MATHEMATICAL COMMUNICATIONS
Math. Commun. **23**(2018), 43–60

Box-counting dimension of solution curves for a class of two-dimensional nonautonomous linear differential systems*

Masakazu Onitsuka and Satoshi Tanaka[†]

Department of Applied Mathematics, Faculty of Science, Okayama University of Science, Ridaichou 1–1, Okayama 700–0005, Japan

Received March 14, 2017; accepted June 12, 2017

Abstract. The two-dimensional linear differential system

$$x' = y, \quad y' = -x - h(t)y$$

is considered on $[t_0, \infty)$, where $h \in C^1[t_0, \infty)$ and $h(t) > 0$ for $t \geq t_0$. The box-counting dimension of graphs of solution curves is calculated. Criteria to obtain the box-counting dimension of spirals are also established.
AMS subject classifications: 34A30, 37C45, 28A80
Key words: linear system, box-counting dimension, spiral

1. Introduction

In this paper, we consider the following two-dimensional linear differential system

$$\begin{aligned} x' &= y, \\ y' &= -x - h(t)y \end{aligned} \tag{1}$$

for $t \geq t_0$, where $h \in C^1[t_0, \infty)$ and $h(t) > 0$ for $t \geq t_0$. This system has the *zero solution* $(x(t), y(t)) \equiv (0, 0)$. Setting $y = x'$, we can rewrite (1) as the damped linear oscillator

$$x'' + h(t)x' + x = 0, \quad t \geq t_0.$$

By a general theory (for example [1, 4]), there exists a unique solution of (1) on $[t_0, \infty)$ with the initial condition $x(t_1) = \alpha$ and $y(t_1) = \beta$ for every α, $\beta \in \mathbf{R}$ and $t_1 \geq t_0$. Hence, we note that every nontrivial solution $(x(t), y(t))$ satisfies $(x(t), y(t)) \neq (0, 0)$ for $t \geq t_0$.

The zero solution $(x(t), y(t)) \equiv (0, 0)$ of (1) is said to be *attractive* if every solution $(x(t), y(t))$ of (1) satisfies $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$. There are a lot of studies of the attractivity to (1) (see, for example, [2, 11, 12, 20, 21]).

Now, we assume that the zero solution of (1) is attractive. Let $(x(t), y(t))$ be a solution of (1). We define the solution curve of $(x(t), y(t))$ on $[t_1, \infty)$ in $\mathbf{R}^2$ by

$$\Gamma_{(x,y;t_1)} = \{(x(t), y(t)) : t \geq t_1\}$$

*This work was supported by JSPS KAKENHI Grant Number 26400182.
[†]Corresponding author. *Email addresses:* `onitsuka@xmath.ous.ac.jp` (M. Onitsuka), `tanaka@xmath.ous.ac.jp` (S. Tanaka)

for each fixed $t_1 \geq t_0$. A curve $\Gamma_{(x,y;t_1)}$ is said to be *simple* if $(x(t), y(t)) \neq (x(s), y(s))$ for t, $s \in [t_1, \infty)$ with $t \neq s$. A simple solution curve $\Gamma_{(x,y;t_1)}$ is said to be *rectifiable* if the length of $\Gamma_{(x,y;t_1)}$ is finite, that is,

$$\int_{t_1}^{\infty} \sqrt{|x'(t)|^2 + |y'(t)|^2}dt < \infty.$$

Otherwise, it is said to be *non-rectifiable*, that is,

$$\int_{t_1}^{\infty} \sqrt{|x'(t)|^2 + |y'(t)|^2}dt = \infty.$$

The rectifiability of solutions to two-dimensional linear differential systems was studied by Miličić and Pašić [8] and Naito and Pašić [9]. Naito, Pašić and Tanaka [10] obtained rectifiable and non-rectifiable results of solutions to half-linear differential systems. Recently, the following Theorem A has been established in [13]. In what follows, the following notation will be used:

$$H(t) = \int_{t_0}^{t} h(s)ds.$$

Theorem A. *Let $h \in C^1[t_0, \infty)$ satisfy $h(t) > 0$ for $t \geq t_0$. Assume that the following conditions* (2) *and* (3) *are satisfied*:

$$\int_{t_0}^{\infty} h(t)dt = \infty; \tag{2}$$

$$\int_{t_0}^{\infty} |2h'(t) + |h(t)|^2|dt < \infty. \tag{3}$$

Then, the zero solution of (1) *is attractive and every nontrivial solution $(x(t), y(t))$ of* (1) *is a spiral, rotating in a clockwise direction for all sufficiently large $t \geq t_0$, and its solution curve $\Gamma_{(x,y;t_0)}$ is simple. Moreover, the following properties (i) and (ii) hold:*

(i) every nontrivial solution of (1) *is rectifiable if*

$$\int_{t_0}^{\infty} e^{-H(t)/2}dt < \infty;$$

(ii) every nontrivial solution of (1) *is non-rectifiable if*

$$\int_{t_0}^{\infty} e^{-H(t)/2}dt = \infty.$$

In the above theorem, we adopt the definition of a spiral, according to a celebrated book by Hartman [4, Chapters VII and VIII] as follows. For every nontrivial solution $(x(t), y(t))$ of (1), we introduce polar coordinates

$$x(t) = r(t)\cos\theta(t), \quad y(t) = r(t)\sin\theta(t),$$

where the amplitude $r(t) > 0$. A nontrivial solution $(x(t), y(t))$ of (1) is said to be a *spiral* if $|\theta(t)| \to \infty$ as $t \to \infty$.

In this paper, we obtain the box-counting dimension of the solution curve $\Gamma_{(x,y;t_1)}$ for a nontrivial solution $(x(t), y(t))$ of (1). For a bounded subset Γ of $\mathbf{R}^2$, we define the *box-counting dimension* (*Minkowski-Bouligand dimension*) of Γ by

$$\dim_{\mathrm{B}} \Gamma = 2 - \lim_{\varepsilon \to +0} \frac{\log |\Gamma_\varepsilon|}{\log \varepsilon},$$

where Γ_ε denotes the ε-neighborhood of Γ defined by

$$\Gamma_\varepsilon = \{(x,y) \in \mathbf{R}^2 : d((x,y),\Gamma) \leq \varepsilon\}, \tag{4}$$

$d((x,y),\Gamma)$ denotes the Euclidean distance from (x,y) to Γ, and $|\Gamma_\varepsilon|$ denotes the two-dimensional Lebesgue measure of Γ_ε. More details on the definition of the box-counting dimension can be found in Falconer [3] and Tricot [22]. If there exist $d \in [0,2]$, $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 \varepsilon^{2-d} \leq |\Gamma_\varepsilon| \leq c_2 \varepsilon^{2-d}$$

for each sufficiently small $\varepsilon > 0$, then $\dim_{\mathrm{B}} \Gamma = d$.

The following result has been established in Tricot [22, §9.1, Theorem].

Proposition 1. *Let Γ be a simple curve of finite length. Then*

$$\lim_{\varepsilon \to +0} \frac{|\Gamma_\varepsilon|}{2\varepsilon} = \mathrm{length}(\Gamma),$$

where $\mathrm{length}(\Gamma)$ *denotes the length of* Γ.

Therefore, if $\mathrm{length}(\Gamma) < \infty$, then $\dim_{\mathrm{B}} \Gamma = 1$.

