MATHEMATICAL COMMUNICATIONS
Math. Commun. **23**(2018), 127–144

On weighted Adams-Bashforth rules

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Received March 7, 2017; accepted September 7, 2017

Abstract. One class of linear multistep methods for solving the Cauchy problems of the form $y' = F(x, y)$, $y(x_0) = y_0$, contains Adams-Bashforth rules of the form $y_{n+1} = y_n + h\sum_{i=0}^{k-1} B_i^{(k)} F(x_{n-i}, y_{n-i})$, where $\{B_i^{(k)}\}_{i=0}^{k-1}$ are fixed numbers. In this paper, we propose an idea for a weighted type of Adams-Bashforth rules for solving the Cauchy problem for singular differential equations,

$$A(x)y' + B(x)y = G(x, y), \quad y(x_0) = y_0,$$

where A and B are two polynomials determining the well-known classical weight functions in the theory of orthogonal polynomials. Some numerical examples are also included.

AMS subject classifications: 65L05, 33C45

Key words: weighted Adams-Bashforth rule, ordinary differential equation, linear multistep method, weight function

1. Introduction

In this paper, we present an idea for constructing weighted Adams-Bashforth rules for solving Cauchy problems for singular differential equations.

There are two main approaches to increase the accuracy of a numerical method for ordinary non-singular differential equations. In the first approach (i.e., multistep methods), the accuracy is increased by considering previous information, while in the second one (i.e., multistage methods or more precisely Runge-Kutta methods), the accuracy is increased by approximating the solution at several internal points.

Multistep methods were originally proposed by Bashforth and Adams [2] (see also [1, 3, 4]), where the approximate solution of the initial value problem

$$\frac{dy}{dx} = F(x, y), \quad y(x_0) = y_0, \tag{1}$$

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is considered as

$$y_{n+1} = y_n + h\sum_{i=0}^{k-1} B_i^{(k)} F(x_{n-i}, y_{n-i}). \tag{2}$$

Many years later, Moulton [12] (see also, [3, 4]) developed a class of implicit multistep methods, the so-called Adams-Moulton methods,

$$y_{n+1} = y_n + h\sum_{i=-1}^{k-1} \alpha_i^{(k)} F(x_{n-i}, y_{n-i}), \tag{3}$$

which have some better characteristics than the previous ones.

The unknown coefficients $B_i^{(k)}$ and $\alpha_i^{(k)}$ in relations (2) and (3) are chosen in such a way that they have the highest possible accuracy order. These formulas are indeed special cases of the so-called linear multistep methods denoted by

$$y_n = \sum_{j=1}^{k_1} \eta_j y_{n-j} + h\sum_{i=0}^{k_2} \gamma_i F(x_{n-i}, y_{n-i}).$$

Other special cases of linear multistep methods were derived by Nyström and Milne [1, 4]. The idea of Predictor-Corrector methods was proposed by Milne [4] in which y_n is predicted by the Adams-Bashforth methods and then corrected by the Adams-Moulton methods.

It is not fair to talk about linear multistep methods without mentioning the name of Germund Dahlquist. In 1956, he [6] established some basic concepts such as consistency, stability and convergence in numerical methods and showed that if a numerical method is consistent and stable, then it is necessarily convergent.

However, it should be noted that the above-mentioned methods are valid only for non-singular problems of type (1). In other words, if equation (1) is considered as an initial value problem on (a, b) in the form

$$A(x)y' = H(x, y), \quad y(a) = y_0, \tag{4}$$

such that

$$A(a) = 0 \quad \text{or} \quad A(b) = 0,$$

then it is no longer possible to use usual Adams-Bashforth methods or other numerical techniques. For this purpose, in this paper, we gave an idea for using a weighted Adams-Bashforth rule.

For constructing these weighted rules we use a similar procedure as in the case of non-weighted formulas. Therefore, in Section 2, we give a short account of constructing the usual Adams-Bashforth methods by using linear difference operators and the backward Newton interpolation formula. Such a procedure is applied in Section 3 for obtaining the weighted rules. By introducing the weighted local truncation error of such rules, we determine their order. Finally, in order to illustrate the efficiency of such weighted rules, we give some numerical examples in Section 4.

2. Computing the usual Adams-Bashforth methods

In this section, we obtain the explicit forms of the coefficients $\{B_i^{(k)}\}_{i=0}^{k-1}$ in (2) using the backward Newton interpolation formula for $F(x,y)=F(x,y(x))$ at equidistant nodes $x_{n-\nu}=x_n-\nu h$, $\nu=0,1,\ldots,k-1$, and in the next section we apply such an approach in order to get the corresponding weighted type of Adams-Bashforth methods. Here we use standard linear difference operators ∇ (the backward-difference operator), E (the shifting operator), and 1 (the identity operator), defined by

$$\nabla f(x)=f(x)-f(x-h), \qquad Ef(x)=f(x+h) \quad \text{and} \quad 1f(x)=f(x).$$

Since $E^\lambda=(1-\nabla)^{-\lambda}$, we have

$$E^\lambda=\sum_{\nu=0}^{+\infty}(-1)^\nu\binom{-\lambda}{\nu}\nabla^\nu=\sum_{\nu=0}^{+\infty}\frac{(\lambda)_\nu}{\nu!}\nabla^\nu, \tag{5}$$

where

$$(\lambda)_\nu=\lambda(\lambda+1)\cdots(\lambda+\nu-1)=\frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}$$

is Pochhammer's symbol. Assuming $F_{n-\nu}\equiv F(x_{n-\nu},y(x_{n-\nu}))$ for $\nu=0,1,\ldots,k-1$ and taking the first k terms of (5) for $x=x_n+\lambda h$ we get

$$F(x,y(x))=E^\lambda F_n=\sum_{\nu=0}^{k-1}\frac{(\lambda)_\nu}{\nu!}\nabla^\nu F_n+r_k(F_n), \tag{6}$$

where $r_k(F_n)$ denotes the corresponding error term. Using (6) we have

$$\begin{aligned}\sum_{\nu=0}^{k-1}\frac{(\lambda)_\nu}{\nu!}\nabla^\nu F_n &= \sum_{\nu=0}^{k-1}\frac{(\lambda)_\nu}{\nu!}\sum_{i=0}^{\nu}(-1)^i\binom{\nu}{i}E^{-i}F_n\\ &=\sum_{i=0}^{k-1}\left(\frac{(-1)^i}{i!}\sum_{\nu=i}^{k-1}\frac{(\lambda)_\nu}{(\nu-i)!}\right)F_{n-i}=\sum_{i=0}^{k-1}C_i^{(k)}(\lambda)F_{n-i},\end{aligned} \tag{7}$$

