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TOTALLY REAL THUE INEQUALITIES OVER IMAGINARY QUADRATIC FIELDS

István Gaál, Borka Jadrijević and László Remete
University of Debrecen, Hungary and University of Split, Croatia

Abstract. Let $F(x, y)$ be an irreducible binary form of degree ≥ 3 with integer coefficients and with real roots. Let M be an imaginary quadratic field with ring of integers $\mathbb{Z}_M$. Let $K > 0$. We describe an efficient method how to reduce the resolution of the relative Thue inequalities

$$|F(x, y)| \leq K \quad (x, y \in \mathbb{Z}_M)$$

to the resolution of absolute Thue inequalities of type

$$|F(x, y)| \leq k \quad (x, y \in \mathbb{Z}).$$

We illustrate our method with an explicit example.

1. Introduction

Let $F(x, y) \in \mathbb{Z}[x, y]$ be an irreducible binary form of degree ≥ 3 and let $a \in \mathbb{Z} \setminus \{0\}$. There is an extensive literature of *Thue equations* of type

$$F(x, y) = a \quad \text{in} \quad x, y \in \mathbb{Z}.$$

In 1909 A. Thue ([10]) proved that these equations admit only finitely many solutions. In 1967 A. Baker ([1]) gave effective upper bounds for the solutions. Later on authors constructed numerical methods to reduce the bounds and to explicitly calculate the solutions, see [6] for a summary.

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Let M be an algebraic number field with ring of integers $\mathbb{Z}_M$. Let $F(x,y) \in \mathbb{Z}_M[x,y]$ be an irreducible binary form of degree $n \geq 3$ and let $\mu \in \mathbb{Z}_M \setminus \{0\}$. As a generalization of Thue equations consider *relative Thue equations* of type

$$F(x,y) = \mu \ \text{ in } \ x,y \in \mathbb{Z}_M.$$

Using Baker's method S. V. Kotov and V. G. Sprindžuk ([8]) were first to give effective upper bounds for the solutions of relative Thue equations. Their theorem has been extended by several authors. Applying Baker's method, reduction and enumeration algorithms I. Gaál and M. Pohst ([7]) gave an efficient algorithm for solving relative Thue equations (see also [6]).

Let M be an imaginary quadratic number field. Assuming in addition that the roots of $F(x,1)$ are all real, in the present paper we give an efficient algorithm to reduce the resolution of *relative Thue inequalities* of the type

$$|F(x,y)| \leq K \ \text{ in } \ x,y \in \mathbb{Z}_M$$

to the resolution of (absolute) *Thue inequalities* of the type

$$|F(x,y)| \leq k \ \text{ in } \ x,y \in \mathbb{Z}.$$

To find the solutions of the above absolute Thue inequality one can use Kash ([4]) or Magma ([2]) which admit efficient algorithms for solving (absolute) Thue equations $F(x,y) = k'$ for $k' \in \mathbb{Z}$ with $|k'| \leq k$. For an efficient method for calculating "small" solutions of Thue inequalities we refer to [9].

Our method is illustrated with an explicit example.

2. The main result

Let $F(x,y)$ be a binary form of degree $n \geq 3$ with rational integer coefficients. Assume that $f(x) = F(x,1)$ has leading coefficient 1 and distinct real roots $\alpha_1, \dots, \alpha_n$. Let $0 < \varepsilon < 1$, $0 < \eta < 1$ and let $K \geq 1$. Set

$$A = \min_{i \neq j} |\alpha_i - \alpha_j|, \qquad B = \min_i \prod_{j \neq i} |\alpha_j - \alpha_i|,$$

$$C = \max\left\{\frac{K}{(1-\varepsilon)^{n-1}B}, 1\right\},$$

$$C_1 = \max\left\{\frac{K^{1/n}}{\varepsilon A},\ (2C)^{1/(n-2)}\right\}, \qquad C_2 = \max\left\{\frac{K^{1/n}}{\varepsilon A},\ C^{1/(n-2)}\right\},$$

$$D = \left(\frac{K}{\eta(1-\varepsilon)^{n-1}AB}\right)^{1/n}, \qquad E = \frac{(1+\eta)^{n-1}K}{(1-\varepsilon)^{n-1}}.$$

