MARKOV RANDOM FIELDS AND ITERATED TORIC FIBRE PRODUCTS

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ABSTRACT. We prove that iterated toric fibre products from a finite collection of toric varieties are defined by binomials of uniformly bounded degree. This implies that Markov random fields built up from a finite collection of finite graphs have uniformly bounded Markov degree.

1. INTRODUCTION AND MAIN RESULTS

The notion of *toric fibre product* of two toric varieties goes back to [Sul07]. It is of relevance in algebraic statistics since it captures algebraically the Markov random field on a graph obtained by glueing two graphs along a common subgraph; see [RS16] and also below. In [Sul07, RS16, RS14] it is proved that under certain conditions, one can explicitly construct a Markov basis for the large Markov random field from bases for the components. For related results see [Shi12, EKS14, KR14].

However, these conditions are not always satisfied. Nevertheless, in [RS16, Conjecture 56] the hope was raised that when building larger graphs by glueing copies from a finite collection of graphs along a common subgraph, there might be a uniform upper bound on the Markov degree of the models thus constructed, independent of how many copies of each graph are used. A special case of this conjecture was proved in the same paper [RS16, Theorem 54]. We prove the conjecture in general, and along the way we link it to recent work [SS17] in *representation stability*. Indeed, an important point we would like to make, apart from proving said conjecture, is that algebraic statistics is a natural source of problems in *asymptotic algebra*, to which ideas from representation stability apply. Our main theorems are reminiscent of Sam's recent stabilisation theorems on equations and higher syzygies for secant varieties of Veronese embeddings [Sam17a, Sam17b].

Markov random fields. Let $G = (N, E)$ be a finite, undirected, simple graph and for each node $j \in N$ let X_j be a random variable taking values in the finite set $[d_j] := \{1, \ldots, d_j\}$. A joint probability distribution on $(X_j)_{j\in N}$ is said to satisfy the *local Markov properties* imposed by the graph if for any two non-neighbours $j, k \in N$ the variables X_j and X_k are conditionally independent given $\{X_l \mid \{j, l\} \in E\}$.

On the other hand, a joint probability distribution f on the X_j is said to *factorise according to* G if for each maximal clique C of G and configuration $\alpha \in \prod_{j\in C}[d_j]$ of the random variables labelled by C there exists an interaction parameter θ^C_α such that for each configuration $\beta \in \prod_{j\in N}[d_j]$ of all random variables of G:

$$f(\beta) = \prod_{C\in \mathrm{mcl}(G)} \theta^C_{\beta|_C}$$

where $\mathrm{mcl}(G)$ is the set of maximal cliques of G, and $\beta|_C$ is the restriction of β to C.

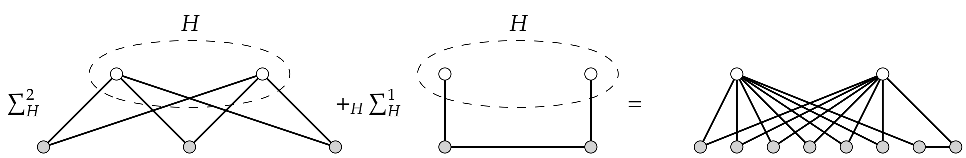
These two notions are connected by the Hammersley-Clifford theorem, which says that a positive joint probability distribution on G factorises according to G if and only if it satisfies the Markov properties; see [HC71] or [Lau98, Theorem 3.9].

The set of all positive joint probability distributions on G that satisfy the Markov properties is therefore a subset of the image of the following map:

$$\varphi_G : \mathbb{C}^{\prod_{C\in\mathrm{mcl}(G)}\prod_{j\in C}[d_j]} \to \mathbb{C}^{\prod_{j\in N}[d_j]}, \qquad (\theta^C_\alpha)_{C,\alpha} \mapsto \left(\prod_{C\in\mathrm{mcl}(G)} \theta^C_{\beta|_C}\right)_\beta$$

It is the ideal I_G of polynomials vanishing on $\mathrm{im}\,\varphi_G$ that is of interest in algebraic statistics. Since the components of φ_G are monomials, I_G is generated by finitely many binomials (differences of two monomials) in the standard coordinates on $\mathbb{C}^{\prod_{j\in N}[d_j]}$, and any finite generating set of binomials can be used to set up a Markov chain for testing whether given observations of the variables X_j are compatible with the assumption that their joint distribution factorises according to the graph G [DS98]. The zero locus of I_G is often called the *graphical model* of G.

Now suppose we have graphs $G_1, \ldots, G_s$ with node sets $N_1, \ldots, N_s$, that $N_i \cap N_k$ equals a fixed set N_0 for all $i \neq k$ in $[s]$, and that the graph induced on N_0 by each G_i is equal to a fixed graph H. Moreover, for each $j \in \bigcup_i N_i$ fix a number d_j of states. We can then glue copies of the G_i along their common subgraph H, by which we mean first taking disjoint copies of the G_i and then identifying the nodes labelled by a fixed $j \in N_0$ across all copies. For $a_1, \ldots, a_s \in \mathbb{Z}_{\geq 0}$, we denote the graph obtained by glueing a_i copies of graph G_i, $i \in [s]$ by $\sum_H^{a_1} G_1 +_H \cdots +_H \sum_H^{a_s} G_s$:



Theorem 1. *Let $G_1, \ldots, G_s$ be graphs with a common subgraph H and a number of states associated to each node. Then there exists a uniform bound $D \in \mathbb{Z}_{\geq 0}$ such that for all multiplicities $a_1, \ldots, a_s$, the ideal I_G of $G = \sum_H^{a_1} G_1 +_H \cdots +_H \sum_H^{a_s} G_s$ is generated by binomials of degree at most D.*

Our proof shows that one needs only finitely many combinatorial types of binomials, independent of $a_1, \ldots, a_s$, to generate I_G. This result is similar in flavour to the Independent Set Theorem from [HS12], where the graph is fixed but the d_j vary. Interestingly, the underlying categories responsible for these two stabilisation phenomena are opposite to each other; see Remark 23.

Example 2. In [RS14] it is proved that the ideal I_G for the complete bipartite graph $G = K_{3,N}$, with two states for each of the random variables, is generated in degree at most 12 for all N. The graph G is obtained by glueing N copies of $K_{3,1}$ along their common subgraph consisting of 3 nodes without any edges.

