Groups and Geometry 2018 Auckland University

Projective planes, Laguerre planes and generalized quadrangles that admit large groups of automorphisms

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24 January 2018

Projective and affine planes

Definition

A **projective plane** $\mathcal{P} = (P, \mathcal{L})$ *consists of a set* P *of points and a set* $\mathcal{L}$ *of lines (where lines are subsets of* P*) such that the following three axioms are satisfied:*

(J) *Two distinct points can be joined by a unique line.*

(I) *Two distinct lines intersect in precisely one point.*

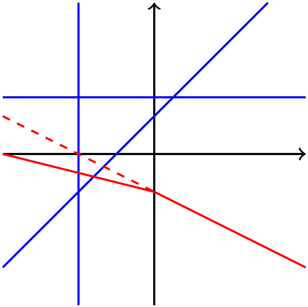
(R) *There are at least four points no three of which are on a line.*

Removing a line and all of its points from a projective plane yields an **affine plane**. Conversely, each affine plane extends to a unique projective plane by adjoining in each line with an 'ideal' point and adding a new line of all ideal points.

Models of projective planes

Desarguesian projective planes are obtained from a 3-dimensional vector space V over a skewfield $\mathbb{F}$. Points are the 1-dimensional vector subspaces of V and lines are the 2-dimensional vector subspaces of V. In case $\mathbb{F}$ is a field one obtains the **Pappian projective plane** over $\mathbb{F}$.

There are many non-Desarguesian projective planes. One of the earliest and very important class is obtain by the (generalized) Moulton planes.

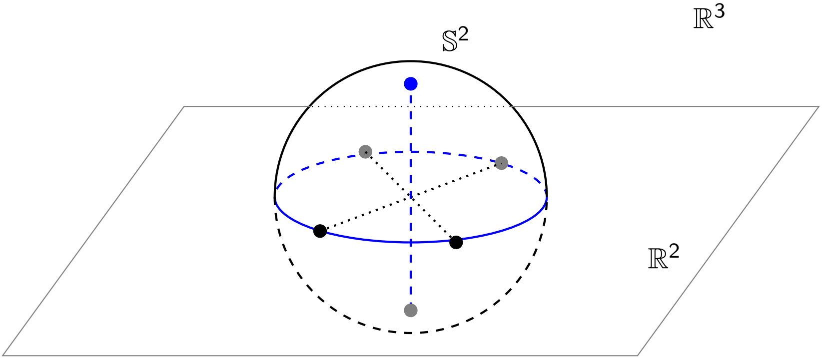


Ways to look at projective planes

Projective and affine planes have been studied under various aspects.

- Configurations that close in them ($\rightarrow$ Desargues and Pappus configurations, ...);
- coordinatizing ternary fields ($\rightarrow$ nearfields, semifields, quasifields, ...);
- linearly transitive groups of central collineations ($\rightarrow$ Lenz-Barlotti classification);
- admitting certain groups of collineations ($\rightarrow$ translation planes, ...);
- finitely many points and lines;
- points and lines sets carrying topologies.

The real Desarguesian plane



The real Desarguesian plane as a topological plane

This model of the real Desarguesian plane on the 2-sphere shows that

- the point set P carries a Hausdorff topology with respect to which P is connected, compact and 2-dimensional;
- the line set $\mathcal{L}$ carries a Hausdorff topology with respect to which $\mathcal{L}$ is connected, compact and 2-dimensional;
- joining two points by a line is continuous with respect to the topologies on $(P \times P) \setminus D_P$ and $\mathcal{L}$;
- intersecting two lines in a point is continuous with respect to the topologies on $(\mathcal{L} \times \mathcal{L}) \setminus D_{\mathcal{L}}$ and P.

Topological projective planes

Definition

A **topological projective plane** *is a projective plane $\mathcal{P} = (P, \mathcal{L})$ where P and $\mathcal{L}$ carry non-discrete Hausdorff topologies such that the two geometric operations described in the axioms (J) and (I) are continuous with respect to induced topologies.*

A topological geometry is called connected or (locally) compact or n-dimensional if the point space has the respective property.

By an n-dimensional projective plane we mean a topological, compact n-dimensional projective plane. Such a plane is connected when $n > 0$.

Theorem (Löwen 1983)

A connected finite-dimensional projective plane is 2-, 4-, 8- or 16-dimensional.

Collineations of projective planes

Definition

A **collineation** *of a projective plane is a permutation of the point set that takes lines to lines.*
An **automorphism** *of a topological projective plane is a continuous collineation.*

Theorem (Löwen 1976)

The automorphism group Σ of a topological, connected, compact projective plane $\mathcal{P} = (P, \mathcal{L})$ is a locally compact group with countable basis when endowed with the compact-open topology. Σ acts a topological transformation group on P and $\mathcal{L}$.