Let $m \geq 1$ be a squarefree positive integer, and set $M = \mathbb{Q}(i\sqrt{m})$. Consider the relative Thue inequality

$$|F(x,y)| \leq K \ \text{ in } \ x,y \in \mathbb{Z}_M. \tag{2.1}$$

If $m \equiv 3 \pmod 4$, then $x, y \in \mathbb{Z}_M$ can be written as

$$x = x_1 + x_2 \frac{1 + i\sqrt{m}}{2} = \frac{(2x_1 + x_2) + x_2 i\sqrt{m}}{2},$$

$$y = y_1 + y_2 \frac{1 + i\sqrt{m}}{2} = \frac{(2y_1 + y_2) + y_2 i\sqrt{m}}{2}$$

with $x_1, x_2, y_1, y_2 \in \mathbb{Z}$.

If $m \equiv 1, 2 \pmod 4$, then $x, y \in \mathbb{Z}_M$ can be written as

$$x = x_1 + x_2 i\sqrt{m}, \quad y = y_1 + y_2 i\sqrt{m}$$

with $x_1, x_2, y_1, y_2 \in \mathbb{Z}$.

THEOREM 2.1. *Let $(x, y) \in \mathbb{Z}_M^2$ be a solution of (2.1). Assume that*

$$|y| > C_1 \quad \text{if} \quad m \equiv 3 \pmod 4, \tag{2.2}$$

$$|y| > C_2 \quad \text{if} \quad m \equiv 1, 2 \pmod 4. \tag{2.3}$$

Then

$$x_2 y_1 = x_1 y_2. \tag{2.4}$$

I. *Further, if $m \equiv 3 \pmod 4$, then the following holds:*

IA1. *If $2y_1 + y_2 = 0$, then $2x_1 + x_2 = 0$ and*

$$|F(x_2, y_2)| \leq \frac{2^n K}{(\sqrt{m})^n}. \tag{2.5}$$

IA2. *If $|2y_1 + y_2| \geq 2D$, then*

$$|F(2x_1 + x_2, 2y_1 + y_2)| \leq 2^n E. \tag{2.6}$$

IB1. *If $y_2 = 0$, then $x_2 = 0$ and*

$$|F(x_1, y_1)| \leq K. \tag{2.7}$$

IB2. *If $|y_2| \geq \frac{2}{\sqrt{m}} D$, then*

$$|F(x_2, y_2)| \leq \frac{2^n}{(\sqrt{m})^n} E. \tag{2.8}$$

II. *If $m \equiv 1, 2 \pmod 4$, then the following holds:*

IIA1. *If $y_1 = 0$, then $x_1 = 0$ and*

$$|F(x_2, y_2)| \leq \frac{K}{(\sqrt{m})^n}. \tag{2.9}$$

IIA2. *If $|y_1| \geq D$, then*

$$|F(x_1, y_1)| \leq E. \tag{2.10}$$

IIB1. *If $y_2 = 0$, then $x_2 = 0$ and*

$$|F(x_1, y_1)| \leq K. \tag{2.11}$$

IIB2. *If* $|y_2| \geq \frac{D}{\sqrt{m}}$, *then*

$$|F(x_2,y_2)| \leq \frac{E}{(\sqrt{m})^n}. \tag{2.12}$$

Our result is a far reaching generalization of an idea of [5].

3. Proof of the main result

In the proof of Theorem 2.1 we shall use the following Lemma.