The box-counting dimension of graphs of solutions to the nonautonomous differential equation was first obtained by Pašić [14]. Thereafter, it was obtained about the nonautonomous second order linear differential equations in [7, 15, 16, 17]. On the other hand, the box-counting dimensions of solution curves to autonomous two-dimensional nonlinear differential systems were established in [18, 19, 23, 24]. Recently, Korkut, Vlah and Županović [6] have considered the equation

$$t^2 x'' + t(2-\mu)x' + (t^2 - \nu^2)x = 0, \tag{5}$$

where μ, $\nu \in \mathbf{R}$, and defined generalized Bessel functions $\widetilde{J}_{\nu,\mu}$ and $\widetilde{Y}_{\nu,\mu}$ by two linearly independent solutions of (5). When $\mu = 1$, equation (5) is known as Bessel's differential equation and Bessel functions J_ν and Y_ν are its two linearly independent solutions. In [6], the relation

$$\widetilde{J}_{\nu,\mu}(t) = t^{\frac{\mu-1}{2}} J_{\widetilde{\nu}}(t), \quad \widetilde{Y}_{\nu,\mu}(t) = t^{\frac{\mu-1}{2}} Y_{\widetilde{\nu}}(t), \quad \widetilde{\nu} = \sqrt{\left(\frac{\mu-1}{2}\right)^2 + \nu^2}$$

is found, and the following result is established.

Theorem B (see [6]). *Let* $\mu \in (0,2)$, $\nu \in \mathbf{R}$ *and* $t_0 > 0$. *Let* $x(t) = \widetilde{J}_{\nu,\mu}(t)$ *or* $\widetilde{Y}_{\nu,\mu}(t)$. *Then the planar curve* $\Gamma = \{(x(t), x'(t)) : t \ge t_0\}$ *satisfies* $\dim_{\mathrm{B}} \Gamma = 4/(4-\mu)$.

It is worth noting that if $x(t) = \widetilde{J}_{\nu,\mu}(t)$ or $\widetilde{Y}_{\nu,\mu}(t)$, then $(x(t), y(t)) := (x(t), x'(t))$ is a solution of the linear differential system

$$\begin{aligned} x' &= y, \\ y' &= -\left(1 - \frac{\nu^2}{t^2}\right)x - \frac{2-\mu}{t}y. \end{aligned} \tag{6}$$

The following two results are the main results of this paper.

Theorem 1. *Let* $h \in C^1[t_0, \infty)$ *satisfy* $h(t) > 0$ *for* $t \ge t_0$. *Assume that* (3) *and the following conditions are satisfied*:

$$\limsup_{t\to\infty} t h(t) < \infty; \tag{7}$$

$$H(t) = 2\alpha \log t + O(1) \quad \textit{as } t \to \infty \quad \textit{for some } \alpha \in (0,1). \tag{8}$$

Then, for every nontrivial solution $(x(t), y(t))$ *of* (1), *there exists* $t_1 \ge t_0$ *such that* $\dim_{\mathrm{B}} \Gamma_{(x,y;t_1)} = 2/(1+\alpha)$.

Here and hereafter, $f(t) = O(1)$ as $t \to \infty$ means that there exist $M > 0$ and t_1 such that $|f(t)| \le M$ for $t \ge t_1$.

Theorem 2. *Let* $h \in C^1[t_0, \infty)$ *satisfy* $h(t) > 0$ *for* $t \ge t_0$. *Assume that* (3) *and the following condition are satisfied*:

$$H(t) = 2 \log t + O(1) \quad \textit{as } t \to \infty. \tag{9}$$

Then, for every nontrivial solution $(x(t), y(t))$ *of* (1), *there exists* $t_1 \ge t_0$ *such that* $\dim_{\mathrm{B}} \Gamma_{(x,y;t_1)} = 1$.

Example 1. *We consider the case where* $h(t) = \lambda t^{-\gamma}$, $\lambda > 0$, $1/2 < \gamma \le 1$ *and* $t_0 = 1$. *It is easy to check that* (2) *and* (3) *are satisfied, and*

$$H(t) = \begin{cases} \dfrac{\lambda}{1-\gamma}(t^{1-\gamma} - 1), & \dfrac{1}{2} < \gamma < 1, \\ \lambda \log t, & \gamma = 1. \end{cases}$$

Theorem A implies that the zero solution of (1) *is attractive and every nontrivial solution* $(x(t), y(t))$ *of* (1) *is a spiral, rotating in a clockwise direction on* $[t_1, \infty)$ *for some* $t_1 \ge t_0$, *and its solution curve* $\Gamma_{(x,y;t_0)}$ *is simple and that every nontrivial solution of* (1) *is rectifiable when either* $1/2 < \gamma < 1$ *or* $\gamma = 1$ *and* $\lambda > 2$, *and every nontrivial solution of* (1) *is non-rectifiable when* $\gamma = 1$ *and* $0 < \lambda \le 2$. *Let* $(x(t), y(t))$ *be a nontrivial solution of* (1). *Therefore, by Proposition 1, if either* $1/2 < \gamma < 1$ *or* $\gamma = 1$ *and* $\lambda > 2$, *then* $\dim_{\mathrm{B}} \Gamma_{(x,y;t_1)} = 1$. *Moreover, Theorem 2 implies that* $\dim_{\mathrm{B}} \Gamma_{(x,y;t_2)} = 1$ *for some* $t_2 \ge t_1$ *when* $\gamma = 1$ *and* $\lambda = 2$. *Applying*

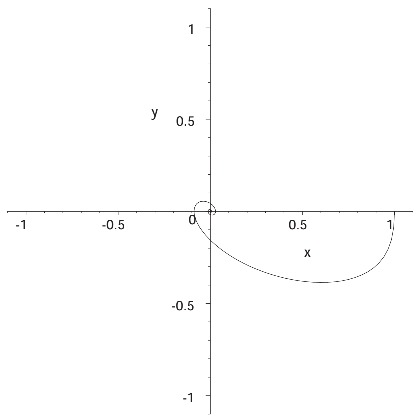
Theorem 1, we conclude that if $\gamma = 1$ *and* $0 < \lambda < 2$*, then there exists* $t_2 \geq t_1$ *such that* $\dim_{\mathrm{B}} \Gamma_{(x,y;t_2)} = 4/(2+\lambda)$.

Now, we set either

$$(x(t), y(t)) = (\widetilde{J}_{0,2-\lambda}(t), \widetilde{J}'_{0,2-\lambda}(t)) \quad \text{or} \quad (x(t), y(t)) = (\widetilde{Y}_{0,2-\lambda}(t), \widetilde{Y}'_{0,2-\lambda}(t)),$$

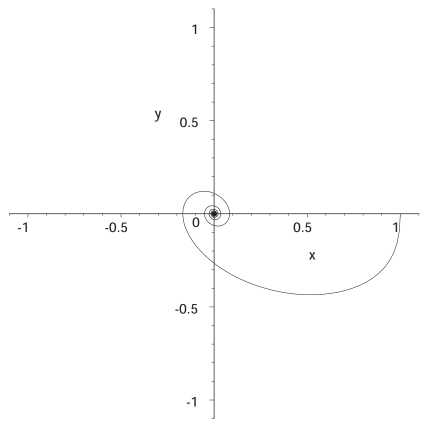
where $0 < \lambda < 2$*. Recalling that* $(\widetilde{J}_{\nu,\mu}(t), \widetilde{J}'_{\nu,\mu}(t))$ *and* $(\widetilde{Y}_{\nu,\mu}(t), \widetilde{Y}'_{\nu,\mu}(t))$ *are solutions of system* (6)*, we find that* $(x(t), y(t))$ *is a solution of* (1) *with* $h(t) = \lambda t^{-1}$.

Here, we give numerical simulations of solution curves.



