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## Notes on integral identities for 3d supersymmetric dualities

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ABSTRACT: Four dimensional  $\mathcal{N} = 2$  Argyres-Douglas theories have been recently conjectured to be described by  $\mathcal{N} = 1$  Lagrangian theories. Such models, once reduced to 3d, should be mirror dual to Lagrangian  $\mathcal{N} = 4$  theories. This has been numerically checked through the matching of the partition functions on the three sphere. In this article, we provide an analytic derivation for this result in the  $A_{2n-1}$  case via hyperbolic hypergeometric integrals. We study the  $D_4$  case as well, commenting on some open questions and possible resolutions. In the second part of the paper we discuss other integral identities leading to the matching of the partition functions in 3d dual pairs involving higher monopole superpotentials.

KEYWORDS: Field Theories in Lower Dimensions, Supersymmetry and Duality

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**1 Introduction**

It has been recently shown that 4d  $\mathcal{N} = 2$  Argyres-Douglas (AD) theories [1] can be obtained by an intricate RG flow structure [2–8]. The analysis starts by considering a 4d  $\mathcal{N} = 2$  SCFT Lagrangian with a gauge group  $G$  and hypermultiplets. Supersymmetry is broken to  $\mathcal{N} = 1$  by coupling the chiral multiplets with some singlets. A nilpotent vev for these singlets triggers an RG flow [2, 3]. In the IR a SCFT can be obtained by iterating a-maximization [9], at the cost of introducing a set of accidental symmetries. The properties of the SCFT under investigations are quite intriguing: it has been conjectured that there are situations with rational central charge that in the IR enhance to AD theories.<sup>1</sup> This led to the conjecture that the  $\mathcal{N} = 1$  theory obtained in this way corresponds to the Lagrangian description of AD theory.

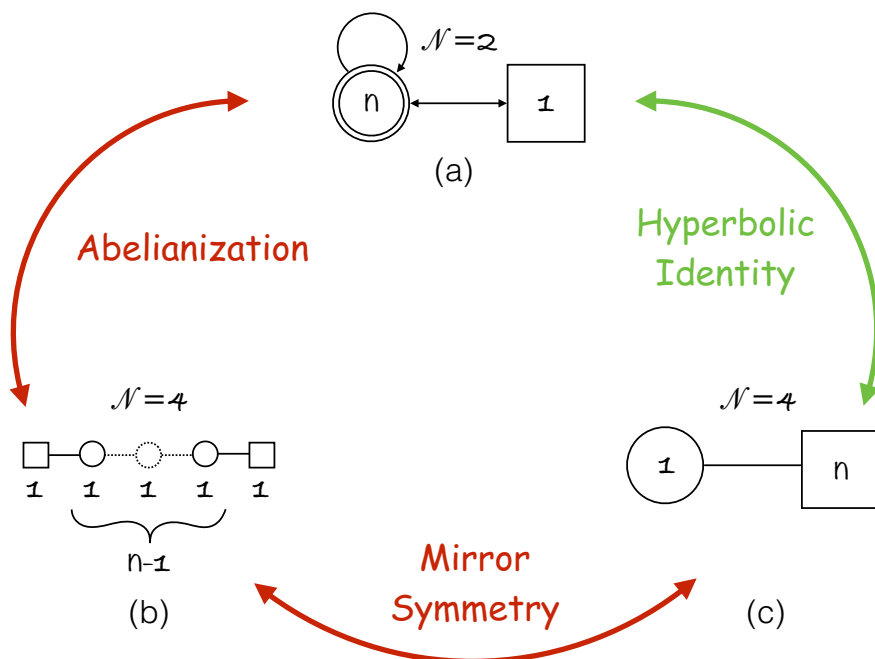
An interesting consequence of the existence of such a Lagrangian formulation is that one can reduce it to 3d and check if it reproduces correctly the expected reduction of AD theories conjectured in [11]. Substantial evidence<sup>2</sup> for the conjectures of [11] has been given in [12] by reducing the 4d superconformal index to the three sphere partition function. In addition, in [13] it was shown that expected RG flows following from the conjectured 3d quivers of [11] were consistent with the form of the 4d index (see also [14]). Finally, the latter result has been extended to some “generalized” AD theories in [15].

