GLASNIK MATEMATIČKI
Vol. 53(73)(2018), 265 – 273

TRINOMIALS $ax^8 + bx + c$ WITH GALOIS GROUPS OF ORDER 1344

Szabolcs Tengely
University of Debrecen, Hungary

Abstract. Bruin and Elkies ([7]) obtained the curve of genus 2 parametrizing trinomials $ax^8 + bx + c$ whose Galois group is contained in $G_{1344} = (\mathbb{Z}/2)^3 \rtimes G_{168}$. They found some rational points of small height and computed the associated trinomials. They conjecture that the only $\mathbb{Q}$-rational points of the hyperelliptic curve

$$Y^2 = 2X^6 + 28X^5 + 196X^4 + 784X^3 + 1715X^2 + 2058X + 2401$$

are given by $(X, Y) = (0, \pm 49), (-1, \pm 38), (-3, \pm 32)$, and $(-7, \pm 196)$. In this paper we prove that the above points are the only S-integral points with $S = \{2, 3, 5, 7, 11, 13, 17, 19\}$.

1. Introduction

In the literature there are many interesting results dealing with trinomials having certain Galois group. Bremner and Spearman ([3]) proved that up to scaling $x^6 + 133x + 209$ is the only irreducible sextic trinomial with Galois group C_6. Brown, Spearman and Yang ([5,6]) characterized rational trinomials with Galois group $A_4, A_4 \times C_2, S_3$ and $C_3 \times S_3$. Brown, Spearman and Yang ([5]) proved that to obtain some cyclic sextic trinomial (other than the previously mentioned $x^6 + 133x + 209$) over some number field K a rational point on the genus 2 curve $Y^2 = X^6 + 105X^4 + 2400X^2 - 19200$ should exist (other than the ones with $X = \pm 4$). Bruin and Elkies ([7]) determined the set of rational points on the hyperelliptic curve $Y^2 = X(81X^5 + 396X^4 +$

2010 *Mathematics Subject Classification.* 11G30, 11Y50.
Key words and phrases. Trinomials, hyperelliptic curves, S-integral points.
The publication is supported by the EFOP-3.6.1-16-2016-00022 project. The project is co-financed by the European Union and the European Social Fund. The research was supported in part by grant K115479 and K128088 (Sz.T.) of the Hungarian National Foundation for Scientific Research.

$738X^3 + 660X^2 + 269X + 48$) via covering techniques and the so-called elliptic Chabauty's method ([8, 9]) and they concluded that every trinomial $ax^7 + bx + c$ over $\mathbb{Q}$ with Galois group contained in G_{168} is equivalent to one of the following trinomials

$$\begin{aligned}&x^7 - 7x + 3,\\ &x^7 - 154x + 99,\\ &37^2x^7 - 28x + 9,\\ &499^2x^7 - 23956x + 3^4 \cdot 113.\end{aligned}$$

They conjecture that the only $\mathbb{Q}$-rational points of the hyperelliptic curve $Y^2 = 2X^6 + 28X^5 + 196X^4 + 784X^3 + 1715X^2 + 2058X + 2401$ are given by $(X, Y) = (0, \pm 49), (-1, \pm 38), (-3, \pm 32)$, and $(-7, \pm 196)$. From the above list of rational points they recover the following degree-8 trinomials with Galois group contained in G_{1344}

$$\begin{aligned}&x^8 + 16x + 28,\\ &x^8 + 576x + 1008,\\ &19^4 \cdot 53x^8 + 19x + 2,\\ &x^8 + 324x + 567.\end{aligned}$$

They remark that the Mordell-Weil group of the Jacobian of the hyperelliptic curve $Y^2 = 2X^6 + 28X^5 + 196X^4 + 784X^3 + 1715X^2 + 2058X + 2401$ has rank 2, so classical Chabauty cannot be applied. To apply elliptic Chabauty one has to find rational points on elliptic curves over a degree 15 extension of $\mathbb{Q}$.

In this paper we provide a partial result related to the above conjecture. We prove the following statement.

