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Modeling PD Closed-loop Control Problems with Fuzzy Differential Equations

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Original scientific paper

This paper reports a fuzzy differential equations approach for the modeling of initial condition uncertainty for a proportional derivative closed-loop control of a direct current motor. Uncertainties are considered on the precision of the sensing devices installed on a driver. The closed-loop system is designed for a plant modeled with fuzzy differential equations. Satisfactory analytic and numerical results for the position regulation problem for ideal case and also considering perturbed initial conditions are reported.

Key words: Fuzzy differential equations, Uncertainty, Mathematical modeling, Control

Sinteza sustava upravljanja s proporcionalno-derivacijskim regulatorom zasnovana na neizrazitim diferencijalnim jednadžbama. U radu je razvijen postupak sinteze proporcionalno-derivacijskog regulatora za upravljanje istosmjernim motorom s neizrazitim (engl. fuzzy) početnim uvjetima zasnovan na neizrazitim diferencijalnim jednadžbama. Pritom je uzeta u obzir nesigurnost određena mjernom preciznošću senzora. U predloženom postupku se zatvoreni regulacijski krug dizajnira korištenjem neizrazitih diferencijalnih jednadžbi. Primjenom projektiranog regulatora na probleme pozicioniranja u idealnom slučaju te u slučaju koji uzima u obzir perturbirane početne uvjete ostvareni su zadovoljavajući analitički i numerički rezultati.

Ključne riječi: neizrazite diferencijalne jednažbe, neodređenost, matematičko modeliranje, upravljanje

1 INTRODUCTION

Many complex industrial processes can not be satisfactory controlled using directly the results of classic and modern control theory either: (i) because its precise structure is unknown, or (ii) because no sufficient mathematical tools for the modeling of such problems exists. Controllers of the proportional-integral-derivative (PID) family are widely used, but PID controllers are not the best tool to control nonlinear systems, due to the fact that they are linear, and real world systems are, indeed, nonlinear. There exist many methodologies and techniques to analyze and design control systems for dynamical, linear, nonlinear, discrete and continuous systems [1, 2]. These methodologies and techniques are successful in applications where a system is well defined and mathematically modeled, but they have failed to deal with the practical aspects of many industrial processes where not all the system's parameters are known, or where the dynamics of such systems can not be fully modeled [3, 4]. An important aspect of those class of systems that is generally not addressed in the design and synthesis of controllers is the system uncertainty induced by errors in measurements made by sensing instruments. This is the motivation to consider in this paper fuzzy differential equations as an alternative to deal with this kind of problems.

Fuzzy control is usually an alternative for PID control when designers consider uncertainties [5]. Fuzzy controllers used in industry have the same structure than a proportional (P), proportional-integral (PI), proportional-derivative (PD) or PID controllers [6], allowing an easy way to consider the nonlinearity and to represent it by rules and fuzzy membership functions into the corresponding fuzzy control law.

As mentioned before, it is a common practice that the parameters that are present in the mathematical modeling process of dynamic systems are assumed as exact values in traditional control techniques, but on real world problems, instead of exact values, the designer usually only has vague, imprecise or incomplete information about those parameters. These imprecisions can be originated from measurements, observation, experimentation, etc. These actions are indeed uncertain themselves. These uncertainties can be modeled by fuzzy theory as in [7, 8], where fuzzy controllers for mechanical systems are designed following the Lyapunov stability theory, and as in other control applications reported in the literature [9–11].

On the other hand, direct current (DC) motors are widely used in many industrial applications, such as electric vehicles, steel rolling mills, electric cranes and robot manipulators, due to its characteristic of being a simple way to implement control systems as it is shown in [12]. Although there have been great efforts to develop accurate mathematical models for DC motors, up to date, they are still susceptible to having parametric uncertainties and other negative effects. Parametric uncertainty is a source of error in many control problems, and in problems of controlling mechanisms that are actuated by motors. Another common source of uncertainty are the measurement instruments with which the initial conditions for a control problem are obtained. That is an additional motivation to consider research about new tools for the mathematical modeling of such kind of systems in this paper.

