A family of uniform lattices acting on a Davis complex with a non-discrete set of covolumes

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CONTENTS

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CHAPTER 1

Introduction

Let G be a locally compact topological group equipped with an appropriate normalisation of the Haar measure μ. A discrete subgroup Γ is a *lattice* in G if the quotient G/Γ admits a finite G-invariant measure, and it is *uniform* if G/Γ is compact.

An interesting question to ask is: what is the set of possible (positive real) covolumes $\mu(G/\Gamma)$ of lattices – uniform or otherwise – in G? This question was first studied in 1945 by Siegel [**20**] for the case where $G = \mathrm{SL}(2, \mathbb{R})$. It has since been pursued by various researchers for the case of a semi-simple algebraic group G over a field. In addition, more recently, this question has been studied by Lubotzky [**2, 12**], Bass [**2**] and Burger and Mozes [**6**] for the automorphism group G of a (product of) tree(s).

Let (W, S) be a Coxeter system with Davis complex Σ. The group of polyhedral automorphisms $\mathrm{Aut}(\Sigma)$ is naturally a locally compact topological group under the compact-open topology (in fact this is true for any connected, locally finite polyhedral complex). In this paper we are interested in determining the set

$$\mathcal{V}_u(\mathrm{Aut}(\Sigma)) := \left\{ \mu(\mathrm{Aut}(\Sigma)/\Gamma) \;\middle|\; \Gamma \text{ is a uniform lattice in } \mathrm{Aut}(\Sigma) \right\}$$

Whilst covolumes of non-uniform lattices may be irrational, Serre's theorem [**19**] tells us immediately that $\mathcal{V}_u(\mathrm{Aut}(\Sigma)) \subseteq \mathbb{Q}^{>0}$.

In the case where (W, S) is the free product of $n \geq 3$ copies of the cyclic group of order two, with Davis complex Σ_n isomorphic to the $(n, 2)$-biregular tree, the result is already known from Levich and Rosenberg's (Chapter 9 of [**16**]) classification of uniform lattices acting on regular and biregular trees. Precisely,

$$\mathcal{V}_u(\mathrm{Aut}(\Sigma_n)) = \left\{ a/b \;\middle|\; a, b \in \mathbb{N} \text{ are coprime, prime divisors of } b \text{ are } < n \right\}$$

We briefly look at this situation in Section 8.1. In Proposition 8.3 we give a new proof of the fact that $\mathcal{V}_u(\mathrm{Aut}(\Sigma_n))$ is a non-discrete set and in Corollary 8.4 provide a new proof of the fact that for any prime $p < n$ and any $\alpha \in \mathbb{N}$, there exists a uniform lattice Γ in $\mathrm{Aut}(\Sigma_n)$ with covolume a/b (in lowest terms) such that b is divisible by p^α.

In the case where Σ is not a tree, not much is known about the set $\mathcal{V}_u(\mathrm{Aut}(\Sigma))$. Thomas [**22**] found a subclass of Davis complexes Σ for which there exists an infinite family of uniform lattices in $\mathrm{Aut}(\Sigma)$ with a set of covolumes converging to that of a non-uniform lattice in $\mathrm{Aut}(\Sigma)$. For a (different) subclass of Davis complexes Σ, White [**25**] constructed an infinite family of uniform lattices in $\mathrm{Aut}(\Sigma)$ with arbitrarily small covolumes.

For certain types of Davis complex Σ it is known that $\mathcal{V}_u(\mathrm{Aut}(\Sigma))$ is a discrete set. Haglund and Paulin [**11**] show that the group $\mathrm{Aut}(\Sigma)$ is non-discrete (as a topological space) iff there exists a non-trivial automorphism of the nerve L as a weighted graph that fixes the star of some vertex in L. We call Σ *flexible* if it satisfies Haglund and Paulin's condition, and *rigid* otherwise. If Σ is rigid then $\mathcal{V}_u(\mathrm{Aut}(\Sigma))$ must be a discrete set. The converse however remains an open problem. That is, for any flexible Davis complex Σ, must $\mathcal{V}_u(\mathrm{Aut}(\Sigma))$ be a non-discrete set?

For example, consider the Coxeter system

$$(W_0, S_0) := \left\langle s_1, s_2, s_3 \;\middle|\; s_1^2 = s_2^2 = s_3^2 = (s_1s_2)^3 = (s_1s_3)^3 = (s_2s_3)^3 = 1 \right\rangle$$

with Davis complex Σ_0 a tesselation of the plane into barycentrically subdivided triangles. We deduce that $\mathcal{V}_u(\mathrm{Aut}(\Sigma_0))$ is a discrete set either by observing that Σ_0 is rigid or alternatively by applying Theorem 2 of [**21**] to Σ_0. This alternate method tells us that if Γ is a uniform lattice in $\mathrm{Aut}(\Sigma_0)$ then its covolume $\mu(\mathrm{Aut}(\Sigma_0)/\Gamma) = a/b$ for coprime $a, b \in \mathbb{N}$ is such that b is indivisible by 2^4, by 3^2 and by any primes other than 2 and 3.

In Section 8.2 we introduce Theorem 8.5, the main result of this paper, which shows that $\mathcal{V}_u(\mathrm{Aut}(\Sigma))$ is a non-discrete set for a particular subclass of Davis complexes Σ. As far as the author is aware, this was only known previously in the case where Σ is a tree. We prove Theorem 8.5 by constructing a family of uniform lattices as fundamental groups of a family of finite complexes of finite groups each with universal cover isomorphic to Σ. Corollary 8.15 extends the construction in Theorem 8.5 and shows that for any $\alpha \in \mathbb{N}$ and any prime $p < j$, where j is the number of free generators in (W, S), there exists a uniform lattice Γ in $\mathrm{Aut}(\Sigma)$ with covolume $\mu(\mathrm{Aut}(\Sigma)/\Gamma) = a/b$ (in lowest terms) such that b is divisible by p^α.

We conclude this paper by looking at regular right-angled buildings. In the case where (W, S) is a right-angled Coxeter system with n generators, it is known [**24**] that there exists a unique regular right-angled building X with $p_i \geq 2$ chambers in each s_i-equivalence class, for all $i \in \{1, ..., n\}$. Using a functor from graphs of groups to complexes of groups, Thomas [**23**] showed that if $m := \max_{1 \leq i \leq n}\{p_i\} > 2$ then

$$\mathcal{V}_u(\mathrm{Aut}(X)) = \left\{ a/b \;\middle|\; a, b \in \mathbb{N} \text{ are coprime, prime divisors of } b \text{ are } < m \right\}$$

In Section 8.3 we look at the situation where (W, S) has $2 \leq j \leq n$ free generators, which we call $s_1, ..., s_j$, with $\max_{1 \leq i \leq j}\{p_i\} > 2$. In Theorem 8.17 we give a new proof of the fact that $\mathcal{V}_u(\mathrm{Aut}(X))$ is a non-discrete set and in Corollary 8.21 provide a new proof of the fact that for any prime $p < \max_{1 \leq i \leq j}\{p_i\}$ and any $\alpha \in \mathbb{N}$, there exists a uniform lattice Γ in $\mathrm{Aut}(X)$ with covolume a/b (in lowest terms) such that b is divisible by p^{α}.

CHAPTER 2

Simplicial, Polyhedral and Cell Complexes

Let $\mathcal{P}(.)$ denote the power set.

Definition 2.1. An *abstract simplicial complex* $\mathcal{X}$ is a family of non-empty finite sets that is closed under the operation of taking non-empty subsets. An element of $\mathcal{X}$ with cardinality $n+1$ is known as an *(abstract) n-face*, and we say that $y \in \mathcal{X}$ is an *(abstract) face of* $z \in \mathcal{X}$ iff $y \subseteq z$.

An *(abstract) n-simplex* is a pair (x, X_x), where $X_x := \mathcal{P}(x) \backslash \varnothing \subseteq \mathcal{X}$ is a subcomplex of $\mathcal{X}$ generated by a single n-face $x \in \mathcal{X}$. Confusingly, depending on the context, the term *(abstract) simplex* may be used to refer to either the face x, the subcomplex X_x or the pair (x, X_x).

We define $\dim(x, X_x) := |x| - 1$ and $\dim(\mathcal{X}) := \max_{x \in \mathcal{X}} \{\dim(x, X_x)\}$.

Note that the 1-skeleton of $\mathcal{X}$ is a (simple, undirected) graph, and the 1-skeleton of X_x is a complete graph.

Remark 2.2. An abstract simplicial complex is a poset under set inclusion.

Definition 2.3. Let $v_o, v_1, ..., v_n \in \mathbb{R}^k$ be affinely independent. Then we may define an *(affine) n-simplex* with vertex set $V := \{v_o, ..., v_n\}$ by

$$\sigma_n := \left\{ \sum_{i=0}^{n} \lambda_i v_i \middle| \sum_{i=0}^{n} \lambda_i = 1, \lambda_i \geq 0 \text{ for } i = 0, ..., n \right\}$$

In other words, σ_n is the smallest convex set in $\mathbb{R}^k$ containing each vertex in V, or the *convex hull* of V.

An *(affine) face* of σ_n is a sub-simplex of σ_n generated by a subset of V.

Definition 2.4. An *(affine) simplicial complex* $\mathcal{K}$ is a set of (affine) simplices in $\mathbb{R}^k$ that is closed under the operation of taking faces, and which has the property that for any pair of simplices $\sigma_1, \sigma_2 \in \mathcal{K}$, then $\sigma_1 \cap \sigma_2$ is either empty or is a face of both σ_1 and σ_2.

Theorem 2.5 (Geometric Realisation). Let $\mathcal{X}$ be an abstract simplicial complex with finite dimension d. Then it is always possible to find an embedding $f : \mathcal{X} \to \mathbb{R}^{2d+1}$ in such a way that for any maximal element $M \in \mathcal{X}$, then $f(M) \subseteq \mathbb{R}^{2d+1}$ is an affinely independent set .

Corollary 2.6. Any abstract simplicial complex $\mathcal{X}$ may be geometrically realised in a natural way as an affine simplicial complex $\mathcal{K}$, where an affine face of $\mathcal{K}$ corresponds with the convex hull of the image under f of an abstract face in $\mathcal{X}$.

Corollary 2.7. An abstract simplicial complex of dimension 1 (i.e. a graph) can always be embedded in $\mathbb{R}^3$.

Definition 2.8. A *(compact, convex) polytope* is the convex hull of a finite set of vertices $v_o, v_1, ..., v_n \in \mathbb{R}^k$. This definition is identical to that of an (affine) simplex, except that we relax the requirement of affinely independent vertices.

Definition 2.9. A *simple polyhedral complex* $\mathcal{K}$ is a set of polytopes in $\mathbb{R}^k$ that is closed under the operation of taking faces, and which has the property that for any pair of polytopes $\sigma_1, \sigma_2 \in \mathcal{K}$, then $\sigma_1 \cap \sigma_2$ is either empty or is a face of both σ_1 and σ_2.

Henceforth, we refer to any topological space homeomorphic to the open n-ball $\mathbb{B}^n$ as an n-cell e^n, and we denote the boundary of $\mathbb{B}^n$ by $\mathbb{S}^{n-1}$.

Definition 2.10. Let A be a (Hausdorff) topological space and let $f : \mathbb{S}^{n-1} \to A$ be a continuous map. We may create a new topological space X by *attaching* (also known as *gluing*) an n-cell to A along f as follows:

$$X := A \cup_f e^n = (A \amalg \mathbb{B}^n)/s \sim f(s), \forall s \in \mathbb{S}^{n-1}$$

Note that we may extend f to a continuous map $f : \overline{\mathbb{B}^n} \to X$ which restricts to a homeomorphism $f|_{\mathbb{B}^n} : \mathbb{B}^n \to e^n$ such that $f(\mathbb{S}^{n-1}) \subseteq A$.

Definition 2.11. A finite dimensional CW or *cell complex* is defined inductively as follows.

We begin with a set of discrete points (which we call the 0-*skeleton* $X^{(0)}$). Then for each successive integer $k = 1, 2, ..., n$ we obtain the k-*skeleton* $X^{(k)}$ by attaching a set of k-cells to the $(k-1)$-skeleton $X^{(k-1)}$. The greatest element $\mathcal{X}$ in the chain of topological spaces $X^{(0)} \subseteq ... \subseteq X^{(n)} := \mathcal{X}$ is an n-dimensional CW complex.

Remark 2.12. We may even define an infinite dimensional CW-complex inductively as the direct limit of its skeleta. However, we require two axioms that are automatically satisfied only in the finite dimensional case: *Closure Finiteness* and *Weak Topology* (hence the nomenclature "CW").

Now let M^n_κ be the complete, simply connected Riemannian n-manifold of constant sectional curvature $\kappa \in \mathbb{R}$ (refer to Chapter 6 and in particular Definition 6.7). For example M^n_0 (resp. M^n_1, M^n_{-1}) is Euclidean space $\mathbb{R}^n$ (resp. the sphere $\mathbb{S}^n$, hyperbolic space $\mathbb{H}^n$) under the standard metric.

Let $m \leq n$ be integers and let $\kappa \in \mathbb{R}$. A *m-plane* is a subspace of M^n_κ isometric to M^m_κ. A *compact convex polyhedron* P is the convex hull of a finite set of points in M^n_κ. If $\kappa > 0$ we require all such points to lie within an open hemisphere. The *dimension* of P is the dimension of the smallest m-plane containing P. The *interior* of P is with respect to this m-plane.

Definition 2.13. Let $\kappa \in \mathbb{R}$. A *M_κ-polyhedral complex* $\mathcal{X}$ is a CW-complex such that for any k-cell $\sigma \in \mathcal{X}$, with attaching map $f : \mathbb{S}^{k-1} \to \mathcal{X}$,

(i) $\mathrm{Int}(\sigma)$ is isometric to the interior of a compact convex polyhedron in M^k_κ; and

(ii) the restriction of f to each open, codimension-1 face of σ is an isometry onto an open cell of $\mathcal{X}$.

A M_0 (resp. M_1, M_{-1})-polyhedral complex is called *piecewise Euclidean* (resp. *spherical*, *hyperbolic*). A *polyhedral complex* is a M_κ-polyhedral complex, for some κ.

Example 2.14. Identifying opposite sides of a rectangle in parallel gives the fundamental polygon $\mathcal{P}$ of a topological 2-torus, see Figure 1. The 2-torus is not a simple polyhedral complex as it involves "self-gluings" of a polygon, but it is a polyhedral complex, with cell structure

$$\mathbb{T}^2 = (\mathbb{S}^1 \vee \mathbb{S}^1) \cup_f e^2 = (e^0 \cup_{g,g'} 2e^1) \cup_f e^2$$

where $g' = g : \mathbb{S}^0 \to e^0$ is the obvious map to the 1-point space and $f : \mathbb{S}^1 \cong \mathcal{P} \to \mathbb{S}^1 \vee \mathbb{S}^1$ is defined by the loop $\alpha\beta\alpha^{-1}\beta^{-1}$ on the fundamental polygon.

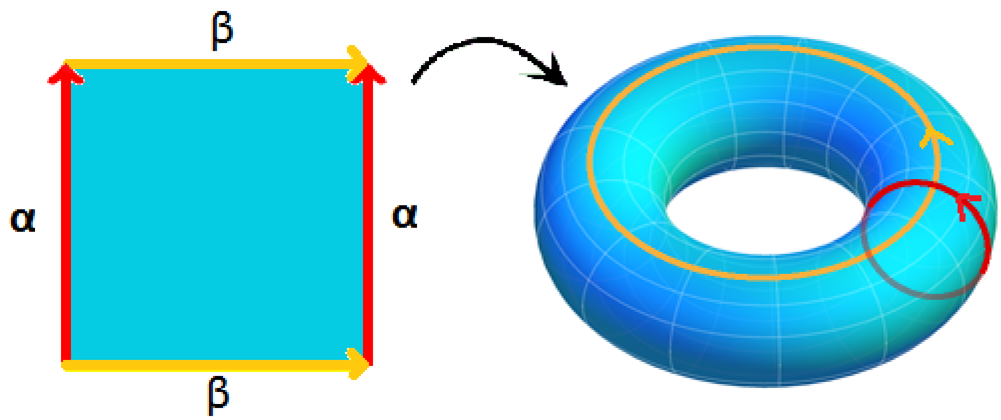

FIGURE 1. The topological 2-torus represented by its fundamental polygon $\mathcal{P}$ and by its Cartesian form respectively.

CHAPTER 3

Group Actions and Topological Groups

Definition 3.1. Let G be a group and X be a set. We say that G *acts on* X if we assign a permutation of X to every $g \in G$ via a group homomorphism $\rho : G \to \mathrm{Sym}[X]$.

Equivalently, we may explicitly represent the *group action* $\bullet : G \times X \to X$ by the map $(g, x) \mapsto g \bullet x$ such that for all $x \in X$ and $g_1, g_2 \in G$, then $1 \bullet x = x$ and $(g_1 g_2) \bullet x = g_1 \bullet (g_2 \bullet x)$.

Definition 3.2. A group G acts *freely* on a set X if $g \bullet x = x$ implies that $g = 1$ for all $x \in X$. That is, only the trivial group element fixes a point in X.

Definition 3.3. A group G acts *faithfully* on a set X if $g \neq h \in G$ implies that there exists an $x \in X$ such that $g \bullet x \neq h \bullet x$. In other words, distinct group elements induce distinct permutations on X.

Example 3.4. Any group G acts freely and faithfully on itself.

Proof. Let $g_1, g_2, h \in G$. If $g_1 \bullet h := g_1 h = h$, then $g_1 = 1$ (right multiply by h^{-1}). Furthermore, if $g_1 \neq g_2$, then $g_1 \bullet h := g_1 h \neq g_2 h =: g_2 \bullet h$. □

Definition 3.5. Let $\bullet$ be the action of a group G on a set X. The *stabiliser* of $x \in X$ is given by $\mathrm{stab}_\bullet(x) := G_x := \{g \in G | g \bullet x = x\}$. It is simple to check that the stabiliser is a subgroup of G.

Definition 3.6. Let $\bullet$ be the action of a group G on a set X. The G-*orbit* of $x \in X$ is given by $G \bullet x := \{g \bullet x | g \in G\}$. We call the set of all G-orbits $X/G := \{G \bullet x | x \in X\}$ the *quotient set* of the action $\bullet$ of G on X. Since every $x \in X$ is contained in exactly one G-orbit, X/G is well-defined.

Definition 3.7. A group G is a *topological group* if it is also a Hausdorff topological space such that the group operations $\cdot : G \times G \to G$ given by $(g, h) \mapsto gh$ and ${}^{-1} : G \to G$ given by $g \mapsto g^{-1}$ are continuous maps for all $g, h \in G$. Intuitively, G has compatible topological and algebraic structures.

Remark 3.8. Any group may be considered a topological group when endowed with the discrete topology.

We now extend Definition 3.6 by including a topology.

Definition 3.9. Let the action $\bullet$ of a topological group G on a topological space X be continuous with respect to the product topology on $G \times X$. If we endow the quotient set X/G with the quotient topology (where two elements in X are equivalent iff they are in the same G-orbit), then we call X/G the *quotient space* of the action $\bullet$ of G on X.

Definition 3.10. Now let (X, d_X) and (Y, d_Y) be metric spaces. A bijective map $f : X \to Y$ is called an *isometry* if it preserves distances, that is if:

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)), \ \ \forall x_1, x_2 \in X$$

Remark 3.11. The set of isometries from a metric space X to itself forms a group structure under composition, which we denote by $\mathrm{Isom}(X)$.

Definition 3.12. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is called a (λ, C)-*quasi-isometry*, for positive constants λ and C, if:

(i) $\forall y \in Y, \exists x \in X$ such that $d_Y(f(x), y) \leq C$ (weak surjectivity), and

(ii) $\lambda^{-1} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq \lambda d_X(x_1, x_2) + C$, $\forall x_1, x_2 \in X$.