The idea has been recently pursued in [5–7] and it has been shown how to recover the results of [11] by reducing the 4d Lagrangian description of AD. The main ideas that allow the authors to obtain the desired result are *abelianization*, sequential confinement, and

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<sup>1</sup>It has been shown that there are also cases with rational charges that do not enhance to  $\mathcal{N} = 2$  [10].

<sup>2</sup>We are grateful to Matthew Buican for precious comments on this issue.



**Figure 1.** Relation between the relevant models describing the 3d  $(A_1, A_{2n-1})$  AD theories reduced to 3d. We also specify the degree of supersymmetry of the quivers under consideration. The double circle denotes an  $SU(n)$  gauge theory while single circles refer to unitary cases with abelian factors. Flavor symmetries are denoted by boxes.

chiral ring stability [16]. Many of these ideas are new and can potentially play a relevant role in the future analysis of 3d dualities.

The relation between the Lagrangian description of the AD theories denoted as  $(A_1, A_{2n-1})$  reduced to 3d and their 3d  $\mathcal{N} = 4$  mirror dual has been discussed in [5, 6]. It has been shown that these two theories can be mapped through an intermediate step, where the natural  $\mathcal{N} = 4$  mirror quiver is obtained through an *abelianization* procedure. In figure 1 we show this chain of dualities involving the three models.

The relation between models (a) and (b) in figure 1 has been numerically checked at the level of the partition function, showing that for small  $n$  (namely  $n = 2, 3$ ) it is possible to prove that the two theories have the same partition function providing a mapping between the R-charges and the real mass parameter [6]. The relation is claimed to work for generic values of the real masses and charges, and after F-maximization, the enhancement to  $\mathcal{N} = 4$  is expected. On the other hand,  $\mathcal{N} = 4$  mirror symmetry maps model (b) to (c), and the equality between the  $\mathcal{N} = 4$  partition functions of the two models have been proven in [17].

One starts from a possible UV completion, that reduces to the model (a) in the IR after an RG flow and to the model (c) after mirror symmetry and a cascade of sequential confinements of the type discussed in [18]. In [6] it has been shown how to connect the two models (a) and (c) in figure 1 in an indirect way. What is actually missing is a direct connection between models (a) and (c) of figure 1.

In this paper we obtain this direct connection by exploiting some mathematical identities among hyperbolic gamma functions and hyperbolic integrals. It turns out that the equivalence of the (squashed) three sphere partition functions of models (a) and (c) in figure 1 can be analytically proven for general  $n$ , hence corroborating the results of [6].

The paper is organized as follows. In section 2 we introduce the necessary main tools for our analysis, the representation of the (squashed) three sphere partition function of 3d  $\mathcal{N} = 2$  theories in terms of hyperbolic gamma functions and hyperbolic hypergeometric integrals. In section 3 we provide a quick review of the derivation of 4d  $\mathcal{N} = 1$  Lagrangian conjectured to enhance to  $(A_1, A_{2n-1})$  in the IR, their reduction to 3d, the abelianization and the mirror description. Then we derive the main result of this paper, the analytic matching of the three sphere partition function between the reduced  $(A_1, A_{2n-1})$  model and its mirror 3d  $\mathcal{N} = 4$  theory with  $U(1)$  gauge group and  $n$  hypermultiplets. In section 4 we speculate on the analogous result for the  $(A_1, D_4)$  case. In this case abelianization works in a different way and we have not been able to provide any analytical proof until now. We explain the nature of the problem and propose another dual description that may play a useful role in the analysis. In section 5 we discuss a further application of the identities among hyperbolic hypergeometric integrals. This is related to some integral identities for the theories with symplectic gauge groups with antisymmetric and fundamental matter. We restrict our analysis to the  $Sp(2)$  case, when the antisymmetric matter disappears. In the subsection 5.1 we show that if we start from the identities among the 3d hyperbolic integrals, it is possible to match the dual phases which discussed in [18], involving higher powers in the monopole superpotential. In the subsection 5.2 we show that considering the further real mass flow one can recover the limiting case of the usual Aharony duality [19] for the  $U(2)$  theory with two flavors. In section 6 we conclude summarizing the main results and mention the open questions.

