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INVERSE COMPONENT CROPPING SEQUENCES AND CONNECTED INVERSE LIMITS OVER INTERVALS

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Abstract. We give a characterization of inverse sequences with upper semicontinuous bonding functions $f_i : [0,1] \multimap [0,1]$ for which the inverse limit of the inverse sequence with f_i^{-1} as bonding functions is connected.

As a byproduct, we obtain another characterization of connected inverse limits of inverse sequences with a single bonding function.

1. Introduction

Connectedness of inverse limits of inverse sequences with upper semicontinuous (set-valued) bonding functions has been one of the most popular subjects of investigation in the theory of such inverse limits. Many papers were published, giving different sufficient or necessary conditions for the inverse limit to be connected (many references may be found in [5]). S. Greenwood and J. Kennedy gave, using so-called component cropping sequences, a characterization of connected inverse limits of inverse sequences of intervals with upper semicontinuous bonding functions whose graphs are connected and surjective ([2]). Also, V. Nall proved the following theorem, giving a characterization of connected inverse limits of inverse sequences of intervals with a single upper semicontinuous bonding function whose graph is surjective ([7]).

Theorem 1.1 ([7, Theorem 3.3, p. 171]). *Let $f : [0,1] \to 2^{[0,1]}$ be an upper semicontinuous function with a surjective graph. The following statements are equivalent.*

1. $\varprojlim\{[0,1], f\}_{i=1}^{\infty}$ *is connected.*
2. $\varprojlim\{[0,1], f^{-1}\}_{i=1}^{\infty}$ *is connected.*

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Then, W. T. Ingram stated the following problem about connections of inverse sequences $\{[0,1], f_i\}_{i=1}^{\infty}$ and $\{[0,1], f_i^{-1}\}_{i=1}^{\infty}$ concerning a possible generalization of Theorem 1.1.

PROBLEM 1.1 ([5, Problem 6.7, p. 78]). *Suppose* (f_i) *is a sequence of upper semicontinuous functions on* $[0,1]$ *with surjective graphs and that* $\varprojlim\{[0,1], f_i\}_{i=1}^{\infty}$ *is connected. Let* (g_i) *be the sequence such that* $g_i = f_i^{-1}$ *for each positive integer* i*. Is* $\varprojlim\{[0,1], g_i\}_{i=1}^{\infty}$ *connected?*

This problem has already been solved. The question was answered in the negative (see [1, 4]). Motivated by the above problem, we give a characterization of inverse sequences $\{[0,1], f_i\}_{i=1}^{\infty}$ for which the inverse limit $\varprojlim\{[0,1], f_i^{-1}\}_{i=1}^{\infty}$ is connected. Moreover, we give a new characterization of connected inverse limits of inverse sequences of intervals with a single bonding function.

2. Definitions and notation

$2^{[0,1]}$ denotes the set of all nonempty closed subsets of $[0,1]$. A function $f : [0,1] \to 2^{[0,1]}$ is *upper semicontinuous at a point* $x \in [0,1]$ provided that if V is any open set in $[0,1]$ containing $f(x)$ then there is an open set U in $[0,1]$ containing x such that $f(t) \subseteq V$ for any $t \in U$; f is called *upper semicontinuous* if it is upper semicontinuous at each point of $[0,1]$.

The *graph* $\Gamma(f)$ *of a function* $f : [0,1] \to 2^{[0,1]}$ is the set of all points $(x,y) \in [0,1] \times [0,1]$ such that $y \in f(x)$. The following is a well known result; the proof can be found in [5, Theorem 1.2, p. 3].

THEOREM 2.1. *Let* $f : [0,1] \to 2^{[0,1]}$ *be a function. Then* f *is upper semicontinuous if and only if its graph* $\Gamma(f)$ *is closed in* $[0,1] \times [0,1]$.

We use $f : [0,1] \multimap [0,1]$ to denote upper semicontinuous functions $f : [0,1] \to 2^{[0,1]}$.

We say that the graph of a function $f : [0,1] \to 2^{[0,1]}$ is *surjective* if for each $y \in [0,1]$ there is a point $x \in [0,1]$ such that $y \in f(x)$.

Let $f : [0,1] \to 2^{[0,1]}$ be a function with a surjective graph. Then we define $f^{-1} : [0,1] \to 2^{[0,1]}$ by $f^{-1}(x) = \{y \in [0,1] \mid x \in f(y)\}$. Note that f is upper semicontinuous if and only if f^{-1} is upper semicontinuous since $\Gamma(f^{-1}) = \{(x,y) \in [0,1] \times [0,1] \mid (y,x) \in \Gamma(f)\}$.

In this paper we deal with *inverse sequences* $\{I_i, f_i\}_{i=1}^{\infty}$, where I_i are closed intervals $[0,1]$ and $f_i : I_{i+1} \multimap I_i$ are upper semicontinuous functions.

The *inverse limit* of an inverse sequence $\{I_i, f_i\}_{i=1}^{\infty}$ is the subspace of the product space $\prod_{i=1}^{\infty} I_i$ consisting of all $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \prod_{i=1}^{\infty} I_i$, such that $x_i \in f_i(x_{i+1})$ for each i. The inverse limit is denoted by $\varprojlim\{I_i, f_i\}_{i=1}^{\infty}$. For more information see [3, 6].

S. Greenwood and J. Kennedy introduced the concept of so-called component cropping sequences (or CC-sequences) to characterize connected inverse limits over intervals, see [2] for more details. Since the concept of the component cropping sequences is new, we describe it in every detail through the following definitions.