The metric spaces (X, d_X) and (Y, d_Y) are called *quasi-isometric* if there exists a (λ, C)-quasi-isometry $f : X \to Y$ for some $\lambda, C > 0$.

Example 3.13. The floor function $f : \mathbb{R}^n \to \mathbb{Z}^n$ given by $(x_1, ..., x_n) \mapsto (\lfloor x_1 \rfloor, ..., \lfloor x_n \rfloor)$, where $\lfloor x_i \rfloor := \max\{z \in \mathbb{Z} \mid z \leq x_i\}$, is a quasi-isometry. We may see this by taking $\lambda = 1$ and $C = \sqrt{n}$ (the length of the longest diagonal of the unit n-hypercube). Intuitively, the space $\mathbb{Z}^n$ "looks like" $\mathbb{R}^n$ when viewed from far away.

Definition 3.14. A *geometry* is a metric space (X, d_X) that is:

(i) *proper* – every closed ball is compact, and

(ii) *geodesic* – any two points in X are connected by a geodesic segment.

Definition 3.15. Let (X, d_X) be a geometry, and let G be a group. The action of G on (X, d_X) is called *geometric* if it is:

(i) *by isometries* – the group action may be interpreted as a homomorphism from $G \to \mathrm{Isom}(X)$,

(ii) *properly discontinuous* – for all compact subsets $K \subseteq X$, the set $\{g \in G \mid gK \cap K \neq \varnothing\}$ is finite, and

(iii) *cocompact* – the quotient space X/G is compact, or equivalently the G-orbit of any $x_0 \in X$ covers X up to some finite distance R.

Remark 3.16. A geometric group action on a geometry is a generalisation of the tesselation-inducing action of a geometric reflection group on $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{H}^n$.

Lemma 3.17 (Milnor-Svarc). Let (X, d_X) be a geometry and let G be a group. If G acts on (X, d_X) geometrically, then G is finitely generated and is quasi-isometric to (X, d_X).

Definition 3.18. A topological space X is *locally compact* if for any $x \in X$, there exists a compact set $K \subseteq X$ and an open set $U \subseteq K$ such that $x \in U$. A topological group that is locally compact as a topological space is known as a *locally compact group*.

Definition 3.19. Let X be a topological space. The smallest σ-algebra $\mathcal{B}$ on X that contains all open sets in X is known as the *Borel algebra*. In other words, $\mathcal{B}$ is generated as σ-algebra by the topology on X. A measure defined on the Borel σ-algebra is called a *Borel measure*.

Definition 3.20. A Borel measure μ on a topological space X is *regular* if for any Borel set $A \in \mathcal{B}$, the measure on A is equal to that of the largest closed set contained in A and the smallest open set containing A. Precisely, $\mu(A) = \sup\{\mu(F) | F \subseteq A, F \text{ is closed}\} = \inf\{\mu(G) | A \subseteq G, G \text{ is open}\}$.

Definition 3.21. Let G be a locally compact group with Borel algebra $\mathcal{B}$. A left-invariant *Haar measure* on G is a regular Borel measure μ such that $\mu(gA) = \mu(A)$ for all $g \in G$ and $A \in \mathcal{B}$. One may define a right-invariant Haar measure analagously.

Theorem 3.22 (Haar-Weil). Let G be a locally compact group. Then there exists a left-invariant (resp. right-invariant) Haar measure μ on G that is unique up to positive scalar multiplication.

Proof. Refer to Theorem 9.2.1. of [**7**]. □

Definition 3.23. Let Γ be a discrete subgroup of a locally compact group G (a subgroup of G which is discrete under the subspace topology). We say that Γ is a *lattice* in G if the quotient space G/Γ admits a finite G-invariant measure, and that it is *uniform* if G/Γ is compact.

Note that a Haar measure μ on G restricts in a natural way to a G-invariant measure on G/Γ, and as such – in a slight abuse of notation – we also refer to this measure by μ. We call $\mu(G/\Gamma)$ the *covolume* of $\Gamma \leq G$.

Proposition 3.24. Let G be a locally compact group acting on a set V with compact open stabilisers and a finite quotient set V/G. Then if a subgroup Γ of G is discrete, each stabiliser $|\Gamma_{v'}|$ is finite.

Proof. Refer to Section 2.1. of [**22**], Section 2.2. of [**21**] and Section 1.9. of [**26**]. □

Theorem 3.25 (**Serre**, [**19**]). Let G be a locally compact group acting on a set V with compact open stabilisers and a finite quotient set V/G. Then there exists a normalisation of the Haar measure μ on G, depending only on the choice of G-set V, such that for each discrete subgroup Γ of G, we have

$$\mu(G/\Gamma) = \sum_{v \in V/\Gamma} \frac{1}{|\Gamma_{v'}|}$$

where for each $v \in V/\Gamma$ we choose a single representative $v' \in V$ of the orbit v. Note that $|\Gamma_{v'}|$ must be independent of the choice of v'.

We now specialise to the case where X is a connected, locally finite polyhedral complex and $G = \mathrm{Aut}(X)$ is the group of polyhedral isometries of X (which preserve the polyhedral structure of X, generalising the notion of a graph automorphism). Let G act on X in the obvious way, and assume that the quotient set X/G is compact.

In Corollary 3.27 we find a criterion to test whether a subgroup of G is a uniform lattice.

Proposition 3.26. Under the compact-open topology, the group G is naturally a Hausdorff, first-countable locally compact group with compact open G-stabilisers of cells in X (in particular, any vertex in X is a 0-cell).

Proof. Refer to Section 2.1. of [**22**], Section 2.2. of [**21**] and Section 1.9. of [**26**]. □

Since all the relevant conditions have now been satisfied by Proposition 3.26, we may equip G with the normalised Haar measure μ as in Theorem 3.25. We apply Theorem 3.25 to the group G acting on the vertex set of X.

Corollary 3.27. Let Γ be a subgroup of G and let $x \in X$. Then Γ is a uniform lattice in G iff the quotient set X/Γ is finite and each vertex stabiliser Γ_x is finite. In addition, the covolume of $\Gamma \leq G$ is given by

$$\mu(G/\Gamma) = \sum_{v \in V(X)/\Gamma} \frac{1}{|\Gamma_{v'}|}$$

where $|\Gamma_{v'}|$ is independent of the choice v' of orbit representative used.

Proof. Refer to Section 2.1. of [**22**], Section 2.2. of [**21**] and Section 1.9. of [**26**]. □

Remark 3.28. We apply these results on polyhedral complexes later in this paper to construct and analyse uniform lattices in Davis complexes.

CHAPTER 4

Coxeter Systems and the Davis Complex

Definition 4.1. Let $M = \{m_{ij}\}_{i,j=1,...,n \in \mathbb{N}}$ be a symmetric $n \times n$ matrix with entries that are identically equal to 1 if they are on the major diagonal and otherwise that are either a positive integer strictly greater than 1 or are equal to ∞.

A *Coxeter group* is a group with a presentation of form

$$W(M) = \langle s_1, s_2, ..., s_n \mid (s_i s_j)^{m_{ij}} = 1, \forall i, j \in \{1, ..., n\} \rangle$$

where we take $g^\infty = 1$ to mean that the group element g has infinite order.

The pair $(W, S) := (W(M), \{s_1, s_2, ..., s_n\})$ is called a *Coxeter system* of *type* M and *rank* n.

Definition 4.2. Let $(W, \{s_1, s_2, ..., s_n\})$ be a Coxeter system of type $M = \{m_{ij}\}_{i,j}$. A generator s_k is *free* if $m_{ik} = \infty$ for all $i \neq k$. A Coxeter system is *free* if all its generators are free.

Definition 4.3. Let (W, S) be a Coxeter system and let $T \subseteq S$. The *special* (also known as *visual* or *standard parabolic*) *subgroup* W_T is the subgroup of W generated by T. Both T and W_T are known as *spherical* if W_T is finite.

Proposition 4.4. Let (W, S) be a Coxeter system and let $T \subseteq S$ be a spherical subset. Then the pair (W_T, T) is itself a Coxeter system.

Proof. Refer to Theorem 4.1.6. of [**8**]. □

Theorem 4.5. Let (W, S) be a Coxeter system of type $M = \{m_{ij}\}$. Then W is finite iff the cosine matrix $C := \{-\cos(\pi/m_{ij})\}$ is positive definite, where we take π/∞ to be 0 in the case that $m_{ij} = \infty$.

Proof. Refer to Theorem 6.12.9. of [**8**]. □

Example 4.6. Let (W, S) be a Coxeter system of type M and rank 3. Without loss of generality, we may write

$$W := \langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^\alpha = (s_1 s_3)^\beta = (s_2 s_3)^\gamma = 1 \rangle$$

for some $\alpha, \beta, \gamma \in (\mathbb{N} \cup \{\infty\})\backslash\{1\}$ such that $\alpha \leq \beta \leq \gamma$. Then W is finite iff (α, β, γ) equals either $(2,3,3)$, $(2,3,4)$, $(2,3,5)$ or $(2,2,k)$ for some $k \in \mathbb{N}\backslash\{1\}$.

Proof. By Theorem 4.5, we know that W is finite iff the cosine matrix

$$C := \begin{pmatrix} 1 & -\cos(\pi/\alpha) & -\cos(\pi/\beta) \\ -\cos(\pi/\alpha) & 1 & -\cos(\pi/\gamma) \\ -\cos(\pi/\beta) & -\cos(\pi/\gamma) & 1 \end{pmatrix}$$

is positive definite.

It is well known that the condition of a matrix being positive definite is equivalent to each of its leading principal minors being positive. Therefore the problem reduces to ensuring that the following conditions hold:

(i) $\det\begin{pmatrix} 1 \end{pmatrix} = 1 > 0$,

(ii) $\det\begin{pmatrix} 1 & -\cos(\pi/\alpha) \\ -\cos(\pi/\alpha) & 1 \end{pmatrix} = 1 - \cos^2(\pi/\alpha) > 0$, and

(iii) $\det\begin{pmatrix} 1 & -\cos(\pi/\alpha) & -\cos(\pi/\beta) \\ -\cos(\pi/\alpha) & 1 & -\cos(\pi/\gamma) \\ -\cos(\pi/\beta) & -\cos(\pi/\gamma) & 1 \end{pmatrix} > 0$.

Condition (i) is trivial, condition (ii) holds as long as $\alpha \neq \infty$, and condition (iii) reduces to:

$$\cos^2(\pi/\alpha) + \cos^2(\pi/\beta) + \cos^2(\pi/\gamma) + 2\cos(\pi/\alpha)\cos(\pi/\beta)\cos(\pi/\gamma) < 1$$

Observe that condition (iii) holds if $(\alpha, \beta, \gamma) = (2,3,5)$, but that it fails if $(\alpha, \beta, \gamma) = (2,3,6), (2,4,4)$ or $(3,3,3)$. Now define a partial ordering on a triple of real numbers by $(a,b,c) \preceq (a',b',c')$ iff $a \leq a'$, $b \leq b'$ and $c \leq c'$.

Since $\alpha \leq \beta \leq \gamma$ are either integers weakly greater than 2 or are ∞, and since the function $\cos(\pi/x)$ is monotonically increasing and non-negative in the interval $x \in [2, \infty]$, condition (iii) must hold if $(\alpha, \beta, \gamma) \preceq (2,3,5)$ and it must fail if $(\alpha, \beta, \gamma) \succeq (2,3,6)$, $(2,4,4)$ or $(3,3,3)$.

Now assume $\alpha = \beta = 2$. Then condition (iii) reduces to $\cos^2(\pi/\gamma) < 1 \Leftrightarrow \gamma \neq \infty, \pm 1/2, \pm 1/3...$, and thus it holds without any further restrictions on γ other than $\gamma \neq \infty$.

To summarise, given that $\alpha \leq \beta \leq \gamma \in (\mathbb{N} \cup \{\infty\})\backslash\{1\}$, then the only possible values for α, β and γ such that W is finite are respectively $(2,3,3)$, $(2,3,4)$, $(2,3,5)$, or $(2,2,k)$ for some $k \in \mathbb{N}\backslash\{1\}$. $\square$

Definition 4.7. Let (W, S) be a Coxeter system. The *nerve* L of (W, S) is the poset of nontrivial, spherical special subgroups of W under set inclusion. The 1-skeleton of L may be interpreted as a simple, undirected graph by identifying each s_i (or equivalently the subgroup $\langle s_i \rangle$ of W generated by s_i) with a vertex and by joining the vertices s_i and s_j with an edge (which we label m_{ij}) iff $m_{ij} < \infty$.

Definition 4.8. Let (W, S) be a Coxeter system with nerve L. The *chamber* K of (W, S) is the topological cone on the barycentric subdivision of L, where we identify the point of the cone with the empty set $\varnothing$. Equivalently, K may be interpreted as the geometric realisation of the abstract simplicial complex $L \cup \varnothing$.

Remark 4.9. The chamber K is contractible (as it is a cone) and compact (as the barycentric subdivision of L is a finite simplicial complex).

Definition 4.10. Let (W, S) be a Coxeter system with chamber K, and let $s \in S$. We define the *s-mirror* of K by

$$K_s := \left\{ \bigcup_{\sigma \in K} \sigma \,\middle|\, \sigma \text{ is a closed simplex in } K, s \in \sigma, \varnothing \notin \sigma \right\}$$

Definition 4.11. Let (W, S) be a Coxeter system. Then the *Davis complex* of (W, S) is the quotient space $\Sigma := W \times K / \sim$, where $(w, k) \sim (w', k')$ iff $k = k'$ and $w^{-1}w' \in W_{\{s \in S | k \in K_s\}}$.

For $w \in W$, the translates (w, K) are known as *chambers* of Σ and the subsets (w, K_s) of the chambers are known as *s-panels* of Σ. Two chambers are *s-adjacent* if their intersection is an s-panel.

Each vertex (w, k) of Σ has a *type* induced by the corresponding element of the poset $L \cup \varnothing$.

Example 4.12. Let (W, S) be a Coxeter system where $W = \langle s_1, s_2, ..., s_n \mid s_1^2 = ... = s_n^2 = 1 \rangle$ for some $n \in N$. We construct the Davis complex Σ_n of (W, S) as follows.

Since $m_{ij} = \infty$ for any $i \neq j$, the nerve L simply consists of n discrete points and the chamber K is isomorphic to the star with n leaves. Since each mirror of K consists of a single vertex, the Davis complex is constructed by gluing each star to precisely n other stars by identifying each leaf with a single other leaf of the same type from another star. That is, the Davis complex Σ_n of (W, S) is isomorphic to the unique $(n, 2)$-biregular tree.

Theorem 4.13 (Moussong). Let (W, S) be a Coxeter system with Davis complex Σ. Then Σ may be endowed with a piecewise Euclidean metric such that Σ is a complete $\mathrm{CAT}(0)$ space.

Proof. Refer to [**14**], Theorem 12.3.3. of [**8**], and to [**22**]. □

Henceforth we assume that the Davis complex Σ is equipped with this piecewise Euclidean metric.

Definition 4.14. Let (W, S) be a Coxeter system with associated Davis complex Σ. There is a *natural (left) group action* $\bullet : W \times \Sigma \to \Sigma$ of a Coxeter group on its Davis complex given by $(g, (w, k)) \mapsto (gw, k)$.

It is clear that this natural action $\bullet$ freely permutes the set of chambers of Σ (by Example 3.4), and that it preserves the type of each vertex in Σ. As such, we may identify the W-orbit classes of Σ with the "type" of their component vertices.

Proposition 4.15. Let (W, S) be a Coxeter system with associated Davis complex Σ. Then the Davis complex Σ is a geometry and furthermore W acts on Σ geometrically.

Proof. See Chapter 7 of [**8**]. □

Corollary 4.16. Any Coxeter group W is quasi-isometric to its Davis complex Σ.

Proof. Follows from the Milnor-Svarc Lemma 3.17 and Proposition 4.15. □

CHAPTER 5

Bass and Serre's theory of graphs of groups

In this chapter we provide a summary of the theory of graphs of groups, otherwise known as Bass-Serre Theory. We go into a fair amount of detail in this chapter, since many of the concepts introduced here for graphs of groups may be generalised to complexes of groups, and so will be important later on in this paper. The material is mainly drawn from [**15**], but we also make reference to [**26**] and to [**1**].

In particular, we describe how one may uniquely associate a "fundamental group" and a "universal covering space" to a given graph of groups, such that the fundamental group acts on the covering space in a sufficiently "nice" manner. In addition to the obvious analogy with algebraic topology, this notion has a very close analogue in the theory of complexes of groups.

Definition 5.1. Let $A = (V, E)$ be a connected, directed graph with end-point maps $o : E \mapsto V$ and $t : E \mapsto V$ that respectively take an edge to its initial or terminal vertex. We interpret an edge of a directed graph as an ordered pair of vertices. If $e := (v_1, v_2)$ is an edge of A, then its *reverse* $\overline{e} := (v_2, v_1)$ is required to also be an edge of A such that $\overline{\overline{e}} = e$, $\overline{e} \neq e$, $o(e) = t(\overline{e})$ and $t(e) = o(\overline{e})$. A *graph of groups* over A consists of both the graph A and a map G which associates:

(i) a group G_v to every vertex $v \in V$,

(ii) a group G_e to every edge $e \in E$, so that $G_e = G_{\overline{e}}$, and

(iii) a monomorphism $\alpha_e : G_e \mapsto G_{t(e)}$ to every edge $e \in E$.

Definition 5.2. Let (G, A) be a graph of groups over the graph $A = (V, E)$, where each vertex group has a presentation $G_v = \langle S_v \mid R_v \rangle$. The *path group* $P(G, A)$ is given by the presentation:

$$\left\langle \bigcup_{v \in V} S_v \bigcup_{e \in E} \{e\} \middle| R_v, e\overline{e} = 1, \alpha_e(g) = e\alpha_{\overline{e}}(g)e^{-1}, \forall v \in V, e \in E, g \in G_e \right\rangle$$

In other words, the path group is generated by the edges in E and (the generators of) each vertex group G_v. Its set of relations dictates that any pair of edges e and $\overline{e}$ are inverses, that the pair of monomorphic images under α of any edge group element mapped to its corresponding adjacent vertex groups are e-conjugates of one another, and finally it includes all of the relations of each vertex group G_v.

Definition 5.3. A *path* in a graph of groups (G, A) is a sequence

$$p := (g_1, e_1, g_2, e_2, ..., e_n, g_{n+1})$$

where $e_i \in E(A)$, $t(e_i) = o(e_{i+1}) := v_{i+1} \in V(A)$ and $g_i \in G_{v_i}$ for all i.

That is, a path in a graph of groups is the same as a path in the underlying graph - an alternating sequence of adjacent vertices and edges - except that we replace each vertex with an element of the corresponding vertex group.

We say that p is a path of *length* n from v_1 to v_{n+1}, and we interpret the corresponding *word* $|p| := g_1 e_1 g_2 e_2 ... e_n g_{n+1}$ as an element of the path group.

Definition 5.4. Let (G, A) be a graph of groups, with $v_1, v_{m+1}, v_{n+1} \in V(A)$ for some integers $0 \leq m \leq n$. Let $p := (g_1, e_1, g_2, ..., e_m, g_{m+1})$ be a path from v_1 to v_{m+1}. We define *path inversion* by

$$p^{-1} := (g_{m+1}^{-1}, \overline{e_m}, ..., g_2^{-1}, \overline{e_1}, g_1^{-1})$$

so that p^{-1} is a path of length m from v_{m+1} to v_1.

