MATHEMATICAL COMMUNICATIONS
Math. Commun. **23**(2018), 145–159

On eqiform Darboux helices in Galilean 3-space

UFUK ÖZTÜRK[1], ESRA BETUL KOÇ ÖZTÜRK[2] AND EMILIJA NEŠOVIĆ[3,*]

[1] *Department of Mathematics, University of Çankırı Karatekin, TR-18 100 Çankırı, Turkey*
[2] *A. H. R. Mah. Çayyolu Cad. Ada Sitesi, B blok No:18/12, TR-18 100 Çankırı, Turkey*
[3] *Department of Mathematics and Informatics, Faculty of Science, University of Kragujevac, Radoja Domanovića 12, SR-34 000 Kragujevac, Serbia*

Received October 26, 2016; accepted June 20, 2017

Abstract. In this paper, we define equiform Darboux helices in a Galilean space $\mathbb{G}_3$ and obtain their explicit parameter equations. We show that equiform Darboux helices have only a non-isotropic axis and characterize equiform Darboux vectors of equiform Darboux helices in terms of equiform rectifying curves. We prove that an equiform Darboux vector of an equiform Darboux helix α is an equiform Darboux helix if an admissible curve α is a rectifying curve. We also prove that there are no equiform curves of constant precession and give some examples of equiform Darboux helices.
AMS subject classifications: 53A20, 53A35

Key words: Galilean 3-space, equiform geometry, Darboux vector

---

1. Introduction

According to [11], Galilean geometry is a real Cayley-Klein geometry with projective signature $(0,0,+,+)$. Galilean space $\mathbb{G}_3$ is a 3-dimensional projective space in which the absolute consists of a real plane ω and a real line $f \in \omega$, together with an elliptic involution I. In non-homogeneous affine coordinates, the similarity group H_8 of the Galilean space $\mathbb{G}_3$ has the following form

$$\begin{cases} \bar{x} = a_{11} + a_{12}x, \\ \bar{y} = a_{21} + a_{22}x + a_{23}\cos\varphi y + a_{23}\sin\varphi z, \\ \bar{z} = a_{31} + a_{32}x - a_{23}\sin\varphi y + a_{23}z\cos\varphi z, \end{cases} \tag{1}$$

where a_{ij} and φ are real numbers. For $a_{12} = a_{23} = 1$, relation (1) defines the group $B_6 \subset H_8$ of isometries of $\mathbb{G}_3$. In particular, if $a_{12} = a_{23} = \alpha \neq 1$, all line segments in the Euclidean plane ($x = const.$) will be mapped into proportional ones with the same coefficient of proportionality equal to α ([13]). The condition $a_{12} = a_{23} = \alpha \neq 1$ defines a subgroup $H_7 \subset H_8$, which preserves the angles between planes and lines. The group H_7 is called equiform group transformations of $\mathbb{G}_3$, which are the compositions of a homothety and an isometry. The geometry of equiform group H_7 is called equiform geometry of the Galilean space $\mathbb{G}_3$.

---

*Corresponding author. *Email addresses:* `ozturkufuk06@gmail.com` (U. Öztürk), `e.betul.e@gmail.com` (E. B. Koç Öztürk), `nesovickg@sbb.rs` (E. Nešović)

Although equiform geometry of curves in Galilean space $\mathbb{G}_3$ has minor importance related to the usual one, equiform helices can be seen as generalizations of ordinary helices and therefore could be of research interest. Equiform geometry of curves in the simple isotropic space $I_3^{(1)}$ and in the double isotropic space $I_3^{(2)}$ is studied in [12]. Some properties of helices in equiform geometry of pseudo-Galilean and Galilean 3-space are obtained in [7, 6, 8]. Equiform geometry of curves in Galilean 4-space is investigated in [2].

In Euclidean 3-space, rectifying curves are defined in [3] as space curves whose position vector always lies in its rectifying plane spanned by the tangent and the binormal vector field of the curve. It is proved in [4] that the Darboux vector of a curve in $\mathbb{E}^3$ with constant curvature κ and a non-constant torsion τ is a rectifying curve. It is shown in [3] that the ratio τ/κ of the torsion τ and the curvature κ of a rectifying curve is a non-constant linear function in na arc-length parameter of the curve. The same property also holds for rectifying curves in Galilean 3-space and pseudo-Galilean 3-space ([9, 10]).

Darboux helices in $\mathbb{E}^3$ are defined in [15] as space curves whose Darboux vector makes a constant angle with some fixed direction (the axis of the helix). However, Darboux helices in equiform geometry of the Galilean space $\mathbb{G}_3$ have not been studied yet.

In this paper, we define equiform Darboux helices in Galilean 3-space as admissible curves whose equiform Darboux vector makes a constant angle with some fixed non-isotropic or isotropic direction. We obtain explicit parameter equations of equiform Darboux helices with a non-isotropic axis and prove that there are no equiform Darboux helices with an isotropic axis. We characterize equiform Darboux vectors of equiform Darboux helices in terms of equiform rectifying curves. We also prove that an equiform Darboux vector of an equiform Darboux helix α is an equiform Darboux helix if an admissible curve α is a rectifying curve. Finally, we prove that there are no equiform curves of constant precession in Galilean 3-space and give some examples of equiform Darboux helices.

2. Preliminaries

In Galilean 3-space, the Galilean scalar product of two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is defined by

$$\langle \mathbf{u},\ \mathbf{v}\rangle = \begin{cases} \quad u_1 v_1 & , \text{ if } u_1 \neq 0 \text{ or } v_1 \neq 0, \\ u_2 v_2 + u_3 v_3 & , \text{ if } u_1 = v_1 = 0. \end{cases} \tag{2}$$

If $\langle \mathbf{u}, \mathbf{v}\rangle = 0$, the vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be orthogonal. The norm of a vector $\mathbf{v}$ is given by $||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$. In particular, if $||\mathbf{v}|| = 1$, the vector $\mathbf{v}$ is called a unit vector.

