,k \le n}.$$

We assume that for any $\mathbf{q} \in U$, the eigenvalues of the Jacobi matrix $D(\mathbf{q})$, are not elements of

$$Q_0 := \{0\} \cup \{\mathbf{q}^\alpha \mid \alpha \in \mathbb{N}^m \setminus \{0\}\}. \tag{B.5}$$

Then the q-Briot-Bouquet Theorem B.3 shows, that for every $\mathbf{q} \in U$, the system of q-difference equations

$$y_j\left(\bar{\mathbf{t}}; \mathbf{q}\right) = H_j(\mathbf{t}, \mathbf{y}(\mathbf{t}); \mathbf{q}) \qquad (1 \le j \le n), \tag{B.6}$$

has an unique converging power series solution vanishing at $\mathbf{t} = \mathbf{0}$,

$$y_j(\mathbf{t}; \mathbf{q}) = \sum_{\alpha \in \mathbb{N}^m \setminus \{0\}} b_\alpha^{(j)}(\mathbf{q}) \mathbf{t}^\alpha \qquad (1 \le j \le n). \tag{B.7}$$

For every $\mathbf{q}_0 \in U$, for $1 \le j \le n$, the series B.7 converges locally uniformly in $(\mathbf{t}, \mathbf{q})$ at $(\mathbf{0}, \mathbf{q}_0)$ on $\mathbf{C}^m \times U$. That is, for every $\mathbf{q}_0 \in U$, there are open environments $Z \subseteq \mathbb{C}^m$ and $V \subseteq U$ of $\mathbf{0}$ and $\mathbf{q}_0$ respectively, such that the series (B.7) converge uniformly on $Z \times V$ in $(\mathbf{t}, \mathbf{q})$, defining analytic functions on this set.

Proof. For every $\mathbf{q} \in U$ and $1 \le j \le n$, since $H_j(\mathbf{t}, \mathbf{y}; \mathbf{q})$ is holomorphic at $(\mathbf{t}, \mathbf{y}, \mathbf{q}) = (\mathbf{0}, \mathbf{0}, \mathbf{q})$ with $H_j(\mathbf{0}, \mathbf{0}; \mathbf{q}) = 0$, we can find a converging power series expansion

$$H_j(\mathbf{t}, \mathbf{y}; \mathbf{q}) = \sum_{\alpha \in \mathbb{N}^m, \beta \in \mathbb{N}^n} C_{(\alpha,\beta)}^{(j)}(\mathbf{q}) \mathbf{t}^\alpha \mathbf{y}^\beta, \tag{B.8}$$

about $(\mathbf{t}, \mathbf{y}) = (\mathbf{0}, \mathbf{0})$, with $C_{(0,0)}^{(j)}(\mathbf{q}) = 0$, for $1 \le j \le n$.
The coefficients $C_{(\alpha,\beta)}^{(j)}(\mathbf{q})$ are holomorphic in $\mathbf{q}$ on U for all $\alpha \in \mathbb{N}^m$ and $\beta \in \mathbb{N}^n$. Substituting formal power series expansions (B.7) into equation (B.6) gives the following recursion for the coefficients $b_\alpha^{(j)}(\mathbf{q})$:

$$(\mathbf{q}^\alpha I_n - D(\mathbf{q})) \begin{pmatrix} b_\alpha^{(1)}(\mathbf{q}) \\ b_\alpha^{(2)}(\mathbf{q}) \\ \vdots \\ b_\alpha^{(n)}(\mathbf{q}) \end{pmatrix} = \begin{pmatrix} M_\alpha \left[\left(C_{(\alpha',\beta)}^{(1)}(\mathbf{q})\right)_{(\alpha',\beta) \in L(\alpha)}; \left(b_{\alpha'}^{(1)}(\mathbf{q})\right)_{\alpha' < \alpha}, \dots, \left(b_{\alpha'}^{(n)}(\mathbf{q})\right)_{\alpha' < \alpha}\right] \\ M_\alpha \left[\left(C_{(\alpha',\beta)}^{(2)}(\mathbf{q})\right)_{(\alpha',\beta) \in L(\alpha)}; \left(b_{\alpha'}^{(1)}(\mathbf{q})\right)_{\alpha' < \alpha}, \dots, \left(b_{\alpha'}^{(n)}(\mathbf{q})\right)_{\alpha' < \alpha}\right] \\ \vdots \\ M_\alpha \left[\left(C_{(\alpha',\beta)}^{(n)}(\mathbf{q})\right)_{(\alpha',\beta) \in L(\alpha)}; \left(b_{\alpha'}^{(1)}(\mathbf{q})\right)_{\alpha' < \alpha}, \dots, \left(b_{\alpha'}^{(n)}(\mathbf{q})\right)_{\alpha' < \alpha}\right] \end{pmatrix}, \tag{B.9}$$

for $\alpha \in \mathbb{N}^m \setminus \{0\}$, where the M_α are polynomials in their inputs with positive coefficients and the sets $L(\alpha)$ are defined by

$$L(\alpha) = \{(\alpha', \beta) \in \mathbb{N}^m \times \mathbb{N}^n : \alpha' \le \alpha, |\beta| \le |\alpha - \alpha'| \text{ and if } \alpha' = 0, \text{ then } |\beta| \ge 2\}.$$

As the eigenvalues of $D(\mathbf{q})$ are not elements of Q_0 for $\mathbf{q} \in U$, we know that, for every

$\alpha \in \mathbb{N}^m \setminus \{0\}$, the matrix $(\mathbf{q}^\alpha I_n - D(\mathbf{q}))$ is invertible for $\mathbf{q} \in U$ and, even stronger,

$$\mathbf{q} \mapsto (\mathbf{q}^\alpha I_n - D(\mathbf{q}))^{-1},$$

is a holomorphic matrix-valued function on U.
Hence this recursion defines unique holomorphic functions $b_\alpha^{(j)}(\mathbf{q})$ on U for $1 \leq j \leq n$ and $\alpha \in \mathbb{N}^m \setminus \{0\}$. Let us take any $\mathbf{q}_0 \in U$ and determine $R_U > 0$ such that

$$B^m_{\max}(\mathbf{q}_0, R_U) \subseteq \overline{B}^m_{\max}(\mathbf{q}_0, R_U) \subseteq U.$$

As $\overline{B}^m_{\max}(\mathbf{q}_0, R_U) \subseteq U$ is compact and the eigenvalues of $D(\mathbf{q})$ are not elements of Q_0 for $\mathbf{q} \in U$, we can obtain the following uniform bound on $\overline{B}^m_{\max}(\mathbf{q}_0, R_U)$,

$$L = \inf_{\mathbf{q} \in \overline{B}^m_{\max}(\mathbf{q}_0, R_U), \alpha \in \mathbb{N}^m \setminus \{0\}} |\det(\mathbf{q}^\alpha I_n - D(\mathbf{q}))| > 0. \tag{B.10}$$

Hence, for every $\mathbf{q} \in \overline{B}^m_{\max}(\mathbf{q}_0, R_U)$, we have

$$\begin{aligned}
\left|\left|(\mathbf{q}^\alpha I_n - D(\mathbf{q}))^{-1}\right|\right|_{\max} &= \left|\left|\frac{\operatorname{adj}(\mathbf{q}^\alpha I_n - D(\mathbf{q}))}{\det(\mathbf{q}^\alpha I_n - D(\mathbf{q}))}\right|\right|_{\max} \\
&= \frac{1}{|\det(\mathbf{q}^\alpha I_n - D(\mathbf{q}))|} \, ||\operatorname{adj}(\mathbf{q}^\alpha I_n - D(\mathbf{q}))||_{\max} \\
&\leq \frac{(n-1)!}{L} \, ||\mathbf{q}^\alpha I_n - D(\mathbf{q})||_{\max}^{n-1} \\
&\leq \frac{(n-1)!}{L} \, (||\mathbf{q}^\alpha I_n||_{\max} + ||(D(\mathbf{q})||_{\max})^{n-1} \\
&\leq \frac{(n-1)!}{L} \, (1 + ||(D(\mathbf{q})||_{\max})^{n-1},
\end{aligned}$$

and, as $||(D(\mathbf{q})||_{\max}$ is clearly uniformly bounded on the compact set $\overline{B}^m_{\max}(\mathbf{q}_0, R_U)$, we have

$$B = \sup_{\mathbf{q} \in \overline{B}^m_{\max}(\mathbf{q}_0, R_U), \alpha \in \mathbb{N}^m \setminus \{0\}} \left|\left|(\mathbf{q}^\alpha I_n - D(\mathbf{q}))^{-1}\right|\right|_{\max} < \infty. \tag{B.11}$$