## 2 Hyperbolic integrals and partition function

In this section we review the mathematical formalism of hyperbolic hypergeometric integrals. Relation of these integrals with the partition function of 3d  $\mathcal{N} = 2$  supersymmetric gauge theories computed from localization on the squashed three sphere  $S_b^3$ , where  $b$  represents the squashing parameter. The partition function corresponds to a matrix integral over the real scalar  $\sigma$  of the  $\mathcal{N} = 2$  vector multiplet, in the Cartan of the gauge group  $G$  [20–22]. It has been shown that the 1-loop contributions of the vector and of the matter fields to the partition function can be formulated in terms of hyperbolic Gamma functions  $\Gamma_h(x)$  represented as follows

$$\Gamma_h(x) \equiv \prod_{m,l=1}^{\infty} \frac{(m+1)\omega_1 + (l+1)\omega_2 - x}{m\omega_1 + l\omega_2 + x}. \tag{2.1}$$

Let us consider one example that will play a prominent role in our analysis, a  $U(n)$  gauge theory with  $f$  pairs of fundamentals and anti-fundamental flavors and one adjoint. The

three sphere partition function of this model corresponds to the following matrix integral

$$\frac{\Gamma_h(\tau)^n}{n!} \int \prod_{i=1}^n \frac{d\sigma_i}{\sqrt{-\omega_1\omega_2}} e^{\frac{2i\pi\lambda\sigma_i}{\omega_1\omega_2}} \prod_{1 \leq i < j \leq n} \frac{\Gamma_h(\tau \pm (\sigma_i - \sigma_j))}{\Gamma_h(\pm(\sigma_i - \sigma_j))} \prod_{i=1}^n \prod_{a,b=1}^f \Gamma_h(\mu_a + \sigma_i; \nu_b - \sigma_i). \quad (2.2)$$

The parameters  $\omega_1$  and  $\omega_2$  are associated to the squashing parameter  $b$  by  $\omega_1 = ib$ ,  $\omega_2 = i/b$ . This can be used to simplify the formula above, fixing  $\omega_1\omega_2 = -1$ . The shorthand notations  $\Gamma_h(x)\Gamma_h(y) = \Gamma_h(x; y)$  and  $\Gamma_h(x)\Gamma_h(-x) = \Gamma_h(\pm x)$  have been used in (2.2). We will adopt the definition  $\omega \equiv (\omega_1 + \omega_2)/2$  in the rest of the paper. We recall also a useful reflection equation, satisfied by the hyperbolic Gamma functions, that plays an important role in our analysis and it implies  $\Gamma(\omega) = 1$  as well,

$$\Gamma_h(2\omega - x)\Gamma_h(x) = 1. \quad (2.3)$$

Let us now explain the various terms appearing in the formula above; the factor  $n!$  corresponds to the dimension of the Weyl group. The functions  $\Gamma_h(x)$  appearing in the numerators correspond to the one loop determinants of the matter fields, while the ones appearing in the denominator are associated to the vector multiplet. The arguments  $x$  in  $\Gamma_h(x)$  represent the linear combination of the real scalars in the vector multiplets of the gauge symmetry, denoted as  $\sigma$ , and of the weakly gauged global symmetries, here denoted as  $\mu, \nu$  and  $\tau$ . They have to be taken in the weight of the representation for each symmetry under which the fields transform. Note that these mass parameters for the weakly gauged global symmetries are generically complex, and the imaginary part represents the R-charge of each multiplet.

The partition function in equation (2.2) contains also the contribution of a Fayet-Iliopoulos (FI) term  $\lambda$ , which is computed as a classical contribution in localization. In (2.2) we omit contributions of Chern-Simons (CS) terms to the partition function because we will not consider them in our analysis.

Hyperbolic hypergeometric integrals as the one in formula (2.2) have been shown to satisfy various classes of integral identities. It is remarkable to note that a given integral can satisfy very different identities, depending on the constraints satisfied by the complex parameters appearing in the argument of  $\Gamma_h(x)$ . These constraints, defined as balancing conditions in the mathematical literature, translate on the physical playground into the presence of non-trivial superpotential interactions (often involving the presence of monopole operators).

It has been shown that a large quantity of such integral identities, most of them are listed in [23], corresponds to the matching of the three sphere partition function of 3d  $\mathcal{N} = 2$  models, and these have been used to corroborate or in other case to derive 3d  $\mathcal{N} = 2$  dualities. However, there exist other identities discussed in [23] that have yet not been associated to any 3d duality. In the following we will focus on some of these identities, discussing their connection with some dualities that appeared in the physics literatures recently.

