

TATE-SHAFAREVICH GROUPS OF CONSTANT ELLIPTIC CURVES AND ISOGENY VOLCANOS

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ABSTRACT. We describe the structure of Tate-Shafarevich groups of a constant elliptic curves over function fields by exploiting the volcano structure of isogeny graphs of elliptic curves over finite fields.

1. INTRODUCTION

Let k be a finite field and E/k an elliptic curve. Let F/k be another curve and set $K = k(F)$ its function field. The base change E/K is usually referred to as a constant elliptic curve over the global function field K . Already since the 60's, from the work of Tate completed by Milne [Mil68], it has been known that the full Birch and Swinnerton-Dyer conjecture holds for such constant elliptic curves and, in particular, that the Tate-Shafarevich group $\text{III}(E/K)$ is finite. But not much information on this group beyond what the Birch and Swinnerton-Dyer conjecture gave was obtained at the time. Later, $\text{III}(E/K)$ was computed for supersingular E and various fields K by Elkies, Shioda and others ([Shi91] and references therein) motivated by the fact that the Mordell-Weil group $E(K)$ with its canonical height pairing gave interesting examples of lattice packings. In those examples, $\text{III}(E/K)$ is a p -group, where p is the characteristic of k .

Parallel to these developments, motivated by applications to computational number theory and cryptography, there have been a number of papers studying the ℓ -isogeny graphs on elliptic curves ([Sut13] and references therein). These graphs are defined by taking as vertices elliptic curves over a field k and edges representing isogenies of degree ℓ between them. A fundamental insight of Kohel (Theorem 2.3 below) was extended and reformulated by Fouquet and Morain [FM02] to the statement that a connected component of the ℓ -isogeny graph for a prime ℓ over a finite field k consisting of ordinary curves has a specific structure referred to as ℓ -volcano graph. We will provide the precise description below.

The purpose of this paper is to bring these two trends together and use the structure of the isogeny graphs to obtain explicit description of the Tate-Shafarevich group $\text{III}(E/K)$ in many instances and, indeed, go further and compute all terms of the descent exact sequence for ℓ -isogenies, when F is also an elliptic curve.

2. PRELIMINARIES

2.1. Isogeny graphs.

Definition 2.1. *A graph with vertex set V is called an ℓ -volcano graph if its vertex set can be partitioned as $V = V_1 \cup V_2 \cdots \cup V_m$, where m is called the height of the volcano, V_1 the base and V_m the crater or top. In addition, the induced graph on V_m is a cycle (the edges on this subgraph are called horizontal), the degree of all vertices not on V_1 is $\ell + 1$, the degree of the vertices in V_1 is 1, for each vertex on $V_i, i < m$, there is a unique edge from it to a vertex in*

V_{i+1} (these are called upward edges) and for each vertex on $V_i, i > 1$, the other edges go to vertices in V_{i-1} (these are called downward edges). The vertices in V_i are said to have height (or level) i .

Remark 2.2. Note that, if we remove the edges of V connecting the vertices of V_m , the connected components of the resulting graph are trees with roots at the the vertices of V_m . We call those the trees of the volcano.

We will assume throughout that ℓ is prime and, not the characteristic of k (the case $\ell = p$ will be briefly mentioned in a comment). As stated above, a connected component of the ℓ -isogeny graph of ordinary curves over the finite field k has the structure of a ℓ -volcano (with a minor modification if the component contains a curve with j -invariant 0 or 1728). Moreover, denoting by $h_\ell(E)$ the height (or level) of E in its connected component of the ℓ -isogeny graph, viewed as a volcano, we have $h_\ell(E) = v_\ell([\text{End}(E) : \mathbb{Z}[\pi]])$. We refer to an isogeny of degree a power of ℓ as upward, downward or horizontal if the corresponding path on the ℓ -isogeny graph is a sequence of edges with the same property.

Theorem 2.3. [Koh96, Propositions 21 and 22] *Let E and E' be isogenous ordinary elliptic curves over k . Then $h_\ell(E) = h_\ell(E')$ if and only if there is an isogeny $E \rightarrow E'$ of degree prime to ℓ .*

2.2. Descent. Briefly, let us recall the descent mechanism and the definition of the Selmer and Tate-Shafarevich groups. For details, see [Sil09, Chapter X].