The Galilean cross product of two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is given by

$$\mathbf{u} \wedge_{\mathbf{G}} \mathbf{v} = \begin{vmatrix} 0 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}. \tag{3}$$

An angle between two unit non-isotropic vectors, a unit non-isotropic and an isotropic vector, or two isotropic vectors, is defined in [1] as follows.

Definition 1. *Let $a=(1,y_2,y_3)$ and $b=(1,z_2,z_3)$ be two unit non-isotropic vectors in general position in $\mathbb{G}_3$. Then an angle φ between a and b is given by*

$$\varphi=\sqrt{(z_2-y_2)^2+(z_3-y_3)^2}.$$

The angle measure θ between the unit non-isotropic vector $a=(1,y_2,y_3)$ and an isotropic vector $c=(0,z_2,z_3)$ is defined by Artykbaev in [1]. Geometrically, such an angle can be interpreted as the length of the projection of the vector $\tilde{a}=(0,y_2,y_3)$ onto the isotropic vector $c=(0,z_2,z_3)$.

Definition 2. *An angle θ between a unit non-isotropic vector $a=(1,y_2,y_3)$ and an isotropic vector $c=(0,z_2,z_3)$ in $\mathbb{G}_3$ is given by*

$$\theta=\frac{y_2z_2+y_3z_3}{\sqrt{z_2^2+z_3^2}}.$$

Definition 3. *An angle ω between two isotropic vectors $c=(0,y_2,y_3)$ and $d=(0,z_2,z_3)$ parallel to the Euclidean plane in $\mathbb{G}_3$ is equal to the Euclidean angle between them. Namely,*

$$\cos\omega=\frac{y_2z_2+y_3z_3}{\sqrt{y_2^2+y_3^2}\sqrt{z_2^2+z_3^2}}.$$

Definition 4. *The curve $\alpha(t)=(x(t),y(t),z(t))$ in the Galilean space $\mathbb{G}_3$ is said to be admissible if it has no inflection points $(\dot{\alpha}(t)\wedge_G\ddot{\alpha}(t)\neq 0)$ and no isotropic tangents $(\dot{x}(t)\neq 0)$.*

The unit speed admissible curve α in $\mathbb{G}_3$ can be written as

$$\alpha(x)=(x,y(x),z(x)),$$

where $s=x$ is the arc-length parameter of α defined by $ds=dx$.

The curvature $\kappa(x)$ and the torsion $\tau(x)$ of the curve $\alpha(x)$ are defined by

$$\kappa(x)=\sqrt{\ddot{y}(x)^2+\ddot{z}(x)^2}, \tag{4}$$

$$\tau(x)=\frac{\ddot{y}(x)\dddot{z}(x)-\dddot{y}(x)\ddot{z}(x)}{\kappa^2(x)}, \tag{5}$$

respectively.

The radius of the curvature of the curve α is given by

$$\rho(x)=\frac{1}{\kappa(x)},$$

where $\kappa(x)\neq 0$, because α has no inflection point.

The Frenet frame $\{t,n,b\}$ of the curve α has the form

$$t(x)=(1,\ \dot{y}(x),\ \dot{z}(x)), \tag{6}$$

$$n(x)=\frac{1}{\kappa(x)}(0,\ddot{y}(x),\ddot{z}(x)), \tag{7}$$

$$b(x)=\frac{1}{\kappa(x)}(0,\ -\ddot{z}(x),\ \ddot{y}(x)). \tag{8}$$

The vectors $t(x)$, $n(x)$ and $b(x)$ are called the *tangent*, the *principal normal* and the *binormal* vector of α, respectively.

The Frenet equations of the unit speed admissible curve α in $\mathbb{G}_3$ read

$$\begin{bmatrix} \dot{t}(x) \\ \dot{n}(x) \\ \dot{b}(x) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(x) & 0 \\ 0 & 0 & \tau(x) \\ 0 & -\tau(x) & 0 \end{bmatrix} \begin{bmatrix} t(x) \\ n(x) \\ b(x) \end{bmatrix}. \tag{9}$$

Also, Frenet's frame vectors of α satisfy the equations

$$t \wedge_{\mathbf{G}} n = b, \quad n \wedge_{\mathbf{G}} b = 0, \quad b \wedge_{\mathbf{G}} t = n. \tag{10}$$

Definition 5. *For an admissible curve $\alpha : I = [a, b] \to \mathbb{G}_3$, the equiform parameter $\sigma(x)$ is defined by ([13])*

$$\sigma(x) = \int_a^x \frac{1}{\rho(t)}\, dt = \int_a^x \kappa(t)\, dt. \tag{11}$$

If $\alpha(\sigma) = (x(\sigma), y(\sigma), z(\sigma))$ is an admissible curve parameterized by an equiformly invariant parameter $\sigma(x)$, then its tangent vector is defined by

$$T(\sigma) = \alpha'(\sigma) = (x'(\sigma), y'(\sigma), z'(\sigma)) = (\frac{1}{\dot{\sigma}(x)}, y'(\sigma), z'(\sigma)).$$