For all $\alpha \in \mathbb{N}^m$ and $\beta \in \mathbb{N}^n$ and $1 \leq j \leq n$, we have a convergent power series expansion

$$C^{(j)}_{(\alpha,\beta)}(\mathbf{q}) = \sum_{\gamma \in \mathbb{N}^m} C^{(j)}_{(\alpha,\beta,\gamma)} (\mathbf{q} - \mathbf{q}_0)^\gamma, \tag{B.12}$$

about $\mathbf{q} = \mathbf{q}_0$.
Even stronger, for $1 \leq j \leq n$, we have a convergent power series expansion,

$$H_j(\mathbf{t}, \mathbf{y}; \mathbf{q}) = \sum_{\alpha, \gamma \in \mathbb{N}^m, \beta \in \mathbb{N}^n} C^{(j)}_{(\alpha,\beta,\gamma)} (\mathbf{q} - \mathbf{q}_0)^\gamma \mathbf{t}^\alpha \mathbf{y}^\beta, \tag{B.13}$$

about $(\mathbf{t}, \mathbf{y}, \mathbf{q}) = (\mathbf{0}, \mathbf{0}, \mathbf{q}_0)$.
For every $1 \leq j \leq n$, we determine an $R_j > 0$, such that, for all $\mathbf{t}, \mathbf{q} \in \mathbb{C}^m$ and $\mathbf{y} \in \mathbb{C}^n$, the

series (B.13) converges if

$$||\mathbf{t}||_{\max} < R_j, \qquad ||\mathbf{q}-\mathbf{q}_0||_{\max} < R_j, \qquad ||\mathbf{y}||_{\max} < R_j. \tag{B.14}$$

We set $R_0 = \min(R_U, R_1, \ldots, R_n)$, take any $0 < R < R_0$ and define

$$M_j = \sup_{\mathbf{q}\in \overline{B}^m_{\max}(\mathbf{q}_0,R),\alpha\in\mathbb{N}^m,\beta\in\mathbb{N}^n} \left|C^{(j)}_{(\alpha,\beta)}(\mathbf{q})\right| R^{|\alpha+\beta|}.$$

Clearly the M_j are finite and we set $M_0 = \max(M_1, \ldots, M_n)$. We define the function

$$G(\mathbf{t}, Y) = M\left(\left(1-\frac{t_1}{R}\right)^{-1}\cdot\ldots\cdot\left(1-\frac{t_m}{R}\right)^{-1}\left(1-\frac{Y}{R}\right)^{-n} - 1 - n\frac{Y}{R}\right).$$

Observe that G is holomorphic at $(\mathbf{t}, Y) = (\mathbf{0}, 0)$ with $G(\mathbf{0}, 0) = 0$ and $\frac{\partial G}{\partial Y}(\mathbf{0}, 0) = 0$. Hence G has a convergent power series expansion

$$G(\mathbf{t}, Y) = \sum_{\substack{\alpha\in\mathbb{N}^m, i\in\mathbb{N}\\ (\alpha,i)\neq(0,0),(0,1)}} C_{(\alpha,i)}\mathbf{t}^\alpha Y^i,$$

around $(\mathbf{t}, Y) = (\mathbf{0}, 0)$.
Let $\alpha \in \mathbb{N}^m, i \in \mathbb{N}$ with $(\alpha, i) \neq (0, 0), (0, 1)$, then we have

$$C_{(\alpha,i)} = \binom{n+i-1}{i}\frac{M_0}{R^{|\alpha|+i}}.$$

Hence, for any $1 \le j \le n$, for $\alpha \in \mathbb{N}^m, \beta \in \mathbb{N}^n$ such that, if $\alpha = 0$, then $|\beta| \ge 2$, we have, by the definition of M_0,

$$\left|C^{(j)}_{(\alpha,\beta)}(\mathbf{q})\right| \le \frac{M_0}{R^{|\alpha+\beta|}} \le C_{(\alpha,|\beta|)},$$

for $\mathbf{q} \in \overline{B}^m_{\max}(\mathbf{q}_0, R)$.
We consider the functional equation

$$Y(\mathbf{t}) = BnG(\mathbf{t}, Y(\mathbf{t})). \tag{B.15}$$

We prove that this equation has an unique solution $Y(\mathbf{t})$ which is holomorphic at $\mathbf{t} = \mathbf{0}$ with $Y(\mathbf{0}) = 0$. For this we apply the implicit function theorem to the function $F(\mathbf{t}, Y) = BnG(\mathbf{t}, Y) - Y$. Observe that $F(\mathbf{0}, 0) = 0$ and

$$\frac{\partial F}{\partial Y}(\mathbf{0}, 0) = -1 \neq 0.$$

Hence we can apply the implicit function theorem and obtain an unique solution $Y(\mathbf{t})$ of the functional equation (B.15) which is holomorphic at $\mathbf{t} = 0$ with $Y(\mathbf{0}) = 0$. Let the Taylor series expansion of $Y(\mathbf{t})$ at $\mathbf{t}=0$ be given by

$$Y(\mathbf{t}) = \sum_{\alpha\in\mathbb{N}^m\setminus\{0\}} B_\alpha \mathbf{t}^\alpha. \tag{B.16}$$

Since Y is a solution of the functional equation (B.15), the coefficients B_α are determined uniquely by the recursion

$$B_\alpha = BnM_\alpha\left(\left(C_{(\alpha',|\beta|)}\right)_{(\alpha',\beta)\in L(\alpha)}; (B_{\alpha'})_{\alpha'<\alpha}, \ldots, (B_{\alpha'})_{\alpha'<\alpha}\right) \tag{B.17}$$

for $\alpha \in \mathbb{N}^m \setminus \{0\}$, where the polynomials M_α are the same as in recursion (B.9).
We prove the following inequality by complete induction with respect to the partial order $\leq$ on $\mathbb{N}^m$,

$$\left|b_\alpha^{(j)}(\mathbf{q})\right| \leq B_\alpha, \tag{B.18}$$

for every $1 \leq j \leq n$ and $\mathbf{q} \in \overline{B}^m_{\max}(\mathbf{q}_0, R)$, for all $\alpha \in \mathbb{N}^m \setminus \{0\}$.
Let us fix a $\mathbf{q} \in \overline{B}^m_{\max}(\mathbf{q}_0, R)$, take any $\alpha \in \mathbb{N}^m \setminus \{0\}$, and assume that for all $\alpha' < \alpha$ inequality (B.18) holds for every $1 \leq j \leq n$. Then we have, by applying inequality (B.4) to equation (B.9),

$$\begin{aligned}
\max_{1\leq i\leq n}\left|b_\alpha^{(i)}(\mathbf{q})\right| \leq & n\left|\left|\left(\mathbf{q}^\alpha I_n - D(\mathbf{q})\right)^{-1}\right|\right|_{\max}\cdot \\
& \max_{1\leq i\leq n}\left|M_\alpha\left[\left(C^{(i)}_{(\alpha',\beta)}(\mathbf{q})\right)_{(\alpha',\beta)\in L(\alpha)}; \left(b^{(1)}_{\alpha'}(\mathbf{q})\right)_{\alpha'<\alpha}, \ldots, \left(b^{(n)}_{\alpha'}(\mathbf{q})\right)_{\alpha'<\alpha}\right]\right| \\
\leq & nB\max_{1\leq i\leq n} M_\alpha\left[\left(\left|C^{(i)}_{(\alpha',\beta)}(\mathbf{q})\right|\right)_{(\alpha',\beta)\in L(\alpha)}; \left(\left|b^{(1)}_{\alpha'}(\mathbf{q})\right|\right)_{\alpha'<\alpha}, \ldots, \left(\left|b^{(n)}_{\alpha'}(\mathbf{q})\right|\right)_{\alpha'<\alpha}\right] \\
\leq & nBM_\alpha\left(\left(C_{(\alpha',|\beta|)}\right)_{(\alpha',\beta)\in L(\alpha)}; (B_{\alpha'})_{\alpha'<\alpha}, \ldots, (B_{\alpha'})_{\alpha'<\alpha}\right) \\
= & B_\alpha,
\end{aligned}$$

where, in the second inequality we used the definition of B (B.11) and the fact that the polynomials M_α have positive coefficients, in the third inequality we use the induction hypothesis (B.18), and in the last equality we used equation (B.17).
By complete induction, we conclude inequality (B.18) holds for all $1 \leq j \leq n$ and $\mathbf{q} \in \overline{B}^m_{\max}(\mathbf{q}_0, R)$, for every $\alpha \in \mathbb{N}^m \setminus \{0\}$. Determine $\rho_0 > 0$, such that the Taylor expansion (B.16) converges if $||\mathbf{t}||_{\max} < \rho_0$. Take any $0 < \rho < \rho_0$, then we have

$$\sum_{\alpha\in\mathbb{N}^m\setminus\{0\}} B_\alpha \rho^{|\alpha|} < \infty. \tag{B.19}$$

