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Perfect 1-error-correcting Hurwitz weight codes*

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Abstract. Let π be a Hurwitz prime and $p = \pi\pi^\star$. In this paper, we construct perfect 1-error-correcting codes in $\mathcal{H}_\pi^n$ for every prime number $p > 3$, where $\mathcal{H}$ denotes the set of Hurwitz integers.

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1. Introduction

Since perfect codes play an important role in coding theory, both for theoretical and practical reasons, many authors have focused on these codes for many years. The first perfect codes which are binary codes were defined by Hamming in [4]. These perfect codes were null-spaces of matrices H and subspaces of vector spaces $\mathbb{Z}_2^n$. Vasil'ev constructed the first non-linear perfect 1-error correcting binary code in [10]. Lindström, 1969, and independently Schönheim, 1968, generalized Vasil'ev's construction to the $q-$ary case in [1, 9]. In 1977, mixed perfect codes were given by Heden in [5]. These codes were not equal to any linear code. As for more recent perfect codes, a generalization of perfect Lee-error-correcting codes and perfect 1-error-correcting Lipschitz weight codes were presented by Heden and Güzeltepe [6, 7].

Let S be the direct product of n copies of the finite field $\mathbb{F}_q$ with q elements and let $C \subset S$. If there is a unique word $e = (0, 0, \dots, \epsilon_i, 0, \dots, 0)$ of weight one to every word $u = (u_1, \dots, u_n) \in S$ such that $u + e \in C$, then C is a perfect 1-error-correcting code in the Hamming metric, where $\epsilon_i \in \mathbb{F}_q$. There are some generalizations of the error-correcting model introduced by Hamming to the cases, when the error ϵ_i belongs to a subset $\mathcal{E}$ of the alphabet used to form the codewords, for example, $\mathcal{E} = \{\mp 1\}$ and base ring $\mathbb{Z}_n$. These codes are called *perfect 1-error-correcting codes* in the Lee metric [8]. In a general case, when $\mathcal{E}$ can be any given subset of the alphabet in use, we call this kind of codes perfect 1-$\mathcal{E}$-error-correcting codes. More information can be found in [6].

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2. Preliminaries

The first codes constructed over Hurwitz integers were presented by Güzeltepe in [2]. The set of Hurwitz intergers is a subset of quaternions. It is a noncommutative ring. The set $H(\mathbb{Z})$ called integer quaternions or Lipschitz integers over the integers $\mathbb{Z}$ is defined as

$$H(\mathbb{Z}) = \{a_0 + a_1e_1 + a_2e_2 + a_3e_3 |\, a_0, a_1, a_2, a_3 \in \mathbb{Z}\},$$

where

$$e_1^2 = e_2^2 = e_3^2 = -1,$$

and

$$e_1e_2 = -e_2e_1 = e_3, \;\; e_2e_3 = -e_3e_2 = e_1, \;\; e_3e_1 = -e_1e_3 = e_2.$$

The set $\mathcal{H} = H(\mathbb{Z}) \cup H(\mathbb{Z} + \frac{1}{2})$ is the set of Hurwitz integers. A Hurwitz prime is an element $\pi = a_0 + a_1e_1 + a_2e_2 + a_3e_3 \in \mathcal{H}$ such that $p = \pi\pi^\star = (a_0 + a_1e_1 + a_2e_2 + a_3e_3)(a_0 - a_1e_1 - a_2e_2 - a_3e_3) = a_0^2 + a_1^2 + a_2^2 + a_3^2$ is an odd prime in $\mathbb{Z}$. The norm of π is defined by $N(\pi) = \pi\pi^\star$.

The elements in the right ideal

$$\langle\pi\rangle = \{\pi\delta |\; \delta \in \mathcal{H}\}$$

define a normal subgroup of the additive group of the ring $\mathcal{H}$. The set of cosets to $\langle\pi\rangle$ in $\mathcal{H}$ defines an Abelian group denoted as $\mathcal{H}_\pi = \mathcal{H}/\langle\pi\rangle$.

