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Direct operational matrix approach for weakly singular Volterra integro-differential equations: application in theory of anomalous diffusion

Khosrow Maleknejad*, Raziyeh Dehbozorgi and Morteza Garshasbi

School of Mathematics, Iran University of Science and Technology, 16 844, Tehran, Iran

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Abstract. In the current paper, we present an efficient direct scheme for weakly singular Volterra integro-differential equations arising in the theory of anomalous diffusion. The behavior of the system demonstrating the anomalous diffusion is significant for small times. The method is based on operational matrices of Chebyshev and Legendre polynomials with some techniques to reduce the total errors of the already existing schemes. The proposed scheme converts these equations into a linear system of algebraic equations. The main advantages of the method are high accuracy, simplicity of performing, and low storage requirement. The main focus of this study is to obtain an analytical explicit expression to estimate the error. Numerical results confirm the superiority and applicability of our scheme in comparison with other methods in the literature.

AMS subject classifications: 65R20, 45J05, 45E10

Key words: Direct numerical scheme, weakly singular, Volterra integro-differential equations, operational matrix, anomalous diffusion, error estimation

1. Introduction

Today a huge variety of research activities deals with the diffusion phenomenon as one of the most important transport mechanisms found in nature. When a diffusion process is modeled at a microscopic level, it is related to the random motion of individual particles and the mean squared displacement is not asymptotically linear in time. In this situation, the diffusion process named anomalous diffusion has many practical applications. A large body of recent achievements has shown that anomalous diffusion models can offer a superior fit to experimental data [2, 12, 14]. For instance, one may found the applications of these models in the context of diffusion on fractal lattices, diffusion on the random network, materials with memory, e.g., viscoelastic materials and heterogeneous media such as a heterogeneous aquifer, soil and underground fluid flow [9, 10, 20]. The present study is devoted to the numerical solution of the one-dimensional generalized Langevin equation (GLE), which arises from mathematical modeling of anomalous diffusion. According to this

*Corresponding author. *Email addresses:* `Maleknejad@iust.ac.ir` (K. Maleknejad), `R_dehbozorgi@mathdep.iust.ac.ir` (R. Dehbozorgi), `M_garshasbi@iust.ac.ir` (M. Garshasbi)

equation, as discussed in [5], the time evolution of the velocity v(t) satisfies the following integro-differential equation

$$v'(t) = \frac{F(t)}{m} + \lambda \int_0^t K(t-\tau)v(\tau)d\tau, \tag{1}$$

where $K(t-\tau)$ is a dissipative memory kernel, m is a particle mass and $F(t)$ is a stochastic force which is zero centered and stationary and λ is a real scaling factor or a physical quantity.

According to [18], one of the important anomalous diffusive kernels is $K(t) = \frac{1}{\sqrt{t}}$. A number of numerical algorithms to approximate the solution of this kind of equations have been listed in [1, 13]. These algorithms include Mckee scheme, spline collocation method, Legendre approximation method, piecewise polynomial collocation method, jumping process by a factor to eliminate singularity at zero, exponentially fitted scheme and alternative methods. The majority of the algorithms mentioned above need a dense discretization of mesh or multiple iterations along with solving large systems of algebraic equations to obtain a proper solution. Thus, the mentioned methods have high computational complexity and storage requirement.

On the other hand, recently operational matrices have attracted researchers' attention as a powerful tool to approximate the solution of integral equations directly (without using any projection methods), e.g. [16, 19, 21]. These methods are one-step approaches, easily implemented, and have low calculation cost due to the sparsity of an operational matrix. However, they also need to be improved for better accuracy. To overcome the shortcoming of these methods and generalize them to each polynomial basis functions and $K(t) = \frac{1}{t^\alpha}, 0 < \alpha < 1$, we introduce a direct operational matrix method with a higher rate of convergence rather than previous works. The method reduces these equations into a linear system of algebraic equations. Although the prevalent method is to employ the almost operational matrix [17, 19, 21], here we use a trick to avoid the error of using this matrix. Finally, we present an explicit analytical formula to estimate the error.

In the present paper, we focus on the following weakly singular Volterra integro-differential equations of the first and second order:

$$\frac{du}{dt} = f(t) + \lambda \int_0^t \frac{u(x)}{(t-x)^\alpha} dx, \tag{2}$$

with an initial condition $u(0) = a$ and

$$\frac{d^2u}{dt^2} = f(t) + \lambda \int_0^t \frac{u(x)}{(t-x)^\alpha} dx, \tag{3}$$

where $f(t)$ are known and with initial conditions $u(0) = a$, $u'(0) = b$ and $t \in [0, 1]$. Note that the behavior of the system demonstrating anomalous diffusion is very important for small times $t \ll 1$, see the discussion in [5, 19]. These models represent some classes of viscoelastic mass media, especially in nano-technology. For more details about the physical background of these mathematical models, see [7, 10].