Take a $1 \leq j \leq n$ and define, for $\alpha \in \mathbb{N}^m \setminus \{0\}$, the function

$$Y_\alpha^{(j)}(\mathbf{t}, \mathbf{q}) = b_\alpha^{(j)}(\mathbf{q})\mathbf{t}^\alpha,$$

which is holomorphic on $U \times \mathbb{C}^m$ and hence, also holomorphic on the compact set

$$S = \overline{B}^m_{\max}(\mathbf{0}, \rho) \times \overline{B}^m_{\max}(\mathbf{q}_0, R). \tag{B.20}$$

Let $||\cdot||^S_\infty$ denote the supremum norm on S, then we have, by inequality (B.18),

$$||Y^{(j)}_\alpha||^S_\infty \leq B_\alpha \rho^{|\alpha|},$$

and therefore, by equation (B.19),

$$\sum_{\alpha \in \mathbb{N}^m \setminus \{0\}} ||Y^{(j)}_\alpha||^S_\infty < \infty.$$

We conclude that

$$y_j(\mathbf{t}; \mathbf{q}) = \sum_{\alpha \in \mathbb{N}^m \setminus \{0\}} Y^{(j)}_\alpha(\mathbf{t}, \mathbf{q}) = \sum_{\alpha \in \mathbb{N}^m \setminus \{0\}} b^{(j)}_\alpha(\mathbf{q}) \mathbf{t}^\alpha, \tag{B.21}$$

converges uniformly on S, defining a complex function holomorphic on the interior of S. □

Remark B.5. It is straightforward to extend Theorem B.4 to include parameters. That is, we write $\mathbf{b} = (b_1, \ldots, b_s)$ for some $s \in \mathbb{N}^*$, set $H = H(\mathbf{t}, \mathbf{y}; \mathbf{q}, \mathbf{b})$ and assume that the conditions in Theorem B.4 hold for all $\mathbf{b}$ in some fixed open set $V \subseteq \mathbb{C}^s$. Then the obtained series $y(\mathbf{t}; \mathbf{q}, \mathbf{b})$ converge locally uniformly in $(\mathbf{t}, \mathbf{q}, \mathbf{b})$ at $(\mathbf{t}, \mathbf{q}, \mathbf{b}) = (\mathbf{0}, \mathbf{q}_0, \mathbf{b}_0)$ for $(\mathbf{q}_0, \mathbf{b}_0) \in U \times V$.

Appendix C

The QRT Mapping

In this section we discuss the *QRT mapping*, first introduced in Quispel, Roberts and Thompson [74]. We denote

$$X = \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix},$$

and take square matrices

$$A_i = \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ \delta_i & \epsilon_i & \zeta_i \\ \kappa_i & \lambda_i & \mu_i \end{pmatrix}. \qquad (i = 1, 2)$$

The QRT mapping is the 18-parameter family of mappings given by

$$\overline{x} = \frac{f_1(y) - x f_2(y)}{f_2(y) - x f_3(y)}, \qquad \overline{y} = \frac{g_1(\overline{x}) - y g_2(\overline{x})}{g_2(\overline{x}) - y g_3(\overline{x})}, \tag{C.1}$$

with

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} = (A_0 X) \times (A_1 X), \qquad g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \\ g_3(x) \end{pmatrix} = \left(A_0^T X\right) \times \left(A_1^T X\right).$$

Such a mapping has an *invariant* given by

$$I(x, y) = \frac{\alpha_0 x^2 y^2 + \beta_0 x^2 y + \gamma_0 x^2 + \delta_0 x y^2 + \epsilon_0 x y + \zeta_0 x + \kappa_0 y^2 + \lambda_0 y + \mu_0}{\alpha_1 x^2 y^2 + \beta_1 x^2 y + \gamma_1 x^2 + \delta_1 x y^2 + \epsilon_1 x y + \zeta_1 x + \kappa_1 y^2 + \lambda_1 y + \mu_1},$$

that is,

$$I\left(\overline{x}, \overline{y}\right) = I(\overline{x}, y) = I(x, y). \tag{C.2}$$

Conversely, the invariant defines the QRT mapping, via the following geometric process. Consider the *pencil of quadratic curves*

$$\{I(x, y) = \lambda\}_{\lambda \in \mathbb{P}}.$$

Given a point $(x_0, y_0) \in \mathbb{P} \times \mathbb{P}$, write $\lambda = I(x_0, y_0)$ and let l_1 be the line $(x, y) = (x, y_0)$ in $\mathbb{P} \times \mathbb{P}$. Consider the curve $\mathcal{C}$ defined by $I(x, y) = \lambda$, then $(\overline{x}_0, y_0)$ is the unique other point on the intersection of $\mathcal{C}$ and l_1 in $\mathbb{P} \times \mathbb{P}$. Similarly let l_2 be the line $(x, y) = (\overline{x}_0, y)$, then $(\overline{x}_0, \overline{y}_0)$ is the unique other point on the intersection of $\mathcal{C}$ and l_2. Of course $(x_0, y_0) \mapsto (\overline{x}_0, \overline{y}_0)$ now coincides with the QRT mapping.

Let us write $P = I(x, y)$ and

$$A = \alpha_0 - \alpha_1 P, \quad B = \beta_0 - \beta_1 P, \quad D = \delta_0 - \delta_1 P, \quad G = \gamma_0 - \gamma_1 P, \quad E = \epsilon_0 - \epsilon_1 P, \tag{C.3a}$$

$$K = \kappa_0 - \kappa_1 P, \quad L = \lambda_0 - \lambda_1 P, \quad Z = \zeta_0 - \zeta_1 P, \quad U = \mu_0 - \mu_1 P, \tag{C.3b}$$

then, by equation (C.2), we have

$$Ax^2y^2 + Bx^2y + Dxy^2 + Gx^2 + Exy + Ky^2 + Zx + Ly + U = 0, \tag{C.4}$$

$$A\overline{x}^2y^2 + B\overline{x}^2y + D\overline{x}y^2 + G\overline{x}^2 + E\overline{x}y + Ky^2 + Z\overline{x} + Ly + U = 0, \tag{C.5}$$

$$A\overline{x}^2\overline{y}^2 + B\overline{x}^2\overline{y} + D\overline{x}\overline{y}^2 + G\overline{x}^2 + E\overline{x}\overline{y} + K\overline{y}^2 + Z\overline{x} + L\overline{y} + U = 0. \tag{C.6}$$

Subtracting equation (C.4) from (C.5) and equation (C.5) from (C.6) we obtain respectively,

$$(\overline{x} - x)\left(A(\overline{x} + x)y^2 + B(\overline{x} + x)y + Dy^2 + G(\overline{x} + x) + Ey + Z\right) = 0,$$
$$(\overline{y} - y)\left(A\overline{x}^2(\overline{y} + y) + B\overline{x}^2 + D\overline{x}(\overline{y} + y) + E\overline{x} + K(\overline{y} + y) + L\right) = 0.$$

Excluding the cases $\overline{x} = x$ and $\overline{y} = y$, we obtain

$$\overline{x} = -x - \frac{Dy^2 + Ey + Z}{Ay^2 + By + G}, \qquad \overline{y} = -y - \frac{B\overline{x}^2 + E\overline{x} + L}{A\overline{x}^2 + D\overline{x} + K}. \tag{C.7}$$

If the various parameters $A, B, \ldots, L, Z$ in this system are plain complex numbers, this has been called the *asymmetric McMillan map* in Iatrou and Roberts [39], as it is indeed an asymmetric extension of the classical *McMillan map* [62]. To summarise the discussion so far, given particular values $(x_0, y_0) \in \mathbb{P} \times \mathbb{P}$, let $(x_n, y_n)_{n \in \mathbb{Z}}$ be the sequence generated by the QRT mapping. Then, by calculating $P = I(x_0, y_0) \in \mathbb{P}$, we can associate a particular asymmetric McMillan map (C.7), specified by parameter values (C.3), such that $(x_n, y_n)_{n \in \mathbb{Z}}$ is generated by this McMillan map. We remark that the associated asymmetric McMillan map depends strongly on the particular initial values $(x_0, y_0) \in \mathbb{P}$ chosen.

Our main use of the above observation is in fact in opposite direction. We interpreted the QRT mapping as a system of q-difference equations on a given q-domain. To construct solutions to this system, we take any q-periodic function $P(t)$ and instead solve the simpler system given by (C.7), subject to $P(t) = I(x(t), y(t))$. The obtained solutions $(x(t), y(t))$ then satisfy the original system of q-difference equations defined by the QRT mapping. The justification of the last step is given by the following lemma.