Σ is a Lie group if $\mathcal{P}$ is 2- or 4-dimensional. In this case Σ has dimension at most $4 \cdot \dim \mathcal{P}$.

Collineations of 2-dimensional projective planes

Theorem (Salzmann 1959)

A collineation of a 2-dimensional projective plane is continuous.

Theorem (Salzmann, Groh et al 1950s-1980s)

All 2-dimensional projective planes whose automorphism groups Σ are at least 2-dimensional are known.

- *If* $\dim \Sigma \geq 5$*, then the plane is the real Desarguesian plane.*
- *If* $\dim \Sigma = 4$*, then the plane is isomorphic to a proper Moulton plane.*

There are 2-dimensional projective planes that admit no automorphism other than the identity.

Ovals in projective planes

Definition

*An **oval** in a projective plane $\mathcal{P}$ is a set $\mathcal{O}$ of points such that*

- *every line in $\mathcal{P}$ intersects $\mathcal{O}$ in no more than two points, and*
- *every point p of $\mathcal{O}$ is contained in precisely one line that intersects $\mathcal{O}$ only in p (tangent line to $\mathcal{O}$ at p).*

An oval $\mathcal{O}$ in a topological projective plane is called topological if the map $L \mapsto L \cap \mathcal{O}$ is continuous.

Every strictly convex, differentiable simply closed curve in the Euclidean plane is a topological oval in the real projective plane

Topological ovals in finite-dimensional projective planes

Theorem (Buchanan, Hähl, Löwen 1980)

- *An oval in a connected, finite-dimensional projective plane is topological if and only if the oval is a closed subset of the point set.*
- *If a connected, finite-dimensional projective plane contains a topological oval then the plane is 2- or 4-dimensional.*

Theorem (Polster, Rosehr, S. 1997)

For any three non-collinear points of a 2-dimensional projective plane there is a topological oval passing through these three points.

Laguerre planes

Definition

A **Laguerre plane** $\mathcal{L} = (P, \mathcal{C}, \mathcal{G})$ *consists of a set* P *of points, a set* $\mathcal{C}$ *of circles and a set* $\mathcal{G}$ *of generators (where circles and generators are both subsets of* P*) such that the following five axioms are satisfied:*

(G) $\mathcal{G}$ *partitions* P.

(C) *Each circle intersects each generator in precisely one point.*

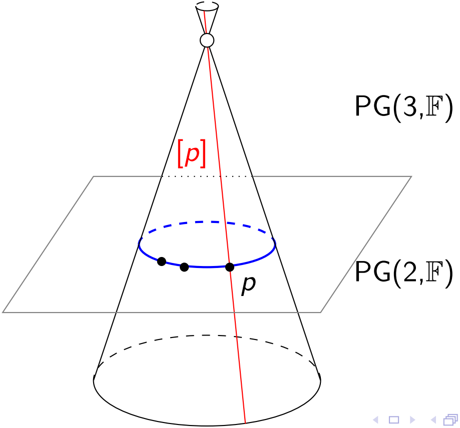
(J) *Three points no two of which are on the same generator can be joined by a unique circle.*

(T) *The circles that touch a circle* C *geometrically at a point* p *of* C *form a partition of* $P \setminus [p]$ *where* $[p]$ *is the generator that contains* p.

(R) *There are at least two circles and there is a circle that contains at least three points nonparallel.*

Models of Laguerre planes

Ovoidal Laguerre planes are obtained as the geometry of non-trivial plane sections of a cone, minus its vertex, over an oval in 3-dimensional projective space over a field $\mathbb{F}$. In case the oval is a conic one obtains the **Miquelian Laguerre plane** over $\mathbb{F}$.



Derived incidence structures

The derived incidence structure $\mathcal{L}_p$ at a point p of a Laguerre plane $\mathcal{L}$ has point set $P \setminus [p]$ and lines the generators of $\mathcal{L}$ not containing p and the circles of $\mathcal{L}$ containing p. This is an affine plane. In fact $\mathcal{L}$ is a Laguerre plane if and only if each $\mathcal{L}_p$ is an affine plane.

Circles not passing through p induce ovals in the projective extension of the affine plane $\mathcal{L}_p$ at p by adding the point at infinity of 'vertical' lines that come from generators of the Laguerre plane. Each of these ovals has the line at infinity as a tangent. In $\mathcal{L}_p$ one has the lines and a collection of parabolic curves.

Conversely, given a collection of lines and parabolic curves one has to amend each member by a point at infinity in order to obtain a Laguerre plane. This may not be possible or may be possible in more than one way.