Theorem 1.1. *Let* $S = \{2, 3, 5, 7, 11, 13, 17, 19\}$. *The only* S*-integral points on the hyperelliptic curve*

$$\mathcal{C}_1 : Y^2 = 2X^6 + 28X^5 + 196X^4 + 784X^3 + 1715X^2 + 2058X + 2401$$

are given by $(X, Y) = (0, \pm 49), (-1, \pm 38), (-3, \pm 32)$, *and* $(-7, \pm 196)$.

The proof is based on techniques developed in [11] for integral points on hyperelliptic curves and [13, 14] for S-integral points.

2. Auxiliary results

We recall some notation and results from [11, 13] related to S-integral points on hyperelliptic curves that will be used later on. Consider the hyperelliptic curve

$$\mathcal{C}: \quad ay^2 = F(x) := x^6 + b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0, \tag{2.1}$$

where $a \neq 0, b_i \in \mathbb{Z}$. Let α be a root of F and $J(\mathbb{Q})$ be the Jacobian of the curve $\mathcal{C}$. We have that

$$x - \alpha = \kappa\xi^2$$

where κ, $\xi \in K = \mathbb{Q}(\alpha)$ and κ comes from a finite set. By knowing the Mordell-Weil group of the curve $\mathcal{C}$ it is possible to provide a method to compute such a finite set. We assume that a rational point P_0 on $\mathcal{C}$ is known. Let $\epsilon_0 = 1$ if P_0 is one of the two points at infinity and $\epsilon_0 = \gamma_0 - \alpha d_0^2$, where $x(P_0) = \gamma_0/d_0^2, \gamma_0 \in \mathbb{Z}$ and $d_0 \in \mathbb{N}$. Every coset of $J(\mathbb{Q})/2J(\mathbb{Q})$ can be represented by a point of the form $\sum_{i=1}^m (P_i - P_0)$ where the set $\{P_1, \dots, P_m\}$ is stable under the action of the Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, and such that all $y(P_i)$ are non-zero. Let $x(P_i) = \gamma_i/d_i^2$, where γ_i is and algebraic integer and $d_i \in \mathbb{N}$. An algebraic number $\epsilon = \epsilon_0^{(m \mod 2)} \prod_{i=1}^m (\gamma_i - \alpha d_i^2)$ is associated to such a coset. The following result is [13, Lemma 3.1.2].

LEMMA 2.1. *Let $\mathcal{E}$ be a set of ϵ associated as above to a complete set of coset representatives for $J(\mathbb{Q})/2J(\mathbb{Q})$. Let Δ be the discriminant of the polynomial F. For each $\epsilon \in \mathcal{E}$ let B_ϵ be the set of square-free rational integers supported only by primes dividing $a\Delta Norm_{K/\mathbb{Q}}(\epsilon)\prod_{p\in S} p$. Let $\mathcal{K} = \{\epsilon b : \epsilon \in \mathcal{E}, b \in B_\epsilon\}$. Then $\mathcal{K}$ is a finite subset of $\mathcal{O}_K$ and if (x, y) is an S-integral point on* (2.1)*, then $x - \alpha = \kappa\xi^2$ for some $\kappa \in \mathcal{K}, \xi \in K$.*

We introduce some notation we need to provide upper bounds for the size of S-integral solutions of hyperelliptic equations. Let α be an algebraic integer of degree at least 3, and let κ be a integer belonging to K. Let α_1, α_2, α_3 be distinct conjugates of α and κ_1, κ_2, κ_3 be the corresponding conjugates of κ. Let

$$K_1 = \mathbb{Q}(\alpha_1, \alpha_2, \sqrt{\kappa_1\kappa_2}), \quad K_2 = \mathbb{Q}(\alpha_1, \alpha_3, \sqrt{\kappa_1\kappa_3}), \quad K_3 = \mathbb{Q}(\alpha_2, \alpha_3, \sqrt{\kappa_2\kappa_3}),$$

and

$$L = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3, \sqrt{\kappa_1\kappa_2}, \sqrt{\kappa_1\kappa_3}).$$