In fact, the integrals on the right-hand sides of equations (2) and (3) can be interpreted as fractional integrals of order $1-\alpha$. Application of the fractional derivative

of order $1-\alpha$ to the equations (2) and (3) reduces these equations to the two-terms fractional differential equations. Although these equations can be solved in explicit integral form in terms of the Mittag-Leffler function [6, 15], obtaining the numerical solution using these methods seems complicated, especially when $f(t)$ does not have a simple form. Moreover, Nemati et al. [17] solved fractional integro-differential equations via a fractional operational matrix. Although they solved this equation in the general case of a fractional order, some modifications may cause better accuracy. Therefore, we prepare a simple scheme to approximate the solution of these equations without using fractional calculus with a higher rate of convergence. The scheme allows us to choose arbitrary polynomial basis functions. Here, we employ Chebyshev and Legendre polynomials.
The rest of this paper is organized as follows: Section 2 is devoted to some basic concepts and denotations about shifted Chebyshev and Legendre polynomials, function approximation and matrix form. Furthermore, we derive an almost operational matrix for desired polynomial basis functions. Section 3 is devoted to the numerical procedure. In Section 4, convergence analysis is given. Finally, in Section 5, numerical results are reported to verify the applicability of the proposed method in comparison with other methods in the literature. Through these examples, the superiority of our scheme with two basis functions is shown.

2. Preliminaries

In this section, we give some basic concepts which will be useful in the sequel. For the sake of a good comparison, we consider two desired polynomial basis functions, but they can be easily generalized to others. Throughout the paper, $\Phi_m(t) = [\phi_0(t), ..., \phi_m(t)]^T$ will be denoted as a vector of basis functions.

Definition 1 (see [4]). *The Chebyshev polynomials of the first kind are defined as*

$$T_n(x) = \cos(n\theta), \quad \theta = \arccos(x),$$

or with its recursive formula they can be interpreted as

$$\begin{aligned} T_0(x) &= 1, \quad T_1(x) = x, \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x). \end{aligned}$$

Definition 2 (see [4]). *The Legendre polynomials are defined by the following recursive formulae:*

$$\begin{aligned} P_0(x) &= 1, \quad P_1(x) = x, \\ P_m(x) &= \frac{2m-1}{m} xP_{m-1}(x) - \frac{m-1}{m} P_{m-2}(x). \end{aligned}$$

These two definitions hold in the interval $[-1, 1]$. Hence, to transfer the interval $x = [-1, 1]$ to $t = [0, 1]$, the map $x = 2t - 1$ can be used.

2.1. Function approximation

Shifted Chebyshev and Legendre polynomials as orthogonal basis functions $\Phi(t)$ form complete sets in $L^2[0,1]$, i.e., a function $f(t) \in L^2[0,1]$ can be represented as

$$f(t) = \sum_{i=0}^{\infty} c_i \phi_i(t) = C^T \Phi(t), \tag{4}$$

where $c_i = \dfrac{\langle f(t), \phi_i(t) \rangle}{\langle \phi_i(t), \phi_i(t) \rangle}$, $C = \{c_i\}_{i=1}^{\infty}$ and $\Phi(t) = \{\phi_i(t)\}_{i=1}^{\infty}$.

Let $X = L^2[0,1]$ and X_m =span$\{\phi_0(t), ..., \phi_m(t)\}$. Since $X_m \subset X$ is a finite dimensional vector space, there exists a unique best approximation $f_m(t) \in X_m$ such that

$$\|f - f_m\|_2 = \inf_{g \in X_m} \|f - g\|_2,$$

where

$$f(t) \simeq f_m(t) = \sum_{i=0}^{m} c_i \phi_i(t) = C^T \Phi_m(t),$$

where $C = [c_0, ..., c_m]^T$ and $\Phi_m(t) = [\phi_0(t), ..., \phi_m(t)]^T$.

2.2. Matrix form

Assume $L_m(x) = [l_0(x), l_1(x), ..., l_m(x)]^T$=$[1, x, x^2, ..., x^m]^T$ to be standard basis functions; then obviously each polynomial basis Φ_m can satisfy the following formula

$$\Phi_m(x) = A L_m(x), \tag{5}$$

where the matrix A is an invertible lower triangle matrix. Since each $\phi_i(x) \in \Phi_m(x)$ is a polynomial of degree i, it can be represented in terms of $l_j(x) = x^j$, $0 \le j \le i$, i.e.,