Given an isogeny $\phi : E \rightarrow E_1$ defined over a field K , we have the exact sequence

$$0 \rightarrow \ker \phi \rightarrow E \rightarrow E_1 \rightarrow 0$$

which gives, by taking cohomology, the Kummer sequence,

$$0 \rightarrow E_1(K)/\phi(E(K)) \rightarrow H^1(K, \ker \phi) \rightarrow H^1(K, E)[\phi] \rightarrow 0.$$

If K is a global field, the corresponding exact sequences for each completion K_v of K can also be considered. By comparing the corresponding sequences, one obtains the descent sequence

$$(2.1) \quad 0 \rightarrow E_1(K)/\phi(E(K)) \rightarrow \text{Sel}^\phi(E_1/K) \rightarrow \text{III}(E/K)[\phi] \rightarrow 0$$

where $\text{Sel}^\phi(E_1/K)$ is the Selmer group consisting of the elements of $H^1(K, \ker \phi)$ whose image in $\prod_v H^1(K_v, \ker \phi)$ is also on the image of $\prod_v E_1(K_v)$ (where the direct product is taken over all places v of K) and

$$\text{III}(E/K) := \ker \left(H^1(K, E) \rightarrow \bigoplus_v H^1(K_v, E) \right)$$

is the Tate-Shafarevich group.

2.3. Constant elliptic curves. If E is an elliptic curve over a finite field k and F is another curve defined over k with function field $K = k(F)$, then a basic fact, central to our approach, is the identification $E(K) = \text{Mor}(F, E)$. If, in addition F is also an elliptic curve with identity 0_F then, evaluating a map $\alpha \in E(K) = \text{Mor}(F, E)$ at 0_F gives a retraction of the inclusion $E(k) \subset E(K)$. This gives a split exact sequence,

$$(2.2) \quad 0 \rightarrow E(k) \rightarrow E(K) \rightarrow \text{Hom}(F, E) \rightarrow 0.$$

The following lemma is central to our determination of the first term in (2.1).

Lemma 2.4. *Suppose $\phi : E \rightarrow E_1$ is an isogeny of ℓ -primary degree. If ϕ is downward, then $\text{Hom}(E, E_1) = \phi \text{End}(E)$. If ϕ has prime degree, then $\text{Hom}(E, E_1) = \phi \text{End}(E)$ if and only if ϕ is downward or $E \simeq E_1$ and ϕ is self dual up to an automorphism.*

Proof. First suppose ϕ is downward and $\alpha \in \text{Hom}(E, E_1)$. We can factor α as $\alpha = \alpha_\ell \circ \alpha'$ with $\alpha' : E \rightarrow E'$ an isogeny of degree prime to ℓ and $\alpha_\ell : E' \rightarrow E_1$ an isogeny of ℓ -primary degree. By Theorem 2.3, $h_\ell(E) = h_\ell(E')$ which is greater than $h_\ell(E_1)$ by assumption. From the structure of the ℓ -sogeny graph (see Remark 2.2) the final edge in any path from E' to E_1 in the ℓ -isogeny graph must be ϕ . Therefore $\alpha_\ell \in \phi \text{Hom}(E', E_1)$ and so $\alpha \in \phi \text{End}(E)$.

For the remainder of the proof we suppose ϕ has degree ℓ . If ϕ is not downward, then it is either upward or horizontal. We consider each case in turn.

First suppose ϕ is upward and let $L = \text{End}(E_1) \otimes \mathbb{Q}$. Then ϕ determines an injective ring homomorphism $\iota : \text{End}(E) \rightarrow L$ by $\gamma \mapsto \phi \gamma \hat{\phi} \otimes \ell^{-1}$. This induces an inclusion $\text{End}(E) \subset \text{End}(E_1) \subset L$ with $[\text{End}(E_1) : \text{End}(E)] = \ell$ since ϕ is upward. Applying the case already proven to the downward isogeny $\hat{\phi}$ we get $\text{Hom}(E_1, E) = \hat{\phi} \text{End}(E_1)$. Dualizing gives $\text{Hom}(E, E_1) = \text{End}(E_1) \phi$. If we also had $\text{Hom}(E, E_1) = \phi \text{End}(E)$, then multiplying by $\hat{\phi}$ on the right would give $\phi \text{End}(E) \hat{\phi} = \text{End}(E_1) \ell$. But, this implies $\iota(\text{End}(E)) = \text{End}(E_1)$, which is a contradiction.