Now let $q := (g'_{m+1}, e_{m+1}, g_{m+2}, ..., e_n, g_{n+1})$ be a path from v_{m+1} to v_{n+1}. We define *concatenation of paths* by

$$p \cdot q := (g_1, e_1, g_2, ..., e_m, g_{m+1}g'_{m+1}, e_{m+1}, g_{m+2}, ..., e_n, g_{n+1})$$

where $g_{m+1}g'_{m+1}$ refers to group multiplication in the vertex group $G_{v_{m+1}}$. Clearly $p \cdot q$ is a path of length n from v_1 to v_{n+1}.

Henceforth, for any pair of vertices a and b in a graph of groups, we define the set $\pi[a, b] := \{|p| \big| p \text{ is a path from } a \text{ to } b\}$.

Definition 5.5. Let (G, A) be a graph of groups. The *fundamental group* $\pi_1(G, A, v_1)$ with *basepoint* $v_1 \in V(A)$ is given by the set

$$\pi[v_1, v_1] = \{|p| \big| p \text{ is a path from } v_1 \text{ to } v_1\}$$

under concatenation and inversion of paths. Or, more precisely, the group multiplication and inversion are given by $|p||q| = |p \cdot q|$ and $|p|^{-1} = |p^{-1}|$ respectively for $|p|, |q| \in \pi_1(G, A, v_1)$.

It is easy to check that $\pi_1(G, A, v_1)$ is indeed a subgroup of the path group, with identity element given by the identity element of the group G_{v_1}.

Proposition 5.6. Let (G, A) be a graph of groups and let Φ denote the edge set of a maximal tree in A. Then we have a group isomorphism $\pi_1(G, A, v_1) \cong P(G, A)/\langle\langle\Phi\rangle\rangle$ for any choice of basepoint $v_1 \in V(A)$.

Proof. Refer to Proposition 5.2 of [**15**]. □

Since the fundamental group is independent of the choice of basepoint up to isomorphism, we denote it by $\pi_1(G, A)$.

Definition 5.7. A path $p := (g_1, e_1, g_2, ..., e_n, g_{n+1})$ in a graph of groups (G, A) is known as *reduced* if:

(i) $g_1 \neq 1$ if $n = 0$, and

(ii) $g_i \notin \alpha_{e_{i-1}}(G_{e_{i-1}})$ if $e_i = \overline{e_{i-1}}$, for any i.

In other words, if a reduced path has length 0, then it cannot consist solely of the identity element of G_{v_1} and furthermore, if a reduced path turns back on itself at some vertex v_i, then the group element chosen to represent G_{v_i} cannot be in the image under $\alpha_{e_{i-1}}$ of $G_{e_{i-1}}$ (the group associated to the edge just traversed).

The corresponding word $|p| := g_1 e_1 g_2 ... e_n g_{n+1}$ is known as *path reduced* (note that this is different to the usual notion of a *reduced word*).

Remark 5.8. Let (G, A) be a graph of groups. It is not always possible to write an element of the path group $P(G, A)$ in the form $|p|$, for some path p in (G, A). Even when it is possible, a given $|p| \in P(G, A)$ does not necessarily uniquely determine a path p in the graph of groups (G, A). For example, the reduced path $p_1 := (g_1, e_1, g_2)$ and the non-reduced path $p_2 := (g_1, e_1, 1, \overline{e_1}, 1, e_1, g_2)$ both correspond with the reduced word $|p_1| = |p_2| = g_1 e_1 g_2 \in P(G, A)$.

Lemma 5.9 (Britton). Let p be a reduced path in a graph of groups (G, A). Then $|p| \neq 1$ in the path group.

Proof. See Theorem 5.1 of [**15**]. □

To circumvent the issue raised in Remark 5.8, we now introduce the notion of an S-reduced path – a generalisation of a reduced path – the purpose of which is to ensure that each element of $\pi[v_1, v_2]$ can be expressed uniquely as a path with certain properties.

Definition 5.10. Let e be an edge in a graph of groups (G, A). Given that $\alpha_{\overline{e}}(G_e)$ is a subgroup of $G_{t(\overline{e})} = G_{o(e)}$ (as α_e is a monomorphism), we arbitrarily choose (via the axiom of choice) a representative for each coset in $\{g\alpha_{\overline{e}}(G_e) \subseteq G_{o(e)} \mid g \in G_{o(e)}\}$ such that the trivial coset representative is the identity. We denote this choice of coset representatives of $\alpha_{\overline{e}}(G_e)$ in $G_{o(e)}$ by S_e, and we call $S := \{S_e\}_{e \in E(A)}$ a *set of (coset) representatives* for (G, A). Note that there are many possible valid choices of S_e and thus S.

For a given choice of S, a path $p := (g_1, e_1, g_2, ..., e_n, g_{n+1})$ in (G, A) is called S-*reduced* if:

(i) $g_i \in S_{e_i}$ for all $1 \leq i \leq n$, and

(ii) $g_i \neq 1$ if $e_i = \overline{e_{i-1}}$, for any i.

Henceforth let S be a given choice of set of representatives for a graph of groups (G, A).

Proposition 5.11. Let v_1 and v_2 be vertices in A. Then each element of $\pi[v_1, v_2]$ may be uniquely represented by an S-reduced path.

Proof. See Lemma 5.1 of [**15**]. □

Definition 5.12. Let v_1 and v_2 be vertices in A. Two elements x and y of $\pi[v_1, v_2]$ are *equivalent*, denoted by $x \sim y$, if $x = yg$ for some $g \in G_{v_2}$ (i.e. if they are equal up to right multiplication by an element of G_{v_2}).

Proposition 5.13. Let v_1 and v be vertices in A. Then every element of $\pi[v_1, v]/\sim$ may be uniquely associated with (in a slight abuse of notation) an S-reduced "path" of form $(s_1, e_1, s_2, ..., s_n, e_n)$, where $o(e_1) = v_1$, $t(e_n) = v$ and $s_i \in G_{v_i}$ for all i.

Proof. Let $x := g_1 e_1 g_2, ...g_n e_n g$ be an arbitrary element of $\pi[v_1, v]$, and denote by $[x]$ the corresponding equivalence class under $\sim$ (i.e. the coset). We must first show that

$$[x] = g_1 e_1 g_2 ...g_n e_n G_v := \{g_1 e_1 g_2 ...g_n e_n h \mid h \in G_v\}$$

$(\subseteq)$ Let $y \in [x]$. Then $y = g_1 e_1 g_2 ...g_n e_n g g'$ for some $g' \in G_v$ (as it can be obtained from x by right multiplication by an element of G_v). Since $gg' \in G_v$, therefore $y \in g_1 e_1 g_2 ...g_n e_n G_v$.

$(\supseteq)$ Let $g_1 e_1 g_2 ...g_n e_n g'' \in g_1 e_1 g_2, ...g_n e_n G_v$. We can right-multiply x by $g^{-1} g'' \in G_v$ to get $g_1 e_1 g_2 ...g_n e_n g''$. Therefore $g_1 e_1 g_2 ...g_n e_n g'' \in [x]$.

Now by Proposition 5.11, we may uniquely associate $x \in \pi[v_1, v]$ with an S-reduced path of form $(s_1, e_1, s_2, ..., s_n, e_n, g''')$, where $o(e_1) = v_1$, $t(e_n) = v$, $g''' \in G_v$ and $s_i \in G_{v_i}$ for all i. Therefore we may uniquely associate $g_1 e_1 g_2 ...g_n e_n G_v = [x] \in \pi[v_1, v]/\sim$ with the S-reduced "path" $(s_1, e_1, s_2, ..., s_n, e_n)$. □

Remark 5.14. Multiplication of "paths" of form $(g_1, e_1, g_2, ..., g_n, e_n)$ is given by concatenation as in Definition 5.4, where the "missing" final group element is implicitly taken to be the identity.

Definition 5.15. The *universal covering space* of a graph of groups (G, A) with basepoint $v_1 \in V(A)$, denoted by $(\widetilde{G, A, v_1})$, is given by the directed graph (V', E') with:

- vertex set given by
$$V' = \bigcup_{v \in V(A)} \pi[v_1, v]/\sim$$
where $x \sim y$ iff $x = yg$ for some $g \in G_v$, such that
- there exists a (directed) edge $(u, w) \in E'$ iff $u^{-1}w$ is a path of length 1, or equivalently (by Proposition 5.13), if their corresponding S-reduced paths are respectively of form $(s_1, e_1, ..., s_n, e_n)$ and $(s_1, e_1, ..., s_n, e_n, s_{n+1}, e_{n+1})$ where the edge e_{n+1} points away from $t(e_n)$ in A. That is, if the paths u from v_1 to $a \in V(A)$ and w from v_1 to $b \in V(A)$ differ in A by only a single (directed) edge (a, b).

Proposition 5.16. The universal cover of a graph of groups (G, A) with basepoint $v_1 \in V(A)$ is a tree.

Proof. We attempt to avoid confusion in this proof by using a different font to distinguish between a "`path`" in the graph $(\widetilde{G, A, v_1})$, as opposed to an S-reduced "path".

Let v be an arbitrary vertex of $(\widetilde{G, A, v_1})$. By Proposition 5.13, we may uniquely associate v with an S-reduced path of form $(s_1, e_1, ..., s_n, e_n)$, where $o(e_1) = v_1$. Note that the trivial path (1) is S-reduced.

Furthermore, by Definition 5.15, any `path` in $(\widetilde{G, A, v_1})$ can be expressed as a sequence of S-reduced paths successively obtained from the preceding path by concatenating them on the right with an S-reduced path of length 1.

(*Existence*) There exists a `path` in $(\widetilde{G, A, v_1})$ between the vertices (1) and $v = (s_1, e_1, ..., s_n, e_n)$, given by
$$\Big((1), (s_1, e_1), (s_1, e_1, s_2, e_2), ..., (s_1, e_1, ..., s_n, e_n)\Big)$$

(*Uniqueness*) Assume that there exists another `path` in $(\widetilde{G, A, v_1})$ between the vertices (1) and v, given by
$$\Big((1), (s'_1, e'_1), (s'_1, e'_1, s'_2, e'_2), ..., (s'_1, e'_1, ..., s'_m, e'_m)\Big)$$
Since v is uniquely associated with an S-reduced path by Proposition 5.13, therefore $n = m$ and $s_1 = s'_1, e_1 = e'_1, s_2 = s'_2, ..., s'_m = s'_n$.

So we have a fixed vertex (1) in the graph $(\widetilde{G, A, v_1})$ that is connected to every other vertex by a unique `path`. Therefore $(\widetilde{G, A, v_1})$ is a tree. □

Proposition 5.17. The universal covering tree of a graph of groups (G, A) is independent of the choice of basepoint, up to graph isomorphism.

Proof. See Section 5.3 of [**15**]. □

Proposition 5.18. The degree of a vertex v in the universal covering tree of a graph of groups (G, A) is given by

$$\sum_{e \in E(A),\ v = o(e)} [G_v : G_e]$$

where $[.]$ denotes the index of a subgroup.

Proof. Refer to Remark 1.18. of [**1**]. □

Henceforth, let (G, A) be a graph of groups with basepoint $v_1 \in V(A)$. For simplicity, we denote the fundamental group $\pi_1(G, A, v_1)$ by π_1 and the universal covering tree $(\widetilde{G, A, v_1})$ by T.

Definition 5.19. The *action* of π_1 on T is given by

$$\begin{aligned} \bullet : \pi_1 \times T &\to T \\ g \bullet [x] &:= [gx] \end{aligned}$$

where $[.]$ denotes the equivalence class under $\sim$.

Claim 5.20. The map $\bullet : \pi_1 \times T \to T$ is indeed an action, and furthermore it acts by directed graph automorphisms (that is, without edge inversions). Therefore π_1 is isomorphic to a subgroup of $\mathrm{Aut}(T)$ if $\bullet$ is faithful.

Proof. Let $[x]$ be a vertex in T and let $g_1, g_2 \in \pi_1$, We must firstly check the conditions for the map $\bullet$ to be an action, namely:

(i) $1 \bullet [x] = [1x] = [x]$, and

(ii) $(g_1 g_2) \bullet [x] = [g_1 g_2 x] = g_1 \bullet [g_2 x] = g_1 \bullet (g_2 \bullet [x])$.

Now let $([x], [y])$ be a (directed) edge in T. By Proposition 5.13 and Definition 5.15, without loss of generality, we may uniquely express $[x]$ and $[y]$ by the S-reduced paths $(s_1, e_1, ..., s_n, e_n)$ and $(s_1, e_1, ..., s_n, e_n, s_{n+1}, e_{n+1})$

respectively, where $o(e_1) = v_1$ and the edge e_{n+1} points away from $t(e_n)$ in A. Also let $g = (g_1, e_1', ..., g_n, e_n', h) \in \pi_1$. Then

$$g \bullet [x] = [gx] = (g_1, e_1', ..., g_n, e_n', hs_1, e_1, ..., s_n, e_n), \text{ and}$$
$$g \bullet [y] = [gy] = (g_1, e_1', ..., g_n, e_n', hs_1, e_1, ..., s_n, e_n, s_{n+1}, e_{n+1})$$

which again differ in A by only a single edge e_{n+1} pointing away from $t(e_n)$ since the path $(g \bullet [x])^{-1}(g \bullet [y]) = (s_{n+1}, e_{n+1})$ has length 1. Therefore there exists an edge from $g \bullet [x]$ to $g \bullet [y]$ in T.

Conversely, assume that the vertices $g' \bullet [x']$ and $g' \bullet [y']$ are adjacent in T, for some $[x'], [y'] \in V(T)$ and $g' \in \pi_1$. Since $g'^{-1} \in \pi_1$, then the vertices $g'^{-1} \bullet (g' \bullet [x']) = [x']$ and $g'^{-1} \bullet (g' \bullet [y']) = [y']$ are also adjacent in T. □

Theorem 5.21. Let $[x] = (s_1, e_1, ..., e_n)$ be a vertex in T, where $o(e_1) = v_1$ and $t(e_n) = v$. Without loss of generality, let $[y] = (s_1, e_1, ..., e_n, s_{n+1}, e_{n+1})$ be an arbitrary vertex in T adjacent to $[x]$. Then for any $x \in [x]$, π_1 acts on T with the following properties:

(i) the quotient space $T/\pi_1 = A$,

(ii) $\pi_{1_{[x]}} := \mathrm{stab}_\bullet([x]) = xG_vx^{-1}$ and

(iii) $\pi_{1_{([x],[y])}} := \mathrm{stab}_\bullet([x]) \cap \mathrm{stab}_\bullet([y]) = (xs_{n+1})(\alpha_{\overline{e_{n+1}}}(G_{e_{n+1}}))(xs_{n+1})^{-1}$

That is, the T-vertex and T-edge stabiliser subgroups of π_1 are respectively conjugates of vertex groups and edge groups in (G, A).

Proof. Refer to Section 5.3 of [**15**]. □

Note that the natural projection map $T \to A$ sends any vertex in T to its terminal vertex in A when considered as a path in (G, A).

Corollary 5.22. Assume that every vertex group of (G, A) as well as A itself is finite, and let μ be the normalisation of the Haar measure on G as in Theorem 3.25. If π_1 acts faithfully on T, then π_1 is a uniform lattice in $\mathrm{Aut}(T)$ with covolume

$$\mu(\mathrm{Aut}(T)/\pi_1) = \sum_{v \in V(A)} \frac{1}{|G_v|}$$

Proof. By construction, the quotient set $V(T)/\pi_1$ is identical to the vertex set of the quotient space T/π_1. Applying part (i) of Theorem 5.21, we deduce that $V(T)/\pi_1 = V(T/\pi_1) = V(A)$.

Fix a $v \in V(T)/\pi_1 = V(A)$, and observe that any representative $v' \in V(T)$ of the orbit v describes a path in (G, A) ending at $v \in V(A)$. Then, by part (ii) of Theorem 5.21, the vertex stabiliser subgroup $\pi_{1_{v'}}$ is a conjugate of the vertex group G_v and therefore has the same (finite) order.

In addition, we know that $T/\pi_1 = A$ is finite by assumption. Then, by Corollary 3.27, π_1 is a uniform lattice in $\mathrm{Aut}(T)$. Furthermore, we deduce that the quotient space $T/\mathrm{Aut}(T)$ is also finite as by Claim 5.20 we may consider π_1 to be a subgroup of $\mathrm{Aut}(T)$.

Observe that any vertex in T has only finitely many adjacent vertices as there are finitely many possible choices for s_{n+1} and e_{n+1} in Definition 5.15 (since A and each G_v is finite by assumption). That is, T is locally finite.

By Proposition 3.26, since our tree T is a connected, locally finite polyhedral complex, we know that $\mathrm{Aut}(T)$ is a locally compact group acting on T with compact open vertex stabilisers.

Having satisfied the conditions, we may now apply Serre's Theorem 3.25 to the group $\mathrm{Aut}(T)$ acting on the set $V(T)$. The covolume of the uniform lattice π_1 in $\mathrm{Aut}(T)$ is given by

$$\mu(\mathrm{Aut}(T)/\pi_1) = \sum_{v \in V(T)/\pi_1} \frac{1}{|{\pi_1}_{v'}|} = \sum_{v \in V(A)} \frac{1}{|G_v|}$$

where v' is chosen as above. $\square$

Theorem 5.23. Let $\{N_v \trianglelefteq G_v\}_{v \in V(A)}$ be a maximal family of normal subgroups of the vertex groups G_v satisfying the condition that for each edge $e \in E(A)$, there exists a unique subgroup $N_e = N_{\overline{e}} \leq G_e$ such that $N_{o(e)} = \alpha_e(N_e)$ and $N_{t(e)} = \alpha_{\overline{e}}(N_e)$. Then each N_v is the kernel of the action of $\pi_1(G, A, v)$ on $(\widetilde{G, A, v})$.

Proof. Refer to Proposition 1.23 of [**1**]. $\square$

Corollary 5.24. If at least one vertex or edge group in (G, A) is trivial, then the action of π_1 on T is faithful.

Proof. Let $G_{v'}$ be a trivial vertex group in (G, A), for some $v' = t(e')$, and let $\{N_v \trianglelefteq G_v\}_{v \in V(A)}$ be a maximal family of normal subgroups satisfying the condition in Theorem 5.23. Since $N_{v'}$ is trivial, then $N_{e'}$ must be trivial as $\alpha_{e'}$ is injective, and then $N_{o(e')} = \alpha_{\overline{e'}}(N_{e'})$ must also be trivial. Iterating, we deduce that N_v is trivial for all $v \in V(A)$ since A is connected. This argument is identical if instead we are given a trivial edge group.

It is easy to show that a group action is faithful iff the corresponding group homomorphism has a trivial kernel. Then by Theorem 5.23, the action of $\pi_1(G, A, v_1)$ on $(\widetilde{G, A, v_1})$ is faithful regardless of choice of basepoint. [Alternatively, one could argue that this is immediate since both $\pi_1(G, A, v_1)$ and $(\widetilde{G, A, v_1})$ are independent of choice of basepoint up to isomorphism]. $\square$

In summary, given a graph of groups (G, A), we have thus far shown how to uniquely (up to isomorphism) construct an associated group called the fundamental group $\pi_1(G, A)$ and an associated tree called the universal covering tree $\widetilde{(G, A)}$.

In addition, we showed that $\pi_1(G, A)$ has a natural action on $\widetilde{(G, A)}$ by directed graph automorphisms (i.e. without edge inversions), with quotient space $\widetilde{(G, A)}/\pi_1(G, A) = A$ and with vertex and edge stabiliser subgroups that are respectively conjugates of vertex groups and edge groups in (G, A).

In fact, in a certain sense, the converse is also true.

Given a group H acting on a tree T by directed graph automorphisms, we may uniquely construct an associated *induced (quotient)* graph of groups (G, A), where the underlying graph A is the quotient graph T/H and the vertex and edge groups are respectively T-vertex and T-edge stabilizer subgroups of H. The precise construction is as follows.