where $\lambda=(x-x_n)/h$ and

$$C_i^{(k)}(\lambda)=\frac{(-1)^i}{i!}\sum_{\nu=i}^{k-1}\frac{(\lambda)_\nu}{(\nu-i)!}=(-1)^i\frac{(\lambda)_i}{i!}\binom{\lambda+k-1}{k-1-i}, \tag{8}$$

because, based on induction, we have

$$\begin{aligned}\sum_{\nu=i}^{k}\frac{(\lambda)_\nu}{(\nu-i)!}&=\sum_{\nu=i}^{k-1}\frac{(\lambda)_\nu}{(\nu-i)!}+\frac{(\lambda)_k}{(k-i)!}\\ &=(\lambda)_i\binom{\lambda+k-1}{k-1-i}+(\lambda)_i\binom{\lambda+k-1}{k-i}=(\lambda)_i\binom{\lambda+k}{k-i}.\end{aligned}$$

Now, integrating (1) over (x_n, x_{n+1}) and approximating $F(x,y)$ by its backward Newton interpolation polynomial (7) yield

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} F(x,y)\mathrm{d}x \approx h\sum_{i=0}^{k-1} B_i^{(k)} F_{n-i}, \tag{9}$$

where

$$B_i^{(k)} = \int_0^1 C_i^{(k)}(\lambda)\mathrm{d}\lambda = \frac{(-1)^i}{i!}\int_0^1 (\lambda)_i \binom{\lambda+k-1}{k-1-i}\mathrm{d}\lambda. \tag{10}$$

Table 1 shows the values $B_i^{(k)}$ in (2) for $k = 2, 3, \ldots, 8$.

k	2	3	4	5	6	7	8
$B_0^{(k)}$	$\frac{3}{2}$	$\frac{23}{12}$	$\frac{55}{24}$	$\frac{1901}{720}$	$\frac{4277}{1440}$	$\frac{198721}{60480}$	$\frac{16083}{4480}$
$B_1^{(k)}$	$-\frac{1}{2}$	$-\frac{4}{3}$	$-\frac{59}{24}$	$-\frac{1387}{360}$	$-\frac{2641}{480}$	$-\frac{18637}{2520}$	$-\frac{1152169}{120960}$
$B_2^{(k)}$		$\frac{5}{12}$	$\frac{37}{24}$	$\frac{109}{30}$	$\frac{4991}{720}$	$\frac{235183}{20160}$	$\frac{242653}{13440}$
$B_3^{(k)}$			$-\frac{3}{8}$	$-\frac{637}{360}$	$-\frac{3649}{720}$	$-\frac{10754}{945}$	$-\frac{296053}{13440}$
$B_4^{(k)}$				$\frac{251}{720}$	$\frac{959}{480}$	$\frac{135713}{20160}$	$\frac{2102243}{120960}$
$B_5^{(k)}$					$-\frac{95}{288}$	$-\frac{5603}{2520}$	$-\frac{115747}{13440}$
$B_6^{(k)}$						$\frac{19087}{60480}$	$\frac{32863}{13440}$
$B_7^{(k)}$							$-\frac{5257}{17280}$

Table 1: *The coefficients of usual Adams-Bashforth formulae*

By assuming that all previous values y_{n-i}, $i = 0, 1, \ldots, k-1$, are exact, i.e., $y_{n-i} = y(x_{n-i})$, $i = 0, 1, \ldots, k-1$, (9) gives the k-step method (2). This k-step method, known also as *the kth-order Adams-Bashforth method*, can be written in the form

$$y_{n+k} - y_{n+k-1} = h\sum_{j=0}^{k-1} \beta_j^{(k)} F_{n+j},$$

where $\beta_j^{(k)} = B_{k-1-j}^{(k)}$, $j = 0, 1, \ldots, k-1$.

According to (6), *the local truncation error* of this method at the point $x_{n+k} \in [a,b]$ can be expressed in the form

$$(T_h)_{n+k} = \frac{y(x_{n+k}) - y(x_{n+k-1})}{h} - \sum_{j=0}^{k-1} \beta_j^{(k)} y'(x_{n+j}) = \int_0^1 r_k(F_{n+k-1})\mathrm{d}\lambda, \tag{11}$$

where $x \mapsto y(x)$ is the exact solution of the Cauchy problem (1). If $y \in C^{k+2}[a,b]$, then (11) can be expressed as (cf. [7, pp. 409–410])

$$(T_h)_{n+k} = C_k y^{(k+1)}(\xi_k)h^k = C_k y^{(k+1)}(x_n)h^k + O(h^{k+1}), \tag{12}$$

where $x_n < \xi_k < x_{n+k-1}$. In the simplest case ($k = 1$), we have the well-known Euler method, $y_{n+1} - y_n = hF_n$. The so-called *error constants* C_k in *the main term of the local truncation error* for $k = 1, 2, 3, 4$ and 5 are

$$C_1 = \frac{1}{2}, \quad C_2 = \frac{5}{12}, \quad C_3 = \frac{3}{8}, \quad C_4 = \frac{251}{720}, \quad C_5 = \frac{95}{288}, \tag{13}$$

respectively. Details on multistep methods, including convergence, stability and estimation of global errors $e_n = y_n - y(x_n)$, can be found in [4, 7, 11].

Remark 1. These coefficients $B_i^{(k)}$ can also be expressed in terms of the first kind Stirling numbers $\mathbf{S}(n, k)$, which are defined by

$$\prod_{i=0}^{n-1} (x - i) = \sum_{k=0}^{n} \mathbf{S}(n, k) x^k,$$

(see [5, 8, 9, 13]). Namely, for each $k \in \mathbb{N}$, coefficients (10) can be explicitly represented in terms of the first kind Stirling numbers as

$$B_i^{(k)} = \sum_{\nu=i}^{k-1} \left(\frac{\displaystyle\sum_{j=0}^{\nu} \frac{(-1)^j}{j+1} \mathbf{S}(\nu, j)}{\displaystyle(-1)^\nu \nu! + \sum_{j=1}^{\nu} i^j (j+1) \mathbf{S}(\nu+1, j+1)} \right), \quad i = 0, 1, \ldots, k-1. \tag{14}$$

3. Weighted type of Adams-Bashforth methods

In this section, we study the Cauchy problem for a special type of differential equations of the first order given on a finite interval, on a half line or on the real line, which can be considered, without loss of generality, as $(-1, 1)$, $(0, +\infty)$, and $(-\infty, +\infty)$. Thus, we consider the following initial value problem on (a, b)

$$A(x)y' + B(x)y = G(x, y), \quad y(x_0) = y_0, \tag{15}$$

where A and B are two polynomials determining the well-known classical weight functions in the theory of orthogonal polynomials (cf. [10, p. 122]). Such polynomials and weight functions are given in Table 2, where $\alpha, \beta, \gamma > -1$.

