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On a one-sample distribution-free test statistic V

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SUMMARY

A table of exact critical values of a one-sample distribution-free test statistic V is presented for selected significance levels and sample sizes $n = 3(1)20$. It is shown that this test is computationally similar to the well-known Wilcoxon rank sum test statistic.

Some key words: Distribution-free tests; Wilcoxon test; Empirical distribution function.

1. INTRODUCTION

Let $X_1, \ldots, X_n$ be independent random variables having the continuous distribution function $F(x)$ and let $F_n(x) = n^{-1}\{\text{number of } j \text{ with } X_j < x\}$ be the empirical distribution function. Let x_i be a real number with $F(x_i) = i/r$. We define

$$V = \sum_{i=1}^{n-1} n\{F_n(x_i) - F(x_i)\} = \sum_{i=1}^{n-1} \{nF_n(x_i) - i\}, \tag{1}$$

Table 1. *Table of critical values V for one-sided and two-sided tests*

	Significance level for one-sided test						
	0·10	0·05	0·025	0·01	0·005	0·001	0·0005
	Significance level for two-sided test						
n	0·20	0·10	0·05	0·02	0·01	0·002	0·001
3	3	3	—	—	—	—	—
4	4	5	5	6	6	—	—
5	5	6	7	8	9	10	10
6	6	8	9	10	11	13	14
7	8	10	11	13	14	16	17
8	9	12	14	16	17	20	21
9	11	14	16	19	21	24	25
10	13	16	19	22	24	28	30
11	15	18	21	25	28	32	34
12	16	21	24	29	31	37	39
13	18	23	27	32	35	42	44
14	20	26	30	36	39	46	49
15	23	29	34	40	44	52	55
16	25	31	37	44	48	57	60
17	27	34	40	48	52	62	66
18	29	37	44	52	57	68	72
19	32	40	48	56	62	73	78
20	34	43	51	61	67	79	84

which is equal to $-nS_n^*$ proposed by Riedwyl (1967) and give a table of exact critical values of V for significance levels 0·0005, 0·001, 0·005, 0·01, 0·025, 0·05, 0·10 and sample sizes $3 \leqslant n \leqslant 20$. Since $V\{\frac{1}{12}n(n^2-1)\}^{-\frac{1}{2}}$ is asymptotically normal with zero mean and variance 1, one can use normal approximations.

2. Computation of the distribution of V

Let $\phi_j(x)$ be the indicator function of the set $\{x|x > X_j\}$, so that

$$nF_n(x) = \sum_{j=i}^{n} \phi_j(x);$$

and let

$$V_j = \sum_{i=1}^{n-1} \{\phi_j(x_i) - (i/n)\}, \tag{2}$$

so that $V = \sum_{j=1}^{n} V_j$. Since X_j lies with probability $1/n$ in each of the intervals $(-\infty, x_1), [x_1, x_2), \ldots,$ $[x_{k-1}, x_k), \ldots, [x_{n-1}, \infty)$, V_j will be, with probability $1/n$, equal to

$$\sum_{i=1}^{k-1}\left(-\frac{i}{n}\right) + \sum_{i=k}^{n-1}\left(1-\frac{i}{n}\right) = (n-k) - \tfrac{1}{2}(n-1) \quad (k = 1, \ldots, n).$$

Since the V_j's are independent, the distribution of V is the nth convolution power of a uniform distribution on the set $\{-\frac{1}{2}(n-1), -\frac{1}{2}(n-1)+1, \ldots, -\frac{1}{2}(n-1)+m, \ldots, \frac{1}{2}(n-1)\}$.

In Table 1 the exact critical values at V are given for selected significance levels. The characteristic function of $V\{\frac{1}{12}n(n^2-1)\}^{-\frac{1}{2}}$ is easily computed to be

$$\begin{aligned} \phi_n(t) &= \left[\frac{\sin\left\{\left(\frac{3n}{n^2-1}\right)^{\frac{1}{2}} t\right\}}{n \sin\left\{\left(\frac{3}{n(n^2-1)}\right)^{\frac{1}{2}} t\right\}}\right]^n \\ &= \left\{1 - \frac{t^2}{2n} + O(n^{-2})\right\}^n \\ &\to e^{-\frac{1}{2}t^2} \end{aligned} \tag{3}$$

as $n \to \infty$.