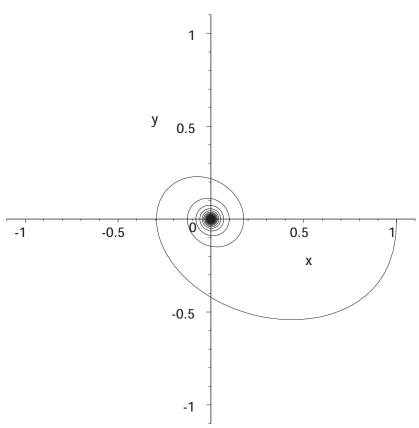
$h(t) = 3t^{-3/4}$

$\dim_{\mathrm{B}} \Gamma_{(x,y;t_1)} = 1$, rectifiable



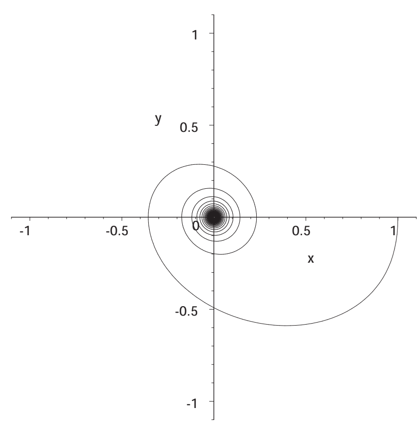
$h(t) = 3t^{-1}$

$\dim_{\mathrm{B}} \Gamma_{(x,y;t_1)} = 1$, rectifiable



$h(t) = 2t^{-1}$

$\dim_{\mathrm{B}} \Gamma_{(x,y;t_2)} = 1$, non-rectifiable



$h(t) = (5/3)t^{-1}$

$\dim_{\mathrm{B}} \Gamma_{(x,y;t_2)} = 12/11$, non-rectifiable

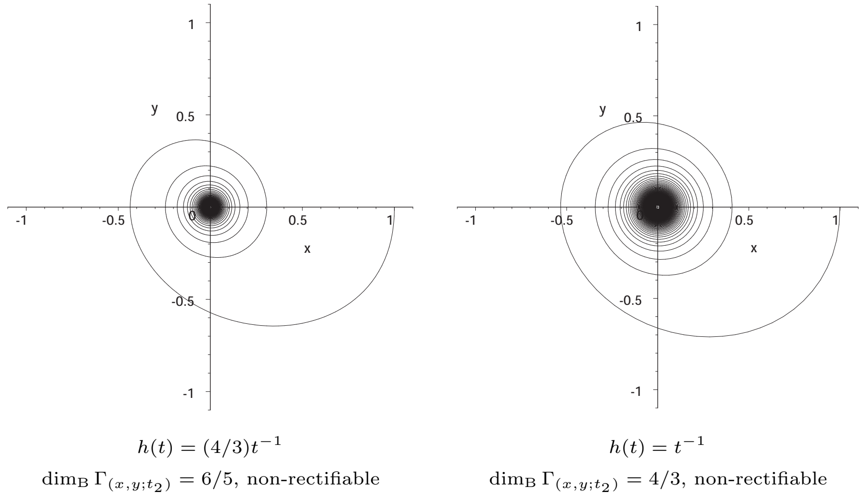


Figure 1: *Solution curves for the case where* $h(t) = \lambda t^{-\gamma}$

The box-counting dimension of the graph of the spiral $r = \varphi^{-\alpha}$, $\varphi \geq \varphi_1 > 0$ in polar coordinates is $2/(1+\alpha)$ when $0 < \alpha < 1$ (see, for example, Tricot [22, §10.4]). Žubrinić and Županović [23, Theorem 5] generalized this fact to the function $r = f(\varphi)$, $\varphi \geq \varphi_1$. Korkut, Vlah, Žubrinić and Županović [5, Therem 2] improved this result. See also Korkut, Vlah and Županović [6, Theorem 2]. In this paper, we give the following alternative criterion of the dimension of spirals.

Theorem 3. *Let* $\varphi_1 > 0$ *and let* $f \in C[\varphi_1, \infty)$ *satisfy* $\lim_{\varphi\to\infty} f(\varphi) = 0$. *Assume that there exist positive constants* $\underline{m}$, $\overline{a}$, M *and* $\alpha \in (0,1)$ *such that for all* $\varphi \geq \varphi_1$

$$\underline{m}\varphi^{-\alpha} \leq f(\varphi),$$
$$0 < f(\varphi) - f(\varphi + 2\pi) \leq \overline{a}\varphi^{-\alpha-1},$$
$$\mathrm{length}(\Gamma(\varphi_1, \varphi)) \leq M\varphi^{1-\alpha}.$$

Let Γ *be the graph of* $r = f(\varphi)$ *in polar coordinates, that is,*

$$\Gamma = \{(f(\varphi)\cos\varphi, f(\varphi)\sin\varphi) : \varphi \geq \varphi_1\}.$$

Then, $\dim_B \Gamma = 2/(1+\alpha)$.

From Theorem 3, we have the following Corollary.

Corollary 1. *Let* $\varphi_1 > 0$ *and let* $f \in C^1[\varphi_1, \infty)$ *satisfy* $\lim_{\varphi\to\infty} f(\varphi) = 0$. *Assume that there exist positive constants* $\underline{m}$, K *and* $\alpha \in (0,1)$ *such that for all* $\varphi \geq \varphi_1$

$$\underline{m}\varphi^{-\alpha} \leq f(\varphi),$$
$$-K\varphi^{-\alpha-1} \leq f'(\varphi) \leq 0.$$

Assume, moreover, that $f'(\varphi) \not\equiv 0$ *on* $[\varphi, \varphi + 2\pi)$ *for each fixed* $\varphi \geq \varphi_1$. *Let* $\Gamma = \{(f(\varphi)\cos\varphi, f(\varphi)\sin\varphi) : \varphi \geq \varphi_1\}$. *Then,* $\dim_B \Gamma = 2/(1+\alpha)$.

The proof of Corollary 1 will be given in Section 2. Using Corollary 1, we prove Theorem 1 in Section 4. Corollary 1 is similar to the criterion by Korkut, Vlah, Žubrinić and Županović [5, Therem 2]. The proof of Theorem 2 in [5] is based on the proof of Theorem 5 in [23]. Žubrinić and Županović employed the radial box dimension to prove Theorem 5 in [23]. On the other hand, the proof of Theorem 3, which will be given in Section 2, is more direct.

The box-counting dimension of the graph of the spiral $r = \varphi^{-1}$, $\varphi \geq \varphi_1 > 0$ in polar coordinates is 1 (see Tricot [22, §10.4]). We generalize this fact as follows.

Theorem 4. *Let $\varphi_1 > 1$ and let $f \in C[\varphi_1, \infty)$ satisfy $\lim_{\varphi\to\infty} f(\varphi) = 0$. Assume that there exist positive constants $\overline{m}$ and M such that for all $\varphi \geq \varphi_1$*

$$\begin{aligned} 0 &< f(\varphi) \leq \overline{m}\varphi^{-1}, \\ 0 &< f(\varphi) - f(\varphi + 2\pi), \\ \operatorname{length}(\Gamma(\varphi_1, \varphi)) &\leq M \log \varphi. \end{aligned}$$

Let $\Gamma = \{(f(\varphi)\cos\varphi, f(\varphi)\sin\varphi) : \varphi \geq \varphi_1\}$. Then, $\dim_{\mathrm{B}} \Gamma = 1$.

The following corollary follows from Theorem 4.

Corollary 2. *Let $\varphi_1 > 1$ and let $f \in C[\varphi_1, \infty)$ satisfy $\lim_{\varphi\to\infty} f(\varphi) = 0$. Assume that there exist positive constants $\overline{m}$ and K such that for all $\varphi \geq \varphi_1$*

$$\begin{aligned} 0 &< f(\varphi) \leq \overline{m}\varphi^{-1}, \\ -K\varphi^{-1} &\leq f'(\varphi) \leq 0. \end{aligned}$$

Assume, moreover, that $f'(\varphi) \not\equiv 0$ on $[\varphi, \varphi + 2\pi)$ for each fixed $\varphi \geq \varphi_1$. Let $\Gamma = \{(f(\varphi)\cos\varphi, f(\varphi)\sin\varphi) : \varphi \geq \varphi_1\}$. Then, $\dim_{\mathrm{B}} \Gamma = 1$.