We derive Theorem 1 from a general stabilisation result on *toric fibre products*, which we introduce next.

Toric fibre products. Fix a ground field K, let r be a natural number, and let $U_1, \ldots, U_r, V_1, \ldots, V_r$ be finite-dimensional vector spaces over K.

Define a bilinear operation

$$\prod_j U_j \times \prod_j V_j \to \prod_j (U_j \otimes V_j), \quad (u, v) \mapsto u * v := (u_1 \otimes v_1, \ldots, u_r \otimes v_r). \tag{1}$$

Definition 3. The *toric fibre product* $X * Y$ of Zariski-closed subsets $X \subseteq \prod_j U_j$ and $Y \subseteq \prod_j V_j$ equals the Zariski-closure of the set $\{u * v \mid u \in X, v \in Y\}$. ♣

Remark 4. In [Sul07], the toric fibre product is defined at the level of ideals: if $(x_i^j)_i$ are coordinate functions on U_j and $(y_k^j)_k$ are coordinate functions on V_j, then $(z_{i,k}^j := x_i^j \otimes y_k^j)_{i,k}$ are coordinate functions on $U_j \otimes V_j$. The ring homomorphism of coordinate rings

$$K[(z_{i,k}^j)_{j,i,k}] = K\left[\prod_j (U_j \otimes V_j)\right] \to K\left[\prod_j U_j \times \prod_j V_j\right] = K[(x_i^j)_{j,i}, (y_k^j)_{j,k}]$$

dual to (1) sends $z_{i,k}^j$ to $x_i^j \cdot y_k^j$. If we compose this homomorphism with the projection modulo the ideal of $X \times Y$, then the kernel of the composition is precisely the toric fibre product of the ideals of X and Y as introduced in [Sul07]. In that paper, multigradings play a crucial role for *computing* toric fibre products of ideals, but do not affect the *definition* of toric fibre products. ♣

The product $*$ is associative and commutative up to reordering tensor factors. We can iterate this construction and form products like $X^{*2} * Y * Z^{*3}$, where Z also lives in a product of r vector spaces W_j. This variety lives in $\prod_j (U_j^{\otimes 2}) \otimes V_j \otimes (W_j^{\otimes 3})$.

We will not be taking toric fibre products of general varieties, but rather *Hadamard-stable* ones. For this, we have to choose coordinates on each U_j, so that $U_j = K^{d_j}$.

Definition 5. On K^d the Hadamard product is defined as $(a_1, \ldots, a_d) \bigcirc (b_1, \ldots, b_d) = (a_1 b_1, \ldots, a_d b_d)$. On $U := U_1 \times \cdots \times U_r$, where $U_j = K^{d_j}$, it is defined component-wise. A set $X \subseteq \prod_j U_j$ is called *Hadamard-stable* if X contains the all-one vector $\mathbf{1}_U$ (the unit element of $\bigcirc$) and if moreover $x \bigcirc z \in X$ for all $x, z \in X$. ♣

Remark 6. By [ES96, Remark after Proposition 2.3], X is Hadamard-stable if and only if its ideal is generated by differences of two monomials.

In particular, the Zariski-closure in U of a subtorus of the $\sum_j d_j$-dimensional torus $\prod_j (K \setminus \{0\})^{d_j}$ is Hadamard-stable. These are the toric varieties from the abstract.

Suppose that we also fix identifications $V_j = K^{d'_j}$ and a corresponding Hadamard multiplication $\prod_j V_j \times \prod_j V_j \to \prod_j V_j$. Equipping the spaces $U_j \otimes V_j$ with the natural coordinates and corresponding Hadamard multiplication, the two operations just defined satisfy $(u \bigcirc u') * (v \bigcirc v') = (u * v) \bigcirc (u' * v')$ as well as $\mathbf{1}_{U_j \otimes V_j} = \mathbf{1}_{U_j} \otimes \mathbf{1}_{V_j}$. Consequently, if both $X \subseteq \prod_j U_j$ and $Y \subseteq \prod_j V_j$ are Zariski-closed and Hadamard-stable, then so is their toric fibre product $X * Y$. We can now formulate our second main result.

Theorem 7. *Let $s, r \in \mathbb{Z}_{\geq 0}$. For each $i \in [s]$ and $j \in [r]$ let $d_{ij} \in \mathbb{Z}_{\geq 0}$ and set $V_{ij} := K^{d_{ij}}$. For each $i \in [s]$, let $X_i \subseteq \prod_j V_{ij}$ be a Hadamard-stable Zariski-closed subset. Then there exists a uniform bound $D \in \mathbb{Z}_{\geq 0}$ such that for any exponents $a_1, \ldots, a_s$ the ideal of $X_1^{*a_1} * \cdots * X_s^{*a_s} \subseteq \prod_{j=1}^r \bigotimes_{i=1}^s V_{ij}^{\otimes a_i}$ is generated by polynomials of degree at most D.*

Remark 8. A straightforward generalisation of this theorem also holds, where each X_i is a closed sub-scheme given by some ideal J_i in the coordinate ring of $\prod_j V_{ij}$. Hadamard-stable then says that that the pull-back of J_i under the Hadamard product lies in the ideal of $X_i \times X_i$, and the toric fibre product is as in Remark 4. Since this generality would slightly obscure our arguments, we have decided to present explicitly the version with Zariski-closed subsets—see also Remark 22.

Also, the theorem remains valid if we remove the condition that the X_i contain the all-one vector, but require only that they be closed under Hadamard-multiplication; see Remark 21. ♣

Organisation of this paper. The remainder of this paper is organised as follows. In Section 2 we introduce the categories of (affine) **Fin**-varieties and, dually, $\mathbf{Fin}^{\mathrm{op}}$-algebras. The point is that, as we will see in Section 4, the iterated toric fibre products together form such a **Fin**-variety (or rather a $\mathbf{Fin}^s$-variety, where $\mathbf{Fin}^s$ is the product category of s copies of **Fin**).

