GLASNIK MATEMATIČKI
Vol. 53(73)(2018), 205 – 220

η-RICCI SOLITONS IN (ε)-ALMOST PARACONTACT METRIC MANIFOLDS

Adara Monica Blaga, Selcen Yüksel Perktaş, Bilal Eftal Acet and Feyza Esra Erdoğan

West University of Timişoara, Romania and Adıyaman University, Turkey

Abstract. The object of this paper is to study η-Ricci solitons on (ε)-almost paracontact metric manifolds. We investigate η-Ricci solitons in the case when its potential vector field is exactly the characteristic vector field ξ of the (ε)-almost paracontact metric manifold and when the potential vector field is torse-forming. We also study Einstein-like and (ε)-para Sasakian manifolds admitting η-Ricci solitons. Finally we obtain some results for η-Ricci solitons on (ε)-almost paracontact metric manifolds with a special view towards parallel symmetric $(0,2)$-tensor fields.

1. Introduction

The notion of Ricci soliton which is a natural generalization of an Einstein metric (i.e. the Ricci tensor S is a constant multiple of g) was introduced by Hamilton ([14]) in 1982. A pseudo-Riemannian manifold (M,g) is called a *Ricci soliton* if it admits a smooth vector field V (potential vector field) on M such that

$$\frac{1}{2}\left(\pounds_V\, g\right)(X,Y) + S(X,Y) + \lambda g(X,Y) = 0, \tag{1.1}$$

where $\pounds_V$ denotes the Lie-derivative in the direction V, λ is a constant and X, Y are arbitrary vector fields on M. A Ricci soliton is said to be shrinking, steady or expanding according to λ being negative, zero or positive respectively. It is obvious that a trivial Ricci soliton is an Einstein manifold with V zero or Killing vector field. Since Ricci solitons are the fixed points of the Ricci flow, they are important in understanding Hamilton's Ricci flow ([13]):

2010 *Mathematics Subject Classification.* 53C15, 53C25, 53C40, 53C42, 53C50.

Key words and phrases. (ε)-almost paracontact metric manifold, (ε)-para Sasakian manifold, Einstein-like manifold, η-Ricci soliton.

$\frac{\partial}{\partial t}g_{ij} = -2S_{ij}$, viewed as a dynamical system, on the space of Riemannian metrics modulo diffeomorphisms and scalings. In differential geometry, the Ricci flow is an intrinsic geometric flow. It can be viewed as a process that deforms the metric of a Riemannian manifold in a way formally analogous to the diffusion of heat, smoothing out the irregularities in the metric.

Geometric flows, especially Ricci flows, have become important tools in theoretical physics. Ricci soliton is known as quasi Einstein metric in physics literature [12] and the solutions of the Einstein field equations correspond to Ricci solitons ([1]). Relation with the string theory and the fact that (1.1) is a particular case of Einstein field equation make the equation of Ricci soliton interesting in theoretical physics.

In spite of introducing and studying firstly in Riemannian geometry, the Ricci soliton equation has recently been investigated in pseudo-Riemannian context, especially in Lorentzian case, see [3, 6, 8, 17].

The concept of η-Ricci soliton was initiated by Cho and Kimura in [9]. For a given 1-form η, an *η-Ricci soliton* is a data (g, V, λ, μ) on a pseudo-Riemannian manifold (M, g) satisfying

$$\frac{1}{2}\left(\pounds_V\, g\right)(X,Y) + S(X,Y) + \lambda g(X,Y) + \mu\eta\otimes\eta\,(X,Y) = 0, \tag{1.2}$$

where V is a vector field, $\pounds_V$ denotes the Lie-derivative in the direction V, S stands for the Ricci tensor field, λ and μ are constants and X, Y are arbitrary vector fields on M. In [7] the authors studied η-Ricci solitons on Hopf hypersurfaces in complex space forms. In the context of paracontact geometry η-Ricci solitons were investigated in [4, 5, 3].

In 1923, Eisenhart ([11]) proved that if a Riemannian manifold admits a second order parallel symmetric covariant tensor which is not a constant multiple of the metric tensor, then the manifold is reducible. In 1925, it was shown by Levy in [15] that a second order parallel symmetric non-degenerate tensor field in a space form is proportional to the metric tensor. Sharma ([21]) studied second order parallel tensors by using Ricci identities. Second order parallel tensors have been studied by various authors in different structures of manifolds, see [10, 11, 15, 16, 21–23].

In 1976, Sāto in [19] introduced the almost paracontact structure as a triple (φ, ξ, η) of a $(1,1)$-tensor field φ, a vector field ξ and a 1-form η satisfying $\varphi^2 = I - \eta\otimes\xi$ and $\eta(\xi) = 1$. The structure is an analogue of the almost contact structure ([18]) and is closely related to almost product structure (in contrast to almost contact structure, which is related to almost complex structure). An almost contact manifold is always odd-dimensional but an almost paracontact manifold could be even-dimensional as well. In 1969, Takahashi ([24]) introduced almost contact manifolds equipped with an associated pseudo-Riemannian metric and, in particular, he studied Sasakian manifolds equipped with an associated pseudo-Riemannian metric. These indefinite almost contact metric manifolds and indefinite Sasakian manifolds are also known as (ε)-almost contact metric manifolds and (ε)-Sasakian manifolds, respectively ([2]). In 1989, Matsumoto ([17]) replaced the structure vector field ξ by $-\xi$ in an almost paracontact manifold and associated a Lorentzian metric with the resulting structure and called it a Lorentzian almost paracontact manifold.