The literature about applications of fuzzy differential equations for modeling uncertainty in dynamical systems is scarce. A fuzzy differential equation is a set of differential equations in which at least one of its components belong to the fuzzy mathematics, that is coefficients, variables, operators and/or initial conditions. Reference [13] is a pioneer work that provides the basic theoretical foundations about fuzzy differential equations and about the approach to solve them. Relevant applications using fuzzy differential equations have been reported in [14] applied to demographic and life expectancy modeling problems, which are also problems that present high parametric uncertainty. A prey-predator population model based on fuzzy differential equations is discussed in [15]. A triangular initial condition fuzzy differential equation problem is reported in [16], and [17] presents examples for the modeling of spring-mass-damper system using fuzzy differential equations. It is important to note that [13] and [16] provide only pure mathematics of fuzzy differential equations. References [15] and [14] induce possible applications about population dynamics and demography. Reference [17] refers to the fuzzy differential equations modeling for a spring-mass-damper system. To the authors' best knowledge, this paper is the first application of fuzzy differential equations in control engineering.

The main objective of this paper is to propose fuzzy differential equations as an alternative approach for dealing with the problem of modeling dynamical systems subject to uncertainty. This considers the fact that:

- Uncertainty is inherent to any real-world system and must be taken into account in controller design in order to achieve good performance, and
- Fuzzy differential equations are a non-explored alternative and a pertinent mathematical tool for modeling this type of problems. The methodology is applied to a DC motor, which it is a well known system, and which allows the reader to focus into the fuzzy differential equations modeling approach [18–20].

Also, this makes the methodology relevant for practical industrial implementation.

The rest of the paper is organized as follows: The problem is stated on Section 2. The foundations of fuzzy differential equations are presented in Section 3. Section 4 describes the controller analysis. The controller synthesis and results are reported in Section 5. Finally, conclusions are presented in Section 6.

2 PROBLEM STATEMENT

A DC motor is a common component in many mechanical dynamic systems. A linear model of a DC motor basically consists of two equations:

1. a mechanical equation, and
2. an electric equation.

The electrical circuit of the armature and the rotational mechanical diagram of a DC motor are shown in Figure 1. The electromagnetic torque T is proportional to the armature current $T = K_e i$, and the induced voltage e is proportional to the mechanical speed of the rotor shaft $e = K_e \dot{q}$.

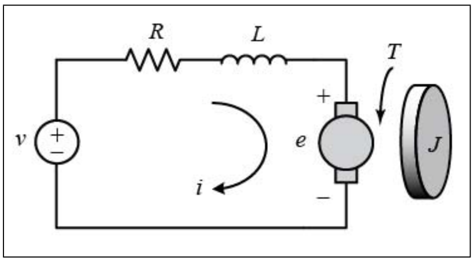

Fig. 1. Electrical armature circuit and rotational mechanical diagram of a DC motor.

The equations obtained by applying Newton's second law and Kirchhoff's voltage law to the circuit depicted in Figure 1 are:

$$J_m \ddot{q} + b\dot{q} = K_t i \tag{1}$$

and

$$L\frac{di}{dt} + Ri = v - K_e \dot{q} \tag{2}$$

respectively, where $\ddot{q}$ is the angular acceleration of the rotor shaft (rad/s^2), $\dot{q}$ is the angular velocity of the rotor shaft (rad/s) and, therefore, q is the angular position of the rotor shaft (rad). J_M is the moment of inertia of the system (kgm^2/s^2), L is the inductance of the armature (H), R is the armature resistance (Ω), K_e is the electromotive force constant (Nm/A), K_t is the constant torque (Nm/A), b is the friction coefficient of the motor and v is the voltage source (v).