Topological Laguerre planes

Definition

A **topological Laguerre plane** *is a Laguerre plane* $\mathcal{L} = (P, \mathcal{C}, \mathcal{G})$ *where* P*,* $\mathcal{L}$ *and* $\mathcal{G}$ *carry non-discrete Hausdorff topologies such that the geometric operations described in the axioms (C), (J), (T) and intersection of circles are continuous with respect to induced topologies.*

By an n-dimensional Laguerre plane where we mean a topological, locally compact n-dimensional Laguerre plane. Such a plane is connected when $n > 0$.

A circle not passing through the point p induces a closed oval in the derived projective plane at p.

Corollary

A connected, finite-dimensional Laguerre plane is 2- or 4-dimensional.

Automorphisms of topological Laguerre planes

Definition

A **(geometric) automorphism** *of a Laguerre plane is a permutation of the point set that takes generators to generators and circles to circles. An* **automorphism** *of a topological Laguerre plane is a continuous geometric automorphism.*

Corollary

A (geometric) automorphism of a 2-dimensional Laguerre plane is continuous.

Theorem (Förtsch 1982)

The automorphism group, endowed with the compact-open topology, of a $2s$*-dimensional Laguerre plane where* $s = 1, 2$ *is a Lie group of dimension at most* $7s$.

2-dimensional Laguerre planes

Theorem (Groh 1970)

The 2-dimensional Laguerre planes are precisely the Laguerre planes on the cylinder $Z = \mathbb{S}^1 \times \mathbb{R}$ all whose circles are graphs of continuous maps $\mathbb{S}^1 \to \mathbb{R}$.

Theorem (Groh 1969)

A 2-dimensional Laguerre plane is ovoidal if and only if the kernel of the plane (consisting of all automorphisms that fix each generator) is 4-dimensional.

There are many constructions for non-ovoidal 2-dimensional Laguerre planes.

There are 2-dimensional Laguerre planes that admit no automorphism other than the identity.

Automorphism groups of 2-dimensional Laguerre planes

Theorem (Löwen, Pfüller, S. 1980s-2000)

All 2-dimensional Laguerre planes whose automorphism groups Σ are at least 5-dimensional or whose kernels Δ are at least 3-dimensional are known.

- *If* $\dim \Sigma \geq 6$*, then the plane is the real classical Laguerre plane.*
- *If* $\dim \Sigma = 5$*, then the plane is isomorphic to an ovoidal plane over a skew parabola.*
- *If* $\dim \Delta = 4$*, then the plane is ovoidal.*

The classification of 2-dimensional Laguerre planes whose automorphism groups Σ are 4-dimensional is almost complete. Those planes where Σ fixes a generator or is point transitive are known. There are three families of planes known where Σ fixes a circle and acts transitively on it.

Central automorphisms and Kleinewillinghöfer types

A **central automorphism** of a Laguerre plane is a (geometric) automorphism that fixes at least one point and induces a central collineation in the derived projective plane at this point.

- Laguerre homology: all points on a circle are fixed.
- Laguerre translation: all points on a generator are fixed and a translation is induced one derived affine plane.
- Laguerre homothety: two non-parallel fixed points.

Kleinewillinghöfer 1979 classified groups of automorphisms according to the linearly transitive groups of central automorphisms contained in them. She obtained a total of 46 combined types (with respect to Laguerre homologies, Laguerre translations and Laguerre homotheties).

Kleinewillinghöfer types of 2-dimensional Laguerre planes

The Kleinewillinghöfer type of a Laguerre plane is the type of its (full) automorphism group.

Theorem (Polster, Schillewaert, S. 2004-2018)

A 2-dim. Laguerre plane is of one of the following 25 combined types

I. *A.1, A.2, B.1, B.3, C.1, E.1, E.4, G.1, H.1, H.11,*

II. *A.1, A.2, E.1, E.4, G.1,*

III. *B.1, B.3, H.1, H.11,*

IV. *A.1, A.2,*

V. *A.1,*

VII. *D.1, D.8, or K.13.*

There are examples of 2-dim. Laguerre planes for each of the above types.

Generalized quadrangles

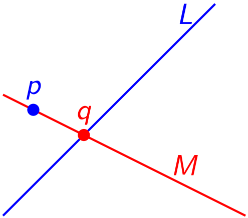
Definition

A **generalized quadrangle** $\mathcal{Q} = (P, \mathcal{L})$ *consists of a point set* P*, a line set* $\mathcal{L}$ *whose elements are subsets of* P *such that*

(J) *Any two distinct points are on at most one line.*

(P) *For every line* L *and every point* p *not on* L*, there exist a unique point* q *on* L *and a unique line* M *through* p *such that* $q \in M$.