Lemma C.1. *The mapping* (C.7)*, with* (C.3) *where* $P = I(x, y)$*, coincides with the QRT mapping* (C.1)*.*

Proof. Note that $P = I(x, y)$ is essentially (C.4). By reversing the above calculation we find that (C.5) and (C.6) hold as well, in particular we recover (C.2). Eliminating P from equation

(C.4) and the first equation in (C.7) gives the time evolution of x in (C.1). Eliminating P from equation (C.5) and the second equation in (C.7) gives the time evolution of y in (C.1). $\square$

C.1 Linearisable QRT mappings

If $A = B = D = 0$, then the autonomous system (C.7) is linear. We therefore consider a special type of QRT mapping, specified by the parameter conditions

$$\alpha_0 = \alpha_1 = \beta_0 = \beta_1 = \delta_0 = \delta_1 = 0. \tag{C.8}$$

We call such a QRT mapping *linearisable*. Equations (C.7) become the following system of linear equations

$$\overline{x} + x + \frac{E}{G}y = -\frac{Z}{G}, \qquad \overline{y} + y + \frac{E}{K}\overline{x} = -\frac{L}{K}. \tag{C.9}$$

"Solving" this system is straightforward, we first look for an equilibrium solution (x_{eq}, y_{eq}), that is, a solution invariant under the time evolution, so

$$2x_{eq} + \frac{E}{G}y_{eq} = -\frac{Z}{G}, \qquad 2y_{eq} + \frac{E}{K}x_{eq} = -\frac{L}{K},$$

which gives

$$x_{eq} = \frac{2KZ - EL}{E^2 - 4GK}, \qquad y_{eq} = -\frac{2GL - EZ}{E^2 - 4GK}. \tag{C.10}$$

Writing $x = x_{eq} + x_h$ and $y = y_{eq} + y_h$, we can rewrite (C.9) as the following homogeneous system,

$$\begin{pmatrix} \overline{x}_h \\ \overline{y}_h \end{pmatrix} = \begin{pmatrix} -1 & -\frac{E}{G} \\ \frac{E}{K} & \frac{E^2}{KG} - 1 \end{pmatrix} \begin{pmatrix} x_h \\ y_h \end{pmatrix}. \tag{C.11}$$

The next step involves diagonalising the matrix

$$M := \begin{pmatrix} -1 & -\frac{E}{G} \\ \frac{E}{K} & \frac{E^2}{KG} - 1 \end{pmatrix}, \tag{C.12}$$

and we hence consider the associated characteristic equation

$$|M - \lambda I| = \lambda^2 + \left(2 - \frac{E^2}{KG}\right)\lambda + 1 = 0, \tag{C.13}$$

which generically does not have a solution in $\mathbb{C}(P)$.
To overcome this limitation we could set $P = c_2\Lambda + c_1 + c_0/\Lambda$ where $\overline{\Lambda} = \Lambda$ for some well chosen c_0, c_1, c_2, to guarantee that equation (C.13) has a root in $\mathbb{C}(\Lambda)$, leading to a nice parameterisation. However the calculations quickly get out of hand, so we illustrate this process by example in (3.2.1). Once the general solution of (C.9) is found, we substitute it into equation (C.4), which forces us to fix the value of one free parameter as is done in equation (3.28). Then, by Lemma C.1, we obtain the generic solution of the QRT mapping subject to conditions (C.8). Note that we assumed $E^2 - 4GK \neq 0$ to obtain the equilibrium

solution (C.10). The case $E^2 - 4GK = 0$ is delicate and requires a separate analysis. We discuss such a case in Section 3.2.3.

Appendix D

Proofs of Results in Section 5.3

In this Appendix we discuss the proofs of Propositions 5.3.1, 5.3.2 and 5.3.3. As the method of proof is identical to that of Proposition 5.2.5, we only discuss the major steps in each case.

D.1 Proposition 5.3.1

We calculate the monodromy corresponding the solution $(f,g)=(f^{(0,1)},g^{(0,1)})$, meromorphic at $t=0$, defined in Proposition 3.1.1, where we assume the corresponding conditions (3.6) on the parameters. We write

$$f(t)=f_0+f_1t+f_2t^2+\dots,\qquad f_0=\frac{b_5b_6-b_7b_8}{b_5b_6(b_7+b_8)-b_7b_8(b_5+b_6)},$$
$$g(t)=g_0+g_1t+g_2t^2+\dots,\qquad g_0=\frac{b_5b_6-b_7b_8}{b_5+b_6-(b_7+b_8)}.$$

Lemma D.1. *Upon fixing nonzero meromorphic functions $s_i^\infty(t)$ on $\mathbb{C}^*$, satisfying*

$$s_i^\infty(qt)=\delta_i\alpha s_i^\infty(t),\quad \alpha=q^{-1}f_0^{-1},\quad \delta_1=\frac{1}{b_5b_6},\quad \delta_2=\frac{1}{b_7b_8},$$

for $i\in\{1,2\}$, there exist unique $\Psi_1^\infty(z,t)$ and $\Psi_2^\infty(z,t)$, such that

$$Y^\infty(z,t)=\Psi^\infty(z,t)\begin{pmatrix}s_1^\infty(t)&0\\0&s_2^\infty(t)\end{pmatrix},$$

defines a fundamental solution of Yamada's Lax pair $\widetilde{L}$ (4.82), where $\Psi_1^\infty(z,t)$ and $\Psi_2^\infty(z,t)$ are holomorphic at $(z,t)=(\infty,0)$, with $\Psi_1^\infty(\infty,0)=1=\Psi_2^\infty(\infty,0)$.

We write

$$D_0^\infty(z)=\left(D_0^{\infty,1}(z),D_0^{\infty,2}(z)\right):=\Psi^\infty(z,0),$$

then

$$D_0^{\infty,1}(qz) = \left(1 - \frac{1}{qb_5 z}\right)\left(1 - \frac{1}{qb_6 z}\right) D_0^{\infty,1}(z),$$
$$D_0^{\infty,2}(qz) = \left(1 - \frac{1}{qb_7 z}\right)\left(1 - \frac{1}{qb_8 z}\right) D_0^{\infty,2}(z),$$

and hence, by Lemma D.1,

$$D_0^{\infty}(z) = \left(\left(\frac{1}{b_5 z}, \frac{1}{b_6 z}; q\right)_\infty, \left(\frac{1}{b_7 z}, \frac{1}{b_8 z}; q\right)_\infty\right).$$

We define

$$\Psi^{\infty,0}(z,t) = \Psi^{\infty}(z,t)P^{\infty}(z)^{-1}, \quad P^{\infty}(z) := \begin{pmatrix} \theta_q(qb_5 z, qb_6 z)^{-1} & 0 \\ 0 & \theta_q(qb_7 z, qb_8 z)^{-1} \end{pmatrix}, \tag{D.1}$$

then

$$\Psi^{\infty,0}(z,0) = ((qb_5 z, qb_6 z; q)_\infty, (qb_7 z, qb_8 z; q)_\infty).$$

Lemma D.2. *Upon fixing a nonzero meromorphic function $s^0(t)$ on $\mathbb{C}^*$, satisfying*

$$s^0(qt) = \alpha t^{-2} s^0(t), \quad \alpha = q^{-1} f_0^{-1},$$

there exists, for any choice of $c_{0,0}^0, c_{0,1}^0 \in \mathbb{C}$, an unique $\Psi(z,t)$, such that

$$Y(z,t) = s^0(t)\Psi(\xi,t), \quad \xi = \frac{z}{t},$$

defines a solution of Yamada's Lax pair L (2.21), where $\Psi(\xi,t)$ holomorphic at $(\xi,t) = (0,0)$, with

$$\Psi(0,t) = c_{0,0}^0 + c_{0,1}^0 t + \mathcal{O}\left(t^2\right). \quad (t \to 0)$$

Furthermore $D_0(z) = \Psi(z,0)$ satisfies the degree two model equation (4.51) with parameter values $\sigma = \sigma_0^{II}$, defined in (5.27).

