GLASNIK MATEMATIČKI
Vol. 53(73)(2018), 153 – 177

SUMS OF MATRIX-VALUED WAVE PACKET FRAMES IN $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$

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Abstract. The purpose of this paper is to first show relations between wave packet frame bounds and the scalars associated with finite sum of matrix-valued wave packet frames for the matrix-valued function space $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. A sufficient condition with explicit wave packet frame bounds for finite sum of matrix-valued wave packet frames in terms of scalars and frame bounds associated with the finite sum of frames is given. An optimal estimate of wave packet frame bounds for the finite sum of matrix-valued wave packet frames is presented. In the second part, we show that the rate of convergence of the frame algorithm can be increased by using frame bounds and scalars associated with the finite sum of frames. Finally, a necessary and sufficient condition for finite sum of matrix-valued wave packet frames in terms of series associated with wave packet vectors is given.

1. Introduction

Cordoba and Fefferman in [6] introduced the concept of wave packet system by applying certain collection of dilations, modulations and translations to the Gaussian function in the study of some classes of singular integral operators. In mathematical physics, a wave packet is a function Ψ defined on a domain $\Omega \subset \mathbb{R}^d$ which is "well localized in phase space". That is, Ψ and its Fourier transform $\widehat{\Psi}$ are both concentrated in reasonably small sets. Wave packets over the real line are generated from classical dilations, translations

2010 *Mathematics Subject Classification.* 42C15, 42C30, 42C40.
Key words and phrases. Bessel sequence, frames, wave packet frames.
The research of Jyoti is supported by Council of Scientific and Industrial Research (CSIR), India (Grant No.: 09/045(1374)/2015-EMR-I). Deepshikha is supported by CSIR, India (Grant No.: 09/045(1352)/2014-EMR-I). Lalit was partially supported by R & D Doctoral Research Programme, University of Delhi (Grant No.: RC/2014/6020).

and modulations of a given window (or scaling) function. Labate et al. in [23] adopted the same expression to describe, more generally, any collection of functions which are obtained by applying the same operations to a finite family of functions in $L^2(\mathbb{R}^d)$. Gabor systems, wavelet systems and the Fourier transform of wavelet systems are special cases of wave packet systems. Lacey and Thiele ([24, 25]) gave applications of wave packet systems in boundedness of the Hilbert transforms. The wave packet systems and related frame properties have been studied by several authors, see [5, 7, 10, 13, 18–20] and references therein.

Recently, Obeidat, Samarah, Casazza and Tremain studied finite sum of Hilbert space frames in [26]. They discussed finite sum of frames which include images of Bessel sequences and frames under bounded linear operators and frame operator on the underlying space. In the present work, we consider a matrix-valued Weyl-Heisenberg wave packet frame of the form

$$\mathcal{W}(\Psi) \equiv \{D_{A_j} T_{Bk} E_{C_m} \Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$$

for the matrix-valued function space $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, see Definition 3.1. Notable contribution in the paper includes necessary and sufficient conditions with explicit frame bounds for the finite sum of matrix-valued wave packet frames for the function space $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ in terms of wave packet frame bounds and scalars associated with the given finite sum of matrix-valued wave packet frames. It is shown that the width of the frame can be decreased by using frame bounds and scalars associated with the finite sum of frames. This is useful in the frame algorithms which depend on the frame bounds. Optimal frame bounds of matrix-valued wave packet frames are discussed.

1.1. *Frames in Hilbert Spaces.* Let $\mathcal{H}$ be a separable complex Hilbert space with an inner product $\langle .,. \rangle$ and norm $\|.\| = \sqrt{\langle .,. \rangle}$. A countable sequence $\{f_k\}_{k\in\mathbb{I}} \subset \mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist positive real numbers $L_o \leq U_o < \infty$ such that

$$L_o\|f\|^2 \leq \sum_{k\in\mathbb{I}} |\langle f, f_k\rangle|^2 \leq U_o\|f\|^2 \text{ for all } f \in \mathcal{H}. \tag{1.1}$$

The scalars L_o and U_o are called *lower frame bound* and *upper frame bound*, respectively. If it is possible to choose $L_o = U_o$, then we say that the frame is *tight*. It is Parseval if $L_o = U_o = 1$. The sequence $\{f_k\}_{k\in\mathbb{I}}$ is called a *Bessel sequence* with *Bessel bound* U_o if the upper inequality in (1.1) holds for all $f \in \mathcal{H}$.

The ratio

$$\frac{U_o - L_o}{U_o + L_o}$$

is called the *width* of the frame. We recall that the width of a frame measures its tightness.

The scalars

$$L_{Opt} = \sup \Big\{ L_o > 0 : L_o \text{ satisfies lower inequality in } \ (1.1) \Big\}$$

and

$$U_{Opt} = \inf \Big\{ U_o > 0 : U_o \text{ satisfies upper inequality in } \ (1.1) \Big\}$$

are called the *best frame bounds* or *optimal frame bounds* of the frame.

Let $\{f_k\}_{k\in\mathbb{I}}$ be a frame for $\mathcal{H}$. The following three operators play an important role in stable reconstruction and analysis of each vector (signal) in the underlying space:

$$T : \ell^2(\mathbb{I}) \to \mathcal{H}, \quad T\{c_k\}_{k\in\mathbb{I}} = \sum_{k\in\mathbb{I}} c_k f_k, \ \{c_k\}_{k\in\mathbb{I}} \in \ell^2(\mathbb{I}) \ (\textit{pre-frame operator}),$$

$$T^* : \mathcal{H} \to \ell^2(\mathbb{I}), \quad T^* f = \{\langle f, f_k\rangle\}_{k\in\mathbb{I}}, \quad f \in \mathcal{H} \quad (\textit{analysis operator}),$$

$$S = TT^* : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{k\in\mathbb{I}} \langle f, f_k\rangle f_k, \ f \in \mathcal{H} \quad (\textit{frame operator}).$$

The frame operator S is a bounded, linear, invertible and positive operator on $\mathcal{H}$. This gives the *reconstruction* of each vector $f \in \mathcal{H}$,

$$f = SS^{-1} f = \sum_{k\in\mathbb{I}} \langle S^{-1} f, f_k\rangle f_k = \sum_{k\in\mathbb{I}} \langle f, S^{-1} f_k\rangle f_k.$$

Thus, a frame for $\mathcal{H}$ allows each vector in $\mathcal{H}$ to be written as a linear combination of the elements in the frame, but the linear independence between the frame elements is not required. More precisely, frames are basis-like building blocks that span a vector space but allow for linear dependency, which is useful to reduce noise, find sparse representations, spherical codes, compressed sensing, signal processing, wavelet analysis etc., see [1]. For recent development in discrete frames in the unitary space $\mathbb{C}^N$, we refer to [11, 12]. Nowadays frames are also studied by using the iterated function systems (IFS) ([15, 27, 28]), reproducing systems ([16]), quantum systems ([29]).

Duffin and Schaeffer in [14] described the frame condition for the first time, in the context of non-harmonic Fourier series, as a method to calculate the coefficients in a linear combination of the vectors of a linearly dependent spanning set, a "*Hilbert space frame*". Few years later, in 1986, frames were reintroduced and brought to life by Daubechies, Grossmann and Meyer ([8]). Since then, the theory of frames began to be studied more widely. The basic theory of frames and their applications in different directions in science and engineering can be found in the books of Casazza and Kutyniok ([1]), Christensen ([4]), Heil ([17]), Daubechies ([9]) and beautiful research tutorials by Casazza ([2]) and Casazza and Lynch ([3]).

1.2. *Related work and Motivation.* Holub proved in [21] that if $\{e_k\}_{k=1}^{\infty}$ is any normalized basis for a separable Hilbert space $\mathcal{H}$ and $\{f_k^*\}_{k=1}^{\infty}$ is the associated dual basis of coefficient functionals, then the sequence $\{f_k+f_k^*\}_{k=1}^{\infty}$ is again a basis for $\mathcal{H}$. Obeidat, Samarah, Casazza and Tremain studied finite sum of Hilbert space frames in [26]. They proved necessary and sufficient conditions on Bessel sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ and bounded linear operators L_1, L_2 on a separable Hilbert space $\mathcal{H}$ such that the sum $\{L_1 f_k + L_2 g_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$. Kaushik, Singh and Virender gave sufficient conditions for finite sum of Gabor frames for $L^2(\mathbb{R})$ in terms of frame bounds and scalars associated with the finite sum of Gabor frames, see Theorem 4.3 of [22]. It will be interesting to find new relations and conditions on scalars and frame bounds associated with the finite sum of frames for the underlying space. The frame algorithm (see Theorem 4.1) gives approximation of each signal (vector) in the space. Furthermore, the convergence rate of the frame algorithm depends on the width of the frame, i.e., on the frame bounds. Naturally, it is an important consideration that to what extent the width of frame can be decreased and range of the optimal frame bounds can be minimized. This is our motivation for writing this paper.

1.3. *Outline.* The rest of the paper is organized as follows: In Section 2, we recapitulate basic facts about matrix-valued wave packet frames in the matrix-valued function space $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. Section 3 starts with a sufficient condition with explicit frame bounds for finite sum of matrix-valued wave packet frames for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, see Theorem 3.2. It is observed that the condition given in Theorem 3.2 is only sufficient but not necessary, see Example 3.3. A necessary and sufficient condition for finite sum of matrix-valued wave packet frames in terms of series associated with wave packet vectors is given in Theorem 3.4. An example spinning off of Theorem 3.4 is given. An optimal estimate of wave packet frame bounds for the finite sum of matrix-valued wave packet frames is presented in Theorem 3.9. In Section 4, we discuss an application of frame bounds associated with sums of matrix-valued wave packet frames in separable Hilbert spaces. It is shown that the rate of approximation in the frame algorithm can be increased by suitable choice of scalars and frame bounds which appear in the finite linear sum of frames, see Example 4.2.