$$\phi_i(x) = \sum_{j=0}^{i} a_{ij} l_j(x) = \sum_{i=0}^{j} a_{ij} x^j,$$

where $a_{ij}, j = 0, ..., i$ are the elements of the $(i+1)$th row of the matrix A. These elements can be easily obtained with the Mathematica's function `Coefficient`.
Here, we introduce a general form of the matrix A for shifted Chebyshev polynomials over the interval $[0,1]$ as follows:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ -1 & 2 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 1 & t_{21} & 2^3 & 0 & \cdots & 0 & \cdots & 0 \\ -1 & t_{31} & t_{32} & 2^5 & \cdots & 0 & \cdots & 0 \\ 1 & t_{41} & t_{42} & t_{43} & 2^7 & 0 & \cdots & 0 \\ \vdots & & & \vdots & & \ddots & & \\ \cos(m\pi) & t_{m1} & t_{m2} & t_{m3} & t_{m4} & & \cdots & 2^{2m-1} \end{pmatrix}_{(m+1)\times(m+1)}, \tag{6}$$

where $t_{i0} = \cos(m\pi)$, $t_{ij} = 4t_{i-1,j-1} - 2t_{i-1,j} - t_{i-2,j}$ with $i = 2, ..., m,\ j = 1, ..., i-2$ and $t_{ij} = 4t_{i-1,j-1} - 2t_{i-1,j}$ for $i = 2, ..., m,\ j = i-1$.

2.3. Almost operational matrix of order α

The operational matrix of integration P for Φ_m is fulfilled in the following expression

$$\int_0^t \Phi_m(x)dx \simeq P\ \Phi_m(t). \tag{7}$$

The operational matrix of integration for shifted Chebyshev polynomials in the interval $[0, 1]$ is as follows [4]:

$$P = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{4} & 0 & \frac{1}{4} & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{3} & \frac{-1}{2} & 0 & \frac{1}{6} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{(-1)^m}{m(m-2)} & 0 & 0 & 0 & \cdots & \frac{-1}{2(m-2)} & 0 & \frac{1}{2m} \\ \frac{(-1)^{m+1}}{(m-1)(m-1)} & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2(m-1)} & 0 \end{pmatrix}, \tag{8}$$

and the operational matrix of integration for shifted Legendre polynomials in the interval $[0, 1]$ is

$$P = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{3} & 0 & \frac{1}{3} & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{-1}{5} & 0 & \frac{1}{5} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{-1}{2m-1} & 0 & \frac{1}{2m-1} \\ \frac{(-1)^m}{m+1} & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2m+1} & 0 \end{pmatrix}. \tag{9}$$

The almost operational matrix of order α, S_α, for Φ_m is defined by the following expression ([19])

$$\int_0^t \frac{\Phi_m(x)}{(t-x)^\alpha}dx \simeq S_\alpha \Phi_m(t). \tag{10}$$

In order to obtain S_α for Φ_m , consider the following well-known formula

$$\int_0^t \frac{x^n}{(t-x)^\alpha}dx = \beta_{n,\alpha}\ t^{n+1-\alpha}, \tag{11}$$

where

$$\beta_{n,\alpha} = \frac{\Gamma(1-\alpha)\Gamma(1+n)}{\Gamma(2-\alpha+n)}.$$

Using the above formula and (5), the almost operational matrix S_α for each polynomial basis function $\Phi_m(x)$ can be obtained as

$$\int_0^t \frac{\Phi_m(x)}{(t-x)^\alpha}dx = \int_0^t \frac{AL_m(x)}{(t-x)^\alpha}dx = E_\alpha t^{1-\alpha} L_m(t) = R_\alpha t^{1-\alpha}\Phi_m(t) \simeq S_\alpha \Phi_m(t), \tag{12}$$

where $R_\alpha = E_\alpha A^{-1}$ and $E_\alpha = AB$, in which $B = diag(\beta_{0,\alpha}, \beta_{1,\alpha}, ..., \beta_{m,\alpha})$.

Remark 1. *It should be pointed out that the prevalent schemes based on operational matrices have used the almost operational matrix S_α [19, 21]. In this study, we use the following relation in the approximation procedure instead of relation (12),*

$$\int_0^t \frac{\Phi_m(x)}{(t-x)^\alpha}dx = E_\alpha t^{1-\alpha} L_m(t), \tag{13}$$

which yields the reduction of the total error. Note that the above relation is precise and so the error of using S_α is omitted. Moreover, E_α is a lower triangle matrix, so the computational complexity of the present scheme is lower than the procedure which is based on S_α.

3. Computational procedure

In this section, we explain how to implement the desired polynomials and its corresponding operational matrices for solving problems (2) and (3). We state how our direct scheme can be used to convert these equations into a linear system of algebraic equations.