### 3 The Lagrangian $(A_1, A_{2n-1})$ theory and its reduction

In this section we review the 3d reduction of the  $(A_1, A_{2n-1})$  AD theories derived in [5, 6] and show the analytic matching of its  $S_b^3$  partition function with the one of its mirror dual. The starting point is the 4d construction of [2–4]. One considers a 4d  $SU(n)$   $\mathcal{N} = 2$  theory with  $2n$  flavors and couple them to a  $2n \times 2n$  singlet  $M$ . The superpotential of the theory is

$$W = \text{tr} Q\Phi\tilde{Q} + \text{tr} \tilde{Q}MQ. \quad (3.1)$$

By assigning a vev to  $\langle M \rangle$  that corresponds to the principal nilpotent orbit of the flavor symmetry and by expanding around this vev it has been shown that the theory flows to an IR fixed point if the contribution of accidental symmetries is included.

By following the prescription discussed in [24], in order to modify a-maximization in presence of accidental symmetries. In this case one should modify the Lagrangian by adding some extra fields as discussed originally in [25]. The explicit use of such extra fields has appeared only very recently in [6], where the authors denoted them as flipping fields. Supplementing this prescription with the chiral ring stability criterion, they obtained the 4d  $\mathcal{N} = 1$  Lagrangian description of AD with superpotential

$$W = \sum_{j=0}^{n-2} \alpha_j \text{tr} q\phi^j \tilde{q} + \sum_{j=2}^n \beta_j \text{tr} \phi^j. \quad (3.2)$$

This is the theory whose central charge coincides with the one obtained in the  $(A_1, A_{2n-1})$  AD theory.

The 3d reduction of the theory mentioned above has been presented in [6]. Note that unlike the discussion of [26, 27], the Kaluza Klein (KK) monopole superpotential which is usually appearing when reducing 4d dualities to 3d ones [28], is not generated here. It is expected that the dimensional reduction of this theory is mirror dual to the reduction of the  $(A_1, A_{2n-1})$  AD theories to 3d discussed in [11].

Mirror symmetry relates theories (b) and (c) in figure 1 [29]. It has been shown after performing F-maximization that theory (a) is effectively equivalent to (b) [6]. The exact R-charge for the adjoint field has been numerically found to be  $r_\phi = 0$ , while the exact R-charge of the fundamentals is  $r_q = r_{\tilde{q}} = \frac{1}{2}$ . The latter corresponds to the free field value, and it is a necessary result for the  $\mathcal{N} = 4$  hypermultiplets.

The net effect of having a zero R-charge for the adjoint is that the  $SU(n)$  gauge symmetry *abelianizes* into a  $U(1)^{n-1}$  quiver. On the partition function this can be understood because the one-loop determinant of the adjoint cancels out the one of the vector multiplet. The abelian quiver is given in (b) with the following  $\mathcal{N} = 4$  superpotential

$$W = \Phi_i(P_i\tilde{P}_i - P_{i+1}\tilde{P}_{i+1}), \quad (3.3)$$

where  $P_i$  and  $\tilde{P}_i$  form the  $\mathcal{N} = 4$  bifundamental hypermultiplets. This quiver is mirror dual to the one in (c), corresponding to an  $\mathcal{N} = 4$  theory with superpotential

$$W = \gamma_n \sum_{i=1}^n Q_i \tilde{Q}_{n-i+1}, \quad (3.4)$$

where  $Q_i$  and  $\tilde{Q}_{n-i+1}$  form the  $\mathcal{N} = 4$  hypermultiplets and  $\gamma_n$  is a singlet.

Another crucial result of [6] has been to obtain such a mirror dual description starting from the UV 3d model, obtained before integrating out the massive deformations and performing mirror symmetry at this stage. This gave rise to a 3d quiver theory with a series of nodes sequentially confining, thanks to a new duality discovered in [18]. The final quiver (c) has been obtained at the end of an intricate cascade of confinements.

As mentioned in the introduction, the direct connection between the models (a) and (c) in figure 1 is still missing. In the following we will show that the equivalence of these two theories can be obtained without any recursion to the ideas of *abelianization* as well as sequential confinement of [6]. This would be a possible equivalence due to the analytical matching of their partition function for generic values of the gauge rank  $n$ . The matching is achieved by elaborating on an integral identity involving hyperbolic gamma functions mentioned in [23].