DEFINITION 2.2. *For each positive integer i, let $I_i = [0,1]$, and let*

$$a_i, a_{i+1}, b_i, b_{i+1} \in [0,1]$$

such that $a_i < b_i$, $a_{i+1} < b_{i+1}$, $[a_i, b_i] \neq [0,1]$, and $[a_{i+1}, b_{i+1}] \neq [0,1]$. Then

1. $T_i(a_i, a_{i+1}, b_i, b_{i+1}) = (I_{i+1} \times (b_i, 1]) \cup ([a_{i+1}, b_{i+1}] \times [a_i, b_i])$,
2. $B_i(a_i, a_{i+1}, b_i, b_{i+1}) = (I_{i+1} \times [0, a_i)) \cup ([a_{i+1}, b_{i+1}] \times [a_i, b_i])$,
3. $L_i(a_i, a_{i+1}, b_i, b_{i+1}) = ([0, a_{i+1}) \times I_i) \cup ([a_{i+1}, b_{i+1}] \times [a_i, b_i])$,
4. $R_i(a_i, a_{i+1}, b_i, b_{i+1}) = ((b_{i+1}, 1] \times I_i) \cup ([a_{i+1}, b_{i+1}] \times [a_i, b_i])$,
5. $TL_i(a_i, a_{i+1}, b_i, b_{i+1}) = (I_{i+1} \times (b_i, 1]) \cup ([0, a_{i+1}) \times I_i) \cup ([a_{i+1}, b_{i+1}] \times [a_i, b_i])$,
6. $TR_i(a_i, a_{i+1}, b_i, b_{i+1}) = (I_{i+1} \times (b_i, 1]) \cup ((b_{i+1}, 1] \times I_i) \cup ([a_{i+1}, b_{i+1}] \times [a_i, b_i])$,
7. $BL_i(a_i, a_{i+1}, b_i, b_{i+1}) = (I_{i+1} \times [0, a_i)) \cup ([0, a_{i+1}) \times I_i) \cup ([a_{i+1}, b_{i+1}] \times [a_i, b_i])$,
8. $BR_i(a_i, a_{i+1}, b_i, b_{i+1}) = (I_{i+1} \times [0, a_i)) \cup ((b_{i+1}, 1] \times I_i) \cup ([a_{i+1}, b_{i+1}] \times [a_i, b_i])$.

REMARK 2.3. If for a $j \in \{i, i+1\}$, $b_j = 1$ then $(b_j, 1] = \emptyset$. Similarly, if for a $j \in \{i, i+1\}$, $a_j = 0$ then $[0, a_j) = \emptyset$.

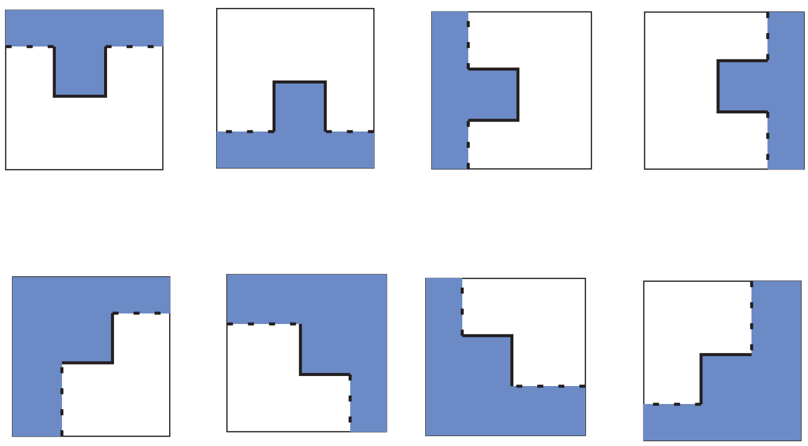

FIGURE 1. The sets defined in Definition 2.2, where for each $j \in \{i, i+1\}$, $a_j \neq 0$ and $b_j \neq 1$.

Definition 2.4. *Suppose that* $\varepsilon > 0$, i *is a positive integer,* I_i *and* I_{i+1} *are both equal to* $[0,1]$, $f_i : I_{i+1} \multimap I_i$ *is an upper semicontinuous function,* $a_i, a_{i+1}, b_i, b_{i+1} \in [0,1]$ *are such elements that*

1. $a_i < b_i$,
2. $a_{i+1} < b_{i+1}$,
3. $[a_i, b_i] \neq I_i$,
4. $[a_{i+1}, b_{i+1}] \neq I_{i+1}$,

and that X_i *denotes one of the following notation* T_i, B_i, L_i, R_i, TL_i, TR_i, BL_i, BR_i *(for example, if* $X_i = TL_i$*, then* $X_i(a_i, a_{i+1}, b_i, b_{i+1})$ *denotes* $TL_i(a_i, a_{i+1}, b_i, b_{i+1})$*).*

Let C'_i *be any connected component (or shortly component) of*

$$\Gamma(f_i) \cap ((a_{i+1} - \varepsilon, b_{i+1} + \varepsilon) \times (a_i - \varepsilon, b_i + \varepsilon))$$

such that C'_i *is a subset of* $X_i(a_i, a_{i+1}, b_i, b_{i+1})$*. If* $C'_i \cap ([a_{i+1}, b_{i+1}] \times [a_i, b_i]) \neq \emptyset$*, then each component* C_i *of* $C'_i \cap ([a_{i+1}, b_{i+1}] \times [a_i, b_i])$ *is called an* X_i-set, framed by $[a_{i+1}, b_{i+1}] \times [a_i, b_i]$. *We denote this by*

$$\Gamma(f_i) \sqsubset_{C_i} X_i(a_i, a_{i+1}, b_i, b_{i+1}).$$

Definition 2.5. *Let* $\{I_i, f_i\}_{i=1}^{\infty}$ *be an inverse sequence of intervals* $I_i = [0,1]$ *and upper semicontinuous functions* $f_i : I_{i+1} \multimap I_i$ *with surjective and connected graphs. Also, let* $(p_1, p_2, p_3, \ldots) \in \varprojlim\{I_i, f_i\}_{i=1}^{\infty}$*, let* m *and* n *be positive integers,* $m + 1 \leq n$*, and for each* $i \in \{m, m+1, m+2, \ldots, n, n+1\}$*, let* $a_i, b_i \in [0,1] = I_i$ *such that* $a_i < b_i$ *and* $[a_i, b_i] \neq [0,1]$*.*