In particular, the equiform Frenet frame $\{T, N, B\}$ of α is related to the Frenet frame $\{t, n, b\}$ as follows

$$T = \rho t, \tag{12}$$

$$N = \rho n, \tag{13}$$

$$B = \rho b. \tag{14}$$

Consequently, the equiform Frenet equations of the curve $\alpha(\sigma)$ are given by

$$\begin{bmatrix} T'(\sigma) \\ N'(\sigma) \\ B'(\sigma) \end{bmatrix} = \begin{bmatrix} \mathcal{K}(\sigma) & 1 & 0 \\ 0 & \mathcal{K}(\sigma) & \mathcal{T}(\sigma) \\ 0 & -\mathcal{T}(\sigma) & \mathcal{K}(\sigma) \end{bmatrix} \begin{bmatrix} T(\sigma) \\ N(\sigma) \\ B(\sigma) \end{bmatrix}, \tag{15}$$

where $\mathcal{K}$ is called the equiform curvature and $\mathcal{T}$ is called the equiform torsion of the curve $\alpha\,(\sigma)$. These are related to the curvature κ and the torsion τ by the equations

$$\mathcal{K} = \frac{d\rho}{dx}, \quad \mathcal{T} = \frac{\tau}{\kappa}. \tag{16}$$

Throughout the next sections, as already used, let the symbol "dot" denote the derivative with respect to arc-length parameter x and "prime" the derivative with respect to equiform parameter σ. Also, let $\mathbb{R}_0$ denote $\mathbb{R}\backslash\{0\}$.

3. Equiform Darboux helices in $\mathbb{G}_3$

In this section, we first obtain an equiform Darboux vector of an admissible curve in Galilean 3-space and then define equiform Darboux helices. We obtain their explicit parameter equations and prove that any equiform Darboux helix does not have isotropic axis. We also characterize equiform Darboux vectors of equiform Darboux helices in terms of equiform rectifying curves and equiform Darboux helices and give some examples.

When an equiform Frenet frame $\{T, N, B\}$ of an admissible curve α moves along α in $\mathbb{G}_3$, there exists an axis of the frame's rotation. The direction of such axis is given by equiform Darboux vector (equiform centrode) $D(\sigma)$ which satisfies equiform Darboux equations

$$T'(\sigma) = D(\sigma) \wedge_G T(\sigma), \tag{17}$$

$$N'(\sigma) = D(\sigma) \wedge_G N(\sigma), \tag{18}$$

$$B'(\sigma) = D(\sigma) \wedge_G B(\sigma). \tag{19}$$

If $\dot{\sigma}(x) = \kappa(x) = constant \neq 0$, then the following vector

$$D(\sigma) = \dot{\sigma}\mathcal{T}(\sigma)T(\sigma) + \dot{\sigma}B(\sigma) \tag{20}$$

defines the equiform Darboux vector, as follows by using relations (10), (11), (12), (13), (14), (17), (18) and (19). We find that if an admissible curve α has an equiform Darboux vector, then its first equiform curvature $\mathcal{K}$ satisfies the relation $\mathcal{K} = \dot{\rho}(x) = 0$.

Definition 6. *An admissible curve $\alpha(\sigma)$ parameterized by an equiform parameter with the equiform curvature $\mathcal{K}(\sigma) = 0$ in $\mathbb{G}_3$ is called an equiform Darboux helix if its unit equiform Darboux vector $D(\sigma)$ makes the constant angle with some unit non-isotropic fixed direction $U \in \mathbb{G}_3$ or with some isotropic fixed direction $W \in \mathbb{G}_3$.*

Let $\alpha(\sigma)$ be an admissible curve parameterized by an equiform parameter with equiform curvature $\mathcal{K}(\sigma) = 0$ given by

$$\alpha(\sigma) = (x(\sigma), y(\sigma), z(\sigma)). \tag{21}$$

By using of relations (6) and (12) we find

$$T(\sigma) = \rho t = (\rho, \rho\dot{y}, \rho\dot{z}).$$

On the other hand,

$$T(\sigma) = \alpha'(\sigma) = (x'(\sigma), y'(\sigma), z'(\sigma)).$$

These two relations imply

$$x'(\sigma) = \rho(x) = constant.$$

Integrating this we find

$$x(\sigma) = \rho\sigma + c_0, \quad c_0 \in \mathbb{R}_0.$$

Up to a translation, we may take $c_0 = 0$. Hence

$$x(\sigma) = \frac{\sigma}{\dot{\sigma}}. \tag{22}$$

Substituting relation (22) in relation (21), we get

$$\alpha(\sigma) = \left(\frac{\sigma}{\dot{\sigma}}, y(\sigma), z(\sigma)\right).$$

Therefore, as $\dot{\sigma}$ is constant, the equiform Frenet frame of α has the form

$$T(\sigma) = \left(\frac{1}{\dot{\sigma}}, y'(\sigma), z'(\sigma)\right), \tag{23}$$

$$N(\sigma) = (0, y''(\sigma), z''(\sigma)), \tag{24}$$

$$B(\sigma) = (0, -z''(\sigma), y''(\sigma)). \tag{25}$$

Substituting (23) and (25) in (20), we find

$$D(\sigma) = (\mathcal{T}, \dot{\sigma}\mathcal{T}y' - \dot{\sigma}z'', \dot{\sigma}\mathcal{T}z' + \dot{\sigma}y''). \tag{26}$$

In order to characterize equiform Darboux helices, we must exclude the case when the equiform Darboux vector is the constant vector, since it trivially makes a constant angle with any fixed direction. In relation to that, assume that equiform Darboux vector $D(\sigma)$ given by relation (26) is the constant vector. Differentiating relation (26) with respect to σ, we obtain

$$D' = (\mathcal{T}', \dot{\sigma}\mathcal{T}'y' + \dot{\sigma}\mathcal{T}y'' - \dot{\sigma}z''', \dot{\sigma}\mathcal{T}'z' + \dot{\sigma}\mathcal{T}z'' + \dot{\sigma}y'''). \tag{27}$$

From relations (15), (24), (25) and $\mathcal{K}(\sigma) = 0$, we have $y''' = -\mathcal{T}z''$ and $z''' = \mathcal{T}y''$. Substituting these two equalities in relation (27), we get

$$0 = D' = (\mathcal{T}', \dot{\sigma}\mathcal{T}'y', \dot{\sigma}\mathcal{T}'z') = \dot{\sigma}\mathcal{T}'T.$$

Since $\dot{\sigma}(x) \neq 0$, it follows $\mathcal{T}(\sigma) = constant$. Therefore, we can give the following lemma.