2. Preliminaries

Throughout the paper, symbols $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of integers, positive integers, real numbers and complex numbers, respectively. Let $d, s, r \in \mathbb{N}$. The complex vector space of all s-by-s complex matrices is denoted by $\mathcal{M}_s(\mathbb{C})$. Matrix-valued functions are denoted by bold letters. We consider the matrix-valued function space

$$L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}) = \Big\{ \mathbf{f}(u) : u \in \mathbb{R}^d,\ f_{ij}(u) \in L^2(\mathbb{R}^d)\ (1 \le i \le s, 1 \le j \le r) \Big\},$$

where

$$\mathbf{f}(u) = \begin{bmatrix} f_{11}(u) & f_{12}(u) & \cdots & f_{1r}(u) \\ f_{21}(u) & f_{22}(u) & \cdots & f_{2r}(u) \\ \vdots & \vdots & \ddots & \vdots \\ f_{s1}(u) & f_{s2}(u) & \cdots & f_{sr}(u) \end{bmatrix}.$$

The norm on $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ is given by

$$\|\mathbf{f}\| = \Big(\sum_{\substack{1\le i\le s \\ 1\le j\le r}} \int_{\mathbb{R}^d} |f_{ij}(u)|^2 du \Big)^{\frac{1}{2}}.$$

The integration of a matrix-valued function $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ is given by

$$\int_{\mathbb{R}^d} \mathbf{f}(u) = \begin{bmatrix} \int_{\mathbb{R}^d} f_{11}(u) & \int_{\mathbb{R}^d} f_{12}(u) & \cdots & \int_{\mathbb{R}^d} f_{1r}(u) \\ \int_{\mathbb{R}^d} f_{21}(u) & \int_{\mathbb{R}^d} f_{22}(u) & \cdots & \int_{\mathbb{R}^d} f_{2r}(u) \\ \vdots & \vdots & \ddots & \vdots \\ \int_{\mathbb{R}^d} f_{s1}(u) & \int_{\mathbb{R}^d} f_{s2}(u) & \cdots & \int_{\mathbb{R}^d} f_{sr}(u) \end{bmatrix}.$$

The matrix-valued inner product on $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ is defined as

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{\mathbb{R}^d} \mathbf{f}(u)\mathbf{g}^*(u),\ \ \mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}),$$

where $*$ denotes the transpose and the complex conjugate. It has the following properties.

1. For every $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$,

$$\langle \mathbf{f}, \mathbf{g} \rangle = \langle \mathbf{g}, \mathbf{f} \rangle^*.$$

2. For every $\mathbf{f}, \mathbf{g}, \mathbf{h} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ and for every $M_1, M_2 \in \mathcal{M}_s(\mathbb{C})$,

$$\langle M_1\mathbf{f} + M_2\mathbf{g}, \mathbf{h} \rangle = M_1\langle \mathbf{f}, \mathbf{h} \rangle + M_2\langle \mathbf{g}, \mathbf{h} \rangle.$$

It is also readily seen that

$$\|\mathbf{f}\|^2 = \mathrm{trace}\langle \mathbf{f}, \mathbf{f} \rangle.$$

We consider the space $\ell^2(\mathbb{Z}^{d+2}, \mathcal{M}_s(\mathbb{C}))$ given by

$$\ell^2(\mathbb{Z}^{d+2}, \mathcal{M}_s(\mathbb{C})) := \Big\{ \{M_k\}_{k\in\mathbb{Z}^{d+2}} \subset \mathcal{M}_s(\mathbb{C}) : \sum_{k\in\mathbb{Z}^{d+2}} \|M_k\|^2 < \infty \Big\}.$$

$\ell^2(\mathbb{Z}^{d+2}, \mathcal{M}_s(\mathbb{C}))$ is a Banach space with respect to the norm given by

$$\|\{M_k\}_{k\in\mathbb{Z}^{d+2}}\| = \Big(\sum_{k\in\mathbb{Z}^{d+2}} \|M_k\|^2\Big)^{\frac{1}{2}}.$$

By $GL_d(\mathbb{R})$ we denote the set of all invertible d-by-d matrices over $\mathbb{R}$. Let $a, b \in \mathbb{R}^d$ and let C be a real d-by-d matrix. For b and u in $\mathbb{R}^d$, $b \cdot u$ denotes the standard inner product of b and u. We consider operators $T_a,\ E_b, D_C : L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}) \to L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ given by

$$\begin{aligned} T_a\mathbf{f}(u) &= \mathbf{f}(u-a) \quad \text{(Translation by } a),\\ E_b\mathbf{f}(u) &= e^{2\pi i b\cdot u}\mathbf{f}(u) \quad \text{(Modulation by } b),\\ D_C\mathbf{f}(u) &= |\det C|^{\frac{1}{2}}\mathbf{f}(Cu) \quad \text{(Dilation by } C). \end{aligned}$$

We end the section by recording two lemmas which give Cauchy-Schwartz type inequality and norm of an operator on $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, respectively.

Lemma 2.1. *For any $\boldsymbol{f}, \boldsymbol{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, we have $|\operatorname{trace}\langle \boldsymbol{f}, \boldsymbol{g}\rangle| \leq \|\boldsymbol{f}\|\|\boldsymbol{g}\|$.*

Lemma 2.2. *Let $U : L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}) \to L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ be a bounded linear operator such that*

$$\langle U\boldsymbol{f}, \boldsymbol{g}\rangle = \langle \boldsymbol{f}, U\boldsymbol{g}\rangle, \ \textit{for all } \boldsymbol{f}, \boldsymbol{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}).$$

Then

$$\|U\| = \sup_{\substack{\|\boldsymbol{f}\|=1\\ \boldsymbol{f}\in L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})}} |\operatorname{trace}\langle U\boldsymbol{f}, \boldsymbol{f}\rangle|.$$

3. Main Results

Let $\Psi \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ and let $\Lambda_1, \Lambda_2 \subset \mathbb{R}$ and $\Lambda^d \subset \mathbb{R}^d$ be countable sets. Let $\{A_j\}_{j\in\Lambda_1} \subset GL_d(\mathbb{R})$, $B \in GL_d(\mathbb{R})$ and $\{C_m\}_{m\in\Lambda_2} \subset \mathbb{R}^d$. A system of the form

$$\mathcal{W}(\Psi) \equiv \{D_{A_j}T_{Bk}E_{C_m}\Psi\}_{j\in\Lambda_1, m\in\Lambda_2, k\in\Lambda^d}$$

is called a *Weyl-Heisenberg matrix-valued wave packet system* in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. In this paper, we consider $\Lambda_1 = \Lambda_2 = \mathbb{Z}, \Lambda^d = \mathbb{Z}^d$ so that the family considered is $\mathcal{W}(\Psi) = \{D_{A_j}T_{Bk}E_{C_m}\Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$.

Definition 3.1. *A family $\mathcal{W}(\Psi) \equiv \{D_{A_j}T_{Bk}E_{C_m}\Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ is a* Weyl-Heisenberg matrix-valued wave packet frame (WHMV wave packet frame, *in short) for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ if there exist positive scalars $L_\Psi, U_\Psi < \infty$ such that for all $\boldsymbol{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$,*

$$L_\Psi\|\boldsymbol{f}\|^2 \leq \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \Big\|\Big\langle \boldsymbol{f}, D_{A_j}T_{Bk}E_{C_m}\Psi\Big\rangle\Big\|^2 \leq U_\Psi\|\boldsymbol{f}\|^2. \tag{3.1}$$

The scalars L_Ψ and U_Ψ are called *lower wave packet frame bound* and *upper wave packet frame bound*, respectively. If the upper inequality in (3.1) is satisfied, then we say that $\mathcal{W}(\Psi)$ is a *WHMV wave packet Bessel sequence* with *Bessel bound* U_Ψ. The *frame operator* $S_\Psi : L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}) \to L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ associated with $\mathcal{W}(\Psi)$ is given by

$$S_\Psi : \mathbf{f} \mapsto \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \langle \mathbf{f}, D_{A_j} T_{Bk} E_{C_m} \Psi \rangle D_{A_j} T_{Bk} E_{C_m} \Psi, \ \mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}).$$

If $\mathcal{W}(\Psi)$ is a frame for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, then S_Ψ is a linear, bounded and invertible operator satisfying

$$\langle S_\Psi \mathbf{f}, \mathbf{g} \rangle = \langle \mathbf{f}, S_\Psi \mathbf{g} \rangle, \ \mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$$

and hence

$$\langle S_\Psi^{-1} \mathbf{f}, \mathbf{g} \rangle = \langle \mathbf{f}, S_\Psi^{-1} \mathbf{g} \rangle, \ \mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}).$$

Let $[p] = \{1, 2, 3, .., p\}$ be a finite subset of $\mathbb{N}$ and let $\{A_j\}_{j\in\mathbb{Z}} \subset GL_d(\mathbb{R})$, $B \in GL_d(\mathbb{R})$, $\{C_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}^d$. For each $\xi \in [p]$, assume that $\mathcal{W}(\Psi_\xi) \equiv \{D_{A_j} T_{Bk} E_{C_m} \Psi_\xi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ is a WHMV wave packet frame for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, where $\Psi_\xi \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$.

The finite sum

$$\Big\{ \sum_{\xi=1}^{p} \alpha_\xi \mathcal{W}(\Psi_\xi) \Big\} \equiv \left\{ \sum_{\xi=1}^{p} \alpha_\xi D_{A_j} T_{Bk} E_{C_m} \Psi_\xi \right\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d},$$

where $\alpha_1, \alpha_2, ..., \alpha_p$ are non-zero scalars, in general, does not constitute a frame for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. The following theorem gives a sufficient condition for the finite sum $\Big\{ \sum_{\xi=1}^{p} \alpha_\xi \mathcal{W}(\Psi_\xi) \Big\}$ in terms of wave packet frame bounds and scalars associated with the finite sum to be a WHMV wave packet frame for the function space $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. An application of this result to the frame algorithm is given in Section 4. In matrix theory this type of condition is known as the dominance property.

