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On arithmetic functions of balancing and Lucas-balancing numbers

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Abstract. For any integers $n \geq 1$ and $k \geq 0$, let $\phi(n)$ and $\sigma_k(n)$ denote the Euler phi function and the sum of the k-th powers of the divisors of n, respectively. In this article, the solutions to some Diophantine equations about these functions of balancing and Lucas-balancing numbers are discussed.

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1. Introduction

A balancing number n and a balancer r are the solutions to a simple Diophantine equation $1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$ [1]. The sequence of balancing numbers $\{B_n\}$ satisfies the recurrence relation $B_{n+1} = 6B_n - B_{n-1}$, $n \geq 1$ with initials $(B_0, B_1) = (0, 1)$. The companion of $\{B_n\}$ is the sequence of Lucas-balancing numbers $\{C_n\}$ that satisfies the same recurrence relation as that of balancing numbers but with different initials $(C_0, C_1) = (1, 3)$ [4]. Further, the closed forms known as Binet formulas for both of these sequences are given by

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}, \quad C_n = \frac{\lambda_1^n + \lambda_2^n}{2},$$

where $\lambda_1 = 3 + 2\sqrt{2}$ and $\lambda_2 = 3 - 2\sqrt{2}$ are the roots of the equation $x^2 - 6x + 1 = 0$. Some more recent developments of balancing and Lucas-balancing numbers can be seen in [2, 3, 6, 8].

Let $\phi(n)$ and $\sigma_k(n)$ denote the Euler phi function and the divisor function of n, respectively. In this study, we examine the solutions to some Diophantine equations relating to these functions.

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2. Auxiliary results

In this section, we discuss some results which are used subsequently.

The following result is found in [5].

Lemma 1. *For any positive integer $k \geq 1$, the period of $\{B_n\}_{n\geq 0}$ modulo 2^k is 2^k.*

The following results are given in [7].

Lemma 2. *Let m and n be positive integers; then $\Big(B_m, B_n\Big) = B_{(m,n)}$, where (x, y) denotes the greatest common divisor of x and y.*

Lemma 3. *Let m and n be positive integers; then $B_m \big| B_n \Leftrightarrow m|n$.*

Lemma 4. *For any odd prime p, $B_{p-1} \equiv 3\big((\frac{p}{8})-1\big) \pmod p$ and $B_{p+1} \equiv 3\big((\frac{p}{8})+1\big) \pmod p$, where $(\frac{m}{n})$ denotes the Legendre symbol.*

Lemma 5. *Let p be a prime; then $p|B_{p-(\frac{8}{p})}$.*

We now discuss some congruence properties for balancing and Lucas-balancing numbers that are useful when proving subsequent results.

Lemma 6. *For any natural number $k \geq 2$, $B_{2^{k-1}} \equiv 2^{k-1} \cdot 3 \pmod{2^{k+1}}$.*

Proof. The method of induction is used to prove this result. For $k = 2$, the result is obvious. Assume that $B_{2^{k-1}} \equiv 3 \cdot 2^{k-1} \pmod{2^{k+1}}$ holds for $k \geq 3$. It follows that $B_{2^{k-1}} = 2^{k-1} \cdot 3u$, where $u \equiv 1 \pmod 4$. From the identity $B_{2m} = B_m(B_{m+1} - B_{m-1})$,

$$B_{2^k} = B_{2\cdot 2^{k-1}} = B_{2^{k-1}} \cdot (B_{2^{k-1}+1} - B_{2^{k-1}-1}) = 3 \cdot 2^{k-1}u \cdot 2v = 3 \cdot 2^k uv,$$

where $B_{2^{k-1}+1} - B_{2^{k-1}-1} = 2v$ for any positive integer v. This completes the proof. □

Lemma 7. *For any odd integer n and $k \geq 1$, if $B_n \equiv 1 \pmod{2^k}$, then $n \equiv 1 \pmod{2^k}$.*

Proof. Consider $B_n \equiv 1 \pmod{2^k}$. Then from Lemma 1, we can write $B_{2^k+1} \equiv 1 \pmod{2^k}$. It follows that $2^k + 1 \equiv 1 \pmod{2^k}$. Since $B_1 = 1$ and n is odd, $n \equiv 1 \pmod{2^k}$. □

Lemma 8. *For any positive integer $k \geq 2$, $C_{2^k} \equiv 1 \pmod{2^{k+4}}$.*

Proof. The proof of this result is analogous to Lemma 6. □

3. Main result

In this section, we prove our main result.