(a, b)	$w(x)$	$A(x)$	$B(x)$
$(-1, 1)$	$(1-x)^\alpha (1+x)^\beta$	$1 - x^2$	$\beta - \alpha - (\alpha + \beta + 2)x$
$(0, +\infty)$	$x^\gamma \mathrm{e}^{-x}$	x	$\gamma + 1 - x$
$(-\infty, +\infty)$	e^{-x^2}	1	$-2x$

Table 2: *Classical weight functions and corresponding polynomials A and B*

Let again $\{x_k\}$ be a system of equidistant nodes with the step h, i.e., $x_k = x_0 + kh \in [a, b]$.

Since the differential equation of the weight function is as (cf. [10, p. 122])

$$(Aw)' = Bw,$$

after multiplying by $w(x)$, our initial value problem (15) becomes

$$A(x)w(x)y' + B(x)w(x)y = w(x)G(x,y), \quad y(x_0) = y_0,$$

which is equivalent to

$$(A(x)w(x)y)' = w(x)G(x,y), \quad y(x_0) = y_0. \tag{16}$$

Now, integrating from both sides of (16) over $[x_n, x_{n+1}]$ yields

$$A(x_{n+1})w(x_{n+1})y(x_{n+1}) - A(x_n)w(x_n)y(x_n) = \int_{x_n}^{x_{n+1}} w(x)G(x,y)\mathrm{d}x. \tag{17}$$

Let $x = x_n + \lambda h$. Similar to relations (6), (7) and (9), the right-hand side of (17) can be written in the form

$$\int_{x_n}^{x_{n+1}} w(x)G(x,y)\mathrm{d}x = h\int_0^1 w(x_n+\lambda h)\left\{\sum_{\nu=0}^{k-1}\frac{(\lambda)_\nu}{\nu!}\nabla^\nu G_n + r_k(G_n)\right\}\mathrm{d}\lambda, \tag{18}$$

and approximated as

$$\begin{aligned}\int_{x_n}^{x_{n+1}} w(x)G(x,y)\mathrm{d}x &\approx h\int_0^1 w(x_n+\lambda h)\sum_{\nu=0}^{k-1}\frac{(\lambda)_\nu}{\nu!}\left(\sum_{i=0}^{\nu}(-1)^i\binom{\nu}{i}G_{n-i}\right)\mathrm{d}\lambda\\ &= h\sum_{i=0}^{k-1}\left(\frac{(-1)^i}{i!}\sum_{\nu=i}^{k-1}\frac{1}{(\nu-i)!}\int_0^1 w(x_n+\lambda h)(\lambda)_\nu \mathrm{d}\lambda\right)G_{n-i}\\ &= h\sum_{i=0}^{k-1}B_i^{(k)}(h,x_n)G_{n-i},\end{aligned}$$

where

$$B_i^{(k)}(h,x_n) = \int_0^1 w(x_n+\lambda h)C_i^{(k)}(\lambda)\mathrm{d}\lambda, \tag{19}$$

$G_{n-i} \equiv G(x_{n-i}, y(x_{n-i}))$, $C_i^{(k)}(\lambda)$ is given by (8) and $r_k(G_n)$ is the error term in the corresponding backward Newton interpolation formula for $G(x,y) = G(x,y(x))$ at equidistant nodes $x_{n-i} = x_n - ih$, $i = 0,1,\ldots,k-1$. Hence, the approximate form of (17) becomes

$$A(x_{n+1})w(x_{n+1})y(x_{n+1}) - A(x_n)w(x_n)y(x_n) = h\sum_{i=0}^{k-1}B_i^{(k)}(h,x_n)G_{n-i}, \tag{20}$$

where the coefficients $B_i^{(k)}(h,x_n)$ depend on h and x_n. As in the case of the standard Adams-Bashforth methods, by assuming that all previous values y_{n-i},

$i = 0, 1, \dots, k-1$, are exact, i.e., $y_{n-i} = y(x_{n-i})$, $i = 0, 1, \dots, k-1$, (20) gives our *weighted k-step method*

$$A(x_{n+1})w(x_{n+1})y_{n+1} - A(x_n)w(x_n)y_n = h\sum_{i=0}^{k-1} B_i^{(k)}(h, x_n)G_{n-i}, \quad n \geq k-1, \tag{21}$$

where $G_{n-i} \equiv G(x_{n-i}, y_{n-i})$, $i = 0, 1, \dots, k-1$.

The mentioned dependence of the coefficients $B_i^{(k)}(h, x_n)$ on the stepsize h and x_n makes these methods fundamentally different from the standard ones.

Similarly to (11), we can here define the corresponding *weighted local truncation error* at the point $x_{n+k} \in [a, b]$ as

$$\begin{aligned}(T_h^w)_{n+k} = \bigg\{&\frac{1}{h}\left[A(x_{n+k})w(x_{n+k})y(x_{n+k}) - A(x_{n+k-1})w(x_{n+k-1})y(x_{n+k-1})\right] \\ &- \sum_{j=0}^{k-1} B_{k-j-1}^{(k)}(h, x_{n+k-1})G(x_{n+j}, y(x_{n+j}))\bigg\}\frac{1}{A(x_{n+k})w(x_{n+k})},\end{aligned}$$

where $x \mapsto y(x)$ is the exact solution of the Cauchy problem (15).

Then, according to (17), with $n := n+k-1$, and using (18) and (20), we obtain

$$(T_h^w)_{n+k} = \frac{1}{A(x_{n+k})w(x_{n+k})h}\int_{x_{n+k-1}}^{x_{n+k}} w(x)r_k(G_{n+k-1})\mathrm{d}x.$$

The first term omitted in the summation on the right-hand side in (18) is a good approximation of the truncation error. We will call this quantity *the main term of the truncation error* and denote by $(\widehat{T}_h^w)_{n+k}$.