The sequence

$$([a_i, b_i])_{i=m}^{n+1}$$

is called a component cropping sequence *over* $[m,n]$ *for* $\{f_i \mid i = 1,2,3,\ldots\}$ *with respect to* $(p_1, p_2, p_3, \ldots)$*, if*

1. $(p_1, p_2, p_3, \ldots) \in \left(\prod_{i=1}^{m-1} I_i\right) \times \left(\prod_{i=m}^{n+1} [a_i, b_i]\right) \times \left(\prod_{i=n+2}^{\infty} I_i\right)$,
2. *for each* $i \in \{m, m+1, m+2, m+3, \ldots, n\}$*, there is a component* C_i *of*

$$\Gamma(f_i) \cap ([a_{i+1}, b_{i+1}] \times [a_i, b_i])$$

such that $(p_{i+1}, p_i) \in C_i$ *and*

(a) $\Gamma(f_m) \sqsubset_{C_m} R_m(a_m, a_{m+1}, b_m, b_{m+1})$ *or* $\Gamma(f_m) \sqsubset_{C_m} L_m(a_m, a_{m+1}, b_m, b_{m+1})$.

(b) • *if* $n = m + 1$*, then*

$$\Gamma(f_{m+1}) \sqsubset_{C_{m+1}} T_{m+1}(a_{m+1}, a_{m+2}, b_{m+1}, b_{m+2})$$

$$\text{if } \Gamma(f_m) \sqsubset_{C_m} L_m(a_m, a_{m+1}, b_m, b_{m+1}), \text{ and}$$

$$\Gamma(f_{m+1}) \sqsubset_{C_{m+1}} B_{m+1}(a_{m+1}, a_{m+2}, b_{m+1}, b_{m+2})$$

$$\text{if } \Gamma(f_m) \sqsubset_{C_m} R_m(a_m, a_{m+1}, b_m, b_{m+1});$$

- *if* $n > m+1$, *then*

$$\Gamma(f_{m+1}) \sqsubset_{C_{m+1}} BR_{m+1}(a_{m+1}, a_{m+2}, b_{m+1}, b_{m+2})$$

or

$$\Gamma(f_{m+1}) \sqsubset_{C_{m+1}} BL_{m+1}(a_{m+1}, a_{m+2}, b_{m+1}, b_{m+2})$$

if $\Gamma(f_m) \sqsubset_{C_m} R_m(a_m, a_{m+1}, b_m, b_{m+1})$, *and*

$$\Gamma(f_{m+1}) \sqsubset_{C_{m+1}} TR_{m+1}(a_{m+1}, a_{m+2}, b_{m+1}, b_{m+2})$$

or

$$\Gamma(f_{m+1}) \sqsubset_{C_{m+1}} TL_{m+1}(a_{m+1}, a_{m+2}, b_{m+1}, b_{m+2})$$

if $\Gamma(f_m) \sqsubset_{C_m} L_m(a_m, a_{m+1}, b_m, b_{m+1})$;

(c) *if* $m+1 \le i < n-1$, *then*

$$\Gamma(f_{i+1}) \sqsubset_{C_{i+1}} BL_{i+1}(a_{i+1}, a_{i+2}, b_{i+1}, b_{i+2})$$

or

$$\Gamma(f_{i+1}) \sqsubset_{C_{i+1}} BR_{i+1}(a_{i+1}, a_{i+2}, b_{i+1}, b_{i+2})$$

if $\Gamma(f_i) \sqsubset_{C_i} BR_i(a_i, a_{i+1}, b_i, b_{i+1})$ *or*

$$\Gamma(f_i) \sqsubset_{C_i} TR_i(a_i, a_{i+1}, b_i, b_{i+1}),$$

and

$$\Gamma(f_{i+1}) \sqsubset_{C_{i+1}} TL_{i+1}(a_{i+1}, a_{i+2}, b_{i+1}, b_{i+2})$$

or

$$\Gamma(f_{i+1}) \sqsubset_{C_{i+1}} TR_{i+1}(a_{i+1}, a_{i+2}, b_{i+1}, b_{i+2})$$

if $\Gamma(f_i) \sqsubset_{C_i} BL_i(a_i, a_{i+1}, b_i, b_{i+1})$ *or*

$$\Gamma(f_i) \sqsubset_{C_i} TL_i(a_i, a_{i+1}, b_i, b_{i+1});$$

(d) *if* $n > m+1$, *then*

$$\Gamma(f_n) \sqsubset_{C_n} B_n(a_n, a_{n+1}, b_n, b_{n+1})$$

if $\Gamma(f_{n-1}) \sqsubset_{C_{n-1}} BR_{n-1}(a_{n-1}, a_n, b_{n-1}, b_n)$ *or* $\Gamma(f_{n-1}) \sqsubset_{C_{n-1}} TR_{n-1}(a_{n-1}, a_n, b_{n-1}, b_n)$, *and*

$$\Gamma(f_n) \sqsubset_{C_n} T_n(a_n, a_{n+1}, b_n, b_{n+1})$$

if $\Gamma(f_{n-1}) \sqsubset_{C_{n-1}} BL_{n-1}(a_{n-1}, a_n, b_{n-1}, b_n)$ *or* $\Gamma(f_{n-1}) \sqsubset_{C_{n-1}} TL_{n-1}(a_{n-1}, a_n, b_{n-1}, b_n)$.

