MATHEMATICAL COMMUNICATIONS
Math. Commun. **23**(2018), 15–27

A family of isospectral fourth order Sturm-Liouville problems and equivalent beam equations

Hanif Mirzaei*

Department of Mathematics, Sahand University of Technology, P.O. Box 51335-1996 Tabriz, Iran

Received November 4, 2015; accepted April 11, 2017

Abstract. In this paper, we consider the class of fourth order Sturm-Liouville equation of the form $y^{(4)}(z) - 2(q(z)y')' + (q^2(z) - q''(z))y(z) = \lambda^2 y(z)$, $0 < z < L$, with boundary conditions $y(z) = y''(z) = 0$ at $z = 0, L$. We prove that this class is equivalent to a second order Sturm-Liouville problem. Using Darboux lemma we obtain the closed form of fourth order Sturm-Liouville equations that is isospectral to a given one. We also obtain the Euler-Bernoulli beam equation equivalent to this class.

AMS subject classifications: 34B24, 58J53, 74K10

Key words: Isospectral, fourth order Sturm-Liouville equation, Euler-Bernoulli beam equation

1. Introduction

Free transversal motion of frequency ω of a thin straight beam of length l is governed by the Euler-Bernoulli equation [5, 11]

$$(EI(x)u''(x))'' = A(x)\rho\omega^2 u(x), \quad 0 < x < l, \tag{1}$$

where $E > 0$ is Young's modulus, $\rho > 0$ is volume mass density, $A(x)$ is the cross-sectional area, and $I(x)$ is the second moment of this area about an axis through the centroid of the cross-section at abscissa x. If we put

$$x = ls, \quad u(x) = u(s), \quad r(s) = \frac{I(x)}{I(x_c)}, \quad a(s) = \frac{A(x)}{A(x_c)}, \quad \lambda^2 = \frac{A(x_c)\rho\omega^2 l^4}{EI(x_c)}, \quad x_c \in [0, l], \tag{2}$$

then equation (1) in dimensionless form becomes

$$(r(s)u''(s))'' = \lambda^2 a(s)u(s), \quad 0 < s < 1, \tag{3}$$

where $r(s), a(s) \in C^2[0, l]$ are positive functions. Using transformation

$$b(z) = \left(\frac{a(s)}{r(s)}\right)^{\frac{1}{4}}, \quad c(z) = [r^3(s)a(s)]^{\frac{1}{4}}, \quad \frac{dz}{ds} = b(z), \tag{4}$$

*Corresponding author. *Email address:* h_mirzaei@sut.ac.ir (H. Mirzaei)

equation (3) becomes

$$y^{(4)}(z) + (A(z)y'(z))' + B(z)y(z) = \lambda^2 y(z), \quad 0 < z < L, \tag{5}$$

where

$$\begin{aligned}
y(z) &= u(s)b(z)c(z),\\
A(z) &= 2\frac{c'^2}{c^2} - 4\frac{c''}{c} - 3\frac{b''}{b} + 2\frac{b'^2}{b^2} - 2\frac{b'c'}{bc},\\
B(z) &= 4\frac{b'^4}{b^4} - \frac{c^{(4)}}{c} + 4\frac{c'^4}{b^4} + 4\frac{c''^2}{c^2} - \frac{b^{(4)}}{b} + 3\frac{b''^2}{b^2} - 10\frac{c'^2c''}{c^3}\\
&\quad + 4\frac{c'c'''}{c^2} + 3\frac{b'b'''}{b^2} - 9\frac{b'^2b''}{b^3} - 2\frac{c'^3b'}{bc^3} + 3\frac{b'b''c'}{b^2c} - \frac{c'b'''}{bc}\\
&\quad - 2\frac{b'^3c'}{b^3c} + 4\frac{b'c'c''}{bc^2} + \frac{c''b''}{bc},
\end{aligned} \tag{6}$$

and $L = \int_0^1 (\frac{a(s)}{r(s)})^{1/4} ds$. For more details on the use of the beam model in vibration analysis, see [5]. Equation (5) is the fourth order Sturm-Liouville equation equivalent to beam equation (3). For equation (3), the most commonly used boundary conditions are

$$\begin{aligned}
&\text{Free} : u'' = (ru'')' = 0\\
&\text{Clamped} : u = u' = 0\\
&\text{Pinned} : u = u'' = 0\\
&\text{Sliding} : u' = (ru'')' = 0
\end{aligned}. \tag{7}$$