Lemma 1. *Let $\alpha(\sigma)$ be an admissible curve in $\mathbb{G}_3$ parameterized by equiform parameter σ with the equiform curvature $\mathcal{K}(\sigma) = 0$ and the equiform torsion $\mathcal{T}(\sigma)$. Then its equiform Darboux vector $D(\sigma)$ is a constant vector if and only if $\mathcal{T}(\sigma) = constant$.*

In the sequel, assume that $\mathcal{T}(\sigma) \neq constant$. Then the equiform Darboux vector of α is a non-constant non-isotropic vector. By using relations (2) and (26), we obtain that the unit equiform Darboux vector D_0 is given by

$$D_0 = \frac{D}{\|D\|} = \left(1, \dot{\sigma}y' - \frac{\dot{\sigma}}{\mathcal{T}}z'', \dot{\sigma}z' + \frac{\dot{\sigma}}{\mathcal{T}}y''\right). \tag{28}$$

In order to find explicit parameter equations of Darboux helices, we consider the next two cases: (A) an axis of an equiform Darboux helix is a non-isotropic vector; (B) an axis of an equiform Darboux helix is a isotropic vector.

(A) an axis of an equiform Darboux helix is a non-isotropic vector;

Let $U=(1,u_2,u_3)$ be a unit non-isotropic fixed axis of an equiform Darboux helix α. According to Definition 6, the unit equiform Darboux vector D_0 of α makes the constant angle φ with an axis U. By using relation (28) and Definition 1, we have

$$\varphi^2=\left(\dot{\sigma}y'-\frac{\dot{\sigma}}{\mathcal{T}}z''-u_2\right)^2+\left(\dot{\sigma}z'+\frac{\dot{\sigma}}{\mathcal{T}}y''-u_3\right)^2=c_1^2,$$

where $c_1\in\mathbb{R}_0$. This implies

$$\dot{\sigma}y'-\frac{\dot{\sigma}}{\mathcal{T}}z''-u_2=c_1\cos\omega, \tag{29}$$

$$\dot{\sigma}z'+\frac{\dot{\sigma}}{\mathcal{T}}y''-u_3=c_1\sin\omega, \tag{30}$$

where $\omega(\sigma)$ is some arbitrary differentiable function. Differentiating both sides of the relations (29) and (30) with respect to σ, we obtain

$$\dot{\sigma}y''-\left(\frac{\dot{\sigma}}{\mathcal{T}}\right)'z''-\frac{\dot{\sigma}}{\mathcal{T}}z'''=-c_1\omega'\sin\omega, \tag{31}$$

$$\dot{\sigma}z''+\left(\frac{\dot{\sigma}}{\mathcal{T}}\right)'y''+\frac{\dot{\sigma}}{\mathcal{T}}y'''=c_1\omega'\cos\omega. \tag{32}$$

If $\mathcal{K}$ is substituted by $\mathcal{K}=0$ in (15) and by using relations (24) and (25), we find

$$y'''=-\mathcal{T}z'',\quad z'''=\mathcal{T}y''. \tag{33}$$

Substituting (33) in (31) and (32), relations (31) and (32) reduce to

$$\left(\frac{\dot{\sigma}}{\mathcal{T}}\right)'z''=c_1\omega'\sin\omega, \tag{34}$$

$$\left(\frac{\dot{\sigma}}{\mathcal{T}}\right)'y''=c_1\omega'\cos\omega. \tag{35}$$

Also, from relation (4) we can write

$$\ddot{y}(x)^2+\ddot{z}(x)^2=\kappa(x)^2=\dot{\sigma}(x)^2, \tag{36}$$

and thus

$$y''^2+z''^2=\frac{1}{\dot{\sigma}^2}. \tag{37}$$

Substituting (34) and (35) in (37) yields

$$\omega'=\frac{1}{c_1}\left(\frac{1}{\mathcal{T}}\right)'. \tag{38}$$

Integrating the last relation, we get

$$\omega=\frac{1}{c_1\mathcal{T}}+c_2,\quad c_2\in\mathbb{R}. \tag{39}$$

By taking $c_2 = 0$ it follows that

$$\omega = \frac{1}{c_1 \mathcal{T}}, \quad c_1 \in \mathbb{R}_0. \tag{40}$$

Substituting (38) in (35) and (34), we find

$$y'' = \frac{1}{\dot{\sigma}} \cos\omega, \quad z'' = \frac{1}{\dot{\sigma}} \sin\omega. \tag{41}$$

Substituting the first equation of (41) in the second equation of (33) gives

$$z''' = \frac{1}{\dot{\sigma}} \mathcal{T} \cos\omega.$$

On the other hand, differentiating the second equation of (41) with respect to σ, we get

$$z''' = \frac{1}{\dot{\sigma}} \omega' \cos\omega.$$

The last two relations and relation (40) imply

$$\mathcal{T} = \frac{1}{\sqrt{2c_1\sigma + c_3}}, \quad c_1 \in \mathbb{R}_0^+, c_3 \in \mathbb{R}. \tag{42}$$

By taking $c_3 = 0$ and by using the relations (40), (41) and (42), we obtain

$$\begin{aligned} z'' &= \frac{1}{\dot{\sigma}} \sin\left(\sqrt{\frac{2\sigma}{c_1}}\right), \\ y'' &= \frac{1}{\dot{\sigma}} \cos\left(\sqrt{\frac{2\sigma}{c_1}}\right). \end{aligned}$$