Now consider the case when ϕ is horizontal. This can only happen when $\text{End}(E) = \mathcal{O}_L$ is the maximal order in $L = \text{End}(E) \otimes \mathbb{Q}$ and $\ell = \mathfrak{p}\mathfrak{q}$ is not inert. Moreover, $\phi = \phi_{\mathfrak{p}}$ or $\phi = \phi_{\mathfrak{q}}$, where for an ideal $\mathfrak{a} \subset \text{End}(E)$, the notation here means that $\ker(\phi_{\mathfrak{a}}) = \{P \in E : \alpha(P) = 0, \text{ for all } \alpha \in \mathfrak{a}\}$. Relabeling if necessary we can suppose $\phi = \phi_{\mathfrak{p}}$. If \mathfrak{p} is principal, then $E_1 \simeq E$ and $\hat{\phi} \in \phi \text{End}(E)$ is equivalent to $\hat{\phi} \in \phi \text{Aut}(E)$. Now suppose these ideals are not principal. Then every endomorphism of E of degree divisible by ℓ is itself divisible by $[\ell]$. Let m be the order of \mathfrak{p} in the class group of \mathcal{O}_L . Then $\phi_{\mathfrak{q}^{m-1}} \in \text{Hom}(E, E_1)$ corresponds to the path along the crater rim opposite from $\phi = \phi_{\mathfrak{p}}$. We claim that $\phi_{\mathfrak{q}^{m-1}} \notin \phi \text{End}(E)$. Indeed, if $\phi_{\mathfrak{q}^{m-1}} = \phi_{\mathfrak{p}} \gamma$ with $\gamma \in \text{End}(E)$, then being an endomorphism of ℓ -primary degree $\gamma = [\ell^r]$ for some r . But this implies $\mathfrak{q}^m = \mathfrak{q}^r \mathfrak{p}^{r+1}$, which contradicts unique factorization of ideals in \mathcal{O}_L . \square

3. CALCULATION OF THE SELMER GROUP

Suppose $\phi : E \rightarrow E_1$ is a separable isogeny of degree n over the finite field k with n relatively prime to the characteristic of k . Let F/k be an elliptic curve and set $K = k(F)$. Set $G_k = \text{Gal}(\bar{k}/k)$.

Proposition 3.1. *The ϕ -Selmer group, $\text{Sel}^\phi(E'/K)$, is the unramified subgroup,*

$$\ker \left(\text{H}^1(K, E[\phi]) \rightarrow \prod_v \text{H}^1(\bar{k}K_v, E[\phi]) \right).$$

Proof. It is enough to show that for any place v of K the image of $E_1(K_v)/\phi(E(K_v))$ in $\text{H}^1(K_v, E[\phi])$ under the map in (2.1) coincides with the kernel of the restriction map

$H^1(K_v, E[\phi]) \rightarrow H^1(\bar{k}K_v, E[\phi])$. Suppose k_v is the residue field of K_v . The inflation-restriction sequence yields a commutative diagram with exact row

$$\begin{array}{ccccccc} E_1(k_v)/\phi(E(k_v)) & \longrightarrow & E_1(K_v)/\phi(E(K_v)) & & & & \\ \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & H^1(k_v, E[\phi]) & \longrightarrow & H^1(K_v, E[\phi]) & \longrightarrow & H^1(\bar{k}K_v, E[\phi]). \end{array}$$

We claim that the arrows emanating from the top left are both isomorphisms, from which the result easily follows.

The top horizontal map is induced by the inclusion $E_1(k_v) \subset E_1(K_v)$. It is an isomorphism since the kernel of the reduction map $E_1(K_v) \rightarrow E_1(k_v)$ is ℓ -divisible for every ℓ not divisible by $p = \text{char}(k)$ [Sil09, Propositions IV.2.3 and VI.2.2] and $\deg(\phi)$ is prime to p . That the vertical arrow is an isomorphism follows from exactness of (2.1) and the fact that $H^1(k, E) = 0$ since every smooth genus 1 curve over k_v has a rational point. \square

Define $r_\phi : \text{Hom}(F, E') \rightarrow \text{Hom}(F[n], E[\phi])$ by $r_\phi(\alpha) = \phi^*\alpha|_{F[n]}$, where the notation means the restriction of the isogeny $\phi^*\alpha$ to $F[n]$.

Lemma 3.2. $r_\phi(\alpha) = (\alpha^*|_{E'[\phi^*]})^D$, the Cartier dual of the morphism of Galois modules obtained by restricting α^* to kernel of ϕ^* .

Proof. The Weil pairing gives isomorphisms $F[n] \ni x \mapsto e_n(x, \bullet) \in \text{Hom}(F[n], \mu_n) = F[n]^D$ and $E[\phi] \ni x \mapsto e_\phi(x, \bullet) \in E'[\phi^*]^D$. The Cartier dual of α^* restricted to $F[n]$ is the map $F[n]^D \ni e_n(x, \bullet) \mapsto e_n(x, \alpha^*(\bullet))$. For $y \in E'[\phi^*]$ functoriality of the Weil pairing gives that

$$e_n^F(x, \alpha^*(y)) = e_n^{E'}(\alpha(x), y) = e_\phi(\phi^*\alpha(x), y).$$

Therefore $(\alpha^*|_{E'[\phi^*]})^D$ is the map sending x to $\phi^*\alpha(x) = r_\phi(\alpha)(x)$. \square

Lemma 3.3. $r_\phi(\alpha) = 0$ if and only if $\alpha \in \phi \text{Hom}(F, E)$.