Let

$$\mathcal{E} = \{\pm1, \pm e_1, \pm e_2, \pm e_3, \frac{1}{2}(\pm1 \pm e_1 \pm e_2 \pm e_3)\}$$

and let $\mathcal{E}_\pi$ denotes the family of cosets to $\langle\pi\rangle$ containing the elements of $\mathcal{E}$. We define the distance between the words $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ in $\mathcal{H}_\pi^n$, $d(u, v) = 1$, if there exists a $j \in \{1, 2, \ldots, n\}$ and an ϵ in $\mathcal{E}_\pi$ such that $u_j = v_j + \epsilon$ and $u_i = v_i$ for $i \neq j$. A perfect 1-error-correcting Hurwitz weight code of length n is a subset C of $\mathcal{H}^n$, such that every element in $\mathcal{H}^n \backslash C$ is at distance one from exactly one word of C.

The following theorem was presented in [3].

Theorem 1. *If π is an odd Hurwitz prime, then the size of $\mathcal{H}_\pi$ is equal to $N(\pi)^2$.*

Here note that if the norm of a Hurwitz integer q is an odd integer, then the element q is called an odd Hurwitz integer.

3. Notations and some lemmas

The symbol $[\cdot]$ denotes the rounding to the closest integer. For example, $\left[\frac{6}{11}\right] = 1$. For Hurwitz integers we can define a rounding operation as follows: *The rounding of Hurwitz integers:*

$$[a_0 + a_1e_1 + a_2e_2 + a_3e_3] = [a_0] + [a_1]\, e_1 + [a_2]\, e_2 + [a_3]\, e_3.$$

Also, the symbol $[[\cdot]]$ denotes the rounding to the closest number in $H(\mathbb{Z}+\frac{1}{2})$. For example,

$$\left[\left[\frac{6}{11}\right]\right]=\frac{1}{2},\ \left[\left[\frac{-3}{11}\right]\right]=-\frac{1}{2},\ \left[\left[\frac{-1}{11}\right]\right]=-\frac{1}{2},\ \left[\left[\frac{12}{11}\right]\right]=\frac{3}{2}.$$

In case $[[a]]$, $a\in\mathbb{Z}$, one can choose any of the closest numbers, for example, $[[\frac{0}{11}]]=\frac{1}{2}$, $[[\frac{0}{11}]]=-\frac{1}{2}$.

Proposition 1. *Let π be a prime in $H(\mathbb{Z})$ and let q be an element in Hurwitz integers $\mathcal{H}$. Then there exist $\gamma\in H(\mathbb{Z})$ and $\delta_1\in\mathcal{H}$ such that*

$$q=\pi\gamma+\delta_1 \textit{ and } N(\delta_1)<N(\pi).$$

Proof. Let $\widetilde{q}=\pi^\star q=s_0+s_1e_1+s_2e_2+s_3e_3$. Then there exists $\gamma\in H(\mathbb{Z})$, such that

$$\gamma=\left[\tfrac{s_0}{N(\pi)}\right]+\left[\tfrac{s_1}{N(\pi)}\right]e_1+\left[\tfrac{s_2}{N(\pi)}\right]e_2+\left[\tfrac{s_3}{N(\pi)}\right]e_3=r_0+r_1e_1+r_2e_2+r_3e_3$$

and $N(\widetilde{q}-\gamma N(\pi))<N(\pi)^2$. Indeed,

$$N(\widetilde{q}-\gamma N(\pi))=t_0^2+t_1^2+t_2^2+t_3^2<4(\frac{N(\pi)}{2})^2=N(\pi)^2,$$

where $t_i=s_i-N(\pi)r_i$ such that $|t_i|<\frac{N(\pi)}{2}$. Hence we obtain that

$$N(\pi)N(\pi^\star)=N(\pi)^2>N(\widetilde{q}-N(\pi)\gamma)=N(\pi^\star q-\pi^\star\pi\gamma)=N(\pi)N(q-\pi\gamma)=N(\pi)N(\delta_1).$$

This completes the proof.