Proposition 1. *Let the exact solution of the singular Cauchy problem* (15) *be sufficiently smooth, as well as the function $x \mapsto g(x) = G(x, y(x))$. Then, the main term of the truncation error at the point x_{n+k} can be expressed in the form*

$$(\widehat{T}_h^w)_{n+k} = \frac{h^k g^{(k)}(\xi_k)}{A(x_{n+k})w(x_{n+k})}\int_0^1 \binom{\lambda+k-1}{k} w(x_{n+k-1}+\lambda h)\mathrm{d}\lambda, \tag{22}$$

where

$$g^{(k)}(x) = A(x)y^{(k+1)} + [B(x) + kA'(x)]\,y^{(k)} + k\Big[B'(x) + \frac{1}{2}(k-1)A''(x)\Big]y^{(k-1)} \tag{23}$$

and $x_{n-1} < \xi_k < x_{n+k-1}$.

Proof. According to (18), we have

$$(\widehat{T}_h^w)_{n+k} = \frac{\nabla^k G_{n+k-1}}{A(x_{n+k})w(x_{n+k})k!}\int_0^1 w(x_{n+k-1}+\lambda h)(\lambda)_k \mathrm{d}\lambda,$$

where the factor in front of the integral can be expressed in terms of divided differences as (cf. [7, p. 410])

$$\frac{\nabla^k g_{n+k-1}}{k!} = h^k\left[x_{n+k-1}, x_{n+k-2}, \dots, x_{n-1}\right]g.$$

On the other hand, supposing that the function $x \mapsto g(x) = G(x, y(x))$ is sufficiently smooth, we can write

$$[x_{n+k-1}, x_{n+k-2}, \ldots, x_{n-1}]\, g = \frac{g^{(k)}(\xi_k)}{k!},$$

where ξ_k is between the smallest and the largest of these points. In order to calculate these derivatives,

$$g'(x) = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}y', \quad g''(x) = \frac{\partial^2 G}{\partial x^2} + \left[2\frac{\partial^2 G}{\partial x \partial y} + \frac{\partial^2 G}{\partial y^2}y'\right]y' + \frac{\partial G}{\partial y}y'', \quad \text{etc.}$$

we use the relation $g(x) = G(x, y(x)) = A(x)y'(x) + B(x)y(x)$. Since $A(x)$ is a polynomial of degree at most two and $B(x)$ is a polynomial of first degree, these derivatives can be calculated much simpler for each $k \geq 0$ in the form (23).

In this way, we obtain (22). $\square$

Formally, (22) is of the same form as (12), i.e., $C_k g^{(k)}(\xi_k)h^k$, where

$$C_k = C_k(h, x_n) = \frac{1}{A(x_{n+k})w(x_{n+k})}\int_0^1 \binom{\lambda + k - 1}{k} w(x_{n+k-1} + \lambda h)\mathrm{d}\lambda$$

and $g^{(k)}$ is given by (23). In the case of standard Adams-Bashforth methods, C_k are the error constants given by (13) and they are independent of the stepsize h and x_n. Also, instead of $g^{(k)}$, there is only the derivative $y^{(k+1)}$ in (12). Because of these differences, the actual order of the weighted methods can be reduced (see examples in Section 4).

In the sequel, we consider the three cases given previously in Table 2.

3.1. Case $(a, b) = (-1, 1)$

Consider the Cauchy problem of Jacobi type

$$(1 - x^2)y' + (\beta - \alpha - (\alpha + \beta + 2)x)\, y = G(x, y), \qquad y(x_0) = y_0,$$

where $x_n = x_0 + nh \in (-1, 1)$.

In this case, relation (20) is reduced to

$$y_{n+1} = d(h, x_n)y_n + h\sum_{i=0}^{k-1} D_i^{(k)}(h, x_n)G_{n-i}, \tag{24}$$

where

$$d(h, x_n) = \frac{A(x_n)w(x_n)}{A(x_{n+1})w(x_{n+1})} = \frac{(1 - x_n)^{\alpha+1}(1 + x_n)^{\beta+1}}{(1 - x_{n+1})^{\alpha+1}(1 + x_{n+1})^{\beta+1}}$$

and

$$D_i^{(k)}(h, x_n) = \frac{1}{A(x_{n+1})w(x_{n+1})}\int_0^1 w(x_n + \lambda h)C_i^{(k)}(\lambda)d\lambda.$$

Putting

$$c(h, x_n) = \frac{(1 - x_n)^{\alpha}(1 + x_n)^{\beta}}{(1 - x_{n+1})^{\alpha+1}(1 + x_{n+1})^{\beta+1}}$$

and

$$\Phi_i^{(k)}(h, x_n) = \int_0^1 (1 - x_n - \lambda h)^{\alpha}(1 + x_n + \lambda h)^{\beta} C_i^{(k)}(\lambda)\mathrm{d}\lambda, \tag{25}$$

rule (24) can be simplified as

$$y_{n+1} = c(h, x_n)\left((1 - x_n^2)y_n + h\sum_{i=0}^{k-1} \Phi_i^{(k)}(h, x_n)G_{n-i}\right). \tag{26}$$

In the special (Legendre) case $\alpha = \beta = 0$, (25) is reduced to

$$\Phi_i^{(k)}(h, x_n) = \int_0^1 C_i^{(k)}(\lambda)d\lambda = B_i^{(k)}, \tag{27}$$

where $B_i^{(k)}$ are the same coefficients as for standard (non-weighted) Adams-Bashforth formulas given by (10).

3.2. Case $(a, b) = (0, \infty)$

Now, consider the Cauchy problem of Laguerre type

$$xy' + (\gamma + 1 - x)y = G(x, y), \quad y(x_0) = y_0,$$

in which $x_n = nh$ for $n = 0, 1, \ldots$, and the main relation (20) is reduced to the corresponding equation (24) with

$$d(h, x_n) = \frac{A(x_n)w(x_n)}{A(x_{n+1})w(x_{n+1})} = \mathrm{e}^h\left(\frac{x_n}{x_{n+1}}\right)^{\gamma+1},$$

and

$$D_i^{(k)}(h, x_n) = \frac{B_i^{(k)}(h, x_n)}{A(x_{n+1})w(x_{n+1})} = \frac{\mathrm{e}^h}{x_{n+1}^{\gamma+1}}\int_0^1 (x_n + \lambda h)^{\gamma}\mathrm{e}^{-\lambda h}C_i^{(k)}(\lambda)\mathrm{d}\lambda.$$

In other words, we have

$$D_i^{(k)}(h, x_n) = x_n^{-(\gamma+1)}d(h, x_n)\Phi_i^{(k)}(h, x_n),$$

such that

$$\Phi_i^{(k)}(h, x_n) = \int_0^1 (x_n + \lambda h)^{\gamma}\mathrm{e}^{-\lambda h}C_i^{(k)}(\lambda)\mathrm{d}\lambda. \tag{28}$$