Proof. If $\alpha = \phi\alpha'$, then $r_\phi(\alpha) = \phi^*\phi\alpha' = n\alpha' = 0$ when restricted to $F[n]$. Conversely, suppose $\phi^*\alpha = 0$ on $F[n]$. Then the isogeny $\phi^*\alpha$ must factor through multiplication by n . This means there is some $\beta \in \text{Hom}(F, E')$ such that $\phi^*\alpha = n\beta = \phi^*\phi\beta$. Therefore the map $\alpha - \phi\beta$ takes values in $E'[\phi^*]$. But a difference of isogenies is either surjective or the zero map. Hence $\alpha = \phi\beta$. \square

Remark 3.4. In the case that $\phi = [\ell]$ is multiplication by ℓ on E , $r_\phi(\alpha) = \ell\alpha \in \text{Hom}(F[\ell^2], E[\ell])$. This can be identified with $\alpha|_{F[\ell]} \in \text{Hom}(F[\ell], E[\ell])$ as follows. Every element of the Galois module $\text{Hom}(F[\ell^2], E[\ell])$ is divisible by ℓ . The map sending $\beta = \ell\beta' \in \text{Hom}(F[\ell^2], E[\ell])$ to $\beta'|_{F[\ell]} \in \text{Hom}(F[\ell], E[\ell])$ is an isomorphism sending $r_\ell(\alpha)$ to $\alpha|_{F[\ell]}$.

Theorem 3.5. *There is an exact and commutative diagram*

$$\begin{array}{ccccccc}
0 & \longrightarrow & E'(k)/\phi(E(k)) & \longrightarrow & E'(k)/\phi(E(k)) & & \\
& & \downarrow & & \downarrow \partial_\phi & & \\
0 & \longrightarrow & E'(K)/\phi(E(K)) & \xrightarrow{\partial_\phi} & \text{Sel}^\phi(E'/K) & \longrightarrow & \text{III}(E/K)[\phi] \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & \frac{\text{Hom}_{G_k}(F, E')}{\phi \text{Hom}_{G_k}(F, E)} & \xrightarrow{r_\phi} & \text{Hom}_{G_k}(F[n], E[\phi]) & \longrightarrow & \text{III}(E/K)[\phi] \longrightarrow 0
\end{array}$$

Proof. Let $G = \text{Gal}(K^{\text{sep}}/\bar{k}K)$. Since $E[\phi]^G = E[\phi]$, the inflation-restriction sequence gives

$$0 \rightarrow H^1(k, E[\phi]) \rightarrow H^1(K, E[\phi]) \rightarrow \text{Hom}_{G_k}(G, E[\phi]) \rightarrow H^2(k, E[\phi]) = 0,$$

where the final term is 0 because k , being a finite field, has cohomological dimension 1. By the Weil conjectures $H^1(k, E) = 0$. So the connecting homomorphism induces an isomorphism $\partial_\phi : E'(k)/\phi(E(k)) \simeq H^1(k, E[\phi])$. By Proposition 3.1 the ϕ -Selmer group and its image in $\text{Hom}_{G_k}(G, E[\phi])$ are the unramified subgroups. The maximal unramified extension of $\bar{k}K$ of exponent ℓ corresponds to multiplication by ℓ on \bar{k} , so every unramified (continuous) homomorphism $G \rightarrow E[\phi]$ factors through $F[\ell]$. This gives an isomorphism $\text{Hom}_{G_k}(G, E[\phi])^{\text{unr}} \rightarrow \text{Hom}_{G_k}(F[\ell], E[\phi])$.