THEOREM 3.2. *For each* $\xi \in [p]$, *let* $\mathcal{W}(\Psi_\xi) = \{D_{A_j} T_{Bk} E_{C_m} \Psi_\xi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ *be a WHMV wave packet frame for* $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ *with frame bounds* L_ξ, U_ξ *and let* $\alpha_1, \alpha_2, ..., \alpha_p$ *be non-zero scalars. If*

$$|\alpha_i|^2 L_i > \sum_{\substack{\xi=1\\ \xi\neq i}}^{p} |\alpha_\xi|^2 U_\xi + \sum_{\substack{\xi,t=1\\ \xi\neq i, t\neq i, \xi\neq t}}^{p} |\alpha_\xi \alpha_t| \sqrt{U_\xi U_t} \tag{3.2}$$

for some $i \in [p]$*, then the finite sum* $\left\{ \sum_{\xi=1}^{p} \alpha_\xi \mathcal{W}(\Psi_\xi) \right\}$ *is a frame for* $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ *with frame bounds*

$$\left(|\alpha_i|\sqrt{L_i} - \sqrt{\sum_{\substack{\xi=1 \\ \xi\neq i}}^{p} |\alpha_\xi|^2 U_\xi + \sum_{\substack{\xi,t=1 \\ \xi\neq i, t\neq i, \xi\neq t}}^{p} |\alpha_\xi \alpha_t| \sqrt{U_\xi U_t}} \right)^2,$$

and

$$\left(|\alpha_i|\sqrt{U_i} + \sqrt{\sum_{\substack{\xi=1 \\ \xi\neq i}}^{p} |\alpha_\xi|^2 U_\xi + \sum_{\substack{\xi,t=1 \\ \xi\neq i, t\neq i, \xi\neq t}}^{p} |\alpha_\xi \alpha_t| \sqrt{U_\xi U_t}} \right)^2.$$

Proof. By using Cauchy-Bunyakovsky-Schwarz inequality, we compute

$$\begin{aligned}
&\sum_{\substack{j,m\in\mathbb{Z} \\ k\in\mathbb{Z}^d}} \Big\| \Big\langle \mathbf{f}, \sum_{\xi=1}^{p} \alpha_\xi D_{A_j} T_{Bk} E_{C_m} \Psi_\xi - \alpha_i D_{A_j} T_{Bk} E_{C_m} \Psi_i \Big\rangle \Big\|^2 \\
&\leq \sum_{\substack{j,m\in\mathbb{Z} \\ k\in\mathbb{Z}^d}} \Big[\sum_{\substack{\xi=1 \\ \xi\neq i}}^{p} |\alpha_\xi|^2 \|\langle \mathbf{f}, D_{A_j} T_{Bk} E_{C_m} \Psi_\xi \rangle\|^2 \\
&\quad + \sum_{\substack{\xi,t=1 \\ \xi\neq i, t\neq i, \xi\neq t}}^{p} |\alpha_\xi \alpha_t| \|\langle \mathbf{f}, D_{A_j} T_{Bk} E_{C_m} \Psi_\xi \rangle\| \|\langle \mathbf{f}, D_{A_j} T_{Bk} E_{C_m} \Psi_t \rangle\| \Big] \\
&= \sum_{\substack{\xi=1 \\ \xi\neq i}}^{p} |\alpha_\xi|^2 \sum_{j,m\in\mathbb{Z}, k\in\mathbb{Z}^d} \|\langle \mathbf{f}, D_{A_j} T_{Bk} E_{C_m} \Psi_\xi \rangle\|^2 \\
&\quad + \sum_{\substack{\xi,t=1 \\ \xi\neq i, t\neq i, \xi\neq t}}^{p} |\alpha_\xi \alpha_t| \sum_{\substack{j,m\in\mathbb{Z} \\ k\in\mathbb{Z}^d}} \|\langle \mathbf{f}, D_{A_j} T_{Bk} E_{C_m} \Psi_\xi \rangle\| \|\langle \mathbf{f}, D_{A_j} T_{Bk} E_{C_m} \Psi_t \rangle\| \\
&\leq \left(\sum_{\substack{\xi=1 \\ \xi\neq i}}^{p} |\alpha_\xi|^2 U_\xi + \sum_{\substack{\xi,t=1 \\ \xi\neq i, t\neq i, \xi\neq t}}^{p} |\alpha_\xi \alpha_t| \sqrt{U_\xi U_t} \right) \|\mathbf{f}\|^2,
\end{aligned}$$

for any $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$.

By using the obtained estimate, for any $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, we compute

$$\begin{aligned}
&\Bigg(\sum_{\substack{j,m\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \Big\|\Big\langle \mathbf{f}, \sum_{\xi=1}^{p} \alpha_\xi D_{A_j} T_{Bk} E_{C_m} \Psi_\xi \Big\rangle\Big\|^2\Bigg)^{1/2} \\
&\quad\le \Bigg(\sum_{\substack{j,m\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \Big\|\Big\langle \mathbf{f}, \alpha_i D_{A_j} T_{Bk} E_{C_m} \Psi_i \Big\rangle\Big\|^2\Bigg)^{1/2} \\
&\qquad + \Bigg(\sum_{\substack{j,m\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \Big\|\Big\langle \mathbf{f}, \sum_{\xi=1}^{p} \alpha_\xi D_{A_j} T_{Bk} E_{C_m} \Psi_\xi - \alpha_i D_{A_j} T_{Bk} E_{C_m} \Psi_i \Big\rangle\Big\|^2\Bigg)^{1/2} \\
&\quad\le |\alpha_i|\sqrt{U_i}\|\mathbf{f}\| + \left(\sqrt{\sum_{\substack{\xi=1\\ \xi\neq i}}^{p} |\alpha_\xi|^2 U_\xi + \sum_{\substack{\xi,t=1\\ \xi\neq i, t\neq i, \xi\neq t}}^{p} |\alpha_\xi \alpha_t| \sqrt{U_\xi U_t}}\right)\|\mathbf{f}\| \\
&\quad= \left(|\alpha_i|\sqrt{U_i} + \sqrt{\sum_{\substack{\xi=1\\ \xi\neq i}}^{p} |\alpha_\xi|^2 U_\xi + \sum_{\substack{\xi,t=1\\ \xi\neq i, t\neq i, \xi\neq t}}^{p} |\alpha_\xi \alpha_t| \sqrt{U_\xi U_t}}\right)\|\mathbf{f}\|.
\end{aligned}$$

This gives, for any $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, that

$$\begin{aligned}
&\sum_{\substack{j,m\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \Big\|\Big\langle \mathbf{f}, \sum_{\xi=1}^{p} \alpha_\xi \mathcal{W}(\Psi_\xi) \Big\rangle\Big\|^2 \\
&\quad\le \left(|\alpha_i|\sqrt{U_i} + \sqrt{\sum_{\substack{\xi=1\\ \xi\neq i}}^{p} |\alpha_\xi|^2 U_\xi + \sum_{\substack{\xi,t=1\\ \xi\neq i, t\neq i, \xi\neq t}}^{p} |\alpha_\xi \alpha_t| \sqrt{U_\xi U_t}}\right)^2 \|\mathbf{f}\|^2.
\end{aligned} \tag{3.3}$$

Next we compute (for any $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$)

$$\begin{aligned}
&\Bigg(\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \Big\|\Big\langle \mathbf{f}, \sum_{\xi=1}^{p} \alpha_\xi D_{A_j} T_{Bk} E_{C_m} \Psi_\xi \Big\rangle\Big\|^2\Bigg)^{1/2} \\
&\quad\ge \Bigg(\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \Big\|\Big\langle \mathbf{f}, \alpha_i D_{A_j} T_{Bk} E_{C_m} \Psi_i \Big\rangle\Big\|^2\Bigg)^{1/2} \\
&\qquad - \Bigg(\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \Big\|\Big\langle \mathbf{f}, \sum_{\xi=1}^{p} \alpha_\xi D_{A_j} T_{Bk} E_{C_m} \Psi_\xi - \alpha_i D_{A_j} T_{Bk} E_{C_m} \Psi_i \Big\rangle\Big\|^2\Bigg)^{1/2}
\end{aligned}$$

$$\geq |\alpha_i|\sqrt{L_i}\|\mathbf{f}\| - \left(\sqrt{\sum_{\substack{\xi=1\\ \xi\neq i}}^{p}|\alpha_\xi|^2 U_\xi + \sum_{\substack{\xi,t=1\\ \xi\neq i, t\neq i, \xi\neq t}}^{p}|\alpha_\xi\alpha_t|\sqrt{U_\xi U_t}}\right)\|\mathbf{f}\|$$

$$= \left(|\alpha_i|\sqrt{L_i} - \sqrt{\sum_{\substack{\xi=1\\ \xi\neq i}}^{p}|\alpha_\xi|^2 U_\xi + \sum_{\substack{\xi,t=1\\ \xi\neq i, t\neq i, \xi\neq t}}^{p}|\alpha_\xi\alpha_t|\sqrt{U_\xi U_t}}\right)\|\mathbf{f}\|.$$

Therefore, for all $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, we have

$$\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}\left\|\left\langle \mathbf{f}, \sum_{\xi=1}^{p}\alpha_\xi \mathcal{W}(\Psi_\xi)\right\rangle\right\|^2$$

$$\geq \left(|\alpha_i|\sqrt{L_i} - \sqrt{\sum_{\substack{\xi=1\\ \xi\neq i}}^{p}|\alpha_\xi|^2 U_\xi + \sum_{\substack{\xi,t=1\\ \xi\neq i, t\neq i, \xi\neq t}}^{p}|\alpha_\xi\alpha_t|\sqrt{U_\xi U_t}}\right)^2 \|\mathbf{f}\|^2. \tag{3.4}$$

Hence, by (3.3) and (3.4), the finite sum $\left\{\sum_{\xi=1}^{p}\alpha_\xi\mathcal{W}(\Psi_\xi)\right\}$ is a WHMV wave packet frame for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ with the required frame bounds. ∎

The following example shows that condition (3.2) given in Theorem 3.2 is only sufficient but not necessary.

Example 3.3. Let $p = 2$ and let τ_1 be the triangle of vertices $(0,0), (1,0)$ and $(0,1)$. Consider a sequence of non-degenerate triangles $\{\tau_j\}_{j\in\mathbb{N}}$ with disjoint interiors satisfying $\bigcup_{j\in\mathbb{N}}\tau_j = \mathbb{R}^2$.

Let $u_j = ({u_j}^{(1)}, {u_j}^{(2)})$, $v_j = ({v_j}^{(1)}, {v_j}^{(2)})$ and $w_j = ({w_j}^{(1)}, {w_j}^{(2)})$ denote the vertices of the triangle τ_j for each $j \in \mathbb{N}$. Then the wave packet system $\{D_{A_j}E_{C_j}T_k\psi\}_{j\in\mathbb{N},k\in\mathbb{Z}^2}$ with $\hat{\psi}(\xi) = \chi_{\tau_1}(\xi)$ for $\xi \in \mathbb{R}^2, C_j = ({A_j}^T)^{-1}u_j$ is a Parseval frame for $L^2(\mathbb{R}^2)$, where (see [23])

$${A_j}^T = \begin{bmatrix} {v_j}^{(1)} - {u_j}^{(1)} & {w_j}^{(1)} - {u_j}^{(1)} \\ {v_j}^{(2)} - {u_j}^{(2)} & {w_j}^{(2)} - {u_j}^{(2)} \end{bmatrix}.$$

Since

$$D_{A_j}T_kE_{C_j}\psi = \exp(-2\pi ik.C_j)D_{A_j}E_{C_j}T_k\psi,$$

we get that the family $\{D_{A_j}T_kE_{C_j}\psi\}_{j\in\mathbb{N},k\in\mathbb{Z}^2}$ is a Parseval frame for $L^2(\mathbb{R}^2)$. Let

$$\Psi = \begin{bmatrix} 0 & \psi_{12} \\ \psi_{21} & 0 \end{bmatrix}, \text{ where } \psi_{12} = \psi_{21} = \psi.$$