We use Lemma D.2 to define a fundamental solution $Y^0(z,t) = s^0(t)\Psi^0(\xi,t)$ of Yamada's Lax pair L (2.21), with

$$\Psi^0(\xi,0) = y^0(\xi; \sigma_0^{\mathrm{II}}),$$

and we define

$$\Psi^{0,\infty}(\xi,t) = \Psi^0(\xi,t)Q(\xi; \sigma_0^{\mathrm{II}}),$$

which gives

$$\Psi^{0,\infty}(\xi,0) = \psi^{\infty}(\xi; \sigma_0^{\mathrm{II}}).$$

Following the matching procedure, as outlined in Section 5.2.5, we find

$$\Psi^{\infty,0}(z,t) = \Psi^{0,\infty}(\xi,t),$$

and hence

$$Y^\infty(z,t) = Y^0(z,t)\mathcal{P}(z,t),$$

where

$$\mathcal{P}(z,t) = Q\left(\frac{z}{t};\sigma_0^{\mathrm{II}}\right)\begin{pmatrix}\theta_q(qb_5z,qb_6z)^{-1} & 0\\ 0 & \theta_q(qb_7z,qb_8z)^{-1}\end{pmatrix}\begin{pmatrix}\frac{s_1^\infty(t)}{s^0(t)} & 0\\ 0 & \frac{s_2^\infty(t)}{s^0(t)}\end{pmatrix}. \tag{D.2}$$

This is consistent with the notation in Section 4.9, where

$$\begin{aligned} c_0^1(t) &:= s^0(t)\Psi_1^0(0,t), & c_0^2(t) &:= s^0(t)\Psi_2^0(0,t),\\ \widetilde{c}_0^1(t) &:= s_1^\infty(t)\Psi_1^\infty(\infty,t), & \widetilde{c}_0^2(t) &:= s_2^\infty(t)\Psi_2^\infty(\infty,t). \end{aligned}$$

Proposition 5.3.1 is now easily derived.

D.2 Proposition 5.3.2

We calculate the monodromy corresponding to the solution $(f,g) = (\check{f}^{(0,1)},\check{g}^{(0,1)})$, meromorphic at $t=\infty$, defined by equation (3.15). We write

$$\begin{aligned} f(t) &= f_0 + f_1t^{-1} + f_2t^{-2} + \ldots, & f_0 &= \frac{b_5+b_6-(b_7+b_8)}{b_5b_6-b_7b_8},\\ g(t) &= g_0 + g_1t^{-1} + g_2t^{-2} + \ldots, & g_0 &= \frac{b_5b_6(b_7+b_8)-b_7b_8(b_5+b_6)}{b_5b_6-b_7b_8}, \end{aligned}$$

and we assume the corresponding conditions (3.6) on the parameters.

Considering the coefficients of the auxiliary equation (4.99) at $z=0$, we have

$$\begin{aligned} \gamma_0(t) &= f_0(f_0g_0-1)qb_5b_6b_7b_8g_0^{-1}t^6 + \mathcal{O}\left(t^5\right),\\ \gamma_1(t) &= f_0(f_0g_0-1)q^2(b_5b_6+b_7b_8)t^6 + \mathcal{O}\left(t^5\right),\\ \gamma_2(t) &= f_0(f_0g_0-1)q^3g_0t^6 + \mathcal{O}\left(t^5\right), \end{aligned}$$

as $t\to\infty$. Hence (4.99) is regular singular at $t=\infty$, with exponents $\{b_5b_6q^{-1}g_0^{-1}, b_7b_8q^{-1}g_0^{-1}\}$.

Lemma D.3. *Upon fixing nonzero meromorphic functions $\widehat{s}_i^0(t)$ on $\mathbb{C}^*$, satisfying*

$$\widehat{s}_i^0(qt) = \widehat{\delta}_i\widehat{\beta}\widehat{s}_i^0(t), \quad \widehat{\beta} = q^{-1}g_0^{-1}, \quad \widehat{\delta}_1 = b_5b_6, \quad \widehat{\delta}_2 = b_7b_8,$$

for $i\in\{1,2\}$, there exist unique $\widehat{\Psi}_1^0(z,t)$ and $\widehat{\Psi}_2^0(z,t)$, such that

$$\widehat{Y}^0(z,t) = \widehat{\Psi}^0(z,t)\begin{pmatrix}\widehat{s}_1^0(t) & 0\\ 0 & \widehat{s}_2^0(t)\end{pmatrix},$$

defines a fundamental solution of Yamada's Lax pair L (2.21), where $\widehat{\Psi}_1^0(z,t)$ and $\widehat{\Psi}_2^0(z,t)$ are holomorphic at $(z,t)=(0,\infty)$, with $\widehat{\Psi}_1^0(0,\infty) = 1 = \widehat{\Psi}_2^0(0,\infty)$.

We write

$$\widehat{D}_0^0(z)=\left(\widehat{D}_0^{0,1}(z),\widehat{D}_0^{0,2}(z)\right):=\widehat{\Psi}^0(z,\infty),$$

then

$$\widehat{D}_0^{0,1}(qz)=\frac{1}{(1-qb_5z)(1-qb_6z)}\widehat{D}_0^{0,1}(z),$$
$$\widehat{D}_0^{0,2}(qz)=\frac{1}{(1-qb_7z)(1-qb_8z)}\widehat{D}_0^{0,2}(z),$$

and hence, by Lemma D.3,

$$\widehat{D}_0^0(z)=\left((qb_5z,qb_6z;q)_\infty,(qb_7z,qb_8z;q)_\infty\right).$$

We define

$$\widehat{\Psi}^{0,\infty}(z,t)=\widehat{\Psi}^0(z,t)P^0(z),\quad P^0(z):=\begin{pmatrix}\theta_q(qb_5z,qb_6z)^{-1} & 0\\ 0 & \theta_q(qb_7z,qb_8z)^{-1}\end{pmatrix}, \tag{D.3}$$

then

$$\widehat{\Psi}^{0,\infty}(z,0)=\left(\left(\frac{1}{b_5z},\frac{1}{b_6z};q\right)_\infty^{-1},\left(\frac{1}{b_7z},\frac{1}{b_8z};q\right)_\infty^{-1}\right).$$

Lemma D.4. *Upon fixing a nonzero meromorphic function $\widehat{s}^\infty(t)$ on $\mathbb{C}^*$, satisfying $\widehat{s}^\infty(qt)=\widehat{\beta}t^2\widehat{s}^\infty(t)$, there exists, for any $\widehat{c}_{0,0},\widehat{c}_{0,1}\in\mathbb{C}$, an unique $\widehat{\psi}(\xi,t)$ such that*

$$\widetilde{y}(z,t)=\widehat{s}^\infty(t)\widehat{\psi}(\xi,t),\quad \xi=\frac{z}{t},$$

defines a solution of $\widetilde{L}$ (4.82), where $\widehat{\psi}(\xi,t)$ is holomorphic at $(\xi,t)=(\infty,\infty)$, with

$$\widehat{\psi}(\infty,t)=\widehat{c}_{0,0}+\widehat{c}_{0,1}t^{-1}+\mathcal{O}\left(t^{-2}\right).\quad (t\to\infty)$$

Furthermore $\widehat{d}_0(\xi)=\widehat{\psi}(\xi,\infty)$ defines a solution of the second order q-difference equation

$$\begin{aligned}d(q\xi)+\left(-(1+q^{-1})+\left(b_1^{-1}+b_2^{-1}+b_3^{-1}+b_4^{-1}\right)q^{-1}\xi^{-1}-q^{-2}\left(\frac{1}{b_5b_6}+\frac{1}{b_7b_8}\right)\xi^{-2}\right)d(z)\\ +q^{-1}(1-b_1^{-1}\xi^{-1})(1-b_2^{-1}\xi^{-1})(1-b_3^{-1}\xi^{-1})(1-b_4^{-1}\xi^{-1})d(\xi/q)=0.\end{aligned} \tag{D.4}$$

Note that equation (D.4), upon scaling

$$d(\xi)=\widehat{s}_*(\xi)y(\xi^{-1}),\quad \widehat{s}_*(\xi)=\left(\frac{1}{b_1\xi},\frac{1}{b_2\xi},\frac{1}{b_3\xi},\frac{1}{b_4\xi};q\right)_\infty,$$

coincides with the degree two model equation (4.51) for $y(z)$, with parameter values $\sigma=\widehat{\sigma}_\infty^{\mathrm{I}}$, as defined in (5.28). We hence use Lemma D.4 to specify an unique fundamental solution $\widehat{Y}^\infty(z,t)=\widehat{s}^\infty(t)\widehat{\Psi}^\infty(\xi,t)$ of $\widetilde{L}$ (4.82), such that

$$\widehat{\Psi}^\infty(\xi,\infty)=\widehat{s}_*(\xi)y^0(\xi^{-1};\widehat{\sigma}_\infty^{\mathrm{I}}).$$