Let S be a finite set of rational primes with $|S| = s$. If $S = \emptyset$, then let $P = 1$, otherwise $P = \max S$. Let d be the degree of L. Let d_1, d_2, d_3 and r_1, r_2, r_3 be the degrees and the unit ranks of K_1, K_2, K_3 respectively. Let R be an upper bound for the regulators of K_1, K_2, K_3 and R_S an upper bound for the respective S_{K_i}-regulators of K_1, K_2, K_3. Let s_i be the number of places in S_{K_i}. Let h_{K_i} be an upper bound for the class numbers of the K_i. For a positive real number a let $\log^*(a) = \max\{1, \log a\}$. Let $c_j^* = \max_{i=1,2,3} c_j(s_i, d_i), j = 1, 2, \dots, 5$, where

$$c_1(s_i, d_i) = \frac{((s_i - 1)!)^2}{2^{s_i - 2} d_i^{s_i - 1}}, \quad c_2(s_i, d_i) = 29e\sqrt{s_i - 2}\, c_1(s_i, d_i) d_i^{s_i - 1} \log^*(d_i),$$

$$c_3(s_i, d_i) = \frac{((s_i - 1)!)^2}{2^{s_i - 1}} \begin{cases} 2/\log 2 & \text{if } d_i = 1, \\ (\log(3d_i))^2 & \text{if } d_i \geq 2, \end{cases}$$

$$c_4(s_i,d_i)=d_i\pi^{s_i-2}c_2(s_i,d_i),\quad c_5(s_i,d_i)=2d_ic_3(s_i,d_i).$$

Let $c_6^*=\max_{i=1,2,3}c_6(r_i,d_i)$, where

$$c_6(r_i,d_i)=\begin{cases}0 & \text{if } r_i=0,\\ 1/d_i & \text{if } r_i=1,\\ 29er_i!\sqrt{r_i-1}\log(d_i) & \text{if } r_i\geq 2.\end{cases}$$

Let

$$N=\max_{1\leq i,j\leq 3}\left|\operatorname*{Norm}_{\mathbb{Q}(\alpha_i,\alpha_j)/\mathbb{Q}}(\kappa_i(\alpha_i-\alpha_j))\right|^2,$$

$$H^*=\max\left\{\pi/d,\frac{\log N}{\min_{1\leq i\leq 3}d_i}+c_6^*R+\mathrm{h}(\kappa)+h\left(\sum_{p\in S}\log p\right)\right\},$$

$$c_7(n,d)=\min\{1.451(30\sqrt{2})^{n+4}(n+1)^{5.5},\pi 2^{6.5n+27}\}d^2\log(ed),$$

$$c_8(n,d)=(16ed)^{2(n+1)}n^{3/2}\log(2nd)\log(2d),$$

$$c_9(n,d)=(2d)^{2n+1}\log(2d)\log^3(3d),$$

$$c_{10}^*=2H^*+2H^*d(s+1)(1+2(c_4^*)^2c_7(s_1+s_2-1,d)R_S^2\times$$
$$\times\log(\sqrt{2}e\max\{(s_1+s_2-2)\pi/\sqrt{2},c_2^*R_S\}),$$

$$c_{11}^*=4d(s+1)H^*(c_4^*)^2c_7(s_1+s_2-1,d)R_S,$$

$$c_{12}^*=2H^*+2H^*d(s+1)+c_{11}^*\log\left(\frac{\max\{c_5^*,1\}}{2\sqrt{2}dH^*}\right),$$

$$c_{13}^*=\log 2+2H^*+4(s_1+s_2-2)H^*(c_1^*)^2c_2^*c_9(s_1+s_2-1,d)R_S^3,$$

$$c_{14}^*=\frac{2H^*d^{s_1+s_2-2}P^d}{\log(2)\log^*(P^d)}(c_1^*)^2c_8(s_1+s_2,d)R_S^2,$$

$$c_{15}^*=2H^*+2H^*d(s+1)+$$
$$+c_{14}^*\log\left(\frac{\max\{c_5^*,1\}e^{(s_1+s_2)(6(s_1+s_2)-1)}d^{3(s_1+s_2-1)}\log(2d)P^{d(s_1+s_2)}}{H^*c_9(s_1+s_2-1,d)}\right).$$

The following result is [13, Theorem 3.7.1].