Definition 5.25. Let H be a group acting on a tree T by directed graph automorphisms. Let $A = T/H$ and consider the projection map $p : T \to A$. Choose (using the axiom of choice) an edge from every H-orbit of edges in T, and define a subtree $S \subseteq T$ as the union of these edges and their adjacent vertices. Then the restriction map $p|_S : E(S) \to E(A)$ must be a bijection. However, it is NOT true that $p|_S : V(S) \to V(A)$ is necessarily a bijection.

As such, we take a set $X \subseteq S$ such that $p(X)$ is a spanning tree of A, ensuring that the map $p|_X : V(X) \to V(A)$ is a bijection.

Let $v \in V(A)$ and $e \in E(A)$. Given that we may now uniquely define the pre-images $p^{-1}(v) \in X$ and $p^{-1}(e) \in S$, then we define the vertex group $G_v := \operatorname{stab}(p^{-1}(v))$ and the edge group $G_e := \operatorname{stab}(p^{-1}(e))$. Finally, define a monomorphism

$$\alpha_e : G_e \to G_{t(e)} \text{ by}$$
$$g \mapsto g_e g g_e^{-1}$$

where $g_e \in H$ is chosen such that $g_e \bullet t(p^{-1}(e)) \in X$. Note that if $t(p^{-1}(e))$ is already in X, then we simply take $g_e = 1$.

Example 5.26. We give an (extremely simple) example of an induced graph of groups. Let $H = \mathbb{Z}^+$ be the group of integers under addition and let T be the regular tree $\mathcal{T}_2$.

The quotient space $\mathcal{T}_2/\mathbb{Z}^+$ consists of a single vertex and loop. Then S is the union of a single edge in $\mathcal{T}_2$ with its two adjacent vertices. Explicitly, $S = \{a, a+1, (a, a+1)\}$ for some $a \in \mathbb{Z}$. The map $p|_S : E(S) \to E(\mathcal{T}_2/\mathbb{Z}^+)$ is a bijection, however $p|_S : V(S) \to V(\mathcal{T}_2/\mathbb{Z}^+)$ is NOT a bijection because S has two vertices whilst $\mathcal{T}_2/\mathbb{Z}^+$ only has one. We take $X = \{a\} \subseteq S$ to be a single point, ensuring that $p(X)$ is a spanning tree of $\mathcal{T}_2/\mathbb{Z}^+$ (again, a single point). Clearly $p|_X : V(X) \to V(\mathcal{T}_2/\mathbb{Z}^+)$ is a bijection.

The lone vertex group, edge group and monomorphism in $\mathcal{T}_2/\mathbb{Z}^+$ are respectively given by $G_v := \mathrm{stab}(a) = \{1\}$, $G_e := \mathrm{stab}((a, a+1)) = \{1\}$ and $\alpha_e = id : G_e \to G_v$.

Theorem 5.27. Let H be a group acting on a tree T by directed graph automorphisms, and let (G, A) be the induced graph of groups. Then $H = \pi_1(G, A)$ and $T = \widetilde{(G, A)}$. More precisely, for any basepoint $v_1 \in V(A)$, there exists a group isomorphism $\rho : H \to \pi_1(G, A, v_1)$ and a graph isomorphism $\phi : T \to \widetilde{(G, A, v_1)}$ such that $\phi(gv) = \rho(g)\phi(v)$ for all $g \in H$ and for all $v \in V(T)$ (preserving the group action).

Proof. Refer to Theorem 3.6 of [**1**]. □

In other words, we have a bijection between the class of graphs of groups and the class of pairs which consist of a group acting on a tree by directed graph automorphisms. In particular, we may recover a graph of groups using only its associated fundamental group and universal covering tree.

CHAPTER 6

Simple complexes of groups

In this chapter we introduce the theory of simple complexes of groups, which is a generalisation of Bass and Serre's theory of graphs of groups. We do not go into nearly as much detail in this chapter as we did with graphs of groups, including only what we need for Chapter 8.

We first recall some definitions taken from III.$\mathcal{C}$ of [**4**] and Section 2.3 of [**22**].

Definition 6.1. A *small category without loops* (a.k.a. a *scwol*) X is a disjoint union of a set of vertices $V(X)$ and a set of (directed) edges $E(X)$, with each edge $e \in E(X)$ oriented with an initial vertex $i(e)$ and a terminal vertex $t(e)$, such that:

(i) $i(e) \neq t(e)$ (i.e. no loops); and

(ii) if e' is another edge with $t(e) = i(e')$ then there exists a composition edge ee' with $i(ee') = i(e)$ and $t(ee') = t(e')$ (i.e. composition of edges).

Definition 6.2. A *morphism* of scwols is a functor between categories. An *automorphism* of a scwol X is a morphism $\phi : X \to X$ with an inverse, or equivalently, an oriention-preserving polyhedral isometry of the geometric realisation of X. An *action* of a group G on a scwol X is a group homomorphism $\rho : G \to \mathrm{Aut}(X)$ such that, for any $g \in G$ and $e \in E(X)$, $g \cdot i(e) \neq t(e)$ and if g fixes $i(e)$ then it must also fix e.

Definition 6.3. A *(simple) complex of groups* $\mathcal{A}$ over a scwol X associates a group $\mathcal{A}_v$ to each vertex $v \in V(X)$ and a monomorphism $\psi_e : \mathcal{A}_{i(e)} \to \mathcal{A}_{t(e)}$ to each edge $e \in E(X)$ such that $\psi_{e'} \circ \psi_e = \psi_{ee'}$ for all edges e and e' satisfying $t(e) = i(e')$.

That is, a complex of groups is a commuting diagram of monomorphisms of groups.

Definition 6.4. Let $\mathcal{A}$ be a complex of groups over a scwol X. Let v be a vertex in X. The *local development* or *star* of $\mathcal{A}$ at v is the polyhedral complex $\mathrm{St}(v)$ given by the affine realisation of the union of the posets

(i) $\mathcal{P} := \Big\{\big(g\psi_e(\mathcal{A}_{i(e)}), i(e)\big) \;\Big|\; g \in \mathcal{A}_v, e \in E(X) \text{ with } t(e) = v\Big\}$ with partial ordering given by $\big(g\psi_e(\mathcal{A}_{i(e)}), i(e)\big) \prec \big(g'\psi_{e'}(\mathcal{A}_{i(e')}), i(e')\big)$ if there exists a $f' \in E(X)$ with $i(f') = i(e')$, $t(f') = i(e)$ and $g^{-1}g' \in \psi_e(\mathcal{A}_{i(e)})$;

(ii) $\mathcal{Q} := \Big\{\big(\mathcal{A}_v, t(e)\big) \;\Big|\; e \in E(X) \text{ with } i(e) = v\Big\}$ with partial ordering given by $\big(\mathcal{A}_v, t(e)\big) \prec \big(\mathcal{A}_v, t(e')\big)$ if there exists a $f \in E(X)$ with $i(f) = t(e')$ and $t(f) = t(e)$; and

(iii) the singleton poset $\big\{(\mathcal{A}_v, v)\big\}$

with the additional requirement that $\beta \prec \big(\mathcal{A}_v, v\big) \prec \alpha$ for any $\alpha \in \mathcal{P}$ and $\beta \in \mathcal{Q}$.

The *link* $\mathrm{Lk}(v)$ of $\mathcal{A}$ at v is defined similarly except that we omit the singleton poset $\big\{(\mathcal{A}_v, v)\big\}$.

In Chapter 8 we will only need to compute the link in certain special cases, which are described in the following example.

Example 6.5. Let $\mathcal{A}$ be a complex of groups over a scwol X. Let v be a vertex in X with local group $\mathcal{A}_v$. Let Φ be the poset of all vertices $\{v_1, ..., v_k\}$ in X adjacent to v where, for any $i, j \in \{1, ..., k\}$, $v_i \prec v_j$ if there exists an edge in X with initial vertex v_j and terminal vertex v_i. We use Definition 6.4 to (combinatorially) compute $\mathrm{Lk}(v)$ of $\mathcal{A}$ in the following cases.

Case I: *$\mathcal{A}_v$ is the trivial group.* Then $\mathcal{P}$ is greatly simplified as all left cosets of monomorphic images of groups in $\mathcal{A}_v$ must be trivial. Hence, as a poset, $\mathrm{Lk}(v)$ is isomorphic to Φ.

Case II: *All edges adjacent to v in X are oriented away from v.* Then $\mathcal{P}$ is empty. Hence, as a poset, $\mathrm{Lk}(v)$ is isomorphic to Φ.

Case III: *Φ has only one element v', the edge in X adjacent to both v and v' is oriented towards v, and $\mathcal{A}_{v'}$ has index n in $\mathcal{A}_v$.* Then $\mathcal{Q}$ is empty and $\mathrm{Lk}(v) = \mathcal{P}$ is the poset consisting of all n left cosets of $\mathcal{A}_{v'}$ in $\mathcal{A}_v$ with no ordering between any of them.

Case IV: *The geometric realisation of Φ has two connected components, Φ_1 and Φ_2.* Then all edges adjacent to v in X must be oriented towards v, so $\mathcal{Q}$ is empty. Let $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) be the sub-complex of groups of $\mathcal{A}$ given by removing all vertices in Φ_1 (resp. Φ_2) and their adjacent edges from the underlying scwol X. Then $\mathrm{Lk}(v)$ of $\mathcal{A}$ is the disjoint union of $\mathrm{Lk}(v)$ of $\mathcal{A}_1$ and $\mathrm{Lk}(v)$ of $\mathcal{A}_2$.

Analogous to the construction in Bass-Serre theory, a group G acting on a connected, simply connected scwol X can always be written in the form of a complex of groups over the quotient scwol $G\backslash X$, with vertex groups as the stabilisers of a (choice of) preimage of the vertices under the natural projection $X \to G\backslash X$ and monomorphisms chosen accordingly. It can be shown that the resulting complex of groups is unique up to a notion of isomorphism (refer to Chapter 3 of [**22**] for more details).

However, the converse no longer holds! That is, it is not always the case that a complex of groups is associated with such a group action. We call a complex of groups $\mathcal{A}$ over a scwol Y that does correspond with such a group action *strictly developable*, and call G the *fundamental group* $\pi_1(\mathcal{A})$, and X the *universal cover* $\widetilde{\mathcal{A}}$, where Y is isomorphic to $G\backslash X$.

For an example of a non-developable complex of groups, refer to Example 12.17. of [**4**].

Despite this complication, we can be sure that all complexes of groups constructed in Chapter 8 are indeed strictly developable by a theorem of Haefliger's, which gives a sufficient condition for strict developability. That condition is satisfied if geodesic triangles in each local development of a complex of groups are "no fatter" than triangles in Euclidean space.

In order to state Haefliger's theorem, we first recall some definitions and results from I.1, I.2 and II.1 of [**4**].

Definition 6.6. Let (X, d) be a metric space. A *geodesic segment* between points $x, y \in X$ is the image of a path of length $d(x, y)$ joining x and y. There may be multiple geodesic segments between two given points. We say that (X, d) is *geodesic* (resp. k-*geodesic* for some $k > 0$) if there exists a geodesic segment joining any pair of points $x, y \in X$ (resp. such that $d(x, y) \leq k$).

Definition 6.7. Let $\kappa \in \mathbb{R}$. The *model space* (M^n_κ, d') is Euclidean space $\mathbb{R}^n$ under the standard metric if $\kappa = 0$, is obtained from the sphere $\mathbb{S}^n$ by multiplying the usual metric by $1/\sqrt{\kappa}$ if $\kappa > 0$ and is obtained from hyperbolic space $\mathbb{H}^n$ by multiplying the usual metric by $1/\sqrt{-\kappa}$ if $\kappa < 0$. Denote the diameter of M^n_κ by D_κ. Take $D_\kappa = \infty$ if $\kappa \leq 0$.

Alternatively M^n_κ can be defined as the complete, simply connected Riemannian n-manifold of constant sectional curvature $\kappa \in \mathbb{R}$. See Chapter 6 of [**4**].

Definition 6.8. Let $\kappa \in \mathbb{R}$ and let $k \leq n$ be integers. A geodesic k-simplex $\Delta^k \subset M^n_\kappa$ is the convex hull of $(k + 1)$ points in general position (that is, they are not all contained in a subspace isometric to M^{n-1}_κ). If $\kappa > 0$ we require all points to lie within an open hemisphere.

Proposition 6.9. Let x, y and z be points in a metric space (X, d). Let $\kappa \in \mathbb{R}$ and assume that $d(x, y)+d(x, z)+d(y, z) < 2D_\kappa$ (note that this condition is vacuous unless $\kappa > 0$). Then there exist points x', y' and z' in M^2_κ such that $d(x, y) = d'(x', y')$, $d(x, z) = d'(x', z')$ and $d(y, z) = d'(y', z')$, which along with a (choice of) three geodesic segments joining the respective points x', y' and z', is called a *comparison triangle* for (x, y, z), and is uniquely defined up to an isometry of M^2_κ.

Proof. Refer to Lemma $I.2.14$ of [**4**]. □

Definition 6.10. Let (X, d) be a metric space. A *geodesic triangle* $\triangle := \triangle(x, y, z)$ consists of three points x, y and z in X along with a (choice of) three geodesic segments joining them. Let $\triangle' := \triangle'(x', y', z')$ be the comparison triangle for (x, y, z) and consider a point p on the geodesic segment $[x, y]$. We say $p' \in [x', y']$ is a *comparison point* for $p \in [x, y]$ if $d(x, p) = d'(x', p')$. Similarly for the other two sides of $\triangle$.

Definition 6.11. Let $\kappa \in \mathbb{R}$. A metric space (X, d) is a $\mathrm{CAT}(\kappa)$ *space* if

(i) $\kappa \leq 0$ and (X, d) is geodesic such that any geodesic triangle $\triangle \in X$ with comparison triangle $\triangle' \in M^2_\kappa$ satisfies $d(x, y) \leq d(x', y')$ for all $x, y \in \triangle$ and all comparison points $x', y' \in \triangle'$; or

(ii) $\kappa > 0$ and (X, d) is D_κ-geodesic such that any geodesic triangle $\triangle \in X$ of perimeter less than $2D_\kappa$ with comparison triangle $\triangle' \in M^2_\kappa$ satisfies $d(x, y) \leq d(x', y')$ for all $x, y \in \triangle$ and all comparison points $x', y' \in \triangle'$.

Definition 6.12. A metric space X has *curvature* $\leq \kappa$ if for any $x \in X$ there exists a small enough $d_x > 0$ such that the ball $B(x, d_x)$ is a $\mathrm{CAT}(\kappa)$ space under the induced metric. If $\kappa \leq 0$ then X is called *non-positively curved*.

Theorem 6.13 (**Haefliger**). A complex of groups is strictly developable if each of its local developments is non-positively curved.

Proof. Refer to Theorem 12.28. of [**4**]. □

Example 6.14. Let (W, S) be a Coxeter system with chamber K and Davis complex Σ. We construct the *chamber complex (of groups)* $\widehat{K}$ of (W, S) over the scwol K by associating to each vertex in K the corresponding spherical subgroup of W with natural inclusions as monomorphisms (inducing an orientation on the edges of K and hence Σ, so they are indeed scwols).

One can check using Definition 6.4 that the set of isometry classes of links of $\widehat{K}$ is the same as that for Σ (interpreted as a complex of groups with all vertex groups trivial over itself as a scwol). Since Σ can be endowed with a $\mathrm{CAT}(0)$ metric (Theorem 4.13 of Moussong [**14**]), Theorem 6.13 ensures that $\widehat{K}$ is strictly developable. The fundamental group of $\widehat{K}$ is isomorphic to W and the universal cover of $\widehat{K}$ is isomorphic to Σ as a polyhedral complex (refer to Section 2.3 of [**22**]). We can then apply Corollary 3.27 of Serre's theorem [**18**] to show that $W \cong \pi_1(\widehat{K})$ is a uniform lattice in $\mathrm{Aut}(\Sigma)$.

Definition 6.15. Let $\mathcal{A}$ be a complex of groups over a scwol X. For any group Γ, we denote by $\Gamma \times \mathcal{A}$ the complex of groups obtained by taking the direct product of Γ with each vertex group in $\mathcal{A}$ along with the corresponding canonical monomorphisms.

We can describe any polyhedral complex combinatorially as a scwol. Fix any $\kappa \in \mathbb{R}$. To any M_κ-polyhedral complex P we take the first barycentric subdivision $\mathrm{BS}(P)$ and define a scwol X with vertex set the (barycentres of) cells in P and edge set the edges in $\mathrm{BS}(P)$ oriented towards cells of smaller dimension. Conversely, for any scwol X', there exists a natural geometric realisation of X' as a piecewise Euclidean (M_0) polyhedral complex P' with each cell isometric to a geodesic Euclidean simplex (see 1.3 of Chapter III.$\mathcal{C}$ of [**4**]). Observe that P' is not necessarily a simplicial complex as the intersection of two distinct cells is not necessarily a single face, rather it is a union of faces. For example, interpret the 2-torus as a polyhedral complex consisting of a square with opposite sides identified in parallel. Its barycentric subdivision is not a simplicial complex.

Proposition 6.16. Let $\mathcal{A}$ be a complex of groups over a scwol X and let $\kappa \in \mathbb{R}$. Consider a geometric realisation of X as a M_κ-polyhedral complex with only finitely many isometry classes of cells. Then there exists an induced M_κ-polyhedral complex structure on each local development of $\mathcal{A}$.

Proof. Refer to 4.14. and 4.16 of Chapter III.$\mathcal{C}$ of [**4**]. □

In Chapter 8 we geometrically realise all of our constructed scwols as piecewise Euclidean polyhedral complexes. We do this by equipping each embedded copy of a chamber (in the sense of Definition 4.8) with Moussong's metric from Theorem 4.13. So we may speak of isometry classes of local developments of our complex of groups constructions.

CHAPTER 7

Buildings

We recall some definitions taken from Chapter 18 of [**8**] and Section 1.4 of [**24**].

Definition 7.1. Let S be a set. A *chamber system* Θ over S is a set with elements called *chambers* together with a family of equivalence relations indexed by S. Two distinct chambers are called *s-adjacent* if they are s-equivalent for some $s \in S$. Let $\boldsymbol{s} = (s_1, ..., s_l)$ be a word in S. A *gallery* in Θ of *type $\boldsymbol{s}$ connecting* the chambers θ_0 and θ_l is a finite sequence of chambers $(\theta_0, \theta_1, ..., \theta_l)$ such that θ_{i-1} is s_i-adjacent to θ_i for all $i = 1, ..., l$.

Definition 7.2. Let (W, S) be a Coxeter system. A *building* of type (W, S) is a chamber system Θ over S, with each s-equivalence class containing at least 2 chambers, together with a "distance" function $d : \Theta \times \Theta \to W$ such that two chambers θ_1 and θ_2 in Θ are connected by a gallery of type $\boldsymbol{s}$ iff $d(\theta_1, \theta_2) =_W \boldsymbol{s}$. The building Θ is *regular* if each s-equivalence class has the same number of chambers, for all $s \in S$.

Example 7.3. Let (W, S) be a Coxeter system. The *abstract Coxeter complex* $\boldsymbol{W}$ is the building of type (W, S) with chambers the elements of W, where for any $s \in S$ two chambers w_1 and w_2 are s-adjacent iff $w_2 = w_1 s$, together with the distance function $d(w_1, w_2) = w_1^{-1} w_2$. In particular, every s-equivalence class of chambers has precisely 2 elements for every $s \in S$.