Integrating the last two equations, we get

$$y(\sigma) = c_4 + c_5\sigma + \frac{3c_1\sqrt{2c_1\sigma}}{\dot{\sigma}} \sin\left(\sqrt{\frac{2\sigma}{c_1}}\right) + (\frac{3c_1^2 - 2c_1\sigma}{\dot{\sigma}}) \cos(\sqrt{\frac{2\sigma}{c_1}}), \tag{43}$$

$$z(\sigma) = c_6 + c_7\sigma - \frac{3c_1\sqrt{2c_1\sigma}}{\dot{\sigma}} \cos\left(\sqrt{\frac{2\sigma}{c_1}}\right) + (\frac{3c_1^2 - 2c_1\sigma}{\dot{\sigma}}) \sin(\sqrt{\frac{2\sigma}{c_1}}), \tag{44}$$

where $c_1 \in \mathbb{R}_0^+$, $c_4, c_5, c_6, c_7 \in \mathbb{R}$. Relations (22), (43) and (44) imply that the parameter equation of α reads

$$\begin{aligned} \alpha(\sigma) = (&\frac{\sigma}{\dot{\sigma}}, c_4 + c_5\sigma + \frac{3c_1\sqrt{2c_1\sigma}}{\dot{\sigma}} \sin(\sqrt{\frac{2\sigma}{c_1}}) + (\frac{3c_1^2 - 2c_1\sigma}{\dot{\sigma}}) \cos(\sqrt{\frac{2\sigma}{c_1}}), \\ &c_6 + c_7\sigma - \frac{3c_1\sqrt{2c_1\sigma}}{\dot{\sigma}} \cos(\sqrt{\frac{2\sigma}{c_1}}) + (\frac{3c_1^2 - 2c_1\sigma}{\dot{\sigma}}) \sin(\sqrt{\frac{2\sigma}{c_1}})), \end{aligned}$$

where $\dot{\sigma} \in \mathbb{R}_0$, $c_1 \in \mathbb{R}_0^+$, $c_4, c_5, c_6, c_7 \in \mathbb{R}$. Therefore, the following theorem is proved.

Theorem 1. *Let α be an admissible curve parameterized by equiform parameter σ, with the equiform curvature $\mathcal{K}(\sigma) = 0$ and the equiform torsion $\mathcal{T}(\sigma) \neq constant$ in $\mathbb{G}_3$. If α is an equiform Darboux helix whose unit Darboux vector makes constant angle $\varphi = c_1$ with a non-isotropic axis, then:*
(i) it has a parameter equation given by

$$\alpha(\sigma) = (\frac{\sigma}{\dot{\sigma}}, c_2 + c_3\sigma + \frac{3c_1\sqrt{2c_1\sigma}}{\dot{\sigma}} \sin(\sqrt{\frac{2\sigma}{c_1}}) + (\frac{3c_1^2 - 2c_1\sigma}{\dot{\sigma}}) \cos(\sqrt{\frac{2\sigma}{c_1}}), \quad (45)$$

$$c_4 + c_5\sigma - \frac{3c_1\sqrt{2c_1\sigma}}{\dot{\sigma}} \cos(\sqrt{\frac{2\sigma}{c_1}}) + (\frac{3c_1^2 - 2c_1\sigma}{\dot{\sigma}}) \sin(\sqrt{\frac{2\sigma}{c_1}})), \quad (46)$$

where $\dot{\sigma} \in \mathbb{R}_0$, $c_1 \in \mathbb{R}_0^+$, $c_2, c_3, c_4, c_5 \in \mathbb{R}$.
(ii) its equiform torsion is given by

$$\mathcal{T}(\sigma) = \frac{1}{\sqrt{2c_1\sigma}}, \quad c_1 \in \mathbb{R}_0^+. \quad (47)$$

Example 1. *Let us consider an admissible curve $\alpha(\sigma)$ in $\mathbb{G}_3$ with parameter equation given by relations (45) and (46) (Figure 1). Putting $\dot{\sigma} = c_1 = 2$, $c_3 = c_5 = 1$ and $c_2 = c_4 = 0$ in (45) and (46), we get*

$$\alpha(\sigma) = (\frac{\sigma}{2}, \sigma + 6\sqrt{\sigma}\sin(\sqrt{\sigma}) + (6-2\sigma)\cos(\sqrt{\sigma}), \sigma - 6\sqrt{\sigma}\cos(\sqrt{\sigma}) + (6-2\sigma)\sin(\sqrt{\sigma})).$$

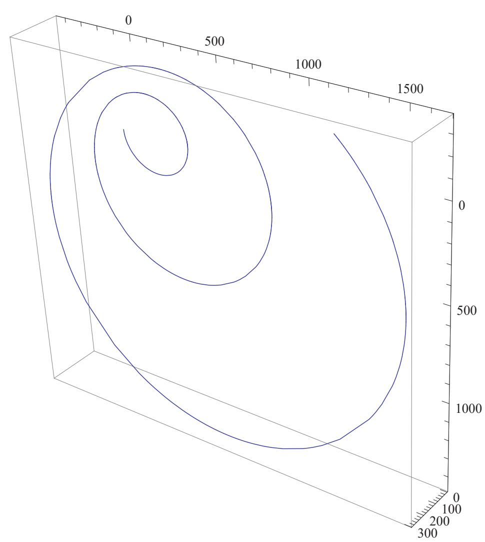

Figure 1: *The equiform Darboux helix α with the non-isotropic axis*

According to relations (23), (24) and (25), the equiform Frenet frame of the curve