Indeed, they sit naturally in a Cartesian product of copies of the **Fin**-variety of rank-one tensors, which, as we prove in Section 3, is Noetherian (Theorem 12). This Noetherianity result is of a similar flavour as the recent result from [SS17] (see also [DK14, Proposition 7.5] which follows the same proof strategy) that any finitely generated $\mathbf{Fin}^{\mathrm{op}}$-*module* is Noetherian; this result played a crucial role in a proof of the *Artinian conjecture*. However, our Noetherianity result concerns certain $\mathbf{Fin}^{\mathrm{op}}$-*algebras* rather than modules, and is more complicated. Finally, in Section 4 we first prove Theorem 7 and then derive Theorem 1 as a corollary.

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2. Affine **Fin**-varieties and $\mathbf{Fin}^{\mathrm{op}}$-algebras

The category **Fin** has as objects all finite sets and as morphisms all maps. Its opposite category is denoted $\mathbf{Fin}^{\mathrm{op}}$. When **C** is another category whose objects are called *somethings*, then a **Fin**-*something* is a covariant functor from **Fin** to **C** and a $\mathbf{Fin}^{\mathrm{op}}$-*something* is a contravariant functor from **Fin** to **C**. The **C**-homomorphism associated to a **Fin**-morphism π is denoted π_* (the *push-forward of* π) in the covariant case and π^* (the *pull-back of* π) in the contravariant case.

More generally, we can replace **Fin** by any category **D**. The **D**-somethings themselves form a category, in which morphisms are natural transformations. In our paper, **D** is always closely related to **Fin** or to $\mathbf{Fin}^s$, the s-fold product of **Fin**.

Here are three instances of **Fin** or $\mathbf{Fin}^{\mathrm{op}}$-somethings crucial to our paper.

Example 9. Fix $n \in \mathbb{Z}_{\geq 0}$. Then the functor $S \mapsto [n]^S$ is a $\mathbf{Fin}^{\mathrm{op}}$-set, which to $\pi \in \mathrm{Hom}_{\mathbf{Fin}}(S,T)$ associates the map $\pi^* : [n]^T \to [n]^S, \alpha \mapsto \alpha\pi$, the composition of α and π.

Building on this example, the functor $A_n : S \mapsto K[x_\alpha \mid \alpha \in [n]^S]$, a polynomial ring in variables labelled by $[n]^S$, is a $\mathbf{Fin}^{\mathrm{op}}$-$K$-algebra, which associates to π the K-algebra homomorphism $\pi^* : A(T) \to A(S), \quad x_\alpha \mapsto x_{\alpha\pi}$.

Third, we define an affine $\mathbf{Fin}$-K-vector space Q_n by $Q_n : S \mapsto (K^n)^{\otimes S}$, the space of $n \times \cdots \times n$-tensors with factors labelled by S, which sends $\pi \in \mathrm{Hom}_{\mathbf{Fin}}(S,T)$ to the linear morphism $\pi_* : (K^n)^{\otimes S} \to (K^n)^{\otimes T}$ determined by

$$\pi_* : \otimes_{i \in S} v_i \mapsto \otimes_{j \in T} \left(\bigodot_{i \in \pi^{-1}(j)} v_i \right),$$

where $\bigodot$ is the Hadamard product in K^n. We follow the natural convention that an empty Hadamard product equals the all-one vector $\mathbf{1} \in K^n$; in particular, this holds in the previous formula for all $j \in T$ that are not in the image of π.

The ring A_n and the space Q_n are related as follows: $Q_n(S)$ has a basis consisting of vectors $e_\alpha := \otimes_{i \in S} e_{\alpha_i}$, $\alpha \in [n]^S$, where $e_1, \ldots, e_n$ is the standard basis of K^n; $A_n(S)$ is the coordinate ring of $Q_n(S)$ generated by the dual basis $(x_\alpha)_{\alpha \in [n]^S}$; and for $\pi \in \mathrm{Hom}_{\mathbf{Fin}}(S,T)$ the pullback $\pi^* : A_n(T) \to A_n(S)$ is the homomorphism of K-algebras dual to the linear map $\pi_* : Q_n(S) \to Q_n(T)$. Indeed, this is verified by the following computation for $\alpha \in [n]^T$:

$$x_\alpha(\pi_* \otimes_{i \in S} v_i) = x_\alpha(\otimes_{j \in T} \left(\bigodot_{i \in \pi^{-1}(j)} v_i \right)) = \prod_{j \in T} \left(\bigodot_{i \in \pi^{-1}(j)} v_i \right)_{\alpha_j} = \prod_{i \in S} (v_i)_{\alpha_{\pi(i)}} = \pi^*(x_\alpha)(\otimes_{i \in S} v_i).$$

This is used in Section 3. ♣

In general, by *algebra* we shall mean an associative, commutative K-algebra with 1, and homomorphisms are required to preserve 1. So a $\mathbf{Fin}^{\mathrm{op}}$-algebra B assigns to each finite set S an algebra and to each map $\pi : S \to T$ an algebra homomorphism $\pi^* : B(T) \to B(S)$. An *ideal* in B is a $\mathbf{Fin}^{\mathrm{op}}$-subset I of B (i.e., $I(S)$ is a subset of $B(S)$ for each finite set S and π^* maps $I(T)$ into $I(S)$) such that each $I(S)$ is an ideal in $B(S)$; then $S \mapsto B(S)/I(S)$ is again a $\mathbf{Fin}^{\mathrm{op}}$-algebra, the *quotient* B/I of B by I.

Given a $\mathbf{Fin}^{\mathrm{op}}$-algebra B, finite sets S_j for j in some index set J, and an element $b_j \in B(S_j)$ for each j, there is a unique smallest ideal I in B such that each $I(S_j)$ contains b_j. This ideal is constructed as:

$$I(S) = \left(\pi^*(b_j) \,\middle|\, j \in J, \pi \in \mathrm{Hom}(S, S_j) \right)$$

This is the ideal *generated* by the b_j. A $\mathbf{Fin}^{\mathrm{op}}$-algebra is called *Noetherian* if each ideal I in it is generated by finitely many elements in various $I(S_j)$, i.e., J can be taken finite.

Example 10. The $\mathbf{Fin}^{\mathrm{op}}$-algebra A_1 is Noetherian. Indeed, $A_1(S)$ is the polynomial ring $K[t]$ in a single variable t for all S, and the homomorphism $A_1(T) \to A_1(S)$ is the identity $K[t] \to K[t]$. So Noetherianity follows from Noetherianity of the algebra $K[t]$.