I) Approximation of equation (2)

To solve this equation, we expand $u'(t)$ with respect to (4)

$$u'(t) = C^T \Phi(t), \tag{14}$$

where

$$C = \{c_i\}_{i=1}^\infty, \qquad \Phi(t) = \{\phi_i(t)\}_{i=1}^\infty.$$

By integrating the above equation, we have

$$u(t) = \int_0^t C^T \Phi(t)dx + u(0). \tag{15}$$

Using the operational matrix P defined in (7) and $u(0) = U_0^T \Phi(t)$ yields

$$u(t) = C^T P\ \Phi(x) + U_0^T \Phi(t). \tag{16}$$

Substituting (16) into (2) and using (13) results in

$$C^T A\ L(t) = f(t) + \lambda(C^T P + U_0^T)E_\alpha\ L(t)t^{1-\alpha}, \tag{17}$$

where $L(t) = \{l_i\}_{i=1}^{\infty} = \{t^{i-1}\}_{i=1}^{\infty}$. By multiplying both sides by $L^T(t)$ and integrating from 0 to 1, we have

$$C^T AD = D_f + \lambda(C^T P + U_0^T)E_\alpha\ D_\alpha, \tag{18}$$

where

$$D = \int_0^1 L(t)L^T(t)dt, \quad D_f = \int_0^1 f(t)L^T(t)dt, \quad D_\alpha = \int_0^1 L(t)L^T(t)t^{1-\alpha}dt. \tag{19}$$

Therefore, via the described denotations

$$C^T = (D_f + \lambda U_0^T E_\alpha D_\alpha)(AD - \lambda P E_\alpha D_\alpha)^{-1}, \tag{20}$$

the unknown $u(t)$ can be immediately obtained from equation (15) as

$$u(t) = [(D_f + \lambda U_0^T E_\alpha D_\alpha)(AD - \lambda P E_\alpha D_\alpha)^{-1}]P\Phi(t) + u(0). \tag{21}$$

Hence the desired approximate solution $u_m(t)$ can be obtained

$$u_m(t) = [(D_f + \lambda U_0^T E_\alpha D_\alpha)(AD - \lambda P E_\alpha D_\alpha)^{-1}]P\Phi_m(t) + u(0), \tag{22}$$

where all above matrices are square matrices of order $m+1$ and the elements of the vector D_f are

$$(D_f)_i = \int_0^1 f(t)t^{i-1}(t)dt, \quad i = 1, ..., m+1.$$

Algorithm

Input: $m \in \mathbb{N},\ \lambda \in \mathbb{R},\ 0 \le \alpha \le 1$ and the functions $f(t)$, $u(0)$.
Step1: Define arbitrary polynomial basis functions $\Phi_m(t)$.
Step2: Define standard polynomials basis functions $L_m(t)$.
Step3: Construct the operational matrix P defined in [4], the matrix A using (5) and the matrix E_α by (12).
Step4: Construct vectors D_f, U_0 and matrices D, D_α using (19).
Step5: Put all of the above parameters into equation (22) to obtain the approximate solution $u_m(t)$.

II) Approximation of equation (3)

In order to solve this equation, we repeat the prescribed procedure. Let us consider $u''(t) = C^T\Phi(t)$ and $u'(0) = U_1^T\Phi(t)$; then we may derive an approximate solution as

$$u(t) = [((D_f + \lambda U_0^T E_\alpha D_\alpha)(AD - \lambda P E_\alpha D_\alpha)^{-1})P^2 + U_1^T P]\Phi(t) + u(0). \tag{23}$$

Remark 2. *It is worthwhile to note that $f(t)$ in equations (2) and (3) is used precisely not approximately in the proposed method.*

The difference and the superiority of our scheme with respect to other methods [19, 21] in terms of accuracy are interpreted in Remarks 1 and 2. This claim is verified in the experimental examples.

4. Error estimation

This section will provide some theorems to indicates how to evaluate error estimation of the scheme. Moreover, one theorem is devoted to providing a sufficient condition for the convergence of the scheme. For convenience, the whole discussion below is stated for equation (2), but it can be generalized to equation (3). Let $e_n(t) := u(t) - u_n(t)$.
First, in order to study the behavior of the approximation error, we represent an analytical error estimator for equation (2). It follows from the definition of the derivation of $u(t)$ (14) for small t and equation (21). In equation (21), the infinite series $\Phi(t)$ (3) at $t = 0$ is as follows:

$$[(D_f + \lambda U_0^T E_\alpha D_\alpha)(AD - \lambda P E_\alpha D_\alpha)^{-1}]P\Phi(0) = 0. \tag{24}$$

Hence, the approximation error for the truncated series of Φ at the nth term in the vicinity of $t = 0$ is as follows:

$$Error = [(D_f + \lambda U_0^T E_\alpha D_\alpha)(AD - \lambda P E_\alpha D_\alpha)^{-1}]P\Phi_n(0). \tag{25}$$

Note that this error estimation is provided for $t \ll 1$, which is significant in the theory of anomalous diffusion. This claim is verified by the following theorem.