Lemma 3.1. *Let* $x, y \in \mathbb{Z}, y \neq 0$. *Assume that*

$$\left|\alpha_{i_0} - \frac{x}{y}\right| \leq \frac{d}{|y|^n}$$

for some i_0 $(1 \leq i_0 \leq n)$ *and* $d > 0$. *If*

$$|y| \geq \left(\frac{d}{\eta A}\right)^{1/n},$$

then

$$|F(x,y)| \leq d(1+\eta)^{n-1} \prod_{j \neq i_0} |\alpha_j - \alpha_{i_0}|.$$

Proof. By our assumption, we have

$$\left|\alpha_j - \frac{x}{y}\right| \leq |\alpha_j - \alpha_{i_0}| + \left|\alpha_{i_0} - \frac{x}{y}\right| \leq (1+\eta)|\alpha_j - \alpha_{i_0}|$$

for $j \neq i_0$. Therefore

$$\prod_{j=1}^{n} \left|\alpha_j - \frac{x}{y}\right| = \left|\alpha_{i_0} - \frac{x}{y}\right| \cdot \prod_{j \neq i_0}^{n} \left|\alpha_j - \frac{x}{y}\right| \leq \frac{d}{|y|^n} \cdot (1+\eta)^{n-1} \cdot \prod_{j \neq i_0} |\alpha_j - \alpha_{i_0}|,$$

which implies our assertion. □

Now we turn to the proof of our main result Theorem 2.1.

Proof. Let $(x, y) \in \mathbb{Z}_M^2$ be an arbitrary solution of (2.1) with $y \neq 0$. Let $\beta_j = x - \alpha_j y, \ j = 1, \ldots, n$, then the inequality (2.1) can be written as

$$|\beta_1 \cdots \beta_n| \leq K. \tag{3.1}$$

Let i_0 be the index with

$$|\beta_{i_0}| = \min_j |\beta_j|.$$

Then $|\beta_{i_0}| \leq K^{\frac{1}{n}}$ and together with (2.2) and (2.3) we get

$$|\beta_j| \geq |\beta_j - \beta_{i_0}| - |\beta_{i_0}| \geq |\alpha_j - \alpha_{i_0}| \cdot |y| - K^{\frac{1}{n}} \geq (1-\varepsilon) \cdot |\alpha_j - \alpha_{i_0}| \cdot |y|$$

for $j \neq i_0$. From the previous inequality and (3.1), we have

$$|\beta_{i_0}| \leq \frac{K}{\prod_{j\neq i_0} |\beta_j|} \leq \frac{c}{|y|^{n-1}} \tag{3.2}$$

with

$$c = \frac{K}{(1-\varepsilon)^{n-1} \prod_{j\neq i_0} |\alpha_j - \alpha_{i_0}|}.$$

By (3.2) we obtain

$$\left|\alpha_{i_0} - \frac{x\overline{y}}{|y|^2}\right| = \left|\alpha_{i_0} - \frac{x}{y}\right| \leq \frac{c}{|y|^n},$$

hence

$$\left|\alpha_{i_0}|y|^2 - x\overline{y}\right| \leq \frac{c}{|y|^{n-2}},$$

which implies

$$|\mathrm{Im}(x\overline{y})| \leq \frac{c}{|y|^{n-2}}.$$

Note that $\frac{c}{|y|^{n-2}} < \frac{1}{2}$ and $\frac{c}{|y|^{n-2}} < 1$ for $m \equiv 3 \pmod 4$ and $m \equiv 1, 2 \pmod 4$, respectively, according to (2.2) and (2.3). Therefore $|\mathrm{Im}(x\overline{y})| = \frac{1}{2}|x_2y_1 - x_1y_2|\sqrt{m} < \frac{1}{2}$ and $|\mathrm{Im}(x\overline{y})| = |x_2y_1 - x_1y_2|\sqrt{m} < 1$ for $m \equiv 3 \pmod 4$ and $m \equiv 1, 2 \pmod 4$, respectively. Hence in both cases we have (2.4).

I. Let $m \equiv 3 \pmod 4$.

IA. The inequality (3.2) implies $|\mathrm{Re}(\beta_{i_0})| \leq \frac{c}{|y|^{n-1}}$, i.e.

$$|(2x_1 + x_2) - \alpha_{i_0}(2y_1 + y_2)| \leq \frac{2c}{|y|^{n-1}}. \tag{3.3}$$

IA1. If $2y_1 + y_2 = 0$, then (3.3) yields $2x_1 + x_2 = 0$, and the inequality (2.1) has the form

$$\left|F\left(\frac{x_2 i\sqrt{m}}{2}, \frac{y_2 i\sqrt{m}}{2}\right)\right| \leq K$$

whence we get (2.5).