Definition 7.4. Let (W, S) be a Coxeter system, with associated abstract Coxeter complex $\boldsymbol{W}$, and let Θ be a building of type (W, S). An *apartment* of Θ is an embedded copy of $\boldsymbol{W}$ in Θ under a W-distance preserving map. There exists a *natural geometric realisation* of Θ, described as follows. Interpret each chamber of Θ as an indexed copy of the "chamber" K of (W, S) (in the sense of Definition 4.8) and similarly interpret each apartment of Θ as a copy of the Davis complex Σ of (W, S). For example the natural geometric realisation of the building $\boldsymbol{W}$ is Σ.

We abuse notation and also refer to the natural geometric realisation of Θ as a *building*.

It follows from the definition of a building Θ that for any two chambers K and K' in Θ then there exists an apartment of Θ containing both of them, and for any two such apartments Σ and Σ' both containing K and K' then there exists an isomorphism $\Sigma \to \Sigma'$ fixing K and K' pointwise (see Chapter IV of [**5**]). This ensures that the natural geometric realisation is well-defined and allows us to generalise Moussong's piecewise Euclidean metric on the Davis complex to any building.

Theorem 7.5. Let (W, S) be a Coxeter system and let Θ be a building of type (W, S). Then (the natural geometric realisation of) Θ may be endowed with a piecewise Euclidean metric such that Θ is a complete $\mathrm{CAT}(0)$ space.

Proof. Moussong [**14**] first showed this in the case of Davis complexes. Refer to Theorems $18.3.1$ and $18.3.9$ of [**8**], and Theorem 7 of [**24**]. □

Henceforth we equip all buildings with this $\mathrm{CAT}(0)$ metric.

Definition 7.6. A Coxeter system (W, S) of type M is *right-angled* if every non-diagonal entry of the matrix M is equal to either 2 or ∞. A building of type (W, S) is *right-angled* if (W, S) is *right-angled*.

Theorem 7.7 (**Classification of regular right-angled buildings**). Let $(W, \{s_1, ..., s_n\})$ be a right-angled Coxeter system and associate an integer $p_i \geq 2$ to generator s_i for every i. Then, up to isomorphism, there exists a unique regular building Σ of type $(W, \{s_1, ..., s_n\})$ such that for all i each s_i-equivalence class contains p_i chambers.

Proof. Refer to Proposition 1.2 of [**11**] and Theorem 8 of [**24**]. □

Example 7.8. Let (W, S) be a right-angled Coxeter system with chamber K. Denote $S := \{s_1, ..., s_n\}$. Let Σ be the regular right-angled building of type (W, S) with $p_i \geq 2$ chambers in each s_i-equivalence class. Let $\{s_{i_1}, ..., s_{i_l}\}$ be an arbitrary spherical subset of S and recall that the vertices of K are labelled by the spherical subsets of S. We generalise Example 6.14 as follows.

Construct the *chamber complex (of groups)* $\widehat{K}$ of (W, S) with integers $\{p_1, ..., p_n\}$ over the scwol K by associating to each vertex $\{s_{i_1}, ..., s_{i_l}\}$ in K the group $\mathcal{C}_{p_{i_1}} \times ... \times \mathcal{C}_{p_{i_l}}$ with natural inclusions as monomorphisms. As in Example 6.14, one can check that $\widehat{K}$ is strictly developable, with universal cover isomorphic to Σ as a polyhedral complex and with fundamental group a uniform lattice in $\mathrm{Aut}(\Sigma)$.

CHAPTER 8

A family of uniform lattices acting on Σ with a non-discrete set of covolumes

Let (W, S) be a Coxeter system of type $M = \{m_{ij}\}_{i,j}$ with nerve L. Denote $S := \{s_1, ..., s_n\}$. Recall that a generator s_k is *free* if $m_{ik} = \infty$ for all $i \neq k$. Let S' denote the subset of free generators of S. If S' is non-empty, without loss of generality write $S' := \{s_1, ..., s_j\}$ for some $1 \leq j \leq n$. Take $j = 0$ to mean that S' is empty.

In this chapter we prove the following theorem.

Theorem 8.1. The set of covolumes of uniform lattices in the automorphism group of a polyhedral complex Σ is non-discrete, where Σ can be any of the following.

(i) The Davis complex of (W, S) where

- $j = n \geq 3$ (i.e. all generators are free).

(ii) The Davis complex of (W, S), where $j \geq 3$, such that the connected components of L either:

- correspond to spherical subgroups of (W, S); or
- are cycle graphs with an even number of edges and constant edge-labels (including at least one such cycle). Distinct cycles are not required to have the same labels or number of edges.

(iii) The regular right-angled building of type (W, S), with $p_i \geq 2$ chambers in each s_i-equivalence class for all $1 \leq i \leq n$, where

- $j \geq 2$;
- (W, S) is right-angled; and
- there exists at least one $i \in \{1, ..., j\}$ with $p_i > 2$.

We will also prove the following corollary, which is obtained from the constructions we use to establish Theorem 8.1.

Corollary 8.2. Let Σ be as in Theorem 8.1. In cases (i) and (ii) (resp. case (iii)), let $p < j$ (resp. $p < \max_{1\leq i\leq j}\{p_i\}$) be prime. Then for arbitrarily large $\alpha \in \mathbb{N}$ there exists a uniform lattice Γ in $\mathrm{Aut}(\Sigma)$ with covolume $\mu(\mathrm{Aut}(\Sigma)/\Gamma) = a/b$ (in lowest terms) such that b is divisible by p^α.

A sketch of the proof of Theorem 8.1 is as follows. We explicitly construct a family of finite graphs (resp. complexes) of finite groups with at least one trivial local group in case (i) (resp. cases (ii) and (iii)). Assuming strict developability, we apply Corollary 3.27 of Serre's theorem [**18**] to confirm that the fundamental groups of our constructions are indeed uniform lattices in the (polyhedral) automorphism groups of their respective universal covers, compute their respective covolumes as the sum of reciprocals of the orders of the vertex groups and observe that this set of covolumes is non-discrete. We check that the set of isometry classes of links of our graph (resp. complex) of groups constructions is the same as that for Σ (interpreted as a complex of groups with all vertex groups trivial over itself as a scwol). Finally, we apply Theorem 6.13 to ensure that all of our constructions are indeed strictly developable, and then use certain case-specific arguments to show that the universal cover of each of our constructions is isomorphic to Σ as a graph (resp. polyhedral complex).

For a precise description of how to apply Corollary 3.27 of Serre's theorem to a graph of groups, refer to Corollaries 5.22 and 5.24 in Chapter 5. It is straightforward to generalise this argument to a complex of groups.

We then prove Corollary 8.2 by generalising our constructions so that the associated set of covolumes converges at a rate of $1/p$ for any such prime p.

A few remarks on notation. It should be assumed that every monomorphism not explicitly defined in our graphs/complexes of groups constructions in Figures 1 to 11 is the natural inclusion. Vertices of the same colour in these diagrams have isometric local developments unless otherwise specified. Let $\mathbf{1}$ denote the trivial group. Let $\mathcal{D}_{2m}$ denote the dihedral group of order $2m$ and let $\mathcal{C}_\beta^\alpha$ denote the direct product of α copies of the cyclic group of order β. Assume the convention that $\mathcal{C}_\beta^0$ is the trivial group. A rational number a/b is said to be in lowest terms if a and b are coprime integers. Let μ be the normalisation of the Haar measure on $\mathrm{Aut}(\Sigma)$ as in Theorem 3.25.

Recall that a convergent sequence $(x_1, x_2, ...) \to x$ of real numbers has a *rate of convergence* of $\lim_{l\to\infty} \frac{|x_{l+1}-x|}{|x_l - x|}$.

8.1. Σ is the Davis complex of a free Coxeter group

In this section we prove case (i) of Theorem 8.1 and Corollary 8.2.

Proposition 8.3. Let $(W, \{s_1, ..., s_n\})$ be a Coxeter system of type $M = \{m_{ij}\}_{i,j}$ such that $m_{ij} = \infty$ for any $i \neq j$. Recall from Example 4.12 that the Davis complex Σ_n of (W, S) is isomorphic to the $(n, 2)$-biregular tree. Then the set of covolumes of uniform lattices in $\mathrm{Aut}(\Sigma_n)$ is non-discrete.

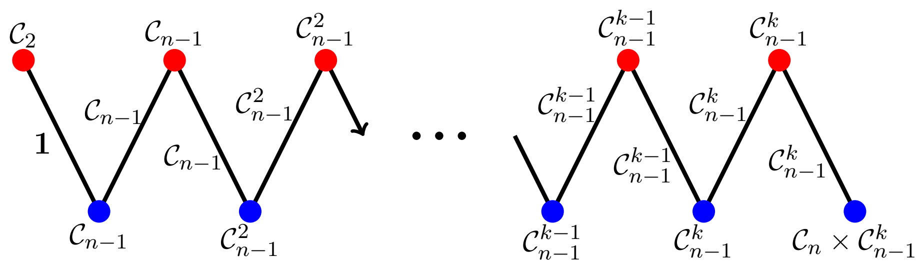

FIGURE 1. A family $\{(G, A)_k\}_{k \in \mathbb{N}}$ of finite graphs of finite groups

Proof. Consider the family $\{(G, A)_k\}_{k \in \mathbb{N}}$ of graphs of groups in Figure 1. Fix $k \in \mathbb{N}$. Since $(G, A)_k$ has a trivial edge group, by Corollary 5.24 the fundamental group $\pi_1(G, A)_k$ acts faithfully on the universal cover $\widetilde{(G, A)}_k$.

For each blue (resp. red) vertex $v \in A_k$, observe that the sum of indices of each adjacent edge group in G_v is n (resp. 2). By Proposition 5.18, blue (resp. red) vertices in A_k correspond with vertices of degree n (resp. 2) in $\widetilde{(G, A)}_k$. In addition, the colour bi-partition of the graph A_k extends to a bipartition of $\widetilde{(G, A)}_k$ (as two vertices are adjacent in the universal cover iff their projections are adjacent in the underlying graph of groups). That is, $\widetilde{(G, A)}_k$ is isomorphic to the unique $(n, 2)$-biregular tree Σ_n.

Having satisfied all the conditions required to apply Corollary 5.22 to $(G, A)_k$, we deduce that $\pi_1(G, A)_k$ is a uniform lattice in $\mathrm{Aut}(\Sigma_n)$ with covolume

$$\mu\big(\,\mathrm{Aut}(\Sigma_n)/\pi_1(G, A)_k\big) = \sum_{v \in V(A_k)} \frac{1}{|G_v|}$$
$$= \frac{1}{2} + 2\Bigg(\sum_{i=1}^{k} \frac{1}{(n-1)^i}\Bigg) + \frac{1}{n(n-1)^k}$$
$$\longrightarrow \frac{1}{2} + \frac{2}{n-2} \quad \text{as} \quad k \longrightarrow \infty$$

with a rate of convergence of $\frac{1}{n-1}$.

Observe that the limit point found above may be realised as the covolume of the uniform lattice $\pi_1(G, A)$ in $\mathrm{Aut}(\Sigma_n)$; see Figure 2. □

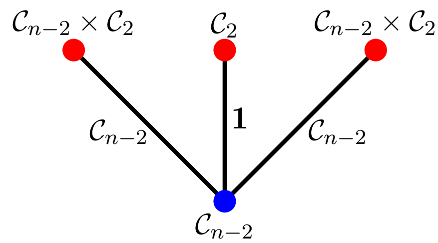

FIGURE 2. A finite graph of finite groups (G, A)

In fact the constructions in the proof of Proposition 8.3 give us a new proof of the following result of Rosenberg's (see Theorem 9.2.1 of [**16**]).

Corollary 8.4. For any $n \geq 3$ let Σ_n be the $(n, 2)$-biregular tree. Let $p < n$ be prime. For any (arbitrarily large) $\alpha \in \mathbb{N}$ there exists a uniform lattice Γ in $\mathrm{Aut}(\Sigma_n)$ with covolume $\mu\big(\mathrm{Aut}(\Sigma_n)/\Gamma\big) = a/b$ (in lowest terms) such that b is divisible by p^α.

Proof. We construct a family of graphs of groups by generalising the construction in Figure 1, which will become the case $p = n - 1$. We give an example of the case where $p = 2$, $n = 5$ and $k = 2$ in Figure 3.

Take any integer k. We will construct a tree A_k^p with blue vertices and red vertices, and a complex of groups (G_k^p, A_k^p). No vertex in A_k^p is adjacent to another of the same colour. We partition (G_k^p, A_k^p) into descending "levels" from $i = 0$ to $i = k + 1$. Level 0 consists of a single red vertex with local group $\mathcal{C}_2$ and a single edge with trivial local group connecting the lone red vertex in level 0 to the lone blue vertex in level 1. Level $k + 1$ consists of $(n - p)^k$ blue vertices each with local group $\mathcal{C}_n \times \mathcal{C}_p^k$, no red vertices and no edges.

Every remaining level i for $i = 1, ..., k$ consists of $(n - p)^{i-1}$ blue vertices, $(n - p)^i$ red vertices, and $2(n - p)^i$ edges connecting each blue vertex on level i with $n - p$ red vertices on level i and each red vertex on level i with one blue vertex on level $i + 1$. Every vertex and edge group on level i, for $i = 1, ..., k$, is $\mathcal{C}_p^i$.

The fundamental group of each graph of groups in $\{(G_k^p, A_k^p)\}_{k\in\mathbb{N}}$ is a uniform lattice in Σ_n by the same argument as in the proof of Proposition 8.3. A simple calculation will show that the corresponding sequence of covolumes converges at a rate of $\frac{1}{p}$ as $k \to \infty$. The sum of reciprocals of orders of the vertex groups (in lowest terms) will therefore have a denominator that is divisible by ever-increasing powers of p as k gets larger. $\square$

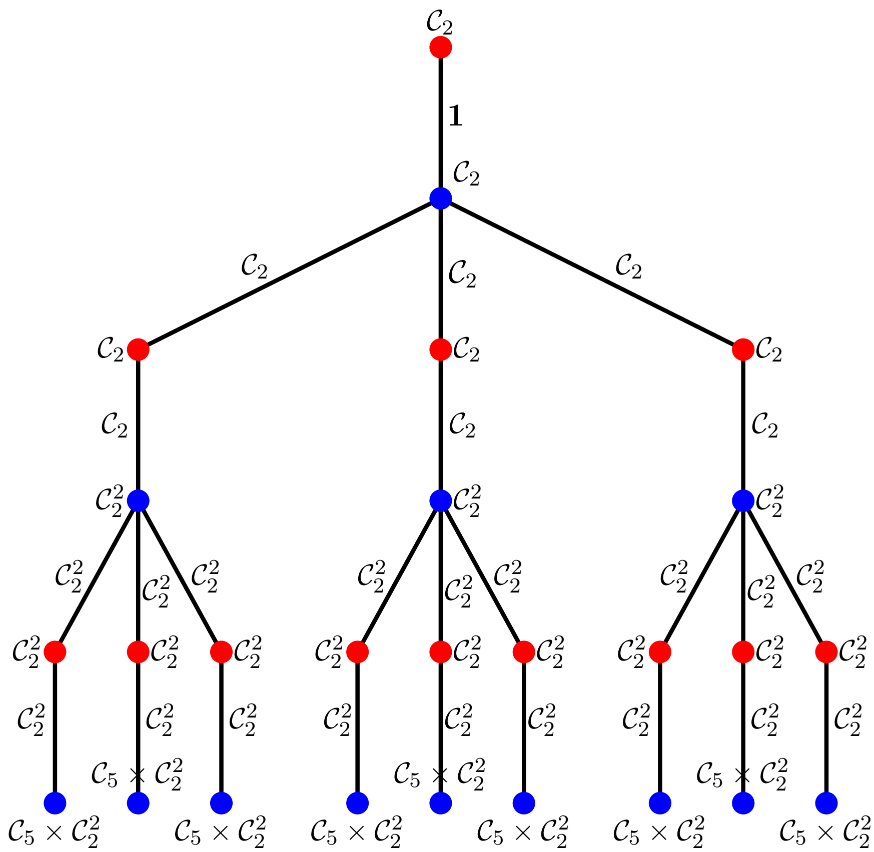

FIGURE 3. The graph of groups (G_2^2, A_2^2) when $n = 5$

8.2. Σ is the Davis complex of a non-free Coxeter group

In this section we prove case (ii) of Theorem 8.1 and Corollary 8.2.

Let (W, S) be any Coxeter system with nerve L. We can uniquely decompose W into a free product $W = *_{i=1,\ldots,\lambda} W_i$ of Coxeter subgroups with respective generating sets disjoint subsets of S. Each W_i is indecomposable if λ is a maximal choice of integer. Observe that this decomposition of W corresponds with a decomposition of L as a disjoint union of its connected components L_i, where L_i is the subgraph of L corresponding to W_i.

Theorem 8.5. Let (W, S) be a Coxeter system with Davis complex Σ and nerve L containing at least four connected components. Completely decompose $W = *_{i=1,\ldots,l+4} W_i$ as a free product of Coxeter subgroups for some non-negative integer l. Then the set of covolumes of uniform lattices in $\mathrm{Aut}(\Sigma)$ is non-discrete if the following conditions hold:

- (at least) one connected component L_{l+1} of L is the $2m$-cycle graph for some $m \geq 2$ with edges labelled by some constant integer $x \geq 2$;
- (at least) three connected components L_{l+2}, L_{l+3} and L_{l+4} of L are lone vertices (that is, S contains at least three free generators); and
- if $l > 0$ each W_i for $1 \leq i \leq l$ is either finite or corresponds to an L_i that is the $2m_i$-cycle graph for some $m_i \geq 2$ with edges labelled by some constant integer $x_i \geq 2$.

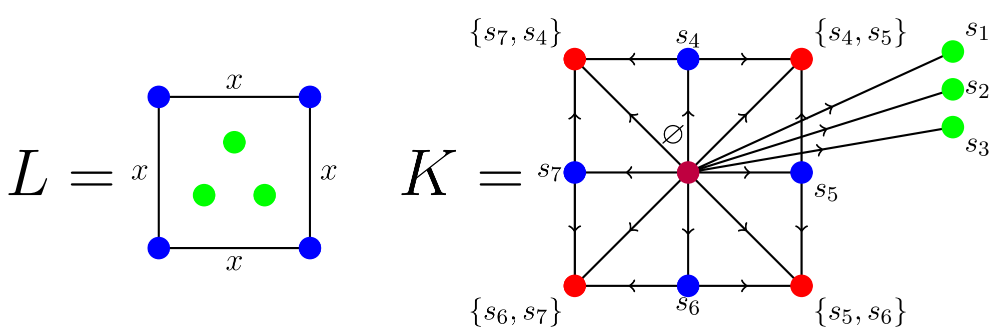

FIGURE 4. The nerve L and chamber K when $l = 0$, $m = 2$

Proof. The structure of the proof is as follows. We consider two cases, $l = 0$ and $l > 0$. When $l = 0$ we construct a family $\{\mathcal{X}_k^{(0)}\}_{k\in\mathbb{N}} \cup \{\mathcal{X}^{(0)}\}$ of complexes of groups. For any $\mathcal{F}$ in this family, we show that the set of isometry classes of links of $\mathcal{F}$ is the same as that of Σ (Proposition 8.7), that $\mathcal{F}$ is strictly developable (Corollary 8.8) and that the universal cover of $\mathcal{F}$ is Σ (Proposition 8.9). We then apply Corollary 3.27 to show that $\pi_1(\mathcal{F})$ is a uniform lattice in $\mathrm{Aut}(\Sigma)$ and compute the corresponding covolume. When $l > 0$ we inductively construct a family $\{\mathcal{X}_k\}_{k\in\mathbb{N}} \cup \{\mathcal{X}\}$ of complexes of groups and modify the argument in the $l = 0$ case to establish the result.

Case. 1: $l = 0$

Take $\{s_1, s_2, s_3\}$ to be the subset of free generators of $S := \{s_1, s_2, ..., s_n\}$.