LEMMA 2.2. *If $x\in\mathbb{Q}\setminus\{0\}$ is a S-integer satisfying $x-\alpha=\kappa\xi^2$ for some $\xi\in K$, then*

$$\mathrm{h}(x)\leq 20\log 2+13\,\mathrm{h}(\kappa)+19\,\mathrm{h}(\alpha)+H^*+$$
$$+8\max\{c_{10}^*/2,c_{13}^*/2,c_{12}^*+c_{11}^*\log c_{11}^*,c_{15}^*+c_{14}^*\log c_{14}^*\}.$$

The previous result provides an upper bound for the size of S-integral solutions, the next one gives lower bound for the size of rational solutions that is not contained in a given set W, the set of known points. This is

[11, Lemma 12.1]. Let P_0 be a fixed rational point on the curve (2.1) and let $\jmath$ be the corresponding Abel-Jacobi map given by

$$\jmath : \mathcal{C} \to J, \quad P \to [P - P_0].$$

Let $D_1, \ldots, D_r$ be generators of the free part of $J(\mathbb{Q})$ and

$$\phi : \mathbb{Z}^r \to J(\mathbb{Q}), \quad (a_1, \ldots, a_r) = \sum_{k=1}^{r} a_k D_k.$$

Lemma 2.3. *Let W be a finite subset of $J(\mathbb{Q})$, and let L be a sublattice of $\mathbb{Z}^r$. Suppose that $\jmath(C(\mathbb{Q})) \subset W + \phi(L)$. Let μ_1 be such that*

$$\mu_1 \le h(D) - \hat{h}(D),$$

where $\hat{h}$ denotes the canonical height and h is an appropriately normalized logarithmic height on J. Let

$$\mu_2 = \max\left\{\sqrt{\hat{h}(w)} \ : \ w \in W\right\}.$$

Let M be the height-pairing matrix for the Mordell–Weil basis $D_1, \ldots, D_r$ and let $\lambda_1, \ldots, \lambda_r$ be its eigenvalues. Let

$$\mu_3 = \min\left\{\sqrt{\lambda_j} \ : \ j = 1, \ldots, r\right\}.$$

Let $m(L)$ be the Euclidean norm of the shortest non-zero vector of L. Then, for any $P \in C(\mathbb{Q})$, either $\jmath(P) \in W$ or

$$h(\jmath(P)) \ge (\mu_3 m(L) - \mu_2)^2 + \mu_1.$$

3. Proof of Theorem 1.1

To obtain an upper bound for the size of the S-integral points we use the following model

$$\mathcal{C}_2 : y^2 = F(x) := x^6 + 20x^4 + 12x^3 + 25x^2 + 24x + 16,$$

which is isomorphic to the curve $\mathcal{C}_1$ over $\mathbb{Z}[\frac{1}{7}]$, hence they have the same S-integral points. As an application of his theory of lower bounds for linear forms in logarithms, Baker ([1]) gave an explicit upper bound for the size of integral solutions of hyperelliptic curves. This result has been improved by many authors (see e.g. [4, 10, 18, 22]). In [11] an improved completely explicit upper bound for integral points were proved combining ideas from [10, 12, 15–17, 22] and in [13, 14] for S-integral points, the main results stated in Section 2. Let α be a root of F. We have that

$$x - \alpha = \kappa \xi^2$$

where κ, $\xi \in K = \mathbb{Q}(\alpha)$ and κ comes from a finite set. An appropriate finite set can be determined using Lemma 2.1. Using MAGMA ([2]) we get that $J(\mathbb{Q})$ is free of rank 2 with Mordell-Weil basis given by

$$D_1 = < x^2 - 2x + 8, 7x - 28 >,$$
$$D_2 = < x^2 + 1/2x + 2, 7/4x + 7 >$$

in Mumford representation, the torsion subgroup is trivial. The MAGMA procedures used to compute these data are based on Stoll's papers [19–21]. We obtain that

$$\mathcal{E} = \{1, \alpha^2 - 2\alpha + 8, 256\alpha^2 + 32\alpha + 32, 256\alpha^4 - 480\alpha^3 + 2016\alpha^2 + 192\alpha + 256\},$$

the discriminant of F is $-2^{24}7^8$ and the primes dividing the norms of the elements of $\mathcal{E}$ are $\{2, 7, 59, 8839\}$.