We say that $\{f_i \mid i = 1, 2, 3, \ldots\}$ admits a component cropping sequence, *if there are*

1. $(p_1, p_2, p_3, \ldots) \in \varprojlim \{I_i, f_i\}_{i=1}^{\infty}$,
2. *positive integers* m *and* n, $m+1 \le n$, *and*

3. *a component cropping sequence over* $[m,n]$ *for* $\{f_i \mid i = 1,2,3,\ldots\}$ *with respect to* $(p_1, p_2, p_3, \ldots)$.

The following theorem is a characterization of connected inverse limits.

THEOREM 2.6 ([2, Theorem 1.6.]). *Let* $\{I_i, f_i\}_{i=1}^{\infty}$ *be an inverse sequence of intervals* $I_i = [0,1]$ *and upper semicontinuous functions* $f_i : I_{i+1} \multimap I_i$ *with surjective and connected graphs. The following statements are equivalent.*

1. *The inverse limit* $\varprojlim \{I_i, f_i\}_{i=1}^{\infty}$ *is not connected.*
2. $\{f_i \mid i = 1,2,3,\ldots\}$ *admits a component cropping sequence.*

3. Main results

In this section we give a characterization of inverse sequences $\{I_i, f_i\}_{i=1}^{\infty}$ for which the inverse limit $\varprojlim \{I_i, f_i^{-1}\}_{i=1}^{\infty}$ is connected.

We start with two examples (also demonstrating how component cropping sequences can be used), where we construct inverse sequences $\{[0,1], f_i\}_{i=1}^{\infty}$ and $\{[0,1], g_i\}_{i=1}^{\infty}$ such that

1. $\varprojlim \{[0,1], f_i\}_{i=1}^{\infty}$ is connected, and
2. $\varprojlim \{[0,1], g_i\}_{i=1}^{\infty}$ is not connected,

giving an alternative solution of Problem 1.1.

EXAMPLE 3.1. Let $\{[0,1], f_i\}_{i=1}^{\infty}$ be the inverse sequence with upper semicontinuous functions on $[0,1]$ defined by: $f_1 : [0,1] \multimap [0,1]$,

$$f_1(x) = \begin{cases} [0,1], & x = 0, \\ \{x, 1\}, & 0 < x \le \frac{1}{4}, \\ \{1\}, & \frac{1}{4} < x \le 1, \end{cases}$$

$f_2 : [0,1] \multimap [0,1]$,

$$f_2(x) = \begin{cases} [0,1], & x = 0, \\ \{1\}, & 0 < x < \frac{1}{4}, \\ \{\frac{1}{3}(x+2), 1\}, & \frac{1}{4} \le x \le 1, \end{cases}$$

and $f_i : [0,1] \multimap [0,1]$, $f_i(x) = \{x\}$, for all $i \ge 3$.

Each point $\mathbf{x} = (x_1, x_2, x_3, \ldots) \in \varprojlim \{[0,1], f_i\}_{i=1}^{\infty}$ has one of the following forms.

(1) If $x_1 \in [0, \frac{1}{4}]$, then the following two cases are possible:
- $\mathbf{x} = (x_1, 0, 0, 0, \ldots)$,
- $\mathbf{x} = (x_1, x_1, 0, 0, \ldots)$.

(2) If $x_1 \in (\frac{1}{4}, 1)$, then $\mathbf{x} = (x_1, 0, 0, 0, \ldots)$.

(3) If $x_1 = 1$, then there are the following possible cases.

- If $x_2 \in [0, \frac{3}{4})$, then $\mathbf{x} = (1, x_2, 0, 0, \ldots)$.
- If $x_2 \in [\frac{3}{4}, 1)$, then the following cases are possible:
  - $\mathbf{x} = (1, x_2, 0, 0, \ldots)$,
  - $\mathbf{x} = (1, x_2, 3x_2 - 2, 3x_2 - 2, \ldots)$.
- If $x_2 = 1$, then $\mathbf{x} = (1, 1, x_3, x_3, \ldots)$, where $x_3 \in [0, 1]$.

Therefore, the inverse limit $\varprojlim\{[0,1], f_i\}_{i=1}^{\infty}$ is the union of the following five arcs:

- the arc $A_1 = \{(t, t, 0, 0, \ldots) \mid t \in [0, \frac{1}{4}]\}$ with endpoints $(\frac{1}{4}, \frac{1}{4}, 0, 0, \ldots)$ and $(0, 0, 0, 0, \ldots)$,
- the arc $A_2 = \{(t, 0, 0, 0, \ldots) \mid t \in [0, 1]\}$ with endpoints $(0, 0, 0, 0, \ldots)$ and $(1, 0, 0, 0, \ldots)$,
- the arc $A_3 = \{(1, t, 0, 0, \ldots) \mid t \in [0, 1]\}$ with endpoints $(1, 0, 0, 0, \ldots)$ and $(1, 1, 0, 0, \ldots)$,
- the arc $A_4 = \{(1, 1, t, t, \ldots) \mid t \in [0, 1]\}$ with endpoints $(1, 1, 0, 0, \ldots)$ and $(1, 1, 1, 1, \ldots)$,
- the arc $A_5 = \{(1, t, 3t - 2, 3t - 2, \ldots) \mid t \in [\frac{3}{4}, 1]\}$ with endpoints $(1, 1, 1, 1, \ldots)$ and $(1, \frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \ldots)$.