In an abuse of notation, we interpret the Davis complex Σ as a complex of groups with all vertex groups trivial over itself as a scwol (induced by the scwol structure of its chamber K, refer to Example 6.14). We colour L, K and Σ by assigning the colour purple to all vertices of type $\varnothing$, green to all vertices of type s_1, s_2 or s_3, blue to all vertices of type s_4, s_5, ... or s_n and red to all vertices of type $\{s_i, s_j\}$ where $4 \leq i, j \leq n$ and $i \neq j$. Equip Σ with the $\mathrm{CAT}(0)$ metric introduced by Moussong in Theorem 4.13.

Definition 8.6. Take any integers $\alpha, \beta \geq 2$. Let $(W^{\alpha,\beta}, \{t_1, ..., t_{2\beta}\})$ be the Coxeter system with nerve $L^{\alpha,\beta}$ the 2β-cycle graph with constant edge labels α, and let $K^{\alpha,\beta}$ be the corresponding chamber. Let $\mathcal{P}^{\alpha,\beta}$ be the 2β-gon of groups where each vertex group is $\mathcal{D}_{2\alpha}$, each edge group is $\mathcal{C}_2$, the face group is trivial and all monomorphisms are natural inclusions. Observe that $\mathcal{P}^{\alpha,\beta}$ is the complex of groups associated with the natural action of $(W^{\alpha,\beta}, \{t_1, ..., t_{2\beta}\})$ on its Davis complex $\Sigma^{\alpha,\beta}$. That is, $\mathcal{P}^{\alpha,\beta}$ is strictly developable. Each polygon in $\mathcal{P}^{\alpha,\beta}$ has a type induced by that of the underlying scwol $K^{\alpha,\beta}$. Colour the vertices of $K^{\alpha,\beta}$, $\Sigma^{\alpha,\beta}$ and $\mathcal{P}^{\alpha,\beta}$ as follows. Assign the colour red to all vertices of type $\varnothing$, blue to all vertices of type t_1, t_2, ... or $t_{2\beta}$ and purple to all vertices of type $\{t_i, t_j\}$ for any $1 \leq i, j \leq 2\beta$ and $i \neq j$. As always, equip $\Sigma^{\alpha,\beta}$ with Moussong's metric.

Now consider the family $\{\mathcal{X}_k^{(0)}\}_{k\in\mathbb{N}} \cup \{\mathcal{X}^{(0)}\}$ of complexes of groups shown in Figures 5 and 6. Each complex of groups in this family consists of "multiples" of copies of $\mathcal{P}^{m,x}$ stacked on top of one another, glued together via additional green vertices. For each $i \in \{0, 1, ..., k\}$ (resp. $j \in \{0, 1\}$), and for any integer k, we call the $i + 1$'th (resp. $j + 1$'th) lowest embedded "multiple" of $\mathcal{P}^{m,x}$ the *i'th platform* $\mathcal{P}_i^{m,x} = \mathcal{C}_2^i \times \mathcal{P}^{m,x}$ (resp. *j'th platform* $\mathcal{P}_j^{m,x} = \mathcal{P}^{m,x}$) of the structure.

Proposition 8.7. Let $\mathcal{F}$ be any complex of groups in the family $\{\mathcal{X}_k^{(0)}\}_{k\in\mathbb{N}} \cup \{\mathcal{X}^{(0)}\}$. Then the set of isometry classes of links of vertices in $\mathcal{F}$ is the same as that of Σ, as illustrated in Figure 7, and in particular is consistent with the colour schemes of $\mathcal{F}$ and Σ. That is, two vertices in either $\mathcal{F}$ or Σ have isometric links iff they have the same colour.

Proof. Consider the Coxeter system $(W^{m,x}, \{t_1, ..., t_{2x}\})$, its Davis complex $\Sigma^{m,x}$ and the associated complex of groups $\mathcal{P}^{m,x}$ (as in Definition 8.6).

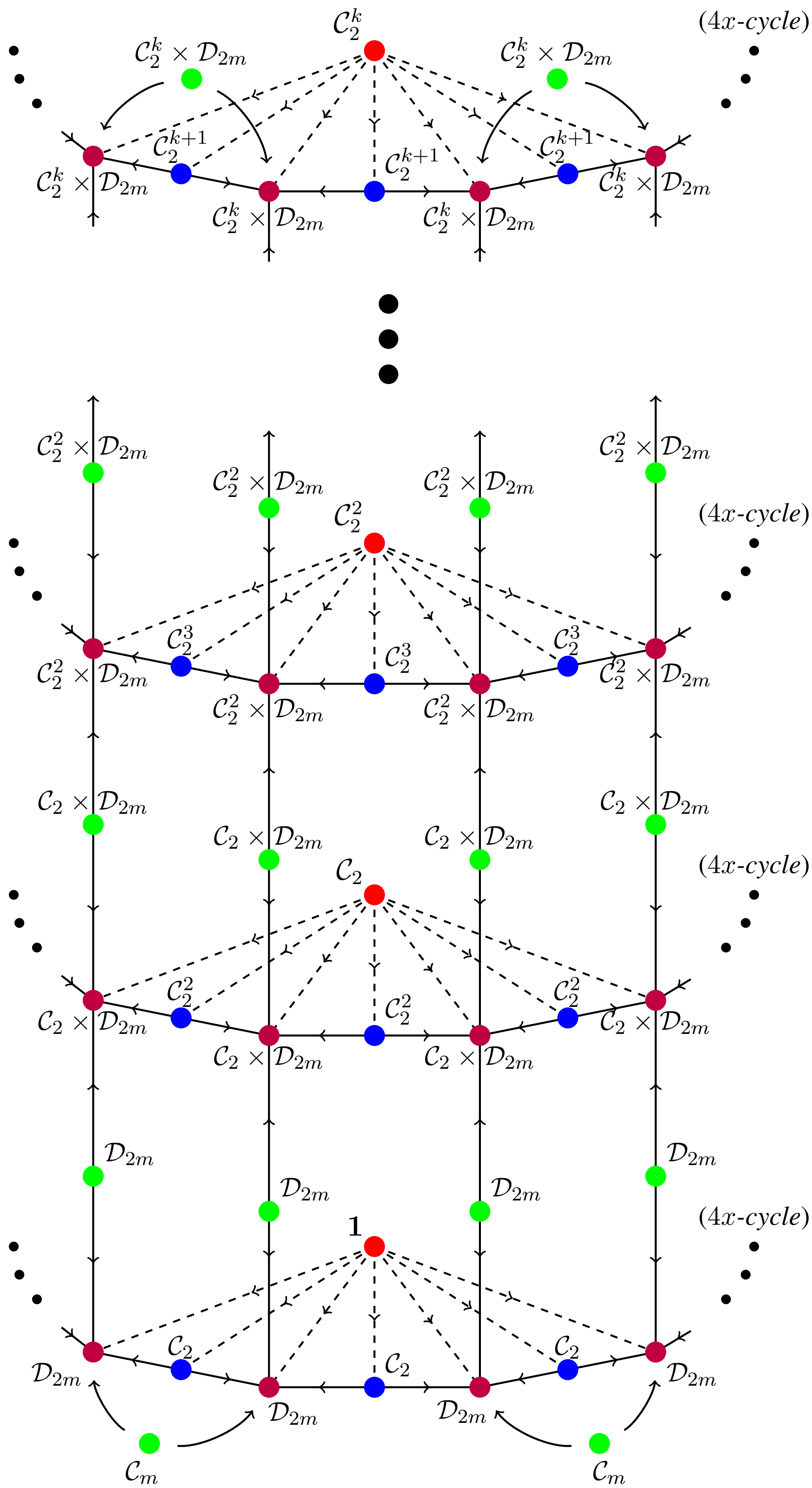

FIGURE 5. A family $\{\mathcal{X}_k^{(0)}\}_{k\in\mathbb{N}}$ of finite complexes of finite groups (with embedded "platforms" $\mathcal{P}_i^{m,x} = \mathcal{C}_2^i \times \mathcal{P}^{m,x}$ for $i = 0, 1, ..., k$)

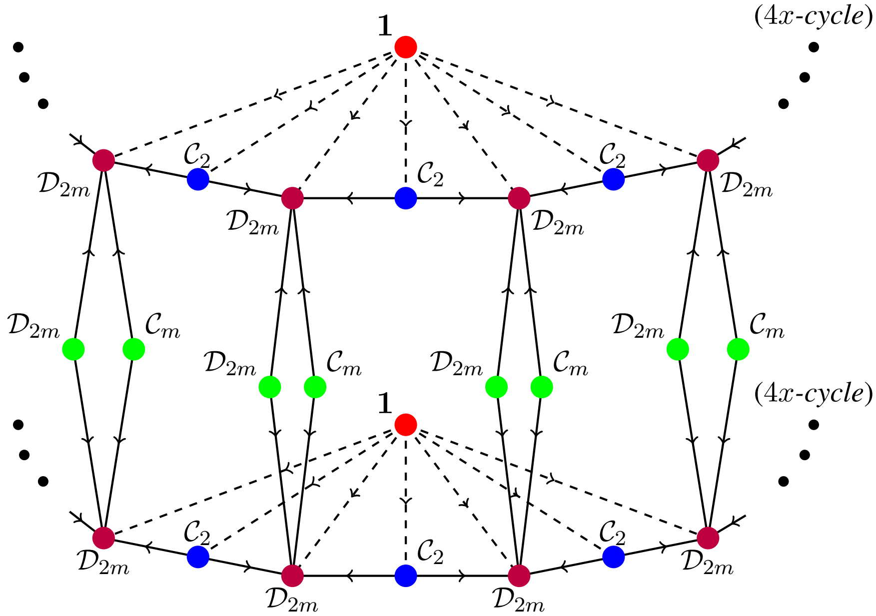

FIGURE 6. A finite complex of finite groups $\mathcal{X}^{(0)}$

Observe that the isometry class of link of each polygon in $\mathcal{P}^{m,x}$ is the same as that of any vertex of the same type in $\Sigma^{m,x}$ since the universal covering map is a type-preserving local isometry.

For any distinct $i, j \in \{1, ..., 2x\}$, define $\phi_{i,j}$ to be the automorphism of $(W^{m,x}, \{t_1, ..., t_{2x}\})$ that swaps the generators t_i and t_j. Observe that $\phi_{i,j}$ induces an automorphism on $\Sigma^{m,x}$ for all $i, j \in \{1, ..., 2x\}$. Exploiting this symmetry means that there are only three inequivalent isometry classes of links of vertices in $\Sigma^{m,x}$; that of vertices of type $\varnothing$ (red vertices), of type t_i (blue vertices) and of type $\{t_i, t_j\}$ for $i \neq j$ (purple vertices).

Interpret $\Sigma^{m,x}$ as a complex of groups with all vertex groups trivial over itself as a scwol. Case I of Example 6.5 tells us that the link of any vertex v in $\Sigma^{m,x}$ is the intersection of $\Sigma^{m,x}$ (as a polyhedral complex) with a 2-sphere of sufficiently small radius centred at v. In particular, the (combinatorial) link of any red vertex in $\Sigma^{m,x}$ is the $4x$-cycle graph with vertices alternating between purple and blue, the link of any blue vertex in $\Sigma^{m,x}$ is the 4-cycle graph with vertices alternating between purple and red, and the link of any purple vertex in $\Sigma^{m,x}$ is the $4m$-cycle graph with vertices alternating between blue and red.

Now consider the complex of groups $\mathcal{F}$. The universal cover of each of its embedded platforms $\mathcal{P}_i^{m,x}$ is $\Sigma^{m,x}$ by the same argument as for $\mathcal{P}^{m,x}$. The link of any red (resp. blue) vertex in $\mathcal{F}$ is the same as that of any red (resp. blue) vertex in $\mathcal{P}^{m,x}$. It remains to show that the link of each green and purple vertex in $\mathcal{F}$ is as in Figure 7. Let v_g (resp. v_p) be (without loss of generality) any green (resp. purple) vertex in $\mathcal{F}$.

Denote the scwol underlying $\mathcal{F}$ by $X_{\mathcal{F}}$. Since both of the edges adjacent to v_g in $X_{\mathcal{F}}$ are oriented towards a purple vertex, Case II of Example 6.5 tells us that $\mathrm{Lk}(v_g)$ is the graph with two purple vertices and no edges.

Let Φ_p be the poset $\{v_{red}, v_{blue}, v'_{blue}, v_{green}, v'_{green}\}$ of all vertices in $X_{\mathcal{F}}$ adjacent to v_p with ordering $v_{blue} \prec v_{red}$ and $v'_{blue} \prec v_{red}$. Observe that the geometric realisation of Φ_p has three connected components, one of which corresponds to the poset of vertices adjacent to any purple vertex in $\mathcal{P}^{m,x}$, and the other two of which are lone green vertices with local groups of index 1 and 2 respectively in $\mathcal{F}_{v_p}$. Cases III and IV of Example 6.5 then tell us that $\mathrm{Lk}(v_p)$ of $\mathcal{F}$ is the disjoint union of the link of any purple vertex in $\mathcal{P}^{m,x}$ with three disconnected green vertices. $\square$

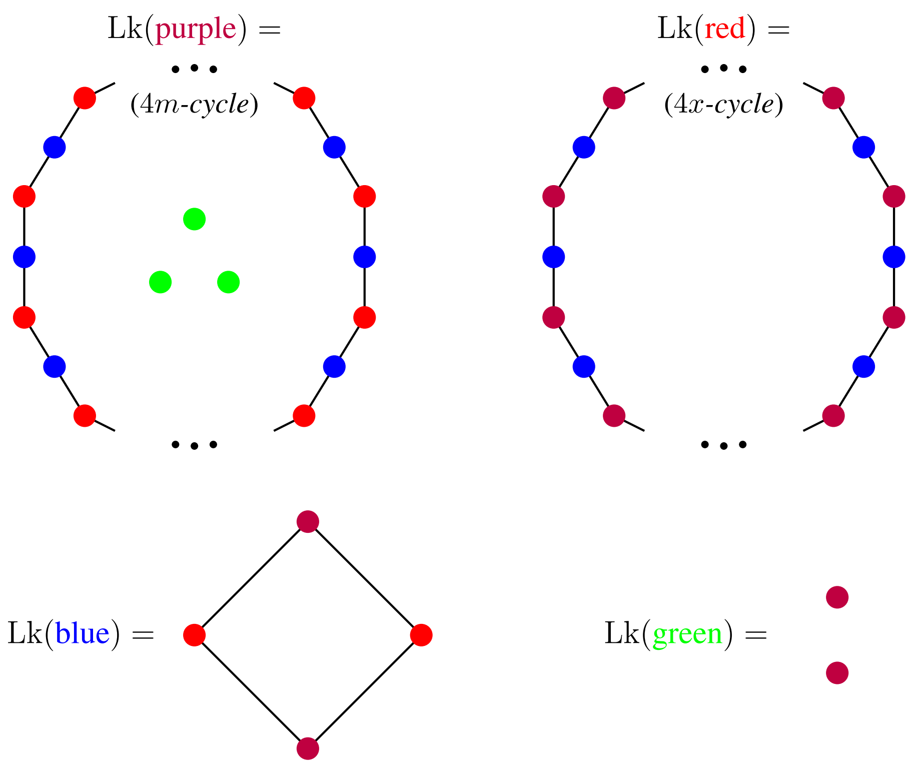

FIGURE 7. The set of isometry classes of links of Σ

Corollary 8.8. Each complex of groups in the family $\{\mathcal{X}_k^{(0)}\}_{k\in\mathbb{N}} \cup \{\mathcal{X}^{(0)}\}$ is strictly developable.

Proof. Proposition 8.7 says that each complex of groups in the family $\{\mathcal{X}_k^{(0)}\}_{k\in\mathbb{N}} \cup \{\mathcal{X}^{(0)}\}$ has a set of isometry classes of local developments identical to that of the Davis complex Σ. Hence they are $\mathrm{CAT}(0)$ under the restriction of Moussong's metric on Σ. The result then follows from Theorem 6.13. $\square$

It is in fact not true, for an arbitrary polyhedral complex, that the local structure (the set of isometry classes of links) uniquely determines the global structure [**13**]. This is an area of current research. So we need the following argument to compute the universal covers.

Proposition 8.9. Each complex of groups in the family $\{\mathcal{X}_k^{(0)}\}_{k\in\mathbb{N}} \cup \{\mathcal{X}^{(0)}\}$ has a universal cover that is isomorphic to Σ as a polyhedral complex.

Proof. It is known [**13**] that, up to isomorphism, there exists a unique polygonal complex Ξ such that the link of each vertex in Ξ is the $2m$-cycle graph and each 2-cell is a regular $2x$-gon for the integers $m, x \geq 2$ (again abusing notation and interpreting Ξ as a complex of groups with all vertex groups trivial over itself as a scwol, its orientation irrelevant).

Observe that the Davis complex Σ_1 of the Coxeter system $(W_1, \{s_4, ..., s_n\})$ with nerve L_1 the $2m$-cycle graph with constant edge labels x is isomorphic to the barycentric subdivision of Ξ under the map sending the purple vertices of type $\varnothing$ to the pre-subdivision vertices of Ξ. So in this particular case the set of isometry classes of links does indeed uniquely determine the polyhedral complex Σ_1.

We may think of Σ as a "$(3,2)$-biregular tree of copies of Σ_1" glued at the purple vertices. If we wish to construct a simply connected polyhedral complex with the same set of isometry classes of links as Σ, we in fact have no choice in how to glue together the copies of Σ_1, for precisely the same reason that there exists a unique $(3,2)$-biregular tree. So Σ is uniquely determined by its set of isometry classes of links, and hence the universal cover of each complex of groups in the family $\{\mathcal{X}_k^{(0)}\}_{k\in\mathbb{N}} \cup \{\mathcal{X}^{(0)}\}$ is isomorphic to Σ as a polyhedral complex. $\square$

Having satisfied all the conditions required to apply Corollary 3.27 to any $\mathcal{X}_k^{(0)}$, we deduce that $\pi_1(\mathcal{X}_k^{(0)})$ is a uniform lattice in $\mathrm{Aut}(\Sigma)$ with covolume

$$\begin{aligned}\mu\big(\,\mathrm{Aut}(\Sigma)/\pi_1(\mathcal{X}_k^{(0)})\big) &= \sum_{v\in V(\mathcal{X}_k^{(0)})} \frac{1}{|(\mathcal{X}_k)_v^{(0)}|} \\ &= \left(1+\frac{2x}{2}+\frac{4x}{2m}\right)\left(\sum_{i=0}^{k}\frac{1}{2^i}\right)+\frac{x}{m}-\frac{x}{2^k\cdot(2m)} \\ &\longrightarrow 2+2x+5x/m \quad \text{as} \quad k\longrightarrow\infty\end{aligned}$$

with a rate of convergence of $\frac{1}{2}$.

Observe that the limit point found above may be realised as the covolume of the uniform lattice $\pi_1(\mathcal{X}^{(0)})$ in $\mathrm{Aut}(\Sigma)$; see Figure 6.

Case. 2: $l > 0$

Recall that the nerve L contains strictly more than four connected components in this case. We construct the relevant family of complexes of groups inductively, beginning with the family $\{\mathcal{X}_k^{(0)}\}_{k\in\mathbb{N}} \cup \{\mathcal{X}^{(0)}\}$ in Figures 5 and 6. Let $L^{(0)}$ and $\Sigma^{(0)}$ be the nerve and Davis complex respectively of (W, S) in the $l = 0$ case.