Equations (1) and (2) can be written in a single equation as

$$J_m\ddot{q} + \left[b + \frac{K_tK_e}{R}\right]\dot{q} + \frac{LK_t}{R}\frac{di}{dt} = \frac{K_tv}{R}. \qquad (3)$$

Considering negligible the engine friction coefficient ($b = 0$) and the armature inductance ($L = 0$) in (3), the model can be simplified to:

$$J_m\ddot{q}(t) + \frac{K_tK_e}{R}\dot{q}(t) = \frac{K_t}{R}v, \qquad (4)$$

with initial conditions $q(0) = 0$, $\dot{q}(0) = 0$. Considering that the control objective is to regulate the angular position $q(t)$ of the DC motor to a constant desired position q_d, the error can be defined as:

$$\varepsilon(t) = q(t) - q_d(t), \qquad (5)$$

and then its time derivative results on:

$$\dot{\varepsilon}(t) = \dot{q}(t) - \dot{q}_d(t). \qquad (6)$$

Note that (6) can be rewritten as:

$$\dot{\varepsilon}(t) = \dot{q}(t) \qquad (7)$$

because $\dot{q}_d(t) = 0$ if q_d is constant.

The actual motor state $q(t)$ and the desired reference state q_d are continuously compared. If the actual state is different from the reference state, then an error signal $\varepsilon(t)$ is generated. This error is used by the controller to change the controllable variables in order to reduce the error so that the system returns to the desired position. Thus, the control objective may be defined as:

$$\lim_{t\to\infty} \varepsilon(t) = 0. \qquad (8)$$

The analysis above presented corresponds to a classical control engineering problem. The problem considered in this paper is focused on the conditions established for $q(0)$ and $\dot{q}(0)$ on (4). On a classical control engineering problem, exact values for initial conditions over $q(t)$ and $\dot{q}(t)$ are assumed known. However, if initial conditions had different or were not reliable, this might change the system's analysis. Moreover, the existence of (4) is subject to initial conditions $q(0) = 0$ and $\dot{q}(0) = 0$.

Traditionally, this issues are studied in control engineering as a robustness problem, adding mathematical complexity to the controller design. The alternative considered in this paper is to replace the model of the DC motor by a model that considers initial conditions as inexactitudes. This is, to model the DC motor via fuzzy differential equations with fuzzy initial conditions. That is, a model of the form

$$\dot{\mathbf{x}} = f(\mathbf{x}, t), \qquad (9)$$

subject to $\mathbf{x}(t) = C$, where C is a fuzzy number.

3 FOUNDATIONS ON FUZZY DIFFERENTIAL EQUATIONS

This section presents definitions that make a generalization of the concept of derivative, and that introduce the concept of fuzzy differential equation.

Consider the following definitions:

Definition 1 *A fuzzy set $\tilde{A}$ is a tuple of two elements defined as $\tilde{A} = \{(x, \mu_A(x))|x \in \mathbb{R}, \mu_A(x) \in [0,1]\}$, where $\mu_A(x)$ is called the membership function of the fuzzy set A [21].*

Definition 2 *Given X any set. The membership function μ_A of a non-empty fuzzy set A is a function: $\mu_A : X \longrightarrow [0,1]$. The function μ_A is interpreted as the degree of membership of each element x to the fuzzy set A, for each $x \in X$ [21].*

From Definitions 1 and 2 can be noted that a fuzzy set allows an element to be part of a set with a different level of membership, i.e. not just belong to a classical set. Note as well that for each element in a range of the definition domain of a fuzzy set, a membership value of that element to a specific set can be obtained. This allows to define the concept of membership function as:

Definition 3 *Let $u : \mathbb{R} \to [0,1]$, the membership function of a fuzzy set over $\mathbb{R}$, $\alpha - cut$ or $\alpha - level$ is defined as the set $[u]^\alpha = \{x \in \mathbb{R} : u(x) \geq \alpha\}$, for each $0 < \alpha \leq 1$. Then support of u to the set $[u]^0 = cl\{x \in \mathbb{R} : u(x) > 0\}$, where cl denote the closure of a subset [21].*

The definition of a fuzzy number, required for the initial condition of (9), can be given as:

Definition 4 *[13] Let u be the membership function of a fuzzy set. It is said that u is a fuzzy number if $u : \mathbb{R} \longrightarrow [0,1]$, and satisfies the following conditions:*

i) u is normal; i.e. there exists al least one $x^* \in \mathbb{R}$ such that $u(x^*) = 1$,

ii) $[u]^\alpha$ is closed $\forall\alpha \in (0,1]$, and

iii) $[u]^0$ is bounded.