Since $A_i \cap A_j \neq \emptyset$ if and only if $|i - j| = 1$ for all $i \neq j$, and since $A_i \cap A_{i+1}$ contains only the common endpoint for each i, it follows that the inverse limit $\varprojlim\{[0,1], f_i\}_{i=1}^{\infty}$ is an arc. Therefore, $\varprojlim\{[0,1], f_i\}_{i=1}^{\infty}$ is connected.

Example 3.2. Let $\{[0,1], g_i\}_{i=1}^{\infty}$ be the inverse sequence with upper semi-continuous functions on $[0,1]$ defined by: $g_1 : [0,1] \multimap [0,1]$,

$$g_1(x) = \begin{cases} \{0, x\}, & 0 \leq x \leq \frac{1}{4}, \\ \{0\}, & \frac{1}{4} < x < 1, \\ [0,1], & x = 1, \end{cases}$$

$g_2 : [0,1] \multimap [0,1]$,

$$g_2(x) = \begin{cases} \{0\}, & 0 \leq x < \frac{3}{4}, \\ \{0, 3x - 2\}, & \frac{3}{4} \leq x < 1, \\ [0,1], & x = 1, \end{cases}$$

and $g_i : [0,1] \multimap [0,1]$, $g_i(x) = \{x\}$, for all $i \geq 3$.

We use Theorem 2.6 to prove that $\varprojlim\{[0,1], g_i\}_{i=1}^{\infty}$ is not connected.

Let $\mathbf{p} = (\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \ldots) \in \varprojlim\{[0,1], g_i\}_{i=1}^{\infty}$. Let $A_1 = [\frac{1}{8}, \frac{3}{8}]$, $A_2 = [\frac{3}{16}, \frac{5}{16}]$, and $A_3 = [\frac{5}{8}, \frac{7}{8}]$. Obviously, $\mathbf{p} \in A_1 \times A_2 \times A_3 \times \prod_{i=4}^{\infty}[0,1]$ and $(A_i)_{i=1}^{3}$ is a component cropping sequence over $[1,2]$ for $\{g_i \mid i = 1, 2, 3, \ldots\}$ with respect to $\mathbf{p}$. Therefore, $\varprojlim\{[0,1], g_i\}_{i=1}^{\infty}$ is not connected by Theorem 2.6 (see Figure 2).

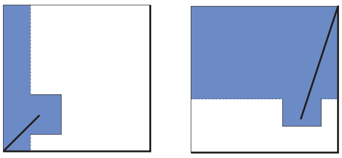

FIGURE 2. The sets $L_1(\frac{1}{8},\frac{3}{16},\frac{3}{8},\frac{5}{16})$ and $T_2(\frac{3}{16},\frac{5}{8},\frac{5}{16},\frac{7}{8})$.

One can easily see that the above examples prove that there is an inverse sequence $\{I_i, f_i\}_{i=1}^{\infty}$ of closed intervals $I_i = [0,1]$ and upper semicontinuous functions $f_i : I_{i+1} \multimap I_i$ with surjective and connected graphs such that

1. $\varprojlim\{[0,1], f_i\}_{i=1}^{\infty}$ is connected, and
2. $\varprojlim\{[0,1], f_i^{-1}\}_{i=1}^{\infty}$ is not connected.

Our next goal is to give a characterization of inverse sequences $\{[0,1], f_i\}_{i=1}^{\infty}$ for which the inverse limit $\varprojlim\{[0,1], f_i^{-1}\}_{i=1}^{\infty}$ is connected. Obviously, the inverse limit $\varprojlim\{[0,1], f_i^{-1}\}_{i=1}^{\infty}$ is connected if and only if $\{f_i^{-1} \mid i = 1,2,3,\ldots\}$ does not admit a component cropping sequence (by Theorem 2.6). This is a characterization of connected inverse limits $\varprojlim\{[0,1], f_i^{-1}\}_{i=1}^{\infty}$ using the inverses f_i^{-1} of the bonding functions f_i. In Theorem 3.5, we give a characterization of connected inverse limits $\varprojlim\{[0,1], f_i^{-1}\}_{i=1}^{\infty}$ using the bonding functions f_i instead of their inverses f_i^{-1}, i.e. we give necessary and sufficient conditions on $\{f_i \mid i = 1,2,3,\ldots\}$ for the inverse limit $\varprojlim\{[0,1], f_i^{-1}\}_{i=1}^{\infty}$ to be connected. First we define inverse component cropping sequences.

DEFINITION 3.3. *Let $\{I_i, f_i\}_{i=1}^{\infty}$ be an inverse sequence of intervals $I_i = [0,1]$ and upper semicontinuous functions $f_i : I_{i+1} \multimap I_i$ with surjective and connected graphs. Also, let $(p_1,p_2,p_3,\ldots) \in \prod_{i=1}^{\infty}[0,1]$, let m and n be positive integers, $m+1 \le n$, for each $i \in \{m, m+1, m+2, \ldots, n\}$, let $a_i, b_i \in [0,1] \subseteq I_i$ such that $a_i < b_i$ and $[a_i,b_i] \ne [0,1]$, and let $a_i', b_i' \in [0,1] = I_{i+1}$ such that $a_i' < b_i'$ and $[a_i', b_i'] \ne [0,1]$.*

The double sequence

$$([a_i,b_i],[a_i',b_i'])_{i=m}^{n}$$

is called an inverse component cropping sequence *over $[m,n]$ for $\{f_i \mid i = 1,2,3,\ldots\}$ with respect to $(p_1,p_2,p_3,\ldots)$, if*