An (ε)-Sasakian manifold is always odd-dimensional. On the other hand, in a Lorentzian almost paracontact manifold given by Matsumoto, the pseudo-Riemannian metric has only index 1 and the structure vector field ξ is always timelike. These circumstances motivated the authors of [25] to associate a pseudo-Riemannian metric, not necessarily Lorentzian, with an almost paracontact structure, and this indefinite almost paracontact metric structure was called an (ε)-almost paracontact structure, where the structure vector field ξ is spacelike or timelike according as $\varepsilon = 1$ or $\varepsilon = -1$ ([26]).

Motivated by these studies, in the present paper we investigate η-Ricci solitons in (ε)-almost paracontact metric manifolds. The paper is organized as follows. Section 2 is devoted to basic concepts on (ε)-almost paracontact metric manifolds. In Section 3, we study η-Ricci solitons in the case when its potential vector field is exactly the characteristic vector field ξ of the Einstein-like (ε)-almost paracontact metric manifold and when the potential vector field is torse-forming in an η-Einstein (ε)-almost paracontact metric manifold. In Section 4, we prove that an (ε)-para Sasakian manifold admitting η-Ricci soliton with a potential vector field pointwise collinear to ξ is an Einstein-like manifold. In Section 5, we give some characterizations for η-Ricci solitons on (ε)-almost paracontact metric manifolds concerning parallel symmetric $(0,2)$-tensor fields.

2. Preliminaries

Let M be an n-dimensional manifold equipped with an *almost paracontact structure* (φ, ξ, η) ([19]) consisting of a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\varphi^2 = I - \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

$$\varphi\xi = 0, \tag{2.3}$$

$$\eta \circ \varphi = 0. \tag{2.4}$$

It is easy to verify that (2.1) and one of (2.2), (2.3) and (2.4) imply the other two equations. If g is a pseudo-Riemannian metric such that

$$g(\varphi X, \varphi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y), \qquad X, Y \in \Gamma(TM), \tag{2.5}$$

where $\varepsilon = \pm 1$, then M is called (ε)-*almost paracontact metric manifold* equipped with an (ε)-*almost paracontact metric structure* $(\varphi, \xi, \eta, g, \varepsilon)$ ([25]). In particular, if $\mathrm{index}(g) = 1$, that is when g is a Lorentzian metric, then the (ε)-almost paracontact metric manifold is called *Lorentzian almost paracontact manifold*. From (2.5) we have

$$g(X, \xi) = \varepsilon \eta(X), \tag{2.6}$$

$$g(X, \varphi Y) = g(\varphi X, Y), \tag{2.7}$$

for all $X, Y \in \Gamma(TM)$. From (2.6) it follows that

$$g(\xi, \xi) = \varepsilon, \tag{2.8}$$

that is, the structure vector field ξ is never lightlike.

Let $(M, \varphi, \xi, \eta, g, \varepsilon)$ be an (ε)-almost paracontact metric manifold (resp. a Lorentzian almost paracontact manifold). If $\varepsilon = 1$, then M is said to be a spacelike (ε)-almost paracontact metric manifold (resp. a spacelike Lorentzian almost paracontact manifold). Similarly, if $\varepsilon = -1$, then M is said to be a timelike (ε)-almost paracontact metric manifold (resp. a timelike Lorentzian almost paracontact manifold) ([25]).

An (ε)-almost paracontact metric structure $(\varphi, \xi, \eta, g, \varepsilon)$ is called (ε)-*para Sasakian structure* if

$$(\nabla_X \varphi) Y = -g(\varphi X, \varphi Y)\xi - \varepsilon \eta(Y) \varphi^2 X, \qquad X, Y \in \Gamma(TM), \tag{2.9}$$

where ∇ is the Levi-Civita connection with respect to g. A manifold endowed with an (ε)-para Sasakian structure is called (ε)-*para Sasakian manifold* ([25]). In an (ε)-para Sasakian manifold, we have

$$\nabla \xi = \varepsilon \varphi \tag{2.10}$$

and the Riemann curvature tensor R and the Ricci tensor S satisfy the following equations ([25]):

$$R(X, Y)\xi = \eta(X) Y - \eta(Y) X, \tag{2.11}$$

$$R(\xi, X) Y = -\varepsilon g(X, Y)\xi + \eta(Y) X, \tag{2.12}$$

$$\eta(R(X, Y) Z) = -\varepsilon \eta(X) g(Y, Z) + \varepsilon \eta(Y) g(X, Z), \tag{2.13}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.14}$$

for all $X, Y, Z \in \Gamma(TM)$.

EXAMPLE 2.1 ([25]). Let $\mathbb{R}^5$ be the 5-dimensional real number space with a coordinate system (x, y, z, t, s). Defining

$$\eta = ds - y dx - t dz, \qquad \xi = \frac{\partial}{\partial s},$$

$$\varphi\left(\frac{\partial}{\partial x}\right) = -\frac{\partial}{\partial x} - y\frac{\partial}{\partial s}, \qquad \varphi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial y},$$

$$\varphi\left(\frac{\partial}{\partial z}\right)=-\frac{\partial}{\partial z}-t\frac{\partial}{\partial s}\,,\qquad \varphi\left(\frac{\partial}{\partial t}\right)=-\frac{\partial}{\partial t}\,,\qquad \varphi\left(\frac{\partial}{\partial s}\right)=0,$$

$$g_1=(dx)^2+(dy)^2+(dz)^2+(dt)^2-\eta\otimes\eta,$$

$$\begin{aligned} g_2 \;=\; & -(dx)^2-(dy)^2+(dz)^2+(dt)^2+(ds)^2 \\ & -t\,(dz\otimes ds+ds\otimes dz)-y\,(dx\otimes ds+ds\otimes dx)\,, \end{aligned}$$

then (φ,ξ,η,g_1) is a timelike Lorentzian almost paracontact structure in $\mathbb{R}^5$, while (φ,ξ,η,g_2) is a spacelike (ε)-almost paracontact structure. Note that $\mathrm{index}(g_2)=3$.