The proofs of Theorem 4 and Corollary 2 will be given in Section 3.

2. Box-counting dimension of spirals

In this section, we prove Theorem 3 and Corollary 1. First, we give a lemma.

Lemma 1. *Let $\varphi_1 > 0$ and let $f \in C[\varphi_1, \infty)$ satisfy $f(\varphi) > 0$ for $\varphi \geq \varphi_1$ and $\lim_{\varphi\to\infty} f(\varphi) = 0$. Assume that there exist positive constants $\overline{a}$ and $\alpha \in (0, 1)$ such that*

$$0 < f(\varphi) - f(\varphi + 2\pi) \leq \overline{a}\varphi^{-\alpha-1}, \quad \varphi \geq \varphi_1.$$

Then, there exists a positive constant $\overline{m}$ such that $f(\varphi) \leq \overline{m}\varphi^{-\alpha}$ for $\varphi \geq \varphi_1$.

Proof. Let $\varphi \geq \varphi_1$. Then, there exist $N \in \mathbf{N} \cup \{0\}$ and $\varphi_0 \in [\varphi_1, \varphi_1 + 2\pi)$ such that $\varphi = \varphi_0 + 2N\pi$. Let $n \in \mathbf{N}$ with $n > N$. It follows that

$$\begin{aligned} f(\varphi) &= f(\varphi_0 + 2N\pi) \\ &= f(\varphi_0 + 2(n+1)\pi) + \sum_{k=N}^{n} [f(\varphi_0 + 2k\pi) - f(\varphi_0 + 2(k+1)\pi)] \\ &\leq f(\varphi_0 + 2(n+1)\pi) + \sum_{k=N}^{n} \overline{a}(\varphi_0 + 2k\pi)^{-\alpha-1}. \end{aligned}$$

Since

$$\frac{(\varphi_0+2k\pi)^{-\alpha-1}}{(\varphi_0+2(k+1)\pi)^{-\alpha-1}} = \left(\frac{\varphi_0+2(k+1)\pi}{\varphi_0+2k\pi}\right)^{\alpha+1} = \left(1+\frac{2\pi}{\varphi_0+2k\pi}\right)^{\alpha+1} \le \left(1+\frac{2\pi}{\varphi_1}\right)^{\alpha+1}, \quad k\in\mathbf{N}\cup\{0\},$$

we have

$$(\varphi_0+2k\pi)^{-\alpha-1} \le M_1(\varphi_0+2(k+1)\pi)^{-\alpha-1}, \quad k\in\mathbf{N}\cup\{0\},$$

where $M_1=[1+(2\pi/\varphi_1)]^{\alpha+1}$. Therefore,

$$\begin{aligned} f(\varphi) &\le f(\varphi_0+2(n+1)\pi)+\sum_{k=N}^{n}\overline{a}M_1(\varphi_0+2(k+1)\pi)^{-\alpha-1} \\ &= f(\varphi_0+2(n+1)\pi)+\overline{a}M_1\sum_{k=N}^{n}\int_k^{k+1}(\varphi_0+2(k+1)\pi)^{-\alpha-1}dt \\ &\le f(\varphi_0+2(n+1)\pi)+\overline{a}M_1\sum_{k=N}^{n}\int_k^{k+1}(\varphi_0+2\pi t)^{-\alpha-1}dt \\ &= f(\varphi_0+2(n+1)\pi)+\overline{a}M_1\int_N^{n+1}(\varphi_0+2\pi t)^{-\alpha-1}dt \\ &= f(\varphi_0+2(n+1)\pi)+\frac{\overline{a}M_1}{2\pi\alpha}\left[(\varphi_0+2N\pi)^{-\alpha}-(\varphi_0+2(n+1)\pi)^{-\alpha}\right]. \end{aligned}$$

Letting $n\to\infty$, we obtain

$$f(\varphi)\le\frac{\overline{a}M_1}{2\pi\alpha}(\varphi_0+2N\pi)^{-\alpha}=\frac{\overline{a}M_1}{2\pi\alpha}\varphi^{-\alpha}.$$

□

Hereafter, in this section, we assume all assumptions of Theorem 3. Then, by Lemma 1, there exists a positive constant $\overline{m}$ such that $f(\varphi)\le\overline{m}\varphi^{-\alpha}$ for $\varphi\ge\varphi_1$.

Let $\varepsilon\in(0,1)$ be sufficiently small. We use the following notation:

$$\begin{aligned} \varphi_2(\varepsilon) &= \left(\frac{2\overline{a}}{\varepsilon}\right)^{\frac{1}{\alpha+1}}; \\ \Gamma(\psi_1,\psi_2) &= \{(f(\varphi)\cos\varphi, f(\varphi)\sin\varphi) : \psi_1\le\varphi<\psi_2\}; \\ T(\Gamma,\varepsilon) &= \Gamma(\varphi_1,\varphi_2(\varepsilon))_\varepsilon; \\ N(\Gamma,\varepsilon) &= \Gamma(\varphi_2(\varepsilon),\infty)_\varepsilon, \end{aligned}$$

where Γ_ε denotes the ε-neighborhood of Γ defined by (4). Then, $\Gamma_\varepsilon=T(\Gamma,\varepsilon)\cup N(\Gamma,\varepsilon)$.

Lemma 2.

$$\{(r\cos\varphi, r\sin\varphi) : 0 \le r \le f(\varphi),\ \varphi \in [\varphi_2(\varepsilon), \varphi_2(\varepsilon)+2\pi)\} \subset N(\Gamma, \varepsilon).$$

Proof. Let

$$(x_0, y_0) \in \{(r\cos\varphi, r\sin\varphi) : 0 \le r \le f(\varphi),\ \varphi \in [\varphi_2(\varepsilon), \varphi_2(\varepsilon)+2\pi)\}.$$

Set $r_0 = \sqrt{x_0^2 + y_0^2}$. Then, there exists $\varphi_0 \ge \varphi_2(\varepsilon)$ such that

$$(x_0, y_0) = (r_0\cos\varphi_0, r_0\sin\varphi_0)$$

and

$$f(\varphi_0 + 2\pi) \le r_0 \le f(\varphi_0).$$

We have

$$0 \le f(\varphi_0) - r_0 \le f(\varphi_0) - f(\varphi_0 + 2\pi) \le \overline{a}\varphi_0^{-\alpha-1} \le \overline{a}(\varphi_2(\varepsilon))^{-\alpha-1} = \frac{\varepsilon}{2}.$$

Therefore,

$$d((x_0, y_0), (f(\varphi_0)\cos\varphi_0, f(\varphi_0)\sin\varphi_0)) = f(\varphi_0) - r_0 < \varepsilon,$$

which means that $(x_0, y_0) \in N(\Gamma, \varepsilon)$. □

Lemma 3.

$$\pi\underline{m}^2\left[(2\overline{a})^{\frac{1}{\alpha+1}} + 2\pi\right]^{-2\alpha}\varepsilon^{\frac{2\alpha}{\alpha+1}} \le |N(\Gamma, \varepsilon)| \le \pi\left[\overline{m}(2\overline{a})^{-\frac{\alpha}{\alpha+1}} + 1\right]^2\varepsilon^{\frac{2\alpha}{\alpha+1}}.$$