α reads

$$T(\sigma) = \left(\frac{1}{2}, 1+\cos\left(\sqrt{\sigma}\right) + \sqrt{\sigma}\sin\left(\sqrt{\sigma}\right), 1+\sin\left(\sqrt{\sigma}\right) - \sqrt{\sigma}\cos\left(\sqrt{\sigma}\right)\right),$$
$$N(\sigma) = \frac{1}{2}(0, \cos(\sqrt{\sigma}), \sin(\sqrt{\sigma})),$$
$$B(\sigma) = \frac{1}{2}(0, -\sin(\sqrt{\sigma}), \cos(\sqrt{\sigma})).$$

From relations (28) and (47), it follows that the unit equiform Darboux vector of α is given by

$$D_0(\sigma) = (1, 2+2\cos(\sqrt{\sigma}), 2+2\sin(\sqrt{\sigma})).$$

Considering the fixed direction $U = (1,2,2)$, one can easily verify that the equiform Darboux vector $D_0(\sigma)$ makes the constant angle $\varphi = 2$ with the fixed direction U. This means that α is the equiform Darboux helix.

Example 2. *Consider an admissible curve $\beta(\sigma)$ in $\mathbb{G}_3$ with parameter equation given by relations (45) and (46) (Figure 2). By setting constants $\dot{\sigma} = 1$, $c_1 = 8$, $c_3 = c_5 = 3$, $c_2 = c_4 = 0$ and substituting them in (45) and (46), we obtain*

$$\beta(\sigma) = (\sigma, 3\sigma + 96\sqrt{\sigma}\sin(\frac{\sqrt{\sigma}}{2}) + (192-16\sigma)\cos(\frac{\sqrt{\sigma}}{2}),$$
$$3\sigma - 96\sqrt{\sigma}\cos(\frac{\sqrt{\sigma}}{2}) + (192-16\sigma)\sin(\frac{\sqrt{\sigma}}{2})).$$

By using relations (23), (24) and (25), we find that the equiform Frenet frame of

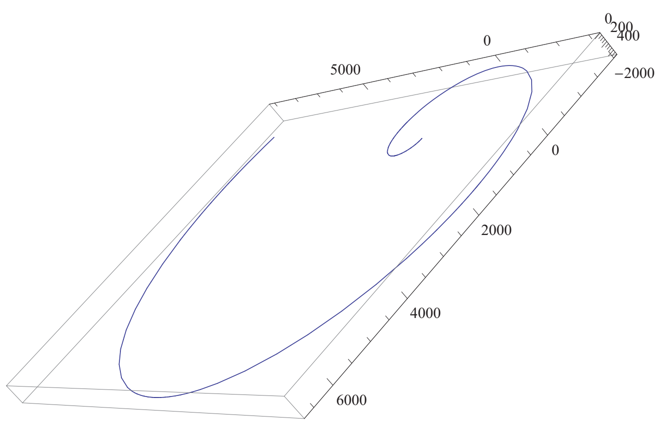

Figure 2: *The equiform Darboux helix β with the non-isotropic axis*

the curve β has the form

$$T(\sigma) = (1, 3+8\cos(\frac{\sqrt{\sigma}}{2}) + 4\sqrt{\sigma}\sin(\frac{\sqrt{\sigma}}{2}), 3+8\sin(\frac{\sqrt{\sigma}}{2}) - 4\sqrt{\sigma}\cos(\frac{\sqrt{\sigma}}{2})),$$
$$N(\sigma) = (0, \cos(\frac{\sqrt{\sigma}}{2}), \sin(\frac{\sqrt{\sigma}}{2})),$$
$$B(\sigma) = (0, -\sin(\frac{\sqrt{\sigma}}{2}), \cos(\frac{\sqrt{\sigma}}{2})).$$

Relations (28) and (47) imply that the unit equiform Darboux vector of β is given by

$$D_0(\sigma) = (1, 3 + 8\cos(\frac{\sqrt{\sigma}}{2}), 3 + 8\sin(\frac{\sqrt{\sigma}}{2})).$$

Considering the fixed direction $U = (1,3,3)$, one can easily verify that the equiform Darboux vector $D_0(\sigma)$ makes the constant angle $\varphi = 8$ with the fixed direction U, which means that β is the equiform Darboux helix.

(B) an axis of the equiform Darboux helix is an isotropic vector;

Let $W = (0, w_2, w_3)$ be an isotropic axis of the equiform Darboux helix $\alpha(\sigma)$. According to Definition 6, the unit equiform Darboux vector $D_0(\sigma)$ of α makes the constant angle θ with an axis W. By using relation (28) and Definition 2, we have

$$\theta = \frac{\left(\dot{\sigma}y' - \frac{\dot{\sigma}}{\mathcal{T}}z''\right)w_2 + \left(\dot{\sigma}z' + \frac{\dot{\sigma}}{\mathcal{T}}y''\right)w_3}{\sqrt{w_2^2 + w_3^2}} = c_1,$$

where $c_1 \in \mathbb{R}_0$. This shows

$$\left(\dot{\sigma}y' + \frac{\dot{\sigma}}{\mathcal{T}}z''\right)w_2 + \left(\dot{\sigma}z' + \frac{\dot{\sigma}}{\mathcal{T}}y''\right)w_3 = c_1\sqrt{w_2^2 + w_3^2} \tag{48}$$

Differentiating both sides of this equality with respect to σ, as $\dot{\sigma}$ is constant, we obtain

$$\left(\dot{\sigma}y'' - (\frac{\dot{\sigma}}{\mathcal{T}})'z'' - \frac{\dot{\sigma}}{\mathcal{T}}z'''\right)w_2 + \left(\dot{\sigma}z'' + (\frac{\dot{\sigma}}{\mathcal{T}})'y'' + \frac{\dot{\sigma}}{\mathcal{T}}y'''\right)w_3 = 0. \tag{49}$$