Simple calculation shows that transformation (4) only preserves the boundary condition on the end to which the beam is clamped. It is well known that beam equation (3) with one of the boundary conditions (7) has an infinite sequence of eigenvalues $\{\lambda_n^2\}_{n=1}^{\infty}$ such that

$$0 \leq \lambda_1^2 < \lambda_2^2 < \lambda_3^2 < ..., \tag{8}$$

and there is no finite accumulation in the spectrum. Note that for free-free, free-sliding and sliding-sliding boundary conditions the beam equation has rigid body modes, see [5]. The eigenvalues of second order Sturm-Liouville problems are simple, but fourth order equation (5) may have eigenvalues with multiplicity two, see Theorem 1, part b. Isospectral problems, i.e. problems that have the same spectrum, for beams and fourth order Sturm-Liouville problems are studied in many papers [1, 2, 4, 6, 8, 12, 13, 14]. Reference [2] appears to be a first study of isospectral sets for fourth order ordinary differential operators, on the finite interval $[0, 1]$. By changing independent and dependent variables Gottlieb [8] found seven different classes of nonuniform beams that are isospectral to the uniform beam. Isospectral operators (second and fourth order) with discontinuous coefficients are found in [14]. For second order Sturm-Liouville problems, Turbowits et al. [12] found a complete characterization of isospectral potentials. Ghanbari [4] used the idea of [12], and factorized the beam operator as a product of the second order differential operator and its adjoint. Note that the coefficients of the factors satisfy a nonlinear ordinary differential equation. He obtained one special solution of this equation. Soh [13] solved the resulting nonlinear differential equation in [4] by the Lie group method and obtained the isospectral fourth order Sturm-Liouville equations and corresponding beam equations. Morassi and et al. [6] consider the class of Euler-Bernoulli beams such that the product between the bending stiffness and the linear mass density is constant. They prove that this class is equivalent to a string, then reduce the string equation to the second order Sturm-Liouville problem and use Darboux's lemma for constructing isospectral beams. In [1], the authors construct families of Euler-Bernoulli Kirchhoff beams which have exactly the same buckling loads of a given beam.

In this paper, we consider a class of the fourth order Sturm-Liouville equation of the form

$$y^{(4)}(z) - 2(q(z)y'(z))' + (q^2(z) - q''(z))y(z) = \lambda^2 y(z), \quad 0 < z < L, \tag{9}$$

with pinned-pinned boundary conditions

$$y(0) = y''(0) = 0, \quad y(L) = y''(L) = 0. \tag{10}$$

Equation (9) together with boundary conditions (10) is called the fourth order Sturm-Liouville problem (FSLP). The study of isospectral problems for this class of FSLP has not attracted that much attention so far. Note that the method of papers [4, 13] cannot be applied to this class of problems. This is due to the fact that the type of an FLSP cannot be changed by a simple reversal of the order of products. In Theorem 1, we prove that this problem is equivalent to a second order Sturm-Liouville problem. Using Darboux's lemma we find a family of isospectral FSLPs to a given one. Also, in Theorems 2 and 3 we obtain the equivalent beam equation for this class of FSLPs.

2. Isospectral fourth order Sturm-Liouville problems

Our analysis consists of three steps:

(i) In Theorem 1, we prove that FSLP (9),(10) is equivalent to a second order Sturm-Liouville problem,

(ii) We apply Darboux's lemma for constructing isospectral problems to the second order Sturm-Liouville problem in step (i),

(iii) Using the equivalence in Theorem 1, we obtain an isospectral FSLP.

Lemma 1 (Darboux's Lemma [3, 5])**.** *Let μ be a real number, and suppose g is a nontrivial solution of the Sturm-Liouville equation*

$$-g'' + \hat{q}g = \mu g, \tag{11}$$

with potential $\widehat{q} = \widehat{q}(z)$. If f is a nontrivial solution of

$$-f'' + \hat{q}f = \lambda f, \tag{12}$$

and $\lambda \neq \mu$, then $y = \frac{1}{g}[g, f] = \frac{1}{g}[gf' - fg']$ is a nontrivial solution of the Sturm-Liouville equation

$$-y'' + \check{q}y = \lambda y, \tag{13}$$

where $\check{q} = \hat{q} - 2(\ln g)''$.