3. η-Ricci solitons on Einstein-like (ε)-almost paracontact metric manifolds

We introduce the following definition analogous to Einstein-like para Sasakian manifolds ([20]).

Definition 3.1. *An (ε)-almost paracontact metric manifold $(M,\varphi,\xi,\eta,g,\varepsilon)$ is said to be Einstein-like if its Ricci tensor S satisfies*

$$S\,(X,Y)=a\,g\,(X,Y)+b\,g\,(\varphi X,Y)+c\,\eta\,(X)\,\eta\,(Y)\,,\quad X,Y\in\Gamma(TM), \tag{3.1}$$

for some real constants a, b and c.

We deduce the following properties.

Proposition 3.2. *In an Einstein-like (ε)-almost paracontact metric manifold $(M,\varphi,\xi,\eta,g,\varepsilon,a,b,c)$ we have*

$$S\,(\varphi X,Y)=S(X,\varphi Y), \tag{3.2}$$

$$S\,(\varphi X,\varphi Y)=S(X,Y)-(\varepsilon a+c)\eta\,(X)\,\eta\,(Y)\,, \tag{3.3}$$

$$S\,(X,\xi)=(\varepsilon a+c)\eta\,(X)\,, \tag{3.4}$$

$$S\,(\xi,\xi)=\varepsilon a+c, \tag{3.5}$$

$$\begin{aligned}(\nabla_X S)\,(Y,Z)=\;& bg((\nabla_X\varphi)Y,Z)\\ &+\varepsilon c\,\{\eta(Y)g(\nabla_X\xi,Z)+\eta(Z)g(\nabla_X\xi,Y)\}\,,\end{aligned} \tag{3.6}$$

$$(\nabla_X Q)\,Y=b(\nabla_X\varphi)Y+\varepsilon c\,\{\eta(Y)\nabla_X\xi+\varepsilon g(\nabla_X\xi,Y)\xi\}\,, \tag{3.7}$$

where Q is the Ricci operator defined by $g(QX,Y)=S(X,Y)$, $X,Y\in\Gamma(TM)$. Moreover, if the manifold is (ε)-para Sasakian, then

$$\varepsilon a+c=1-n, \tag{3.8}$$

$$r=na+b\,\mathrm{trace}(\varphi)+\varepsilon c, \tag{3.9}$$

where r is the scalar curvature.

Remark that the Ricci operator Q of an Einstein-like (ε)-almost paracontact metric manifold is of the form

$$Q = aI + b\varphi + \varepsilon c\eta \otimes \xi$$

and the structure vector field ξ is an eigenvector of Q with the corresponding eigenvalue $a + \varepsilon c$.

Let $(M, \varphi, \xi, \eta, g, \varepsilon, a, b, c)$ be an Einstein-like (ε)-almost paracontact metric manifold admitting the *η-Ricci soliton* (g, ξ, λ, μ):

$$\frac{1}{2}\pounds_\xi g + S + \lambda g + \mu\eta \otimes \eta = 0, \tag{3.10}$$

with λ and μ real constants. Replacing (3.1) in the last equation we get

$$\begin{aligned} &g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) \\ &\qquad + 2\left\{(a+\lambda)\, g(X,Y) + bg(\varphi X, Y) + (c+\mu)\eta(X)\eta(Y)\right\} = 0, \end{aligned} \tag{3.11}$$

for all $X, Y \in \Gamma(TM)$. If we take $X = Y = \xi$ in (3.11) we have

$$\varepsilon(a+\lambda) + c + \mu = 0, \tag{3.12}$$

by virtue of (2.2) and (2.8).

Using (3.12) and taking $Y = \xi$ in (3.11), we obtain

$$g(\nabla_\xi \xi, X) = 0, \tag{3.13}$$

which implies $\nabla_\xi \xi = 0$. So we easily see that

$$(\nabla_\xi \varphi)\, \xi = 0 \quad \text{and} \quad \nabla_\xi \eta = 0. \tag{3.14}$$

Also from (3.6), (3.7) and (3.14) we get

$$(\nabla_\xi S)\,(Y, Z) = bg((\nabla_\xi \varphi)Y, Z) \tag{3.15}$$

and

$$\nabla_\xi Q = b\nabla_\xi \varphi. \tag{3.16}$$

Hence we have the following result.

Proposition 3.3. *Let $(M, \varphi, \xi, \eta, g, \varepsilon, a, b, c)$ be an Einstein-like (ε)-almost paracontact metric manifold admitting an η-Ricci soliton (g, ξ, λ, μ). Then*

i) $\varepsilon(a+\lambda) + c + \mu = 0$,
ii) *ξ is a geodesic vector field,*
iii) $(\nabla_\xi \varphi)\, \xi = 0$ *and* $\nabla_\xi \eta = 0$,
iv) $(\nabla_\xi S)\,(Y, Z) = bg((\nabla_\xi \varphi)Y, Z)$ *and* $\nabla_\xi Q = b\nabla_\xi \varphi$.

Moreover, if the manifold is (ε)-para Sasakian, then

$$\nabla_\xi S = 0 \quad \textit{and} \quad \nabla_\xi Q = 0.$$

A vector field ξ is called *torse-forming* if

$$\nabla_X \xi = fX + w(X)\xi, \tag{3.17}$$

is satisfied for some smooth function f and a 1-form w.