According to relation (33), we have

$$y''' = -\mathcal{T}z'', \quad z''' = \mathcal{T}y''. \tag{50}$$

Substituting (50) in (49) yields

$$(\frac{\dot{\sigma}}{\mathcal{T}})'(-w_2z'' + w_3y'') = 0. \tag{51}$$

Since $(\frac{\dot{\sigma}}{\mathcal{T}})' \neq 0$, it follows

$$-w_2z'' + w_3y'' = 0. \tag{52}$$

We also have

$$y'' = \frac{1}{\dot{\sigma}^2}\ddot{y}, \quad z'' = \frac{1}{\dot{\sigma}^2}\ddot{z}.$$

Substituting this in relation (52), we get

$$\ddot{z} = \frac{w_3}{w_2}\ddot{y}. \tag{53}$$

Differentiating the previous relation with respect to x, we find

$$\dddot{z} = \frac{w_3}{w_2}\dddot{y}. \tag{54}$$

From relations (5), (53) and (54), we find $\tau = 0$. Substituting this in (16), we get $\mathcal{T}(\sigma) = 0$, which is a contradiction. Therefore, the following theorem is proved.

Theorem 2. *If $\alpha(\sigma)$ is an equiform Darboux helix with an equiform curvature $\mathcal{K}(\sigma) = 0$ and an equiform torsion $\mathcal{T}(\sigma) \neq constant$ in Galilean space $\mathbb{G}_3$, then its axis can not be of isotropic direction.*

Next we show that there exists a simple relationship between equiform Darboux vectors and equiform rectifying curves.

Theorem 3. *The equiform Darboux vector $D(\sigma)$ of an equiform Darboux helix $\alpha(\sigma)$ in $\mathbb{G}_3$ with the equiform curvature $\mathcal{K}(\sigma) = 0$ and the equiform torsion $\mathcal{T}(\sigma) \neq constant$ is an equiform rectifying curve.*

Proof. Assume that the equiform Darboux vector D of α is given by relation (20). Substituting (12), (14) and (16) in (20), we easily get

$$D(\sigma(x)) = \rho\tau(x)t(x) + \rho\kappa b(x), \quad \rho = constant \neq 0, \tag{55}$$

where x is the arc-length parameter of admissible curve $\alpha(x)$ with the Frenet frame $\{t, n, b\}$. Substituting relations (6) and (8) in this equation, we get

$$D(x) = (\rho\tau, \rho\tau\dot{y} - \rho\ddot{z}, \rho\tau\dot{z} + \rho\ddot{y}). \tag{56}$$

Denote by u the arc-length parameter of the curve D. Then u is given by $u = \rho\tau(x)$. Differentiating both sides of (55), we find

$$D'(u) = t(x(u)).$$

Denote by $\{t_D, n_D, b_D\}$ the Frenet frame of D. It can be easily verified that

$$t_D = D'(u) = t, \quad n_D = n, \quad b_D = b. \tag{57}$$

The Frenet curvature κ_D and the Frenet torsion τ_D of D have the forms

$$\kappa_D = \frac{\kappa}{\rho\dot{\tau}}, \quad \tau_D = \frac{\tau}{\rho\dot{\tau}}, \tag{58}$$

respectively. Denote by $\{T_D, N_D, B_D\}$ the equiform Frenet frame of D. Then it is related to the Frenet frame $\{t_D, n_D, b_D\}$ as follows

$$T_D = \frac{1}{\kappa_D}t_D, \quad N_D = \frac{1}{\kappa_D}n_D, \quad B_D = \frac{1}{\kappa_D}b_D. \tag{59}$$

By using (55), (57), (58), (59) and $\rho\kappa = 1$, we get

$$D = \tau_D T_D + \kappa_D B_D. \tag{60}$$

Since the position vector of the equiform Darboux vector $D(\sigma)$ lies in its equiform rectifying plane $span\{T_D, B_D\}$, it follows that the equiform Darboux vector $D(\sigma)$ is the equiform rectifying curve. □

Now we can ask the following question: "Can an equiform Darboux vector of an equiform Darboux helix α with equiform curvature $\mathcal{K} = 0$ and equiform torsion $\mathcal{T} \neq constant$ be an equiform Darboux helix?" The answer is given in the next theorem.

Theorem 4. *An equiform Darboux vector $D(\sigma)$ of an equiform Darboux helix $\alpha(\sigma)$ in $\mathbb{G}_3$ with the equiform curvature $\mathcal{K}(\sigma) = 0$ and the equiform torsion $\mathcal{T}(\sigma) \neq$ constant is the equiform Darboux helix if an admissible curve $\alpha(x)$ is a rectifying curve.*

Proof. Assume that $\alpha(\sigma)$ is the equiform Darboux helix parameterized by equiform parameter σ whose equiform Darboux vector $D(\sigma)$ makes the constant angle with some fixed non-isotropic direction. Denote by D_D the equiform Darboux vector of D. Then D_D can be written as

$$D_D(\omega) = a(\omega)T_D(\omega) + b(\omega)N_D(\omega) + c(\omega)B_D(\omega), \tag{61}$$

where $\{T_D, N_D, B_D\}$ is the equiform Frenet frame of D and $a(\omega), b(\omega), c(\omega)$ are some differentiable functions in equiform parameter ω of D. Denote by $\mathcal{K}_D$ and $\mathcal{T}_D$ the equiform curvature and the equiform torsion of D, respectively.