To prove the equivalence of the partitions functions in the models (a) and (c)  $Z_{S_b^3}^{(a)} = Z_{S_b^3}^{(c)}$ , we consider the *Theorem 5.6.8* of [23] which states the following identity

$$\begin{aligned} & \frac{\Gamma_h(\tau)^n}{n!} \int \prod_{1 \leq j < k \leq n} \frac{\Gamma_h(\tau \pm (x_j - x_k))}{\Gamma_h(\pm(x_j - x_k))} \prod_{i=1}^n \Gamma_h(\mu - x_i; \nu + x_i) e^{i\pi \lambda x_i} dx_i \\ &= \prod_{j=0}^{n-1} \Gamma_h \left( (j+1)\tau; j\tau + \mu + \nu; \omega - j\tau - \frac{\mu + \nu}{2} \pm \frac{\lambda}{2} \right) e^{\frac{i\pi n \lambda}{2}(\mu - \nu)}. \end{aligned} \quad (3.5)$$

Our first step would be to modify the  $\Gamma_h$  functions appearing on the r.h.s. of equation (3.5). This modification will be done by considering the following identities, calculated through the reflection relation (2.3)

$$\begin{aligned} \prod_{j=0}^{n-1} \Gamma_h((j+1)\tau) &= \frac{\Gamma_h(\tau)}{\prod_{j=1}^{n-1} \Gamma_h(2\omega - (j+1)\tau)} = \frac{\Gamma_h(\tau)}{\prod_{j=2}^n \Gamma_h(2\omega - j\tau)}, \\ \prod_{j=0}^{n-1} \Gamma_h(j\tau + \mu + \nu) &= \frac{\Gamma_h((n-1)\tau + \mu + \nu)}{\prod_{j=0}^{n-2} \Gamma_h(2\omega - j\tau - \mu - \nu)}. \end{aligned} \quad (3.6)$$

The second step is integrating both sides of identity (3.5) over  $\int d\eta$  where the paramters are related as  $\eta = \frac{\lambda}{2}$ . On the field theory side this corresponds to turning on a vector multiplet (i.e. gauging) for the topological symmetry. This gauging modifies the gauge group on the l.h.s. of equation (3.5), converting the  $U(n)$  factor into  $SU(n)$  as done in [28, 30, 31]. This can be seen by noting that the integral over  $\eta$  on the l.h.s. corresponds to  $\delta\left(\sum_{i=1}^n x_i\right)$ . On the r.h.s. of equation (3.5) the integration over  $\eta$  leaves a  $U(1)$  gauge group with  $n$  pairs of fundamentals and anti-fundamentals, corresponding to the fields originally charged under the topological symmetry. We arrive at the following equality

$$\begin{aligned} & \frac{\Gamma_h(\tau)^{n-1}}{n!} \prod_{j=2}^n \Gamma_h(2\omega - j\tau) \prod_{j=0}^{n-2} \Gamma_h(2\omega - j\tau - \mu - \nu) \\ & \times \int \prod_{1 \leq j < k \leq n} \frac{\Gamma_h(\tau \pm (x_j - x_k))}{\Gamma_h(\pm(x_j - x_k))} \prod_{i=1}^n \Gamma_h(\mu - x_i; \nu + x_i) \delta\left(\sum_{l=1}^n x_l\right) dx_i \\ &= \Gamma_h((n-1)\tau + \mu + \nu) \int d\eta \prod_{j=1}^n \Gamma_h\left(\omega - (j-1)\tau - \frac{\mu + \nu}{2} \pm \eta\right) e^{i\pi n \eta(\mu - \nu)}. \end{aligned} \quad (3.7)$$

Observe that the l.h.s. is the 3d  $\mathcal{N} = 2$   $SU(n)$  theory with one flavor, one adjoint and the  $\alpha_j$  and  $\beta_j$  singlets with superpotential (3.2). Indeed the one loop determinants of  $\alpha_j$  and  $\beta_j$ , and the zero roots of the  $SU(n)$  adjoint appear in front of the integral, and the integrand has the flavors  $q$  and  $\tilde{q}$  with real masses  $\mu$  and  $\nu$ , respectively. The real mass for the adjoint has been identified with  $\tau$ .

The complex parameter  $\mu$ ,  $\nu$  and  $\tau$  are unconstrained<sup>3</sup> and can be expressed as

$$\mu = m_q + \omega\Delta_q, \quad \nu = m_{\tilde{q}} + \omega\Delta_{\tilde{q}}, \quad \tau = m_\phi + \omega\Delta_\phi, \quad (3.8)$$

where  $m_i$  refers to the real mass of each field while  $\Delta_i$  to its R-charge.