Taking the inner product with ξ we have

$$0 = g(\nabla_X \xi, \xi) = \varepsilon\left(f\eta(X) + w(X)\right),$$

for all $X \in \Gamma(TM)$, which implies

$$w = -f\eta. \tag{3.18}$$

It follows

$$\nabla_X \xi = f\left(X - \eta(X)\xi\right) = f\varphi^2 X. \tag{3.19}$$

Now assume that $(M, \varphi, \xi, \eta, g, \varepsilon, a, b, c)$ is an Einstein-like (ε)-almost paracontact metric manifold admitting an η-Ricci soliton (g, ξ, λ, μ) and that the potential vector field ξ is torse-forming. Replacing (3.19) in (3.11) we obtain, for all $X, Y \in \Gamma(TM)$

$$0 = (f + a + \lambda)\left\{g(X,Y) - \varepsilon\eta(X)\eta(Y)\right\} + bg(\varphi X, Y)$$

and

$$0 = g((f + a + \lambda)\varphi X + bX, \varphi Y),$$

which implies

$$0 = (f + a + \lambda)\varphi^2 X + b\varphi X, \tag{3.20}$$

that is

$$b\varphi X = -(f + a + \lambda)X + (f + a + \lambda)\eta(X)\xi. \tag{3.21}$$

So we have the following statement.

THEOREM 3.4. *Let $(M, \varphi, \xi, \eta, g, \varepsilon, a, b, c)$ be an Einstein-like (ε)-almost paracontact metric manifold admitting an η-Ricci soliton (g, ξ, λ, μ) with torse-forming potential vector field. Then M is an η-Einstein manifold.*

In the remaining part of this section, we shall consider M an *η-Einstein manifold* (that is an Einstein-like (ε)-almost paracontact metric manifold with $b = 0$), admitting an η-Ricci soliton (g, ξ, λ, μ) with torse-forming potential vector field ξ. Using (3.20) we have

$$f = -a - \lambda.$$

So we can write

$$\nabla_X \xi = -(a + \lambda)\left(X - \eta(X)\xi\right) = -(a + \lambda)\varphi^2 X.$$

By using (3.19) we obtain

$$R(X,Y)\xi = (a + \lambda)^2\{\eta(X)Y - \eta(Y)X\} \tag{3.22}$$

and

$$S(X,\xi)=(a+\lambda)^2(1-n)\eta(X), \tag{3.23}$$

for all $X, Y \in \Gamma(TM)$. From (3.4) and (3.23) we get

$$(\varepsilon a+c)\eta(X)=(a+\lambda)^2(1-n)\eta(X),$$

which implies

$$c=-\varepsilon a+(a+\lambda)^2(1-n). \tag{3.24}$$

Also by using (3.12) in the last equation we get

$$\mu=-\varepsilon\left(\lambda+\varepsilon(a+\lambda)^2(1-n)\right). \tag{3.25}$$

So we have the following result.

THEOREM 3.5. *Let $(M,\varphi,\xi,\eta,g,\varepsilon,a,c)$ be an η-Einstein (ε)-almost paracontact metric manifold admitting an η-Ricci soliton (g,ξ,λ,μ) with torse-forming potential vector field. Then f is a constant function and*

$$c=-\varepsilon a+(a+\lambda)^2(1-n),$$
$$\mu=-\varepsilon\left(\lambda+\varepsilon(a+\lambda)^2(1-n)\right).$$

Assuming $a=0$ we get the following statement.

THEOREM 3.6. *If $(M,\varphi,\xi,\eta,g,\varepsilon,0,0,c)$ is an n-dimensional $(n>1)$ non-Ricci flat η-Einstein (ε)-almost paracontact metric manifold admitting the torse-forming Ricci soliton (g,ξ,λ), then*

$$\lambda=\frac{\varepsilon}{n-1} \quad \text{and} \quad c=-\frac{1}{n-1}.$$

Moreover, the soliton is expanding (resp. shrinking) if M is spacelike (resp. timelike).

PROPOSITION 3.7. *In an η-Einstein (ε)-almost paracontact metric manifold admitting an η-Ricci soliton with torse-forming potential vector field, we have*

$$\begin{aligned}&(\nabla_X S)(Y,Z)\\&=-c\varepsilon\,(a+\lambda)\,\{\eta(Y)g(X,Z)+\eta(Z)g(X,Y)-2\varepsilon\eta(X)\eta(Y)\eta(Z)\}\end{aligned} \tag{3.26}$$

and

$$(\nabla_X Q)\,Y=-c\,(a+\lambda)\,\{\eta(Y)X+\varepsilon g(X,Y)\xi-2\eta(X)\eta(Y)\xi\}. \tag{3.27}$$

THEOREM 3.8. *Let $(M,\varphi,\xi,\eta,g,\varepsilon,a,c)$ be an η-Einstein (ε)-almost paracontact metric manifold admitting an η-Ricci soliton (g,ξ,λ,μ) with torse-forming potential vector field. If $f\neq 0$ and the Ricci operator Q is Codazzi, then M is an Einstein manifold and ξ is a Killing vector field.*

PROOF. From the condition

$$(\nabla_X Q)\, Y = (\nabla_Y Q)\, X, \tag{3.28}$$

for all $X, Y \in \Gamma(TM)$, using (3.27) we get

$$c(a+\lambda)\{\eta(X)Y - \eta(Y)X\} = 0,$$

for all $X, Y \in \Gamma(TM)$. Since $a+\lambda = -f \neq 0$ we obtain $c = 0$ and hence, M is an Einstein manifold.