For $n \geq 2$ the $\mathbf{Fin}^{\mathrm{op}}$-algebra A_n is *not* Noetherian. For instance, consider the monomials

$u_2 := x_{21}x_{12} \in A_n([2])$, $u_3 := x_{211}x_{121}x_{112} \in A_n([3])$, $u_4 := x_{2111}x_{1211}x_{1121}x_{1112} \in A_n([4])$, and so on. For any map $\pi : [k] \to [l]$ with $k > l$, by the pigeon hole principle there are two indices $i, j \in [k]$ such that $\pi(i) = \pi(j) =: m \in [l]$. Then $\pi^* x_{1\cdots 121 \cdots 1}$, where the 2 is in the m-th position, is a variable with at least two indices equal to 2. Since u_k contains no such variable, $\pi^* u_l$ does not divide u_k. So $u_2, u_3, \ldots$ generates a non-finitely generated monomial $\mathbf{Fin}^{\mathrm{op}}$-ideal in A_n. (On the other hand, for each d the piece of A_n of homogeneous polynomials of degree at most d *is* Noetherian as a $\mathbf{Fin}^{\mathrm{op}}$-module, see [DK14, Proposition 7.5].) ♣

We shall see in the following section that certain interesting quotients of each A_n *are* Noetherian.

3. Rank-one tensors form a Noetherian **Fin**-variety

Let $Q_n^{\leq 1}(S)$ be the variety of *rank-one tensors*, i.e., those of the form $\otimes_{i \in S} v_i$ for vectors $v_i \in K^n$. We claim that this defines a Zariski-closed **Fin**-subvariety $Q_n^{\leq 1}$ of Q_n.

For this, we must verify that for a map $\pi : S \to T$ the map $Q_n(S) \to Q_n(T)$ dual to the algebra homomorphism $A_n(T) \to A_n(S)$ sends $Q_n^{\leq 1}(S)$ into $Q_n^{\leq 1}(T)$. And indeed, in Example 9 we have seen that this map sends

$$\otimes_{i \in S} v_i \mapsto \otimes_{j \in T} \left(\bigodot_{i \in \pi^{-1}(j)} v_i \right).$$

It is well known that (if K is infinite) the ideal in $A_n(S)$ of $Q_n^{\leq 1}(S)$ equals the ideal $I_n(S)$ generated by all binomials constructed as follows. Partition S into two parts S_1, S_2, let $\alpha_i, \beta_i \in [n]^{S_i}$ and write $\alpha_1 \| \alpha_2$ for the element of $[n]^S$ which equals α_i on S_i. Then we have the binomial

$$x_{\alpha_1\|\alpha_2} x_{\beta_1\|\beta_2} - x_{\alpha_1\|\beta_2} x_{\beta_1\|\alpha_2} \in I_n(S), \tag{2}$$

and $I_n(S)$ is the ideal generated by these for all partitions and all $\alpha_1, \alpha_2, \beta_1, \beta_2$. The functor $S \mapsto I_n(S)$ is an ideal in the $\mathbf{Fin}^{\mathrm{op}}$-algebra A_n; for infinite K this follows from the computation above, and for arbitrary K it follows since the binomials above are mapped to binomials by pull-backs of maps $T \to S$ in **Fin**. Moreover, I_n is finitely generated (see also [DK14, Lemma 7.4]):

Lemma 11. *The ideal I_n in the $\mathbf{Fin}^{\mathrm{op}}$-algebra A_n is finitely generated.*

Proof. In the determinantal equation (2), if there exist distinct $j, l \in S_1$ such that $\alpha_1(j) = \alpha_2(l)$ and $\beta_1(j) = \beta_2(l)$, then the equation comes from an equation in $I_n(S \setminus \{j\})$ via the map $S \to S \setminus \{j\}$ that is the identity on $S \setminus \{j\}$ and maps $j \to l$. By the pigeon hole principle this happens when $|S_1| > n^2$. Similarly for $|S_2| > n^2$. Hence I_n is certainly generated by $I_n([2n^2 - 1])$. □

The main result in this section is the following.

Theorem 12. *For each $n \in \mathbb{Z}_{\geq 0}$ the coordinate ring A_n/I_n of the **Fin**-variety $Q_n^{\leq 1}$ of rank-one tensors is a Noetherian $\mathbf{Fin}^{\mathrm{op}}$-algebra.*

Our proof follows the general technique from [SS17], namely, to pass to a suitable category close to **Fin** that allows for a Gröbner basis argument. However, the relevant well-partial-orderedness proved below is new and quite subtle. We use the category **OS** from [SS17] (also implicit in [DK14, Section 7]) defined as follows.

Definition 13. The objects of the category **OS** ("ordered-surjective") are all finite sets equipped with a linear order and the morphisms $\pi : S \to T$ are all surjective maps with the additional property that the function $T \to S,\ j \mapsto \min \pi^{-1}(j)$ is strictly increasing. ♣

Any **Fin**-algebra is also an **OS**-algebra, and **OS**-Noetherianity implies **Fin**-Noetherianity. So to prove Theorem 12 we set out to prove the stronger statement that A_n/I_n is, in fact, **OS**-Noetherian.

We get a more concrete grip on the K-algebra A_n/I_n through the following construction. Let M_n denote the (Abelian) $\mathbf{Fin}^{\mathrm{op}}$-monoid defined by

$$M_n(S) := \left\{ \alpha \in \mathbb{Z}_{\geq 0}^{[n]\times S} \mid \forall j,l \in S : \sum_{i=1}^{n} \alpha_{ij} = \sum_{i=1}^{n} \alpha_{il} \right\},$$

in which the multiplication is given by addition, and where the pull-back of a map $\pi : S \to T$ is the map $\pi^* : M_n(T) \to M_n(S)$ sending $(\alpha_{ij})_{i\in[n],j\in T}$ to $(\alpha_{i\pi(j)})_{i\in[n],j\in S}$. So elements of $M_n(S)$ are matrices with nonnegative integral entries and constant column sum. Let KM_n denote the **Fin**-algebra sending S to the monoid K-algebra $KM_n(S)$. The following proposition is a reformulation of a well-known fact.