1. *for each positive integer i, $p_{2i-1} = p_{2i+2}$ and $(p_{2i}, p_{2i-1}) \in \Gamma(f_i)$,*
2. *for each $i \in \{m, m+1, m+2, \ldots, n-1\}$, $[a_{i+1}', b_{i+1}'] = [a_i, b_i]$,*

3. *for each* $i \in \{m, m+1, m+2, m+3, \dots, n\}$*, there is a component* C_i *of*

$$\Gamma(f_i) \cap ([a'_i, b'_i] \times [a_i, b_i])$$

such that $(p_{2i}, p_{2i-1}) \in C_i$ *and*

(a) $\Gamma(f_m) \sqsubset_{C_m} T_m(a_m, a'_m, b_m, b'_m)$ *or* $\Gamma(f_m) \sqsubset_{C_m} B_m(a_m, a'_m, b_m, b'_m)$.

(b)
- *if* $n = m+1$*, then*

$$\Gamma(f_{m+1}) \sqsubset_{C_{m+1}} R_{m+1}(a_{m+1}, a'_{m+1}, b_{m+1}, b'_{m+1})$$

if $\Gamma(f_m) \sqsubset_{C_m} B_m(a_m, a'_m, b_m, b'_m)$*, and*

$$\Gamma(f_{m+1}) \sqsubset_{C_{m+1}} L_{m+1}(a_{m+1}, a'_{m+1}, b_{m+1}, b'_{m+1})$$

if $\Gamma(f_m) \sqsubset_{C_m} T_m(a_m, a'_m, b_m, b'_m)$*;*

- *if* $n > m+1$*, then*

$$\Gamma(f_{m+1}) \sqsubset_{C_{m+1}} TL_{m+1}(a_{m+1}, a'_{m+1}, b_{m+1}, b'_{m+1})$$

or

$$\Gamma(f_{m+1}) \sqsubset_{C_{m+1}} BL_{m+1}(a_{m+1}, a'_{m+1}, b_{m+1}, b'_{m+1})$$

if $\Gamma(f_m) \sqsubset_{C_m} T_m(a_m, a'_m, b_m, b'_m)$*, and*

$$\Gamma(f_{m+1}) \sqsubset_{C_{m+1}} TR_{m+1}(a_{m+1}, a'_{m+1}, b_{m+1}, b'_{m+1})$$

or

$$\Gamma(f_{m+1}) \sqsubset_{C_{m+1}} BR_{m+1}(a_{m+1}, a'_{m+1}, b_{m+1}, b'_{m+1})$$

if $\Gamma(f_m) \sqsubset_{C_m} B_m(a_m, a'_m, b_m, b'_m)$*;*

(c) *if* $m+1 \leq i < n-1$*, then*

$$\Gamma(f_{i+1}) \sqsubset_{C_{i+1}} TL_{i+1}(a_{i+1}, a'_{i+1}, b_{i+1}, b'_{i+1})$$

or

$$\Gamma(f_{i+1}) \sqsubset_{C_{i+1}} BL_{i+1}(a_{i+1}, a'_{i+1}, b_{i+1}, b'_{i+1})$$

if $\Gamma(f_i) \sqsubset_{C_i} TL_i(a_i, a'_i, b_i, b'_i)$ *or* $\Gamma(f_i) \sqsubset_{C_i} TR_i(a_i, a'_i, b_i, b'_i)$*, and*

$$\Gamma(f_{i+1}) \sqsubset_{C_{i+1}} TR_{i+1}(a_{i+1}, a'_{i+1}, b_{i+1}, b'_{i+1})$$

or

$$\Gamma(f_{i+1}) \sqsubset_{C_{i+1}} BR_{i+1}(a_{i+1}, a'_{i+1}, b_{i+1}, b'_{i+1})$$

if $\Gamma(f_i) \sqsubset_{C_i} BL_i(a_i, a'_i, b_i, b'_i)$ *or* $\Gamma(f_i) \sqsubset_{C_i} BR_i(a_i, a'_i, b_i, b'_i)$*;*

(d) *if* $n > m+1$, *then*

$$\Gamma(f_n) \sqsubset_{C_n} L_n(a_n, a'_n, b_n, b'_n)$$

if $\Gamma(f_{n-1}) \sqsubset_{C_{n-1}} TL_{n-1}(a_{n-1}, a'_{n-1}, b_{n-1}, b'_{n-1})$ *or* $\Gamma(f_{n-1}) \sqsubset_{C_{n-1}} TR_{n-1}(a_{n-1}, a'_{n-1}, b_{n-1}, b'_{n-1})$, *and*

$$\Gamma(f_n) \sqsubset_{C_n} R_n(a_n, a'_n, b_n, b'_n)$$

if $\Gamma(f_{n-1}) \sqsubset_{C_{n-1}} BL_{n-1}(a_{n-1}, a'_{n-1}, b_{n-1}, b'_{n-1})$ *or* $\Gamma(f_{n-1}) \sqsubset_{C_{n-1}} BR_{n-1}(a_{n-1}, a'_{n-1}, b_{n-1}, b'_{n-1})$.

We say that $\{f_i \mid i = 1,2,3,\ldots\}$ admits an inverse component cropping sequence*, if there are*

1. $(p_1, p_2, p_3, \ldots) \in \prod_{i=1}^{\infty}[0,1]$,
2. *positive integers* m *and* n, $m+1 \leq n$, *and*
3. *an inverse component cropping sequence over* $[m,n]$ *for* $\{f_i \mid i = 1,2,3,\ldots\}$ *with respect to* $(p_1, p_2, p_3, \ldots)$*, see Figure 3.*

We use the following lemma in the proof of Theorem 3.5.