Hence, the relation corresponding to (20) takes the form

$$y_{n+1} = \mathrm{e}^h\left(\frac{x_n}{x_{n+1}}\right)^{\gamma+1}y_n + \frac{\mathrm{e}^h h}{x_{n+1}^{\gamma+1}}\sum_{i=0}^{k-1}\Phi_i^{(k)}(h, x_n)G_{n-i}. \tag{29}$$

Note that when $\gamma = 0$, the coefficients (28) are independent of x_n and

$$\Phi_i^{(k)}(h, x_n) = \Phi_i^{(k)}(h) = \int_0^1 \mathrm{e}^{-\lambda h} C_i^{(k)}(\lambda)\mathrm{d}\lambda. \tag{30}$$

For instance, for $k = 1(1)5$ and $i = 0, 1, \dots, k-1$, relation (30) gives

$\underline{k = 1}$:

$$\Phi_0^{(1)}(h) = \frac{1 - \mathrm{e}^{-h}}{h};$$

$\underline{k = 2}$:

$$\Phi_0^{(2)}(h) = \frac{1 + h - \mathrm{e}^{-h}(1 + 2h)}{h^2},$$

$$\Phi_1^{(2)}(h) = -\frac{1 - \mathrm{e}^{-h}(1 + h)}{h^2};$$

$\underline{k = 3}$:

$$\Phi_0^{(3)}(h) = \frac{2 + 3h + 2h^2 - \mathrm{e}^{-h}(2 + 5h + 6h^2)}{2h^3},$$

$$\Phi_1^{(3)}(h) = -\frac{2(1 + h) - \mathrm{e}^{-h}(2 + 4h + 3h^2)}{h^3},$$

$$\Phi_2^{(3)}(h) = \frac{2 + h - \mathrm{e}^{-h}(2 + 3h + 2h^2)}{2h^3};$$

$\underline{k = 4}$:

$$\Phi_0^{(4)}(h) = \frac{6 + 12h + 11h^2 + 6h^3 - 2\mathrm{e}^{-h}(3 + 9h + 13h^2 + 12h^3)}{6h^4},$$

$$\Phi_1^{(4)}(h) = -\frac{2(3 + 5h + 3h^2) - \mathrm{e}^{-h}(6 + 16h + 19h^2 + 12h^3)}{2h^4},$$

$$\Phi_2^{(4)}(h) = \frac{6 + 8h + 3h^2 - 2\mathrm{e}^{-h}(3 + 7h + 7h^2 + 4h^3)}{2h^4},$$

$$\Phi_3^{(4)}(h) = -\frac{2(3 + 3h + h^2) - \mathrm{e}^{-h}(6 + 12h + 11h^2 + 6h^3)}{6h^4};$$

$\underline{k = 5}$:

$$\Phi_0^{(5)}(h) = \frac{12 + 30h + 35h^2 + 25h^3 + 12h^4 - \mathrm{e}^{-h}(12 + 42h + 71h^2 + 77h^3 + 60h^4)}{12h^5},$$

$$\Phi_1^{(5)}(h) = -\frac{2(12 + 27h + 26h^2 + 12h^3) - \mathrm{e}^{-h}(24 + 78h + 118h^2 + 107h^3 + 60h^4)}{6h^5},$$

$$\Phi_2^{(5)}(h) = \frac{12 + 24h + 19h^2 + 6h^3 - \mathrm{e}^{-h}(12 + 36h + 49h^2 + 39h^3 + 20h^4)}{2h^5},$$

$$\Phi_3^{(5)}(h) = -\frac{2(12 + 21h + 14h^2 + 4h^3) - \mathrm{e}^{-h}(24 + 66h + 82h^2 + 61h^3 + 30h^4)}{6h^5},$$

$$\Phi_4^{(5)}(h) = \frac{12 + 18h + 11h^2 + 3h^3 - \mathrm{e}^{-h}(12 + 30h + 35h^2 + 25h^3 + 12h^4)}{12h^5}.$$

Remark 2. According to (30), it is clear that $\lim\limits_{h\to 0}\Phi_i^{(k)}(h)=B_i^{(k)}$, $i=0,1,\ldots,k-1$, where $B_i^{(k)}$ are the coefficients of standard (non-weighted) Adams-Bashforth formulas given by (10).

Remark 3. Consider the Cauchy-Laguerre problem $xy'+(1-x)y=y$ (i.e., $G(x,y)=y$), which is simplified as

$$y'=y,\quad y(0)=1,$$

with the exact solution $y=\mathrm{e}^x$. Considering the simplest method (for $k=1$) gives

$$y_{n+1}=\frac{\mathrm{e}^h}{1+\frac{h}{x_n}}\left(y_n+\frac{h}{x_n}\Phi_0^{(1)}(h)G_n\right)=\frac{\mathrm{e}^h}{x_{n+1}}\left(x_ny_n+(1-\mathrm{e}^{-h})G(x_n,y_n)\right).$$

Now, substituting $x_n=nh$ in the above relation yields

$$y_{n+1}=\frac{(1+nh)\mathrm{e}^h-1}{(n+1)h}y_n,\quad \text{with}\quad y_0=1.$$

For example, we have

$$y_1=\frac{\mathrm{e}^h-1}{h},\quad y_2=\frac{(1+h)\mathrm{e}^h-1}{2h}\cdot\frac{\mathrm{e}^h-1}{h},$$
$$y_3=\frac{(1+2h)\mathrm{e}^h-1}{3h}\cdot\frac{(1+h)\mathrm{e}^h-1}{2h}\cdot\frac{\mathrm{e}^h-1}{h},$$

or, in general,

$$y_n=\prod_{\nu=1}^{n}\frac{[1+(\nu-1)h]\mathrm{e}^h-1}{\nu h}.$$

The method is convergent, i.e.,

$$\lim_{\substack{n\to+\infty\\(nh=x=\text{const})}} y_n=\lim_{n\to+\infty}\prod_{\nu=1}^{n}\frac{\left[1+(\nu-1)\dfrac{x}{n}\right]\mathrm{e}^{x/n}-1}{\dfrac{\nu x}{n}}=\mathrm{e}^x. \tag{31}$$