According to the Remark at page 42 in [13] we only need to compute bounds for some of these possible values. In our case only 4 values remain

$$\kappa_1 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 59 \cdot 8839,$$
$$\kappa_2 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 59 \cdot 8839 \cdot (\alpha^2 - 2\alpha + 8),$$
$$\kappa_3 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 59 \cdot 8839 \cdot (256\alpha^2 + 32\alpha + 32),$$
$$\kappa_4 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 59 \cdot 8839 \cdot$$
$$\cdot (256\alpha^4 - 480\alpha^3 + 2016\alpha^2 + 192\alpha + 256).$$

For these values we have the following bounds

κ	κ_1	κ_2	κ_3	κ_4
Bound for the S-regulator	3.102×10^{123}	3.102×10^{123}	1.001×10^{292}	9.457×10^{292}
S-unit rank	64	64	113	113
bound for $h(x)$	1.741×10^{1792}	1.741×10^{1792}	3.449×10^{4165}	3.449×10^{4165}

It means that if (x, y) is an S-integral point on the curve $\mathcal{C}_2$ with $x = x_1/x_2, x_1, x_2 \in \mathbb{Z}, \gcd(x_1, x_2) = 1$, then Lemma 2.2 implies that

$$\max\{|x_1|, |x_2|\} \leq \exp(3.449 \times 10^{4165}),$$

here we used the MAGMA code `upperbounds.m` written by Gallegos-Ruiz to obtain bounds for the solutions. We note that the total running time of the calculations was 30.6 hours on an Intel Core i7-6700HQ 2.6GHz PC.

Let W be the image of the set of these known rational points in $J(\mathbb{Q})$, that is $W = \{0 \cdot D_1 + 0 \cdot D_2, -4 \cdot D_1 + 3 \cdot D_2, -5 \cdot D_1 + 0 \cdot D_2, -2 \cdot D_1 + 1 \cdot D_2, -1 \cdot D_1 - 1 \cdot D_2, -3 \cdot D_1 - 1 \cdot D_2, -4 \cdot D_1 + 1 \cdot D_2, -1 \cdot D_1 - 3 \cdot D_2\}$. Applying the Mordell-Weil sieve explained in [11] we obtain that $\jmath(C(\mathbb{Q})) \subseteq W + BJ(\mathbb{Q})$, where

$$B = 2^4 \cdot 3^4 \cdot 5^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41$$
$$\cdot 43 \cdot 47 \cdot 53 \cdot 59 \cdot 61 \cdot 67 \cdot 79 \cdot 83 \cdot 103 \cdot 107 \cdot 163 \cdot 167 \cdot 179 \cdot 181.$$

For this computation, we used information modulo good primes $p < 50000$ such that $\#J(\mathbb{F}_p)$ is 300-smooth. The total running time of this calculations was 34 minutes on an Intel Core i7-6700HQ 2.6GHz PC. We have that to 3 decimal places

$$\mu_1 = -7.873, \quad \mu_2 = 1.921, \quad \mu_3 = 0.283.$$

We apply Lemma 2.3 successively to primes of good reduction that satisfy the conditions of the lemma and Criteria (I)(IV) ([11, p. 878]). Using the first 50000 primes we obtain that a lower bound for the size of $\jmath(P)$ for P in the set of unknown rational points is

$$3.483 \times 10^{672}$$

and

$$\begin{aligned} B_1 = {} & 7563170114517001337699926872933929 4555 \\ & 38174684977550374967399628867322197 8757 \\ & 26365989785325666235115888371369266 7920 \\ & 793326000000. \end{aligned}$$

We replace B by B_1 and start to sieve using primes that did not satisfied the criteria in the first application. After the second turn we have that the bound is

$$6.945 \times 10^{2510}$$

and the new value of B is of size 4.87×10^{567}. By applying the Mordell-Weil sieve using the first 50000 primes two more times we get that

$$h(\jmath(P)) \geq 2.157 \times 10^{9124}$$

for an unknown rational point P. Hence

$$h(x) \geq 1.079 \times 10^{9124}.$$

The total running time of this calculations was 21.8 hours on an Intel Core i7-6700HQ 2.6GHz PC. It contradicts the bound obtained earlier, hence the only S-integral points with $S = \{2, 3, 5, 7, 11, 13, 17, 19\}$ on the hyperelliptic curve $\mathcal{C}_1$ are given by

$$(X, Y) = (0, \pm 49), (-1, \pm 38), (-3, \pm 32), (-7, \pm 196).$$

Acknowledgements.