To transition from $(\xi,t)=(\infty,\infty)$ to $(\xi,t)=(0,\infty)$ in the (ξ,t) plane via $t=\infty$, we set

$$\widehat{\Psi}^{\infty,0}(\xi,t)=\theta_q(qb_1\xi,qb_2\xi,qb_3\xi,qb_4\xi)^{-1}\widehat{\Psi}^{\infty}(\xi,t)Q(\xi^{-1};\widehat{\sigma}^{\mathrm{I}}_{\infty}),$$

which gives

$$\widehat{\Psi}^{\infty,0}(\xi,\infty)=(qb_1\xi,qb_2\xi,qb_3\xi,qb_4\xi;q)_{\infty}^{-1}\psi^{\infty}(\xi^{-1};\widehat{\sigma}^{\mathrm{I}}_{\infty}).$$

Via the matching procedure, as outlined in Section 5.2.5, we find

$$\widehat{\Psi}^{0,\infty}(z,t)=\widehat{\Psi}^{\infty,0}(\xi,t),$$

and hence

$$\widehat{Y}^{\infty}(z,t)=\widehat{Y}^{0}(z,t)\widehat{\mathcal{P}}(z,t),$$

where

$$\widehat{\mathcal{P}}(z,t)=\theta_q\left(qb_1\frac{z}{t},qb_2\frac{z}{t},qb_3\frac{z}{t},qb_4\frac{z}{t}\right)\begin{pmatrix}\frac{\widehat{s}^{\infty}(t)}{\widehat{s}^{0}_1(t)} & 0\\ 0 & \frac{\widehat{s}^{\infty}(t)}{\widehat{s}^{0}_2(t)}\end{pmatrix}\widehat{P}^0\left(z\right)Q\left(\frac{t}{z};\widehat{\sigma}^{\mathrm{I}}_{\infty}\right)^{-1}.$$

This is consistent with the notation in Section 4.9, where

$$c_0^1(t):=\widehat{s}^0_1(t)\widehat{\Psi}^0_1(0,t),\qquad c_0^2(t):=\widehat{s}^0_2(t)\widehat{\Psi}^0_2(0,t),$$
$$\widetilde{c}_0^1(t):=\widehat{s}^{\infty}(t)\widehat{\Psi}^{\infty}_1(\infty,t),\qquad \widetilde{c}_0^2(t):=\widehat{s}^{\infty}(t)\widehat{\Psi}^{\infty}_2(\infty,t).$$

Proposition 5.3.2 is now easily derived.

D.3 Proposition 5.3.3

We calculate the monodromy corresponding to the solution $(f,g)=(\check{f}^{(1,1)},\check{g}^{(1,1)})$, meromorphic at $t=\infty$, defined by equation (3.15). We write

$$f(t)=f_{-1}t+f_0+f_1t^{-1}+\dots,\qquad f_{-1}=\frac{b_1b_2-b_3b_4}{b_1b_2(b_3+b_4)-b_3b_4(b_1+b_2)},$$
$$g(t)=g_{-1}t+g_0+g_1t^{-1}+\dots,\qquad g_{-1}=\frac{b_1b_2-b_3b_4}{b_1+b_2-(b_3+b_4)},$$

and we assume the corresponding conditions (3.11) on the parameters.

Consider the coefficients of the auxiliary equation (4.99) at $z=0$, we have

$$\gamma_0(t)=qg_{-1}^2(f_{-1}g_{-1}-1)t^8+\mathcal{O}\left(t^7\right),$$
$$\gamma_1(t)=qg_{-1}^2(f_{-1}g_{-1}-1)q(q+1)f_{-1}t^9+\mathcal{O}\left(t^8\right),$$
$$\gamma_2(t)=qg_{-1}^2(f_{-1}g_{-1}-1)q^4f_{-1}^2t^{10}+\mathcal{O}\left(t^9\right),$$

as $t\to\infty$. We therefore rescale

$$c_0(t)=\widehat{s}^0(t)\widehat{c}_0(t),\quad \widehat{s}^0(qt)=\widehat{\alpha}t^{-1}\widehat{s}^0(t),\quad \widehat{\alpha}=-q^{-1}f_{-1}^{-1},$$

then the rescaled equation for $\widehat{c}_0(t)$ is regular singular at $t=\infty$, with exponents $\{1,q^{-1}\}$, and hence with resonance. A direct calculation shows that $t=\infty$ in in fact an apparent singularity. As usual we leave it to the reader to choose $\widehat{s}^0(t)$, nonzero and meromorphic on $\mathbb{C}^*$, at their pleasure.

Lemma D.5. *For any choice of $\widehat{c}_{0,0},\widehat{c}_{0,1}\in\mathbb{C}$, there exists an unique $\widehat{\psi}(z,t)$ such that*

$$Y(z,t)=\widehat{s}^0(t)\widehat{\psi}(z,t),$$

defines a solution of Yamada's Lax pair L (2.21), where $\widehat{\psi}(z,t)$ holomorphic at $(z,t)=(0,\infty)$ with

$$\widehat{\psi}(0,t)=\widehat{c}_{0,0}+\widehat{c}_{0,1}t^{-1}+\mathcal{O}\left(t^{-2}\right). \quad (t\to\infty)$$

Furthermore, $d_0(z)=\widehat{\psi}(z,\infty)$ defines a solution of the second order q-difference equation

$$\begin{aligned}&(1-qb_5z)(1-qb_6z)(1-qb_7z)(1-qb_8z)d_0(qz)\\&\quad+\left[-(1+q)+q(b_5+b_6+b_7+b_8)z-(b_1b_2+b_3b_4)qz^2\right]d_0(z)+qd_0(z/q)=0.\end{aligned}\tag{D.5}$$

Note that equation (D.5), upon scaling,

$$d_0(z)=s_*(z)y(z),\quad s_*(z):=(qb_5z,qb_6z,qb_7z,qb_8z;q)_\infty,$$

coincides with the degree two model equation (4.51) for $y(z)$, with parameter values $\sigma=\widehat{\sigma}_0^{\mathrm{II}}$, defined in (5.29). We hence specify an unique fundamental solution $\widehat{Y}^0(z,t)=\widehat{s}^0(t)\widehat{\Psi}^0(z,t)$ of Yamada's Lax pair, such that

$$\widehat{\Psi}^0(z,\infty)=s_*(z)y^0(z;\widehat{\sigma}_0^{\mathrm{II}}).$$

We now make the transition from $(z,t)=(0,\infty)$ to $(\xi,t)=(0,\infty)$ via the line $t=\infty$ in the (z,t) plane, by setting

$$\widehat{\Psi}^{0,\infty}(z,t)=\theta_q(qb_5z,qb_6z,qb_7z,qb_8z)^{-1}\widehat{\Psi}^0(z,t)Q(z;\widehat{\sigma}_0^{\mathrm{II}}),$$

which gives

$$\widehat{\Psi}^{0,\infty}(z,\infty)=\left(\frac{1}{b_5z},\frac{1}{b_6z},\frac{1}{b_7z},\frac{1}{b_8z};q\right)_\infty^{-1}\psi^\infty(z;\widehat{\sigma}_0^{\mathrm{II}}).$$

Lemma D.6. *Upon fixing meromorphic functions $\widehat{s}_i^\infty(t)$ on $\mathbb{C}^*$, satisfying*

$$\widehat{s}_i^\infty(qt)=\widehat{\delta}_i\widehat{\alpha}t\widehat{s}_i^\infty(t),\quad \widehat{\delta}_1=\frac{1}{b_1b_2},\quad \widehat{\delta}_2=\frac{1}{b_3b_4},$$

for $i\in\{1,2\}$, there exist unique $\widehat{\Psi}_1^\infty(\xi,t)$ and $\widehat{\Psi}_2^\infty(\xi,t)$ such that

$$\widehat{Y}^\infty(z,t)=\widehat{\Psi}^\infty(\xi,t)\begin{pmatrix}\widehat{s}_1^\infty(t)&0\\0&\widehat{s}_2^\infty(t)\end{pmatrix},$$

defines a fundamental solution of Yamada's Lax pair $\widetilde{L}$ (4.82), where $\widehat{\Psi}_1^\infty(\xi,t)$ and $\widehat{\Psi}_2^\infty(\xi,t)$

are holomorphic at $(\xi,t)=(\infty,\infty)$*, with* $\widehat{\Psi}_1^{\infty}(\infty,\infty)=1=\widehat{\Psi}_2^{\infty}(\infty,\infty)$.