IA2. If $2y_1 + y_2 \neq 0$, then

$$|(2x_1 + x_2) - \alpha_{i_0}(2y_1 + y_2)| \leq \frac{2c}{|y|^{n-1}} = \frac{2c}{\left|\dfrac{(2y_1 + y_2) + y_2 i\sqrt{m}}{2}\right|^{n-1}}$$
$$\leq \frac{2^n c}{|2y_1 + y_2|^{n-1}}.$$

Since we have assumed

$$|2y_1 + y_2| \geq \left(\frac{2^n c}{\eta A}\right)^{1/n},$$

Lemma 3.1 implies

$$|F(2x_1+x_2,2y_1+y_2)|\le 2^n c(1+\eta)^{n-1}\prod_{j\ne i_0}|\alpha_j-\alpha_{i_0}|$$

whence we get (2.6).

IB. By the inequality (3.2), we have $|\mathrm{Im}(\beta_{i_0})|\le \frac{c}{|y|^{n-1}}$, i.e.

$$\sqrt{m}|x_2-\alpha_{i_0}y_2|\le \frac{2c}{|y|^{n-1}}. \tag{3.4}$$

IB1. If $y_2=0$, then (3.4) implies $x_2=0$ and the inequality (2.1) has the form

$$\left|F\left(\frac{2x_1}{2},\frac{2y_1}{2}\right)\right|\le K$$

whence we get (2.7).

IB2. If $y_2\ne 0$, then

$$|x_2-\alpha_{i_0}y_2|\le \frac{2c}{\sqrt{m}|y|^{n-1}}=\frac{2c}{\sqrt{m}\left|\dfrac{(2y_1+y_2)+y_2 i\sqrt{m}}{2}\right|^{n-1}}\le \frac{2^n c}{(\sqrt{m})^n|y_2|^{n-1}}.$$

Since

$$|y_2|\ge\left(\frac{2^n c}{(\sqrt{m})^n\eta A}\right)^{1/n},$$

Lemma 3.1 implies

$$|F(x_2,y_2)|\le \frac{2^n c}{(\sqrt{m})^n}(1+\eta)^{n-1}\prod_{j\ne i_0}|\alpha_j-\alpha_{i_0}|$$

which implies (2.8).

II. Let $m\equiv 1,2 \pmod 4$.

IIA. The inequality (3.2) implies $|\mathrm{Re}(\beta_{i_0})|\le \frac{c}{|y|^{n-1}}$, i.e.

$$|x_1-\alpha_{i_0}y_1|\le \frac{c}{|y|^{n-1}}. \tag{3.5}$$

IIA1. If $y_1=0$, then (3.5) yields $x_1=0$ and the inequality (2.1) has the form

$$|F(i\sqrt{m}x_2,i\sqrt{m}y_2)|\le K,$$

whence we get (2.9).

IIA2. If $y_1\ne 0$, then

$$|x_1-\alpha_{i_0}y_1|\le \frac{c}{|y|^{n-1}}=\frac{c}{|y_1+i\sqrt{m}y_2|^{n-1}}\le \frac{c}{|y_1|^{n-1}}.$$

Since we have assumed

$$|y_1|\ge\left(\frac{c}{\eta A}\right)^{1/n},$$

Lemma 3.1 implies

$$|F(x_1, y_1)| \le c(1+\eta)^{n-1} \prod_{j \ne i_0} |\alpha_j - \alpha_{i_0}|$$

whence we get (2.10).

IIB. By the inequality (3.2) we have $|\mathrm{Im}(\beta_{i_0})| \le \frac{c}{|y|^{n-1}}$, i.e.

$$\sqrt{m}|x_2 - \alpha_{i_0} y_2| \le \frac{c}{|y|^{n-1}}. \tag{3.6}$$

IIB1. If $y_2 = 0$, then (3.6) implies $x_2 = 0$ and the inequality (2.1) has the form

$$|F(x_1, y_1)| \le K$$

which is just our assertion (2.11).