□

Proposition 2. *Let π be a prime in $H(\mathbb{Z})$ and let q be an element in Hurwitz integers $\mathcal{H}$. Then there exist $\gamma\in H(\mathbb{Z}+\frac{1}{2})$ and $\delta_2\in\mathcal{H}$ such that*

$$q=\pi\gamma+\delta_2 \text{ and } \mathrm{N}(\delta_2)<\mathrm{N}(\pi).$$

Proof. Let $\widetilde{q}=\pi^\star q=s_0+s_1e_1+s_2e_2+s_3e_3$. Then there exists $\gamma\in H(\mathbb{Z}+\frac{1}{2})$, such that

$$\gamma=\left[\left[\frac{s_0}{N(\pi)}\right]\right]+\left[\left[\frac{s_1}{N(\pi)}\right]\right]e_1+\left[\left[\frac{s_2}{N(\pi)}\right]\right]e_2+\left[\left[\frac{s_3}{N(\pi)}\right]\right]e_3=r_0+r_1e_1+r_2e_2+r_3e_3$$

and $N(\widetilde{q}-\gamma N(\pi))<N(\pi)^2$. The remainder of the proof is similar to the proof of Proposition 1.

□

We now define some sets as follows

$$A_i=\{q\in H(\mathbb{Z}):N(q)=i\}.$$

Let π be a prime in $H(\mathbb{Z})$ and let $q \in A_i$. From now on, w will denote $\frac{1}{2}(1+e_1+e_2+e_3)$. Using Proposition 1 and Proposition 2, we get

$$q = \pi\gamma_1 + \delta_1, \quad \text{with} \quad N(\delta_1) < N(\pi)$$

and

$$q = \pi\gamma_2 + \delta_2, \quad \text{with} \quad N(\delta_2) < N(\pi),$$

respectively. Under the above conditions, we define

$$A_{i_1} = \{q \in A_i : \quad N(\delta_1) \neq N(\delta_2) \text{ and } N(\delta_1), N(\delta_2) \geq i\}$$

and

$$A_{i_2} = \{q \in A_i : \ N(\delta_1) = N(\delta_2) = i\}.$$

It is obvious that

$$A_i = A_{i_1} \cup A_{i_2} \quad \text{and} \quad A_{i_1} \cap A_{i_2} = \emptyset.$$

Note that for any $q \in A_i$, if $N(\delta_1) < i$ or $N(\delta_2) < i$, then we obtain that $A_{i_1} = A_{i_2} = \emptyset$.

Proposition 3. *For all i, $|A_i| = 24k$, where $k \in \mathbb{Z}$.*

Proof. It is clear that $A_1 = \mathcal{E}$ and $|\mathcal{E}| = 24$. Let $q_1 \in A_s$, $1 \leq s \leq i$; then $N(q_1) = s$. If $\mu \in \mathcal{E}$, then $q_1\mu \in A_s$ since $N(q_1\mu) = N(q_1)N(\mu) = s$. Recall that the norm N is a multiplicative norm. Hence, we get $|q_1\mathcal{E}| = 24$. Let $q_2 \notin q_1\mathcal{E}, q_2 \in A_s$. Then $|q_2\mathcal{E}| = 24$. In this way, let us assume that the number of distinct elements in A_s is k_1. Hence, we obtain

$$A_s = q_1\mathcal{E} \cup q_2\mathcal{E} \cup \cdots \cup q_{k_1}\mathcal{E} \text{ and, therefore } |A_s| = 24k_1.$$

This completes the proof. □

4. Construction of necessary partitions

Under the above conditions, we now give a partition of $\mathcal{H}_\pi$.

Definition 1. *Let π be a prime in $\mathcal{H}$ with $N(\pi) = p$, where $p > 3$ is a prime integer. We define the set P_{i_1} as*

$$P_{i_1} = A_{i_1}.$$

Let $q \in A_{i_2}$. Let us assume that $q = \pi\gamma_1 + \delta_1$, $q = \pi\gamma_2 + \delta_2$, where δ_1 and δ_2 were defined in Proposition 1 and Proposition 2, respectively. Then the set P_{i_2} is constructed by $\delta_1 \in P_{i_2}, \delta_2 \notin P_{i_2}$; in other words,

$$P_{i_2} \underset{\neq}{\subset} A_{i_2} = \{q \in A_{i_2} : \ q = \delta_1, \ N(\delta_1) = N(\delta_2) = i\} - \delta_2\mathcal{E}.$$

Then, the partition of $\mathcal{H}_\pi$ is obtained by

$$P_i = P_{i_1} \cup P_{i_2}.$$

If $P_j = \emptyset$, then P_j is not in the partition; that is, the index i is not ordered. For example, in Example 2, P_4 does not appear in the partition.