Proof. Set

$$r_*(\varepsilon) = \min_{\psi\in[\varphi_2(\varepsilon), \varphi_2(\varepsilon)+2\pi]} f(\psi), \quad r^*(\varepsilon) = \max_{\psi\in[\varphi_2(\varepsilon), \varphi_2(\varepsilon)+2\pi]} f(\psi),$$

and

$$A = \{(r\cos\varphi, r\sin\varphi) : 0 \le r \le f(\varphi),\ \varphi \in [\varphi_2(\varepsilon), \varphi_2(\varepsilon)+2\pi)\}.$$

Then, we easily find that

$$\{(r\cos\varphi, r\sin\varphi) : 0 \le r \le r_*(\varepsilon),\ \varphi \in \mathbf{R}\} \subset A.$$

Therefore, Lemma 2 implies that

$$\begin{aligned}
|N(\Gamma, \varepsilon)| &\ge |A| \\
&\ge \pi(r_*(\varepsilon))^2 \\
&\ge \pi\left(\min_{\psi\in[\varphi_2(\varepsilon), \varphi_2(\varepsilon)+2\pi]} \underline{m}\psi^{-\alpha}\right)^2 \\
&= \pi\underline{m}^2(\varphi_2(\varepsilon) + 2\pi)^{-2\alpha} \\
&= \pi\underline{m}^2\left[(2\overline{a})^{\frac{1}{\alpha+1}} + 2\pi\varepsilon^{\frac{1}{\alpha+1}}\right]^{-2\alpha}\varepsilon^{\frac{2\alpha}{\alpha+1}} \\
&\ge \pi\underline{m}^2\left[(2\overline{a})^{\frac{1}{\alpha+1}} + 2\pi\right]^{-2\alpha}\varepsilon^{\frac{2\alpha}{\alpha+1}},
\end{aligned}$$

since $\varepsilon \in (0,1)$.

Let $(x,y) \in N(\Gamma,\varepsilon)$. Then, there exists $(x_0,y_0) \in \Gamma(\varphi_2(\varepsilon),\infty)$ and

$$d((x,y),(x_0,y_0)) < \varepsilon.$$

Hence,

$$d((x,y),(0,0)) \le d((x,y),(x_0,y_0)) + d((x_0,y_0),(0,0)) < \varepsilon + r^*(\varepsilon).$$

It follows that

$$\begin{aligned}|N(\Gamma,\varepsilon)| &\le \pi(\varepsilon + r^*(\varepsilon))^2 \\ &\le \pi\left(\varepsilon + \max_{\psi\in[\varphi_2(\varepsilon),\varphi_2(\varepsilon)+2\pi]} \overline{m}\psi^{-\alpha}\right)^2 \\ &= \pi\left[\varepsilon + \overline{m}(\varphi_2(\varepsilon))^{-\alpha}\right]^2 \\ &= \pi\left[\varepsilon^{\frac{1}{\alpha+1}} + \overline{m}(2\overline{a})^{-\frac{\alpha}{\alpha+1}}\right]^2 \varepsilon^{\frac{2\alpha}{\alpha+1}} \\ &\le \pi\left[1 + \overline{m}(2\overline{a})^{-\frac{\alpha}{\alpha+1}}\right]^2 \varepsilon^{\frac{2\alpha}{\alpha+1}}.\end{aligned}$$

□

Lemma 4. *Let x, $y \in C[a,b]$ and let*

$$G = \{(x(s),y(s)) : a \le s \le b\}.$$

Assume that $(x(s),y(s)) \neq (x(t),y(t))$ for $a \le s < t \le b$. Then,

$$|G_\varepsilon| \le 4\pi\varepsilon \operatorname{length}(G) + 4\pi\varepsilon^2.$$

Proof. The proof is similar to the proof of Lemma 26 in [17]. Let $\varepsilon > 0$. Set $s_1 = a$ and

$$s_{i+1} = \max\{s \in [s_i,b] : d((x(t),y(t)),(x(s_i),y(s_i))) \le \varepsilon,\ t \in [s_i,s]\}$$

for $i = 1,2,\cdots$. Then, there exists $n \ge 2$ such that $s_n = b$. Set $N = \max\{i \in \mathbf{N} : s_i < b\}$. We find that $N \ge 1$,

$$a = s_1 < s_2 < \cdots < s_i < s_{i+1} < \cdots < s_N < s_{N+1} = b,$$

and if $N \ge 2$, then

$$d((x(s_i),y(s_i)),(x(s_{i+1}),y(s_{i+1}))) = \varepsilon, \quad i = 1,2,\cdots,N-1.$$

We will prove that

$$G_\varepsilon \subset \bigcup_{i=1}^{N} B_{2\varepsilon}(x(s_i),y(s_i)), \tag{10}$$

where

$$B_{2\varepsilon}(x_0,y_0) = \{(x,y) \in \mathbf{R}^2 : d((x_0,y_0),(x,y)) \le 2\varepsilon\}.$$

Let $(x_1, y_1) \in G_\varepsilon$. Then, there exists $\sigma \in [a, b]$ such that

$$d((x_1, y_1), (x(\sigma), y(\sigma))) \leq \varepsilon.$$

Because of the definition of s_i, we find that $\sigma \in [s_k, s_{k+1}]$ for some $k \in \{1, 2, \cdots, N\}$, which implies that

$$d((x(\sigma), y(\sigma)), (x(s_k), y(s_k))) \leq \varepsilon.$$

Hence, it follows that

$$\begin{aligned} &d((x_1, y_1), (x(s_k), y(s_k))) \\ &\qquad \leq d((x_1, y_1), (x(\sigma), y(\sigma))) + d((x(\sigma), y(\sigma)), (x(s_k), y(s_k))) \leq 2\varepsilon, \end{aligned}$$

which means that $(x_1, y_1) \in B_{2\varepsilon}(x(s_k), y(s_k))$. Therefore, we obtain (10). By (10), we conclude that

$$|G_\varepsilon| \leq \sum_{i=1}^{N} |B_{2\varepsilon}(x(s_i), y(s_i))| = 4N\pi\varepsilon^2. \tag{11}$$

When $N = 1$, from (11) it follows that

$$|G_\varepsilon| \leq 4\pi\varepsilon^2 \leq 4\pi\varepsilon \, \text{length}(G) + 4\pi\varepsilon^2.$$

Now, we assume that $N \geq 2$. We observe that

$$\begin{aligned} \text{length}(G) &\geq \sum_{i=1}^{N} d((x(s_i), y(s_i)), (x(s_{i+1}), y(s_{i+1}))) \\ &\geq \sum_{i=1}^{N-1} d((x(s_i), y(s_i)), (x(s_{i+1}), y(s_{i+1}))) \\ &= (N-1)\varepsilon, \end{aligned}$$

that is,

$$N\varepsilon \leq \text{length}(G) + \varepsilon. \tag{12}$$

Combining (11) with (12), we obtain $|G_\varepsilon| \leq 4\pi\varepsilon \, \text{length}(G) + 4\pi\varepsilon^2$. □

Lemma 5.

$$|T(\Gamma, \varepsilon)| \leq 4\pi \left[M(2\overline{a})^{\frac{1-\alpha}{\alpha+1}} + 1 \right] \varepsilon^{\frac{2\alpha}{\alpha+1}}.$$

Proof. From Lemma 4, it follows that

$$\begin{aligned} |T(\Gamma, \varepsilon)| &\leq 4\pi\varepsilon \, \text{length}(\Gamma(\varphi_1, \varphi_2(\varepsilon))) + 4\pi\varepsilon^2 \\ &\leq 4\pi\varepsilon M(\varphi_2(\varepsilon))^{1-\alpha} + 4\pi\varepsilon^2 \\ &= 4\pi M(2\overline{a})^{\frac{1-\alpha}{\alpha+1}} \varepsilon^{\frac{2\alpha}{\alpha+1}} + 4\pi\varepsilon^2 \\ &= 4\pi \left[M(2\overline{a})^{\frac{1-\alpha}{\alpha+1}} + \varepsilon^{\frac{2}{\alpha+1}} \right] \varepsilon^{\frac{2\alpha}{\alpha+1}} \\ &\leq 4\pi \left[M(2\overline{a})^{\frac{1-\alpha}{\alpha+1}} + 1 \right] \varepsilon^{\frac{2\alpha}{\alpha+1}}. \end{aligned}$$

□

Now, we are ready to prove Theorem 3.