Theorem 1. *Consider equation (2) with the initial condition $u(0) = a$; then we have the following error bound:*

$$|e_n(t)| \leq e_n(0)\ e^{\left(\frac{|\lambda|}{(1-\alpha)(2-\alpha))}\right)} \tag{26}$$

Proof. Since $e'_n(t) = u'(t) - u'_n(t)$, we have

$$\begin{aligned} |e_n(t)|' &\leq |e'_n(t)| \qquad\qquad a.e. \\ &\leq |\lambda| \left| \int_0^t \frac{(u(x) - u_n(x))}{(t-x)^\alpha} dx \right| \\ &\leq |\lambda| \frac{t^{1-\alpha}}{1-\alpha} |e_n(t)|, \end{aligned} \tag{27}$$

Now using Grönwall's inequality [8], we have

$$|e_n(t)| \leq e_n(0)\ e^{\left(\frac{|\lambda| t^{2-\alpha}}{(1-\alpha)(2-\alpha)}\right)}.$$

Since $0 \leq \alpha < 1$ and $0 \leq t \leq 1$, the result is obtained. ☐

It is worth mentioning that from relations (25) and (26), one can conclude that the error bound for every $t \in [0,1]$ is related to the error estimation introduced in (25). This result follows from $e_n(0)$ in (26) being equal to the *Error* (25). This result ensures that the *Error* defined in (25) works for all small t.
Now we introduce a sufficient hypothesis for the convergence of the proposed scheme for equation (2). To this end the following lemmas may be helpful. Let the norm $\|.\|$ be an infinity norm.

Definition 3 (see [3]). *A function $f : [0,1] \to \mathcal{R}$ belongs to Sobolev space $W^{m,p}[0,1]$, if all distributional derivative (weak derivative) of f, $f^{(i)}$, lies in $L^p[0,1]$ for all $0 \le i \le k$ with the norm*

$$\| f \|_{W^{m,p}} = \sum_{i=0}^{m} \|f^{(i)}\|_{L^p},$$

where $\|.\|_{L^p}$ denotes the Lebesgue norm.

Lemma 1 (see [3]). *If $f(t) \in W^{k,\infty}[0,1]$ and $f_N(t) = \sum_{r=0}^{N} c_r \Phi_r(t) = \mathbf{C}^T \Phi(t)$ are the best approximation polynomials of $f(t)$, then $\|f(t) - f_N(t)\| = O(N^{-k})$.*

Lemma 2. *Under the hypothesis of the prescribed equation (2), the approximation error of u_n fulfills the following expression:*

$$\|u(t) - u_n(t)\| \le \|u'(t) - u'_n(t)\|. \tag{28}$$

Proof. Since $u(t) = \int_0^t u'(t)dt + u(0)$, then regarding the triangular inequality, norm inequality and $t \in [0,1]$, one can conclude that

$$\begin{aligned} \|u(t) - u_n(t)\| &\le \int_0^t \|u'(t) - u'_n(t)\|dt + \|u(0) - u_n(0)\| \\ &\le \|u'(t) - u'_n(t)\| + \|u(0) - u_n(0)\|. \end{aligned} \tag{29}$$

On the other hand, since $u(0) = a$ is a scalar, in regard to the prescribed lemma, one can eliminate the second part of the above equation,

$$\|u(t) - u_n(t)\| \le \|u'(t) - u'_n(t)\|. \tag{30}$$

□

Theorem 2. *The proposed scheme is convergent for equation (2) if the parameters λ and α satisfy in $|\lambda|(\frac{1}{1-\alpha}) < 1$.*

Proof. Define $Ku(t) = \int_0^t K(t,x)u(x)dx$, where $K(t,x) = \dfrac{1}{(t-x)^\alpha}$, $0 < \alpha < 1$. Since in the proposed method $f(t)$ and $K(t,x)$ have not been approximated, thus from equations (2), we obtain

$$\begin{aligned} \|u'(t) - u'_n(t)\| = &\ \Big\|f(t) + \lambda \int_0^t \frac{u(x)}{(t-x)^\alpha}dx - f(t) - \lambda \int_0^t \frac{u_n(x)}{(t-x)^\alpha}dx\Big\| \\ = &\ |\lambda| \Big\| \int_0^t \frac{(u(x) - u_n(x))}{(t-x)^\alpha}dx\Big\| \le |\lambda|\|K\|\|u(t) - u_n(t)\|, \end{aligned} \tag{31}$$

where

$$\|K\| = \max_{t\in[0,1]} \int_0^t |K(t,x)|dx.$$

Then by using (30), it is easy to see that $\|u(t) - u_n(t)\|(1 - |\lambda|(\frac{1}{1-\alpha})) \le 0$. So if $|\lambda|(\frac{1}{1-\alpha}) < 1$, one can conclude $\|u(t) - u_n(t)\| \to 0$, as n tends to infinity. □

5. Numerical experiments

In order to demonstrate the performance and the accuracy of our scheme, we consider the following examples.