Moreover, writing (3.7) for $Y = \xi$ we obtain

$$\nabla_X \xi = 0.$$

Therefore, $L_\xi g = 0$, hence ξ is Killing vector field. □

Let us remark the following particular cases:

Case I: $f = -1$. In this case, ξ is an irrotational vector field and we have

$$\begin{aligned}
&\nabla\xi = -I + \eta \otimes \xi,\\
&\lambda = 1 - a,\\
&\mu = -\varepsilon\,(1+\varepsilon c)\,,\\
&R(X,Y)\xi = \eta(X)Y - \eta(Y)X.
\end{aligned}$$

Since $\eta \neq 0$, from (3.18) it is easy to see that ξ can not be a concurrent vector field.

Case II: $f = 0$. In this case, ξ is a recurrent vector field and we have

$$\begin{aligned}
&\nabla\xi = 0,\\
&\lambda = -a,\\
&\mu = \varepsilon a = -c,\\
&R(X,Y)\xi = 0.
\end{aligned}$$

Furthermore, S and Q are ∇-parallel.

We end these considerations by giving two examples of η-Ricci solitons on a timelike Lorentzian almost paracontact manifold.

EXAMPLE 3.9. Let $M = \mathbb{R}^3$ and (x,y,z) be the standard coordinates in $\mathbb{R}^3$. Set

$$\varphi_0 := -\frac{\partial}{\partial x} \otimes dx - \frac{\partial}{\partial y} \otimes dy, \quad \xi_0 := \frac{\partial}{\partial z}, \quad \eta_0 := dz,$$
$$g_0 := dx \otimes dx + e^{2x} dy \otimes dy - dz \otimes dz$$

and consider the orthonormal system of vector fields

$$E_1 := \frac{\partial}{\partial x}, \quad E_2 := e^{-x}\frac{\partial}{\partial y}, \quad E_3 := \frac{\partial}{\partial z}.$$

It follows

$$\nabla_{E_1}E_1 = 0, \quad \nabla_{E_1}E_2 = 0, \quad \nabla_{E_1}E_3 = 0, \quad \nabla_{E_2}E_1 = E_2, \quad \nabla_{E_2}E_2 = -E_1,$$
$$\nabla_{E_2}E_3 = 0, \quad \nabla_{E_3}E_1 = 0, \quad \nabla_{E_3}E_2 = 0, \quad \nabla_{E_3}E_3 = 0.$$

Then the Riemann and the Ricci curvature tensor fields are given by:

$$R(E_1,E_2)E_2=-E_1,\quad R(E_1,E_3)E_3=0,\quad R(E_2,E_1)E_1=-E_2,$$
$$R(E_2,E_3)E_3=0,\quad R(E_3,E_1)E_1=0,\quad R(E_3,E_2)E_2=0,$$
$$S(E_1,E_1)=-1,\quad S(E_2,E_2)=-1,\quad S(E_3,E_3)=0.$$

In this case, for $\lambda=1$ and $\mu=1$, the data (g_0,ξ_0,λ,μ) is an η_0-Ricci soliton on the timelike Lorentzian almost paracontact manifold $(\mathbb{R}^3,\varphi_0,\xi_0,\eta_0,g_0)$.

EXAMPLE 3.10. Let $(\mathbb{R}^3,\varphi_0,\xi_0,\eta_0,g_0)$ be the timelike Lorentzian almost paracontact manifold considered in Example 3.9. Then the data $(g_0,\xi,\lambda=-1,\mu=1)$ is a shrinking η-Ricci soliton on $(\mathbb{R}^3,\varphi_0,\xi_0,\eta_0,g_0)$, where

$$\xi=\frac{\partial}{\partial x}+2\sqrt{2}e^{-x}\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}$$

and

$$\eta=\sqrt{2}dx+e^{x}dy.$$

4. η-Ricci solitons on (ε)-para Sasakian manifolds

Let $(M,\varphi,\xi,\eta,g,\varepsilon)$ be an (ε)-para Sasakian manifold admitting an η-Ricci soliton (g,V,λ,μ) and assume that the potential vector field V is pointwise collinear with the structure vector field ξ, that is, $V=k\xi$, for k a smooth function on M. Then from (1.2) and (2.6) we have

$$\begin{aligned}\varepsilon(Xk)\eta(Y)+\varepsilon(Yk)\eta(X)+2\varepsilon kg(\varphi X,Y)&\\ +2S(X,Y)+2\lambda g(X,Y)+2\mu\eta(X)\eta(Y)&=0,\end{aligned}\tag{4.1}$$

for all $X,Y\in\Gamma(TM)$. Taking $Y=\xi$ in (4.1) and using (2.14) we get

$$\varepsilon\,(Xk)+\{\varepsilon\,(\xi k)-2(n-1)+2\varepsilon\lambda+2\mu\}\,\eta(X)=0.\tag{4.2}$$

If we replace X by ξ in the last equation we obtain

$$\xi k=\varepsilon(n-1)-\lambda-\varepsilon\mu.\tag{4.3}$$

Using (4.3) in (4.2) gives

$$Xk=(\varepsilon(n-1)-\lambda-\varepsilon\mu)\,\eta(X).$$

We conclude that k is constant if $\varepsilon(n-1)=\lambda+\varepsilon\mu$ and in this case, from (4.1) we get

$$S(X,Y)=-\lambda g(X,Y)-\varepsilon kg(\varphi X,Y)-\mu\eta(X)\eta(Y).$$

So we have the following statement.