$\mathscr{F}$ denotes the space of all fuzzy numbers in $\mathbb{R}$. This space has been studied by several authors [22–25]. From Definition 4 it is clear that a fuzzy number can have different geometrical representations (e.g. triangular). To understand the arithmetic operations that can be computed over fuzzy numbers is important to define the concept of closed interval for a fuzzy set as:

Definition 5 *If $u \in \mathscr{F}$, then the α-cut $[u]^\alpha$ is a closed interval denoted by $[u_L^\alpha(t), u_R^\alpha(t)]$, where u_L^α and u_R^α are the lower and upper ends of $[u]^\alpha$ respectively.*

For $u, v \in \mathscr{F}$ and $\lambda \in \mathbb{R}$, the sum operations $u \oplus v$ and product operation $\lambda \cdot u$ are defined as:

$$(u \oplus v)(x) = \sup_{x_1+x_2=x} \min\{u(x_1), v(x_2)\} \tag{10}$$

and

$$(\lambda \cdot u)(x) = \begin{cases} u(\frac{x}{\lambda}) & if \quad \lambda \neq 0 \\ \chi_{\{0\}}(x) & if \quad \lambda = 0 \end{cases} \tag{11}$$

respectively, where $\chi_{\{0\}}$ is the characteristic function of 0 (zero).

The definition of differentiability in the fuzzy sense was first introduced by [26] as:

Definition 6 *Let $u, v \in \mathscr{F}$. If there exists $w \in \mathscr{F}$ such that $u = v \oplus w$, then w is called the H-difference of u and v, and it is denoted by $u \ominus v$ [26].*

Definition 6 is based on the H-set difference defined as:

Definition 7 *[26] Let $F : T \to \mathscr{F}$ and $t_0 \in T \subseteq \mathbb{R}$. The function F is said differentiable on t_0 if:*

(I) There exist an element $F'(t_0) \in \mathscr{F}$ such that, for all $h > 0$ sufficiently close to zero, there exist $F(t_0+h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0-h)$ and limits

$$\lim_{h\to 0^+} \frac{F(t_0+h) \ominus F(t_0)}{h} = \lim_{h\to 0^+} \frac{F(t_0) \ominus F(t_0-h)}{h}$$

are equal to $F'(t_0)$, or
(II) There exist an element $F'(t_0) \in \mathscr{F}$ such that, for all $h < 0$ sufficiently close to zero, there exist $F(t_0+h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0-h)$ and limits

$$\lim_{h\to 0^-} \frac{F(t_0+h) \ominus F(t_0)}{h} = \lim_{h\to 0^-} \frac{F(t_0) \ominus F(t_0-h)}{h}$$

are equal to $F'(t_0)$.

It should be noted that if F is differentiable in the first form (I), then it is not differentiable in the second form (II), and vice versa, which is resumed in the following theorem:

Theorem 1 *[13, 27] Let $F : T \to \mathscr{F}$, y $[F(t)]^\alpha = [F_L^\alpha(t), F_R^\alpha(t)]$, for each $\alpha \in [0,1]$. Then:*

(i) If F is differentiable from the first form (I) then $F_L^\alpha(t)$ and $F_R^\alpha(t)$ are differentiable functions and

$$[F'(t)]^\alpha = [(F_L^\alpha(t))', (F_R^\alpha(t))'], \tag{12}$$

or
(ii) If F is differentiable from the second form (II) then $F_L^\alpha(t)$ y $F_R^\alpha(t)$ are differentiable functions and

$$[F'(t)]^\alpha = [(F_R^\alpha(t))', (F_L^\alpha(t))']. \tag{13}$$

Proof:

(i) If $h > 0$ and $\alpha \in [0,1]$, then $[F(t+h) \ominus F(t)]^\alpha = [F_L^\alpha(t+h) \ominus F_L^\alpha(t), F_R^\alpha(t+h) \ominus F_R^\alpha(t)]$. Dividing both sides of the equation by h is obtained:

$$\begin{aligned} &\left[\frac{F(t+h) \ominus F(t)}{h}\right]^\alpha = \\ &\quad \left[\frac{F_L^\alpha(t+h) \ominus F_L^\alpha(t)}{h}, \frac{F_R^\alpha(t+h) \ominus F_R^\alpha(t)}{h}\right]. \end{aligned} \tag{14}$$

