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Measure pseudo affine-periodic solutions of semilinear differential equations*

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Abstract. In this paper, we introduce the concept of a pseudo affine-periodic function via measure theory, that is, a measure pseudo (Q, T)-affine-periodic function. The existence and uniqueness of a measure pseudo (Q, T)-affine-periodic solution for semilinear differential equations are investigated. The working tools are based on the Banach contraction mapping principle and the Leray-Schauder alternative theorem. Finally, an example is presented to illustrate the main findings.

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Key words: Measure pseudo affine-periodic function, measure theory, exponential dichotomy, Banach contraction mapping principle, Leray-Schauder alternative theorem

1. Introduction

The problem of the existence and uniqueness of a periodic solution of differential equations has been the main subject of investigation. Many authors have made important contributions to this theory by using various methods and techniques, such as the fixed-point theorem, the Kaplan-Yorke method and coincidence degree theory [7, 12, 16, 20, 24]. However, real systems usually exhibit internal variations or external perturbations which are only approximately periodic. Recently, the concept of an affine-periodic solution was proposed [22, 25]; this solution is a kind of periodic or quasi-periodic solution with symmetry. For more details on the applications to differential equations or difference equations, one can see [5, 6, 15, 17, 21, 23].

On the other hand, Blot, Cieutat and Ezzinbi [3, 4] use the results of measure theory to establish μ-ergodicity and introduce the new concepts of a μ-pseudo almost periodic functionand a μ-pseudo almost automorphic function. Subsequently, μ-ergodicity is generalized into (μ, ν)-ergodicity by Diagana et al. [10]. In this paper, we give the concept of a measure pseudo affine-periodic function by (μ, ν)-ergodicity, and investigate the existence and uniqueness of a measure pseudo affine-periodic solution for semilinear differential equations.

The paper is organized as follows. In Section 2, the measure pseudo (Q, T)-affine-periodic function is introduced and composition theorems are given. Section 3

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is devoted to the applications to differential equations under Lipschitz perturbation and non-Lipschitz perturbation, respectively. In Section 4, an example is presented to illustrate the main findings.

2. Measure pseudo affine-periodicity

Consider the following system:

$$x' = f(t,x), \tag{1}$$

where $f(t,x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the affine symmetry

$$f(t+T,x) = Qf(t,Q^{-1}x), \tag{2}$$

where $Q \in GL(\mathbb{R}^n)$, $T > 0$ is a constant.

Definition 1 (see [6])**.** *If $f(t,x)$ satisfies affine symmetry* (2)*, then* (1) *is said to be a (Q,T)-affine-periodic system.*

Definition 2 (see [6])**.** *The solution $x(t)$ of* (1) *is said to be a (Q,T)-affine-periodic solution if $x(t)$ satisfies $x(t+T) = Qx(t)$ for all $t \in \mathbb{R}$.*

Remark 1. *Note that if $Q = I$ (identity matrix), $Q = -I$, $Q^N = I$, $Q \in SO(n)$; then (Q,T)-affine-periodic solution $x(t)$ defined in Definition 2 is just T-periodic, anti-periodic, harmonic and quasi-periodic respectively. One can see [6] for more details.*

For $T > 0$, define

$$\begin{aligned} C_T(\mathbb{R},\mathbb{R}^n) &= \{x \in C(\mathbb{R},\mathbb{R}^n) : x(t+T) = Qx(t) \text{ for all } t \in \mathbb{R}\}, \\ C_0(\mathbb{R},\mathbb{R}^n) &= \{x \in C(\mathbb{R},\mathbb{R}^n) : \lim_{|t|\to+\infty} |x(t)| = 0\}, \\ PAP_0(\mathbb{R},\mathbb{R}^n) &= \left\{x \in C(\mathbb{R},\mathbb{R}^n) : \lim_{r\to+\infty} \frac{1}{2r}\int_{-r}^{r} |x(t)|dt = 0\right\}. \end{aligned}$$

Definition 3. *A function $f \in C(\mathbb{R},\mathbb{R}^n)$ is called asymptotically (Q,T)-affine-periodic if there exists $g \in C_T(\mathbb{R},\mathbb{R}^n), \varphi \in C_0(\mathbb{R},\mathbb{R}^n)$ such that $f = g + \varphi$. The collection of those functions is denoted by $AAP_T(\mathbb{R},\mathbb{R}^n)$.*

Definition 4 (see [6])**.** *A function $f \in C(\mathbb{R},\mathbb{R}^n)$ is said to be pseudo (Q,T)-affine-periodic if there exists $g \in C_T(\mathbb{R},\mathbb{R}^n), \varphi \in PAP_0(\mathbb{R},\mathbb{R}^n)$ such that $f = g + \varphi$. The collection of those functions is denoted by $PAP_T(\mathbb{R},\mathbb{R}^n)$.*

Let U be the set of all functions $\rho : \mathbb{R} \to (0,+\infty)$ which are positive and locally integrable over $\mathbb{R}$. For $r > 0$ and each $\rho \in U$, set $m(r,\rho) := \int_{-r}^{r} \rho(t)\mathrm{d}t$ and $U_\infty := \{\rho \in U : \lim_{r\to+\infty} m(r,\rho) = +\infty\}$. For $\rho_1, \rho_2 \in U_\infty$, define the weighted ergodic space [9]

$$WPAA_0(\mathbb{R},\mathbb{R}^n,\rho_1,\rho_2) := \left\{f \in C(\mathbb{R},\mathbb{R}^n) : \lim_{r\to+\infty} \frac{1}{m(r,\rho_2)}\int_{-r}^{r} \rho_1(t)|f(t)|dt = 0\right\}.$$

Definition 5. *Let $\rho_1, \rho_2 \in U_\infty$. A function $f \in C(\mathbb{R}, \mathbb{R}^n)$ is called weighted pseudo (Q,T)-affine-periodic if it can be decomposed as $f = g + \varphi$, where $g \in C_T(\mathbb{R}, \mathbb{R}^n)$ and $\varphi \in WPAA_0(\mathbb{R}, \mathbb{R}^n, \rho_1, \rho_2)$. Denote by $WPAP_T(\mathbb{R}, \mathbb{R}^n, \rho_1, \rho_2)$ the set of such functions.*

Next, we introduce the concept of a measure pseudo (Q,T)-affine-periodic function by the results of measure theory. $\mathcal{B}$ denotes the Lebesgue σ-field of $\mathbb{R}$, $\mathcal{M}$ stands for the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a,b]) < +\infty$ for all $a, b \in \mathbb{R}$ $(a \leq b)$.

Definition 6 (see [10]). *Let $\mu, \nu \in \mathcal{M}$, the measures μ and ν are equivalent at infinity, written $\mu \sim \nu$, if there exist constants $c_0, c_1 > 0$ and a bounded interval $I \subset \mathbb{R}$ (eventually $\emptyset$) such that*

$$c_0 \nu(A) \leq \mu(A) \leq c_1 \nu(A)$$

for all $A \in \mathcal{B}$ satisfying $A \cap I = \emptyset$.

Definition 7 (see [10]). *Let $\mu, \nu \in \mathcal{M}$. A function $f \in C(\mathbb{R}, \mathbb{R}^n)$ is said to be (μ,ν)-ergodic if*

$$\lim_{r \to +\infty} \frac{1}{\nu([-r,r])} \int_{[-r,r]} |f(t)| d\mu(t) = 0.$$

Denote by $\mathcal{E}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ the set of such functions.