Indeed, equations (12) and (13) are isospectral. In [7], Darboux's lemma is applied twice and obtain the closed form of potential q isospectral to $\hat{q}$, and the corresponding rod equation with cross-sectional area $a(z)$ as follows

$$q(z) = \hat{q} - 2\frac{d^2}{dz^2}(\ln(1 + \alpha \int_0^x g_m^2(t))), \quad m = 1, 2, \ldots \tag{14}$$

$$a(z) = \hat{a}(z) - \frac{\alpha g_m(z)[g_m, \hat{a}]}{\lambda_m(1 + \alpha \int_0^z g_m^2(t))}, \quad m = 1, 2, \ldots \tag{15}$$

where (λ_m, g_m) is the mth eigenpair of equation (11) such that $\int_0^L {g_m}^2(t)dt = 1$, $\hat{a}(z)$ is the cross-sectional area of rod equation corresponding to potential $\hat{q}$, and $\alpha > -1$ is a constant real number.

Theorem 1. *a) Let $(\lambda^2, y(z)), \lambda > 0$, be a simple eigenpair of FSLP*

$$\begin{cases} y^{(4)} - 2(q(z)y')' + (q^2(z) - q''(z))y = \lambda^2 y, & 0 < z < L \\ y(0) = y''(0) = 0, \quad y(L) = y''(L) = 0 \end{cases}. \tag{16}$$

Then only one of the pairs $(\lambda, y(z))$ or $(-\lambda, y(z))$ is an eigenpair of the second order Sturm-Liouville problem

$$\begin{cases} y''(z) + (\lambda - q(z))y = 0, & 0 < z < L, \\ y(0) = 0, \quad y(L) = 0 \end{cases}, \tag{17}$$

where $q \in C^2[0, L]$. Conversely, if $(\lambda, y(z))$ is an eigenpair of (17), then $(\lambda^2, y(z))$ is an eigenvalue of (16) with real number λ.

b) If $(\lambda^2, y(z)), \lambda > 0$, is an eigenpair of FSLP (16) with multiplicity two, then both λ and $-\lambda$ are the eigenvalues of problem (17).

Proof. Let (λ, y) be an eigenpair of (17). Clearly, λ is a real number since problem (17) is selfadjoint. By assumption $q \in C^2[0, 1]$ and from equation (17) we conclude that $y \in C^4[0, L]$. By differentiating twice, equation (17) implies that

$$y^{(4)}(z) - q''y - 2q'y' + (\lambda - q)y'' = 0.$$

Using equation (17) and replacing y'' with $(q - \lambda)y$ we have

$$y^{(4)}(z) - q''y - 2q'y' + 2\lambda qy - q^2 y - \lambda^2 y = 0.$$

Using equation (17) again and replacing λy with $qy - y''$ we obtain

$$y^{(4)}(z) - q''y - 2q'y' + q^2y - 2qy'' - \lambda^2 y = 0.$$

We can write the last equation as follows

$$y^{(4)} - 2(q(z)y')' + (q^2(z) - q''(z))y = \lambda^2 y. \tag{18}$$

By taking the limit of equation (17) as $z \to 0^+$ and using boundary conditions, we have $y''(0) = 0$. Similarly, $y''(L) = 0$. Thus (λ^2, y) is an eigenpair of (16). Conversely, let (λ^2, y) be a simple eigenpair of (16). We can factorize the equation (16) as follows

$$(D^2 - q - \lambda)(D^2 - q + \lambda)y = 0, \quad \lambda > 0. \tag{19}$$

Define $\chi(z) = (D^2 - q + \lambda)y$. We consider the two cases:

Case I: $\chi(z) \equiv 0$. By the lemma in the Appendix, this case is satisfied for eigenvalues λ_n with sufficiently large indices n. Thus $y''(z) + (\lambda - q(z))y = 0$; therefore (λ, y) is an eigenpair of (17).

Case II: $\chi(z) \neq 0$, (this case may occur for lower indices of eigenvalues λ_n). By definition of $\chi(z)$ and boundary conditions (16) we have $\chi(0) = \chi(L) = 0$. Using (19) we have

$$\chi'' + (-\lambda - q)\chi = 0;$$

thus $\chi''(0) = 0,\ \chi''(L) = 0$, and $(-\lambda, \chi)$ is an eigenpair of (17). Multiplying equation (19) from the left by $(D^2 - q + \lambda)$ we obtain

$$\begin{gathered}(D^2 - q + \lambda)(D^2 - q - \lambda)\chi = 0,\\ \chi^{(4)} - 2(q(z)\chi')' + (q^2(z) - q''(z))\chi = \lambda^2 \chi.\end{gathered}$$

Therefore (λ^2, χ) and (λ^2, y) are eigenpairs of problem (16). By assuming (λ^2, y) is a simple eigenpair of (16), there exists a real number $\beta \neq 0$ such that $\chi = \beta y$. Thus we have

$$\begin{cases}\beta y = y'' + (\lambda - q)y, \\ \beta y'' + \beta(-\lambda - q)y = 0,\end{cases} \Longrightarrow (\beta - 2\lambda)y = 0. \tag{20}$$

Since $y \neq 0$, then $\beta = 2\lambda$ and using equation $y'' + (-\lambda - q)y = 0$ we imply that in case II $(-\lambda, y)$ is an eigenpair of problem (17).