Proposition 14. *The* $\mathbf{Fin}^{\mathrm{op}}$*-algebra* A_n/I_n *is isomorphic to the* $\mathbf{Fin}^{\mathrm{op}}$*-algebra* KM_n *(and the same is true when both are regarded as* $\mathbf{OS}^{\mathrm{op}}$*-algebras).*

Proof. For each finite set S, the K-algebra homomorphism $\Phi_S : A_n(S) \to KM_n(S)$ that sends $x_\alpha, \alpha \in [n]^S$ to the $[n] \times S$ matrix in $M_n(S)$ with a 1 at the positions $(\alpha_j, j),\ j \in S$ and zeroes elsewhere is surjective and has as kernel the ideal $I_n(S)$. Moreover, if $\pi : S \to T$ is a morphism in **Fin**, then we have $\Phi_T\pi^* = \pi^*\Phi_S$, i.e., the $(\Phi_S)_S$ define a natural transformation. □

Choose any monomial order $>$ on $\mathbb{Z}^n_{\geq 0}$, i.e., a well-order such that $a > b$ implies $a + c > b + c$ for every $a, b, c \in \mathbb{Z}^n_{\geq 0}$. Then for each object S in **OS** we define a linear order $>$ on $M_n(S)$, as follows: $\alpha > \beta$ if $\alpha \neq \beta$ and the smallest $j \in S$ with $\alpha_{\cdot,j} \neq \beta_{\cdot,j}$ (i.e., the j-th column of α is not equal to that of β) satisfies $\alpha_{\cdot,j} > \beta_{\cdot,j}$ in the chosen monomial order on $\mathbb{Z}^n_{\geq 0}$. A straightforward verification shows that this is a monomial order on $M_n(S)$ (we call the elements of $M_n(S)$ monomials, even though $KM_n(S)$ is not a polynomial ring). Moreover, for various S, these orders are interrelated as follows.

Lemma 15. *For any* $\pi \in \mathrm{Hom}_{\mathbf{OS}}(S,T)$ *and* $\alpha, \beta \in M_n(T)$*, we have* $\alpha > \beta \Rightarrow \pi^*\alpha > \pi^*\beta$*.*

Proof. If $j \in T$ is the smallest column index where α and β differ, then $\alpha' := \pi^*\alpha$ and $\beta' := \pi^*\beta$ differ in column $l := \min \pi^{-1}(j)$, where they equal $\alpha_{\cdot,j}$ and $\beta_{\cdot j}$, respectively, and the former is larger than the latter. Furthermore, if l' is the smallest position where α', β' differ, then $\alpha_{\cdot,\pi(l')} \neq \beta_{\cdot,\pi(l')}$ and hence $\pi(l') \geq j$ and hence $l' = \min \pi^{-1}(\pi(l')) \geq \min \pi^{-1}(j) = l$. Hence in fact $l = l'$ and $\pi^*\alpha > \pi^*\beta$. □

In addition to the well-order $\leq$ on each individual $M_n(S)$, we also need the following partial order $|$ on the union of all of them.

Definition 16. Let S, T be objects in **OS**. We say that $\alpha \in M_n(T)$ *divides* $\beta \in M_n(S)$ if there exist a $\pi \in \mathrm{Hom}_{OS}(S, T)$ and a $\gamma \in M_n(S)$ such that $\beta = \gamma + \pi^*\alpha$. In this case, we write $\alpha|\beta$. ♣

The key combinatorial property of the relation just defined is the following.

Proposition 17. *The relation | is a well-quasi-order, that is, for any sequence* $\alpha^{(1)} \in M_n(S_1), \alpha^{(2)} \in M_n(S_2), \ldots$ *there exist* $i < j$ *such that* $\alpha^{(i)}|\alpha^{(j)}$.

Proof. First, to each $\alpha \in M_n(S)$ we associate the monomial ideal $P(\alpha)$ in the polynomial ring $R := K[z_1, \ldots, z_n]$ (here K is but a place holder) generated by the monomials $z^{\alpha_{\cdot,j}}, j \in S$. The crucial fact that we will use about monomial ideals in R is that in any sequence $P_1, P_2, \ldots$ of such ideals there exist $i < j$ such that $P_i \supseteq P_j$—in other words, monomial ideals are well-quasi-ordered with respect to reverse inclusion [Mac01].

To prove the proposition, suppose, on the contrary, that there exists a sequence as above with $\alpha^{(i)} \not| \alpha^{(j)}$ for all $i < j$. Such a sequence is called *bad*. Then by basic properties of well-quasi-orders, some bad sequence exists in which moreover

$$P(\alpha^{(1)}) \supseteq P(\alpha^{(2)}) \supseteq \ldots \tag{3}$$

Among all bad sequences *with this additional property* choose one in which, for each $j = 1, 2, \ldots$, the cardinality $|S_j|$ is minimal among all bad sequences starting with $\alpha^{(1)}, \ldots, \alpha^{(j-1)}$.

Write $\alpha^{(j)} = (\gamma^{(j)}|\beta^{(j)})$, where $\beta^{(j)} \in \mathbb{Z}^n_{\geq 0}$ is the last column (the one labelled by the largest element of S_j), and $\gamma^{(j)}$ is the remainder. By Dickson's lemma, there exists a subsequence $j_1 < j_2 < \ldots$ such that $\beta^{(j_1)}, \beta^{(j_2)}, \ldots$ increase weakly in the coordinatewise ordering on $\mathbb{Z}^n_{\geq 0}$. By restricting to a further subsequence, we may moreover assume that also

$$P(\gamma^{(j_1)}) \supseteq P(\gamma^{(j_2)}) \supseteq \ldots \tag{4}$$

Then consider the new sequence

$$\alpha^{(1)}, \ldots, \alpha^{(j_1-1)}, \gamma^{(j_1)}, \gamma^{(j_2)}, \ldots$$

By (4) and (3), and since $P(\alpha^{(j)}) \supseteq P(\gamma^{(j)})$, this sequence also satisfies (3). We claim that, furthermore, it is bad.