Definition 8. *Let $\mu, \nu \in \mathcal{M}$. A function $f \in C(\mathbb{R}, \mathbb{R}^n)$ is said to be measure pseudo (Q,T)-affine-periodic if it can be decomposed as $f = g + \varphi$, where $g \in C_T(\mathbb{R}, \mathbb{R}^n)$ and $\varphi \in \mathcal{E}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$. Denote by $MPAP_T(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ the collection of such functions.*

Remark 2. (*i*) *If μ, ν are the Lebesgue measures, then the measure pseudo (Q,T)-affine-periodic function $MPAP_T(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ is a pseudo (Q,T)-affine-periodic function $PAP_T(\mathbb{R}, \mathbb{R}^n)$.*

(*ii*) *Let $\rho_1(t) > 0$, $\rho_2(t) > 0$ a.e. on $\mathbb{R}$ for the Lebesgue measure, μ, ν denote the positive measure defined by*

$$\mu(A) = \int_A \rho_1(t) dt, \qquad \nu(A) = \int_A \rho_2(t) dt \quad \text{for } \ A \in \mathcal{B},$$

where dt denotes the Lebesgue measure on $\mathbb{R}$; then the measure pseudo (Q,T)-affine-periodic function $MPAP_T(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ is a weighted pseudo (Q,T)-affine-periodic function $WPAP_T(\mathbb{R}, \mathbb{R}^n, \rho_1, \rho_2)$.

(*iii*) *It is not difficult to see that*

$$\begin{aligned} &C_T(\mathbb{R}, \mathbb{R}^n) \subset AAP_T(\mathbb{R}, \mathbb{R}^n) \subset PAP_T(\mathbb{R}, \mathbb{R}^n) \subset WPAP_T(\mathbb{R}, \mathbb{R}^n, \rho_1, \rho_2) \\ &\quad \subset MPAP_T(\mathbb{R}, \mathbb{R}^n, \mu, \nu) \subset BC(\mathbb{R}, \mathbb{R}^n), \end{aligned}$$

where $BC(\mathbb{R}, \mathbb{R}^n)$ denotes the Banach space of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$.

Definition 9. *Let $\mu, \nu \in \mathcal{M}$. A function $f \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is said to be measure pseudo (Q,T)-affine-periodic if it can be decomposed as $f(t,x) = g(t,x) + \varphi(t,x)$, where $g(t+T,x) = Qg(t, Q^{-1}x)$ and φ satisfies*

$$\lim_{r\to+\infty} \frac{1}{\nu([-r,r])}\int_{[-r,r]} |\varphi(t,x)| d\mu(t) = 0 \ \text{ uniformly for } x \text{ in any bounded subset of } \mathbb{R}^n.$$

Denote by $MPAP_T(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n, \mu, \nu)$ the collection of such functions.

For $\mu \in \mathcal{M}$, $\tau \in \mathbb{R}$, we denote by μ_τ the positive measure on $(\mathbb{R}, \mathcal{B})$ defined by

$$\mu_\tau(A) = \mu(\{a + \tau : a \in A\}) \quad \text{ for } A \in \mathcal{B}. \tag{3}$$

In this paper, we formulate the following hypotheses:

(M_1) Let $\mu, \nu \in \mathcal{M}$ such that

$$\limsup_{r\to+\infty} \frac{\mu([-r,r])}{\nu([-r,r])} < +\infty.$$

(M_2) Let $\mu, \nu \in \mathcal{M}$ such that for all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that $\mu_\tau(A) \le \beta\mu(A)$, $\nu_\tau(A) \le \beta\nu(A)$ if $A \in \mathcal{B}$ satisfies $A \cap I = \emptyset$.

Similarly to the proof in [6], one has

Lemma 1. *Let $\mu, \nu \in \mathcal{M}$ satisfy (M_2); then $\mathcal{E}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ and $MPAP_T(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ are translation invariants.*

Lemma 2. *Let $\mu, \nu \in \mathcal{M}$ satisfy (M_1) and (M_2); then $MPAP_T(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ is a Banach space when endowed with the supremum norm $\|x\| = \sup_{t\in\mathbb{R}} |x(t)|$.*

Similarly to the proof of [4], we give the composition theorem of a measure pseudo (Q,T)-affine-periodic function.

Definition 10. *We denote by $\mathcal{UC}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ the set of all continuous functions $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ which are uniformly continuous in the second variable, i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|f(t,x) - f(t,y)| \le \varepsilon, \quad \text{for all } t \in \mathbb{R} \text{ and } x, y \in \mathbb{R}^n \text{ with } |x - y| < \delta.$$

Theorem 1. *Let $\mu, \nu \in \mathcal{M}$, $f \in MPAP_T(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n, \mu, \nu) \cap \mathcal{UC}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, and the following condition holds:*

$(\mathcal{J})$ For all bounded subset D of $\mathbb{R}^n$, f is bounded on $\mathbb{R} \times D$.

Then $f(\cdot, x(\cdot)) \in MPAP_T(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ if $x \in MPAP_T(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$.

In what follows following, we establish another composition theorem of the measure pseudo (Q,T)-affine-periodic function which weakens the assumptions on f.

Let $p \in [1, \infty)$. The space $BS^p(\mathbb{R}, \mathbb{R}^n)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to \mathbb{R}^n$ such that

$f^b \in L^\infty(\mathbb{R}, L^p([0,1];\mathbb{R}^n))$, where f^b is the Bochner transform of f defined by $f^b(t,s) := f(t+s), t \in \mathbb{R}, s \in [0,1]$. $BS^p(\mathbb{R},\mathbb{R}^n)$ is a Banach space with the norm [19]

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} |f(\tau)|^p \mathrm{d}\tau \right)^{1/p}.$$

It is obvious that $L^p(\mathbb{R},\mathbb{R}^n) \subset BS^p(\mathbb{R},\mathbb{R}^n) \subset L^p_{loc}(\mathbb{R},\mathbb{R}^n)$ and $BS^p(\mathbb{R},\mathbb{R}^n) \subset BS^q(\mathbb{R},\mathbb{R}^n)$ for $p \ge q \ge 1$.

Definition 11. *For each $p \in [1,\infty)$, we denote by $\mathcal{UC}^p(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$ the set of all continuous functions $f : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ with the property that for every $\varepsilon > 0$, there exist a function $L_f \in BS^p(\mathbb{R},\mathbb{R}^+)$ and $\delta > 0$ such that*

$$|f(t,x) - f(t,y)| \le L_f(t)\varepsilon, \quad \text{for all } t \in \mathbb{R} \text{ and } x, y \in \mathbb{R}^n \text{ with } |x-y| < \delta.$$

Theorem 2. *Assume that (M_1) holds, $f = g + \varphi \in MPAP_T(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n,\mu,\nu) \cap \mathcal{UC}^p(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$, where $g \in C_T(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n)$, $\varphi \in \mathcal{E}(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n,\mu,\nu)$ and*

$(\mathcal{I}_1)$ *There exists a positive number M such that*

$$\limsup_{r\to+\infty} \frac{1}{\nu([-r,r])} \int_{[-r,r]} L_f(t) d\mu(t) \le M.$$

$(\mathcal{I}_2)$ *$g(t,x)$ is uniformly continuous in any bounded subset of $\mathbb{R}^n$ uniformly for $t \in \mathbb{R}$.*

Then $f(\cdot,x(\cdot)) \in MPAP_T(\mathbb{R},\mathbb{R}^n,\mu,\nu)$ if $x \in MPAP_T(\mathbb{R},\mathbb{R}^n,\mu,\nu)$.

The proof is similar to [11, Theorem 3.3], here we omit it.

3. $MPAP_T$ solutions of differential equations

Let $X(t)$ be a fundamental matrix solution of homogeneous linear differential equations:

$$x' = A(t)x, \quad t \in \mathbb{R}, \tag{4}$$

with initial value $X(0) = I$, where $A(t) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is continuous and ensures the uniqueness of solutions of (4) with respective to the initial value.

Definition 12 (see [8])**.** *It is said that there exists an exponential dichotomy of (4) if there exist a projection P and constants $K, L, \alpha, \beta > 0$ such that*

$$|X(t)PX^{-1}(s)| \le Ke^{-\alpha(t-s)}, \quad t \ge s,$$
$$|X(t)(I-P)X^{-1}(s)| \le Le^{-\beta(s-t)}, \quad s \ge t,$$

where $|\cdot|$ is the Euclidean norm.