It remains to show that the square involving r_ϕ commutes. For this suppose $(\alpha : F \rightarrow E') \in E'(K)$ is a morphism. Adjusting by $E'(k)$ if necessary we may assume $\alpha \in \text{Hom}(F, E')$ is an isogeny. The fibered product $F' = F \times_{E'} E$ along α and ϕ corresponds to an extension L/K for which there is a point $\beta \in E(L)$ such that $\phi\beta = \alpha$. The cocycle $\eta : \text{Gal}(\bar{K}/K) \rightarrow E[\phi]$ given by $\sigma \mapsto \sigma(\beta) - \beta$ represents $\partial_\phi(\alpha)$. The action of σ is translation by some $P_\sigma \in \ker(\psi)$ so that $(\sigma\beta)(x) = \beta(x + P_\sigma)$. Then $\sigma\beta - \beta = \beta(P_\sigma)$. We have a commutative diagram

$$\begin{array}{ccccc}
F & \xrightarrow{\psi^*} & F' & \xrightarrow{\psi} & F \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \alpha \\
E' & \xrightarrow{\phi^*} & E & \xrightarrow{\phi} & E'
\end{array}$$

where the rows are multiplication by $n = \text{deg}(\phi)$. The cocycle η factors through $F[n]$ and from the commutativity of the diagram we see that the map $\sigma \mapsto \beta(P_\sigma)$ is given by $r_\phi(\alpha) : F[n] \rightarrow E[\phi]$. \square

When ϕ is multiplication by n the theorem together with Remark 3.4 gives the following.

Corollary 3.6. *Suppose E, F are elliptic curves over the finite field k and n is an integer not divisible by the characteristic of k . With $K = k(F)$, there is an exact sequence*

$$0 \rightarrow \frac{\text{Hom}_{G_k}(F, E)}{n \text{Hom}_{G_k}(F, E)} \rightarrow \text{Hom}_{G_k}(F[n], E[n]) \rightarrow \text{III}(E/K)[n] \rightarrow 0.$$

If E and F are isogenous ordinary elliptic curves, then $\text{III}(E/K)[n]$ has rank at most 2 over \mathbb{Z}/n .

This says that the nontrivial elements of $\text{III}(E/K)[n]$ arise from Galois-module morphisms $F[n] \rightarrow E[n]$ that do not come from an isogeny $F \rightarrow E$. From this one sees quite directly how Tate's isogeny theorem is equivalent to the finiteness of $\text{III}(E/K)[n^\infty]$.

Corollary 3.7. *Suppose $\phi : E \rightarrow E_1$ is an isogeny with kernel generated by a k -rational point of order n . Then $\text{Sel}^\phi(E/K)$ sits in a split exact sequence*

$$0 \rightarrow k^\times/k^{\times n} \rightarrow \text{Sel}^\phi(E/K) \rightarrow F[n](k) \rightarrow 0.$$

If $E[n] \subset E(k)$, then $\text{Sel}^n(E/K)$ and sits in a split exact sequence

$$0 \rightarrow k^\times/k^{\times n} \oplus k^\times/k^{\times n} \rightarrow \text{Sel}^n(E/K) \rightarrow F[n](k) \oplus F[n](k) \rightarrow 0.$$

Theorem 3.8. *Suppose $\phi : E \rightarrow E_1$ is an isogeny of prime degree ℓ , defined over k whose kernel is generated by a k -rational point, where E, E_1 are ordinary and let F be in the same component of the ℓ -isogeny graph as E and let $K = k(F)$. Then the following are equivalent*

- (1) $h_\ell(E_1) < h_\ell(E) \leq h_\ell(F)$;
- (2) $\text{Hom}(F, E_1) = \phi \text{Hom}(F, E)$;
- (3) $E_1(K)/\phi(E(K)) = E_1(k)/\phi(E(k))$;
- (4) $\text{III}(E/K)[\phi] = \text{III}(E/K)[\ell]$ has rank 2.

Denote by σ the triple

$$(\text{rk } E_1(K)/\phi(E(K)), \text{rk } \text{Sel}^\phi(E/K), \text{rk } \text{III}(E/K)[\phi])$$

where rk is the rank over \mathbb{Z}/ℓ . When the above equivalent conditions hold $\sigma = (1, 3, 2)$. Otherwise $\sigma = (2, 2, 0)$ if $h_\ell(F) = 0$, $\sigma = (3, 3, 0)$ if $h_\ell(E) < h_\ell(E_1) \leq h_\ell(F)$ and $\sigma = (2, 3, 1)$, otherwise.

Proof. The equivalence of (2) and (3) follows immediately from the split exact sequences (2.2) for E and E_1 .

Suppose $h_\ell(E_1) < h_\ell(E) \leq h_\ell(F)$. Let $\alpha \in \text{Hom}(F, E_1)$ be any isogeny. We can factor α as $\alpha_\ell \circ \alpha'$ with $\alpha' : F \rightarrow F'$ of degree prime to ℓ and $\alpha_\ell : F' \rightarrow E_1$ of ℓ -primary degree. By Theorem 2.3 $h_\ell(F) = h_\ell(F')$. Thus α_ℓ is a path on the ℓ -volcano consisting of downward edges, so on one the trees (see remark 2.2) and thus must factor through ϕ . Since α_ℓ has ℓ -primary degree this implies that α_ℓ , and hence also α , lies in $\phi \text{Hom}(F, E)$.