The author would like to thank the anonymous reviewers for their valuable comments and suggestions to improve the quality of the paper.

References

[1] A. Baker, *Bounds for the solutions of the hyperelliptic equation*, Proc. Cambridge Philos. Soc., **65** (1969), 439–444.

[2] W. Bosma, J. Cannon and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), 235–265.

[3] A. Bremner and B. K. Spearman, *Cyclic sextic trinomials* x^6+Ax+B, Int. J. Number Theory **6** (2010), 161–167.

[4] B. Brindza, *On S-integral solutions of the equation* $y^m = f(x)$, Acta Math. Hungar. **44** (1984), 133–139.

[5] S. C. Brown, B. K. Spearman and Q. Yang, *On the Galois groups of sextic trinomials*, JP J. Algebra Number Theory Appl. **18** (2010), 67–77.

[6] S. C. Brown, B. K. Spearman and Q. Yang, *On sextic trinomials with Galois group C_6, S_3 or $C_3 \times S_3$*, J. Algebra Appl. **12** (2013), 1250128, 9 pp.

[7] N. Bruin and N. D. Elkies, *Trinomials $ax^7 + bx + c$ and $ax^8 + bx + c$ with Galois groups of order 168 and* $8 \cdot 168$, Algorithmic number theory (Sydney, 2002), Lecture Notes in Comput. Sci. 2369, Springer, Berlin, 2002, 172–188.

[8] N. Bruin, Chabauty methods and covering techniques applied to generalized Fermat equations, Dissertation, University of Leiden, Leiden, 1999. CWI Tract 133, Stichting Mathematisch Centrum Centrum voor Wiskunde en Informatica, Amsterdam, 2002.

[9] N. Bruin, *Chabauty methods using elliptic curves*, J. Reine Angew. Math. **562** (2003) 27–49.

[10] Y. Bugeaud, *Bounds for the solutions of superelliptic equations*, Compositio Math. **107** (1997), 187–219.

[11] Y. Bugeaud, M. Mignotte, S. Siksek, M. Stoll, and Sz. Tengely, *Integral points on hyperelliptic curves*, Algebra Number Theory **2** (2008), 859–885.

[12] Y. Bugeaud, M. Mignotte, and S. Siksek, *Classical and modular approaches to exponential Diophantine equations. I. Fibonacci and Lucas perfect powers*, Ann. of Math. (2) **163** (2006), 969–1018.

[13] H. R. Gallegos-Ruiz, S-integral points on hyperelliptic curves, PhD thesis, University of Warwick, 2010.

[14] H. R. Gallegos-Ruiz, *S-integral points on hyperelliptic curves*, Int. J. Number Theory **7** (2011), 803–824.

[15] E. Landau, Verallgemeinerung eines Pólyaschen satzes auf algebraische zahlkörper, 1918.

[16] E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers. II.* Izv. Ross. Akad. Nauk Ser. Mat. **64** (2000), 125–180.

[17] A. Pethő and B. M. M. de Weger, *Products of prime powers in binary recurrence sequences. I. The hyperbolic case, with an application to the generalized Ramanujan-Nagell equation*, Math. Comp. **47** (1986), 713–727.

[18] V. G. Sprindžuk, *The arithmetic structure of integer polynomials and class numbers,* Analytic number theory, mathematical analysis and their applications (dedicated to I. M. Vinogradov on his 85th birthday), Trudy Mat. Inst. Steklov. **143** (1977), 152–174.

[19] M. Stoll, *On the height constant for curves of genus two*, Acta Arith. **90** (1999), 183–201.

[20] M. Stoll, *Implementing 2-descent for Jacobians of hyperelliptic curves*, Acta Arith. **98** (2001), 245–277.

[21] M. Stoll, *On the height constant for curves of genus two. II*, Acta Arith. **104** (2002), 165–182.

[22] P. M. Voutier, *An upper bound for the size of integral solutions to* $Y^m = f(X)$, J. Number Theory **53** (1995), 247–271.

Sz. Tengely
Institute of Mathematics
University of Debrecen
P.O.Box 12, 4010 Debrecen
Hungary
E-mail: tengely@science.unideb.hu

Received: 7.11.2017.
Revised: 28.2.2018.