Suppose, for instance, that $\gamma^{(j_1)}|\gamma^{(j_2)}$. Set $a_i := \max S_{j_i}$ for $i = 1, 2$. Then there exists a $\pi \in \mathrm{Hom}_{\mathbf{OS}}(S_{j_2} \setminus \{a_2\}, S_{j_1} \setminus \{a_1\})$ such that $\gamma^{(j_2)} - \pi^*\gamma^{(j_1)} \in M_n(S_{j_2} \setminus \{a_2\})$. But then extend π to an element π of $\mathrm{Hom}_{\mathbf{OS}}(S_{j_2}, S_{j_1})$ by setting $\pi(a_2) := a_1$; since $\beta^{(j_1)}$ is coordinate-wise smaller than $\beta^{(j_2)}$ we find that $\alpha^{(j_2)} - \pi^*\alpha^{(j_1)} \in M_n(S_{j_2})$, so $\alpha^{(j_1)}|\alpha^{(j_2)}$, in contradiction to the badness of the original sequence.

On the other hand, suppose for instance that $\alpha^{(1)}|\gamma^{(j_2)}$ and write $a_2 := \max S_{j_2}$. Then there exists a $\pi \in \mathrm{Hom}_{\mathbf{OS}}(S_{j_2} \setminus \{a_2\}, S_1)$ such that $\gamma^{(j_2)} - \pi^*(\alpha^{(1)}) \in M_n(S_{j_2} \setminus \{a_2\})$. Now, *and this is why we required that* (3) *holds*, since $P(\alpha^{(j_2)}) \subseteq P(\alpha^{(1)})$, there exists an element $s \in S_1$ such that the column $\beta^{(j_2)}$ is coordinatewise at least as large as the s-th column of $\alpha^{(1)}$. Extend π to an element of $\mathrm{Hom}_{\mathbf{OS}}(S_{j_2}, S_1)$ by setting $\pi(a_2) = s$. Since a_2 is the maximal element of S_{j_2}, this does not destroy the property that the function $\min \pi^{-1}(.)$ be increasing in its argument. Moreover, this π has the property that $\alpha^{(j_2)} - \pi^*\alpha^{(1)} \in M_n(S_{j_2})$, again a contradiction.

Since we have found a bad sequence satisfying (3) but with strictly smaller underlying set at the j_1-st position, we have arrived at a contradiction. □

Next we use a Gröbner basis argument.

Proof of Theorem 12. We prove the stronger statement that KM_n is Noetherian as an **OS**-algebra. Let P be any ideal in $A_n/I_n = KM_n$. For each object S in **OS**, let $L(S) \subseteq M_n(S)$ denote the set of leading terms of nonzero elements of $P(S)$ relative to the ordering $>$. Proposition 17 implies that there exists a finite collection $S_1, \ldots, S_N$ and $\alpha^{(j)} \in L(S_j)$ such that each element of each $L(S)$ is divisible by some $\alpha^{(j)}$. Correspondingly, there exist elements $f_j \in P(S_j)$ with leading monomial $\alpha^{(j)}$ and leading coefficient 1. To see that the f_j generate P, suppose that there exists an S such that $P(S)$ is not contained in the ideal generated by the f_j, and let $g \in P(S)$ have minimal leading term β among all elements of $P(S)$ not in the ideal generated by the f_j; without loss of generality g has leading coefficient 1. By construction, there exists some j and some $\pi \in \mathrm{Hom}_{\mathbf{OS}}(S, S_j)$ such that $\beta - \pi^*\alpha^{(j)} \in M_n(S)$. But now, by Lemma 15, we find that the leading monomial of $\pi^* f_j$ equals $\pi^*\alpha^{(j)}$, hence subtracting a monomial times $\pi^* f_j$ from g we obtain an element of $P(S)$ with smaller leading monomial that is not in the ideal generated by the f_j—a contradiction. □

Below, we need the following generalisation of Theorem 12.

Theorem 18. *For any $n_1, \ldots, n_r \in (\mathbb{Z})_{\geq 0}$ the* **Fin***-algebra (or* **OS***-algebra) $(A_{n_1}/I_{n_1}) \otimes \cdots \otimes (A_n/I_{n_1})$ is Noetherian.*

Proof. This algebra is isomorphic to $B := K(M_{n_1} \times \cdots \times M_{n_r})$. There is a natural embedding $\iota : M_{n_1} \times \cdots \times M_{n_r} \to M_{n_1+\ldots+n_s} =: M_n$ by forming a block matrix; its image consists of block matrices with constant partial column sums. And while a subalgebra of a Noetherian algebra is not necessarily Noetherian, this is true in the current setting.

The crucial point is that if $\alpha_i \in (M_{n_1} \times \cdots \times M_{n_r})(S_i)$ for $i = 1, 2$, then *a priori* $\iota(\alpha_1)|\iota(\alpha_2)$ only means that $\iota(\alpha_2) - \pi^*\iota(\alpha_1) \in M_n(S_2)$; but since both summands have constant partial column sums, so does their difference, so in fact, the difference lies in the image of ι. With this observation, the proof above for the case where $r = 1$ goes through unaltered for arbitrary r. □

Remark 19. Similar arguments for passing to sub-algebras are also used in [HS12] and [Dra10]. ♣

4. Proofs of the main results

In this section we prove Theorems 1 and 7.

Toric fibre products. To prove Theorem 7, we work with a product of s copies of the category **Fin**; one for each of the varieties X_i whose iterated fibre products are under consideration. Let $s, r \in \mathbb{Z}_{\geq 0}$. For each $i \in [s]$ and $j \in [r]$ let $d_{ij} \in \mathbb{Z}_{\geq 0}$ and set

$V_{ij} := K^{d_{ij}}$. Consider the $\mathbf{Fin}^s$-variety $\mathcal{V}$ that assigns to an s-tuple $S = (S_1, \dots, S_s)$ the product

$$\prod_{j=1}^{r} \bigotimes_{i=1}^{s} V_{ij}^{\otimes S_i}$$

and to a morphism $\pi = (\pi_1, \dots, \pi_s) : S \to T := (T_1, \dots, T_s)$ in $\mathbf{Fin}^s$ the linear map $\mathcal{V}(S) \to \mathcal{V}(T)$ determined by

$$\left(\otimes_i \otimes_{k \in S_i} v_{ijk}\right)_{j \in [r]} \mapsto \left(\otimes_i \otimes_{l \in T_i} \left(\bigcirc_{k \in \pi_i^{-1}(l)} v_{ijk}\right)\right)_{j \in [r]}, \tag{5}$$

where the Hadamard product $\bigcirc$ is the one on V_{ij}. Let $Q^{\leq 1}(S)$ be the Zariski-closed subset of $\mathcal{V}(S)$ consisting of r-tuples of tensors of rank at most one; thus $Q^{\leq 1}$ is a $\mathbf{Fin}^s$-subvariety of $\mathcal{V}$.