(R) *Every point is on at least two lines, and every line contains at least two points.*



Models of generalized quadrangles

Let φ be a symmetric bilinear form of Witt index 2 in $\mathbb{F}^{n+1}$ where $\mathbb{F}$ is a field of characteristic $\neq 2$ and $n \geq 4$. Let Q be the associated quadric in n-dimensional projective space $\mathrm{PG}(n, \mathbb{F})$ over $\mathbb{F}$. If $\mathcal{L}$ is the set of lines of $\mathrm{PG}(n, \mathbb{F})$ entirely contained in Q (2-dimensional totally isotropic subspaces of $\mathbb{F}^{n+1}$ with respect to φ), then $\mathrm{Q}(n, \mathbb{F}, \varphi) = (Q, \mathcal{L})$ is a generalized quadrangle, called the **orthogonal quadrangle (over $\mathbb{F}$)**.

Let ρ be a symplectic polarity in a 3-dimensional projective space $\mathrm{PG}(3, \mathbb{F})$ over $\mathbb{F}$. If P is the point set of $\mathrm{PG}(3, \mathbb{F})$ and $\mathcal{L}$ is the set of lines of $\mathrm{PG}(3, \mathbb{F})$ fixed under ρ, then $\mathrm{W}(\mathbb{F}) = (P, \mathcal{L})$ is a generalized quadrangle, called the **symplectic quadrangle (over $\mathbb{F}$)**.

The generalized quadrangles $\mathrm{W}(\mathbb{F})$ and $\mathrm{Q}(4, \mathbb{F})$ are dual to each other but are not isomorphic.

Topological generalized quadrangles

Definition

A **topological generalized quadrangle** *is a generalized quadrangle $\mathcal{Q} = (P, \mathcal{L})$ where P and $\mathcal{L}$ carry non-discrete Hausdorff topologies such that the two geometric operations described in axiom (P) are continuous with respect to induced topologies.*

A **compact generalized quadrangle with topological parameters** (s, t) *is a generalized quadrangle such that all lines are homotopy equivalent to s-spheres and all line pencils are homotopy equivalent to t-spheres.*

Theorem (Knarr 1990, Kramer 1994)

If a compact generalized quadrangle has topological parameters (s, t) such that $s, t > 1$, then either $s + t$ is odd or $s = t \in \{2, 4\}$.

Theorem (Forst 1981)

The compact 3-dimensional generalized quadrangles are precisely the compact generalized quadrangles with topological parameter 1.

Lie geometry

Definition

The Lie geometry *of a Laguerre plane $\mathcal{L}$ has points the points of $\mathcal{L}$ plus the circles of $\mathcal{L}$ plus one additional point ω at infinity. The lines of the Lie geometry are the generators amended by ω and the extended tangent pencils, that is, the collections of all circles that touch a given circle at a point p together with the point p. Incidence is the natural one.*

Definition

A generalized quadrangle $\mathcal{Q}$ is called **antiregular** *if for each triple of pairwise non-collinear points p, q, r the set $p^\perp \cap q^\perp \cap r^\perp$, where $x^\perp$ is the set of all points collinear with x, is either empty or contains precisely two points.*

Theorem (Forst 1981, Schroth 1995)

The Lie geometry of a $2s$-dimensional Laguerre plane, $s = 1, 2$, is an antiregular compact generalized quadrangle with topological parameter s.

Derivation of an antiregular generalized quadrangle

Definition

The **derivation** $\mathcal{Q}_p$ **at a point** *p of an antiregular generalized quadrangle $\mathcal{Q}$ has point set $p^\perp \setminus \{p\}$ and circle set whose members are of the form $p^\perp \cap q^\perp$ for points q not collinear with p.*

For every point p of an antiregular generalized quadrangle $\mathcal{Q}$ the derivation $\mathcal{Q}_p$ is a Laguerre plane.

Theorem (Schroth 1995)

Each derivation of a compact antiregular generalized quadrangle with topological parameter s is a $2s$-dimensional Laguerre plane.
Up to duality every compact generalized quadrangle with topological parameter s is the Lie geometry of a $2s$-dimensional Laguerre plane.

3-dimensional generalized quadrangles

Corollary (Forst 1981)

The automorphism group of a compact 3-dimensional generalized quadrangle has dimension at most 10.

Theorem (Schroth 2000)

If the automorphism group of a non-Miquelian 2-dimensional Laguerre plane $\mathcal{L}$ has dimension at least 4, then the automorphism group Σ of the generalized quadrangle $\mathcal{Q}$ obtained as the Lie geometry of $\mathcal{L}$ has dimension 4 or 5.

Suppose Σ is the automorphism group of a compact 3-dimensional generalized quadrangle $\mathcal{Q}$. If $\dim \Sigma > 5$, then $\mathcal{Q}$ is the real orthogonal quadrangle or its dual.