Example 1. *Let* $\pi = 3 + e_1 + e_2$. *Then*

$$\begin{aligned} P_1 &= P_{1_1} \cup P_{1_2} = \mathcal{E} \cup \emptyset = \mathcal{E}, \\ P_2 &= P_{2_1} \cup P_{2_2} = (1+e_1)\mathcal{E} \cup \emptyset = (1+e_1)\,\mathcal{E} \end{aligned}$$

and

$$P_3 = P_{3_1} \cup P_{3_2},$$

where $P_{3_1} = A_{3_1} = (1+e_1+e_2)\mathcal{E} \cup (1+e_1+e_3)\mathcal{E}$,

$$\begin{aligned} P_{3_2} &= A_{3_2} - \left(\frac{1}{2}(-1+e_1-e_2-3e_3)\right)\mathcal{E} \\ &= \left\{ \begin{array}{l} \pm(1+e_2+e_3), \pm(1+e_1-e_2), \pm(1-e_1-e_3), \pm(e_1+e_2-e_3), \\ \pm\left(\frac{1}{2}-\frac{e_1}{2}-\frac{3e_2}{2}-\frac{e_3}{2}\right), \pm\left(\frac{1}{2}-\frac{3e_1}{2}+\frac{e_2}{2}+\frac{e_3}{2}\right), \\ \pm\left(\frac{1}{2}+\frac{e_1}{2}-\frac{e_2}{2}+\frac{3e_3}{2}\right), \pm\left(\frac{3}{2}+\frac{e_1}{2}+\frac{e_2}{2}-\frac{e_3}{2}\right), \\ \pm\left(\frac{3}{2}-\frac{e_1}{2}-\frac{e_2}{2}+\frac{e_3}{2}\right), \pm\left(\frac{1}{2}-\frac{e_1}{2}+\frac{3e_2}{2}-\frac{e_3}{2}\right), \\ \pm\left(\frac{1}{2}+\frac{e_1}{2}-\frac{e_2}{2}-\frac{3e_3}{2}\right), \pm\left(\frac{1}{2}+\frac{3e_1}{2}+\frac{e_2}{2}+\frac{e_3}{2}\right), \\ \pm(e_1+e_2+e_3), \pm(1-e_2+e_3), \pm(1+e_1-e_3), \pm(1-e_1+e_2), \\ \pm\left(\frac{3}{2}-\frac{e_1}{2}-\frac{e_2}{2}-\frac{e_3}{2}\right), \pm\left(\frac{1}{2}-\frac{e_1}{2}+\frac{e_2}{2}+\frac{3e_3}{2}\right), \\ \pm\left(\frac{1}{2}+\frac{3e_1}{2}-\frac{e_2}{2}+\frac{e_3}{2}\right), \pm\left(\frac{1}{2}+\frac{e_1}{2}+\frac{3e_2}{2}-\frac{e_3}{2}\right), \\ \pm\left(\frac{3}{2}+\frac{e_1}{2}+\frac{e_2}{2}+\frac{e_3}{2}\right), \pm\left(\frac{1}{2}+\frac{e_1}{2}-\frac{3e_2}{2}-\frac{e_3}{2}\right), \\ \pm\left(\frac{1}{2}-\frac{e_1}{2}+\frac{e_2}{2}-\frac{3e_3}{2}\right), \pm\left(\frac{1}{2}-\frac{3e_1}{2}-\frac{e_2}{2}+\frac{e_3}{2}\right), \end{array} \right\} \\ &\quad - \left(\frac{1}{2}(-1+e_1-e_2-3e_3)\right)\mathcal{E} \\ &= \left\{ \begin{array}{l} \pm(1+e_2+e_3), \pm(1+e_1-e_2), \pm(1-e_1-e_3), \pm(e_1+e_2-e_3), \\ \pm\left(\frac{1}{2}-\frac{e_1}{2}-\frac{3e_2}{2}-\frac{e_3}{2}\right), \pm\left(\frac{1}{2}-\frac{3e_1}{2}+\frac{e_2}{2}+\frac{e_3}{2}\right), \\ \pm\left(\frac{1}{2}+\frac{e_1}{2}-\frac{e_2}{2}+\frac{3e_3}{2}\right), \pm\left(\frac{3}{2}+\frac{e_1}{2}+\frac{e_2}{2}-\frac{e_3}{2}\right), \\ \pm\left(\frac{3}{2}-\frac{e_1}{2}-\frac{e_2}{2}+\frac{e_3}{2}\right), \pm\left(\frac{1}{2}-\frac{e_1}{2}+\frac{3e_2}{2}-\frac{e_3}{2}\right), \\ \pm\left(\frac{1}{2}+\frac{e_1}{2}-\frac{e_2}{2}-\frac{3e_3}{2}\right), \pm\left(\frac{1}{2}+\frac{3e_1}{2}+\frac{e_2}{2}+\frac{e_3}{2}\right), \end{array} \right\} \\ &= (1+e_2+e_3)\ \mathcal{E} \cup (e_1+e_2+e_3)\mathcal{E} - (\frac{1}{2}(-1+e_1-e_2-3e_3))\mathcal{E} \end{aligned}$$