We write

$$\widehat{D}_0^{\infty}(\xi)=\left(\widehat{D}_0^{\infty,1}(\xi),\widehat{D}_0^{\infty,2}(\xi)\right):=\widehat{\Psi}^{\infty}(\xi,0),$$

then

$$\begin{aligned}\widehat{D}_0^{\infty,1}(\xi)&=\left(1-\frac{1}{b_1\xi}\right)\left(1-\frac{1}{b_2\xi}\right)\widehat{D}_0^{\infty,1}(\xi/q),\\ \widehat{D}_0^{\infty,2}(\xi)&=\left(1-\frac{1}{b_3\xi}\right)\left(1-\frac{1}{b_4\xi}\right)\widehat{D}_0^{\infty,2}(\xi/q),\end{aligned}$$

and hence, by Lemma D.6,

$$\widehat{D}_0^{\infty}(\xi)=\left(\left(\frac{1}{b_1\xi},\frac{1}{b_2\xi};q\right)_\infty,\left(\frac{1}{b_3\xi},\frac{1}{b_4\xi};q\right)_\infty\right).$$

We define

$$\widehat{\Psi}^{\infty,0}(\xi,t)=\widehat{\Psi}^{\infty}(\xi,t)P^{\infty}(\xi)^{-1},\quad P^{\infty}(\xi):=\begin{pmatrix}\theta_q(qb_1\xi,qb_2\xi) & 0\\ 0 & \theta_q(qb_3\xi,qb_4\xi)\end{pmatrix},\tag{D.6}$$

then

$$\widehat{\Psi}^{\infty,0}(\xi,0)=\left((qb_1\xi,qb_2\xi;q)_\infty^{-1},(qb_3\xi,qb_4\xi;q)_\infty^{-1}\right).$$

Via the matching procedure, as outlined in Section 5.2.5, we find

$$\widehat{\Psi}^{0,\infty}(z,t)=\widehat{\Psi}^{\infty,0}(\xi,t),$$

and hence

$$\widehat{Y}^{\infty}(z,t)=\widehat{Y}^{0}(z,t)\widehat{\mathcal{P}}(z,t),$$

where

$$\widehat{\mathcal{P}}(z,t)=\theta_q(qb_5z,qb_6z,qb_7z,qb_8z)^{-1}Q\left(z;\widehat{\sigma}_0^{\mathrm{II}}\right)\widehat{P}^{\infty}\left(\frac{z}{t}\right)\begin{pmatrix}\frac{\widehat{s}_1^{\infty}(t)}{\widehat{s}^{0}(t)} & 0\\ 0 & \frac{\widehat{s}_2^{\infty}(t)}{\widehat{s}^{0}(t)}\end{pmatrix}.$$

This is consistent with the notation in Section 4.9, where

$$\begin{aligned}c_0^1(t)&:=\widehat{s}^0(t)\widehat{\Psi}_1^0(0,t), & c_0^2(t)&:=\widehat{s}^0(t)\widehat{\Psi}_2^0(0,t),\\ \widetilde{c}_0^1(t)&:=\widehat{s}_1^{\infty}(t)\widehat{\Psi}_1^{\infty}(\infty,t), & \widetilde{c}_0^2(t)&:=\widehat{s}_2^{\infty}(t)\widehat{\Psi}_2^{\infty}(\infty,t).\end{aligned}$$

Proposition 5.3.3 is now easily derived.

Bibliography

[1] C.R. Adams. On the Linear Ordinary q-Difference Equation. *Annals of Mathematics*, 30(1/4):195–205, 1928.

[2] P. Bachmann. *Die analytische Zahlentheorie.* Leipzig B.G. Teubner, 1894.

[3] D. C. Barnett, R.G. Halburd, W. Morgan, and R.J. Korhonen. Nevanlinna theory for the q-difference operator and meromorphic solutions of q-difference equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 137(3):457–474, 2007.

[4] M.A. Bershtein and A.I. Shchechkin. Bilinear equations on Painlevé τ functions from CFT. *Commun. Math. Phys.*, 339(3):1021–1061, 2015.

[5] G.D. Birkhoff. The Generalized Riemann Problem for Linear Differential Equations and the Allied Problems for Linear Difference and q-Difference Equations. *Proceedings of the American Academy of Arts and Sciences*, 49(9):521–568, 1913.

[6] P. Boutroux. Recherches sur les transcendantes de M. Painlevé et l'étude asymptotique des équations differentielles du second ordre. *Ann. École Norm*, 30:255–375, 1913.

[7] E. Brézin and V.A. Kazakov. Exactly solvable field theories of closed strings. *Phys. Lett.*, 236:144–150, 1990.

[8] C. Briot and J. Bouquet. Propriétés des fonctions définie par des équations différentielles. *J. l'Ecole Polytech.*, 21:133–198, 1856.

[9] R.D. Carmichael. The General Theory of Linear q-Difference Equations. *American Journal of Mathematics*, 34(2):147–168, 1912.

[10] P.A. Clarkson. Painlevé transcendents. In *NIST handbook of mathematical functions*, pages 723–740. U.S. Dept. Commerce, Washington, DC, 2010.

[11] B. Dubrovin and M. Mazzocco. Monodromy of certain Painlevé-VI transcendents and reflection groups. *Invent. Math.*, 141(1):55–147, 2000.

[12] J. J. Duistermaat and N. Joshi. Okamoto's space for the first Painlevé equation in Boutroux coordinates. *Arch. Ration. Mech. Anal.*, 202(3):707–785, 2011.

[13] J.J. Duistermaat. *Discrete integrable systems.* Springer Monographs in Mathematics. Springer, New York, 2010. QRT maps and elliptic surfaces.

[14] A. Fokas, A. Its, A. Kapaev, and V. Novokshenov. *Painlevé transcendents*, volume 128 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2006. The Riemann-Hilbert approach.

[15] R. Fuchs. Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegenen wesentlich singulären Stellen. *Math. Ann.*, 63(3):301–321, 1907.

[16] O. Gamayun, N. Iorgov, and O. Lisovyy. Conformal field theory of Painlevé VI. *J. High Energy Phys.*, JHEP10(2012)(38):25p, 2012.

[17] B. Gambier. Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est a points critiques fixes. *Acta Math.*, 33(1):1–55, 1910.

[18] G. Gasper and M. Rahman. *Basic hypergeometric series*, volume 35 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1990. With a foreword by Richard Askey.

[19] P. Gavrylenko and O. Lisovyy. Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions. *ArXiv e-prints*, August 2016.

[20] B. Grammaticos and A. Ramani. On a novel q-discrete analogue of the Painlevé VI equation. *Phys. Lett. A*, 257(5-6):288–292, 1999.

[21] B. Grammaticos and A. Ramani. Discrete Painlevé equations: a review. In *Discrete integrable systems*, volume 644 of *Lecture Notes in Phys.*, pages 245–321. Springer, Berlin, 2004.

[22] B. Grammaticos, A. Ramani, and V. Papageorgiou. Do integrable mappings have the Painlevé property? *Phys. Rev. Lett.*, 67(14):1825–1828, 1991.

[23] D.P. Gupta, M.E.H. Ismail, and D.R. Masson. Contiguous relations, basic hypergeometric functions, and orthogonal polynomials. III. Associated continuous dual q-Hahn polynomials. *J. Comput. Appl. Math.*, 68(1-2):115–149, 1996.

[24] D. Guzzetti. The elliptic representation of the general Painlevé VI equation. *Commun. Pure Appl. Math.*, 55(10):1280–1363, 2002.

[25] D. Guzzetti. The logarithmic asymptotics of the sixth Painlevé equation. *J. Phys. A: Math. Theor.*, 41(20):205201(46p), 2008.

[26] D. Guzzetti. Solving the sixth Painlevé equation: towards the classification of all the critical behaviors and the connection formulae. *Int. Math. Res. Not.*, 6:1352–1413, 2012.

[27] D. Guzzetti. Tabulation of Painlevé 6 transcendents. *Nonlinearity*, 25(12):3235–3276, 2012.

[28] D. Guzzetti. A review of the sixth Painlevé equation. *Constr. Approx.*, 41(3):495–527, 2015.

[29] R.G. Halburd and R.J. Korhonen. Meromorphic solutions of difference equations, integrability and the discrete Painlevé equations. *J. Phys. A*, 40(6):R1–R38, 2007.

[30] G.H. Hardy. *Orders of infinity.* Cambridge University Press, 1910.

[31] R. Hartshorne. *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[32] M. Hay, P. Howes, N. Nakazono, and Y. Shi. A systematic approach to reductions of type-Q ABS equations. *J. Phys. A*, 48(9):095201, 24, 2015.

[33] E. Heine. Über die Reihe $1+\frac{(q^\alpha-1)(q^\beta-1)}{(q-1)(q^\gamma-1)}x+\frac{(q^\alpha-1)(q^{\alpha+1}-1)(q^\beta-1)(q^{\beta_1}-1)}{(q-1)(q^2-1)(q^\gamma-1)(q^{\gamma+1}-1)}+\cdots$. (Aus einem Schreiben an Lejeune Dirichlet). *J. Reine Angew. Math.*, 32:210–212, 1846.

[34] E. Heine. Untersuchungen über die Reihe. *J. Reine Angew. Math.*, 34:285–328, 1847.