Theorem 1. *The following statements hold:*

(*i*) *The only solutions to the equation* $\phi(|B_n|) = 2^m$ *are obtained for* $n = \pm 1, \pm 2, \pm 4$.

(*ii*) *The only solutions to the equation* $\phi(|C_n|) = 2^m$ *are obtained for* $n = 0, \pm 1, \pm 2$.

(*iii*) *The only solutions to the equation* $\sigma(|B_n|) = 2^m$ *are obtained for* $n = \pm 1$.

(*iv*) *The only solutions to the equation* $\sigma(|C_n|) = 2^m$ *are obtained for* $n = 0, \pm 1$.

Proof. In order to prove (*i*), we first show that if $\phi(B_n) = 2^m$; then 2 is the only prime factor of n. Assume that the above statement does not hold, that is, there exists a prime $p > 2$ such that $p|n$ and hence using Lemma 3, $B_p|B_n$, it follows that

$$\phi(B_p)|\phi(B_n) = 2^m.$$

Therefore, $\phi(B_p) = 2^{m_1}$ for some $m_1 \leq m$, and it follows that

$$B_p = 2^t p_1 p_2 \cdots p_s, \text{ for t} \geq 1, \text{ p} \geq 1, \tag{1}$$

where $p_1, p_2, \cdots, p_s$ are distinct Fermat primes. Since B_p is co-prime to B_1 and B_2 for $p > 2$, then $2 \nmid B_p$ and $3 \nmid B_p$. This forces $t = 0$ and $p_1 > 3$ and hence $p_i > 3$ for all $i = 1, 2, \ldots, s$.

Let us consider $p_i = 2^{2^{e_i}} + 1$ for $e_i \geq 1$. Since $p_1 \equiv 5 \pmod 8$, from Lemma 4, it follows that p_1 divides $B_{p_1 - 1}$ and hence $p_1|(B_p, B_{p_1-1})$. Using Lemma 2, $p_1|B_{(p,\ p_1-1)}$. It follows that $p|p_1 - 1 = 2^{2^{e_1}}$ forces $p = 2$ and hence $e_1 = 0$, which is a contradiction.

Assume that $n = 2^u 3^v$. To show $u \leq 2$, assume that $u > 2$. Then

$$235416 = B_8|B_n,$$

which concludes $3|73728 = \phi(235416)|\phi(B_n) = 2^m$, which is a contradiction. To show that $v = 0$, assuming $v > 0$ we get $35 = B_3|B_n$; therefore

$$3|24 = \phi(35)|\phi(B_n) = 2^m,$$

which is again a contradiction. The above discussion concludes $n|2^2$ and the result follows.

In order to prove (*ii*), we proceed as follows. Since $\phi(1) = 1 = 2^0$, $\phi(3) = 2$ and $\phi(17) = 16 = 2^4$, it follows that the solution to (*ii*) are the elements from the set $n = \{0, 1, 2\}$. In order to prove identity (*ii*) completely, we need to show that these are the only solutions. If possible, let $\phi(C_n) = 2^m$ for $n \geq 3$; it follows that

$$C_n = 2^l p_1 \ldots p_k,$$

where $l \geq 0$ and $p_1 < \ldots < p_k$ are Fermat primes. Since $C_n^2 = 8B_n^2 + 1$, so C_n are odd, which forces l to be zero.