Hence we obtain the partition of $H_{3+e_1+e_2}$ as P_1, P_2, P_3. The sizes of P_1, P_2, P_3 are $24, 24, 72$, respectively. Also $\mathcal{E}_\pi = \{P_1, P_2, P_3\}$.

In order to assist the reader in understanding the method, we give some specific examples.

- *Take the element $q = 1 + e_1 + e_2 \in A_3$. Using Proposition 1 and Proposition 2, we obtain*

$$\begin{aligned} \delta_1 &= q \\ \delta_2 &= \frac{1}{2} - \frac{1}{2}e_1 - \frac{3}{2}e_2 + \frac{3}{2}e_3 \end{aligned}$$

  It is obvious that $N(\delta_1) \neq N(\delta_2)$. So $1 + e_1 + e_2 \in P_{3_1}$.

- *Take the element $q = 1 + e_2 + e_3 \in A_3$. Using Proposition 1 and Proposition 2, we obtain*

$$\begin{aligned} \delta_1 &= q \\ \delta_2 &= -\frac{1}{2} + \frac{1}{2}e_1 - \frac{1}{2}e_2 - \frac{3}{2}e_3 \end{aligned}$$

  It is obvious that $N(\delta_1) = N(\delta_2)$. So $1 + e_2 + e_3 \in P_{3_2}$. It should also be noted that if an element q_1

$$q_1 \in (-\frac{1}{2} + \frac{1}{2}e_1 - \frac{1}{2}e_2 - \frac{3}{2}e_3)\mathcal{E} = \delta_2\mathcal{E},$$

  then $q_1 \notin P_{3_2}$.

- *Take the element $q = \frac{1}{2} + \frac{3}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 \in A_3$. Using Proposition 1 and Proposition 2, we obtain*

$$\begin{aligned} \delta_1 &= q \\ \delta_2 &= -e_1 - e_2 - e_3 \end{aligned}$$

  It is obvious that $N(\delta_1) = N(\delta_2)$. So $\frac{1}{2} + \frac{3}{2}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3 \in P_{3_2}$.