Consider nonhomogeneous linear differential equations:

$$x' = A(t)x + f(t), \quad t \in \mathbb{R}, \tag{5}$$

where $f : \mathbb{R} \to \mathbb{R}^n$ is a bounded and continuous function. We have the following results on the existence of bounded solutions of (5).

Lemma 3 (see [8]). *If* (4) *has an exponential dichotomy with projection* P*, then* (5) *has the following bounded solution:*

$$x(t)=\int_{-\infty}^{t}X(t)PX^{-1}(s)f(s)ds-\int_{t}^{+\infty}X(t)(I-P)X^{-1}(s)f(s)ds.$$

Consider semilinear differential equations:

$$x'=A(t)x+f(t,x(t)),\qquad t\in\mathbb{R}, \tag{6}$$

where $f:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous, $A(t):\mathbb{R}\to\mathbb{R}^{n\times n}$ is continuous and $f(t,x)$ is a measure pseudo (Q,T)-affine-periodic function.

To study (6), we require the following assumptions:

(H_1) (4) has an exponential dichotomy with projection P and constants $K,L,\alpha,\beta>0$.

(H_2) (4) is a (Q,T)-affine-periodic system, that is, $A(t+T)=QA(t)Q^{-1}$.

(H_3) $f(t,x)=g(t,x)+\varphi(t,x)\in MPAP_T(\mathbb{R}\times\mathbb{R}^n,\mathbb{R}^n,\mu,\nu)$.

3.1. Lipschitz case

In this subsection, if f satisfies the Lipschitz condition, we investigate the existence and uniqueness of $MPAP_T(\mathbb{R},\mathbb{R}^n,\mu,\nu)$ solution of (6), i.e., the following (H_4) holds:

(H_4) There exists a constant $L_f>0$ such that

$$|f(t,x)-f(t,y)|\le L_f|x-y|,\qquad x,\ y\in\mathbb{R}^n,\quad t\in\mathbb{R}.$$

Lemma 4 (see [6]). *If* (H_2) *holds and* $f\in C_T(\mathbb{R},\mathbb{R}^n)$*, then*

$$\int_{-\infty}^{t}X(t)PX^{-1}(s)f(s)ds-\int_{t}^{+\infty}X(t)(I-P)X^{-1}(s)f(s)ds\in C_T(\mathbb{R},\mathbb{R}^n).$$

Theorem 3. *Assume that* (M_1), (M_2), (H_1)-(H_4) *hold; then* (6) *has a unique solution* $x\in MPAP_T(\mathbb{R},\mathbb{R}^n,\mu,\nu)$ *if* $\dfrac{KL_f}{\alpha}+\dfrac{LL_f}{\beta}<1$.

Proof. For $y\in MPAP_T(\mathbb{R},\mathbb{R}^n,\mu,\nu)$, we consider the following equation:

$$x'=A(t)x+f(t,y(t)),\qquad t\in\mathbb{R}. \tag{7}$$

By Lemma 3, it has the following bounded solution:

$$x(t)=\int_{-\infty}^{t}X(t)PX^{-1}(s)f(s,y(s))ds-\int_{t}^{+\infty}X(t)(I-P)X^{-1}(s)f(s,y(s))ds.$$

Define the map $\mathcal{F}:MPAP_T(\mathbb{R},\mathbb{R}^n,\mu,\nu)\to MPAP_T(\mathbb{R},\mathbb{R}^n,\mu,\nu)$ by

$$(\mathcal{F}y)(t)=\int_{-\infty}^{t}X(t)PX^{-1}(s)f(s,y(s))ds-\int_{t}^{+\infty}X(t)(I-P)X^{-1}(s)f(s,y(s))ds. \tag{8}$$

It is not difficult to see that $\mathcal{F}$ is well defined. In fact, for $y \in MPAP_T(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$, $f(\cdot, y(\cdot)) \in MPAP_T(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ by Theorem 1; then let $f(\cdot, y(\cdot)) = f_1(\cdot) + f_2(\cdot)$, where $f_1 \in C_T(\mathbb{R}, \mathbb{R}^n)$, $f_2 \in \mathcal{E}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$. Then

$$(\mathcal{F}y)(t) := \Lambda_1(t) + \Lambda_2(t),$$

where

$$\Lambda_1(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)f_1(s)ds - \int_t^{+\infty} X(t)(I-P)X^{-1}(s)f_1(s)ds,$$
$$\Lambda_2(t) = \int_{-\infty}^{t} X(t)PX^{-1}(s)f_2(s)ds - \int_t^{+\infty} X(t)(I-P)X^{-1}(s)f_2(s)ds.$$

By Lemma 4, $\Lambda_1 \in C_T(\mathbb{R}, \mathbb{R}^n)$. Next, we will show that $\Lambda_2 \in \mathcal{E}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$. For $r > 0$, one has

$$\begin{aligned}
&\frac{1}{\nu([-r,r])}\int_{[-r,r]} |\Lambda_2(t)| d\mu(t)\\
&\quad\le \frac{1}{\nu([-r,r])}\int_{[-r,r]} \left|\int_{-\infty}^{t} X(t)PX^{-1}(s)f_2(s)ds\right| d\mu(t)\\
&\qquad + \frac{1}{\nu([-r,r])}\int_{[-r,r]} \left|\int_{t}^{+\infty} X(t)(I-P)X^{-1}(s)f_2(s)ds\right| d\mu(t)\\
&\quad\le \frac{K}{\nu([-r,r])}\int_{[-r,r]} \int_{-\infty}^{t} e^{-\alpha(t-s)}\left|f_2(s)\right| ds d\mu(t)\\
&\qquad + \frac{L}{\nu([-r,r])}\int_{[-r,r]} \int_{t}^{+\infty} e^{\beta(t-s)}\left|f_2(s)\right| ds d\mu(t)\\
&\quad= \frac{K}{\nu([-r,r])}\int_{[-r,r]} \int_{0}^{+\infty} e^{-\alpha s}\left|f_2(t-s)\right| ds d\mu(t)\\
&\qquad + \frac{L}{\nu([-r,r])}\int_{[-r,r]} \int_{-\infty}^{0} e^{\beta s}\left|f_2(t-s)\right| ds d\mu(t)\\
&\quad= K\int_0^{+\infty} e^{-\alpha s}\left(\frac{1}{\nu([-r,r])}\int_{[-r,r]} |f_2(t-s)| d\mu(t)\right) ds\\
&\qquad + L\int_{-\infty}^{0} e^{\beta s}\left(\frac{1}{\nu([-r,r])}\int_{[-r,r]} |f_2(t-s)| d\mu(t)\right) ds\\
&\quad= K\int_0^{+\infty} e^{-\alpha s}\Phi_r(s)ds + L\int_{-\infty}^{0} e^{\beta s}\Phi_r(s)ds,
\end{aligned}$$

where

$$\Phi_r(s) = \frac{1}{\nu([-r,r])}\int_{[-r,r]} |f_2(t-s)| d\mu(t).$$

Since (M_2) holds, from Lemma 1 it follows that $f_2(\cdot - s) \in \mathcal{E}(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ for $s \in \mathbb{R}$. Hence $\Phi_r(s) \to 0$ as $r \to +\infty$. Note that Φ_r is bounded by (M_1) and $e^{-\alpha s}$, $e^{\beta s}$ are integrable on $[0,\infty)$, $[-\infty,0)$, respectively. From Lebesgue dominated convergence theorem, it follows that

$$\lim_{r\to+\infty}\left(K\int_0^{+\infty}e^{-\alpha s}\Phi_r(s)ds+L\int_{-\infty}^{0}e^{\beta s}\Phi_r(s)ds\right)=0.$$

Then

$$\lim_{r\to+\infty}\frac{1}{\nu([-r,r])}\int_{[-r,r]}|\Lambda_2(t)|d\mu(t)=0,$$

so $\Lambda_2\in\mathcal{E}(\mathbb{R},\mathbb{R}^n,\mu,\nu)$. Hence $\mathcal{F}$ is well defined.