THEOREM 4.1. *Let $(M,\varphi,\xi,\eta,g,\varepsilon)$ be an (ε)-para Sasakian manifold. If M admits an η-Ricci soliton (g,V,λ,μ) and V is pointwise collinear with the structure vector field ξ, then V is a constant multiple of ξ provided $\varepsilon(n-1)=\lambda+\varepsilon\mu$ and M is an Einstein-like manifold.*

Remark 4.2. Under the hypotheses of Theorem 4.1, if $R(\xi,\cdot)\cdot S=0$, then V is a constant multiple of ξ. Indeed, the condition on S is

$$S(R(\xi,X)Y,Z)+S(Y,R(\xi,X)Z)=0,$$

for all $X,Y,Z\in\Gamma(TM)$. Using (1.2), (2.10) and (2.12) we obtain

$$(\varepsilon(n-1)-\lambda)\{\eta(Y)g(X,Z)+\eta(Z)g(X,Y)\}$$
$$-\varepsilon\{\eta(Y)g(\varphi X,Z)+\eta(Z)g(\varphi X,Y)\}-2\mu\eta(X)\eta(Y)\eta(Z)=0,$$

for all $X,Y,Z\in\Gamma(TM)$ and taking $X=Y=Z=\xi$ we get

$$\varepsilon(n-1)-\lambda-\varepsilon\mu=0.$$

Assuming $V=\xi$ we get the following statement.

Theorem 4.3. *If $(M,\varphi,\xi,\eta,g,\varepsilon)$ is an n-dimensional $(n>1)$ (ε)-para Sasakian manifold admitting the Ricci soliton (g,ξ,λ), then*

$$\lambda=\frac{\varepsilon}{n-1}.$$

Moreover, the soliton is expanding (resp. shrinking) if M is spacelike (resp. timelike).

Remark 4.4. If we assume that M is an Einstein-like (ε)-para Sasakian manifold and $V=\xi$, we have

$$\frac{1}{2}\left(\pounds_\xi\, g\right)(X,Y)+S(X,Y)+\lambda g(X,Y)+\mu\eta(X)\eta(Y)$$
$$=(\varepsilon+b)g(\varphi X,Y)+(a+\lambda)g(X,Y)+(c+\mu)\eta(X)\eta(Y),$$

which implies that if

$$\varepsilon+b=0,\quad a+\lambda=0,\quad c+\mu=0,$$

then $(g,\xi,-a,-c)$ is an η-Ricci soliton on M.

5. Parallel symmetric $(0,2)$-tensor fields on (ε)-almost paracontact metric manifolds

Let α be a $(0,2)$-tensor field which is assumed to be parallel with respect to Levi-Civita connection ∇, that is $\nabla\alpha=0$. Applying the Ricci identity

$$\nabla^2\alpha(X,Y;Z,W)-\nabla^2\alpha(X,Y;W,Z)=0,$$

we have ([21])

$$\alpha(R(X,Y)Z,W)+\alpha(R(X,Y)W,Z)=0. \tag{5.1}$$

Taking $Z=W=\xi$ and using the symmetry property of α, we write

$$\alpha(R(X,Y)\xi,\xi)=0. \tag{5.2}$$

Assume that $(M, \varphi, \xi, \eta, g, \varepsilon)$ is an (ε)-almost paracontact metric manifold with torse-forming characteristic vector field. Then from (3.19) we have

$$R(X,Y)\xi = f^2\{\eta(X)Y - \eta(Y)X\} + X(f)\varphi^2 Y - Y(f)\varphi^2 X. \tag{5.3}$$

Replacing (5.3) in (5.2) we get

$$\begin{aligned} &f^2\{\eta(X)\alpha(Y,\xi) - \eta(Y)\alpha(X,\xi)\} \\ &\quad + X(f)\alpha\left(\varphi^2 Y,\xi\right) - Y(f)\alpha\left(\varphi^2 X,\xi\right) = 0. \end{aligned} \tag{5.4}$$

If we take $X = \xi$ in (5.4) we obtain

$$\left(f^2 + \xi(f)\right)\{\alpha(Y,\xi) - \eta(Y)\alpha(\xi,\xi)\} = 0. \tag{5.5}$$

Let $f^2 + \xi(f) \neq 0$; then we have

$$\alpha(Y,\xi) = \eta(Y)\alpha(\xi,\xi). \tag{5.6}$$

Definition 5.1. *An (ε)-almost paracontact metric manifold $(M, \varphi, \xi, \eta, g, \varepsilon)$ with torse-forming characteristic vector field is called regular if $f^2 + \xi(f) \neq 0$.*

Since α is a parallel $(0,2)$-tensor field, then $\alpha(\xi,\xi)$ is a constant. Taking the covariant derivative of (5.6) with respect to X we derive

$$\alpha(\nabla_X Y,\xi) + f\{\alpha(X,Y) - \eta(X)\eta(Y)\alpha(\xi,\xi)\} = X(\eta(Y))\alpha(\xi,\xi), \tag{5.7}$$

which implies

$$\begin{aligned} f\{\alpha(X,Y) - \eta(X)\eta(Y)\alpha(\xi,\xi)\} &= \varepsilon\{X(g(Y,\xi) - g(\nabla_X Y,\xi)\}\alpha(\xi,\xi) \\ &= \varepsilon g(Y,\nabla_X\xi)\alpha(\xi,\xi) \\ &= \varepsilon f\{g(X,Y) - \varepsilon\eta(X)\eta(Y)\}\alpha(\xi,\xi) \end{aligned}$$

and we obtain

$$\alpha(X,Y) = \varepsilon g(X,Y)\alpha(\xi,\xi). \tag{5.8}$$

Therefore we obtain the following statement.

Theorem 5.2. *A symmetric parallel second order covariant tensor in a regular (ε)-almost paracontact metric manifold with torse-forming characteristic vector field is a constant multiple of the metric tensor.*

Applying this result to solitons, we deduce the following statement.