Example 2. *Let $\pi = 2 + 2e_1 + 2e_2 + e_3$. Then*

$$\begin{aligned} P_1 &= P_{1_1} \cup P_{1_2} = \mathcal{E} \cup \emptyset = \mathcal{E}, \\ P_2 &= P_{2_1} \cup P_{2_2} = (1 + e_1)\mathcal{E} \cup \emptyset = (1 + e_1)\,\mathcal{E}, \\ P_{3_1} &= (1 + e_1 + e_2)\,\mathcal{E} \cup (1 + e_1 + e_3)\,\mathcal{E} \cup (1 + e_2 + e_3)\,\mathcal{E} \cup (e_1 + e_2 + e_3)\,\mathcal{E}, \\ P_{3_2} &= \emptyset, \\ P_3 &= P_{3_1} \cup P_{3_2}, \end{aligned}$$

$P_4 = \emptyset$ since either $N(\delta_1) < 4$ or $N(\delta_2) < 4$ for any element $q \in A_4$. For example, let $q = 2 \in A_4$. Then we get $\delta_1 = q$, $\delta_2 = -\frac{3}{2} + \frac{1}{2}e_1 - \frac{1}{2}e_2 + \frac{1}{2}e_3$. It is obvious that $N(\delta_2) = 3 < 4$. So $2 \notin P_4$.
$P_5 = (2 + e_1)\mathcal{E}$.

Note that if $q \in A_5, q \notin P_5$, then we obtain that either $N(\delta_1) < 5$ or $N(\delta_2) < 5$. For example, let $q = 2+e_2$. Then we get $\delta_1 = q = 2+e_2$, $\delta_2 = -\frac{3}{2}+\frac{1}{2}e_1+\frac{1}{2}e_2+\frac{1}{2}e_3$. It is obvious that $N(\delta_2) = 3 < 5$. So $2+e_2 \notin P_5$.

Hence we obtain the partition of $H_{2+2e_1+2e_2+e_3}$ as P_1, P_2, P_3, P_5. The sizes of P_1, P_2, P_3, P_5 are $24, 24, 96, 24$, respectively. Also $\mathcal{E}_\pi = \{P_1, P_2, P_3, P_5\}$.

Observe that $P_1 = \mathcal{E}$, $P_2 = (1+e_1)\mathcal{E}, P_3$ appear in all partitions. P_3 are not fixed. P_4 did not appear in all partitions.

Proof of the following theorem is straightforward from the partition of $\mathcal{H}_\pi$.

Theorem 2. *Assume that π is a prime in $H(\mathbb{Z})$ with $N(\pi) > 3$. Let the partition of $\mathcal{H}_\pi$ be $P_{i_1} = \mathcal{E}, P_{i_2} = (1+e_1)\mathcal{E}, P_{i_3}, \ldots, P_{i_n}$. Let $g_1 = 1 \in P_{i_1} = \mathcal{E}, g_2 = (1+e_1) \in P_{i_2} = (1+e_1)\mathcal{E}, g_3 \in P_{i_3}, \ldots, g_n \in P_{i_n}$. Then the null-space C of the matrix*

$$H = \begin{pmatrix} 1, & 1+e_1, & g_3, & g_4, & \cdots, & g_n \end{pmatrix}$$

is a perfect 1-error-correcting Hurwitz weight code with the length $n = |\mathcal{E}_\pi|$ in $\mathcal{H}_\pi^n$, in the metric defined by $\mathcal{E}\pi$.

Example 3. *Let $\pi = 2+2e_1+2e_2+e_3$. The null-space of the following matrix H*

$$H = \begin{pmatrix} 1, & 1+e_1, & 1+e_1+e_2, & 2+e_1 \end{pmatrix}$$

is a perfect 1-error-correcting Hurwitz weight code with the length $n = |\mathcal{E}_\pi| = 4$. After a transmission, let the received vector $r = \begin{pmatrix} -1-e_1, & 1+e_1, & 1+e_2, & 1 \end{pmatrix}$. Then the syndrome of r is

$$\begin{aligned} S = Hr^{tr} &= \begin{pmatrix} 1, & 1+e_1, & 1+e_1+e_2, & 2+e_1 \end{pmatrix} \begin{pmatrix} -1-e_1, & 1+e_1, & 1+e_2, & 1 \end{pmatrix}^{tr} \\ &= e_1(1+e_1). \end{aligned}$$

This shows that the error has appeared in the second coordinate position since $1+e_1 \in P_2$. The value of the error is e_1. Hence we obtain the corrected vector as

$$c = r - e = \begin{pmatrix} -1-e_1, & 1+e_1, & 1+e_2, & 1 \end{pmatrix} - \begin{pmatrix} 0, & e_1, & 0, & 0 \end{pmatrix} = \begin{pmatrix} -1-e_1, & 1, & 1+e_2, & 1 \end{pmatrix}.$$