Let $x,y\in MPAP_T(\mathbb{R},\mathbb{R}^n,\mu,\nu)$. One has

$$\begin{aligned}|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)|&\le\left|\int_{-\infty}^{t}X(t)PX^{-1}(s)[f(s,x(s))-f(s,y(s))]ds\right|\\&\quad+\left|\int_{t}^{+\infty}X(t)PX^{-1}(s)[f(s,x(s))-f(s,y(s))]ds\right|\\&\le\int_{-\infty}^{t}Ke^{-\alpha(t-s)}\left|f(s,x(s))-f(s,y(s))\right|ds\\&\quad+\int_{t}^{+\infty}Le^{\beta(t-s)}\left|f(s,x(s))-f(s,y(s))\right|ds\\&\le\int_{-\infty}^{t}KL_fe^{-\alpha(t-s)}\left|x(s)-y(s)\right|ds\\&\quad+\int_{t}^{+\infty}LL_fe^{\beta(t-s)}\left|x(s)-y(s)\right|ds\\&\le\left(\frac{KL_f}{\alpha}+\frac{LL_f}{\beta}\right)\|x-y\|.\end{aligned}$$

By the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point $y\in MPAP_T(\mathbb{R},\mathbb{R}^n,\mu,\nu)$, which is the unique $MPAP_T$ solution to (6). □

Next, we establish a version of Theorem 3 which enables us to consider locally Lipschitz perturbations for (6), that is, the following condition (H_5) is satisfied:

(H_5) There exists a continuous and nondecreasing function $L_f:[0,+\infty)\to[0,+\infty)$ such that for each number $\lambda>0$, and $x,y\in\mathbb{R}^n$, $|x|\le\lambda$, $|y|\le\lambda$, we have

$$|f(t,x)-f(t,y)|\le L_f(\lambda)|x-y|,\qquad t\in\mathbb{R}.$$

Theorem 4. *Assume that (M_1), (M_2), (H_1)–(H_3), (H_5) are fulfilled if there exists $\lambda>0$ such that*

$$\Theta:=\frac{KL_f(\lambda)}{\alpha}+\frac{LL_f(\lambda)}{\beta}+\frac{1}{\lambda}\left(\frac{K}{\alpha}+\frac{L}{\beta}\right)\sup_{t\in\mathbb{R}}|f(t,0)|<1;$$

then (6) has a unique solution $x\in MPAP_T(\mathbb{R},\mathbb{R}^n,\mu,\nu)$ with $\|x\|\le\lambda$.

Proof. Let

$$\Omega = \{x \in MPAP_T(\mathbb{R}, \mathbb{R}^n, \mu, \nu) : \|x\| \leq \lambda\}.$$

It is clear that Ω is a convex and closed vector subspace of $BC(\mathbb{R}, \mathbb{R}^n)$. For $y \in MPAP_T(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$, define the map $\mathcal{F}$ as (8). It is not difficult to see that $\mathcal{F}$ maps Ω into Ω. In fact, for $y \in \Omega$, one has

$$\begin{aligned}
|\mathcal{F}y(t)| &\leq \int_{-\infty}^{t} Ke^{-\alpha(t-s)}|f(s, y(s))|ds + \int_{t}^{+\infty} Le^{\beta(t-s)}|f(s, y(s))|ds \\
&\leq \int_{-\infty}^{t} Ke^{-\alpha(t-s)}L_f(\lambda)|y(s)|ds + \int_{-\infty}^{t} Ke^{-\alpha(t-s)}|f(s, 0)|ds \\
&\quad + \int_{t}^{+\infty} Le^{\beta(t-s)}L_f(\lambda)|y(s)|ds + \int_{t}^{+\infty} Le^{\beta(t-s)}|f(s, 0)|ds \\
&\leq \left(\frac{KL_f(\lambda)}{\alpha} + \frac{LL_f(\lambda)}{\beta}\right)\lambda + \left(\frac{K}{\alpha} + \frac{L}{\beta}\right)\sup_{t\in\mathbb{R}}|f(t, 0)| \\
&\leq \lambda.
\end{aligned}$$

On the other hand, for $x, y \in \Omega$, we have

$$\begin{aligned}
|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)| \leq &KL_f(\lambda)\int_{-\infty}^{t} e^{-\alpha(t-s)}|x(s) - y(s)|ds \\
&+ LL_f(\lambda)\int_{t}^{+\infty} e^{\beta(t-s)}|x(s) - y(s)|ds \\
\leq &\left(\frac{KL_f(\lambda)}{\alpha} + \frac{LL_f(\lambda)}{\beta}\right)\|x - y\|,
\end{aligned}$$

which means that

$$\|\mathcal{F}x - \mathcal{F}y\| \leq \left(\frac{KL_f(\lambda)}{\alpha} + \frac{LL_f(\lambda)}{\beta}\right)\|x - y\|.$$

Since $\Theta < 1$, it follows that $\mathcal{F}$ is a contraction in Ω. By the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point $y \in MPAP_T(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$, which is the unique $MPAP_T$ solution to (6). $\square$

Theorem 5. *Assume that* (M_1), (M_2), $(\mathcal{I}_1)$, $(\mathcal{I}_2)$, (H_1)-(H_3) *hold and the following condition is satisfied:*

(H_6) *There exists a function* $L_f \in BS^p(\mathbb{R}, \mathbb{R}^+)$, $p \geq 1$ *such that*

$$|f(t, x) - f(t, y)| \leq L_f(t)|x - y|, \quad x,\ y \in \mathbb{R}^n, \quad t \in \mathbb{R}.$$

Then (6) *has a unique solution* $x \in MPAP_T(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ *if*

$$\left(\frac{K}{1 - e^{-\alpha}} + \frac{L}{1 - e^{-\beta}}\right)\|L_f\|_{S^1} < 1.$$

Proof. Define the map $\mathcal{F}$ as in (8). It is easy to see that $\mathcal{F}$ is well defined by Theorem 2. Let $x, y \in MPAP_T(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$. One has

$$
\begin{aligned}
&|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)| \\
&\qquad \le \int_{-\infty}^{t} Ke^{-\alpha(t-s)} \left| f(s, x(s)) - f(s, y(s)) \right| ds \\
&\qquad\quad + \int_{t}^{+\infty} Le^{\beta(t-s)} \left| f(s, x(s)) - f(s, y(s)) \right| ds \\
&\qquad \le \left(\int_{-\infty}^{t} KL_f(s) e^{-\alpha(t-s)} ds + \int_{t}^{+\infty} LL_f(s) e^{\beta(t-s)} ds \right) \cdot \|x - y\| \\
&\qquad = \left(\int_{0}^{+\infty} KL_f(t-s) e^{-\alpha s} ds + \int_{-\infty}^{0} LL_f(t-s) e^{\beta s} ds \right) \cdot \|x - y\| \\
&\qquad \le \left(\sum_{k=0}^{+\infty} e^{-\alpha k} \int_{k}^{k+1} KL_f(t-s) ds + \sum_{k=-\infty}^{0} e^{\beta k} \int_{k-1}^{k} LL_f(t-s) ds \right) \cdot \|x - y\| \\
&\qquad \le \left(\frac{K}{1 - e^{-\alpha}} + \frac{L}{1 - e^{-\beta}} \right) \|L_f\|_{S^1} \cdot \|x - y\|.
\end{aligned}
$$

Thus by the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $MPAP_T(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$, which is the unique $MPAP_T$ solution of (6). $\square$

3.2. Non-Lipschitz case

In this subsection, we study the existence of the $MPAP_T(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ solution of (6), when f does not satisfy the Lipschitz condition. First, we recall a useful compactness criterion and the nonlinear Leray-Schauder alternative theorem.