Now we prove part b. Since λ^2 is of multiplicity two, there are two independent eigenfunctions y_1 and y_2 corresponding to λ^2 such that

$$\begin{aligned}y_1^{(4)} - 2(q(z)y_1')' + (q^2(z) - q''(z))y_1 &= \lambda^2 y_1,\\ y_2^{(4)} - 2(q(z)y_2')' + (q^2(z) - q''(z))y_2 &= \lambda^2 y_2.\end{aligned}$$

By factorization of the above equation we have

$$(D^2 - q - \lambda)(D^2 - q + \lambda)y_1 = 0, \quad (D^2 - q - \lambda)(D^2 - q + \lambda)y_2 = 0.$$

Define $\psi(z) = (D^2 - q + \lambda)y_1$ and $\varphi(z) = (D^2 - q + \lambda)y_2$. If $\psi(z) \equiv 0$ and $\varphi(z) \equiv 0$, then λ is an eigenvalue of problem (17) with multiplicity two, this is a contradiction since the eigenvalues of problem (17) are simple. If $\psi(z) \neq 0$ and $\varphi(z) \neq 0$, then $-\lambda$ is an eigenvalue of problem (17) with multiplicity two; this is a contradictioin. Thus only one of the functions $\psi(z)$ and $\varphi(z)$ must be zero. We suppose that $\psi(z) \equiv 0$ and $\varphi(z) \neq 0$. From $\psi(z) \equiv 0$, we conclude that λ is an eigenvalue of problem (17). From $\varphi(z) \neq 0$, we conclude that $-\lambda$ is an eigenvalue of problem (17). □

Combining Theorem 1 and Darboux's lemma, we come to the following corollary:

Corollary 1. *For any given $q(z) \in C^2[0, L]$, FSLP (16) is isospectral to the family of the following FSLPs*

$$\begin{cases} y^{(4)} - 2(q_{m,\alpha}(z)y')' + (q_{m,\alpha}^2(z) - q_{m,\alpha}''(z))y = \lambda^2 y, \quad 0 < z < L \\ y(0) = y''(0) = 0, \quad y(L) = y''(L) = 0 \end{cases}, \tag{21}$$

where $q_{m,\alpha}$ is given by (14).

Up to now, we find the family of isospectral FSLPs (21) analytically. For applying analytical results we require the m-th eigenfunction of problem (17). With this we can determine the function $q_{m,\alpha}$. However, this can only be done analytically for constant potentials q. In our example, we will use a numerical approximation of g_m instead. This numerical approximation will then be smoothed by the polynomial least square approximation. We use the Matslise package [10] to solve the Sturm-Liouville problem numerically and we use Maple for polynomial least square smoothing and subsequent symbolic computation. We present this numerical experiment as Example 2.

Example 1. *We apply the obtained results, and construct a family of isospectral FSLPs. If we choose $q = 0$, then we have the following FSLP*

$$\begin{cases} y^{(4)} = \lambda^2 y, \quad 0 < z < 1 \\ y(0) = y''(0) = 0, \quad y(1) = y''(1) = 0 \end{cases}. \tag{22}$$

By Theorem 1 this problem is equivalent to the problem

$$\begin{cases} y'' = \lambda y, \quad 0 < z < 1 \\ y(0) = 0, \quad y(1) = 0 \end{cases}. \tag{23}$$

Using Darboux's lemma problem (23) is isospectral to the following two parameters problem

$$\begin{cases} w'' + (\lambda - q_{m,\alpha}(z))w = 0, \quad 0 < z < 1 \\ w(0) = 0, \quad w(1) = 0 \end{cases}. \tag{24}$$

where

$$\begin{aligned} q_{m,\alpha} &= -2(\ln(1 + 2\alpha \int_0^z \sin^2(m\pi t)dt))'' \\ &= -4\alpha \frac{-\alpha + \alpha\cos(2m\pi z) + m\pi(1 + \alpha z)\sin(2m\pi z)}{(1 + \alpha z - \frac{\alpha}{2m\pi}\sin(2m\pi z))^2}. \end{aligned}$$