Consider the subgroup W_1 of (W, S) with corresponding nerve L_1 satisfying either Condition 8.10 or 8.11 below. Denote the Davis complex of W_1 by Σ_1. As previously, we abuse notation and interpret Σ_1 as a complex of groups with all vertex groups trivial over itself as a scwol. The *opposite complex of groups* Σ_1^{op} is obtained by reversing the direction of all the arrows in Σ_1. Assign the colour purple to all vertices of type $\varnothing$ in Σ_1 (and Σ_1^{op}). Recall from Definition 8.6 the colour scheme of the complex of groups $\mathcal{P}^{m_1,x_1}$.

Let $L^{(1)} := L^{(0)} \sqcup L_1$ have corresponding Davis complex $\Sigma^{(1)}$, where $\sqcup$ denotes disjoint union. We show how to construct a family $\{\mathcal{X}_k^{(1)}\}_{k\in\mathbb{N}}$ of complexes of groups with sums of reciprocals of the orders of the vertex groups converging to that of a complex of groups $\mathcal{X}^{(1)}$, all with universal cover isomorphic to $\Sigma^{(1)}$ as a polyhedral complex.

Condition 8.10. W_1 is a spherical subgroup of (W, S).

Let c_1 be the lowest common multiple of the integers $|W_1|$ and $2x$. For each $i \in \{0, 1, ..., k\}$ (resp. $j \in \{0, 1\}$) let A_i (resp. A_j) be the set of all c_1 purple vertices on the i'th (resp. j'th) platforms of $c_1/(2x)$ copies of $\mathcal{X}_k^{(0)}$ (resp. $\mathcal{X}^{(0)}$), let B_i (resp. B_j) be the set of all c_1 purple vertices of $c_1/|W_1|$ copies of $\mathcal{C}_2^i \times \Sigma_1^{op}$ (resp. Σ_1^{op}) and let $\phi_i : A_i \to B_i$ (resp. $\phi_j : A_j \to B_j$) be any choice of bijection. Fix an integer k. We construct a complex of groups $\mathcal{X}_k^{(1)}$ (resp. $\mathcal{X}^{(1)}$) by first taking the disjoint union of $c_1/(2x)$ copies of $\mathcal{X}_k^{(0)}$ (resp. $\mathcal{X}^{(0)}$) with $c_1/|W_1|$ copies of $\sqcup_{i\in\{0,1,\dots,k\}}\mathcal{C}_2^i \times \Sigma_1^{op}$ (resp. $\sqcup_{j\in\{0,1\}}\Sigma_1^{op}$), and then identifying pairs of purple vertices under each bijection in the family $\{\phi_i \mid i \in \{0,1,...,k\}\}$ (resp. $\{\phi_j \mid j \in \{0,1\}\}$).

Condition 8.11. L_1 is the $2m_1$-cycle graph with edges labelled by a constant integer $x_1 \geq 2$.

Let c_1 be the lowest common multiple of the integers $2x_1$ and $2x$. For each $i \in \{0,1,...,k\}$ (resp. $j \in \{0,1\}$) recall that A_i (resp. A_j) is the set of all c_1 purple vertices on the i'th (resp. j'th) platforms of $c_1/(2x)$ copies of $\mathcal{X}_k^{(0)}$ (resp. $\mathcal{X}^{(0)}$), let C_i (resp. C_j) be the set of all c_1 purple vertices of $c_1/(2x_1)$ copies of $\mathcal{P}_i^{m_1,x_1} = \mathcal{C}_2^i \times \mathcal{P}^{m_1,x_1}$ (resp. $\mathcal{P}_j^{m_1,x_1} = \mathcal{P}^{m_1,x_1}$) and let $\psi_i : A_i \to C_i$ (resp. $\psi_j : A_j \to C_j$) be any choice of bijection. Fix an integer k. We construct a complex of groups $\mathcal{X}_k^{(1)}$ (resp. $\mathcal{X}^{(1)}$) by first taking the disjoint union of $c_1/(2x)$ copies of $\mathcal{X}_k^{(0)}$ (resp. $\mathcal{X}^{(0)}$) with $c_1/(2x_1)$ copies of $\sqcup_{i\in\{0,1,\dots,k\}}\mathcal{P}_i^{m_1,x_1}$ (resp. $\sqcup_{j\in\{0,1\}}\mathcal{P}_j^{m_1,x_1}$), and then identifying pairs of purple vertices under each bijection in the family $\{\psi_i \mid i \in \{0,1,...,k\}\}$ (resp. $\{\psi_j \mid j \in \{0,1\}\}$).

Proposition 8.12. Each complex of groups in the family $\{\mathcal{X}_k^{(1)}\}_{k\in\mathbb{N}}\cup\{\mathcal{X}^{(1)}\}$ has a set of isometry classes of links identical to that of the Davis complex $\Sigma^{(1)}$.

Proof. We use a similar argument as in Proposition 8.7. Let $\mathcal{F}$ be any complex of groups in the family $\{\mathcal{X}_k^{(0)}\}_{k\in\mathbb{N}} \cup \{\mathcal{X}^{(0)}\}$ and let $\mathcal{G}$ be any complex of groups in the family $\{\mathcal{X}_k^{(1)}\}_{k\in\mathbb{N}} \cup \{\mathcal{X}^{(1)}\}$.

Assume Condition 8.10 (resp. Condition 8.11) holds. It is not difficult to see that the links of the green, blue and red vertices in $\mathcal{G}$ are isometric to those of the corresponding coloured vertices in $\mathcal{F}$, and it follows from Case IV of Example 6.5 that the link of any purple vertex in $\mathcal{G}$ is the disjoint union of the link of any purple vertex in $\mathcal{F}$ with the link of any purple vertex in Σ_1 (resp. $\mathcal{P}^{m_1,x_1}$). Finally, the links of the remaining vertices in $\mathcal{G}$ are isometric to the links of the corresponding (non-specified but non-purple) coloured vertices in Σ_1 (resp. $\mathcal{P}^{m_1,x_1}$). □

Corollary 8.13. Each complex of groups in the family $\{\mathcal{X}_k^{(1)}\}_{k\in\mathbb{N}} \cup \{\mathcal{X}^{(1)}\}$ is strictly developable.

Proof. The result follows from Proposition 8.12 and Theorem 6.13, since each link of $\Sigma^{(1)}$ is $\mathrm{CAT}(0)$ under Moussong's metric. □

Proposition 8.14. Each complex of groups in the family $\{\mathcal{X}_k^{(1)}\}_{k\in\mathbb{N}}\cup\{\mathcal{X}^{(1)}\}$ has a universal cover that is isomorphic to $\Sigma^{(1)}$ as a polyhedral complex.

Proof. Recall from Definition 8.6 that the complex of groups $\mathcal{P}^{m_1,x_1}$ is strictly developable. Interpret the universal cover Σ^{m_1,x_1} of $\mathcal{P}^{m_1,x_1}$ as a complex of groups with all vertex groups trivial over itself as a scwol.

Recall from Proposition 8.9 that $\Sigma^{(0)}$ is uniquely determined by its set of isometry classes of links. In fact, this is also true of Σ^{m_1,x_1} and the finite Davis complex Σ_1. For Σ^{m_1,x_1} it follows from the proof of Proposition 8.9 and for Σ_1 it follows from the fact that any finite Coxeter group naturally acts on a sphere of appropriate dimension, inducing a tesselation by spherical simplicies, with corresponding Davis complex the cone on this tesselation.

Assume that Condition 8.10 (resp. Condition 8.11) holds. Observe that $\Sigma^{(1)}$ can be thought of as a "$(\Sigma^{(0)},\Sigma_1)$-biregular tree" (resp. "$(\Sigma^{(0)},\Sigma^{m_1,x_1})$-biregular tree") glued together at the purple vertices. By a similar argument as in the proof of Proposition 8.9, in the construction of $\Sigma^{(1)}$ we have no choice in how to glue together the copies of $\Sigma^{(0)}$ and Σ_1 (resp. Σ^{m_1,x_1}) since $\Sigma^{(1)}$ is simply connected. □

We complete the proof of Theorem 8.5 by iterating the above construction for the sequence of subgroups $(W_1,W_2,\dots,W_l)$ of (W,S) in order to obtain a family $\{\mathcal{X}_k := \mathcal{X}_k^{(l)}\}_{k\in\mathbb{N}}$ of complexes of groups with sums of reciprocals of the orders of the vertex groups converging to that of a complex of groups $\mathcal{X} := \mathcal{X}^{(l)}$, all with universal cover isomorphic to $\Sigma=\Sigma^{(l)}$ as a polyhedral complex. The construction of $\mathcal{X}_k^{(r)}$ (resp. $\mathcal{X}^{(r)}$) from $\mathcal{X}_k^{(r-1)}$ (resp. $\mathcal{X}^{(r-1)}$) for any integer $1<r\le l$ is the same as the construction of $\mathcal{X}_k^{(1)}$ (resp. $\mathcal{X}^{(1)}$) from $\mathcal{X}_k^{(0)}$ (resp. $\mathcal{X}^{(0)}$) except that we take c_r to be the lowest common multiple of the integers $|W_r|$ and c_{r-1} if Condition 8.10 holds, or alternatively the lowest common multiple of the integers $2x_r$ and c_{r-1} if Condition 8.11 holds. □

By modifying the family $\{\mathcal{X}_k\}_{k\in\mathbb{N}}$ of complexes of groups described in the proof of Theorem 8.5, we arrive at the following result.

Corollary 8.15. Let (W,S) be a Coxeter system with $j\ge 3$ free generators as in the statement of Theorem 8.5. Let Σ be the Davis complex of (W,S) and let $p<j$ be prime. For any (arbitrarily large) $\alpha\in\mathbb{N}$ there exists a uniform lattice Γ in $\mathrm{Aut}(\Sigma)$ with covolume $\mu\big(\mathrm{Aut}(\Sigma)/\Gamma\big)=a/b$ (in lowest terms) such that b is divisible by p^α.

Proof. For any $k \in \mathbb{N}$, construct a complex of groups ${\mathcal{X}'}_k^{(0)}$ by replacing the i'th platform in $\mathcal{X}_k^{(0)}$ with $\mathcal{C}_p^i \times \mathcal{P}^{m,x}$ for each $i \in \{1, 2, ..., k\}$ and adjusting each green vertex group immediately above platform i from $\mathcal{C}_2^i \times \mathcal{D}_{2m}$ to $\mathcal{C}_p^i \times \mathcal{D}_{2m}$ accordingly. All else is left the same. We then construct a complex of groups ${\mathcal{X}'}_k$ from ${\mathcal{X}'}_k^{(0)}$ by exactly the same iterative process as we obtained $\mathcal{X}_k$ from $\mathcal{X}_k^{(0)}$ in the proof of Theorem 8.5 (with the only difference being that there are now $l - p + 2$ iterations rather than l).

The fundamental group of ${\mathcal{X}'}_k$ is a uniform lattice in Σ by the same argument as in the proof of Theorem 8.5. A simple calculation will show that the corresponding sequence of covolumes converges at a rate of $\frac{1}{p}$ as $k \to \infty$. The sum of reciprocals of orders of the vertex groups (in lowest terms) will therefore have a denominator that is divisible by ever-increasing powers of p as k gets larger. $\square$

We conclude this section with some observations and open questions.

We could define an infinite complex of finite groups $\mathcal{X}_\infty$ by taking the obvious limit of the family $\{\mathcal{X}_k\}_{k\in\mathbb{N}}$ as $k \to \infty$. This would give us a family $\{\pi_1(\mathcal{X}_k)\}_{k\in\mathbb{N}}$ of uniform lattices in $\mathrm{Aut}(\Sigma)$ with a set of covolumes converging to that of the (necessarily) non-uniform lattice $\pi_1(\mathcal{X}_\infty)$. In fact, this is what Thomas proved for a different subclass of Davis complexes in [**22**].

In the proof of Theorem 8.5, observe that the limit point

$$\lim_{k\to\infty} \mu\big(\mathrm{Aut}(\Sigma)/\pi_1(\mathcal{X}_k)\big) = \mu\big(\mathrm{Aut}(\Sigma)/\pi_1(\mathcal{X}_\infty)\big)$$

is a rational number. One wonders if it would be possible to find a similarly convergent family of complexes of groups $\{\mathcal{Y}_k\}_{k\in\mathbb{N}} \cup \{\mathcal{Y}_\infty\}$, sharing all the same properties of $\{\mathcal{X}_k\}_{k\in\mathbb{N}} \cup \{\mathcal{X}_\infty\}$, except that the limit point $\mu\big(\mathrm{Aut}(\Sigma)/\pi_1(\mathcal{Y}_\infty)\big)$ is irrational? Non-uniform lattices with irrational covolumes have been studied by Farb and Hruska [**10**] in the case of trees.

Now recall the following definition.

Definition 8.16. An *end* of a graph is an equivalence class of embedded rays under the equivalence relation that two rays r_1 and r_2 are equivalent iff there exists a ray r_3 that intersects with each of r_1 and r_2 at infinitely many vertices. An *end* of a finitely generated group with a given generating set is an end of its Cayley graph.

It is well-known (Theorem 8.32 of [**4**]) that the number of ends of an infinite finitely generated group is a quasi-isometric invariant, hence independent of the choice of generating set, which can take only the values 1, 2 or ∞.

Theorem $8.7.4.$ of [**8**] characterises the number of ends of a Coxeter group. Precisely, an infinite Coxeter system (W, S) has exactly 1 end iff the *punctured nerve* $L \setminus \sigma_T$ is connected for each spherical subset T of S, where σ_T denotes the simplex associated to T in the nerve. (W, S) has exactly 2 ends iff there exists a decomposition $(W, S) = (W_0 \times W_1, S_0 \cup S_1)$ where W_0 is the infinite dihedral group and W_1 is a finite Coxeter subgroup of W.

Now let (W, S) be a Coxeter system as in either Proposition 8.3 or Theorem 8.5. Recall that S has $j \geq 3$ free generators $\{s_1, ..., s_j\}$. Assume that we can decompose $(W, S) = (W_0 \times W_1, S_0 \cup S_1)$, where W_1 is finite and W_0 is (without loss of generality) the infinite dihedral group $\langle s_1, s_2\rangle$. Then s_3 must commute with both s_1 and s_2 in (W, S), a contradiction. Moreover observe that the punctured nerve $L \setminus \sigma_{\{s_1\}}$ is disconnected. Hence (W, S) has infinitely many ends. This raises the question: *does there exist a Coxeter system with* 1 *end that has a non-discrete set of covolumes of uniform lattices acting on its Davis complex?*

8.3. Σ is a regular right-angled building

In this section we prove case (iii) of Theorem 8.1 and Corollary 8.2.

Recall that $S := \{s_1, ..., s_n\}$.

Theorem 8.17. Let (W, S) be a right-angled Coxeter system with (at least) two free generators, which we call s_1 and s_2. Let Σ be the unique regular right-angled building of type (W, S) with $p_i \geq 2$ chambers in each s_i-equivalence class, for all $i \in \{1, ..., n\}$. Then the set of covolumes of uniform lattices in $\mathrm{Aut}(\Sigma)$ is non-discrete if either p_1 or p_2 is strictly greater than 2.

Proof. Without loss of generality assume $p_1 \geq p_2$. The structure of the proof is as follows. We consider two cases, $p_1 > p_2 = 2$ and $p_1 \geq p_2 > 2$. If $p_1 > p_2 = 2$ (resp. $p_1 \geq p_2 > 2$) we construct a family $\{\mathcal{A}_k\}_{k\in\mathbb{N}} \cup \{\mathcal{A}\}$ (resp. $\{\mathcal{B}_k\}_{k\in\mathbb{N}} \cup \{\mathcal{B}\}$) of complexes of groups. For any $\mathcal{J}$ in either family, we show that the set of isometry classes of links of $\mathcal{J}$ is the same as that of Σ (Proposition 8.18), that $\mathcal{J}$ is strictly developable (Corollary 8.19) and that the universal cover of $\mathcal{J}$ is Σ (Proposition 8.20). We then apply Corollary 3.27 to show that $\pi_1(\mathcal{J})$ is a uniform lattice in $\mathrm{Aut}(\Sigma)$ and compute the corresponding set of covolumes to establish the result.

As previously, we abuse notation and interpret Σ as a complex of groups with all vertex groups trivial over itself as a scwol. We colour Σ by assigning the colour red to all vertices of type s_1, green to all vertices of type s_2, purple to all vertices of type $\varnothing$ and blue to all remaining vertices.

Assume $n > 2$. Let S' be the subset $\{s_3, ..., s_n\}$ of S. Let K' denote the chamber of $(W_{S'}, S')$ and let Σ' be the unique regular right-angled building of type $(W_{S'}, S')$ with $p_i \geq 2$ chambers in each s_i-equivalence class, for all $i \in \{3, ..., n\}$. In the sense of Example 7.8, let $\widehat{K'}$ be the chamber complex of $(W_{S'}, S')$ with integers $\{p_3, ..., p_n\}$. That is, $\widehat{K'}$ is a complex of groups over the scwol K' with local groups that are direct products of the cyclic groups $\mathcal{C}_{p_i}$ for $i \in \{3, ..., n\}$. If $n = 2$ we take $\widehat{K'}$ to be the complex of groups consisting only of one vertex with trivial local group. Recall from Example 7.8 that the universal cover of $\widehat{K'}$ is isomorphic to Σ' as a polyhedral complex. We colour $\widehat{K'}$ completely blue except for its unique vertex of type $\varnothing$ which we colour purple. This is consistent with the colouring of Σ.

In Figures 8, 9, 10 and 11 we construct a family $\cup_{k\in\mathbb{N}}\{\mathcal{A}_k, \mathcal{B}_k\} \cup \{\mathcal{A}, \mathcal{B}\}$ of complexes of groups that each include multiple embedded copies of $\mathcal{C}^i_{p-1} \times \widehat{K'}$ for some i in $\{0, 1, ..., k\}$. We colour such embedded copies of "multiples" of $\widehat{K'}$ accordingly, that is, all vertices are blue other than those of type $\varnothing$ which we colour purple. We also use the colour red (resp. green) to denote vertices of type s_1 (resp. s_2), that is, with pre-image in the universal cover of degree p_1 (resp. p_2).

A quick remark on how to interpret Figure 11. For any integer k, the complex of groups $\mathcal{B}_k$ consists of "multiples" of copies of an embedded "block" that are stacked below one another, glued together via additional green vertices with local groups that are even powers of $\mathcal{C}_{p_2-1}$. Each embedded block is a $\mathcal{C}^2_{p_2-1}$-multiple of the block immediately above it. We draw two such blocks in Figure 11 - the "base" case and the $\mathcal{C}^{2k}_{p_2-1}$-multiple.

Let Ω denote the sum of reciprocals of orders of the vertex groups in $\widehat{K'}$.

Proposition 8.18. Assuming $p_1 > p_2 = 2$ (resp. $p_1 \geq p_2 > 2$), then each complex of groups in the family $\{\mathcal{A}_k\}_{k\in\mathbb{N}} \cup \{\mathcal{A}\}$ (resp. $\{\mathcal{B}_k\}_{k\in\mathbb{N}} \cup \{\mathcal{B}\}$) has a set of isometry classes of links identical to that of the building Σ.

Proof. We use a similar argument as in Proposition 8.7. Let $\mathcal{J}$ be any complex of groups in the family $\cup_{k\in\mathbb{N}}\{\mathcal{A}_k, \mathcal{B}_k\} \cup \{\mathcal{A}, \mathcal{B}\}$.

It follows from Cases III and IV of Example 6.5 that the link of each red (resp. green) vertex in $\mathcal{J}$ is p_1 (resp. p_2) disconnected purple vertices and that the link of each purple vertex in $\mathcal{J}$ is the disjoint union of the link of the unique purple vertex of type $\varnothing$ in $\widehat{K'}$ with one lone red vertex and one lone green vertex. The link of each of the remaining (blue) vertices in $\mathcal{J}$ is isometric to the link of any vertex of the same type in $\widehat{K'}$.