Let $(X, \|\cdot\|)$ be a Banach space and $h^* : \mathbb{R} \to \mathbb{R}$ a continuous nondecreasing function such that $h^*(t) \ge 1$ for all $t \in \mathbb{R}$, and $h^*(t) \to +\infty$ as $|t| \to +\infty$. Define

$$C_{h^*}(\mathbb{R}, X) := \{u \in C(\mathbb{R}, X) : \lim_{|t| \to +\infty} \frac{u(t)}{h^*(t)} = 0\}$$

endowed with the norm $\|u\|_{h^*} = \sup_{t \in \mathbb{R}} (\|u(t)\| / h^*(t))$.

Lemma 5 (see [14]). *A set $K \subseteq C_{h^*}(\mathbb{R}, X)$ is relatively compact in $C_{h^*}(\mathbb{R}, X)$ if it verifies the following conditions:*

(c_1) *The set $K(t) := \{u(t) : u \in K\}$ is relatively compact in X for each $t \in \mathbb{R}$.*

(c_2) *The set K is equicontinuous.*

(c_3) *For each $\varepsilon > 0$, there exists $\vartheta > 0$ such that $\|u(t)\| \le \varepsilon h^*(t)$ for all $u \in K$ and all $|t| > \vartheta$.*

Theorem 6 (Leray-Schauder alternative theorem, see ([13]). *Let Ω be a closed convex subset of a Banach space X such that $0 \in \Omega$. Let $\mathcal{F} : \Omega \to \Omega$ be a completely continuous map. Then the set $\{x \in \Omega : \ x = \lambda \mathcal{F}(x), \ 0 < \lambda < 1\}$ is unbounded or the map $\mathcal{F}$ has a fixed point in Ω.*

Now, we are in a position to establish the following result of the existence of $MPAP_T(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ solutions. The result is based upon the nonlinear Leray-Schauder alternative theorem. For more details, see [1].

Theorem 7. *Assume (M_1), (M_2), (H_1)-(H_3), $(\mathcal{J})$ hold, and $f \in \mathcal{UC}(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ satisfies the following conditions:*

$(\mathcal{K}_1)$ *There exists a continuous nondecreasing function $W : [0, +\infty) \to [0, +\infty)$ such that $\|f(t, x(t))\| \leq W(\|x\|)$ for all $t \in \mathbb{R}, x \in \mathbb{R}^n$.*

$(\mathcal{K}_2)$ *For each $\varpi \geq 0$,*

$$\lim_{|t| \to +\infty} \frac{1}{h^*(t)} \left(\int_{-\infty}^{t} e^{-\alpha(t-s)} W(\varpi h^*(s)) ds + \int_{t}^{+\infty} e^{-\beta(t-s)} W(\varpi h^*(s)) ds \right) = 0.$$

$(\mathcal{K}_3)$ *For each $\varepsilon > 0$, there exists $\delta > 0$ such that for $x, y \in C_{h^*}(\mathbb{R}, \mathbb{R}^n)$, $\|x-y\|_{h^*} \leq \delta$ implies that for all $t \in \mathbb{R}$,*

$$\int_{-\infty}^{t} e^{-\alpha(t-s)} \|f(s, x(s)) - f(s, y(s))\| ds + \int_{t}^{+\infty} e^{-\beta(t-s)} \|f(s, x(s)) - f(s, y(s))\| ds \leq \varepsilon.$$

$(\mathcal{K}_4)$ *For all $a, b \in \mathbb{R}, a \leq b$ and $\lambda \geq 0$, the set $\{f(s, x(s)) : a \leq s \leq b, x \in C_{h^*}(\mathbb{R}, \mathbb{R}^n), \|x\|_{h^*} \leq \lambda\}$ is relatively compact in $\mathbb{R}^n$.*

$(\mathcal{K}_5)$ $\liminf_{\lambda \to +\infty} \frac{\lambda}{\Phi(\lambda)} > 1$, *where for $t \in \mathbb{R}$, $\lambda \geq 0$, $\Phi(\lambda)$ defined by*

$$\Phi(\lambda) := \left\| \int_{-\infty}^{t} K e^{-\alpha(t-s)} W(\lambda h^*(s)) ds + \int_{t}^{+\infty} L e^{-\beta(t-s)} W(\lambda h^*(s)) ds \right\|_{h^*}.$$

Then (6) has a solution $x \in MPAP_T(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$.

Proof. Define $\Gamma : C_{h^*}(\mathbb{R}, \mathbb{R}^n) \to C(\mathbb{R}, \mathbb{R}^n)$ by

$$(\Gamma x)(t) = \int_{-\infty}^{t} X(t) P X^{-1}(s) f(s, x(s)) ds - \int_{t}^{+\infty} X(t)(I-P) X^{-1}(s) f(s, x(s)) ds, \ t \in \mathbb{R}.$$

Next, we prove that Γ has a fixed point in $MPAP_T(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ and divide the proof into several steps.

(i) For $x \in C_{h^*}(\mathbb{R}, \mathbb{R}^n)$, by $(\mathcal{K}_1)$, one has

$$\begin{aligned}
&\frac{\|\Gamma x(t)\|}{h^*(t)} \\
&\leq \int_{-\infty}^{t} \|X(t) P X^{-1}(s)\| \|f(s, x(s))\| ds + \int_{t}^{+\infty} \|X(t)(I-P) X^{-1}(s)\| \|f(s, x(s))\| ds \\
&\leq \int_{-\infty}^{t} K e^{-\alpha(t-s)} \|f(s, x(s))\| ds + \int_{t}^{+\infty} L e^{-\beta(t-s)} \|f(s, x(s))\| ds \\
&\leq \int_{-\infty}^{t} K e^{-\alpha(t-s)} W(\|x(s)\|) ds + \int_{t}^{+\infty} L e^{-\beta(t-s)} W(\|x(s)\|) ds \\
&\leq \int_{-\infty}^{t} K e^{-\alpha(t-s)} W(\|x\|_{h^*} h^*(s)) ds + \int_{t}^{+\infty} L e^{-\beta(t-s)} W(\|u\|_{h^*} h^*(s)) ds.
\end{aligned}$$

It follows from $(\mathcal{K}_2)$ that $\Gamma : C_{h^*}(\mathbb{R},\mathbb{R}^n) \to C_{h^*}(\mathbb{R},\mathbb{R}^n)$.

(ii) Γ is continuous. In fact, for each $\varepsilon > 0$, by $(\mathcal{K}_3)$, there exits $\delta > 0$, for $x, y \in C_{h^*}(\mathbb{R},\mathbb{R}^n)$ and $\|x-y\|_{h^*} \le \delta$, one has

$$\begin{aligned}\|\Gamma x - \Gamma y\| &\le \int_{-\infty}^{t} \|X(t)PX^{-1}(s)\|\|f(s,x(s)) - f(s,y(s))\|ds \\ &\quad + \int_{t}^{+\infty} \|X(t)(I-P)X^{-1}(s)\|\|f(s,x(s)) - f(s,y(s))\|ds \\ &\le \int_{-\infty}^{t} Ke^{-\alpha(t-s)}\|f(s,x(s)) - f(s,y(s))\|ds \\ &\quad + \int_{t}^{+\infty} Le^{-\beta(t-s)}\|f(s,x(s)) - f(s,y(s))\|ds \\ &\le \max(K,L)\varepsilon, \quad \text{for all } \ t \in \mathbb{R}.\end{aligned}$$

Taking into account that $h^*(t) \ge 1$, we have

$$\frac{\|\Gamma x - \Gamma y\|}{h^*(t)} \le \max(K,L)\varepsilon,$$

which implies that $\|\Gamma x - \Gamma y\|_{h^*} \le \max(K,L)\varepsilon$, so Γ is continuous.