Theorem 3. *Assume that π is a prime in $H(\mathbb{Z})$ with $N(\pi) > 3$. Let the partition of $\mathcal{H}_\pi$ be*

$$\begin{aligned} P_{i_1} &= \mathcal{E}, \\ P_{i_2} &= (1+e_1)\mathcal{E}, \\ P_{i_3} &= d_{11}\mathcal{E} \cup d_{12}\mathcal{E}\cup, \ldots, \cup d_{1n_1}\mathcal{E}, \\ P_{i_4} &= d_{21}\mathcal{E} \cup d_{22}\mathcal{E}\cup, \ldots, \cup d_{2n_2}\mathcal{E}, \\ &\vdots \\ P_{i_m} &= d_{m1}\mathcal{E} \cup d_{m2}\mathcal{E}\cup, \ldots, \cup d_{mn_m}\mathcal{E}. \end{aligned}$$

Then the null-space C of the matrix

$$H = (1,\, 1+e_1,\, d_{11},\, \ldots, d_{1n_1}, d_{21}, \ldots d_{2n_2}, \ldots, d_{m1}, \ldots d_{mn_m})$$

is a perfect 1-error-correcting Hurwitz weight code with the length $n = \frac{N(\pi)^2-1}{24}$ in $\mathcal{H}_\pi^n$, in a metric that it is defined as follows:

the distance between the words $u = (u_1, u_2, \ldots, u_n)$ *and* $v = (v_1, v_2, \ldots, v_n)$ *in* $\mathcal{H}_\pi^n$*,* $d(u, v) = 1$*, if there exist a* $j \in \{1, 2, \ldots, n\}$ *and an* $\epsilon \in d_{rs}\mathcal{E}$ *such that* $u_j = v_j + \epsilon$ *and* $u_i = v_i$ *for* $i \neq j$*.*

A perfect 1-error-correcting Hurwitz weight code of length n *is a subset* C *of* $\mathcal{H}^n$ *such that every element in* $\mathcal{H}^n \backslash C$ *is at distance one from exactly one word of* C*.*

Example 4. *Let* $\pi = 2 + 2e_1 + 2e_2 + e_3$*. Then the partition given in Example 2 is*

$$\begin{aligned} P_1 &= \mathcal{E}, \\ P_2 &= (1 + e_1)\,\mathcal{E}, \\ P_3 &= (1 + e_1 + e_2)\,\mathcal{E} \cup (1 + e_1 + e_3)\,\mathcal{E} \cup (1 + e_2 + e_3)\,\mathcal{E} \cup (e_1 + e_2 + e_3)\,\mathcal{E}, \\ P_4 &= \emptyset, \\ P_5 &= (2 + e_1)\,\mathcal{E}, \end{aligned}$$

that is, $d_{11} = 1 + e_1 + e_2, d_{12} = 1 + e_1 + e_3, d_{13} = 1 + e_2 + e_3, d_{14} = e_1 + e_1 + e_3,$ $d_{31} = 2 + e_1$*. The null-space of the following matrix* H

$$H = \big(1,\, 1 + e_1,\, 1 + e_1 + e_2,\, 1 + e_1 + e_3,\, 1 + e_2 + e_3,\, e_1 + e_2 + e_3,\, 2 + e_1\big)$$

is a perfect 1-error-correcting Hurwitz weight code with the length $n = \frac{N(\pi^2)-1}{24} = \frac{169-1}{24} = 7$*.*

The perfect 1-error-correcting code C *is the null-space in* $\mathcal{H}_\pi^7$ *of* H*, the number of codewords of* C *is* $|C| = 169^6$*. The number of errors for which the error is at distance one from exactly one word of* C *is* $7 \cdot 24 \cdot 169^6$*. Considering the well-known sphere packing bound, the code* C *is perfect since it satisfies the bound, that is,*

$$|H_{2+2e_1+2e_2+e_3}|^7 = 169^7 = 7 \cdot 24 \cdot 169^6 + 169^6.$$

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