Lemma 3.4. *Let* $\{I_i, f_i\}_{i=1}^{\infty}$ *be an inverse sequence of intervals* $I_i = [0,1]$ *and upper semicontinuous functions* $f_i : I_{i+1} \multimap I_i$ *with surjective and connected graphs. The following statements are equivalent.*

1. $\{f_i^{-1} \mid i = 1,2,3,\ldots\}$ *admits a component cropping sequence.*
2. $\{f_i \mid i = 1,2,3,\ldots\}$ *admits an inverse component cropping sequence.*

Proof. Suppose $\{f_i^{-1} \mid i = 1,2,3,\ldots\}$ admits a component cropping sequence. Let

1. $(p_1, p_2, p_3, \ldots) \in \varprojlim\{I_i, f_i^{-1}\}_{i=1}^{\infty}$,
2. m and n be positive integers, $m+1 \leq n$, and
3. $([a_i, b_i])_{i=m}^{n+1}$ be a component cropping sequence over $[m,n]$ for $\{f_i^{-1}|i = 1,2,3,\ldots\}$ with respect to $(p_1, p_2, p_3, \ldots)$.

Next, let

1. for each positive integer i, $q_{2i} = p_i$ and $q_{2i-1} = p_{i+1}$,
2. $(c_m, c_{m+1}, c_{m+2}, \ldots, c_n) = (a_{m+1}, a_{m+2}, a_{m+3}, \ldots, a_{n+1})$,
3. $(d_m, d_{m+1}, d_{m+2}, \ldots, d_n) = (b_{m+1}, b_{m+2}, b_{m+3}, \ldots, b_{n+1})$,
4. $(c'_m, c'_{m+1}, c'_{m+2}, \ldots, c'_n) = (a_m, a_{m+1}, a_{m+2}, \ldots, a_n)$, and
5. $(d'_m, d'_{m+1}, d'_{m+2}, \ldots, d'_n) = (b_m, b_{m+1}, b_{m+2}, \ldots, b_n)$.

It follows that $([c_i, d_i], [c'_i, d'_i])_{i=m}^{n}$ satisfies all of the conditions of Definition 3.3 for $(q_1, q_2, q_3, \ldots)$ and $\{f_i \mid i = 1,2,3,\ldots\}$, and it is therefore an inverse component cropping sequence over $[m,n]$ for $\{f_i \mid i = 1,2,3,\ldots\}$ with respect to $(q_1, q_2, q_3, \ldots)$.

Conversely, suppose that $\{f_i \mid i = 1,2,3,\ldots\}$ admits an inverse component cropping sequence. Let

1. $(p_1, p_2, p_3, \ldots) \in \prod_{i=1}^{\infty}[0,1]$,

2. m and n be positive integers, $m+1 \leq n$, and
3. $([a_i, b_i], [a'_i, b'_i])_{i=m}^{n}$ be an inverse component cropping sequence over $[m, n]$ for $\{f_i \mid i = 1, 2, 3, \ldots\}$ with respect to $(p_1, p_2, p_3, \ldots)$.

Next, let

1. $q_i = p_{2i}$ for each positive integer i,

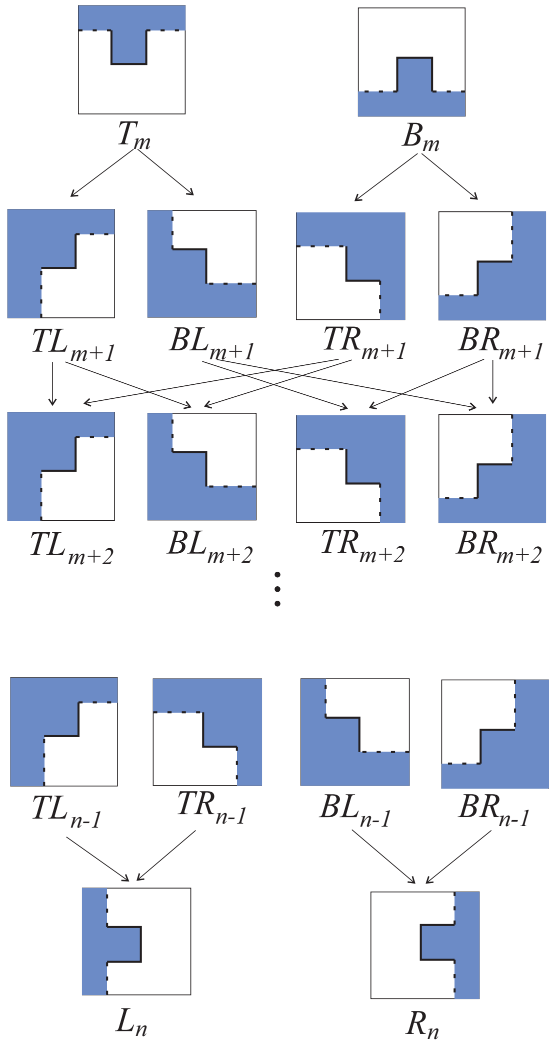

FIGURE 3. Inverse component cropping sequences.

2. for each $i \in \{m, m+1, m+2, \ldots, n\}$, $[c_i, d_i] = [a_i', b_i']$, and $[c_{n+1}, d_{n+1}] = [a_n, b_n]$.