(iii) Γ is completely continuous. Set $B_\lambda(Z)$ for the closed ball with the center at 0 and radius λ in the space Z. Let $V = \Gamma(B_\lambda(C_{h^*}(\mathbb{R},\mathbb{R}^n)))$ and $y = \Gamma(x)$ for $x \in B_\lambda(C_{h^*}(\mathbb{R},\mathbb{R}^n))$.

Initially, we prove that V is a relatively compact subset of $\mathbb{R}^n$ for each $t \in \mathbb{R}$. Let $\varepsilon > 0$. Since $h^*(t) \to \infty$ as $|t| \to +\infty$, it follow $(\mathcal{K}_2)$ that there exists $a \ge 0$ such that

$$\int_{a}^{+\infty} e^{-\alpha s} W(\lambda h^*(t-s))ds + \int_{-\infty}^{-a} e^{-\beta s} W(\lambda h^*(t-s))ds \le \varepsilon.$$

Since

$$\begin{aligned}y(t) &= \int_{-\infty}^{t} X(t)PX^{-1}(s)f(s,x(s))ds - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(s)f(s,x(s))ds \\ &= \int_{0}^{a} X(t)PX^{-1}(t-s)f(t-s,x(t-s))ds \\ &\quad + \int_{a}^{+\infty} X(t)PX^{-1}(t-s)f(t-s,x(t-s))ds \\ &\quad - \int_{-a}^{0} X(t)(I-P)X^{-1}(t-s)f(t-s,x(t-s))ds \\ &\quad - \int_{-\infty}^{-a} X(t)(I-P)X^{-1}(t-s)f(t-s,x(t-s))ds,\end{aligned}$$

and

$$\begin{aligned}
&\Big\|\int_a^{+\infty} X(t)PX^{-1}(t-s)f(t-s,x(t-s))ds \\
&\quad -\int_{-\infty}^{-a} X(t)(I-P)X^{-1}(t-s)f(t-s,x(t-s))ds\Big\| \\
&\le \int_a^{+\infty} Ke^{-\alpha s}W(\varpi h^*(t-s))ds + \int_{-\infty}^{-a} Le^{-\beta s}W(\varpi h^*(t-s))ds \\
&\le \max(K,L)\varepsilon,
\end{aligned}$$

hence $y(t) \in a\overline{c_0(N_1)} + a\overline{c_0(N_2)} + B_\varepsilon(\mathbb{R}^n)$, where $c_0(N_1)$, $c_0(N_2)$ denote the convex hull of N_1, N_2, respectively, and

$$\begin{aligned}
&N_1 = \{X(t)PX^{-1}(t-s)f(\xi,x(\xi)) : 0 \le s \le a, t-a \le \xi \le t, \|x\|_{h^*} \le \lambda\}, \\
&N_2 = \{X(t)(I-P)X^{-1}(t-s)f(\xi,x(\xi)) : -a \le s \le 0, t \le \xi \le t+a, \|x\|_{h^*} \le \lambda\}.
\end{aligned}$$

Using the fact that fundamental matrix solution $X(t)$ is continuous and $(\mathcal{K}_4)$, we infer that N_1, N_2 are relatively compact sets, and $V(t) \subset a\overline{c_0(N_1)}+a\overline{c_0(N_2)}+B_\varepsilon(\mathbb{R}^n)$ is also a relatively compact set.

Next, we show that V is equicontinuous. In fact, for each $\varepsilon > 0$, we can choose $a > 0$, $\delta_1 > 0$ such that

$$\begin{aligned}
&\Big\|\int_a^{+\infty} [X(t+\tau)PX^{-1}(t-\sigma) - X(t)PX^{-1}(t-\sigma)]f(t-\sigma,x(t-\sigma))d\sigma \\
&\quad + \int_0^{\tau} X(t+\tau)PX^{-1}(t+\tau-\sigma)f(t+\tau-\sigma,x(t+\tau-\sigma))d\sigma \\
&\quad - \Big(\int_{-\infty}^{-a}[X(t+\tau)(I-P)X^{-1}(t-\sigma) - X(t)(I-P)X^{-1}(t-\sigma)]f(t-\sigma,x(t-\sigma))d\sigma \\
&\quad - \int_0^{\tau} X(t+\tau)(I-P)X^{-1}(t+\tau-\sigma)f(t+\tau-\sigma,x(t+\tau-\sigma))d\sigma\Big)\Big\| \\
&\le \int_a^{+\infty}[Ke^{-\alpha(\tau+\sigma)} + Ke^{-\alpha\sigma}]W(\lambda h^*(t-\sigma))d\sigma + \int_0^{\tau} Ke^{-\alpha\sigma}W(\lambda h^*(t+\tau-\sigma))d\sigma \\
&\quad + \int_{-\infty}^{-a}[Le^{-\beta(\tau+\sigma)} + Le^{-\beta\sigma}]W(\lambda h^*(t-\sigma))d\sigma + \int_0^{\tau} Le^{-\beta\sigma}W(\lambda h^*(t+\tau-\sigma))d\sigma \\
&\le \frac{\varepsilon}{3}, \qquad \text{for } \ \tau \le \delta_1.
\end{aligned}$$

Moreover, since $\{f(t-\sigma,x(t-\sigma)) : 0 < \sigma < a, x \in B_\lambda(C_{h^*}(\mathbb{R},\mathbb{R}^n))\}$ is a relatively compact set and $X(t)$ is continuous, we can choose $\delta_2 > 0$ such that

$$\big\|[X(t+\tau)PX^{-1}(t-\sigma) - X(t)PX^{-1}(t-\sigma)]f(t-\sigma,x(t-\sigma))\big\| \le \frac{\varepsilon}{3a}, \text{for } \tau \le \delta_2,$$

$$\big\|[X(t+\tau)(I-P)X^{-1}(t-\sigma) - X(t)(I-P)X^{-1}(t-\sigma)]f(t-\sigma,x(t-\sigma))\big\| \le \frac{\varepsilon}{3a}, \quad \text{for } \tau \le \delta_3.$$

Note that

$$
\begin{aligned}
&y(t+\tau)-y(t)\\
&=\int_{-\infty}^{t+\tau} X(t+\tau)PX^{-1}(s)f(s,x(s))ds-\int_{-\infty}^{t} X(t)PX^{-1}(s)f(s,x(s))ds\\
&\quad-\bigg(\int_{t+\tau}^{+\infty} X(t+\tau)(I-P)X^{-1}(s)f(s,x(s))ds\\
&\quad-\int_{t}^{+\infty} X(t)(I-P)X^{-1}(s)f(s,x(s))ds\bigg)\\
&=\int_{-\infty}^{t}[X(t+\tau)PX^{-1}(s)-X(t)PX^{-1}(s)]f(s,x(s))ds\\
&\quad+\int_{t}^{t+\tau} X(t+\tau)PX^{-1}(s)f(s,x(s))ds\\
&\quad-\bigg(\int_{t}^{+\infty}[X(t+\tau)(I-P)X^{-1}(s)-X(t)(I-P)X^{-1}(s)]f(s,x(s))ds\\
&\quad-\int_{t}^{t+\tau} X(t+\tau)(I-P)X^{-1}(s)f(s,x(s))ds\bigg)\\
&=\int_{0}^{a}[X(t+\tau)PX^{-1}(t-\sigma)-X(t)PX^{-1}(t-\sigma)]f(t-\sigma,x(t-\sigma))d\sigma\\
&\quad+\int_{a}^{+\infty}[X(t+\tau)PX^{-1}(t-\sigma)-X(t)PX^{-1}(t-\sigma)]f(t-\sigma,x(t-\sigma))d\sigma\\
&\quad+\int_{0}^{\tau} X(t+\tau)PX^{-1}(t+\tau-\sigma)f(t+\tau-\sigma,x(t+\tau-\sigma))d\sigma\\
&\quad-\bigg(\int_{-a}^{0}[X(t+\tau)(I-P)X^{-1}(t-\sigma)-X(t)(I-P)X^{-1}(t-\sigma)]f(t-\sigma,x(t-\sigma))d\sigma\\
&\quad+\int_{-\infty}^{-a}[X(t+\tau)(I-P)X^{-1}(t-\sigma)-X(t)(I-P)X^{-1}(t-\sigma)]f(t-\sigma,x(t-\sigma))d\sigma\\
&\quad-\int_{0}^{\tau} X(t+\tau)(I-P)X^{-1}(t+\tau-\sigma)f(t+\tau-\sigma,x(t+\tau-\sigma))d\sigma\bigg).
\end{aligned}
$$

Then we have $\|y(t+\tau)-y(t)\|\le\varepsilon$ for τ small enough and independent of $x\in B_\lambda(C_{h^*}(\mathbb{R},\mathbb{R}^n))$.