By Theorem 1, problem (22) is isospectral to

$$\begin{cases} w^{(4)} - 2(q_{m,\alpha}(z)w')' + (q_{m,\alpha}^2(z) - q_{m,\alpha}''(z))w = \lambda^2 w \\ w(0) = w''(0) = 0, \quad w(1) = w''(1) = 0 \end{cases}. \tag{25}$$

In Tables 1 and 2, for different values of α and m, the first four eigenvalues of problems (22) and (25) are listed (Note that $\alpha = 0$ denotes problem (22)). By comparing the eigenvalues it is obvious that problems (22) and (25) are isospectral. We compute the eigenvalues by the method presented in [9].

n	$\alpha = 0$	$\alpha = -0.5$	$\alpha = 0.5$	$\alpha = 0.9$
1	97.4091	97.4091	97.4091	97.4091
2	1558.5454	1558.5455	1558.5455	1558.5455
3	7890.1364	7890.1364	7890.1364	7890.1364
4	24936.7273	24936.7273	24936.7273	24936.7273

Table 1: *Eigenvalues of problem (22) and reconstructed isospectral problem (25) for* $m = 1$

n	$\alpha = -0.5$	$\alpha = 0.5$	$\alpha = 0.9$
1	97.4091	97.4091	97.4091
2	1558.5455	1558.5455	1558.5455
3	7890.1364	7890.1364	7890.1364
4	24936.7275	24936.7274	24936.7275

Table 2: *Eigenvalues of reconstructed isospectral problem (25) for* $m = 2$

Example 2. *Consider the following FSLP on interval* $[0, 5]$,

$$\begin{cases} y^{(4)} - (0.02z^2 y')' + (0.0001z^4 - 0.02)y = \lambda^2 y, \quad 0 \le z \le 5 \\ y(0) = y''(0) = 0, \quad y(5) = y''(5) = 0 \end{cases}. \tag{26}$$

By Theorem 1 this problem is equivalent to

$$\begin{cases} y'' + (\lambda - 0.01z^2)y = 0, \quad 0 < z < 5 \\ y(0) = 0, \quad y(5) = 0 \end{cases}. \tag{27}$$

By using Darboux's lemma and Theorem 1, we find the following problems isospectral to problem (26).

$$\begin{cases} w^{(4)} - 2(q_{m,\alpha}(z)w')' + (q_{m,\alpha}^2(z) - q_{m,\alpha}''(z))w = \lambda^2 w, \\ w(0) = w''(0) = 0, \quad w(5) = w''(5) = 0 \end{cases}. \tag{28}$$

where

$$q_{m,\alpha} = 0.01z^2 - 2(\ln(1 + \alpha \int_0^z g_m^2(t)dt))'', \quad \alpha > -1.$$

For constructing isospectral problems by Darboux's lemma we require the $m-$th eigenfunction of problem (27). We use the Matslise package to solve Sturm-Liouville problem (27) numerically and we use Maple for a polynomial least square approximation of eigenfunctions g_1 and g_2:

$$g_1 \simeq \frac{17}{16582} + \frac{199}{482}z + \frac{253}{9662}z^2 - \frac{527}{9578}z^3 + \frac{95}{11186}z^4 - \frac{22}{59813}z^5,$$
$$g_2 \simeq \frac{1069}{1330}z - \frac{2809}{13694}z^3 - \frac{115}{3687}z^4 + \frac{149}{3584}z^5 - \frac{131}{16138}z^6 + \frac{21}{43135}z^7.$$

For different values of α and m, the eigenvalues of isospectral problems (26) and (28) are listed in Table 3. The eigenvalues computed by the method presented in [9]. By comparing the eigenvalues it is obvious that problems (26) and (28) are isospectral, and the results in Table 3 confirm the analytical results.

	Eigenvalues for $m=1$					Eigenvalues for $m=2$		
n	$\alpha=0$	$\alpha=-0.5$	$\alpha=0.5$	$\alpha=0.9$		$\alpha=-0.5$	$\alpha=0.5$	$\alpha=0.9$
1	0.2150	0.2152	0.2151	0.2152		0.2150	0.2150	0.2150
2	2.7548	2.7540	2.7545	2.7541		2.7538	2.7551	2.7550
3	13.2153	13.2156	13.2146	13.2139		13.2154	13.2153	13.2151
4	40.9508	40.9514	40.9508	40.9509		40.9600	40.9464	40.9441
5	99.0535	99.0540	99.0539	99.0544		99.0592	99.0518	99.0514

Table 3: *Eigenvalues of problem (26) and reconstructed isospectral problem (28)*

3. Beam equation equivalent to FSLP (16)

In Section 1, for Euler-Bernoulli beam equation (1) we find the equivalent fourth order Sturm-Liouville equation (5). In Theorems 2 and 3, we find the Euler-Bernoulli beam equation equivalent to the FSLP of the form (16).