Then, $\Psi \in L^2(\mathbb{R}^2, \mathbb{C}^{2\times 2})$ and for all $\mathbf{f} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \in L^2(\mathbb{R}^2, \mathbb{C}^{2\times 2})$, we have

$$\begin{aligned}
&\sum_{j\in\mathbb{N}, k\in\mathbb{Z}^2} \left\| \left\langle D_{A_j} T_k E_{C_j} \Psi, \mathbf{f} \right\rangle \right\|^2 \\
&\quad = \sum_{j\in\mathbb{N}, k\in\mathbb{Z}^2} \Big[\Big| \int_{\mathbb{R}^2} D_{A_j} T_k E_{C_j} \psi \overline{f_{12}} \Big|^2 + \Big| \int_{\mathbb{R}^2} D_{A_j} T_k E_{C_j} \psi \overline{f_{22}} \Big|^2 \\
&\qquad + \Big| \int_{\mathbb{R}^2} D_{A_j} T_k E_{C_j} \psi \overline{f_{11}} \Big|^2 \quad + \Big| \int_{\mathbb{R}^2} D_{A_j} T_k E_{C_j} \psi \overline{f_{21}} \Big|^2 \Big] \\
&\quad = \|f_{12}\|^2 + \|f_{22}\|^2 + \|f_{11}\|^2 + \|f_{21}\|^2 \\
&\quad = \|\mathbf{f}\|^2.
\end{aligned}$$

Hence, $\{D_{A_j} T_k E_{C_j} \Psi\}_{j\in\mathbb{N}, k\in\mathbb{Z}^2}$ is a Parseval WHMV wave packet frame for $L^2(\mathbb{R}^2, \mathbb{C}^{2\times 2})$.

Choose $\alpha_1 = \alpha_2 = 1$ and $\Psi_1 = \Psi_2 = \Psi$. Then, the family

$$\mathcal{W}(\Psi_\xi) = \{D_{A_j} T_k E_{C_j} \Psi_\xi\}_{j\in\mathbb{N}, k\in\mathbb{Z}^2}$$

is a WHMV wave packet frame for $L^2(\mathbb{R}^2, \mathbb{C}^{2\times 2})$ with frame bounds $L_\xi = 1$, $U_\xi = 1$ ($\xi \in \{1, 2\}$). Hence the finite sum $\Big\{ \sum_{\xi=1}^{2} \alpha_\xi \mathcal{W}(\Psi_\xi) \Big\}$ is a WHMV wave packet frame for $L^2(\mathbb{R}^2, \mathbb{C}^{2\times 2})$ with frame bounds $L_o = U_o = 4$.

The estimate in inequality (3.2) for $i = 1$ is given by

$$1 = |\alpha_1|^2 L_1 > \sum_{\substack{\xi=1 \\ \xi\neq 1}}^{2} |\alpha_\xi|^2 U_\xi + \sum_{\substack{\xi,t=1 \\ \xi\neq 1, t\neq 1, \xi\neq t}}^{2} |\alpha_\xi \alpha_t| \sqrt{U_\xi U_t} = |\alpha_2|^2 U_2 = 1,$$

a contradiction. Similarly, for $i = 2$, we arrive at a contradiction.

Kaushik, Singh and Virender in [22] studied finite sum of Gabor frames in $L^2(\mathbb{R})$. They observed that if some scalar multiple of a series associated with a Gabor frame is dominated by the series associated with the finite sum of Gabor frames, then the finite sum constitutes a Gabor frame for the underlying space and vice-versa, see Theorem 4.2 of [22]. The following theorem generalizes this result in the context of WHMV wave packet frames. This is an adaption of [22, Theorem 4.2].

THEOREM 3.4. *For each $\xi \in [p]$, let $\mathcal{W}(\Psi_\xi) = \{D_{A_j} T_{Bk} E_{C_m} \Psi_\xi\}_{j,m\in\mathbb{Z}, k\in\mathbb{Z}^d}$ be a WHMV wave packet frame for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ with frame bounds L_ξ, U_ξ, and let $\alpha_1, \alpha_2, ..., \alpha_p$ be any non-zero scalars. Then, the finite sum $\Big\{ \sum_{\xi=1}^{p} \alpha_\xi \mathcal{W}(\Psi_\xi) \Big\}$ is a frame for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ if and only if there exists $\beta > 0$*

and $i \in [p]$ *such that*

$$\beta \sum_{\substack{j,m\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \left\| \left\langle \boldsymbol{f}, D_{A_j} T_{Bk} E_{C_m} \Psi_i \right\rangle \right\|^2 \leq \sum_{\substack{j,m\in\mathbb{Z}\\ k\in\mathbb{Z}^d}} \left\| \left\langle \boldsymbol{f}, \sum_{\xi=1}^{p} \alpha_\xi \mathcal{W}(\Psi_\xi) \right\rangle \right\|^2, \tag{3.5}$$

for all $\boldsymbol{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$.

Proof. Suppose (3.5) holds. Then, for any $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, we have

$$\begin{aligned}
\beta L_i \|\mathbf{f}\|^2 &\leq \beta \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\| \left\langle \mathbf{f},\ D_{A_j} T_{Bk} E_{C_m} \Psi_i \right\rangle \right\|^2 \\
&\leq \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\| \left\langle \mathbf{f}, \sum_{\xi=1}^{p} \alpha_\xi \mathcal{W}(\Psi_\xi) \right\rangle \right\|^2 \\
&= \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\| \left\langle \mathbf{f},\ \sum_{\xi=1}^{p} \alpha_\xi D_{A_j} T_{Bk} E_{C_m} \Psi_\xi \right\rangle \right\|^2 \\
&\leq \left(\sum_{\xi=1}^{p} |\alpha_\xi|^2\ U_\xi \right) p\ \|\mathbf{f}\|^2.
\end{aligned}$$

Hence the finite sum $\left\{ \sum_{\xi=1}^{p} \alpha_\xi \mathcal{W}(\Psi_\xi) \right\}$ is a WHMV wave packet frame for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ with frame bounds βL_i and $\left(\sum_{\xi=1}^{p} |\alpha_\xi|^2\ U_\xi \right) p$.

Conversely, let $\left\{ \sum_{\xi=1}^{p} \alpha_\xi \mathcal{W}(\Psi_\xi) \right\}$ be a frame for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ with frame bounds L_o, U_o. Then for any (but fixed) $i \in [p]$, we have

$$\frac{1}{U_i} \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\| \left\langle \mathbf{f},\ D_{A_j} T_{Bk} E_{C_m} \Psi_i \right\rangle \right\|^2 \leq \|\mathbf{f}\|^2,\ \ \mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}).$$

Therefore, for $\beta = \frac{L_o}{U_i} > 0$, we have

$$\begin{aligned}
\beta &\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\| \left\langle \mathbf{f},\ D_{A_j} T_{Bk} E_{C_m} \Psi_i \right\rangle \right\|^2 \\
&= \frac{L_o}{U_i} \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\| \left\langle \mathbf{f},\ D_{A_j} T_{Bk} E_{C_m} \Psi_i \right\rangle \right\|^2 \\
&\leq L_o \|\mathbf{f}\|^2 \\
&\leq \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\| \left\langle \mathbf{f}, \sum_{\xi=1}^{p} \alpha_\xi \mathcal{W}(\Psi_\xi) \right\rangle \right\|^2
\end{aligned}$$

for all $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. The proof is complete. □

The following example illustrates Theorem 3.4.

EXAMPLE 3.5. Consider the Parseval frame $\{D_{A_j}T_kE_{C_j}\psi\}_{j\in\mathbb{N},k\in\mathbb{Z}^2}$ for $L^2(\mathbb{R}^2)$ given in Example 3.3.

(I): Let

$$\Psi_1 = \begin{bmatrix}\psi\\0\\0\end{bmatrix} \in L^2(\mathbb{R}^2,\mathbb{C}^{3\times 1}).$$

Then, for all $\mathbf{f} = \begin{bmatrix}f_{11}\\f_{21}\\f_{31}\end{bmatrix} \in L^2(\mathbb{R}^2,\mathbb{C}^{3\times 1})$, we have

$$\begin{aligned}
&\sum_{j\in\mathbb{N},k\in\mathbb{Z}^2}\left\|\left\langle D_{A_j}T_kE_{C_j}\Psi_1,\mathbf{f}\right\rangle\right\|^2\\
&\quad=\sum_{j\in\mathbb{N},k\in\mathbb{Z}^2}\Big[\big|\int_{\mathbb{R}^2}D_{A_j}T_kE_{C_j}\psi\overline{f_{11}}\big|^2+\big|\int_{\mathbb{R}^2}D_{A_j}T_kE_{C_j}\psi\overline{f_{21}}\big|^2\\
&\qquad+\big|\int_{\mathbb{R}^2}D_{A_j}T_kE_{C_j}\psi\overline{f_{31}}\big|^2\Big]\\
&\quad=\|f_{11}\|^2+\|f_{21}\|^2+\|f_{31}\|^2\\
&\quad=\|\mathbf{f}\|^2.
\end{aligned}$$

Hence, $\{D_{A_j}T_kE_{C_j}\Psi_1\}_{j\in\mathbb{N},k\in\mathbb{Z}^2}$ is a Parseval WHMV wave packet frame for $L^2(\mathbb{R}^2,\mathbb{C}^{3\times 1})$. Similarly, we can show that $\{D_{A_j}T_kE_{C_j}\Psi_2\}_{j\in\mathbb{N},k\in\mathbb{Z}^2}$ is a Parseval WHMV wave packet frame for $L^2(\mathbb{R}^2,\mathbb{C}^{3\times 1})$, where $\Psi_2 = \begin{bmatrix}0\\0\\\psi\end{bmatrix}$.