In order to prove (31) we define a sequence $\{a_n\}_{n\in\mathbb{N}}$ by

$$a_n=\sum_{\nu=1}^{n}\log\left\{\frac{\left[1+(\nu-1)\dfrac{x}{n}\right]\mathrm{e}^{x/n}-1}{\dfrac{\nu x}{n}}\right\}^n$$

and apply the well known Stolz-Cesàro theorem. Namely, if we prove the convergence

$$\lim_{n\to+\infty}\frac{a_n-a_{n-1}}{n-(n-1)}=\lim_{n\to+\infty}\log\left\{\frac{\left[1+\left(1-\dfrac{1}{n}\right)x\right]\mathrm{e}^{x/n}-1}{x}\right\}^n=L,$$

then the limit $\lim_{n\to+\infty} \frac{a_n}{n}$ also exists and it is equal to L. Since

$$\frac{1}{x}\left\{\left[1+\left(1-\frac{1}{n}\right)x\right]\mathrm{e}^{x/n}-1\right\}=1+\frac{x}{n}+O(n^{-2}),$$

we conclude that $L=\log \mathrm{e}^x = x$. Therefore, we obtain (31), because

$$\begin{aligned}\lim_{n\to+\infty}\frac{a_n}{n} &= \lim_{n\to+\infty}\sum_{\nu=1}^{n}\log\left\{\frac{\left[1+(\nu-1)\dfrac{x}{n}\right]\mathrm{e}^{x/n}-1}{\dfrac{\nu x}{n}}\right\}\\ &= \lim_{n\to+\infty}\log\left\{\prod_{\nu=1}^{n}\frac{\left[1+(\nu-1)\dfrac{x}{n}\right]\mathrm{e}^{x/n}-1}{\dfrac{\nu x}{n}}\right\}=x.\end{aligned}$$

3.3. Case $(a,b)=(-\infty,\infty)$

Now, consider the Cauchy problem of Hermite type

$$y'-2xy=G(x,y),\quad y(x_0)=y_0,$$

in which $x_n = x_0 + nh \in (-\infty,\infty)$ and the main relation (20) is reduced to the corresponding equation (24) with

$$d(h,x_n)=\frac{\mathrm{e}^{-x_n^2}}{\mathrm{e}^{-x_{n+1}^2}}=\mathrm{e}^{x_{n+1}^2-x_n^2}=\mathrm{e}^{h(2x_n+h)},$$

and

$$D_i^{(k)}(h,x_n)=\frac{B_i^{(k)}(h,x_n)}{\mathrm{e}^{-x_{n+1}^2}}=\mathrm{e}^{x_{n+1}^2}\int_0^1 \mathrm{e}^{-(x_n+\lambda h)^2}C_i^{(k)}(\lambda)\mathrm{d}\lambda.$$

In other words, we have $D_i^{(k)}(h,x_n)=d(h,x_n)\Phi_i^{(k)}(h,x_n)$, such that

$$\Phi_i^{(k)}(h,x_n)=\int_0^1 \mathrm{e}^{-(2x_n\lambda h+\lambda^2h^2)}C_i^{(k)}(\lambda)\mathrm{d}\lambda.$$

Hence, the relation corresponding to (20) takes the form

$$y_{n+1}=\mathrm{e}^{h(2x_n+h)}\left(y_n+\sum_{i=0}^{k-1}\Phi_i^{(k)}(h,x_n)G_{n-i}\right). \tag{32}$$

Remark 4. As in the case of non-weighted methods, in applications of these methods for $k>1$, we need the additional starting values $y_i = s_i(h)$, $i=1,\dots,k-1$, such that $\lim_{h\to0} = y_0$ (cf. [11, pp. 32–36]).

4. Numerical examples

In order to illustrate the efficiency of our method, in this section we give two numerical examples for singular Cauchy problems on $(0,\infty)$ and $(-1,1)$. In particular, the weighted Adams-Bashforth methods with respect to the standard Laguerre weight given in Subsection 3.2 have the simplest form and they can find adequate application in solving weighted singular Cauchy problems. The third case when $(a,b)=(-\infty,\infty)$ is not interesting for applications because equation (15) is not singular in a finite domain.

Example 1. We first consider a singular Cauchy problem

$$(1-x^2)y' - 3xy = \frac{y^2\left(\left(1-x^2\right)\tan(x)+4x+1\right)\sec(x)}{x-1}, \quad y(-1)=2\cos 1.$$

Here,

$$G(x,y) = \frac{y^2\left(\left(1-x^2\right)\tan(x)+4x+1\right)\sec(x)}{x-1} + xy$$

and the exact solution of this problem is given by $y(x)=(1-x)\cos x$.

In order to solve the problem for $x\in[-1,0]$, we apply the k-step method (26), with $\alpha=\beta=0$ (Legendre case) and $\Phi_i^{(k)}(h,x_n)$ given by (27). For the sake of simplicity, in the case $k>1$, for starting values we use the exact values. Otherwise, some other ways must be applied (see Remark 4).

Relative errors obtained by this k-step method when $k=1,2,\ldots,5$, for $h=0.02$ and $h=0.01$, are displayed in log-scale in Figures 1 and 2, respectively.

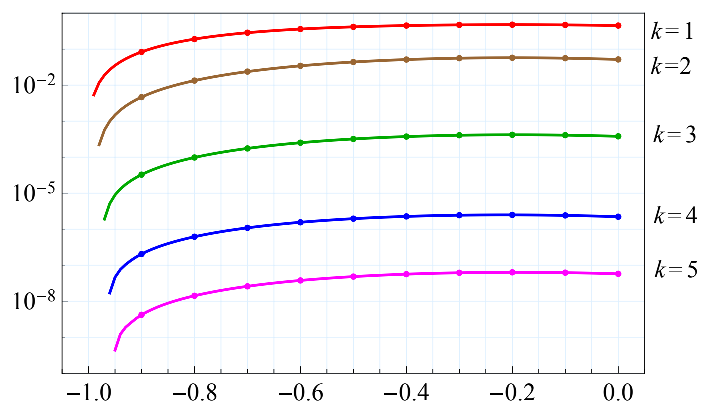

Figure 1: *Relative errors for methods (26) in Example 1 for $h=0.02$ and $k=1,2,\ldots,5$*

We consider now the actual errors, $|y_{n+k}-y(x_{n+k})|$ at a fixed point $x=x_{n+k}$ obtained by using the k-step method (26) for different stepsize h and different k. We take $x=-0.5$ and $h=0.05$, 0.02 and 0.01. The corresponding errors are presented in Table 3. Numbers in parentheses indicate decimal exponents.