Example 1. *Consider equation (2), when $\lambda = 1$, $\alpha = 0.5$, $u(0) = 0$ and*

$$f(t) = \frac{1}{1+t} + 4\sqrt{t} - 2\sqrt{1+t}\ ln(1 + 2\sqrt{\frac{t}{1+t}} + 2t(1 + \sqrt{\frac{t}{1+t}})). \qquad (32)$$

The exact solution can be derived as $u(t) = ln(1 + t)$.

This problem has been studied in [19]. Sing and Postnikov applied the Bernstein operational matrix method with a various degree of this polynomial. Table 1 demonstrates the results for shifted Chebyshev and Legendre polynomials. As can be seen, they are in a close contest. In Figure 1, the difference between the approximate and the exact solution is shown for a various degree of Legendre polynomials. For convenience due to comparison of errors with the method in [19], they are multiplied by different factors varying between 10^2 to 10^6 as labeled in Figure 1. Furthermore, Figure 2 presents a comparison between the approximation errors of Chebyshev polynomials of degree 5 multiplied by 10^2 and the Bernstein polynomials of degree 5 [19]. It is clear that the accuracy of the solution is increased using the present method rather than the Bernstein approach.

m	2	4	6	8	10
$E_\infty(Ch)$	$1.8e-03$	$1.47e-05$	$2.91e-07$	$6.68e-09$	$1.87e-10$
$E_\infty(Le)$	$6.37e-04$	$1.27e-05$	$2.81e-07$	$6.56e-09$	$1.86e-10$

Table 1: *Absolute error of the present method for Chebyshev and Legendre polynomials of various degree m*

Example 2. *Consider the following example of equation (3), when $\lambda = 1$, $\alpha = 0.5$, $u(0) = 1$, $u'(0) = 1$ and the force*

$$f(t) = e^t[1 - \sqrt{\pi}\ Erf(\sqrt{t})],$$

which has the exact solution $u(t) = e^t$ ([11]).

A comparison of approximation errors for Chebyshev and Legendre polynomials of degree 5 is again provided to verify the superiority of our scheme compared to the method in [19] in Figure 3. Since these two polynomial basis functions are almost equal, one may conclude that in the presented method, any kind of these polynomials may be used. It can be considered as a benefit of the present scheme. The absolute error of the approximate solution is shown for various m of shifted Chebyshev and Legendre polynomials in Table 2.

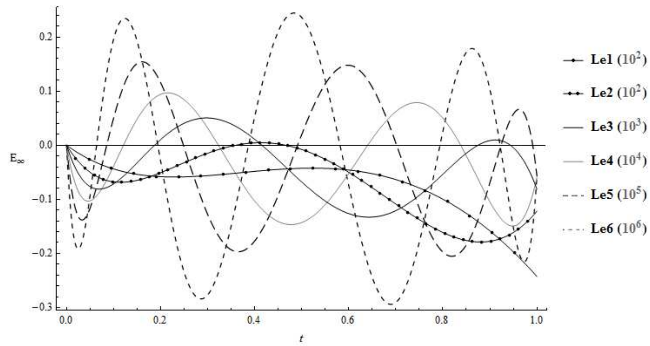

Figure 1: *Approximation error of various degree of Legendre polynomials multiplied by a factor*

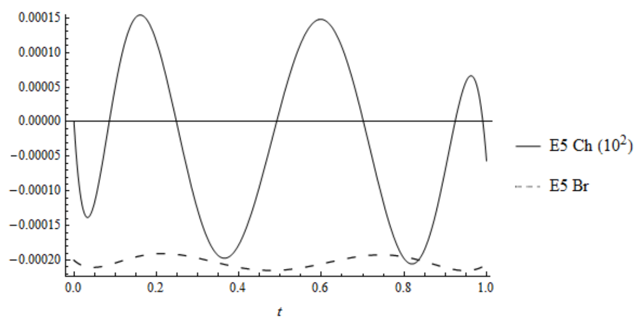

Figure 2: *Comparison between the approximation error of Chebyshev polynomials of degree 5 multiplied by* 10^2 *(—) and the Bernstein polynomials (- -)*