Choose $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = \frac{1}{3}$. Then, $\left\{\sum\limits_{\xi=1}^{2}\alpha_\xi\mathcal{W}(\Psi_\xi)\right\}$ is equal to $\{D_{A_j}T_kE_{C_j}\Psi\}_{j\in\mathbb{N},k\in\mathbb{Z}^2}$, where $\Psi = \begin{bmatrix}\frac{1}{2}\psi\\0\\\frac{1}{3}\psi\end{bmatrix}$. For any $\mathbf{f} = \begin{bmatrix}f_{11}\\f_{21}\\f_{31}\end{bmatrix} \in L^2(\mathbb{R}^2,\mathbb{C}^{3\times 1})$, we compute

$$\begin{aligned}
&\sum_{j\in\mathbb{N},k\in\mathbb{Z}^2}\left\|\left\langle \sum_{\xi=1}^{2}\alpha_\xi\mathcal{W}(\Psi_\xi),\mathbf{f}\right\rangle\right\|^2\\
&\quad=\sum_{j\in\mathbb{N},k\in\mathbb{Z}^2}\left\|\left\langle D_{A_j}T_kE_{C_j}\Psi,\mathbf{f}\right\rangle\right\|^2\\
&\quad=\frac{1}{4}\sum_{j\in\mathbb{N},k\in\mathbb{Z}^2}\Big[\big|\int_{\mathbb{R}^2}D_{A_j}T_kE_{C_j}\psi\overline{f_{11}}\big|^2+\big|\int_{\mathbb{R}^2}D_{A_j}T_kE_{C_j}\psi\overline{f_{21}}\big|^2
\end{aligned}$$

$$+ \big| \int_{\mathbb{R}^2} D_{A_j} T_k E_{C_j} \psi \overline{f_{31}} \big|^2 \Big] + \frac{1}{9} \sum_{j\in\mathbb{N},k\in\mathbb{Z}^2} \Big[\big| \int_{\mathbb{R}^2} D_{A_j} T_k E_{C_j} \psi \overline{f_{11}} \big|^2$$
$$+ \big| \int_{\mathbb{R}^2} D_{A_j} T_k E_{C_j} \psi \overline{f_{21}} \big|^2 + \big| \int_{\mathbb{R}^2} D_{A_j} T_k E_{C_j} \psi \overline{f_{31}} \big|^2 \Big]$$
$$= \frac{1}{4} \Big[\|f_{11}\|^2 + \|f_{21}\|^2 + \|f_{31}\|^2 \Big] + \frac{1}{9} \Big[\|f_{11}\|^2 + \|f_{21}\|^2 + \|f_{31}\|^2 \Big]$$
$$= \frac{13}{36} \|\mathbf{f}\|^2.$$

Choose $\beta = \frac{12}{36}$ and $i = 1$. Then, for all $\mathbf{f} \in L^2(\mathbb{R}^2, \mathbb{C}^{3\times 1})$, we have

$$\begin{aligned}\beta \sum_{j\in\mathbb{N},k\in\mathbb{Z}^2} \Big\| \Big\langle D_{A_j} T_k E_{C_j} \Psi_i, \mathbf{f} \Big\rangle \Big\|^2 &= \frac{12}{36} \|\mathbf{f}\|^2 \\ &\leq \frac{13}{36} \|\mathbf{f}\|^2 \\ &= \sum_{j\in\mathbb{N},k\in\mathbb{Z}^2} \Big\| \Big\langle \sum_{\xi=1}^{2} \alpha_\xi \mathcal{W}(\Psi_\xi), \mathbf{f} \Big\rangle \Big\|^2.\end{aligned}$$

Hence, by Theorem 3.4, the finite sum $\Big\{ \sum_{\xi=1}^{2} \alpha_\xi \mathcal{W}(\Psi_\xi) \Big\}$ is a WHMV wave packet frame for $L^2(\mathbb{R}^2, \mathbb{C}^{3\times 1})$.

(II): Let $\Psi_1 \in L^2(\mathbb{R}^2, \mathbb{C}^{2\times 2})$ be given by $\Psi_1 = \begin{bmatrix} 0 & \psi \\ \psi & 0 \end{bmatrix}$. Then, for all $\mathbf{f} \in L^2(\mathbb{R}^2, \mathbb{C}^{2\times 2})$, we have (see Example 3.3)

$$\sum_{j\in\mathbb{N},k\in\mathbb{Z}^2} \Big\| \Big\langle D_{A_j} T_k E_{C_j} \Psi_1, \mathbf{f} \Big\rangle \Big\|^2 = \|\mathbf{f}\|^2.$$

Hence, $\{D_{A_j} T_k E_{C_j} \Psi_1\}_{j\in\mathbb{N},k\in\mathbb{Z}^2}$ is a Parseval WHMV wave packet frame for $L^2(\mathbb{R}^2, \mathbb{C}^{2\times 2})$. Similarly, $\{D_{A_j} T_k E_{C_j} \Psi_2\}_{j\in\mathbb{N},k\in\mathbb{Z}^2}$ is also a Parseval WHMV wave packet frame for $L^2(\mathbb{R}^2, \mathbb{C}^{2\times 2})$, where $\Psi_2 = \begin{bmatrix} \psi & 0 \\ 0 & \psi \end{bmatrix}$.

Choose $\alpha_1 = \alpha_2 = 1$. Then, $\Big\{ \sum_{\xi=1}^{2} \alpha_\xi \mathcal{W}(\Psi_\xi) \Big\} \equiv \{D_{A_j} T_k E_{C_j} \Psi_o\}_{j\in\mathbb{N},k\in\mathbb{Z}^2}$, where $\Psi_o = \begin{bmatrix} \psi & \psi \\ \psi & \psi \end{bmatrix}$. For every $\beta > 0$ and for any $i \in \{1, 2\}$, we have

$$\beta \sum_{j\in\mathbb{N},k\in\mathbb{Z}^2} \Big\| \Big\langle D_{A_j} T_k E_{C_j} \Psi_i, \mathbf{f} \Big\rangle \Big\|^2 = \beta \|\mathbf{f}\|^2 \text{ for all } \mathbf{f} \in L^2(\mathbb{R}^2, \mathbb{C}^{2\times 2}). \tag{3.6}$$

Let $0 \neq f \in L^2(\mathbb{R}^2)$ be arbitrary but fixed. Then, for $\mathbf{0} \neq \mathbf{f}_o = \begin{bmatrix} -f & f \\ 0 & 0 \end{bmatrix} \in L^2(\mathbb{R}^2, \mathbb{C}^{2\times 2})$, we compute

$$
\begin{aligned}
&\sum_{j\in\mathbb{N},k\in\mathbb{Z}^2} \Big\| \Big\langle \sum_{\xi=1}^{2} \alpha_\xi \mathcal{W}(\Psi_\xi), \mathbf{f}_o \Big\rangle \Big\|^2 \\
&\quad = \sum_{j\in\mathbb{N},k\in\mathbb{Z}^2} \Big\| \Big\langle D_{A_j} T_k E_{C_j} \Psi_o, \mathbf{f}_o \Big\rangle \Big\|^2 \\
&\quad = \sum_{j\in\mathbb{N},k\in\mathbb{Z}^2} \Big\| \int_{\mathbb{R}^2} \begin{bmatrix} D_{A_j} T_k E_{C_j} \psi & D_{A_j} T_k E_{C_j} \psi \\ D_{A_j} T_k E_{C_j} \psi & D_{A_j} T_k E_{C_j} \psi \end{bmatrix} \begin{bmatrix} -\overline{f} & 0 \\ \overline{f} & 0 \end{bmatrix} \Big\|^2 \\
&\quad = \sum_{j\in\mathbb{N},k\in\mathbb{Z}^2} \Big\| \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Big\|^2 \\
&\quad = 0.
\end{aligned} \tag{3.7}
$$

By (3.6) and (3.7), we conclude that condition (3.5) of Theorem 3.4 is not satisfied. Hence, the finite sum $\Big\{ \sum_{\xi=1}^{2} \alpha_\xi \mathcal{W}(\Psi_\xi) \Big\}$ is not a WHMV wave packet frame for $L^2(\mathbb{R}^2, \mathbb{C}^{2\times 2})$.

Next, we discuss optimal bounds of a WHMV wave packet frame in $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. We first state and prove the following two lemmas.

Lemma 3.6. *Let $A : L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}) \to L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ be a linear operator satisfying the following conditions.*

1. $\langle A\boldsymbol{f}, \boldsymbol{g}\rangle = \langle \boldsymbol{f}, A\boldsymbol{g}\rangle$ *for every* $\boldsymbol{f}, \boldsymbol{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$.
2. $\operatorname{trace}\langle A\boldsymbol{f}, \boldsymbol{f}\rangle \geq 0$ *for every* $\boldsymbol{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$.

Then, for any $\boldsymbol{f}, \boldsymbol{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$,

$$
|\operatorname{trace}\langle A\boldsymbol{f}, \boldsymbol{g}\rangle|^2 \leq \operatorname{trace}\langle A\boldsymbol{f}, \boldsymbol{f}\rangle \ \operatorname{trace}\langle A\boldsymbol{g}, \boldsymbol{g}\rangle. \tag{3.8}
$$

Proof. Let $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ be arbitrary but fixed. First, we consider the case in which at least one of $\operatorname{trace}\langle A\mathbf{f}, \mathbf{f}\rangle$ and $\operatorname{trace}\langle A\mathbf{g}, \mathbf{g}\rangle$ is non-zero. Without loss of generality, assume that $\operatorname{trace}\langle A\mathbf{g}, \mathbf{g}\rangle$ is non-zero. Then, for every scalar α, we have

$$
\begin{aligned}
&0 \leq \operatorname{trace}\langle A(\mathbf{f} + \alpha\mathbf{g}), \mathbf{f} + \alpha\mathbf{g}\rangle \\
&\quad = \operatorname{trace}\langle A\mathbf{f}, \mathbf{f}\rangle + \overline{\alpha} \ \operatorname{trace}\langle A\mathbf{f}, \mathbf{g}\rangle + \alpha \ \operatorname{trace}\langle A\mathbf{g}, \mathbf{f}\rangle + \alpha\overline{\alpha} \ \operatorname{trace}\langle A\mathbf{g}, \mathbf{g}\rangle.
\end{aligned} \tag{3.9}
$$

Choose $\alpha = -\frac{\operatorname{trace}\langle A\mathbf{f}, \mathbf{g}\rangle}{\operatorname{trace}\langle A\mathbf{g}, \mathbf{g}\rangle}$. Then, by using (3.9) we have

$$
0 \leq \operatorname{trace}\langle A\mathbf{f}, \mathbf{f}\rangle - \frac{|\operatorname{trace}\langle A\mathbf{f}, \mathbf{g}\rangle|^2}{\operatorname{trace}\langle A\mathbf{g}, \mathbf{g}\rangle}.
$$

This gives (3.8).