We will now prove that the r.h.s. of equation (3.7) corresponds to the theory obtained after sequential confinement and mirror symmetry. The dual theory is compatible with the superpotential (3.4). Our approach to prove this equality begins by studying the relation among the parameters  $\mu$ ,  $\nu$  and  $\tau$ . Each superpotential term has R-charge 2 and global charges 0. In the case at hand the charges of the field  $\gamma_n$  can be read from the partition function and we have

$$\mu_{\gamma_n} = (n-1)\tau + \mu + \nu \quad (3.9)$$

while the  $j$ -th quark  $Q_j$  and antiquarks  $\tilde{Q}_j$  have charge

$$\begin{aligned} \mu_{Q_j} &= \omega - (j-1)\tau - \frac{\mu + \nu}{2} + \eta, \\ \mu_{\tilde{Q}_j} &= \omega - (j-1)\tau - \frac{\mu + \nu}{2} - \eta. \end{aligned}$$

In this way each superpotential term is associated to the following combination as expected

$$\mu_{\gamma_n} + \mu_{Q_j} + \mu_{\tilde{Q}_{n+1-j}} = 2\omega. \quad (3.10)$$

The identity (3.7) can be used to prove the results of [6] with respect of the enhancement of supersymmetry when  $\Delta_\phi = 0$  and  $\Delta_q = \Delta_{\tilde{q}} = 1/2$ . In this case we can set  $\tau = 0$  and  $\mu = \frac{\omega}{2} + b$ ,  $\nu = \frac{\omega}{2} - b$ .

Therefore, the r.h.s. of equation (3.7) becomes

$$\int d\eta e^{i\pi 2nb\eta} \Gamma_h\left(\frac{\omega}{2} \pm \eta\right)^n, \quad (3.11)$$

where the FI terms is  $nb$ . On the other hand, the limit  $\tau \rightarrow 0$  on the l.h.s. is obtained using the following identity [6]

$$\lim_{\tau \rightarrow 0} \Gamma_h(\tau) \Gamma_h(2\omega - j\tau) = j, \quad (3.12)$$

such that the final contribution of the first two terms in equation (3.7) is  $n!$ , which cancels the measure factor of  $SU(n)$ . In the integrand the limit  $\tau \rightarrow 0$  *abelianizes* the gauge group, and one ends up with the partition function of model (b) as expected. We conclude that the equivalence between the two sides of (3.7) in this limit corresponds to the equivalence between the partition function of the two  $\mathcal{N} = 4$  mirror dual phases.

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<sup>3</sup>This translates on the field theory side into the absence of a superpotential for the KK monopoles.



We can also use the identity (3.7) to study the  $\mathcal{N} = 2$  case discussed in [6]. On the field theory side the difference consists of keeping the interaction

$$\alpha_n q \phi^{n-1} \tilde{q}, \tag{3.13}$$

in the 3d UV Lagrangian. By performing F-maximization in this case the field  $\alpha_n$  should not hit the unitary bound. On the dual side the field  $\gamma_n$  is massive, because of a mass term of the form  $\gamma_n \alpha_n$ . Indeed the duality maps naturally the gauge invariant combination  $q \phi^{n-1} \tilde{q}$  to the singlet  $\gamma_n$ . When  $\alpha_n$  appears on one side corresponds to  $\gamma_n$  disappearing on the other side, this fact is common in the Seiberg like dualities.

We can describe this mechanism on  $Z_{S^3}$  by exploiting the relation (3.7). The field  $\alpha_n$  contributes to  $Z_{S^3}$  with its one loop determinant. It corresponds to multiply both sides of equation (3.7) by  $\Gamma_h(2\omega - (n - 1)\tau - \mu - \nu)$ .

Using the reflection relation (2.3) on the r.h.s. of equation (3.7) we obtain

$$\Gamma_h(2\omega - (n - 1)\tau - \mu - \nu) \Gamma_h((n - 1)\tau + \mu + \nu) = 1. \tag{3.14}$$

The final result corresponds to the identity between the partition functions of the expected  $\mathcal{N} = 2$  dual theories which discussed in [6].