Example 3. *In the following test problem, we consider equation (2) with* $\lambda = \frac{-1}{2},\ \alpha = \frac{1}{3},\ u(0) = 0$ *and the exact solution* $u(t) = t^2$*. Hence, the force function is*

$$f(t) = 2t + \frac{27t^{\frac{8}{3}}}{8}. \tag{33}$$

The proposed method results in the exact solution for $m = 2$. It shows that our proposed method highly agrees with the exact solution and low computational complexity. The cause of the present approach superiority compared with the other existing methods is stated in Remarks 1 and 2. Furthermore, the condition of Theorem 2 is not provided but our method is convergent. It indicates that the convergence necessary condition is not as rigorous as Theorem 2.
We also consider various α and λ with an exact solution as a polynomial function of degree N. Our observations clarify that the high accuracy occurs when $m = N$ with at most error estimation 10^{-13}. For instance, Figure 5 depicts the absolute error when in equation (3) we put $\lambda = -0.2,\ \alpha = 0.2,\ u(0) = 0,\ u'(0) = 0$ and $u(t) = t^2$.

Example 4. *As another test problem, consider equation (2) with* $\lambda = \frac{1}{2}, \alpha = 0.4,$

m	2	4	6	8	10
$E_\infty(Ch)$	$1.74e-03$	$1.36e-06$	$9.12e-10$	$2.74e-10$	$8.49e-10$
$E_\infty(Le)$	$2.0e-04$	$1.84e-07$	$1.39e-10$	$2.81e-10$	$8.4e-10$

Table 2: *Absolute error of the present method for various m for shifted Chebyshev and Legendre polynomials for Example 2*

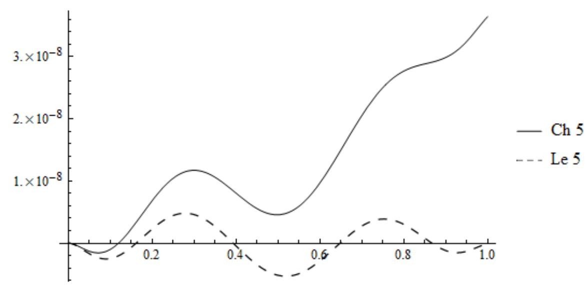

Figure 3: *Approximation error of Chebyshev (—) and Legendre (- -) polynomials for Example 2*

$u(0) = 0$ *and*

$$f(t) = \cos(t) + \frac{25}{48}\, t^{\frac{8}{5}}\, F(1; \frac{13}{10}, \frac{9}{5}; -\frac{t^2}{4}),$$

where F(a;b;c) is a hypergeometric function and the exact solution is $u(t) = \sin(t)$.

Table 3 demonstrates the absolute error of the proposed approach for Chebyshev polynomials of various degree m.

m	2	4	6	8	10
$E_\infty(Ch)$	$5.61e-04$	$6.92e-07$	$3.12e-09$	$1.74e-09$	$7.45e-10$

Table 3: *Absolute error of the present method for Chebyshev polynomials of various degree m for Example 4*

Example 5. *As a test problem, consider equation (3) with* $\lambda = -1, \alpha = 0.3$, $u(0) = 0$, $u'(0) = 0$ *and*

$$f(t) \simeq \frac{-1}{(t+1)^2} + 0.840336\, t^{1.7}\, F(1, 1; 2.7; -t).$$

The exact solution can be derived as $u(t) = ln(1+t)$. Figure 6 depicts the approximation error graph of a various degree of Legendre polynomials. For better visualisation, the approximation errors are multiplied by a factor which varies between 10^4 and 10^{11}. Moreover, the analytical error estimation obtained in (25) is demonstrated in Table 4 at $t = 0.1$.

Example 6. *As the final example, consider the following equation ([17])*

$$u'(t) = f(t) + p(t)u(t) + \int_0^t \frac{u(x)}{(t-x)^{0.5}},$$

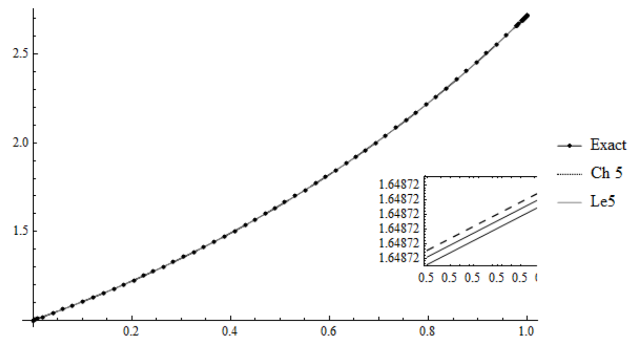

Figure 4: *Comparison of the exact and approximation solution of shifted Chebyshev (- -) and Legendre (—) for Example 2*