Let us now consider the cases $h_\ell(E) < h_\ell(E_1), h_\ell(F)$ or $h_\ell(E) > h_\ell(E_1), h_\ell(F)$. In either case there is an isogeny $\alpha : F \rightarrow E_1$ with $v_\ell(\deg(\alpha)) = |h_\ell(F) - h_\ell(E_1)|$ which never reaches the level of E . This isogeny can be constructed as follows. Take F' on the tree containing E_1 at the level of F which, in addition, is a child of E_1 if $h(E_1) < h(F)$. Then by Theorem 2.3 there is an isogeny $F \rightarrow F'$ of degree prime to ℓ . There is also a path on the tree from F' to E_1 not going through E which defines an isogeny and the isogeny claimed earlier is the composition of those two. This isogeny cannot possibly lie in $\phi \text{Hom}(F, E)$, since any $\gamma \in \text{Hom}(F, E)$ has $v_\ell(\deg(\gamma)) \geq |h_\ell(F) - h_\ell(E)| > |h_\ell(F) - h_\ell(E_1)|$. Thus (2) does not hold.

It remains to consider the case when $h_\ell(E_1) \geq h_\ell(E)$. Lemma 2.4 handles this in the case $F = E$ or is in the branch of the isogeny graph below E . Otherwise, it is clear that there is a path on the isogeny graph from F to E_1 not passing through E , showing that 2,3 do not hold and, from this and the descent sequence 2.1, it follows that 4 does not hold either.

To complete the proof, let $\sigma = (a, b, c)$ and note that $a + c = b$ and that $b = 2$ if $h_\ell(F) = 0$ and $b = 3$, otherwise. Also, $a \geq 1$ because of the contribution of torsion. In order to have $a = 1$ we must have that all isogenies $F \rightarrow E_1$ factor through E . This can only be the case when $h_\ell(F) \geq h_\ell(E) > h_\ell(E_1)$. From this we can already conclude that $\sigma = (2, 2, 0)$ if $h_\ell(F) = 0$. We also get that $\sigma = (1, 3, 2)$ if $h_\ell(F) \geq h_\ell(E) > h_\ell(E_1)$.

We claim that $a = 3$ precisely when $h_\ell(F) \geq h_\ell(E_1) > h_\ell(E)$. Just as in the previous argument, an isogeny $F \rightarrow E$ must factor through E_1 via $\hat{\phi}$ and so an isogeny in $\phi(E(K))$ must factor through $\phi \circ \hat{\phi} = [\ell]$ and $E_1(K)/\phi(E(K)) = E_1(K)/\ell(E_1(K))$ which has rank 3. Hence $\sigma = (3, 3, 0)$.

If $h_\ell(E_1) = h_\ell(E)$, then $a = 2$, so $\sigma = (2, 3, 1)$ unless $h_\ell(F) = 0$. If $0 < h_\ell(F) \leq h_\ell(E) < h_\ell(E_1)$, then $\sigma = (2, 3, 1)$ as we can find a path from E_1 to some E' at the level of F not passing through E and we can then use an isogeny $F \rightarrow E'$ of degree prime to ℓ . Similarly, for $0 < h_\ell(F) \leq h_\ell(E_1) < h_\ell(E)$, $\sigma = (2, 3, 1)$ also. These are all the possibilities since $|h_\ell(E) - h_\ell(E_1)| \leq 1$.

□

Remark 3.9. *If there is an isogeny $\phi : E \rightarrow E_1$ of degree ℓ whose kernel is not generated by a k -rational point, then the ℓ -isogeny graph has height $m = 0$. In this case, one verifies that $\sigma = (1, 1, 0)$.*

4. CALCULATION OF III

Theorem 4.1. *Suppose that E, F are isogenous ordinary elliptic curves over k and set $\mathcal{O} = \text{End}(E) \cap \text{End}(F)$, the intersection taking place in $\mathbb{Q}(\pi)$. There is an isomorphism of groups $\text{III}(E/k(F)) \simeq \mathcal{O}/\mathbb{Z}[\pi] \times \mathcal{O}/\mathbb{Z}[\pi]$. In particular, for any prime ℓ we have $v_\ell(\#\text{III}(E/k(F))) = 2 \min\{h_\ell(E), h_\ell(F)\}$.*

Remark 4.2. *This generalizes the observation of Milne [Mil68, p 102] that $\#\text{III}(E/k(E)) = [\text{End}(E) : \mathbb{Z}[\pi]]^2$.*