Theorem 5.3. *Let $(M, \varphi, \xi, \eta, g, \varepsilon)$ be a regular (ε)-almost paracontact metric manifold with torse-forming characteristic vector field. Then $\alpha := \frac{1}{2}(\pounds_\xi g) + S + \mu\eta\otimes\eta$ (with a real constant μ) is parallel if and only if $(g, \xi, \lambda = -\varepsilon\alpha(\xi,\xi), \mu)$ is an η-Ricci soliton on M.*

Assume that $(M,\varphi,\xi,\eta,g,\varepsilon,a,b,c)$ is an Einstein-like (ε)-almost paracontact metric manifold with torse-forming characteristic vector field. Then

$$\begin{aligned}\frac{1}{2}\left(\pounds_{\xi}\, g\right)(X,Y)+S(X,Y)+\mu\eta(X)\eta(Y)\\ =(f+a)g(X,Y)+bg(\varphi X,Y)+(c+\mu-\varepsilon f)\eta(X)\eta(Y)\end{aligned} \tag{5.9}$$

and we can state the following corollary.

Corollary 5.4. *If $(M,\varphi,\xi,\eta,g,\varepsilon,a,b,c)$ is a regular Einstein-like (ε)-almost paracontact metric manifold with torse-forming characteristic vector field, then $\alpha:=\frac{1}{2}\left(\pounds_{\xi}\, g\right)+S+\mu\eta\otimes\eta$ (with a real constant μ) is parallel if and only if $(g,\xi,\lambda=-(a+\varepsilon(c+\mu)),\mu)$ is an η-Ricci soliton on M.*

Proof. From (5.9) we get $\alpha(\xi,\xi)=\varepsilon(a+\varepsilon c)+\mu$, so

$$\lambda=-\varepsilon\alpha(\xi,\xi)=-(a+\varepsilon(c+\mu)).$$

□

Remark 5.5. If $(M,\varphi,\xi,\eta,g,\varepsilon,a,b,c)$ is a regular Einstein-like (ε)-almost paracontact metric manifold with torse-forming characteristic vector field, then $\alpha:=\frac{1}{2}\left(\pounds_{\xi}\, g\right)+S$ is parallel if and only if $(g,\xi,\lambda=-(a+\varepsilon c))$ is an expanding (resp. shrinking) Ricci soliton on M provided $a+\varepsilon c<0$ (resp. $a+\varepsilon c>0$).

Assume that $(M,\varphi,\xi,\eta,g,\varepsilon)$ is an (ε)-para Sasakian manifold. From (2.11) and (5.2) we have

$$\eta(X)\alpha(Y,\xi)-\eta(Y)\alpha(X,\xi)=0. \tag{5.10}$$

Taking $X=\xi$ and $Y=\varphi^2 Z$ in the last equation we obtain

$$0=\alpha(\varphi^2 Z,\xi)=\alpha(Z,\xi)-\eta(Z)\alpha(\xi,\xi), \tag{5.11}$$

for all $Z\in\Gamma(TM)$.

Since α is a parallel $(0,2)$-tensor field, then $\alpha(\xi,\xi)$ is a constant. Taking the covariant derivative of (5.11) with respect to X we derive

$$\alpha(\nabla_X Z,\xi)+\varepsilon\alpha(Z,\varphi X)=X(\eta(Z))\alpha(\xi,\xi), \tag{5.12}$$

which implies

$$\alpha(\varphi X,Z)=\varepsilon g(\varphi X,Z)\alpha(\xi,\xi).$$

Taking $X=\varphi Y$ in the last equation we get

$$\alpha(Y,Z)=\varepsilon g(Y,Z)\alpha\left(\xi,\xi\right). \tag{5.13}$$

Therefore we obtain the following result.

Proposition 5.6. *On an (ε)-para Sasakian manifold, any parallel symmetric (0,2)-tensor field is a constant multiple of the metric.*

Applying this result to solitons, we deduce the following statement.

THEOREM 5.7. *Let* $(M, \varphi, \xi, \eta, g, \varepsilon)$ *be an* (ε)*-para Sasakian manifold. Then* $\alpha := \frac{1}{2}(\pounds_\xi g) + S + \mu\eta \otimes \eta$ *(with a real constant* μ*) is parallel if and only if* $(g, \xi, \lambda = -\varepsilon\alpha(\xi, \xi), \mu)$ *is an* η*-Ricci soliton on* M.

Assume that $(M, \varphi, \xi, \eta, g, \varepsilon, a, b, c)$ is an Einstein-like (ε)-para Sasakian manifold. Then

$$\begin{aligned}\frac{1}{2}(\pounds_\xi g)(X, Y) + S(X, Y) + \mu\eta(X)\eta(Y) \\ = ag(X, Y) + (\varepsilon + b)g(\varphi X, Y) + (c + \mu)\eta(X)\eta(Y)\end{aligned} \tag{5.14}$$

and we can state the following corollary.

COROLLARY 5.8. *If* $(M, \varphi, \xi, \eta, g, \varepsilon, a, b, c)$ *is an Einstein-like* (ε)*-para Sasakian manifold, then* $\alpha := \frac{1}{2}(\pounds_\xi g) + S + \mu\eta \otimes \eta$ *(with a real constant* μ*) is parallel if and only if* $(g, \xi, \lambda = -(a+\varepsilon(c+\mu)), \mu)$ *is an* η*-Ricci soliton on* M.