IIB2. If $y_2 \ne 0$, then

$$|x_2 - \alpha_{i_0} y_2| \le \frac{c}{\sqrt{m}|y|^{n-1}} = \frac{c}{|y_1 + i\sqrt{m}y_2|^{n-1}} \le \frac{c}{(\sqrt{m})^n |y_2|^{n-1}}.$$

Since

$$|y_2| \ge \left(\frac{c}{(\sqrt{m})^n \eta A} \right)^{1/n},$$

Lemma 3.1 implies

$$|F(x_2, y_2)| \le \frac{c}{(\sqrt{m})^n} (1+\eta)^{n-1} \prod_{j \ne i_0} |\alpha_j - \alpha_{i_0}|$$

whence we get (2.12). ∎

4. How to apply Theorem 2.1

In this section we give useful hints for a practical application of Theorem 2.1.

Using the same notation let us consider again the relative Thue inequality (2.1). We describe our algorithm in the case I (for $m \equiv 3 \pmod 4$) since the case II is completely similar.

1. If $|y| \le C_1$ then we have only finitely many possible values for y and hence for y_1, y_2, as well. For each possible y and for all integers $\mu \in \mathbb{Z}_M$ with $|\mu| \le K$ we calculate the roots of the equation $F(x, y) - \mu = 0$ in x. For such a root x we calculate the corresponding x_1, x_2. If x_1, x_2 are integers, then $x \in \mathbb{Z}_M$ and (x, y) is a solutions of (2.1).

   Alternatively, by $|\beta_{i_0}| \le K^{\frac{1}{n}}$ we obtain $|x| \le K^{\frac{1}{n}} + \max |\alpha_j| \cdot C_1$. We can simply enumerate and test the finitely many possible values of x_1, x_2 and y_1, y_2.
2. Assume that $|y| > C_1$.
   (a) If $|2y_1 + y_2| < 2D$, then

(i) If $|y_2| < 2D/\sqrt{m}$, then we have only finitely many values for y_1, y_2, we proceed as in 1.

(ii) If $|y_2| \geq 2D/\sqrt{m}$, then we use IB2. We solve $F(x_2, y_2) = k$ for all $k \in \mathbb{Z}$ with $|k| \leq 2^n E/(\sqrt{m})^n$. We determine the possible values of y_1 which satisfy $|2y_1 + y_2| < 2D$. We substitute x_2, y_1, y_2 into $x_2 y_1 = x_1 y_2$ to see if there exist corresponding integer x_1.

(b) If $|2y_1 + y_2| \geq 2D$, then we use IA2. We calculate the solutions $X = 2x_1 + x_2, Y = 2y_1 + y_2$ of $F(X, Y) = k$ for all $k \in \mathbb{Z}$ with $|k| \leq 2^n E$.

(i) If $|y_2| < 2D/\sqrt{m}$ then there are only finitely many possible values for y_2. We determine y_1 from Y. Using $X = 2x_1+x_2$ we set $x_2 = X - 2x_1$, substitute $x_2 = X - 2x_1, y_1, y_2$ into $x_2 y_1 = x_1 y_2$ and test if there is a corresponding x_1 in $\mathbb{Z}$.

(ii) If $|y_2| \geq 2D/\sqrt{m}$ we use IB2. We solve $F(x_2, y_2) = k$ for $|k| \leq 2^n E/(\sqrt{m})^n$. We determine x_1, y_1 from x_2, y_2 and X, Y.

For solving absolute Thue equations $F(x, y) = k$ for certain values $k \in \mathbb{Z}$ one can efficiently apply Kash ([4]) and Magma ([2]).

We remark that an appropriate choice of the parameters ε, η of Thereom 2.1 makes the resolution much easier. It is worthy to keep C_1, C_2 and also D small, to avoid extensive tests of small possible solutions. On the other hand, if E is small, then there are fewer Thue equations (over $\mathbb{Z}$) to be solved. Of course we can not make all these constants simultaneously small, therefore we need to make a compromise, taking into consideration also the value of K (which also determines the number of Thue equations to be solved). Usually it is worthy to try several values of ε, η before we start solving (2.1).