Taking the limit as $h \to 0$, and Definition 7, is obtained that
$[F'(t)]^\alpha = [(F_R^\alpha(t))', (F_L^\alpha(t))']$. The case $[F(t) \ominus F(t+h)]^\alpha$ can be demonstrated similarly.
(ii) If $h < 0$ and $\alpha \in [0,1]$, then $[F(t+h) \ominus F(t)]^\alpha = [F_L^\alpha(t+h) \ominus F_L^\alpha(t), [F_R^\alpha(t+h) \ominus F_R^\alpha(t)]$. Dividing both sides of the equation by h is obtained:

$$\begin{aligned} &\left[\frac{F(t+h) \ominus F(t)}{h}\right]^\alpha = \\ &\quad \left[\frac{F_R^\alpha(t+h) \ominus F_R^\alpha(t)}{h}, \frac{F_L^\alpha(t+h) \ominus F_L^\alpha(t)}{h}\right]. \end{aligned} \tag{15}$$

Taking the limit as $h \to 0$, and Definition 7, is obtained that
$[F'(t)]^\alpha = [(F_R^\alpha(t))', (F_L^\alpha(t))']$. The case $[F(t) \ominus F(t+h)]^\alpha$ can be demonstrated similarly. ■

A similar proof of Theorem 1 can be found in [27].

4 MODELING CLOSED-LOOP CONTROL SYSTEMS WITH FUZZY DIFFERENTIAL EQUATIONS

This section presents the modeling of a closed-loop control system with fuzzy differential equations. Considering the problem established in Section 2 and the mathematical foundations from Section 3.

Consider the differential equation with real coefficients:

$$q^{(n)}(t) + a_1 q^{(n-1)}(t) + \ldots + a_n q(t) = \tau(t), \tag{16}$$

where a_i, $i = 1 \ldots n$ are real numbers and $q(t)$ denotes the system's function, $\tau(t)$ is the control action, t is time, and $q^{(i)}(t)$ are the i-*th* derivatives of $q(t)$.

Given the control action $\tau(t)$, which is formed by system's functions $q^{(i)}(t)$, $i = 1 \ldots n$ [28], then:

$$\tau(t) = -k_{ob} \sum_{j=0}^{r} k_{p_j} q^{(j)}(t), \tag{17}$$

where k_{p_j}, $j = 1 \ldots r$ denote the controller parameters, r the order of the fuzzy controller, and k_{ob} is a real constant.

If $q_d(t)$ indicates the desired position, the error is defined as in (5).

Note that, due to the structure of (17) [28], this approach only allows to design proportional (P) and proportional-derivative (PD) controllers.

Take the differential equation (16), with $n = 2$ and $r = 1$, such that the differential equation is:

$$\ddot{q}(t) + a_1\dot{q}(t) = \tau(t), \tag{18}$$

given $r = 1$. It can be observed that the control action is governed by a proportional derivative-controller (PD), and thus (17) is written as

$$\tau(t) = -k_{ob}(k_{p_0}\varepsilon(t) + k_{p_1}\dot{\varepsilon}(t)). \tag{19}$$

Substituting (19) in (18) is obtained that:

$$\ddot{q}(t) + a_1\dot{q}(t) = -k_{ob}(k_{p_0}\varepsilon(t) + k_{p_1}\dot{\varepsilon}(t)), \tag{20}$$

and using (5)-(6) results on the equation:

$$\ddot{q}(t) + a_1\dot{q}(t) = -k_{ob}k_{p_0}(q(t) - q_d) + k_{ob}k_{p_1}(\dot{q}(t) - \dot{q}_d). \tag{21}$$