PROOF. From (5.14) we get $\alpha(\xi, \xi) = \varepsilon(a + \varepsilon c) + \mu$, so

$$\lambda = -\varepsilon\alpha(\xi, \xi) = -(a + \varepsilon(c + \mu)).$$

□

REMARK 5.9. From Corollary 5.4 and Corollary 5.8 we notice that the parallelism of the symmetric $(0, 2)$-tensor field $\alpha := \frac{1}{2}(\pounds_\xi g) + S + \mu\eta \otimes \eta$ on an Einstein-like (ε)-almost paracontact metric manifold $(M, \varphi, \xi, \eta, g, \varepsilon, a, b, c)$ which either is regular with torse-forming characteristic vector field or is (ε)-para Sasakian, yields the same η-Ricci soliton (which depends only on the constants a, c and μ).

ACKNOWLEDGEMENTS.
The authors wish to thank to the referees for their valuable suggestions that definitely improved the paper.

REFERENCES

[1] M. M. Akbar and E. Woolgar, *Ricci solitons and Einstein-scalar field theory*, Classical Quantum Gravity **26** (2009), 055015, 14pp.
[2] A. Bejancu and K. L. Duggal, *Real hypersurfaces of indefinite Kaehler manifolds*, Internat. J. Math. Math. Sci. **16** (1993), 545–556.
[3] A. M. Blaga, *η-Ricci solitons on Lorentzian para-Sasakian manifolds*, Filomat **30** (2016), 489–496.
[4] A. M. Blaga, *η-Ricci solitons on para-Kenmotsu manifolds*, Balkan J. Geom. Appl. **20** (2015), 1–13.
[5] A. M. Blaga and M. Crasmareanu, *Torse-forming η-Ricci solitons in almost paracontact η-Einstein geometry*, Filomat **31** (2017), 499–504.
[6] M. Brozos-Vazquez, G. Calvaruso, E. Garcia-Rio and S. Gavino-Fernandez, *Three-dimensional Lorentzian homogeneous Ricci solitons*, Israel J. Math. **188** (2012), 385–403.
[7] C. Calin and M. Crasmareanu, *η-Ricci solitons on Hopf Hypersurfaces in complex space forms*, Rev. Roumaine Math. Pures Appl. **57** (2012), 55–63.

[8] J. S. Case, *Singularity theorems and the Lorentzian splitting theorem for the Bakry Emery Ricci tensor*, J. Geom. Phys. **60** (2010), 477–490.
[9] J. T. Cho and M. Kimura, *Ricci solitons and real hypersurfaces in a complex space form*, Tôhoku Math. J. (2) **61** (2009), 205–212.
[10] U. C. De, *Second order parallel tensors on P-Sasakian manifolds*, Publ. Math. Debrecen **49** (1996), 33–37.
[11] L. P. Eisenhart, *Symmetric tensors of the second order whose first covariant derivatives are zero*, Trans. Amer. Math. Soc. **25** (1923), 297–306.
[12] D. H. Friedan, *Nonlinear models in* $2+\varepsilon$ *dimensions*, Ann. Phys. **163** (1985), 318–419.
[13] R. S. Hamilton, *The Ricci flow on surfaces*, in: Mathematics and general relativity, Amer. Math. Soc. Providence, 1988, 237–262.
[14] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), 255–306.
[15] H. Levy, *Symmetric tensors of the second order whose covariant derivatives vanish*, Ann. of Math. (2) **27** (1925), 91–98.
[16] Z. Li, *Second order parallel tensors on P-Sasakian manifolds with a coefficient k*, Soochow J. Math. **23** (1997), 97–102.
[17] K. Matsumoto, *On Lorentzian paracontact manifolds*, Bull. Yamagata Univ. Natur. Sci. **12** (1989), 151–156.
[18] S. Sasaki, *On differentiable manifolds with certain structures which are closely related to almost contact structure I*, Tôhoku Math. J. (2) **12** (1960), 459–476.
[19] I. Satō, *On a structure similar to the almost contact structure*, Tensor (N.S.) **30** (1976), 219–224.
[20] R. Sharma, *On Einstein-like p-Sasakian manifold*, Mat. Vesnik **6(19)(34)** (1982), 177–184.
[21] R. Sharma, *Second order parallel tensor in real and complex space forms*, Internat. J. Math. Math. Sci. **12** (1989), 787–790.
[22] R. Sharma, *Second order parallel tensors on contact manifolds*, Algebras Groups Geom. **7** (1990), 145–152.
[23] R. Sharma, *Second order parallel tensors on contact manifolds. II*, C. R. Math. Rep. Acad. Sci. Canada **13** (1991), 259–264.
[24] T. Takahashi, *Sasakian manifold with pseudo-Riemannian metric*, Tôhoku Math. J. (2) **21** (1969), 271–290.
[25] M. M. Tripathi, E. Kılıç, S. Y. Perktaş and S. Keleş, *Indefinite almost paracontact metric manifolds*, Int. J. Math. Math. Sci. (2010), 846195, 19 pp.
[26] S. Y. Perktaş, E. Kılıç, M. M. Tripathi, and S. Keleş, *On* (ε)*-para Sasakian* 3*-manifolds*, Int. J. Pure Appl. Math. **77** (2012), 485–499.

A. M. Blaga
Department of Mathematics
West University of Timişoara
300223 Timişoara
România
E-mail: adarablaga@yahoo.com

S. Yüksel Perktaş
Department of Mathematics
Adıyaman University
02040 Adıyaman
Turkey
E-mail: sperktas@adiyaman.edu.tr

B. E. Acet
Department of Mathematics
Adıyaman University
02040 Adıyaman
Turkey
E-mail: eacet@adiyaman.edu.tr

F. E. Erdoğan
Department of Elementary Education
Adıyaman University
02040 Adıyaman
Turkey
E-mail: ferdogan@adiyaman.edu.tr

Received: 9.8.2017.
Revised: 25.11.2017.