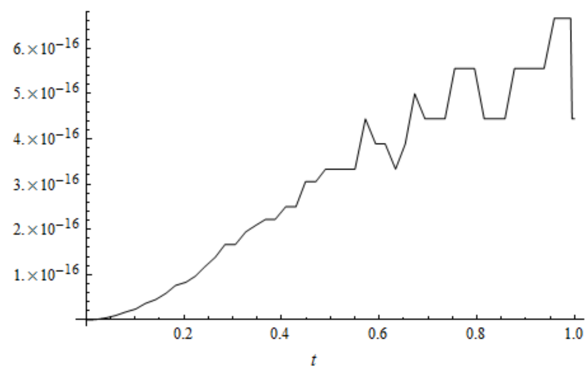

Figure 5: *Approximation error of the method when in equation (3) we have* $\lambda = -0.2, \alpha = 0.2, u(0) = 0, u'(0) = 0$ *and* $u(t) = t^2$

where $f(t) = 2t$ *and* $p(t) = \frac{-16}{15}t^{0.5}$ *with initial condition* $u(0) = 0$.

The exact solution is $u(t) = t^2$.
As can be seen, in this example the term $p(t)u(t)$ is added to the right-hand side of equation (2). Thus the computational procedure changes a bit changes as follows. Again consider relations (14)-(16). Equation (17) converts to

$$C^T A\ L(t) = f(t) + p(t)(C^T P + U_0^T)AL(t) + \lambda(C^T P + U_0^T)E_\alpha\ L(t)t^{1-\alpha}. \quad (34)$$

Owing to the initial condition $u(0) = 0$, one can eliminate any expression that has U_0. So

$$C^T A\ L(t) = f(t) + p(t)C^T PAL(t) + \lambda C^T PE_\alpha\ L(t)t^{1-\alpha}. \quad (35)$$

By following relations (18) and (20), we obtain the unknown approximate solution $u(t)$ as

$$u(t) = [(D_f)(AD - \lambda PE_\alpha D_\alpha - PAQ_1)^{-1}]P\Phi(t),$$

m	2	4	6	8	10
$Error(Le)$	$1.23e-04$	$2.96e-06$	$1.44e-08$	$1.13e-09$	$2.26e-11$

Table 4: *Error estimation obtained from (25) at the point t=0.1 for a various degree of Legendre polynomials*

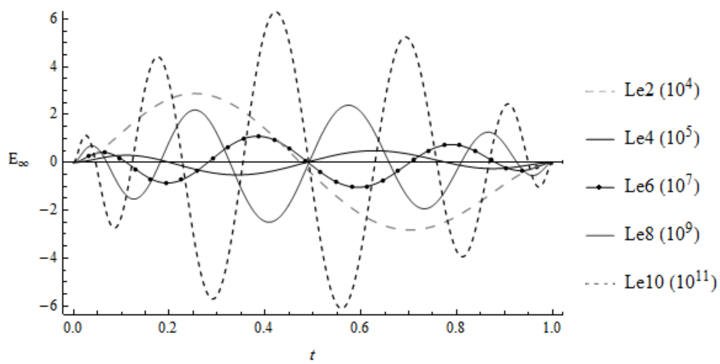

Figure 6: *Approximation error of Legendre polynomials for a various degree (m=2,4,6,8,10) for Example 5*

where all above matrices are defined before, except the matrix Q_1,

$$Q_1 = \int_0^1 p(t)L(t)L^T(t)dt.$$

The present scheme presents the exact solution by employing only $m = 2$. Recently, Nemati et al. [17] solved this problem with fractional operational matrix method based on Chebyshev polynomials of the second kind. Their results are shown in Table 5. It confirms the superiority of our scheme compared with those using fractional calculus.

m	2	4	6	8
Method in [17]	$1.068e-03$	$5.149e-05$	$1.702e-5$	$6.561e-6$

Table 5: *L^2- error norm of the fractional operational matrix method [17] for various m via Chebyshev polynomials of the second kind*

6. Conclusions

Several physical problems of anomalous relaxation or diffusion processes are reduced to weakly singular integro-differential equations. The proposed approach transforms these types of equations into linear systems of equations based on the operational matrix. The main benefits of the method are low cost of setting up the equations without using any projection methods and reduction of the total error by using the precise form of a weakly singular kernel and nonhomogeneous function in the approximation procedure. Other advantages of this procedure are its simplicity and reliability. This approach has the ability to be conducted based on any desired polynomial basis functions. Furthermore, an explicit formula for error estimation in a vicinity of the beginning of the process and an error bound for the scheme can be obtained. Throughout the experimental examples, one can observe that a small size of the operational matrix and basis functions are required to obtain a favorable approximate solution. Further research into the fractional integro-differential equations with a weakly singular kernel via the presented scheme is in progress.

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