Finally, by $(\mathcal{K}_2)$, for $|t|\to+\infty$, one has

$$
\frac{\|y(t)\|}{h^*(t)}\le\int_{-\infty}^{t}Ke^{-\alpha(t-s)}W(\|x\|_{h^*}h^*(s))ds+\int_{t}^{+\infty}Le^{-\beta(t-s)}W(\|x\|_{h^*}h^*(s))ds\to 0,
$$

and this convergence is independent of $x\in B_\lambda(C_{h^*}(\mathbb{R},\mathbb{R}^n))$. Hence, by Lemma 5 V is a relatively compact set in $C_{h^*}(\mathbb{R},\mathbb{R}^n)$.

(iv) If x^λ is a solution of the equation $x^\lambda = \lambda\Gamma(x^\lambda)$ for some $0 < \lambda < 1$, then

$$\begin{aligned}\|x^\lambda\| &= \lambda\left\|\int_{-\infty}^{t} X(t)PX^{-1}(s)f(s,x^\lambda(s))ds - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(s)f(s,x^\lambda(s))ds\right\| \\ &\le \int_{-\infty}^{t} Ke^{-\alpha(t-s)}W(\|x^\lambda\|_{h^*}h^*(s))ds + \int_{t}^{+\infty} Le^{-\beta(t-s)}W(\|x^\lambda\|_{h^*}h^*(s))ds \\ &\le \Phi(\|x^\lambda\|_{h^*})h^*(t).\end{aligned}$$

Hence, one has

$$\frac{\|x^\lambda\|_{h^*}}{\Phi(\|x^\lambda\|_{h^*})} \le 1$$

and by $(\mathcal{K}_5)$, we conclude that the set $\{x^\lambda : x^\lambda = \lambda\Gamma(x^\lambda), \lambda \in (0,1)\}$ is bounded.

(v) We claim that there exists $\lambda_0 > 0$ such that $\mathcal{F}(B_{\lambda_0}(C_{h^*}(\mathbb{R},\mathbb{R}^n))) \subset B_{\lambda_0}(C_{h^*}(\mathbb{R},\mathbb{R}^n))$. If the assertion is false, then for all $\lambda > 0$, we can choose $x^\lambda \in B_\lambda(C_{h^*}(\mathbb{R},\mathbb{R}^n))$ such that $\|\mathcal{F}x^\lambda\|_{h^*} > \lambda$. Similar to the proof of (iv), we deduce that

$$\frac{\lambda}{\Phi(\lambda)} \le 1.$$

Then

$$\liminf_{\xi\to+\infty} \frac{\lambda}{\Phi(\lambda)} \le 1,$$

which contradicts condition $(\mathcal{K}_5)$ establishing the desired assertion.

(vi) It is not difficult to see that $\Gamma(MPAP_T(\mathbb{R},\mathbb{R}^n,\mu,\nu)) \subseteq MPAP_T(\mathbb{R},\mathbb{R}^n,\mu,\nu)$ by Theorem 1 and Theorem 3 . Consequently, with Step (v), we infer that

$$\mathcal{F}(B_{\lambda_0}(C_{h^*}(\mathbb{R},\mathbb{R}^n)) \cap MPAP_T(\mathbb{R},\mathbb{R}^n,\mu,\nu)) \subseteq B_{\lambda_0}(C_{h^*}(\mathbb{R},\mathbb{R}^n)) \cap MPAP_T(\mathbb{R},\mathbb{R}^n,\mu,\nu);$$

hence we derive the following conclusion:

$$\begin{aligned}&\mathcal{F}\left(\overline{B_{\lambda_0}(C_{h^*}(\mathbb{R},\mathbb{R}^n)) \cap MPAP_T(\mathbb{R},\mathbb{R}^n,\mu,\nu)}^{C_{h^*}(\mathbb{R},\mathbb{R}^n)}\right) \\ &\quad\subseteq \overline{\mathcal{F}(B_{\lambda_0}(C_{h^*}(\mathbb{R},\mathbb{R}^n)) \cap MPAP_T(\mathbb{R},\mathbb{R}^n,\mu,\nu))}^{C_{h^*}(\mathbb{R},\mathbb{R}^n)} \\ &\quad\subseteq \overline{B_{\lambda_0}(C_{h^*}(\mathbb{R},\mathbb{R}^n)) \cap MPAP_T(\mathbb{R},\mathbb{R}^n,\mu,\nu)}^{C_{h^*}(\mathbb{R},\mathbb{R}^n)}.\end{aligned}$$

Thus, we consider

$$\mathcal{F} : \overline{B}^{C_{h^*}(\mathbb{R},\mathbb{R}^n)} \to \overline{B}^{C_{h^*}(\mathbb{R},\mathbb{R}^n)},$$

where $B = B_{\lambda_0}(C_{h^*}(\mathbb{R},\mathbb{R}^n)) \cap MPAP_T(\mathbb{R},\mathbb{R}^n,\mu,\nu)$, and $\overline{B}^{C_{h^*}(\mathbb{R},\mathbb{R}^n)}$ denotes the closure of a set B in the space $C_{h^*}(\mathbb{R},\mathbb{R}^n)$. Using (i)-(iii), we have that $\mathcal{F}$ is completely continuous. By (iv) and Theorem 6, we deduce that Γ has a fixed point $x \in \overline{B_{\lambda_0}(C_{h^*}(\mathbb{R},\mathbb{R}^n)) \cap MPAP_T(\mathbb{R},\mathbb{R}^n,\mu,\nu)}^{C_{h^*}(\mathbb{R},\mathbb{R}^n)}$.

(vii) Finally, we show that x (the fixed point of $\mathcal{F}$ given in (vi)) is measure pseudo (Q,T)-affine-periodic. Indeed, let x_n be a sequence in $B_{\lambda_0}(C_{h^*}(\mathbb{R},\mathbb{R}^n)) \cap MPAP_T(\mathbb{R},\mathbb{R}^n,\mu,\nu)$ such that it converges to x in the norm $C_{h^*}(\mathbb{R},\mathbb{R}^n)$. For $\varepsilon > 0$, let $\delta > 0$ be the constant in $(\mathcal{K}_3)$. There exists $n_0 \in \mathbb{N}$ such that $\|x_n - x\|_{h^*} \leq \delta$ for all $n \geq n_0$. For $n \geq n_0$,

$$\begin{aligned}\|\Gamma x_n - \Gamma x\| &\leq \int_{-\infty}^{t} K e^{-\alpha(t-s)} \|f(s, x_n(s)) - f(s, x(s))\| ds \\ &\quad + \int_{t}^{+\infty} L e^{-\beta(t-s)} \|f(s, x_n(s)) - f(s, x(s))\| ds \\ &\leq \max(K, L)\varepsilon,\end{aligned}$$

which implies that Γx_n converges to $\Gamma x = x$ uniformly in $\mathbb{R}$. This implies that $x \in MPAP_T(\mathbb{R}, \mathbb{R}^n, \mu, \nu)$ and completes the proof. $\square$