Remark 4.3. *It would be interesting to find a meaningful interpretation of the isomorphism of the theorem and, in particular, identify a natural symplectic pairing on $\mathcal{O}/\mathbb{Z}[\pi] \times \mathcal{O}/\mathbb{Z}[\pi]$ which corresponds to the Cassels-Tate pairing.*

Proof. Let ℓ be a prime. Since $\text{Hom}_{G_k}(F, E)/\ell$ has dimension 2 over \mathbb{F}_ℓ , Corollary 3.6 shows that $\text{III}(E/k(F))[\ell]$ has dimension 0 or 2. Therefore $\text{III}(E/k(F))[\ell^\infty]$ is a product of two cyclic groups. Since $\mathcal{O}/\mathbb{Z}[\pi]$ is cyclic and $v_\ell([\mathcal{O} : \mathbb{Z}[\pi]]) = \min\{h_\ell(E), h_\ell(F)\}$ we see that the first statement of the theorem follows from the claim that, for each ℓ , $v_\ell(\#\text{III}(E/k(F))[\ell^\infty]) = 2 \min\{h_\ell(E), h_\ell(F)\}$. This holds trivially for the p -part as $\text{III}(E/k(F))[p] = 0$ and the p -isogeny graph has just one level (see Section 5).

Note that $\text{III}(E/k(F)) \simeq \text{III}(F/k(E))$, since both are isomorphic to the Brauer group of the surface $E \times F$ [Tat64, Theorem 3.1]. By swapping the roles of E and F if necessary we may therefore assume that $h_\ell(E) \leq h_\ell(F)$. Let E' be the unique elliptic curve lying below F in the ℓ -isogeny graph (i.e., such that there is a downward isogeny $F \rightarrow E'$, possibly the identity) with $h_\ell(E') = h_\ell(E)$. Since $h_\ell(E) = h_\ell(E')$, Theorem 2.3 implies that there is an isogeny $\alpha : E' \rightarrow E$ of degree prime to ℓ . This isogeny induces an isomorphism $\text{III}(E'/k(F))[\ell^\infty] \simeq \text{III}(E/k(F))[\ell^\infty]$. So, replacing E by E' if needed, we are reduced to the case that E lies below F in the ℓ -isogeny graph.

Suppose the characteristic polynomial of Frobenius on F/k factors as $x^2 + tx + q = (x - a)(x - b)$. By [Mil68, Theorem 3] we have

$$(4.1) \quad \#\text{III}(E/k(F)) \cdot R_{E/k(F)} = q \left(1 - \frac{a}{b}\right) \left(1 - \frac{b}{a}\right),$$

where $R_{E/k(F)} = |\det \text{Tr}(\alpha_i \beta_j)|$, for any bases $\{\alpha_i\}$ and $\{\beta_i\}$ of the free \mathbb{Z} -modules $\text{Hom}(F, E)$ and $\text{Hom}(E, F)$. By Lemma 2.4 we have $\text{Hom}(F, E) = \phi \text{End}(F)$. It follows that $R_{E/k(F)} = \deg(\phi)^2 R_{F/k(F)}$. Comparing (4.1) for $F/k(F)$ and $E/k(F)$ shows that $\#\text{III}(F/k(F)) = \deg(\phi)^2 \#\text{III}(E/k(F)) = \ell^{2(h_\ell(F) - h_\ell(E))} \#\text{III}(E/k(F))$. It thus suffices to verify the claim when $F = E$.

Let $\mathfrak{f}, \mathfrak{g} \in \mathbb{Z}$ be the conductors of the orders $\mathbb{Z}[\pi]$ and $\text{End}(F)$ in the quadratic imaginary field $L = \text{End}(F) \otimes \mathbb{Q}$, so that $\mathbb{Z}[\pi] = \mathbb{Z}[\mathfrak{f}D]$, $\text{End}(F) = \mathbb{Z}[\mathfrak{g}D]$ and $\mathcal{O}_L = \mathbb{Z}[D]$, where $D = \frac{d_L + \sqrt{d_L}}{2}$, d_L is the fundamental discriminant. In particular $\text{End}(F)/\mathbb{Z}[\pi] \simeq \mathbb{Z}/n\mathbb{Z}$ where $n = \mathfrak{f}/\mathfrak{g}$. Then

$$R_{F/k(F)} = \left| \det \begin{bmatrix} \text{Tr}([1]) & \text{Tr}(\mathfrak{g}D) \\ \text{Tr}(\mathfrak{g}D) & \text{Tr}(\mathfrak{g}^2 D^2) \end{bmatrix} \right|.$$