It follows that $([c_i, d_i])_{i=m}^{n+1}$ satisfies all of the conditions of Definition 2.4 for $(q_1, q_2, q_3, \ldots)$ and $\{f_i^{-1} \mid i = 1, 2, 3, \ldots\}$, and it is therefore a component cropping sequence over $[m, n]$ for $\{f_i^{-1} \mid i = 1, 2, 3, \ldots\}$ with respect to $(q_1, q_2, q_3, \ldots)$. □

Theorem 3.5. *Let $\{I_i, f_i\}_{i=1}^{\infty}$ be an inverse sequence of intervals $I_i = [0, 1]$ and upper semicontinuous functions $f_i : I_{i+1} \multimap I_i$ with surjective and connected graphs. The following statements are equivalent.*

1. $\underset{\longleftarrow}{\lim}\{I_i, f_i^{-1}\}_{i=1}^{\infty}$ *is connected.*
2. $\{f_i \mid i = 1, 2, 3, \ldots\}$ *does not admit an inverse component cropping sequence.*

Proof. The inverse limit $\underset{\longleftarrow}{\lim}\{I_i, f_i^{-1}\}_{i=1}^{\infty}$ is connected if and only if $\{f_i^{-1} \mid i = 1, 2, 3, \ldots\}$ does not admit a component cropping sequence by Theorem 2.6.

$\{f_i^{-1} \mid i = 1, 2, 3, \ldots\}$ does not admit a component cropping sequence if and only if $\{f_i \mid i = 1, 2, 3, \ldots\}$ does not admit an inverse component cropping sequence by Lemma 3.4. □

In the following example we demonstrate how to recognize an inverse component cropping sequence.

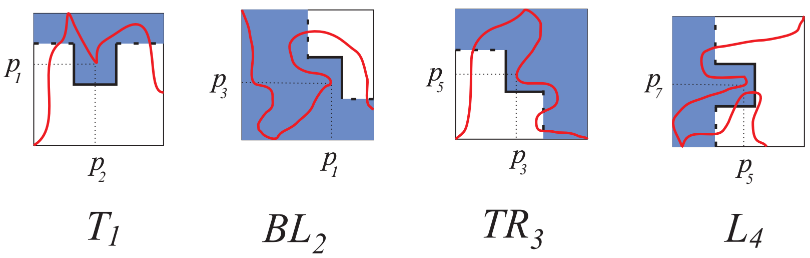

Figure 4. The graphs $\Gamma(f_1)$, $\Gamma(f_2)$, $\Gamma(f_3)$ and $\Gamma(f_4)$.

Example 3.6. Let $\{I_i, f_i\}_{i=1}^{\infty}$ be an inverse sequence of intervals $I_i = [0, 1]$ and upper semicontinuous functions $f_i : I_{i+1} \multimap I_i$ with surjective and connected graphs. The graphs of f_1, f_2, f_3 and f_4 are pictured in Figure 4. For each positive integer $i > 4$, let $f_i(x) = \{x\}$ for each $x \in I_{i+1}$.

As seen in the figure, there is an inverse component cropping sequence over $[1, 4]$ for $\{f_i \mid i = 1, 2, 3, \ldots\}$ with respect to $(p_1, p_2, p_3, p_1, p_5, p_3, p_7, p_5, p_7,$

$p_7, \ldots)$. Therefore the inverse limit $\varprojlim \{I_i, f_i^{-1}\}_{i=1}^{\infty}$ is not connected by Theorem 3.5.

In the following theorem we prove that for the inverse limits of inverse sequences with a single bonding function, the existence of a component cropping sequence is equivalent to existence of an inverse component cropping sequence.

THEOREM 3.7. *Let* $\{I_i, f\}_{i=1}^{\infty}$ *be an inverse sequence of intervals* $I_i = [0,1]$ *and a single upper semicontinuous bonding function* $f : I_{i+1} \multimap I_i$ *with a surjective and connected graph. The following statements are equivalent.*

1. $\{f \mid i = 1, 2, 3, \ldots\}$ *admits a component cropping sequence.*
2. $\{f \mid i = 1, 2, 3, \ldots\}$ *admits an inverse component cropping sequence.*

PROOF. $\{f \mid i = 1, 2, 3, \ldots\}$ admits a component cropping sequence if and only if $\varprojlim \{I_i, f\}_{i=1}^{\infty}$ is not connected by Theorem 2.6. $\varprojlim \{I_i, f\}_{i=1}^{\infty}$ is not connected if and only if $\varprojlim \{I_i, f^{-1}\}_{i=1}^{\infty}$ is not connected by Theorem 1.1. $\varprojlim \{I_i, f^{-1}\}_{i=1}^{\infty}$ is not connected if and only if $\{f \mid i = 1, 2, 3, \ldots\}$ admits an inverse component cropping sequence by Theorem 3.5. □

The following corollary gives another characterization of connected inverse limits of inverse sequences of intervals with a single bonding function.

COROLLARY 3.8. *Let* $\{I_i, f\}_{i=1}^{\infty}$ *be an inverse sequence of intervals* $I_i = [0,1]$ *and a single upper semicontinuous bonding function* $f : I_{i+1} \multimap I_i$ *with a surjective and connected graph. The following statements are equivalent.*

1. $\varprojlim \{I_i, f\}_{i=1}^{\infty}$ *is connected.*
2. $\{f \mid i = 1, 2, 3, \ldots\}$ *does not admit an inverse component cropping sequence.*

PROOF. $\varprojlim \{I_i, f\}_{i=1}^{\infty}$ is connected if and only if $\{f \mid i = 1, 2, 3, \ldots\}$ does not admit a component cropping sequence by Theorem 2.6. $\{f \mid i = 1, 2, 3, \ldots\}$ does not admit a component cropping sequence if and only if $\{f \mid i = 1, 2, 3, \ldots\}$ does not admit an inverse component cropping sequence by Theorem 3.7. □

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