For each $i \in [s]$ let $X_i \subseteq \prod_{j=1}^{r} V_{ij}$ be a Hadamard-stable Zariski-closed subset. Then for any tuple $S = (S_1, \dots, S_s)$ in $\mathbf{Fin}^s$ the variety $\mathcal{X}(S) := X_1^{*S_1} * \cdots * X_s^{*S_s}$ is a Zariski-closed subset of $\mathcal{V}(S)$.

Lemma 20. *The association $S \mapsto \mathcal{X}(S)$ defines a $\mathbf{Fin}^s$-closed subvariety of $Q^{\leq 1}$.*

Proof. From the definition of $*$ in (1) it is clear that the elements of $\mathcal{X}(S)$ are r-tuples of tensors of rank at most 1. Furthermore, for a morphism $S \to T$ in $\mathbf{Fin}^s$ the linear map $\mathcal{V}(S) \to \mathcal{V}(T)$ from (5) sends $\mathcal{X}(S)$ into $\mathcal{X}(T)$—here we use that if $(v_{i,j,k})_{j \in [r]} \in X_i$ for each $k \in \pi^{-1}(l)$, then also $(\bigcirc_{k \in \pi^{-1}(l)} v_{ijk})_{j \in [r]} \in X_i$ since X_i is Hadamard-stable. □

Now Theorem 7 follows once we know that the coordinate ring of $Q^{\leq 1}$ is a Noetherian $(\mathbf{Fin}^s)^{\mathrm{op}}$-algebra. For $s = 1$ and $r = 1$ this is Theorem 12 with n equal to d_{11}. For $s = 1$ and general r, this is Theorem 18 with n_j equal to d_{1j}.

For $r = 1$ and general s, Theorem 7 follows from a $\mathbf{Fin}^s$-analogue of Theorem 12, which is proved as follows. The coordinate ring of $Q^{\leq 1}(S_1, \dots, S_s)$ is the subring of $K(M_{d_{11}}(S_1) \times \cdots \times M_{d_{s1}}(S_s))$ spanned by the monomials corresponding to s-tuples of matrices with the *same* constant column sum. Using Proposition 17 and the fact that a finite product of well-quasi-ordered sets is well-quasi-ordered one finds that the natural $\mathbf{Fin}^s$-analogue on $M_{d_{11}} \times \cdots \times M_{d_{s1}}$ of the division relation $|$ is a well-quasi-order; and this implies, once again, that the coordinate ring of $Q^{\leq 1}$ is a Noetherian $(\mathbf{OS}^s)^{\mathrm{op}}$-algebra.

Finally, for general s and general r, the result follows as in the proof of Theorem 18. This proves the Theorem 7 in full generality. □

Remark 21. The only place where we used that the X_i contain the all-one vector is in the proof of Lemma 20 when $\pi^{-1}(l)$ happens to be empty. If we do not require this, then the conclusion of Theorem 7 still holds, since one can work directly with the category $\mathbf{OS}^s$ in which morphisms π are surjective.

Remark 22. If we replace the X_i by Hadamard-stable closed subschemes rather than subvarieties, then $S \mapsto \mathcal{X}(S)$ is still a $\mathbf{Fin}^s$-closed subscheme of $Q^{\leq 1}$, and since the coordinate ring of the latter is Noetherian, the proof goes through unaltered.

Remark 23. In the Independent Set Theorem from [HS12], the graph $G = (N, E)$ is fixed but the state space sizes d_j grow unboundedly for j in an independent set $T \subseteq N$ and are fixed for $j \notin T$. In this case, given a T-tuple of maps $(\pi_j : S_j \to P_j)_{j\in T}$ of finite sets, where S_j is thought of as the state space of $j \in T$ in the smaller model and P_j as the state space in the larger model, we obtain a natural map from the larger model into the smaller model. Hence then the graphical model is naturally a $(\mathbf{Fin}^{\mathrm{op}})^T$-variety and its coordinate ring is a $\mathbf{Fin}^T$-algebra. Note the reversal of the roles of these two categories compared to Lemma 20.

Markov random fields. Given a finite (undirected, simple) graph $G = (N, E)$ with a number d_j of states attached to each node $j \in N$, the graphical model is $X_G := \overline{\operatorname{im} \varphi_G}$, where φ_G is the parameterisation

$$\varphi_G : \mathbb{C}^{\prod_{C\in \mathrm{mcl}(G)} \prod_{j\in C}[d_j]} \to \mathbb{C}^{\prod_{j\in N}[d_j]}, \qquad (\theta^C_\alpha)_{C,\alpha} \mapsto \left(\prod_{C\in \mathrm{mcl}(G)} \theta^C_{\beta|_C}\right)_\beta .$$

Lemma 24. *For any finite graph G, the graphical model X_G is Hadamard-closed.*

Proof. The parameterisation φ sends the all-one vector in the domain to the all-one vector $\mathbf{1}$ in the target space, so $\mathbf{1} \in \operatorname{im} \varphi$. Moreover, if θ, θ' are two parameter vectors, then $\varphi(\theta \bigcirc \theta') = \varphi(\theta) \bigcirc \varphi(\theta')$, so $\operatorname{im} \varphi$ is Hadamard-closed. Then so is its closure. □

Following [Sul07], we relate graph glueing to toric fibre products. We are given finite graphs $G_1, \dots, G_s$ with node sets $N_1, \dots, N_s$ such that $N_i \cap N_k = N_0$ for all $i \neq k$ in $[s]$ and such that each G_i induces the same graph H on N_0. Moreover, for each $j \in \bigcup_i N_i$ we fix a number d_j of states.