Theorem 2. *The Euler-Bernoulli beam equation*

$$(I(s)u'')'' = \lambda^2\gamma(s)u(s), \quad 0 < s < l, \quad I(s)\gamma(s) = constant, \tag{29}$$

with pinned-pinned boundary conditions can be reduced to an FSLP of the form (16).

Proof. Using transformation (4) for equation (29) we have

$$b(z) = \sqrt{\gamma(s)}, \quad c^2(z) = \frac{1}{\sqrt{\gamma(s)}}.$$

Thus, $b(z) = \frac{1}{c^2(z)}$, $\frac{dz}{ds} = b(z)$, and equation (29) is equivalent to the fourth order Sturm-Liouville equation as follows

$$y^{(4)}(z) + [(2\frac{c''}{c} - 4\frac{c'^2}{c^2})y']' + [-8\frac{c'^4}{c^4} + \frac{c^{(4)}}{c} - 4\frac{c''^2}{c^2} - 6\frac{c'c''}{c^2} + 18\frac{c'^2c''}{c^3}]y = \lambda^2 y. \tag{30}$$

Simple calculation shows that equation (30) is of the form (16) with $q = -\frac{c''}{c} + 2\frac{c'^2}{c^2}$. In general, transformation (4) only preserves the clamped-clamped boundary conditions, but for equations of the form (29) this transformation preserves the pinned-pinned boundary conditions, too. We have $u(s) = c(z)y(z)$; thus

$$u(0) = y(0) = 0, \quad u(l) = y(L) = 0, \quad u''(s) = \frac{1}{c^3}(-2\frac{c'^2}{c^2}y + \frac{c''}{c}y + y'').$$

Therefore, $u''(0) = y''(0) = 0, \quad u''(l) = y''(L) = 0$. Thus problem (29) with pinned-pinned boundary conditions is equivalent to FSLP of the form (16). □

Now, there are two fundamental questions: Is there any beam equation equivalent to FSLP (16)? If there is, how can we find this Euler Bernoulli beam equation? In order to answer these questions we state the following theorem.

Theorem 3. *If FSLP (16) is equivalent to the second order Sturm-Liouville problem* $y'' + (\lambda - q)y = 0, \lambda > 0$*, then there exists a beam equation equivalent to FSLP (16) which is of the form (29).*

Proof. We suppose that FSLP (16) is equivalent to

$$\begin{cases} y'' + (\lambda - q)y = 0, \quad \lambda > 0 \\ y(0) = 0, \quad y(L) = 0 \end{cases}. \tag{31}$$

Problem (31) is equivalent to rod equation

$$\begin{cases} (A(z)w')' + \lambda A(z)w(z) = 0 \\ w(0) = 0, \quad w(L) = 0 \end{cases}, \tag{32}$$

where $A(z) = a^2(z)$ and $q = \frac{a''(z)}{a(z)}, \quad w(z) = \frac{y(z)}{a(z)}$, see [5]. Rod equation (32) is equivalent to the string problem

$$v''(s) + \lambda\rho(s)v(s) = 0, \quad 0 < s < l, \quad v(0) = 0, \quad v(l) = 0. \tag{33}$$

This equivalence can be obtained by changing variable $s(z) = \int_0^z \frac{dt}{A(t)}$, where

$$\rho(s) = a^4(z), \quad l = \int_0^L \frac{1}{A(t)}dt. \tag{34}$$

By Proposition 1 in [6], string problem (33) is equivalent to the beam problem

$$\begin{cases} (I(s)v'')'' = \lambda^2\gamma(s)v(s), \quad 0 < s < l, \quad \gamma(s) = \rho(s) = \frac{1}{I(s)}, \\ v(0) = v''(0) = 0, \quad v(l) = v''(l) = 0, \end{cases} \tag{35}$$

Thus, there exists a beam equation equivalent to FSLP (16). □

In practice, for finding the beam equation equivalent to a given FSLP of the form (16) first we solve the ordinary differential equation $a'' - qa = 0$, and construct rod equation (32). Second, we find the corresponding string equation (33), and then construct the equivalent beam equation. We have done this in the following example.