Proof of Theorem 3. Since

$$|\Gamma_\varepsilon| \ge |N(\Gamma,\varepsilon)|$$

and

$$|\Gamma_\varepsilon| \le |T(\Gamma,\varepsilon)| + |N(\Gamma,\varepsilon)|,$$

Lemmas 3 and 5 imply that there exist positive constants C_1 and C_2 such that

$$C_1\varepsilon^{\frac{2\alpha}{\alpha+1}} \le |\Gamma_\varepsilon| \le C_2\varepsilon^{\frac{2\alpha}{\alpha+1}}$$

for all sufficiently small $\varepsilon \in (0,1)$. Consequently, $\dim_{\mathrm{B}}\Gamma = 2/(1+\alpha)$. □

Proof of Corollary 1. Let $\varphi \ge \varphi_1$ be fixed. Since $f'(\varphi) \le 0$ and $f'(\varphi) \not\equiv 0$ on $[\varphi, \varphi+2\pi)$, we have

$$0 > \int_{\varphi}^{\varphi+2\pi} f'(\psi)d\psi = f(\varphi+2\pi) - f(\varphi).$$

By the mean value theorem, there exists $c \in (\varphi, \varphi+2\pi)$ such that

$$\frac{f(\varphi+2\pi)-f(\varphi)}{2\pi} = f'(c),$$

which implies that

$$f(\varphi) - f(\varphi+2\pi) = -2\pi f'(c) \le 2\pi K c^{-\alpha-1} \le 2\pi K \varphi^{-\alpha-1}.$$

Then, by Lemma 1, there exists a positive constant $\overline{m}$ such that $f(\psi) \le \overline{m}\psi^{-\alpha}$ for $\psi \ge \varphi_1$. Therefore,

$$\begin{aligned}
\operatorname{length}(\Gamma(\varphi_1,\varphi)) &= \int_{\varphi_1}^{\varphi} \sqrt{(f(\psi))^2 + (f'(\psi))^2}d\psi \\
&\le \int_{\varphi_1}^{\varphi} \sqrt{(\overline{m}\psi^{-\alpha})^2 + (K\psi^{-\alpha-1})^2}d\psi \\
&= \int_{\varphi_1}^{\varphi} \psi^{-\alpha}\sqrt{\overline{m}^2 + K^2\psi^{-2}}d\psi \\
&\le \sqrt{\overline{m}^2 + K^2\varphi_1^{-2}} \int_{\varphi_1}^{\varphi} \psi^{-\alpha}d\psi \\
&= \frac{\sqrt{\overline{m}^2 + K^2\varphi_1^{-2}}}{1-\alpha}(\varphi^{1-\alpha} - \varphi_1^{1-\alpha}) \\
&\le \frac{\sqrt{\overline{m}^2 + K^2\varphi_1^{-2}}}{1-\alpha}\varphi^{1-\alpha}.
\end{aligned}$$

Theorem 3 implies that $\dim_{\mathrm{B}}\Gamma = 2/(1+\alpha)$. □

3. Spiral with the box-counting dimension one

In this section, we prove Theorem 4 and assume all assumptions of Theorem 4. Let $\varepsilon \in (0, \varphi_1^{-2})$ be sufficiently small. We use the following notation:

$$T_1(\Gamma, \varepsilon) = \Gamma(\varphi_1, \varepsilon^{-1/2})_\varepsilon;$$

$$N_1(\Gamma, \varepsilon) = \Gamma(\varepsilon^{-1/2}, \infty)_\varepsilon,$$

where $\Gamma(\psi_1, \psi_2) = \{(f(\varphi)\cos\varphi, f(\varphi)\sin\varphi) : \psi_1 \leq \varphi < \psi_2\}$. In the same way as in the proof of Lemma 3, we have the following result.

Lemma 6. $|N_1(\Gamma, \varepsilon)| \leq \pi(\overline{m} + 1)^2\varepsilon$.

Lemma 7. $|T_1(\Gamma, \varepsilon)| \leq -2\pi M\varepsilon \log\varepsilon + 4\pi\varepsilon^2$.

Proof. By Lemma 4, we find that

$$\begin{aligned} |T_1(\Gamma, \varepsilon)| &\leq 4\pi\varepsilon \,\mathrm{length}(\Gamma(\varphi_1, \varepsilon^{-1/2})) + 4\pi\varepsilon^2 \\ &\leq 4\pi M\varepsilon \log\varepsilon^{-1/2} + 4\pi\varepsilon^2 \\ &= -2\pi M\varepsilon \log\varepsilon + 4\pi\varepsilon^2. \end{aligned}$$

□

The following inequality has been obtained in Tricot [22, §9.1].

Lemma 8. *Let G be a curve in $\mathbf{R}^2$ and let* $\mathrm{diam}(G)$ *be the largest distance between each two points in G, that is,*

$$\mathrm{diam}(G) = \sup_{z,w\in G} d(z, w).$$

Assume that $\mathrm{diam}(G) < \infty$. *Then,*

$$|G_\varepsilon| \geq 2\varepsilon \,\mathrm{diam}(G) + \pi\varepsilon^2.$$

Now, we give the proof of Theorem 4.

Proof of Theorem 4. Since the distance between two points

$$(f(\varphi_1)\cos\varphi_1, f(\varphi_1)\sin\varphi_1)$$

and

$$(f(\varphi_1 + \pi)\cos(\varphi_1 + \pi), f(\varphi_1 + \pi)\sin(\varphi_1 + \pi))$$

is equal to $f(\varphi_1) + f(\varphi_1 + \pi)$, we have

$$\mathrm{diam}(\Gamma) \geq f(\varphi_1) + f(\varphi_1 + \pi).$$

Hence, from Lemma 8, it follows that

$$|\Gamma_\varepsilon| \geq 2\varepsilon \operatorname{diam}(\Gamma) + \pi\varepsilon^2 \geq 2(f(\varphi_1) + f(\varphi_1 + \pi))\varepsilon,$$

which implies that

$$\begin{aligned}\liminf_{\varepsilon \to +0} \frac{\log|\Gamma_\varepsilon|}{\log \varepsilon} &\geq \liminf_{\varepsilon \to +0} \frac{\log(f(\varphi_1) + f(\varphi_1 + \pi))\varepsilon}{\log \varepsilon} \\ &= \liminf_{\varepsilon \to +0} \left(\frac{\log(f(\varphi_1) + f(\varphi_1 + \pi))}{\log \varepsilon} + 1 \right) = 1.\end{aligned}$$