3. Approximation

For sample sizes $n > 20$, one approximates the distribution of V by the normal distribution. Using a continuity correction, we have

$$z = \frac{|V| - \frac{1}{2}}{\{\frac{1}{12}n(n^2-1)\}^{\frac{1}{2}}}.$$

Table 2 demonstrates the good agreement already for $n = 10$.

Table 2. *Exact and normal approximate values of* pr $(V \geqslant k)$ *for* $n = 10$

k	Exact	Normal approximation
30	0·000 324	0·000 581
25	0·002 820	0·003 495
20	0·015 103	0·015 902
15	0·055 552	0·055 201
10	0·150 113	0·147 799
5	0·312 553	0·310 147
0	0·521 623	0·521 950

4. Calculation of V

Let the variables X_j $(1 \leqslant j \leqslant n)$ and the quantities x_i $(1 < i \leqslant n-1)$ of §2 be ordered according to increasing magnitude and let ρ_i be the rank of x_i in this ordering. Then we have

$$V = \sum_{i=1}^{n-1} \rho_i - n(n-1).$$

This shows V to be a one-sample analogue of the Wilcoxon (1945) rank sum test statistic, the set of quantiles playing the role of the second sample set.

5. Example

Wetherill (1967, p. 129) tests the one-sided hypothesis that nine observations come from a known normal distribution with mean 40 and standard deviation 1·15. The observations and the underlined quantiles x_i $(1 \leqslant i \leqslant n-1)$ with their ranks are, in increasing order,

$$\frac{38{\cdot}596}{1},\ \frac{39{\cdot}121}{2},\ 39{\cdot}4,\ \frac{39{\cdot}505}{4},\ 39{\cdot}6,\ 39{\cdot}8,\ \frac{39{\cdot}839}{7},\ \frac{40{\cdot}161}{8},\ 40{\cdot}2,\ \frac{40{\cdot}495}{10},\ \frac{40{\cdot}879}{11},$$

$$40{\cdot}9,\ 40{\cdot}9,\ 41{\cdot}4,\ \frac{41{\cdot}404}{15},\ 41{\cdot}8,\ 43{\cdot}6.$$

The $|V| = 14$ is just significant at the one-sided significance level of 5 %, Table 1. Without a table we would calculate $z = 1{\cdot}743$ which is also significant compared with the 5 % quantile of the standard cumulative normal distribution. A classical $t = 1{\cdot}91$ on 8 degrees of freedom is significant too ($t_{0.95,8} = 1{\cdot}86$).

6. Discussion

The test statistic V is an alternative test to competing methods in the one-sample case as the Wilcoxon rank sum test statistic is for the two-sample case. We think that the time used for the calculating will be particularly small.

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On the power of Jonckheere's k-sample test against ordered alternatives

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Summary

The power of Jonckheere's test for trend is considered for a particular class of nonparametric alternatives. A recursive formula is developed for computing the exact distribution of the test statistic. The mean and variance of the test statistic are derived. An approximation is developed based on the asymptotic distribution of the test statistic. Tables of exact and approximate power are given.

Some key words: Nonparametric test for trend; Jonckheere's test; Power of tests under Lehmann alternatives; Asymptotic distribution of nonparametric tests.

1. Introduction

The test procedure discussed here was proposed by Terpstra (1952) and independently by Jonckheere (1954), but is known as Jonckheere's test in the literature.

Assume that we are given random samples of size n from each of k populations. Denote by X_{ij} the jth observation from the ith population $(i = 1, \ldots, k; j = 1, \ldots, n)$. Denote by $F_i(\cdot)$ the continuous cumulative distribution function of X_{ij}. We wish to test the null hypothesis

$$H: F_1(u) = F_2(u) = \ldots = F_k(u) \quad (\text{all } u),$$