Next, we discuss the case $\text{trace}\langle A\mathbf{f}, \mathbf{f}\rangle = 0 = \text{trace}\langle A\mathbf{g}, \mathbf{g}\rangle$. We compute

$$\begin{aligned}
&\text{trace}\langle A(\mathbf{f}+\mathbf{g}), \mathbf{f}+\mathbf{g}\rangle - \text{trace}\langle A(\mathbf{f}-\mathbf{g}), \mathbf{f}-\mathbf{g}\rangle \\
&\quad = 2\ \text{trace}\langle A\mathbf{f}, \mathbf{g}\rangle + 2\,\text{trace}\langle A\mathbf{g}, \mathbf{f}\rangle \\
&\quad = 2\ \text{trace}\langle A\mathbf{f}, \mathbf{g}\rangle + 2\,\text{trace}\langle \mathbf{g}, A\mathbf{f}\rangle \\
&\quad = 2\ \text{trace}\langle A\mathbf{f}, \mathbf{g}\rangle + 2\,\text{trace}\langle A\mathbf{f}, \mathbf{g}\rangle^{*} \\
&\quad = 2\ \text{trace}\langle A\mathbf{f}, \mathbf{g}\rangle + 2\overline{\text{trace}\langle A\mathbf{f}, \mathbf{g}\rangle} \\
&\quad = 4\ \text{Real}\Big(\text{trace}\langle A\mathbf{f}, \mathbf{g}\rangle\Big).
\end{aligned}$$

Similarly, we obtain

$$\text{trace}\langle A(\mathbf{f}+i\mathbf{g}), \mathbf{f}+i\mathbf{g}\rangle - \text{trace}\langle A(\mathbf{f}-i\mathbf{g}), \mathbf{f}-i\mathbf{g}\rangle = 4\ \text{Im}\Big(\text{trace}\langle A\mathbf{f}, \mathbf{g}\rangle\Big).$$

Therefore

$$\begin{aligned}
\text{trace}\langle A\mathbf{f}, \mathbf{g}\rangle = \frac{1}{4}\Big(&\text{trace}\langle A(\mathbf{f}+\mathbf{g}), \mathbf{f}+\mathbf{g}\rangle - \text{trace}\langle A(\mathbf{f}-\mathbf{g}), \mathbf{f}-\mathbf{g}\rangle \\
&+ i\ \text{trace}\langle A(\mathbf{f}+i\mathbf{g}), \mathbf{f}+i\mathbf{g}\rangle \\
&- i\ \text{trace}\langle A(\mathbf{f}-i\mathbf{g}), \mathbf{f}-i\mathbf{g}\rangle\Big).
\end{aligned} \tag{3.10}$$

Now

$$\begin{aligned}
&\text{trace}\langle A(\mathbf{f}+\mathbf{g}), \mathbf{f}+\mathbf{g}\rangle + \text{trace}\langle A(\mathbf{f}-\mathbf{g}), \mathbf{f}-\mathbf{g}\rangle \\
&\quad = \text{trace}\langle A\mathbf{f}, \mathbf{f}\rangle + \text{trace}\langle A\mathbf{f}, \mathbf{g}\rangle + \text{trace}\langle A\mathbf{g}, \mathbf{f}\rangle + \text{trace}\langle A\mathbf{g}, \mathbf{g}\rangle \\
&\qquad + \text{trace}\langle A\mathbf{f}, \mathbf{f}\rangle - \text{trace}\langle A\mathbf{f}, \mathbf{g}\rangle - \text{trace}\langle A\mathbf{g}, \mathbf{f}\rangle + \text{trace}\langle A\mathbf{g}, \mathbf{g}\rangle \\
&\quad = 0.
\end{aligned}$$

Therefore, by using condition (2.), we have

$$\text{trace}\langle A(\mathbf{f}+\mathbf{g}), \mathbf{f}+\mathbf{g}\rangle = 0 = \text{trace}\langle A(\mathbf{f}-\mathbf{g}), \mathbf{f}-\mathbf{g}\rangle. \tag{3.11}$$

Similarly,

$$\text{trace}\langle A(\mathbf{f}+i\mathbf{g}), \mathbf{f}+i\mathbf{g}\rangle + \text{trace}\langle A(\mathbf{f}-i\mathbf{g}), \mathbf{f}-i\mathbf{g}\rangle = 0,$$

implies

$$\text{trace}\langle A(\mathbf{f}+i\mathbf{g}), \mathbf{f}+i\mathbf{g}\rangle = 0 = \text{trace}\langle A(\mathbf{f}-i\mathbf{g}), \mathbf{f}-i\mathbf{g}\rangle. \tag{3.12}$$

Therefore, applying (3.11) and (3.12) in (3.10), we have

$$\text{trace}\langle A\mathbf{f}, \mathbf{g}\rangle = 0 \ \text{ for every } \ \mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}).$$

Hence inequality (3.8) is proved. The proof is complete. ∎

LEMMA 3.7. *Let* $A, B : L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}) \to L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ *be bounded linear operators and suppose that*

1. $\langle A\boldsymbol{f}, \boldsymbol{g}\rangle = \langle \boldsymbol{f}, A\boldsymbol{g}\rangle$ *and* $\langle AB\boldsymbol{f}, \boldsymbol{g}\rangle = \langle \boldsymbol{f}, AB\boldsymbol{g}\rangle$ *for every* $\boldsymbol{f}, \boldsymbol{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$,
2. $\text{trace}\langle A\boldsymbol{f}, \boldsymbol{f}\rangle \geq 0$ *for every* $\boldsymbol{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$.

Then

$$|\operatorname{trace}\langle AB\boldsymbol{f},\boldsymbol{f}\rangle| \le \|B\| \ \operatorname{trace}\langle A\boldsymbol{f},\boldsymbol{f}\rangle \ \textit{for every}\ \boldsymbol{f}\in L^2(\mathbb{R}^d,\mathbb{C}^{s\times r}). \tag{3.13}$$

Proof. By Lemma 3.6 and the inequality of Arithmetic and Geometric mean, we have

$$\begin{aligned} |\operatorname{trace}\langle A\mathbf{f},\mathbf{g}\rangle| &\le \sqrt{\operatorname{trace}\langle A\mathbf{f},\mathbf{f}\rangle \operatorname{trace}\langle A\mathbf{g},\mathbf{g}\rangle} \\ &\le \frac{1}{2}\Big(\operatorname{trace}\langle A\mathbf{f},\mathbf{f}\rangle + \operatorname{trace}\langle A\mathbf{g},\mathbf{g}\rangle\Big) \end{aligned} \tag{3.14}$$

for any $\mathbf{f},\mathbf{g} \in L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$. For any positive integer n and for any $\mathbf{f},\mathbf{g} \in L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$, we compute

$$\begin{aligned} \langle AB^n\mathbf{f},\mathbf{g}\rangle &= \langle B^{n-1}\mathbf{f}, AB\mathbf{g}\rangle \\ &= \langle AB^{n-1}\mathbf{f}, B\mathbf{g}\rangle \\ &= \langle B^{n-2}\mathbf{f}, AB^2\mathbf{g}\rangle \\ &= \cdots \\ &= \langle \mathbf{f}, AB^n\mathbf{g}\rangle. \end{aligned} \tag{3.15}$$

By (3.14) and (3.15), we have

$$\begin{aligned} |\operatorname{trace}\langle AB^n\mathbf{f},\mathbf{f}\rangle| &= |\operatorname{trace}\langle \mathbf{f}, AB^n\mathbf{f}\rangle| \\ &= |\operatorname{trace}\langle A\mathbf{f}, B^n\mathbf{f}\rangle| \\ &\le \frac{1}{2}\Big[\operatorname{trace}\langle A\mathbf{f},\mathbf{f}\rangle + \operatorname{trace}\langle AB^n\mathbf{f}, B^n\mathbf{f}\rangle\Big] \\ &= \frac{1}{2}\Big[\operatorname{trace}\langle A\mathbf{f},\mathbf{f}\rangle + \operatorname{trace}\langle B^n\mathbf{f}, AB^n\mathbf{f}\rangle\Big] \\ &= \frac{1}{2}\Big[\operatorname{trace}\langle A\mathbf{f},\mathbf{f}\rangle + \operatorname{trace}\langle AB^{2n}\mathbf{f},\mathbf{f}\rangle\Big], \end{aligned} \tag{3.16}$$

for any $\mathbf{f} \in L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$.

By using (3.16), for any $\mathbf{f} \in L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$, we compute

$$\begin{aligned} &|\operatorname{trace}\langle AB\mathbf{f},\mathbf{f}\rangle| \\ &\quad\le \frac{1}{2}\Big(\operatorname{trace}\langle A\mathbf{f},\mathbf{f}\rangle + \operatorname{trace}\langle AB^2\mathbf{f},\mathbf{f}\rangle\Big) \\ &\quad\le \frac{1}{2}\Big(\operatorname{trace}\langle A\mathbf{f},\mathbf{f}\rangle + \frac{1}{2}\Big(\operatorname{trace}\langle A\mathbf{f},\mathbf{f}\rangle + \operatorname{trace}\langle AB^4\mathbf{f},\mathbf{f}\rangle\Big)\Big) \\ &\quad\le \quad \cdots \\ &\quad\le \Big(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}\Big)\operatorname{trace}\langle A\mathbf{f},\mathbf{f}\rangle + \frac{1}{2^n}\operatorname{trace}\langle AB^{2^n}\mathbf{f},\mathbf{f}\rangle. \end{aligned} \tag{3.17}$$

If $\|B\| = 1$, then by using Lemma 2.1, we have that for any $\mathbf{f} \in L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})$

$$|\operatorname{trace}\langle AB^{2^n}\mathbf{f},\mathbf{f}\rangle| \le \|AB^{2^n}\mathbf{f}\|\|\mathbf{f}\| \le \|A\|\|\mathbf{f}\|^2.$$

Therefore, the last term in (3.17) tends to 0 as $n \to \infty$ and we obtain

$$|\operatorname{trace}\langle AB\mathbf{f}, \mathbf{f}\rangle| \leq \operatorname{trace}\langle A\mathbf{f}, \mathbf{f}\rangle$$

for all $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. If $\|B\| \neq 1$, we get (3.13) by replacing B by $B/\|B\|$. □

Lemma 3.7 immediately yields the following proposition.

PROPOSITION 3.8. *Let* $A, B : L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}) \to L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ *be bounded linear operators and suppose that*

1. $\langle A\boldsymbol{f}, \boldsymbol{g}\rangle = \langle \boldsymbol{f}, A\boldsymbol{g}\rangle$ *and* $\langle B\boldsymbol{f}, \boldsymbol{g}\rangle = \langle \boldsymbol{f}, B\boldsymbol{g}\rangle$ *for every* $\boldsymbol{f}, \boldsymbol{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$.
2. $\operatorname{trace}\langle A\boldsymbol{f}, \boldsymbol{f}\rangle \geq 0$ *and* $\operatorname{trace}\langle B\boldsymbol{f}, \boldsymbol{f}\rangle \geq 0$ *for every* $\boldsymbol{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$.
3. $AB=BA$.