Unlike the common results of control engineering, a differential equation system, parameterized from a fuzzy initial condition, allowing to model the uncertainty in the initial condition $q(0) = \tilde{0}$, where $\tilde{0}$ is a fuzzy number. By Definition 5, the function $q(t)$ is now a fuzzy function $\tilde{q}(t)$ and has the following property:

$$[\tilde{q}^{(i)}]^\alpha = [(q_L^\alpha(t))^{(i)}, (q_R^\alpha(t))^{(i)}]. \tag{22}$$

Using Definition 5, Theorem 1 and [27] are obtained two systems with two closed-loop differential equations each one. Next it is reported the only one system that have a physical interpretation for the case of study.

$$\begin{aligned} \ddot{q_L^\alpha}(t) + (a_1 + k_{ob}k_{p_1})\dot{q_L^\alpha}(t) + k_{ob}k_{p_0}q_L^\alpha(t) = \\ k_{ob}k_{p_0}q_d + k_{ob}k_{p_1}\dot{q}_d, \end{aligned} \tag{23}$$

$$\begin{aligned} \ddot{q_R^\alpha}(t) + (a_1 + k_{ob}k_{p_1})\dot{q_R^\alpha}(t) + k_{ob}k_{p_0}q_R^\alpha(t) = \\ k_{ob}k_{p_0}q_d + k_{ob}k_{p_1}\dot{q}_d. \end{aligned} \tag{24}$$

5 RESULTS

A simplification is made in order to obtain numerical results for the model (23)-(24) developed in Section 4. From the left side of (20) and (4): $a_1 = \frac{K_tK_e}{J_mR}$, $k_{ob} = \frac{K_t}{J_mR}$, $k_{p_0} = K_p$, $k_{p_1} = K_d$. Where K_p is the proportional gain, and K_d is the derivative gain of the controller. As an illustrative example, consider the nominal parameters for the dynamic model (4) of the DC motor shown in Table 1. These parameters correspond to the QUANSER QNET DC Motor Control Trainer for NI ELVIS [29].

Table 1. Parameters of the DC motor

. Description	Notation	Value	Unit
Moment of Inertia	J_m	1.8×10^{-6}	kgm^2
Constant torque	K_t	0.03	Vs/rad
Electromotive force constant	K_e	0.03	Nm/A
Resistance	R	5.7	Ohm

The PD gains $K_p = 5$ and $K_d = 0.1$ are tuned using the Zeigler-Nishols technique like in [29]. The control objective is to regulate the system in the desired position $q_d = 1$. In the following subsections, results are presented first considering a uncertainty free case, and subsequently considering that initial conditions are affected by an external disturbance.

5.1 Undisturbed case

Considering that the system parameters from Table 1 and the controller gains, (21) can be written as:

$$\ddot{q}(t) + 433.29\dot{q}(t) + 1.6248 \times 10^4 q(t) = 1.6248 \times 10^4. \tag{25}$$

With initial conditions $q(0) = \tilde{0}$, $\dot{q}(0) = \tilde{0}$ (note that $\tilde{0}$ is a fuzzy number), and with $\tilde{0}^\alpha = [0_L^\alpha, 0_R^\alpha]$, and solving (25) with (23) - (24) results, for each $\alpha \in [0, 1]$:

$$\ddot{q_L^\alpha}(t) + 433.29\dot{q_L^\alpha}(t) + 1.6248 \times 10^4 q_L^\alpha(t) = 1.6248 \times 10^4, \tag{26}$$

$$\ddot{q_R^\alpha}(t) + 433.29\dot{q_R^\alpha}(t) + 1.6248 \times 10^4 q_R^\alpha(t) = 1.6248 \times 10^4. \tag{27}$$

Solving (26)-(27), $q_L^\alpha(t)$ and $q_R^\alpha(t)$ are obtained. Figure 2 depicts the solution of (26)-(27) for $\alpha \in [0, 1]$, while Figure 3 (that is a projection of Figure 2) results in a band. This band correspond to the concept of *footprint of uncertainty* (FOU) of a type-2 fuzzy set [11], which means that the fuzzy differential equations model (26)-(27) deals with uncertainty as it was expected. All trajectories for the system (25)-(27) are actually represented in this footprint of uncertainty. It can also be noted that $q_L^\alpha(t)$, $q_R^\alpha(t)$ and $q(t)$ reach the desired reference q_d; that is $\lim_{t\to\infty}(q(t) - q_d) = 0$. In other words $\lim_{t\to\infty}\varepsilon(t) = 0$, and therefore the position regulation control problem has been solved.