For each $\beta_0 \in \prod_{j\in N_0}[d_j]$ and each $i \in [s]$ set $V_{i,\beta_0} := \mathbb{C}^{\prod_{j\in N_i\setminus N_0}[d_j]}$, which we interpret as the ambient space of the part of the probability table of the variables X_j, $j \in N_i$ where we have fixed the states of the variables in N_0 to β_0—up to scaling, these are the conditional joint probabilities for the X_j, $j \in N_i \setminus N_0$ given that the X_j, $j \in N_0$ are in joint state β_0. For $\beta \in \prod_{j\in N_i}[d_j]$ write $\beta = \beta_0\|\beta'$ where β_0, β' are the restrictions of β to N_0 and $N_i \setminus N_0$, respectively. For each maximal clique C in G_i define $C_0 = C \cap N_0$ and $C' = C \setminus N_0$. Correspondingly, decompose $\alpha \in \prod_{j\in C}[d_j]$ as $\alpha = \alpha_0\|\alpha'$, where α_0, α' are the restrictions of α to C_0 and C', respectively.

Then the graphical model $X_{G_i} := \overline{\operatorname{im} \varphi_{G_i}}$ is the closure of the image of the parameterisation

$$\varphi_i : \mathbb{C}^{\prod_{C\in \mathrm{mcl}(G_i)} \prod_{j\in C}[d_j]} \to \prod_{\beta_0\in \prod_{j\in N_0}[d_j]} V_{i,\beta_0}, \qquad (\theta^C_\alpha)_{C,\alpha} \mapsto \left(\left(\prod_{C\in \mathrm{mcl}(G)} \theta^C_{(\beta_0|_{C_0}) \,\|\, (\beta'|_{C'})}\right)_{\beta'}\right)_{\beta_0} .$$

Setting $r := \prod_{j\in N_0} d_j$, we are exactly in the setting of the previous sections: for each $i, k \in [s]$, we have the bilinear map

$$* : \prod_{\beta_0} V_{i,\beta_0} \times \prod_{\beta_0} V_{k,\beta_0} \to \prod_{\beta_0} (V_{i,\beta_0} \otimes V_{k,\beta_0}), \qquad ((v_{i,\beta_0})_{\beta_0}, (v_{k,\beta_0})_{\beta_0}) \mapsto (v_{i,\beta_0} \otimes v_{k,\beta_0})_{\beta_0},$$

and we can take iterated products of this type. The space on the right is naturally isomorphic to $\mathbb{C}^{\prod_{j\in N_1\cup N_2}[d_j]}$, the space of probability tables for the joint distribution of the variables labelled by the vertices in the glued graph $G_i +_H G_k$. Under this identification we have the following.

Proposition 25. *For* $G := \sum_H^{a_1} G_1 +_H \cdots +_H \sum_H^{a_s} G_s$ *we have* $X_G = X_{G_1}^{*a_1} * \cdots * X_{G_s}^{*a_s}$.

Proof. It suffices to prove this for the gluing of two graphs. Note that a clique in $G := G_1 +_H G_2$ is contained entirely in either G_1 or G_2, or in both but then already in H. Let θ, η be a parameter vectors in the domains of $\varphi_{G_1}, \varphi_{G_2}$, respectively. Then

$$\begin{aligned}
&\varphi_{G_1}(\theta) * \varphi_{G_2}(\eta)\\
&= \left(\left(\prod_{C\in\mathrm{mcl}(G_1)} \theta^C_{(\beta_0|_{C_0})\,\|\,(\beta'|_{C'})}\right)_{\beta'}\right)_{\beta_0} * \left(\left(\prod_{C\in\mathrm{mcl}(G_2)} \eta^C_{(\beta_0|_{C_0})\,\|\,(\beta'|_{C'})}\right)_{\beta'}\right)_{\beta_0}\\
&= \left(\left(\prod_{C\in\mathrm{mcl}(G_1)} \theta^C_{(\beta_0|_{C_0})\,\|\,(\beta'|_{C'})}\right)_{\beta'\in\prod_{j\in N_1\setminus N_0}[d_j]} \otimes \left(\prod_{C\in\mathrm{mcl}(G_2)} \eta^C_{(\beta_0|_{C_0})\,\|\,(\beta'|_{C'})}\right)_{\beta'\in\prod_{j\in N_2\setminus N_0}[d_j]}\right)_{\beta_0}\\
&= \left(\left(\prod_{C\in\mathrm{mcl}(G_1)} \theta^C_{(\beta_0|_{C_0})\,\|\,(\beta'|_{C'})} \cdot \prod_{C\in\mathrm{mcl}(G_2)} \eta^C_{(\beta_0|_{C_0})\,\|\,(\beta'|_{C'})}\right)_{\beta'\in\prod_{j\in (N_1\cup N_2)\setminus N_0}[d_j]}\right)_{\beta_0}\\
&= \left(\left(\prod_{C\in\mathrm{mcl}(G)} \mu^C_{(\beta_0|_{C_0})\,\|\,(\beta'|_{C'})}\right)_{\beta'\in\prod_{j\in (N_1\cup N_2)\setminus N_0}[d_j]}\right)_{\beta_0} = \varphi_G(\mu),
\end{aligned}$$

where, for $C \in \mathrm{mcl}(G)$ and $\alpha \in \prod_{j\in C}[d_j]$, the parameter μ^C_α is defined as

$$\mu^C_\alpha := \begin{cases} \theta^C_\alpha & \text{if } C \subseteq N_1 \text{ and } C \not\subseteq N_0,\\ \eta^C_\alpha & \text{if } C \subseteq N_2 \text{ and } C \not\subseteq N_0, \text{ and}\\ \theta^C_\alpha\, \eta^C_\alpha & \text{if } C \subseteq N_0. \end{cases}$$

This computation proves that $X_{G_1} * X_{G_2} \subseteq X_G$. Conversely, given any parameter vector μ for G, we can let θ be the restriction of μ to maximal cliques of the first and third type above, and set η^C_α equal to μ^C_α if C is of the second type above and equal to 1 if it is of the third type. This yields the opposite inclusion. □

Proof of Theorem 1. By Proposition 25, the ideal I_G is the ideal of the iterated toric fibre product $X_{G_1}^{*a_1} * \cdots * X_{G_s}^{*a_s}$. By Lemma 24, each of the varieties X_{G_i} is Hadamard closed. Hence Theorem 7 applies, and I_G is generated by polynomials of degree less than some D, which is independent of $a_1, \dots, a_s$. Then it is also generated by the binomials of at most degree D. □

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