By using equiform Darboux equations (17), (18), (19) and equiform Frenet equations (15), we find

$$a = \tau_D, \quad b = 0, \quad c = \kappa_D, \quad \mathcal{K}_D = 0, \tag{62}$$

where κ_D and τ_D are the Frenet curvature and the Frenet torsion of D, respectively. Let u be the arc-length parameter of D. According to relation (55), the arc-length parameter u is given by $u = \rho\tau(x)$. As $\rho_D = 1/\kappa_D = \rho\dot{\tau}/\kappa = \rho^2\dot{\tau}$ by using (58) and by (16), we get

$$\mathcal{K}_D = \frac{d\rho_D}{du} = \frac{d(\rho^2\dot{\tau})}{du} = \rho^2\frac{d\dot{\tau}}{dx}\frac{dx}{du} = \frac{\rho\ddot{\tau}}{\dot{\tau}}.$$

Thus $\mathcal{K}_D = 0$ shows $\ddot{\tau} = 0$. Consequently, $\tau(x) = a_1x + a_2$, $a_1 \in \mathbb{R}_0$, $a_2 \in \mathbb{R}$. Since $\mathcal{K} = 0$, we have $\kappa(x) = \textit{constant} \neq 0$. Therefore,

$$\frac{\tau(x)}{\kappa(x)} = c_1x + c_2,$$

$c_1 \in \mathbb{R}_0$, $c_2 \in \mathbb{R}$, which means that an admissible curve $\alpha(x)$ is a rectifying curve. Substituting (62) in (61), we find

$$D_D = \tau_D T_D + \kappa_D B_D. \tag{63}$$

Relations (63) and (60) show

$$D_D = D.$$

By assumption, the equiform Darboux vector $D(\sigma)$ makes the constant angle with some fixed non-isotropic direction. Hence D_D also makes the constant angle with the same direction, which means that equiform Darboux vector D is the equiform Darboux helix. □

4. Equiform curves of constant precession in $\mathbb{G}_3$

According to Scofield [14], the Euclidean curves of constant precession are defined as the curves whose Darboux vector makes the constant angle with some fixed direction and rotate about it with constant speed. Therefore, such curves are Darboux helices whose Darboux vector has constant speed.

In this section, we show that there are no equiform curves of constant precession in Galilean 3-space. Let $\alpha(\sigma)$ be the equiform Darboux helix with the equiform Darboux vector $D(\sigma)$ given by relation (20). Assume that the equiform Darboux vector $D(\sigma)$ of α has the constant speed, i.e. $||D'(\sigma)|| = constant \neq 0$. The last condition and the relation (27) give $||D'(\sigma)|| = |\mathcal{T}'(\sigma)| = constant \neq 0$. On the other hand, according to Theorem 1, we have $\mathcal{T}(\sigma) = \frac{1}{\sqrt{2c_1\sigma}}$. Differentiating the last relation with respect to σ, we get

$$\mathcal{T}'(\sigma) = -\frac{1}{2\sigma\sqrt{2c_1\sigma}} \neq constant.$$

This proves the last theorem.

Theorem 5. *There are no equiform curves of constant precession in* $\mathbb{G}_3$.

Acknowledgement

The third author was supported by the Serbian Ministry of Education, Science and Technological Development (grant number #174012).

References

[1] A. Artykbaev, *Total angle about the vertex of a cone in Galilean space*, Math. Notes, **43**(1988), 379–382.
[2] M. E. Aydin, M. Ergüt, *The equiform differential geometry of curves in 4-dimensional Galilean space* $\mathbb{G}_4$, Stud. Univ. Babes-Bolyai Math. **58**(2013), 393–400.
[3] B. Y. Chen, *When does the position vector of a space curve lie in its rectifying plane*? Amer. Math. Monthly **110**(2003), 147–152.
[4] B. Y. Chen, F. Dillen, *Rectifying curves as centrodes and extremal curves*, Bull. Inst. Math. Academia Sinica **33**(2005), 77–90.
[5] B. Divjak, Ž. Milin-Šipuš, *Special curves on ruled surfaces in Galilean and pseudo-Galilean spaces*, Acta Math. Hungar. **98**(2003), 203–215.
[6] Z. Erjavec, *On generalization of helices in the Galilean and the pseudo-Galilean space*, J. Math. Research **6**(2014), 39–50.
[7] Z. Erjavec, B. Divjak, *The equiform differential geometry of curves in the pseudo-Galilean space*, Math. Commun. **13**(2008), 321–332.
[8] Z. Erjavec, B. Divjak, D. Horvat, *The general solutions of Frenet's system in the equiform geometry of the Galilean, pseudo-Galilean, simple isotropic and double isotropic space*, Int. Math. Forum, **6**(2011), 837–856.
[9] H. Öztekin, *Normal and rectifying curves in Galilean space* $\mathbb{G}_3$, Proc. of IAM **5**(2016), 98–109.
[10] H. Öztekin, A. O. Ögrenmis, *Normal and rectifying curves in pseudo-Galilean space* $\mathbb{G}_3^1$ *and their applications*, J. Math. Comput. Sci. **2**(2012), 91–100.
[11] O. Röschel, *Die Geometrie des Galileischen Raumes*, Habilitationsschrift, Institut für Math. und Angew. Geometrie, Leoben, 1984.
[12] B. J. Pavković, *Equiform geometry of curves in the isotropic spaces* $I_3^{(1)}$ *and* $I_3^{(2)}$, Rad JAZU (1986), 39–44.

[13] B. J. Pavković, I. Kamenarović, *The equiform differential geometry of curves in the Galilean space* $\mathbb{G}_3$, Glasnik Mat. **22**(1987), 449–457.

[14] P. D. Scofield, *Curves of constant precession*, Amer. Math. Monthly **102**(1995), 531–537.

[15] E. Ziplar, A. Senol, Y. Yayli, *On Darboux helices in Euclidean* 3*-space*, GJSFR **12**(2012), 72–80.