[35] J. Hietarinta and C.M. Viallet. Singularity Confinement and Chaos in Discrete Systems. *Phys. Rev. Lett.*, 81(2):325–328, 1998.

[36] E. Hille. *Ordinary differential equations in the complex domain.* Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1976. Pure and Applied Mathematics.

[37] A. Hinkkanen and I. Laine. The meromorphic nature of the sixth Painlevé transcendents. *J. Anal. Math.*, 94:319–342, 2004.

[38] P. Howes and N. Joshi. Global asymptotics of the second Painlevé equation in Okamoto's space. *Constr. Approx.*, 39(1):11–41, 2014.

[39] A. Iatrou and J.A.G. Roberts. Integrable mappings of the plane preserving biquadratic invariant curves. *J. Phys. A: Math. Gen.*, 34(34):6617, 2001.

[40] A.R. Its and A.A. Kapaev. Quasi-linear Stokes phenomenon for the second Painlevé transcendent. *Nonlinearity*, 16(1):363–386, 2003.

[41] K. Iwasaki, H. Kimura, S. Shimomura, and M. Yoshida. *From Gauss to Painlevé.* Aspects of Mathematics, E16. Friedr. Vieweg & Sohn, Braunschweig, 1991. A modern theory of special functions.

[42] M. Jimbo. Monodromy problem and the boundary condition for some Painlevé equations. *Publ. Res. Inst. Math. Sci.*, 18:1137–1161, 1982.

[43] M. Jimbo and H. Sakai. A q-analogue of the sixth Painlevé equation. *Lett. Math. Phys.*, 38:145–154, 1996.

[44] N. Joshi and M. Kruskal. An asymptotic approach to the connection problem for the first and the second Painlevé equations. *Phys. Lett. A*, 130(3):129–137, 1988.

[45] N. Joshi and M.D. Kruskal. A direct proof that solutions of the six Painlevé equations have no movable singularities except poles. *Stud. Appl. Math.*, 93(3):187–207, 1994.

[46] N. Joshi and S.B. Lobb. Singular dynamics of a q-difference Painlevé equation in its initial-value space. *J. Phys. A*, 49(1):014002, 24, 2016.

[47] N. Joshi and C.J. Lustri. Stokes phenomena in discrete Painlevé I. *Proc. A.*, 471(2177):20140874, 22, 2015.

[48] N. Joshi, C.J. Lustri, and S. Luu. Stokes Phenomena in Discrete Painlevé II. *ArXiv e-prints*, June 2016.

[49] N. Joshi and M. Radnović. Asymptotic behavior of the fourth Painlevé transcendents in the space of initial values. *Constr. Approx.*, 44(2):195–231, 2016.

[50] N. Joshi and P. Roffelsen. Analytic solutions of q-$P(A_1)$ near its critical points. *Nonlinearity*, 29(12):3696–742, 2016.

[51] Victor G. Kac. *Infinite-dimensional Lie algebras.* Cambridge University Press, Cambridge, third edition, 1990.

[52] K. Kajiwara, M. Noumi, and Y. Yamada. Geometric Aspects of Painlevé Equations. *ArXiv e-prints*, September 2015.

[53] K. Kaneko. Painlevé VI transcendents which are meromorphic at a fixed singularity. *Proc. Japan Acad. Ser. A Math. Sci.*, 82(5):71–76, 2006.

[54] A.A. Kapaev. Quasi-linear stokes phenomenon for the Painlevé first equation. *J. Phys. A*, 37(46):11149–11167, 2004.

[55] A.V. Kitaev. Asymptotic description of solutions of the fourth Painlevé equation on analogues of Stokes's rays. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 169(Voprosy Kvant. Teor. Polya i Statist. Fiz. 8):84–90, 187, 1988.

[56] R. Koekoek, P.A. Lesky, and R.F. Swarttouw. *Hypergeometric orthogonal polynomials and their q-analogues.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010. With a foreword by Tom H. Koornwinder.

[57] E. Landau. *Handbuch der Lehre von der Verteilung der Primzahlen.* Leipzig B.G. Teubner, 1909.

[58] J. Le Caine. The linear q-difference equation of the second order. *Amer. J. Math.*, 65:585–600, 1943.

[59] O. Lisovyy and Y. Tykhyy. Algebraic solutions of the sixth Painlevé equation. *J. Geom. Phys.*, 85:124–163, 2014.

[60] E. Looijenga. Rational surfaces with an anticanonical cycle. *Ann. of Math. (2)*, 114(2):267–322, 1981.

[61] T. Mano. Asymptotic behaviour around a boundary point of the q-Painlevé VI equation and its connection problem. *Nonlinearity*, 23(7):1585–1608, 2010.

[62] E.M. McMillan. A problem in the stability of periodic systems. *Topics in Modern Physics, Colorado University Press, Boulder*, pages 219–244, 1971.

[63] M. Noumi. *Painlevé equations through symmetry*, volume 223 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2004. Translated from the 2000 Japanese original by the author.

[64] Y. Ohyama. Analytic solutions to the sixth q-Painlevé equation around the origin. In *Expansion of integrable systems*, RIMS Kôkyûroku Bessatsu, B13, pages 45–52. Res. Inst. Math. Sci. (RIMS), Kyoto, 2009.

[65] Y. Ohyama. Meromorphic solutions to the q-Painlevé equations around the origin. *Journal of Physics: Conference Series*, 597(1):012063, 2015.

[66] K. Okamoto. Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé. *Japan. J. Math. (N.S.)*, 5(1):1–79, 1979.

[67] C.M. Ormerod and E.M. Rains. An elliptic Garnier system. *ArXiv e-prints*, July 2016.

[68] C.M. Ormerod and E.M. Rains. Commutation Relations and Discrete Garnier Systems. *ArXiv e-prints*, January 2016.

[69] P. Painlevé. Memoire sur les équations différentielles dont l'intégrale générale est uniforme. *Bull. Soc. Math. Phys. France*, 28:201–261, 1900.

[70] P. Painlevé. Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme. *Acta Math.*, 25(1):1–85, 1902.

[71] H. Poincaré. Sur une classe nouvelle de transcendantes uniformes. *J. Math. Pures Appl.*, 4(6):313–366, 1890.

[72] C. Praagman. Fundamental solutions for meromorphic linear difference equations in the complex plane, and related problems. *J. Reine Angew. Math.*, 369:101–109, 1986.

[73] X. Qi and L. Yang. Properties of meromorphic solutions of q-difference equations. *Electron. J. Differential Equations*, pages No. 59, 9, 2015.

[74] G.R.W. Quispel, J.A.G. Roberts, and C.J. Thompson. Integrable mappings and soliton equations. *Phys. Lett.*, 126:419–421, 1988.

[75] E.M. Rains. An isomonodromy interpretation of the hypergeometric solution of the elliptic Painlevé equation (and generalizations). *SIGMA Symmetry Integrability Geom. Methods Appl.*, 7:Paper 088, 24, 2011.

[76] J.P. Ramis, J. Sauloy, and C. Zhang. *Local Analytic Classification of Q-difference Equations*. Amer Mathematical Society, 2013.

[77] H. Sakai. Rational surfaces associated with affine root systems and geometry of the Painlevé equations. *Comm. Math. Phys.*, 220(1):165–229, 2001.

[78] J. Sauloy. *Théorie de Galois des équations aux q-différences fuchsiennes.* PhD thesis, Paul Sabatier University, 1999.

[79] J. Sauloy. Systèmes aux q-différences singuliers réguliers: classification, matrice de connexion et monodromie. *Ann. Inst. Fourier (Grenoble)*, 50(4):1021–1071, 2000.

[80] J. Sauloy. Galois theory of q-difference equations: the "analytical" approach. In *Differential equations and the Stokes phenomenon*, pages 277–292. World Sci. Publ., River Edge, NJ, 2002.

[81] J. Shohat. A differential equation for orthogonal polynomials. *Duke Math. J.*, 5(2):401–417, 1939.

[82] H. Tahara and H. Yamazawa. q-analogue of summability of formal solutions of some linear q-difference-differential equations. *Opuscula Math.*, 35(5):713–738, 2015.

[83] M. van der Put and M.F. Singer. *Galois theory of difference equations*, volume 1666 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1997.

[84] I. Vinogradow. On Waring's problem. *Ann. of Math. (2)*, 36(2):395–405, 1935.

[85] Y. Yamada. Lax formalism for q-Painlevé equations with affine weyl group symmetry of type $E_n^{(1)}$. *Int. Math. Res. Not.*, 2011(17):3823–38, 2011.

[86] X.M. Zheng and Z.X. Chen. Some properties of meromorphic solutions of q-difference equations. *J. Math. Anal. Appl.*, 361(2):472–480, 2010.