Note that the effect of reducing the step-size h to the accuracy of the solution is greater if k is higher. Assuming an asymptotic relation in the form

$$e(h,k,x) = |y_{n+k}^h - y(x_{n+k})| \approx C_k h^{\alpha_k}, \tag{33}$$

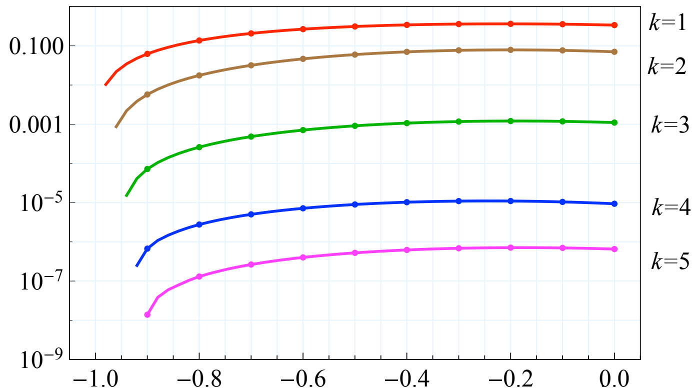

Figure 2: *Relative errors for methods (26) in Example 1 for* $h = 0.01$ *and* $k = 1, 2, \ldots, 5$

h	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
0.05	2.27(−1)	1.14(−1)	4.76(−3)	6.68(−5)	1.45(−5)
0.02	4.10(−1)	7.87(−2)	1.20(−3)	1.18(−5)	6.89(−7)
0.01	5.46(−1)	5.77(−2)	4.21(−4)	2.57(−6)	6.34(−8)

Table 3: *Absolute errors in the obtained sequences* $\{y_{n+k}\}_n$ *at the point* $x = x_{n+k} = -0.5$*, using the* k*-step method (26) for* $k = 1, 2, \ldots, 5$ *and* $h = 0.05$, 0.02 *and* 0.01

where $x_{n+k} = -1 + (n+k)h = x = \text{const}$, and C_k and α_k are some constants, we can calculate the following quotient for two different steps h_1 and h_2,

$$\frac{e(h_1, k, x)}{e(h_2, k, x)} \approx \left(\frac{h_1}{h_2}\right)^{\alpha_k}.$$

Therefore,

$$\alpha_k = \frac{\log(e(h_1, k, x)/e(h_2, k, x)}{\log(h_1/h_2)}. \tag{34}$$

These values are presented in Table 4 for $h_1/h_2 = 2$ and $h_1/h_2 = 5$. As we can see, the obtained values for the exponents α_k are very similar in these two cases.

h_1/h_2	α_1	α_2	α_3	α_4	α_5
2	−0.41	0.45	1.51	2.20	3.44
5	−0.54	0.42	1.51	2.02	3.38

Table 4: *The parameters* α_k *obtained from (34) for* $k = 1, 2, \ldots, 5$ *and different stepsizes*

As we can see, the actual order of the method is reduced approximately for one order of magnitude. This effect is mentioned in Section 3 after Proposition 1.

Example 2. Here we consider again the Cauchy problem of Jacobi type

$$(1 - x^2)y' - 2xy = 1 - x - 4x^2 - 5x^3 + xy, \quad y(-1) = 1,$$

whose exact solution is $y(x) = x^2 + x + 1$. According to Proposition 1, in this simple case, we have

$$g(x) = G(x, y(x)) = -4x^3 - 3x^2 + 1,$$

as well as

$$g'(x) = -12x^2 - 6x, \quad g''(x) = -24x - 6, \quad g'''(x) = -24, \quad g^{(iv)}(x) = 0.$$

Now, we apply the k-step method (26) for $k = 1(1)4$, $h = 0.05$ and $h = 0.01$ (see Tables 5 and 6), and for starting values (when $k > 1$) we use the exact values of the solution. In these tables, we only give the relative errors of the obtained values for $x = -0.9, -0.8, -0.7, -0.6, -0.5$ (m.p. is machine precision).

x	$k=1$	$k=2$	$k=3$	$k=4$
-0.9	5.62(−2)	5.09(−3)		
-0.8	4.88(−2)	6.37(−3)	3.17(−4)	m.p.
-0.7	4.37(−2)	7.03(−3)	4.46(−4)	m.p.
-0.6	3.80(−2)	7.29(−3)	5.62(−4)	m.p.
-0.5	3.18(−2)	7.23(−3)	6.42(−4)	m.p.

Table 5: *Relative errors in the obtained approximate sequences* $\{y_{n+k}\}_n$ *using* k*-step methods* (26) *for* $k = 1(1)4$ *and* $h = 0.05$

x	$k=1$	$k=2$	$k=3$	$k=4$
-0.9	9.81(−3)	2.61(−4)	3.19(−6)	m.p.
-0.8	9.19(−3)	2.82(−4)	3.92(−6)	m.p.
-0.7	8.37(−3)	2.94(−4)	4.53(−6)	m.p.
-0.6	7.33(−3)	2.96(−4)	5.11(−6)	m.p.
-0.5	6.15(−3)	2.89(−4)	5.62(−6)	m.p.

Table 6: *Relative errors in the obtained approximate sequences* $\{y_{n+k}\}_n$ *using* k*-step methods* (26) *for* $k = 1(1)4$ *and* $h = 0.01$

As in Example 1, we consider asymptotic relation (33) at $x = x_{n+k} = -0.5$, when $k = 1, 2, 3$ and $h_1 = 0.05$ and $h_2 = 0.01$. The values of the corresponding exponent (34) are presented in Table 7.

k	$k=1$	$k=2$	$k=3$
$h_1/h_2 = 5$	$\alpha_1 = 1.00$	$\alpha_2 = 2.00$	$\alpha_3 = 2.94$

Table 7: *The parameters* α_k *obtained from (34) for* $k = 1, 2, 3$ *and* $h_1/h_2 = 5$

As we can see, in this polynomial case, there is not previously mentioned defect in the order. Note that the local truncation error (22) is equal to zero for each $k \geq 4$, because of $g^{(k)}(x) = 0$. Also, we see that the actual errors in Tables 5 and 6 for $k = 4$ are on the level of machine precision.