Now, write $p_i = 2^{2^{e_i}} + 1$, $i = 1, 2, \ldots, k$. For $n \geq 3$, $C_n \geq 99$ and hence $p_i > 3$. It can be observed that $p_i \equiv 5$ or $1 \pmod 8$. For $p_i \equiv 5 \pmod 8$, $\left(\frac{p_i}{8}\right) = -1 = \left(\frac{8}{p_i}\right)$. As $8B_n^2 = C_n^2 - 1 \equiv -1 \pmod{p_i}$, it follows that $\left(\frac{8}{p_i}\right) = 1$, a contradiction.

On the other hand, for $p_i \equiv 1 \pmod 8$, $\left(\frac{p_i}{8}\right) = 1 = \left(\frac{8}{p_i}\right)$. Again, the identity $8B_n^2 \equiv -1 \pmod{p_i}$ gives $\left(\frac{8}{p_i}\right) = -1$, also a contradiction and hence the result follows.

Further, clearly $n = 1$ is a solution for $\sigma(|B_n|) = 2^m$ as $\sigma(B_1) = 1$. Now it remains to show that there is no other solution except $n = 1$. If possible, let there exist a solution to $\sigma(B_n) = 2^m$ with $n \geq 2$. For $\sigma(B_n) = 2^m$, let $B_n = q_1 \dots q_k$, where $q_1 < q_2 < \dots < q_k$ are Mersenne primes. Let $q_i = 2^{p_i} - 1$ for $p_i \geq 2$. In particular, $q_i \equiv -1 \pmod 8$ and it follows that $\left(\frac{8}{q_i}\right) = \left(\frac{2}{q_i}\right)^3 = -1$. From Lemma 5, $B_{p-\left(\frac{8}{p}\right)} \equiv 0 \pmod p$, it follows that $q_i|B_{q_i+1}$ and hence $q_i|(B_n, B_{q_i+1})$. It is well known that if $q|n$, then $B_q|B_n$. Therefore, $q_i|(B_q, B_{q_i+1}) = B_{(q,q_i+1)}$ implies $q_i|q_i + 1 = 2^{p_i}$, again a contradiction and (iii) follows.

Since $\sigma(1) = 1 = 2^0, \sigma(3) = 4 = 2^2$ it follows that $n = 0, 1$ are the solutions to $\sigma(|C_n|) = 2^m$. To prove that these are the only ones, assume that there exists a solution with $n \geq 2$. For $\sigma(C_n) = 2^m$, let $C_n = q_1 \dots q_k$, where $q_1 < q_2 < \dots < q_k$ are Mersenne primes. Let $q_i = 2^{p_i} - 1$ for $p_i \geq 2$. Assume that $p_1 > 2$. Since $C_p^2 \equiv 9 \pmod{2^{p_1+1}}$, then

$$C_p^2 - 9 = 8(B_p^2 - 1) \equiv 0 \pmod{2^{p_1+1}},$$

which gives

$$B_p^2 \equiv 1 \pmod{2^{p_1+1}}.$$

It follows that $B_p \equiv \pm 1 \pmod{2^{p_1}}$ and hence using Lemma 7, $p \equiv 1 (\bmod\ 2^{p_1})$. In particular,

$$p \geq 2^{p_i} + 1. \tag{2}$$

Further, since $q_1|C_p$, then $8B_p^2 \equiv -1 \pmod{q_1}$, which implies that $q_1|B_{q_1-1}$. As $B_{2p} = 2B_pC_p$, $q_1|B_{2p}$. Therefore,

$$q_1|(B_{2p}, B_{q_1-1}) = B_{(2p,q_1-1)}.$$

For $p_1 > 2$, $q_1 > 3$ and hence $2p|q_1 - 1 = 2^{p_1} - 2 = 2(2^{p_1-1} - 1)$. In particular,

$$p \leq 2^{p_i-1} - 1, \tag{3}$$

which is a contradiction to (2). Thus, $p_i \leq 2$, which follows (iv). This completes the proof of the theorem. □

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