4. Example

Consider the following perturbed second differential equation for small $\varepsilon_1, \varepsilon_2$:

$$u'' - (1+a)u' + (1-b)u - \varepsilon_1 e^{-t} - \frac{\varepsilon_2}{\sqrt{1+t^2}} \sin u = 0, \qquad t \in \mathbb{R}, \tag{9}$$

where $a, b \in \mathbb{R}$. Using the transformations $u = x_1$ and $x_1' = x_2$, we can transfer (9):

$$x' = Ax + \Theta(t)x + g(t, x), \qquad t \in \mathbb{R}, \tag{10}$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \qquad \Theta(t)x = \begin{pmatrix} 0 & 0 \\ b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon_1 e^{-t} \end{pmatrix}.$$

Since the real part of eigenvalues of A is $\frac{1}{2}$, then $x' = Ax$ admits an exponential dichotomy with $K = L = 1$, $\alpha = \beta = \frac{1}{2}$, that is, (H_1), (H_2) hold. On the other hand, let $f_1(t, x) = \Theta(t)x$. Then for each $T > 0$, we have $f_1(t+T, x) = Qf_1(t, Q^{-1}x)$ with

$$Q = \begin{pmatrix} e^{-T} & 0 \\ 0 & e^{-T} \end{pmatrix},$$

so $\Theta(t)x \in C_T(\mathbb{R} \times \mathbb{R}^2, \mathbb{R}^2)$.

(i) Let

$$g(t, x) = \begin{pmatrix} 0 \\ \dfrac{\varepsilon_2}{\sqrt{1+t^2}} \sin x_1 \end{pmatrix}.$$

Consider the measure μ, where its Radon-Nikodym derivative is

$$\rho_1(t) = e^{\sin t}, \qquad t \in \mathbb{R},$$

and the measure ν, where its Radon-Nikodym derivative is

$$\rho_2(t) = \begin{cases} e^t & \text{if } t \leq 0 \\ 1 & \text{if } t > 0. \end{cases}$$

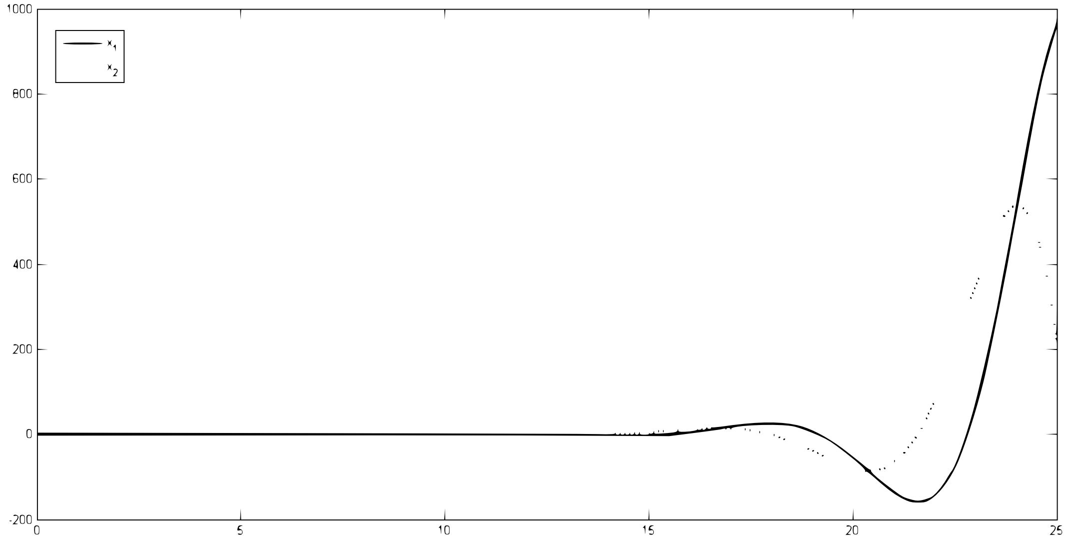

Figure 1: *The unique $MPAP_T$ solution of* (9)

Then from [2], $\mu, \nu \in \mathcal{M}$ satisfy (M_1), (M_2). It is not difficult to see that $g(t,x) \in \mathcal{E}(\mathbb{R}\times\mathbb{R}^2, \mathbb{R}^2, \mu, \nu)$, that is, $f(t,x) = \Theta(t)x + g(t,x) \in MPAP_T(\mathbb{R}\times\mathbb{R}^2, \mathbb{R}^2, \mu, \nu)$. By Theorem 3, (10) has a unique solution $x \in MPAP_T(\mathbb{R}, \mathbb{R}^2, \mu, \nu)$ if $\max(a,b) + \varepsilon_2 < 1/4$. Fig. 1 illustrates the unique $MPAP_T$ solution of (9) and Fig. 2 depicts the phase portrait of (9), where $a = 0.008, b = 0.001, \varepsilon_1 = 0.005, \varepsilon_2 = 0.2$.

(ii) Let

$$g(t,x) = \begin{pmatrix} 0 \\ a(t)e^{-t}\sin x_1 \end{pmatrix},$$

where

$$a(t) = \begin{cases} \sin\left(\dfrac{1}{\cos n + \cos \pi n + 2}\right), & t \in (n-\varepsilon, n+\varepsilon), n \in \mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases}$$

for some small $\varepsilon \in (0, \eta)$, $\eta = \min\left\{\dfrac{1}{2}, \dfrac{1}{4}(1 - e^{-\frac{1}{2}} - 2\max(a,b))\right\}$. By [18, Example 2.3], $a(t) \in BS^2(\mathbb{R}, \mathbb{R})$. Then (H_6) holds with $L_f(t) = \max(a,b) + |a(t)|$. Consider the measures $\mu = \nu$ and their Radon-Nikodym derivative given by $\rho(t) = e^t$. Then $g(t,x) \in \mathcal{E}(\mathbb{R}\times\mathbb{R}^2, \mathbb{R}^2, \mu, \nu)$, whence $f \in MPAP_T(\mathbb{R}\times\mathbb{R}^2, \mathbb{R}^2, \mu, \nu)$.

In addition, since

$$|||a(\cdot)|||_{S^1} = \sup_{t\in\mathbb{R}} \int_t^{t+1} |a(s)|ds \leq 2\varepsilon,$$

$$\left(\frac{K}{1-e^{-\alpha}} + \frac{L}{1-e^{-\beta}}\right)\|L_f\|_{S^1} \leq \frac{2}{1-e^{\frac{1}{2}}} \cdot (\max(a,b) + 2\varepsilon) < 1.$$

By Theorem 5, (10) has a unique solution $x \in MPAP_T(\mathbb{R}, \mathbb{R}^2, \mu, \nu)$ if $\max(a,b) < (1 - e^{-\frac{1}{2}})/2$.

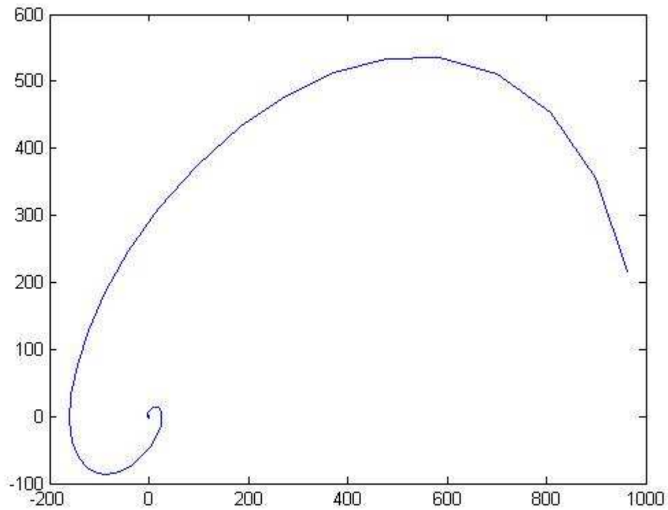

Figure 2: *The phase portrait of* (9)

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