5.2 Disturbed case

A white gaussian noise signal is added to the input signal in order to verify the closed-loop system robustness. This added signal simulates an external disturbance perturbing the system.

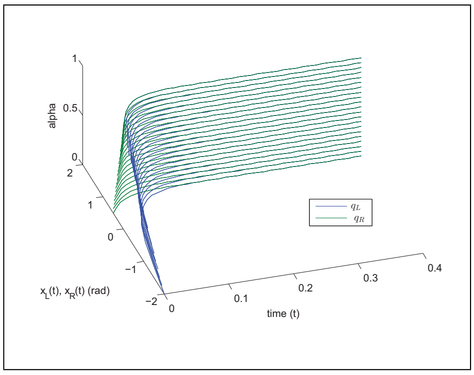

Fig. 2. Trajectories given by Eq. (25).

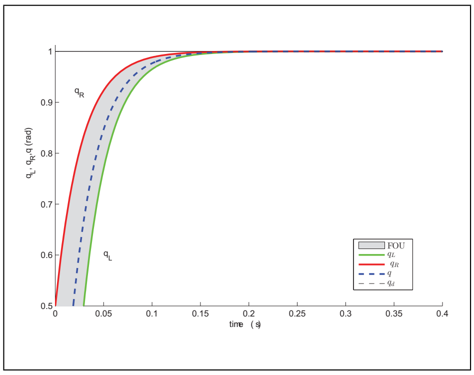

Fig. 3. Trajectories for the system given by Eqs. (26)-(27).

The fuzzy differential equation (25) for the disturbed case, can be expressed as:

$$\ddot{q}(t)+433.29\dot{q}(t)+1.6248\times10^4 q(t) = 1.6248\times10^4+\omega(t), \quad (28)$$

where $\omega(t)$ represents the added noise, and the height of the PSD (Power spectral density) of the white noise is 0.0001.

The obtained trajectories are shown in Figure 4, where it can be seen that the dynamics corresponds to $\lim_{t\to\infty}(q(t)-q_d) = 0$ and therefore to $\lim_{t\to\infty}\varepsilon(t) = 0$. It can be seen that the solution of the disturbed system for all $\alpha \in [0,1]$ remains bounded within the footprint of uncertainty. Thus, the position regulation control problem is solved even under conditions of uncertainty due to the fuzzy differential equation modeling.

6 CONCLUSIONS

This paper proposes and demonstrates that fuzzy differential equations are an effective tool for modeling dynamic systems, particularly for the modeling of closed-loop control systems and systems subject to uncertainties.

The numerical results allow to relate the solutions obtained following the Kaleva theorem 1 [13] to the concept

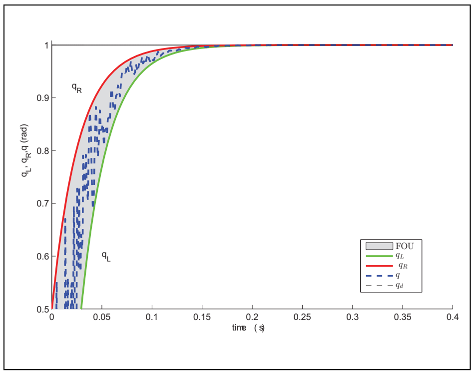

Fig. 4. Trajectories for Eq. (28).

of footprint of uncertainty of the type-2 fuzzy systems [11]. Furthermore, the obtained results suggest the existence of a relationship between the results obtained by type-2 fuzzy controllers [8, 9] and results obtained by modeling using fuzzy differential equations. Future work will explore this possible relationship.

The results show the effects of external disturbances into the system dynamics and open new possible research directions. It will be necessary to test the methodology with systems with a more challenging behavior, e.g. (i) modeling and control of nonsmooth systems, (ii) modeling of complex dynamical systems and, in general (iii) the modeling of uncertain systems. All of these are real world problems subject to uncertainty.

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