It is clear from the construction of the building Σ, and the fact that the chamber complex $\widehat{K'}$ has universal cover Σ', that the link of each vertex in $\mathcal{J}$ is isometric to the link of any vertex of the same type in Σ. □

Corollary 8.19. Each complex of groups in the family $\cup_{k\in\mathbb{N}}\{\mathcal{A}_k, \mathcal{B}_k\} \cup \{\mathcal{A}, \mathcal{B}\}$ is strictly developable.

Proof. The result follows from Proposition 8.18 and Theorem 6.13, since each link of Σ is $\mathrm{CAT}(0)$ under the restriction of the metric in Theorem 7.5. □

Proposition 8.20. Assuming $p_1 > p_2 = 2$ (resp. $p_1 \geq p_2 > 2$), then each complex of groups in the family $\{\mathcal{A}_k\}_{k\in\mathbb{N}} \cup \{\mathcal{A}\}$ (resp. $\{\mathcal{B}_k\}_{k\in\mathbb{N}} \cup \{\mathcal{B}\}$) has a universal cover that is isomorphic to Σ as a polyhedral complex.

Proof. Recall from Theorem 7.7 that there exists a unique regular right-angled building Σ such that for all $i \in \{1, ..., n\}$ each s_i-equivalence class contains p_i chambers. Hence Σ is uniquely determined by its set of isometry classes of links. The result then follows from Proposition 8.18 and Corollary 8.19. □

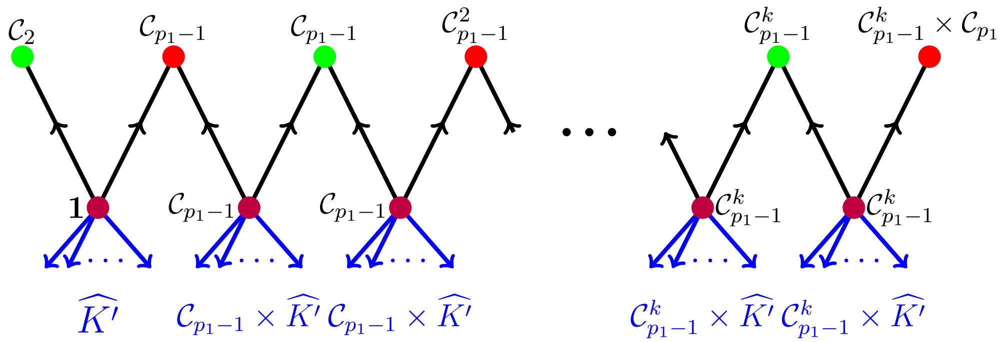

FIGURE 8. A family $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ of finite complexes of finite groups (constructed in the proof of Corollary 8.21 as $\{\mathcal{A}_k^{p_1-1}\}_{k\in\mathbb{N}}$ when $p_2 = 2$)

<u>Case. 1</u>: $p_1 > p_2 = 2$

Having satisfied all the conditions required to apply Corollary 3.27 to $\mathcal{A}_k$ for any $k \in \mathbb{N}$, therefore $\pi_1(\mathcal{A}_k)$ is a uniform lattice in $\mathrm{Aut}(\Sigma)$ with covolume

$$\mu\big(\mathrm{Aut}(\Sigma)/\pi_1(\mathcal{A}_k)\big) = \sum_{v\in V(\mathcal{A}_k)} \frac{1}{|(\mathcal{A}_k)_v|}$$

$$= \left(1 + \sum_{i=1}^{k} \frac{2}{(p_1-1)^i}\right)\Omega + \frac{1}{2} + \sum_{i=1}^{k} \frac{2}{(p_1-1)^i} + \frac{1}{p_1(p_1-1)^k}$$

$$\longrightarrow \left(1 + \frac{2}{p_1-2}\right)\Omega + \frac{1}{2} + \frac{2}{p_1-2} \quad \text{as} \quad k \longrightarrow \infty$$

with a rate of convergence of $\frac{1}{p_1-1}$. This limit point may be realised as the covolume of the uniform lattice $\pi_1(\mathcal{A})$ in $\mathrm{Aut}(\Sigma)$; see Figure 9.

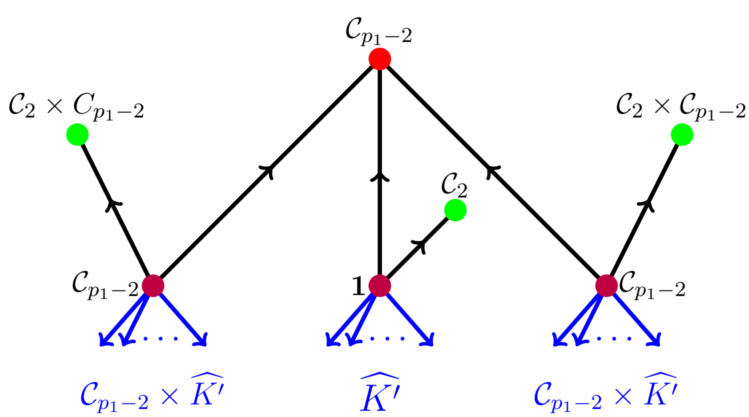

FIGURE 9. A finite complex of finite groups $\mathcal{A}$

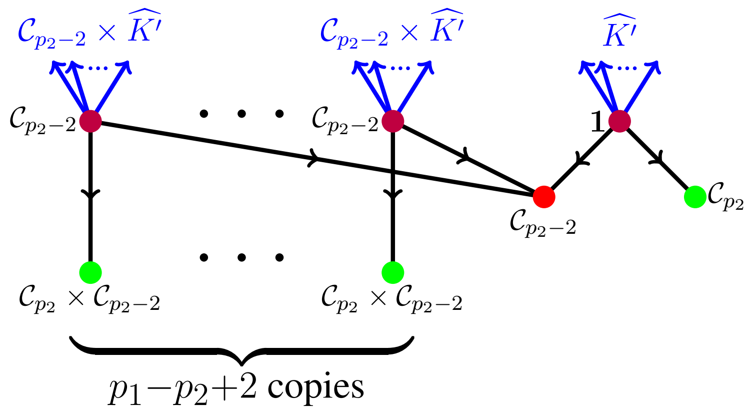

FIGURE 10. A finite complex of finite groups $\mathcal{B}$

Case. 2: $p_1 \geq p_2 > 2$

Observe that the limit point found above may be realised as the covolume of the uniform lattice $\pi_1(\mathcal{B})$ in $\mathrm{Aut}(\Sigma)$; see Figure 10.

Use the Euclidean algorithm to write $p_1 = (p_2 - 1)q + r$ for some non-negative integers q and r.

Consider the family $\{\mathcal{B}_k\}_{k\in\mathbb{N}} \cup \{\mathcal{B}\}$ of complexes of groups shown in Figures 10 and 11. Having satisfied all the conditions required to apply Corollary 3.27 to $\mathcal{B}_k$ for any $k \in \mathbb{N}$, we deduce that $\pi_1(\mathcal{B}_k)$ is a uniform lattice in $\mathrm{Aut}(\Sigma)$ with covolume

$$\mu\big(\mathrm{Aut}(\Sigma)/\pi_1(\mathcal{B}_k)\big) = \sum_{v\in V(\mathcal{B}_k)} \frac{1}{|(\mathcal{B}_k)_v|}$$

$$= \left[\left(q + \frac{r+q}{p_2-1} + \frac{r}{(p_2-1)^2}\right)\Omega + 1 + \frac{q}{p_2-1} + \frac{r+1}{(p_2-1)^2}\right]\left(\sum_{i=0}^{k} \frac{1}{(p_2-1)^{2i}}\right)$$
$$+ \frac{1}{p_2} - 1 - \frac{1}{p_2(p_2-1)^{2k+1}}$$
$$\longrightarrow \frac{p_1(p_2-1+1)\Omega + (p_2-1)^2 + p_1 + 1}{(p_2-1)^2 - 1} + \frac{1}{p_2} - 1 \quad \text{as} \quad k \longrightarrow \infty$$
$$= \frac{p_1 p_2 \Omega + p_1 + p_2}{p_2(p_2-2)}$$

with a rate of convergence of $\frac{1}{p_2-1}$. □

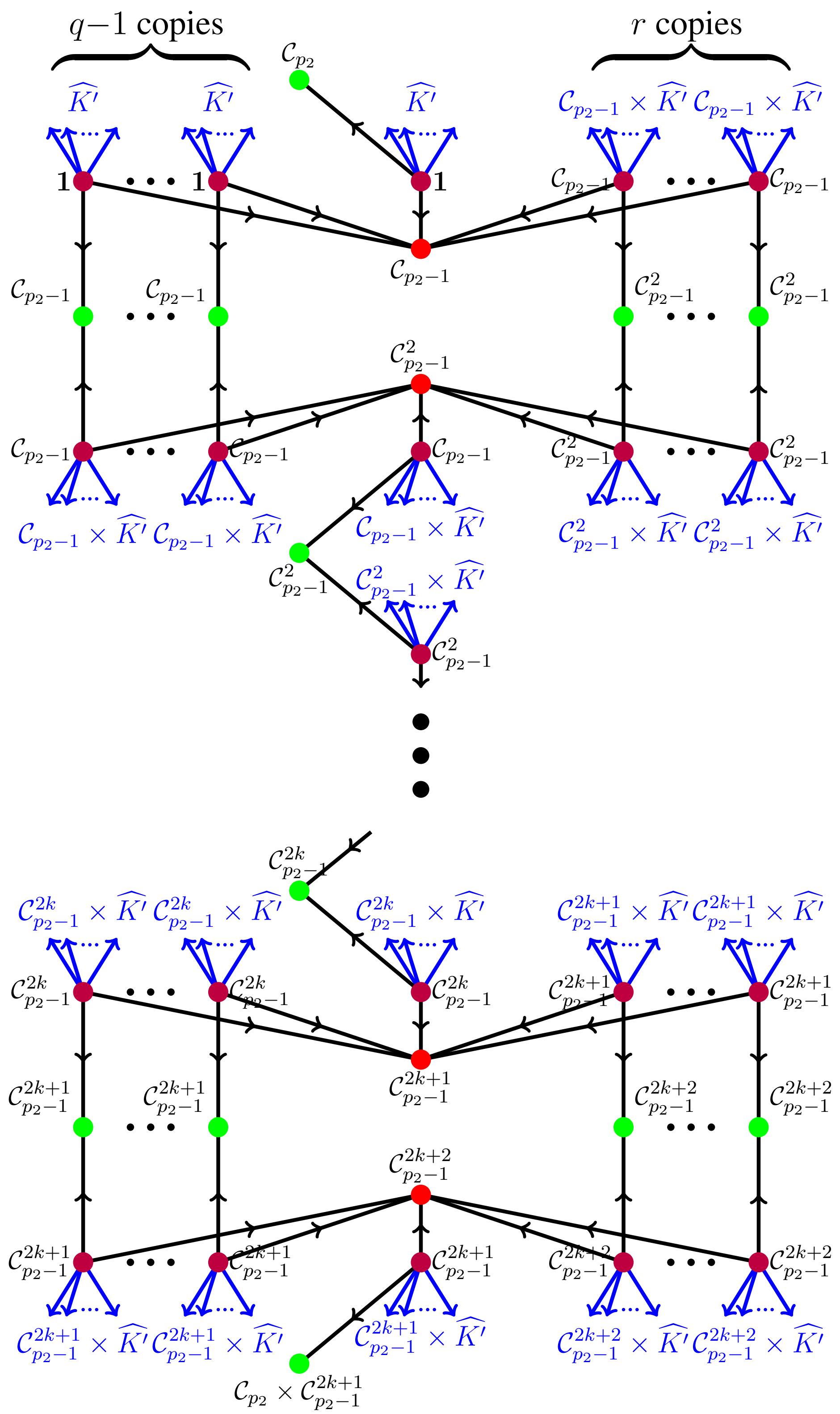

FIGURE 11. A family $\{\mathcal{B}_k\}_{k\in\mathbb{N}}$ of finite complexes of finite groups

Generalising the family $\{\mathcal{A}_k\}_{k\in\mathbb{N}}$ of complexes of groups in Figure 8 gives us a new proof of the following (already known) result. Our argument is similar to the proof of Corollary 8.4.

Corollary 8.21. Let (W, S) be a right-angled Coxeter system with free generators $\{s_1, ..., s_j\}$ for $j \geq 2$. Let Σ be the regular right-angled building of type (W, S) as in Theorem 8.17 and let $p < \max_{1\leq i\leq j}\{p_i\}$ be prime. For any $\alpha \in \mathbb{N}$ there exists a uniform lattice Γ in $\mathrm{Aut}(\Sigma)$ with covolume $\mu\big(\mathrm{Aut}(\Sigma)/\Gamma\big) = a/b$ (in lowest terms) such that b is divisible by p^α.

Proof. Without loss of generality let $p_1 = \max_{1\leq i\leq j}\{p_i\}$. We call a set of edges $1-1$ if they have no pairwise shared vertices and they form a bijection between two given sets of vertices.

Take any $k \in \mathbb{N}$. We first construct a complex of groups $\mathcal{H}_k^p$ with underlying scwol a tree with blue, red and purple vertices. Figure 12 depicts the case where $p_1 - p = 3$ and $k = 2$. No vertex is adjacent to another of the same colour. Edges always point away from purple vertices. Partition $\mathcal{H}_k^p$ into descending "levels" from $i = 0$ to $i = k + 1$.

Level 0 consists of a single green vertex with local group $\mathcal{C}_{p_2}$ and a single edge from the lone purple vertex with trivial vertex group in level 1 to the lone green vertex in level 0. Level $k+1$ consists of $1-1$ edges from $(p_1-p)^k$ purple vertices each with local group $\mathcal{C}_p^k \times \mathcal{C}_{p_2-1}^k$ to the same number of red vertices each with local group $\mathcal{C}_p^k \times \mathcal{C}_{p_2-1}^k \times \mathcal{C}_{p_1}$.

Every remaining level i for $i \in \{1, ..., k\}$ consists of:

- $(p_1-p)^{i-1}$ purple vertices each with local group $\mathcal{C}_p^{i-1}\times\mathcal{C}_{p_2-1}^{i-1}$, which we say are of *type* I;
- $(p_1-p)^i$ purple vertices each with local group $\mathcal{C}_p^i\times\mathcal{C}_{p_2-1}^{i-1}$, which we say are of *type* II;
- $(p_1-p)^{i-1}$ red vertices each with local group $\mathcal{C}_p^i \times \mathcal{C}_{p_2-1}^{i-1}$;
- $(p_1-p)^i$ green vertices each with local group $\mathcal{C}_p^i \times \mathcal{C}_{p_2-1}^i$;
- $1-1$ edges from each type I purple vertex to each red vertex;
- $1-1$ edges from each type II purple vertex to each green vertex;
- edges to each red vertex from (disjoint sets of) $p_1 - p$ type II purple vertices; and
- $1-1$ edges from each type I purple vertex in level $i+1$ to each green vertex in level i.

Recall the definition of the chamber complex $\widehat{K'}$ from the proof of Theorem 8.17. We construct $\mathcal{A}_k^p$ from $\mathcal{H}_k^p$ by gluing a multiple of $\widehat{K'}$ to each purple vertex, as in Figure 8, which depicts the case where $p_1 - p = 1$, $p_2 = 2$ and k is arbitrary.

The fundamental group of each complex of groups in $\{\mathcal{A}_k^p\}_{k\in\mathbb{N}}$ is a uniform lattice in Σ by the same argument as in the proof of Theorem 8.17. A simple calculation will show that the corresponding sequence of covolumes converges at a rate of $\frac{1}{p(p_2-1)}$ as $k \to \infty$. The sum of reciprocals of orders of the vertex groups (in lowest terms) will therefore have a denominator that is divisible by ever-increasing powers of p as k gets larger. $\square$

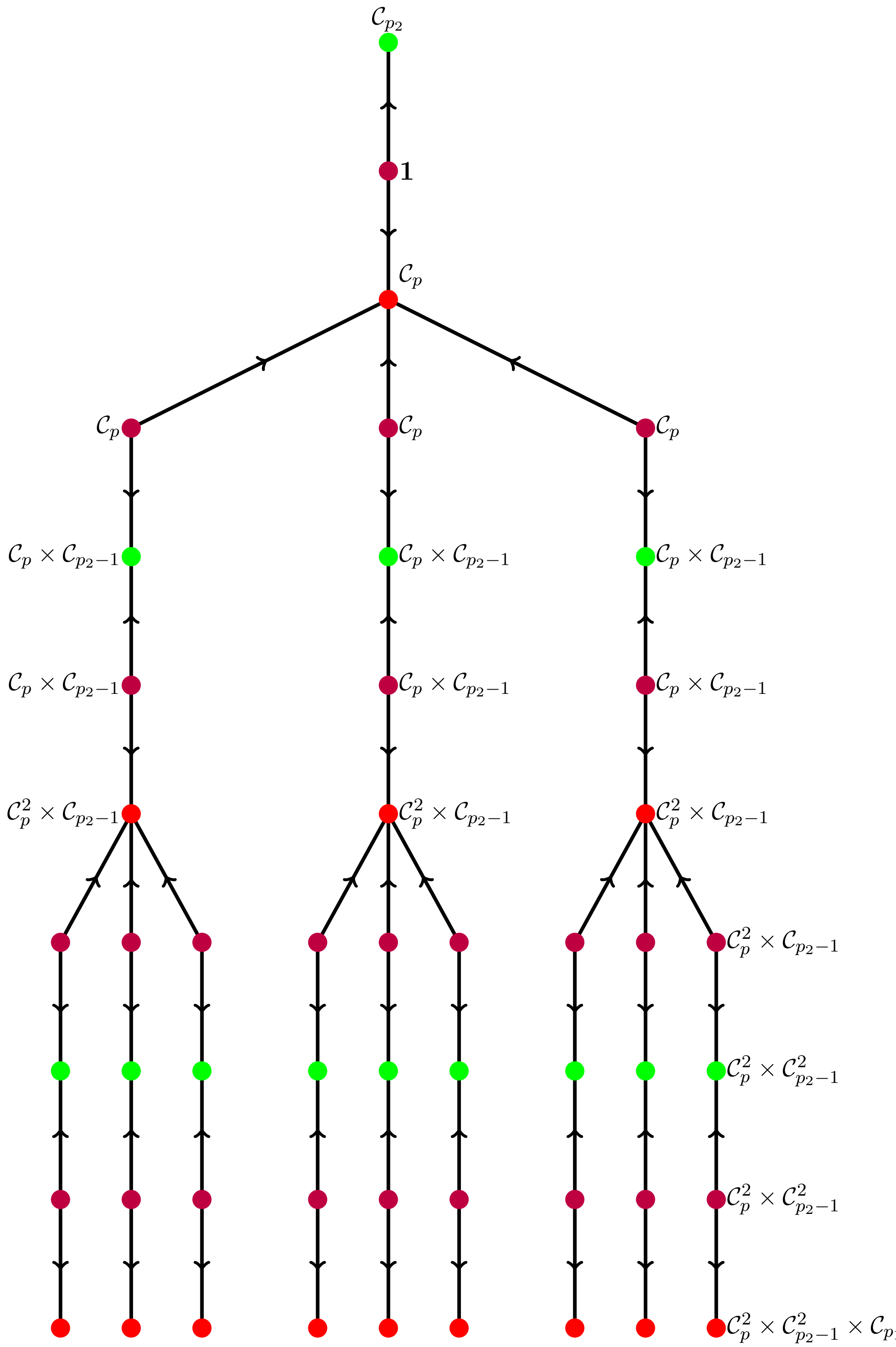

FIGURE 12. The complex of groups $\mathcal{H}_2^p$ with $p_1 - p = 3$ (vertices in any horizontal line have the same vertex groups, but due to space constraints we do not label them all)

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