By Lemmas 6 and 7, we conclude that

$$\begin{aligned}|\Gamma_\varepsilon| &\leq |T_1(\Gamma, \varepsilon)| + |N_1(\Gamma, \varepsilon)| \\ &\leq -2\pi M \varepsilon \log \varepsilon + 4\pi\varepsilon^2 + \pi(\overline{m} + 1)^2 \varepsilon \\ &= [-2\pi M \log \varepsilon + 4\pi\varepsilon + \pi(\overline{m} + 1)^2]\varepsilon \\ &\leq [-2\pi M \log \varepsilon + 4\pi + \pi(\overline{m} + 1)^2]\varepsilon,\end{aligned}$$

since $\varepsilon \in (0, 1)$. Therefore,

$$|\Gamma_\varepsilon| \leq (-c_1 \log \varepsilon + c_2)\varepsilon$$

for some $c_1 > 0$ and $c_2 > 0$, which implies that

$$\begin{aligned}\limsup_{\varepsilon \to +0} \frac{\log|\Gamma_\varepsilon|}{\log \varepsilon} &\leq \limsup_{\varepsilon \to +0} \frac{\log(-c_1 \log \varepsilon + c_2)\varepsilon}{\log \varepsilon} \\ &= \limsup_{\varepsilon \to +0} \left(\frac{\log(-c_1 \log \varepsilon + c_2)}{\log \varepsilon} + 1 \right) = 1.\end{aligned}$$

Consequently, $\dim_{\mathrm{B}} \Gamma = 1$. □

Proof of Corollary 2. Let $\varphi \geq \varphi_1$ be fixed. By the same argument as in the proof of Corollary 1, we find that $0 < f(\varphi) - f(\varphi + 2\pi)$. We observe that

$$\begin{aligned}\operatorname{length}(\Gamma(\varphi_1, \varphi)) &= \int_{\varphi_1}^{\varphi} \sqrt{(f(\psi))^2 + (f'(\psi))^2}\, d\psi \\ &\leq \int_{\varphi_1}^{\varphi} \sqrt{(\overline{m}\psi^{-1})^2 + (K\psi^{-1})^2}\, d\psi \\ &= \sqrt{\overline{m}^2 + K^2} \int_{\varphi_1}^{\varphi} \psi^{-1}\, d\psi \\ &= \sqrt{\overline{m}^2 + K^2} (\log \varphi - \log \varphi_1) \\ &\leq \sqrt{\overline{m}^2 + K^2} \log \varphi,\end{aligned}$$

since $\varphi_1 > 1$. Applying Theorem 4, we conclude that $\dim_{\mathrm{B}} \Gamma = 1$. □

4. Box-counting dimension of solution curves

In this section, we give proofs of Theorems 1 and 2.

For each solution $(x(t), y(t))$ of (1), we use the following notation:

$$r(t) = \sqrt{|x(t)|^2 + |y(t)|^2}.$$

The following Lemmas 9, 10 and 11 have been obtained in [13, Lemmas 2.2, 3.1 and 4.2].

Lemma 9. *Let $(x(t), y(t))$ be a nontrivial solution of (1). Assume that (3) is satisfied. Then, there exist a constant $C > 0$ and a function $\delta \in C[t_0, \infty)$ such that $\lim_{t\to\infty} \delta(t) = 0$ and*

$$[r(t)]^2 = e^{-H(t)}[C + \delta(t)], \quad t \geq t_0.$$

Lemma 10. *Let $(x(t), y(t))$ be a nontrivial solution of (1). If $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$, then*

$$\begin{cases} r'(t) = -h(t)r(t)\sin^2\theta(t), \\ \theta'(t) = -1 - \dfrac{1}{2}h(t)\sin 2\theta(t). \end{cases}$$

Lemma 11. *If (3) is satisfied, then $\lim_{t\to\infty} h(t) = 0$.*

Proof of Theorem 1. Let $(x(t), y(t))$ be a nontrivial solution of (1). We note that (2) holds, by (8). From Theorem A, it follows that $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$, $(x(t), y(t))$ is a spiral rotating in a clockwise direction on $[t_1, \infty)$ for some $t_1 \geq t_0$ and $\Gamma_{(x,y;t_0)}$ is simple. By l'Hopital's rule and Lemmas 10 and 11, we have

$$\lim_{t\to\infty} \frac{\theta(t)}{t} = \lim_{t\to\infty} \theta'(t) = -1. \tag{13}$$

Since

$$t^\alpha r(t) = t^\alpha e^{-H(t)/2}\sqrt{e^{H(t)}[r(t)]^2} = e^{-\frac{1}{2}(H(t) - 2\alpha\log t)}\sqrt{e^{H(t)}[r(t)]^2},$$

Lemma 9 and (8) imply that

$$0 < \liminf_{t\to\infty} t^\alpha r(t) \leq \limsup_{t\to\infty} t^\alpha r(t) < \infty. \tag{14}$$

By (13), (14) and (7), there exist $t_2 \geq \max\{t_1, 1\}$, $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$ such that for $t \geq t_2$

$$-\frac{3}{2}t \leq \theta(t) \leq -\frac{1}{2}t, \tag{15}$$

$$-\frac{3}{2} \leq \theta'(t) \leq -\frac{1}{2}, \tag{16}$$

$$C_1 \leq t^\alpha r(t) \leq C_2, \tag{17}$$

$$th(t) \leq C_3. \tag{18}$$

In view of (15), we note that $\lim_{t\to\infty}\theta(t) = -\infty$. Set $\eta(t) = -\theta(t)$. Then η is positive and strictly increasing on $[t_2,\infty)$. Hence, η has the inverse function η^{-1}. Set $\varphi_2 = \eta(t_2) > 0$ and $f(\varphi) = r(\eta^{-1}(\varphi))$ on $[\varphi_2,\infty)$. Since $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$, we have $\lim_{t\to\infty} r(t) = 0$, and hence, $\lim_{\varphi\to\infty} f(\varphi) = 0$. From (15) and (17), it follows that

$$\varphi^\alpha f(\varphi) = \varphi^\alpha r(\eta^{-1}(\varphi)) = (\eta(t))^\alpha r(t) = \left(\frac{-\theta(t)}{t}\right)^\alpha t^\alpha r(t) \ge \frac{C_1}{2^\alpha}, \quad \varphi \ge \varphi_2,$$

where $t = \eta^{-1}(\varphi)$. By (16) and Lemma 10, we find that

$$f'(\varphi) = r'(\eta^{-1}(\varphi))\frac{1}{\eta'(\eta^{-1}(\varphi))} = -\frac{r'(t)}{\theta'(t)} = \frac{h(t)r(t)\sin^2\theta(t)}{\theta'(t)} \le 0, \quad \varphi \ge \varphi_2, \tag{19}$$

where $t = \eta^{-1}(\varphi)$. We conclude that $f'(\varphi) \not\equiv 0$ on $[\varphi, \varphi+2\pi)$ for each fixed $\varphi \ge \varphi_2$. Indeed, if $f'(\varphi) \equiv 0$ on $[\varphi,\varphi+2\pi)$ for some $\varphi \ge \varphi_2$, then, by (19), $\sin^2\theta(t) \equiv 0$ on $I := [\eta^{-1}(\varphi), \eta^{-1}(\varphi+2\pi))$, that is, $\theta'(t) \equiv 0$ on I. This contradicts (16). Combining (15), (17), (18) with (19), we find that

$$\begin{aligned} -\varphi^{\alpha+1} f'(\varphi) &= (\eta(t))^{\alpha+1}\frac{h(t)r(t)\sin^2\theta(t)}{-\theta'(t)} \\ &= \left(\frac{-\theta(t)}{t}\right)^{\alpha+1}\frac{t^{\alpha+1}h(t)r(t)\sin^2\theta(t)}{-\theta'(t)} \\ &\le \left(\frac{3}{2}\right)^{\alpha+1} 2C_2C_3, \quad \varphi \ge \varphi_2, \end{aligned}$$

where $t = \eta^{-1}(\varphi)$. Set

$$\Gamma = \{(f(\varphi)\cos\varphi, f(\varphi)\sin\varphi) : \varphi \ge \varphi_2\}.$$