Example 3. *In this example, we find the beam equation equivalent to the following FSLP*

$$\begin{cases} y^{(4)} - (0.02z^2y')' + (0.0001z^4 - 0.02)y = \lambda^2 y, \quad 0 < z < 5 \\ y(0) = y''(0) = 0, \qquad y(5) = y''(5) = 0 \end{cases}. \tag{36}$$

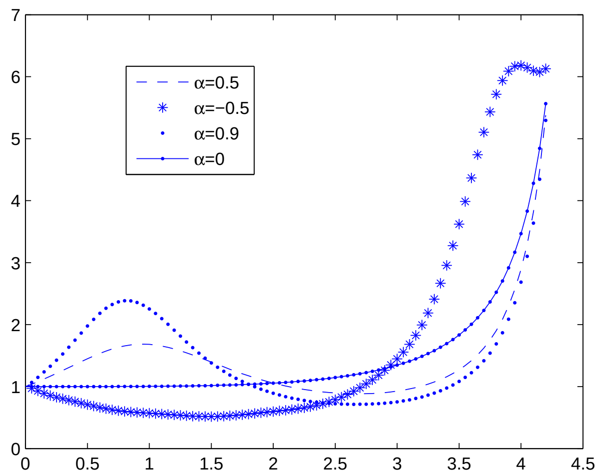

Figure 1: *Linear mass density of isospectral beams corresponding to g_1*

By Theorem 1, this problem is equivalent to the problem

$$\begin{cases} y'' + (\lambda - 0.01z^2)y = 0, \quad 0 < z < 5 \\ y(0) = 0, \quad y(5) = 0 \end{cases}. \tag{37}$$

Theorem 3 states that if the eigenvalues of problem (37) are positive, then we can find a beam equation equivalent to FSLP (36). By the Matslise package the eigenvalues of problem (37) are as follows

$$\lambda_1 = 0.463736, \quad \lambda_2 = 1.659762, \quad \lambda_3 = 3.635292, \ldots$$

Thus the eigenvalues are positive and by Theorem 3, problem (36) is equivalent to a beam equation. Using the Runge-Kutta method and then applying the least squares method we obtain a polynomial least squares approximation for the problem $a'' - 0.01z^2a = 0, \quad a(0) = 1, a'(0) = 0,$ *as follows*

$$a_0(z) \simeq \frac{16921}{16920} - \frac{23}{26805}z + \frac{73}{30503}z^2 - \frac{83}{32139}z^3 + \frac{125}{58146}z^4 - \frac{17}{51916}z^5 + \frac{5}{147791}z^6.$$

From (34) we obtain linear mass density $\gamma_0(s)$ for an equivalent beam equation. By substituting g_1, g_2 and $a_0(z)$ in (15) we find new cross section $a_1(z)$ and $a_2(z)$.

Having $a_1(z)$ and $a_2(z)$ from (34) we obtain new isospectral beam equations with linear mass density, $\gamma_1(s)$ and $\gamma_2(s)$. In Figures 1 and 2, $\gamma_1(s)$ and $\gamma_2(s)$ are plotted for different values of α. Note that for $\alpha = 0$, we have $\gamma_1 = \gamma_2 = \gamma_0$. In Table 4, we compute the eigenvalues of constructed beams. Numerical results confirm the analytical results and isospectrality of the constructed beams.

	Eigenvalues for $m = 1$					Eigenvalues for $m = 2$		
n	$\alpha = 0$	$\alpha = -0.5$	$\alpha = 0.5$	$\alpha = 0.9$		$\alpha = -0.5$	$\alpha = 0.5$	$\alpha = 0.9$
1	0.2150	0.2157	0.2151	0.2151		0.2150	0.2150	0.2150
2	2.7545	2.7589	2.7532	2.7540		2.7565	2.7548	2.7553
3	13.2141	13.2660	13.2117	13.2209		13.2245	13.2153	13.2194
4	40.9467	41.1181	40.9485	40.9768		41.0118	40.9364	40.9415
5	99.0315	99.0413	99.0594	99.0924		98.9849	99.0153	98.9765