5. An example

Let $M = \mathbb{Q}(i\sqrt{5})$, and let

$$F(x, y) = x^4 - 9x^3y - 21x^2y^2 + 88xy^3 + 48y^4$$

and consider the solutions of

$$|F(x, y)| \leq 20 \quad \text{in} \quad x, y \in \mathbb{Z}_M. \tag{5.1}$$

The polynomial $F(x, y)$ is irreducible and the roots of $F(x, 1)$ are approximately

$$-3.4271, -0.49938, 2.7581, 10.1684.$$

We may set $A = 2.9278$, $B = 101.7426$. Further, let $\varepsilon = 0.1$ and $\eta = 0.1$. We are in case II. Calculating the constants, Theorem 2.1 gives:

Assume $|y| > 7.2229$. Then:

IIA1. If $y_1 = 0$, then $x_1 = 0$ and $|F(x_2, y_2)| \leq 0.8000$.

IIA2. If $|y_1| \geq 0.9796$, then $|F(x_1, y_1)| \leq 36.5157$.

IIB1. If $y_2 = 0$, then $x_2 = 0$ and $|F(x_1, y_1)| \leq 20$.
IIB2. If $|y_2| \geq 0.4381$, then $|F(x_2, y_2)| \leq 1.4606$.

First we consider the values with $|y| \leq C_2 = 7.2229$. We have $|x| \leq 20^{\frac{1}{4}} + \max|\alpha_j| \cdot C_2 = 75.64$. Enumerating and testing all possible $x = x_1 + i\sqrt{5}x_2$ and $y = y_1 + i\sqrt{5}y_2$ satisfying these bounds we obtain the solutions $(x_1, x_2, y_1, y_2) = (0,0,0,0), (1,0,0,0), (2,0,0,0), (1,0,-2,0), (2,0,-4,0)$, up to sign.

If $y_1 = 0$ then by IIA1 we have $x_1 = 0$ and $|F(x_2, y_2)| \leq 0.8$, whence $|F(x_2, y_2)| = 0$, $x_2 = 0, y_2 = 0$.

If $y_2 = 0$ then by IIB1 we have $x_2 = 0$ and $|F(x_1, y_1)| \leq 20$. Using Magma we solve $F(x_1, y_1) = k$ for $-20 \leq k \leq 20$. We obtain the solutions $(x_1, y_1) = (0,0), (1,0), (1,-2), (2,0), (2,-4)$, up to sign. These bring the above solutions (x_1, x_2, y_1, y_2) again.

From now on we assume that $y_1 \neq 0$ and $y_2 \neq 0$.

If $|y_1| \leq 0.9796$ and $|y_2| \leq 0.4381$ then by IIA2 we have $|F(x_1, y_1)| \leq 36.5157$ and by IIB2 we have $|F(x_2, y_2)| \leq 1.4606$. In addition to the above calculation we solve $F(x_1, y_1) = k$ for $21 \leq |k| \leq 36$ but we do not get any further solutions. Hence the solutions of $|F(x_1, y_1)| \leq 36.5157$ are $(x_1, y_1) = (0,0), (1,0), (1,-2), (2,0), (2,-4)$, up to sign. Also the solutions of $|F(x_2, y_2)| \leq 1.4606$ are $(x_1, y_1) = (0,0), (1,0), (1,-2)$, up to sign. Testing these possible (x_1, x_2, y_1, y_2) we do not get any new solutions.

If either $|y_1| < 0.9796$ or $|y_2| < 0.4381$ then $y_1 = 0$ or $y_2 = 0$ which cases we have already considered.

Hence all solutions of (5.1) are $(x, y) = (0,0), (1,0), (2,0), (1,-2), (2,-4)$, up to sign. The calculation takes just a few seconds.

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I. Gaál
University of Debrecen
Mathematical Institute
H–4002 Debrecen Pf.400.
Hungary
E-mail: gaal.istvan@unideb.hu

B. Jadrijević
University of Split
Faculty of Science
Ruđera Boškovića 33, 21000 Split
Croatia
E-mail: borka@pmfst.hr

L. Remete
University of Debrecen
Mathematical Institute
H–4002 Debrecen Pf.400.
Hungary
E-mail: remete.laszlo@science.unideb.hu

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