Corollary 1 implies that $\dim_{\mathrm{B}}\Gamma = 2/(1+\alpha)$. Since

$$\begin{aligned} \Gamma_{(x,-y;t_2)} &= \{(x(t), -y(t)) : t \ge t_2\} \\ &= \{(r(t)\cos\theta(t), -r(t)\sin\theta(t)) : t \ge t_2\} \\ &= \{(r(\eta^{-1}(\varphi))\cos\theta(\eta^{-1}(\varphi)), -r(\eta^{-1}(\varphi))\sin\theta(\eta^{-1}(\varphi))) : \varphi \ge \varphi_2\} \\ &= \{(f(\varphi)\cos(-\varphi), -f(\varphi)\sin(-\varphi)) : \varphi \ge \varphi_2\} \\ &= \{(f(\varphi)\cos\varphi, f(\varphi)\sin\varphi) : \varphi \ge \varphi_2\} \\ &= \Gamma, \end{aligned}$$

we have $\dim_{\mathrm{B}}\Gamma_{(x,-y;t_2)} = 2/(1+\alpha)$. Since $\Gamma_{(x,y;t_2)}$ and $\Gamma_{(x,-y;t_2)}$ are symmetric, we conclude that

$$\dim_{\mathrm{B}}\Gamma_{(x,y;t_2)} = \dim_{\mathrm{B}}\Gamma_{(x,-y;t_2)} = \dim_{\mathrm{B}}\Gamma = \frac{2}{1+\alpha}.$$

□

Proof of Theorem 2. Let $(x(t), y(t))$ be a nontrivial solution of (1). Using (9), we have (2). Hence, from Theorem A, it follows that $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$, $(x(t), y(t))$ is a spiral rotating in a clockwise direction on $[t_1, \infty)$ for some $t_1 \geq t_0$ and $\Gamma_{(x,y;t_0)}$ is simple. By the same argument as in the proof of Theorem 1 and noting Lemma 11, there exist $t_2 \geq \max\{t_1, 1\}$, $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$ such that (15), (16) and the following (20) and (21) hold for $t \geq t_2$

$$C_1 \leq tr(t) \leq C_2, \tag{20}$$

$$h(t) \leq C_3. \tag{21}$$

Set $\eta(t) = -\theta(t)$. Then, η has the inverse function η^{-1}. Set $\varphi_2 = \eta(t_2) > 0$ and $f(\varphi) = r(\eta^{-1}(\varphi))$ on $[\varphi_2, \infty)$. Then, $\lim_{\varphi\to\infty} f(\varphi) = 0$. We observe that

$$\varphi f(\varphi) = \varphi r(\eta^{-1}(\varphi)) = \left(\frac{-\theta(t)}{t}\right) tr(t) \leq \frac{3C_2}{2}, \quad \varphi \geq \varphi_2,$$

where $t = \eta^{-1}(\varphi)$. In the same way as in the poof of Theorem 1, using (15), (16), (19), (20) and (21), we conclude that $f'(\varphi) \leq 0$ for $\varphi \geq \varphi_2$, $f'(\varphi) \not\equiv 0$ on $[\varphi, \varphi + 2\pi)$ for each fixed $\varphi \geq \varphi_2$, and that

$$-\varphi f'(\varphi) = \left(\frac{-\theta(t)}{t}\right) \frac{h(t)tr(t)\sin^2\theta(t)}{-\theta'(t)} \leq 3C_2C_3, \quad \varphi \geq \varphi_2,$$

where $t = \eta^{-1}(\varphi)$. Corollary 2 implies that $\dim_B \Gamma = 1$. Hence, $\dim_B \Gamma_{(x,y;t_2)} = 1$. □

References

[1] W. A. Coppel, *Stability and asymptotic behavior of differential equations*, D. C. Heath and Co., Boston, 1965.

[2] L. H. Duc, A. Ilchmann, S. Siegmund, P. Taraba, *On stability of linear time-varying second-order differential equations*, Quart. Appl. Math. **64**(2006), 137–151.

[3] K. Falconer, *Fractal Geometry. Mathematical Fondations and Applications*, John Willey-Sons, 1999.

[4] P. Hartman, *Ordinary Differential Equations*, Classics in Applied Mathematics, Vol. 38, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002. Corrected reprint of the second (1982) edition. With a foreword by Peter Bates.

[5] L. Korkut, D. Vlah, D. Žubrinić, V. Županović, *Wavy spirals and their fractal connection with chirps*, Math. Commun. **21**(2016), 251–271.

[6] L. Korkut, D. Vlah, V. Županović, *Fractal properties of Bessel functions*, Appl. Math. Comput. **283**(2016), 55–69.

[7] M. K. Kwong, M. Pašić, J. S. W. Wong, *Rectifiable oscillations in second order linear differential equations*, J. Differential Equations **245**(2008), 2333–2351.

[8] S. Miličić, M. Pašić, *Nonautonomous differential equations in Banach space and nonrectifiable attractivity in two-dimensional linear differential systems*, Abstr. Appl. Anal. **2013**(2013), Art. ID 935089, 10 pp.

[9] Y. Naito, M. Pašić, *Characterization for rectifiable and nonrectifiable attractivity of nonautonomous systems of linear differential equations*, Int. J. Differ. Equ. **2013**(2013), Art. ID 740980, 11 pp.

[10] Y. Naito, M. Pašić, S. Tanaka, *Rectifiable and nonrectifiable solution curves of half-linear differential systems*, Math. Slovaca, to appear.

[11] M. Onitsuka, *Non-uniform asymptotic stability for the damped linear oscillator*, Nonlinear Anal. **72**(2010), 1266–1274.

[12] M. Onitsuka, *Uniform asymptotic stability for damped linear oscillators with variable parameters*, Appl. Math. Comput. **218**(2011), 1436–1442.

[13] M. Onitsuka, S. Tanaka, *Rectifiability of solutions for a class of two-dimensional linear differential systems*, Mediterr. J. Math. **14**(2017), Art. 51, 11 pp.

[14] M. Pašić, *Minkowski-Bouligand dimension of solutions of the one-dimensional p-Laplacian*, J. Differential Equations **190**(2003), 268–305.

[15] M. Pašić, *Fractal oscillations for a class of second-order linear differential equations of Euler type*, J. Math. Anal. Appl. **341**(2008), 211–223.

[16] M. Pašić, S. Tanaka, *Fractal oscillations of self-adjoint and damped linear differential equations of second-order*, Appl. Math. Comput. **218**(2011), 2281–2293.

[17] M. Pašić, S. Tanaka, *Fractal oscillations of chirp functions and applications to second-order linear differential equations*, Int. J. Differ. Equ. **2013**(2013), Art. ID 857410, 11 pp.

[18] M. Pašić, D. Žubrinić, V. Županović, *Oscillatory and phase dimensions of solutions of some second-order differential equations*, Bull. Sci. Math. **133**(2009), 859–874.

[19] G. Radunović, D. Žubrinić, V. Županović, *Fractal analysis of Hopf bifurcation at infinity*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **22**(2012), 1230043, 15 pp.

[20] R. A. Smith, *Asymptotic stability of* $x'' + a(t)x' + x = 0$, Quart. J. Math. Oxford Ser. **12**(1961), 123–126.

[21] J. Sugie, M. Onitsuka, *Integral conditions on the uniform asymptotic stability for two-dimensional linear systems with time-varying coefficients*, Proc. Amer. Math. Soc. **138**(2010), 2493–2503.

[22] C. Tricot, *Curves and Fractal Dimension*, Springer-Verlag, New York, 1995.

[23] D. Žubrinić, V. Županović, *Fractal analysis of spiral trajectories of some planar vector fields*, Bull. Sci. Math. **129**(2005), 457–485.

[24] D. Žubrinić, V. Županović, *Poincaré map in fractal analysis of spiral trajectories of planar vector fields*, Bull. Belg. Math. Soc. Simon Stevin **15**(2008), 947–960.