Example 3. Now we consider the Cauchy problem of Laguerre type

$$xy' + (1-x)y = \frac{3x^2+1}{(x^2+1)^2}\mathrm{e}^{-x}y^2, \quad y(0)=1,$$

whose exact solution is $y = (x^2+1)\mathrm{e}^x$. We apply the k-step method (29) for $k = 1(1)6$. The corresponding relative errors for $h = 0.05$ are given in Table 8, and for $h = 0.01$ in Table 9. In these tables, we only give relative errors of the obtained values for $x = 0(0.1)1$. As in Example 1, for starting values we use the exact values of solution.

x	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
0.1	5.66(−2)	3.74(−3)				
0.2	1.18(−1)	1.43(−2)	7.17(−4)	1.89(−5)		
0.3	1.81(−1)	2.62(−2)	1.62(−3)	7.55(−5)	2.89(−6)	6.33(−8)
0.4	2.45(−1)	3.94(−2)	2.62(−3)	1.32(−4)	5.83(−6)	2.39(−7)
0.5	3.08(−1)	5.42(−2)	3.76(−3)	1.98(−4)	9.39(−6)	4.16(−7)
0.6	3.70(−1)	7.10(−2)	5.10(−3)	2.74(−4)	1.33(−5)	6.11(−7)
0.7	4.30(−1)	9.01(−2)	6.66(−3)	3.63(−4)	1.79(−5)	8.48(−7)
0.8	4.88(−1)	1.11(−1)	8.49(−3)	4.67(−4)	2.33(−5)	1.11(−6)
0.9	5.42(−1)	1.35(−1)	1.06(−2)	5.88(−4)	2.95(−5)	1.42(−6)
1.0	5.93(−1)	1.62(−1)	1.31(−2)	7.29(−4)	3.67(−5)	1.77(−6)

Table 8: *Relative errors in the obtained sequences* $\{y_{n+k}\}_n$ *using* k*-step methods* (29) *for* $k = 1, 2, \ldots, 6$ *and* $h = 0.05$

x	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
0.1	5.26(−2)	1.86(−3)	2.63(−5)	2.92(−7)	2.97(−9)	2.85(−11)
0.2	1.05(−1)	4.17(−3)	6.20(−5)	7.37(−7)	8.17(−9)	8.70(−11)
0.3	1.58(−1)	6.73(−3)	1.02(−4)	1.23(−6)	1.39(−8)	1.51(−10)
0.4	2.12(−1)	9.66(−3)	1.47(−4)	1.80(−6)	2.04(−8)	2.24(−10)
0.5	2.67(−1)	1.31(−2)	2.00(−4)	2.45(−6)	2.80(−8)	3.09(−10)
0.6	3.23(−1)	1.71(−2)	2.63(−4)	3.23(−6)	3.70(−8)	4.10(−10)
0.7	3.80(−1)	2.18(−2)	3.38(−4)	4.16(−6)	4.76(−8)	5.28(−10)
0.8	4.35(−1)	2.73(−2)	4.26(−4)	5.25(−6)	6.02(−8)	6.67(−10)
0.9	4.89(−1)	3.38(−2)	5.30(−4)	6.53(−6)	7.49(−8)	8.32(−10)
1.0	5.41(−1)	4.12(−2)	6.51(−4)	8.03(−6)	9.22(−8)	1.02(−9)

Table 9: *Relative errors in the obtained sequences* $\{y_{n+k}\}_n$ *using* k*-step methods* (29) *for* $k = 1, 2, \ldots, 6$ *and* $h = 0.01$

Using Proposition 1 we determine the main term of the truncation error at the point $x_{n+k} = x$, for example, when $h = 0.01$ and $x = 0.5$.

Since

$$\int_0^1 \binom{\lambda+k-1}{k}\mathrm{e}^{-(x-h+\lambda h)}\mathrm{d}\lambda = \mathrm{e}^{-x}Q_k, \quad Q_k = \frac{1}{k!}\int_0^1 (\lambda)_k \mathrm{e}^{-(\lambda-1)h}\mathrm{d}\lambda,$$

we first calculate the values: $Q_1 = 0.501671$, $Q_2 = 0.41792$, $Q_3 = 0.376058$, $Q_4 = 0.34955$, $Q_5 = 0.330639$, and $Q_6 = 0.316389$. The corresponding derivatives

$$g^{(k)}(x) = xy^{(k+1)} + (k+1-x)y^{(k)} - ky^{(k-1)}$$

are

$$g'(x) = \mathrm{e}^x\left(3x^2+6x+1\right), \qquad g''(x) = \mathrm{e}^x\left(3x^2+12x+7\right),$$
$$g'''(x) = \mathrm{e}^x\left(3x^2+18x+19\right), \qquad g^{(iv)}(x) = \mathrm{e}^x\left(3x^2+24x+37\right),$$
$$g^{(v)}(x) = \mathrm{e}^x\left(3x^2+30x+61\right), \qquad g^{(vi)}(x) = \mathrm{e}^x\left(3x^2+36x+91\right),$$

where $\xi_k \in \big(x-(k+1)h, x-h\big)$, $k = 1,2,\ldots,6$.

Now, taking $x = x_{n+k} = (n+k)h = 0.5$, $h = 0.01$, and $\xi_k = x - h = 0.49$ in (22), we obtain an approximation of the main term $(\widehat{T}_h^w)_{n+k}$ in the form

$$(\widehat{T}_h^w)_{n+k} \approx \frac{h^k g^{(k)}(x-h)Q_k}{x}, \qquad k = 1,2,\ldots,6,$$

whose numerical values for $k = 1,2,\ldots,6$, after dividing by $y(0.5) = 2.0609$, are: 3.70(−2), 9.00(−4), 1.70(−5), 2.74(−7), 4.00(−9), 5.48(−11), respectively. As expected, the actual global errors from Table 9 (the row referring to $x = 0.5$) are larger compared to the corresponding local truncation errors.

x	α_1	α_2	α_3	α_4	α_5	α_6
0.5	0.08	0.95	1.92	2.87	3.81	4.75
1.0	0.05	0.93	1.94	2.90	3.86	4.80

Table 10: *The parameters α_k obtained from (34) for $k = 1,2,\ldots,6$ and $h1/h2 = 2$ at two points $x = 0.5$ and $x = 1.0$.*

Finally, as in Example 1, we assume the behavior of the actual errors in the form (33), where $x_{n+k} = (n+k)h = x = \text{const}$. We compare actual errors obtained for $h_1 = 0.02$ and $h_2 = 0.01$ at two points $x = 0.5$ and $x = 1.0$. The results for α_k, $k = 0,1,\ldots,6$ are presented in Table 10. As we can see, the obtained values of the exponents α_k at these points are very close, but again with a defect of one order in its magnitude!

Acknowledgements

The authors are deeply grateful to the anonymous referee for his/her comments and constructive suggestions for improvements of this paper.

The work of M. Masjed-Jamei has been supported by the Alexander von Humboldt Foundation under the grant number: Ref 3.4–IRN–1128637–GF–E.

The work of G.V. Milovanović was supported in part by the Serbian Academy of Sciences and Arts (No. Φ-96) and by the Serbian Ministry of Education, Science and Technological Development (No. #OI 174015).

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