For any integer h we have

$$\left| \det \begin{bmatrix} \text{Tr}([1]) & \text{Tr}(hD) \\ \text{Tr}(hD) & \text{Tr}(h^2 D^2) \end{bmatrix} \right| = \left| \begin{bmatrix} 2 & hd_L \\ hd_L & h^2(d_L^2 + d_L)/2 \end{bmatrix} \right| = h^2 d_L$$

As $\pi = \mathfrak{f}D$ we have

$$\mathfrak{f}^2 d_L = \left| \det \begin{bmatrix} \text{Tr}([1]) & \text{Tr}(\pi) \\ \text{Tr}(\pi) & \text{Tr}(\pi^2) \end{bmatrix} \right| = |2(a^2 + b^2) - (a+b)^2| = (a-b)(b-a) = q \left(1 - \frac{b}{a}\right) \left(1 - \frac{a}{b}\right)$$

Comparing with (4.1) we see that $\#\text{III}(F/k(F)) = [\text{End}(F) : \mathbb{Z}[\pi]]^2$, as required. \square

Corollary 4.4. *E has height m in the ℓ -isogeny graph if and only if $\text{III}(E/k(E))[\ell^\infty] \simeq \mathbb{Z}/\ell^m \times \mathbb{Z}/\ell^m$.*

Corollary 4.5. ([MMS⁺08, Theorem 5]) *If E has full rational ℓ^m -torsion, then E has height at least m in the ℓ -isogeny graph.*

Proof. If $E(k)$ contains the ℓ^m -torsion, then by Corollary 3.6 we have that $\text{III}(E/k(E))[\ell^m] \simeq \mathbb{Z}/\ell^m \mathbb{Z} \times \mathbb{Z}/\ell^m \mathbb{Z}$. So the result follows from the previous corollary. \square

Corollary 4.6. *Any downward isogeny of degree ℓ^n must annihilate the ℓ^n -torsion in $\text{III}(E/k(E))$.*

Proof. Suppose E is at level $m > 0$ in the ℓ -isogeny graph. Then $\text{III}(E/k(E))[\ell^\infty] \simeq \mathbb{Z}/\ell^m \mathbb{Z} \times \mathbb{Z}/\ell^m \mathbb{Z}$. By Theorem 3.8 any composition of m downward ℓ -isogenies must kill $\text{III}(E/k(E))[\ell^\infty]$, but each step can only kill ℓ -torsion. Hence each step must kill all of the ℓ -torsion. \square

5. CONCLUDING REMARKS

The questions addressed in this paper can be considered also for supersingular curves and/or in the case $\ell = p$ although (apart from the supersingular $\ell = p$ case) the situation is much simpler.

If E is ordinary and $\ell = p$, there are only two degree p isogenies from E , the Frobenius and Verschiebung. The component of the isogeny graph containing E is thus a cycle consisting of the conjugates of E . The Selmer group and $E_1(K)/\phi(E(K))$ are one dimensional if ϕ is Frobenius or if ϕ is Verschiebung and the p -torsion of E is not defined over K . When ϕ is Verschiebung and the p -torsion of E is defined over K , both these groups are two dimensional. The p -torsion of $\text{III}(E/K)$ is trivial in both cases. See [Ulm91, Proposition 3.3], but note

that the Selmer group in the case $A = 1$ of the Verschiebung descent has dimension one more than reported there.

Now consider the case when E, F are isogenous supersingular curves over k . If $\mathbb{F}_{p^2} \subset k$, then $\text{Hom}_{G_k}(F, E)$ has rank 4 and so Corollary 3.6 shows that $\text{III}(E/k(F))$ is a p -group. If $\mathbb{F}_{p^2} \not\subset k$, then the connected components of the ℓ -isogeny graph of E are volcanoes with at most two levels (for $\ell = 2$, [DG16]). Moreover, if there are two levels, then $\ell = 2$. Indeed, if E is supersingular and $E[\ell] \subset E(k)$, then $\ell | (q - 1)$. If, in addition, $\mathbb{F}_{p^2} \not\subset k$ then $\#E(k) = q + 1$ and we get that $\ell | (q + 1)$, so $\ell = 2$. The analogues of Theorems 3.8 and 4.1 hold when $\mathbb{F}_{p^2} \not\subset k$, with $\text{End}(E)$ an order of the quadratic field $\mathbb{Q}(\sqrt{-p})$.

Finally, we remark that, as noted in [Mil68], if E, F are non-isogenous elliptic curves over k , then it follows from the Birch-Swinnerton-Dyer formula that

$$\#\text{III}(E/k(F)) = (\#E(k) - \#F(k))^2.$$

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