Then

$$\operatorname{trace}\langle AB\boldsymbol{f}, \boldsymbol{f}\rangle \geq 0 \text{ for every } \boldsymbol{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}). \tag{3.18}$$

PROOF. We may assume, without loss of generality, that $0 \leq \operatorname{trace}\langle (I - B)\mathbf{f}, \mathbf{f}\rangle \leq \operatorname{trace}\langle I\mathbf{f}, \mathbf{f}\rangle$, where I is the identity operator on $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. Since otherwise, we can replace B by βB with an appropriate positive factor β.

By using Lemma 2.2 and Lemma 2.1, we have

$$\begin{aligned}
\|I - B\| &= \sup_{\|\mathbf{f}\|=1, \mathbf{f}\in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})} |\operatorname{trace}\langle (I - B)\mathbf{f}, \mathbf{f}\rangle| \\
&\leq \sup_{\|\mathbf{f}\|=1, \mathbf{f}\in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})} |\operatorname{trace}\langle I\mathbf{f}, \mathbf{f}\rangle| \\
&\leq \sup_{\|\mathbf{f}\|=1, \mathbf{f}\in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})} \|I\mathbf{f}\|\|\mathbf{f}\| \\
&= 1.
\end{aligned}$$

Since

$$A(I - B) = A - AB = A - BA = (I - B)A$$

and

$$\langle A(I - B)\mathbf{f}, \mathbf{g}\rangle = \langle (I - B)\mathbf{f}, A\mathbf{g}\rangle = \langle \mathbf{f}, (I - B)A\mathbf{g}\rangle = \langle \mathbf{f}, A(I - B)\mathbf{g}\rangle$$

for all $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. By Lemma 3.7, we have

$$\begin{aligned}
\operatorname{trace}\langle A(I - B)\mathbf{f}, \mathbf{f}\rangle &\leq \|I - B\| \ \operatorname{trace}\langle A\mathbf{f}, \mathbf{f}\rangle \\
&\leq \operatorname{trace}\langle A\mathbf{f}, \mathbf{f}\rangle,
\end{aligned}$$

for all $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. This gives $\operatorname{trace}\langle AB\mathbf{f}, \mathbf{f}\rangle \geq 0$ for all $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. □

Let $\{D_{A_j}T_{Bk}E_{C_m}\Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ be a WHMV wave packet frame for the function space $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ with frame operator S_Ψ and lower and upper frame bounds L_Ψ, U_Ψ, respectively. Then, for any $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, we have

$$\begin{aligned}
&\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\| \left\langle \mathbf{f}, S_\Psi^{-1} D_{A_j} T_{Bk} E_{C_m} \Psi \right\rangle \right\|^2 \\
&\quad = \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\| \left\langle S_\Psi^{-1}\mathbf{f}, D_{A_j} T_{Bk} E_{C_m} \Psi \right\rangle \right\|^2 \\
&\quad \le U_\Psi \|S_\Psi^{-1}\mathbf{f}\|^2 \\
&\quad \le U_\Psi \|S_\Psi^{-1}\|^2 \|\mathbf{f}\|^2.
\end{aligned}$$

Therefore, $\{S_\Psi^{-1} D_{A_j} T_{Bk} E_{C_m} \Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ is a Bessel sequence and hence, the frame operator for $\{S_\Psi^{-1} D_{A_j} T_{Bk} E_{C_m} \Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ is well defined.

Now

$$\begin{aligned}
&\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\langle \mathbf{f}, S_\Psi^{-1} D_{A_j} T_{Bk} E_{C_m} \Psi \right\rangle S_\Psi^{-1} D_{A_j} T_{Bk} E_{C_m} \Psi \\
&\quad = S_\Psi^{-1} \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\langle S_\Psi^{-1}\mathbf{f}, D_{A_j} T_{Bk} E_{C_m} \Psi \right\rangle D_{A_j} T_{Bk} E_{C_m} \Psi \\
&\quad = S_\Psi^{-1} S_\Psi S_\Psi^{-1} \mathbf{f} \\
&\quad = S_\Psi^{-1}\mathbf{f}, \ \mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}).
\end{aligned}$$

Therefore, the frame operator for $\{S_\Psi^{-1} D_{A_j} T_{Bk} E_{C_m} \Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ is S_Ψ^{-1}.

The inequality in (3.1) can be rewritten as

$$L_\Psi \mathrm{trace}\langle \mathbf{f}, \mathbf{f}\rangle \le \mathrm{trace}\langle S_\Psi \mathbf{f}, \mathbf{f}\rangle \le U_\Psi \mathrm{trace}\langle \mathbf{f}, \mathbf{f}\rangle, \ \mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}). \tag{3.19}$$

One may observe that the operators $S_\Psi - L_\Psi I$ and S_Ψ^{-1} satisfy the following:

1. $\langle (S_\Psi - L_\Psi I)\mathbf{f}, \mathbf{g}\rangle = \langle \mathbf{f}, (S_\Psi - L_\Psi I)\mathbf{g}\rangle$ and $\langle S_\Psi^{-1}\mathbf{f}, \mathbf{g}\rangle = \langle \mathbf{f}, S_\Psi^{-1}\mathbf{g}\rangle$, $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$.
2. $\mathrm{trace}\langle (S_\Psi - L_\Psi I)\mathbf{f}, \mathbf{f}\rangle \ge 0$ (by (3.19)) and for $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$,
$$\mathrm{trace}\langle S_\Psi^{-1}\mathbf{f}, \mathbf{f}\rangle = \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\| \left\langle \mathbf{f}, S_\Psi^{-1} D_{A_j} T_{Bk} E_{C_m} \Psi \right\rangle \right\|^2 \ge 0.$$
3. $(S_\Psi - L_\Psi I) S_\Psi^{-1} = I - L_\Psi S_\Psi^{-1} = S_\Psi^{-1}(S_\Psi - L_\Psi I)$.

By Proposition 3.8, we have

$$\mathrm{trace}\langle (S_\Psi - L_\Psi I) S_\Psi^{-1}\mathbf{f}, \mathbf{f}\rangle \ge 0 \text{ for all } \mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}).$$

That is

$$\mathrm{trace}\langle S_\Psi^{-1}\mathbf{f}, \mathbf{f}\rangle \le {L_\Psi}^{-1}\, \mathrm{trace}\langle \mathbf{f}, \mathbf{f}\rangle \text{ for all } \mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}). \tag{3.20}$$

Similarly

$${U_\Psi}^{-1} \mathrm{trace}\langle \mathbf{f}, \mathbf{f}\rangle \le \mathrm{trace}\langle S_\Psi^{-1}\mathbf{f}, \mathbf{f}\rangle \text{ for all } \mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r}). \tag{3.21}$$

By (3.20) and (3.21), for any $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$, we have

$$U_\Psi{}^{-1}\|\mathbf{f}\|^2 \leq \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\|\left\langle \mathbf{f}, S_\Psi^{-1} D_{A_j} T_{Bk} E_{C_m}\Psi \right\rangle\right\|^2 \leq L_\Psi{}^{-1}\|\mathbf{f}\|^2.$$

Hence, $\{S_\Psi^{-1} D_{A_j} T_{Bk} E_{C_m}\Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ is frame for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ with frame bounds $U_\Psi{}^{-1}$ and $L_\Psi{}^{-1}$. Furthermore, if L_{Opt_Ψ} is the optimal lower frame bound for $\{D_{A_j} T_{Bk} E_{C_m}\Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ and the optimal upper frame bound for $\{S_\Psi^{-1} D_{A_j} T_{Bk} E_{C_m}\Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ is $U_\Psi^o < L_{Opt_\Psi}^{-1}$. Since the family $\{S_\Psi^{-1} D_{A_j} T_{Bk} E_{C_m}\Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ is a frame with frame operator S_Ψ^{-1}, we get that

$$\{D_{A_j} T_{Bk} E_{C_m}\Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} = \Big\{(S_\Psi^{-1})^{-1} S_\Psi^{-1} D_{A_j} T_{Bk} E_{C_m}\Psi\Big\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$$

has the lower frame bound $(U_\Psi^o)^{-1} > L_{Opt_\Psi}$, which is a contradiction. Therefore, the family $\{S_\Psi^{-1} D_{A_j} T_{Bk} E_{C_m}\Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ has the optimal upper bound $L_{Opt_\Psi}^{-1}$. Similarly, $U_{Opt_\Psi}^{-1}$ is the optimal lower bound for the family $\{S_\Psi^{-1} D_{A_j} T_{Bk} E_{C_m}\Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$.

Next we observe that as in the case of ordinary frames, matrix-valued wave packet optimal frames bounds can be expressed in terms of norm of the frame operator. Indeed, by the definition of optimal upper frame bound and Lemma 2.2, we have

$$\begin{aligned} U_{Opt_\Psi} &= \sup_{\|\mathbf{f}\|=1, \mathbf{f}\in L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})} \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left\|\left\langle D_{A_j} T_{Bk} E_{C_m}\Psi, \mathbf{f}\right\rangle\right\|^2 \\ &= \sup_{\|\mathbf{f}\|=1, \mathbf{f}\in L^2(\mathbb{R}^d,\mathbb{C}^{s\times r})} |\operatorname{trace}\langle S_\Psi \mathbf{f}, \mathbf{f}\rangle| \\ &= \|S_\Psi\|. \end{aligned}$$

This gives the upper optimal frame bound. Now, by part (i), $L_{Opt_\Psi}^{-1}$ is the optimal upper bound for the frame $\{S_\Psi^{-1} D_{A_j} T_{Bk} E_{C_m}\Psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ which has the frame operator S_Ψ^{-1}. Applying the same argument as above, we obtain $L_{Opt_\Psi}^{-1} = \|S_\Psi^{-1}\|$, that is, $L_{Opt_\Psi} = \|S_\Psi^{-1}\|^{-1}$.

The next theorem gives the range of optimal frame bounds associated with the finite sum $\Big\{\sum_{\xi=1}^p \alpha_\xi \mathcal{W}(\Psi_\xi)\Big\}$.