Table 4: *Eigenvalues of reconstructed beams isospectral to problem (25)*

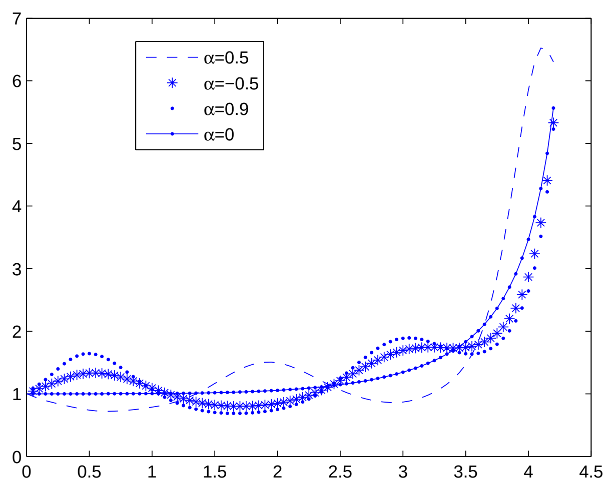

Figure 2: *Linear mass density of isospectral beams corresponding to g_2*

4. Conclusion

In this paper, we have shown that a class of fourth order Sturm-Liouville problem is equivalent to the second order Sturm-Liouville problem. Moreover, by using Darboux's lemma we construct an isospectral FSLP to a given one. Finally, we obtain the beam equation equivalent to FSLP (16), and construct beam equations having the same spectrum.

5. Appendix

Lemma 2. *Let (λ_n, y_n) be an eigenpair of FSLPs (9) and (10). Then for a sufficiently large number n, $\chi_n(z) = (D^2 - q + \lambda_n)y_n$ is zero.*

Proof. By assumption (λ_n, y_n) is an eigenpair of FSLP (9) and (10). Thus

$$y_n^{(4)}(z) - 2(q(z)y_n'(z))' + (q^2(z) - q''(z))y_n(z) = \lambda_n^2 y_n(z).$$

We can factorize this equation as follows

$$(D^2 - q - \lambda_n)(D^2 - q + \lambda_n)y_n = 0.$$

Therefore we have

$$\chi_n'' - (q + \lambda_n)\chi_n = 0, \tag{38}$$

by boundary conditions (10), $\chi_n(0) = \chi_n(L) = 0$. Multiplying equation (38) by χ_n and integrating by parts in $[0, L]$ we find

$$\int_0^L {\chi_n'}^2 dz + \int_0^L (\lambda_n + q)\chi_n^2 dz = 0, \tag{39}$$

by (8), for a sufficiently large number n, $\lambda_n + q > 0$. Thus equation (39) is satisfied only if $\chi_n = 0$. □

References

[1] I. Calio, G. M. L. Gladwell, A. Morassi, *Families of beams with a given buckling spectrum*, Inverse Problems **27**(2011), 045006.

[2] L. F. Caudill, P. A. Perry, A. W. Schueller, *Isospectral sets for fourth-order ordinary differential operators*, SIAM J. Math. Anal. **29**(1998), 935–966.

[3] G. Darboux, *Sur la representation spherique des surfaces*, C. R. Math. Acad. Sci. Paris **94**(1882), 1343–1345.

[4] K. Ghanbari, *On the Isospectral beams*, Electron. J. Differ. Equ. Conf. **12**(2005), 57–64.

[5] M. L. G. Gladwell, *Inverse problems in vibration*, Kluwer Academic, New York, 2004.

[6] M. L. G. Gladwell, A. Morassi, *A family of isospectral EulerBernoulli beams*, Inverse Problems **26**(2010), 035006, 1–12.

[7] M. L. G. Gladwell, A. Morassi, *On isospectral rods, horns and strings*, Inverse Problems **11**(1995), 533–554.

[8] H. P. W. Gottlieb, *Isospectral Euler-Bernoulli beams with continuous density and rigidity functions* Proc. R.Soc. Lond. A **413**(1987), 235–250.

[9] A. Iserles, S. P. Norset, *On the solution of linear differential equations in Lie groups*, Numerical Analysis Report, University of Cambridge, Cambridge, 1997.

[10] V. Ledoux, M. V. Daele, G. V. Berghe, *Matslise: A Matlab package for the Numerical Solution of Sturm-Liouville and Schrodinger equations*, ACM Trans. Math. Software **31**(2005), 532–554.

[11] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, Dover, New York, 1994.

[12] J. POSCHEL, E. TRUBOWITZ, *Inverse Spectral Theory* , UK: Academic, London, 1987.
[13] C. W. SOH, *Isospectral Euler-Bernoulli beams via factorization and the Lie method*, Int. J. Non-Linear Mech. **44**(2009), 396–403.
[14] G. SUBRAMANIAN, A. RAMAN, *Isospectral systems for tapered beams*, J. Sound Vib. **198**(1996), 257–266.