Theorem 3.9. *For each $\xi \in [p]$, let $\mathcal{W}(\Psi_\xi) = \{D_{A_j} T_{Bk} E_{C_m}\Psi_\xi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ be a WHMV wave packet frame for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ with the frame operator S_ξ. Let $\alpha_1, \alpha_2, ..., \alpha_p$ be non-zero scalars. If the finite sum $\Big\{\sum\limits_{\xi=1}^p \alpha_\xi \mathcal{W}(\Psi_\xi)\Big\}$ is a* *frame for $L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ with frame operator S_Ψ, then*

$$\begin{aligned}(3.22)\qquad &\Big(\min_{1\leq\xi\leq p}\|S_\xi{}^{-1}\|^{-1}\Big)\Big(\min_{1\leq\xi\leq p}|\alpha_\xi|\Big)^2\\ &\quad - p(p-1)\Big(\max_{1\leq\xi\leq p}\|S_\xi\|\Big)\Big(\max_{1\leq\xi\leq p}|\alpha_\xi|\Big)^2 \leq \|S_\Psi\|\end{aligned}$$

and

$$(3.23)\qquad p^2\Big(\max_{1\leq\xi\leq p}\|S_\xi\|\Big)\Big(\max_{1\leq\xi\leq p}|\alpha_\xi|\Big)^2 \geq \|S_\Psi^{-1}\|^{-1}.$$

PROOF. We compute

$$\begin{aligned}
&\|S_\Psi^{-1}\|^{-1}\|\mathbf{f}\|^2\\
&\leq \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}\Big\|\Big\langle \mathbf{f}, \sum_{\xi=1}^{p}\alpha_\xi\mathcal{W}(\Psi_\xi)\Big\rangle\Big\|^2\\
&= \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}\Big\|\Big\langle \mathbf{f}, \sum_{\xi=1}^{p}\alpha_\xi D_{A_j}T_{Bk}E_{C_m}\Psi_\xi\Big\rangle\Big\|^2\\
&\leq \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}\Big[\sum_{\xi=1}^{p}|\alpha_\xi|^2\|\langle \mathbf{f}, D_{A_j}T_{Bk}E_{C_m}\Psi_\xi\rangle\|^2\\
&\qquad + \sum_{\xi,t=1,\xi\neq t}^{p}|\alpha_\xi\alpha_t|\|\langle \mathbf{f}, D_{A_j}T_{Bk}E_{C_m}\Psi_\xi\rangle\|\|\langle \mathbf{f}, D_{A_j}T_{Bk}E_{C_m}\Psi_t\rangle\|\Big]\\
&= \sum_{\xi=1}^{p}|\alpha_\xi|^2\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}\|\langle \mathbf{f}, D_{A_j}T_{Bk}E_{C_m}\Psi_\xi\rangle\|^2\\
&\qquad + \sum_{\xi,t=1,\xi\neq t}^{p}|\alpha_\xi\alpha_t|\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}\|\langle \mathbf{f}, D_{A_j}T_{Bk}E_{C_m}\Psi_\xi\rangle\|\|\langle \mathbf{f}, D_{A_j}T_{Bk}E_{C_m}\Psi_t\rangle\|\\
&\leq \Big(\sum_{\xi=1}^{p}|\alpha_\xi|^2\|S_\xi\| + \sum_{\xi,t=1,\xi\neq t}^{p}|\alpha_\xi\alpha_t|\sqrt{\|S_\xi\|\|S_t\|}\Big)\|\mathbf{f}\|^2\\
&\leq \Big[\Big(\max_{1\leq\xi\leq p}\|S_\xi\|\Big)\sum_{\xi,t=1}^{p}|\alpha_\xi\alpha_t|\Big]\|\mathbf{f}\|^2\\
&\leq \Big[p^2\Big(\max_{1\leq\xi\leq p}\|S_\xi\|\Big)\Big(\max_{1\leq\xi\leq p}|\alpha_\xi|\Big)^2\Big]\|\mathbf{f}\|^2
\end{aligned}$$

for all $\mathbf{f}\in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$. This gives (3.23).

To prove inequality (3.22), for any $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^{s\times r})$ we compute

$$\begin{aligned}
&\|S_\Psi\| \|\mathbf{f}\|^2 \\
&\geq \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \Big\| \Big\langle \mathbf{f}\, , \sum_{\xi=1}^{p} \alpha_\xi D_{A_j} T_{Bk} E_{C_m} \Psi_\xi \Big\rangle \Big\|^2 \\
&\geq \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \Big[\sum_{\xi=1}^{p} |\alpha_\xi|^2 \, \|\langle \mathbf{f}, D_{A_j} T_{Bk} E_{C_m} \Psi_\xi\rangle\|^2 \\
&\quad - \sum_{\xi,t=1,\xi\neq t}^{p} |\alpha_\xi \ \alpha_t| \, \|\langle \mathbf{f}, D_{A_j} T_{Bk} E_{C_m} \Psi_\xi\rangle\| \, \|\langle \mathbf{f}, D_{A_j} T_{Bk} E_{C_m} \Psi_t\rangle\| \ \Big] \\
&= \sum_{\xi=1}^{p} |\alpha_\xi|^2 \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \|\langle \mathbf{f}, D_{A_j} T_{Bk} E_{C_m} \Psi_\xi\rangle\|^2 \\
&\quad - \sum_{\xi,t=1,\xi\neq t}^{p} |\alpha_\xi \ \alpha_t| \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \|\langle \mathbf{f}, D_{A_j} T_{Bk} E_{C_m} \Psi_\xi\rangle\| \, \|\langle \mathbf{f}, D_{A_j} T_{Bk} E_{C_m} \Psi_t\rangle\| \\
&\geq \Big[\Big(\min_{1\leq\xi\leq p} \|S_\xi{}^{-1}\|^{-1} \Big) \sum_{\xi=1}^{p} |\alpha_\xi|^2 - \Big(\max_{1\leq\xi\leq p} \|S_\xi\| \Big) \sum_{\xi,t=1,\xi\neq t}^{p} |\alpha_\xi \alpha_t| \Big] \|\mathbf{f}\|^2 \\
&\geq \Big[\Big(\min_{1\leq\xi\leq p} \|S_\xi{}^{-1}\|^{-1} \Big) \Big(\min_{1\leq\xi\leq p} |\alpha_\xi| \Big)^2 \\
&\quad - p(p-1) \Big(\max_{1\leq\xi\leq p} \|S_\xi\| \Big) \Big(\max_{1\leq\xi\leq p} |\alpha_\xi| \Big)^2 \Big] \|\mathbf{f}\|^2.
\end{aligned}$$

This gives (3.22). The theorem is proved. ∎

4. Application to the Frame Algorithm

Frame algorithms are used in approximations of signals (vectors) in the underlying space. One of the variants of classical algorithm, known as frame algorithm, which depends on the width of frames (i.e., on the knowledge of frame bounds) is given in the following theorem.

Theorem 4.1 (Frame Algorithm, [4]). *Let $\{f_k\}_{k=1}^m$ be a frame for a finite dimensional Hilbert space $\mathcal{V}$ with frame bounds L, U and frame operator S. For a given $f \in \mathcal{V}$, define $\{g_k\}_{k=0}^\infty \subset \mathcal{V}$ by*

$$g_0 = 0, \quad g_k = g_{k-1} + \frac{2}{U+L} S(f - g_{k-1}) \qquad (k \in \mathbb{N}).$$

Then

$$\|f - g_k\| \leq \Big(\frac{U-L}{U+L} \Big)^k \|f\| \qquad (k \in \mathbb{N}). \tag{4.1}$$

Theorem 3.2 is also true for ordinary frames in a separable Hilbert space $\mathcal{H}$. The following example shows that the width of the frame can be reduced by using frame bounds and scalars associated with the finite sum of frames. That is, the rate of convergence (in approximation of signals) can be increased.

EXAMPLE 4.2. Let $\mathcal{V} = \mathbb{C}^2$ and let $\{f_1 = (1,1), f_2 = (0,\sqrt{2}), f_3 = (\sqrt{2},0)\} \subset \mathbb{C}^2$. Then, $\{f_k\}_{k=1}^3$ is a frame for $\mathbb{C}^2$ with frame bounds $L_2 = 1$, $U_2 = 4$. The width of the frame $\{f_k\}_{k=1}^3$ is given by

$$\Delta_\star \equiv \frac{U_2 - L_2}{U_2 + L_2} = \frac{3}{5} = 0.6. \tag{4.2}$$

Consider the Parseval frame $\{h_k\}_{k=1}^3 = \Big\{h_1 = (1,0), h_2 = (0,1), h_3 = (0,0)\Big\}$ for $\mathbb{C}^2$. That is, frame bounds of $\{h_k\}_{k=1}^3$ are $L_1 = U_1 = 1$. The finite linear sum $\sum_{\xi=1}^3 \alpha_\xi f_{k,\xi}$ of $\{f_k\}_{k=1}^3$ and $\{h_k\}_{k=1}^3$ is given by

$$\sum_{\xi=1}^{3} \alpha_\xi f_{k,\xi} = \alpha_1 h_k + \alpha_2 f_k \quad \Big(\text{where } f_{k,1} = h_k \text{ and } \ f_{k,2} = f_k,\ k \in \{1,2,3\}\Big).$$

By Theorem 3.2, the finite sum $\sum_{\xi=1}^3 \alpha_\xi f_{k,\xi}$ is a frame for $\mathbb{C}^2$ with frame bounds

$$\mathrm{L}_o = (|\alpha_1|\sqrt{L_1} - |\alpha_2|\sqrt{U_2})^2 = (|\alpha_1| - 2|\alpha_2|)^2$$

and

$$\mathrm{U}_o = (|\alpha_1|\sqrt{U_1} + |\alpha_2|\sqrt{U_2})^2 = (|\alpha_1| + 2|\alpha_2|)^2,$$

whenever $|\alpha_1|^2 = |\alpha_1|^2 L_1 > |\alpha_2|^2 U_2 = 4|\alpha_2|^2$.

Choose $\alpha_1 = 10$ and $\alpha_2 = \frac{1}{10}$. Then, the width of the frame $\sum\limits_{\xi=1}^{3} \alpha_\xi f_{k,\xi}$ is given by

$$\Delta_{\star\star} \equiv \frac{\mathrm{U}_o - \mathrm{L}_o}{\mathrm{U}_o + \mathrm{L}_o} = \frac{8|\alpha_1|\ |\alpha_2|}{2|\alpha_1|^2 + 8|\alpha_2|^2} = 0.04. \tag{4.3}$$

By (4.2) and (4.3), one may observe that there is considerable decrease in the width of the frame associated with frame bounds of finite sum of frames. More precisely, the rate of convergence in the frame algorithm can be increased by using suitable scalars and frame bounds of finite sum of frames. A comparison between the width of a given frame $\{f_k\}_{k=1}^3$ and width of finite sum of frames $\sum_{\xi=1}^3 \alpha_\xi f_{k,\xi}$ for five levels is illustrated in Table 1.

Table 1

Level	Convergence rate with respect to $(\Delta_\star)^k$	Convergence rate with respect to $(\Delta_{\star\star})^k$
k=1	6×10^{-1}	4×10^{-2}
k=2	36×10^{-2}	16×10^{-4}
k=3	216×10^{-3}	64×10^{-6}
k=4	1296×10^{-4}	256×10^{-8}
k=5	7776×10^{-5}	1024×10^{-10}

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Received: 30.10.2017.