#### 4 Comments on the $(A_1, D_4)$ model in 3d

Another class of AD theories with an  $\mathcal{N} = 1$  Lagrangian description is denoted by  $(A_1, D_{2n})$ . In the case of even  $n$  this theory has been constructed starting from the superpotential (3.1), but with a vev  $\langle M \rangle$  corresponding to a non-principal nilpotent orbit of the flavor symmetry group. The final theory is  $SU(n)$  SQCD with two flavors, an adjoint and a set of  $\alpha_j$  and  $\beta_j$  fields interacting through the superpotential

$$W = \sum_{j=0}^{n-2} \alpha_j \text{tr } q \phi^j \tilde{q} + \text{tr } p \phi \tilde{p} + \sum_{j=2}^n \beta_j \text{tr } \phi^j. \tag{4.1}$$

This model can be reduced to 3d, but in this case a monopole superpotential is generated [7]. The reduction has been studied for the  $(A_1, D_4)$  case and it has been shown that the theory is dual to an abelian gauge theory. In this case the *abelianization* is not as simple as in the  $(A_1, A_{2n-1})$  case, essentially because the  $R$ -charge of the adjoint does not vanish. In this case we have not been able to find any exact relation reproducing the matching of the original partition function with the mirror dual theory.

One possible way to have an analytical proof of the *abelianization* at the level of the partition function consists of considering an  $SU(2) \times U(1)$  quiver with one bifundamental flavor connecting the two gauge groups and two flavors in the  $SU(2)$  sector, with superpotential

$$W = M \text{tr } q_{12} q_{21} + \text{tr } q_{21} q_{12} q_{2A} q_{A2} + s \text{tr } q_{2B} q_{B2} + T_{U(1)}^-. \tag{4.2}$$

Here the indices 1 and 2 refer to the  $U(1)$  and to the  $SU(2)$  gauge groups, while  $A$  and  $B$  label the two flavors. The field  $T_-$  corresponds to the anti-monopole of the  $U(1)$  gauge group.

This theory is dual to the model discussed above and can be shown as follows; first we dualize the  $U(1)$  node: it has two flavors and its dual is just given by the meson  $\Phi_{22} = q_{21}q_{12}$  interacting with a singlet  $S_+$  (having the same charges of the monopole  $T_{U(1)}^-$  of the electric theory). This is one of the dualities derived in [18, 32]. The dual theory corresponds to  $SU(2)$  with 2 flavors and superpotential

$$W = \text{tr } \Phi_{22}q_{2A}q_{A2} + s \text{tr } q_{2B}q_{B2} + S_+ \det \Phi_{22} + M \text{tr } \Phi_{22}. \tag{4.3}$$

The field  $\Phi_{22}$  is a composite bifundamental field. It is made out of a singlet  $\sim \phi \times I_2$  and an adjoint  $\Phi$ . By using the matrix identity  $\det \Phi_{22} = -\frac{1}{2}\text{tr } \Phi^2 + \phi^2$  and by integrating out the massive fields  $\phi$  and  $M$  we can rewrite equation (4.3) as

$$W = \text{tr } \Phi_{22}q_{2A}q_{A2} + s \text{tr } q_{2B}q_{B2} + S_+ \text{tr } \Phi^2, \tag{4.4}$$

corresponding to the superpotential of the  $(A_1, D_4)$  theory reduced to 3d, in absence of the KK monopole superpotential. This term can be turned on in both the phases without spoiling the duality just performed.

It should be possible directly prove of the *abelianization* starting from the original  $SU(2) \times U(1)$  quiver. In this case the absence of adjoint matter simplifies the problem and it allows to use a larger web of 3d  $\mathcal{N} = 2$  dualities. We hope to come back to this issue in the future.

## 5 Further applications

In this section we discuss some further examples of integral identities involving hyperbolic hypergeometric integrals. The identities that we will discuss are listed in [23] and they represent symplectic gauge groups with matter fields in the fundamental and in the antisymmetric representations. Here we restrict our analysis to the case of  $SP(2) = SU(2)$  gauge group, where the antisymmetric field disappears. We show that in the case with six fundamentals the integral identity reduces to a modification of Aharony duality studied in [18], with a quadratic monopole superpotential. We recover the identity for a limiting case of Aharony duality for the case of four fundamentals.

### 5.1 An exact relation for a higher power monopole duality

Let us discuss one of the new dualities found in [18], involving power monopole superpotentials. We show that, when the gauge group is  $U(2)$ , the matching of the partition functions of the dual theories can be derived from *Formula 5.3.7* of [23]:

$$\begin{aligned} & \frac{\Gamma_h(\tau)^n}{2^n n!} \int \prod_{i=1}^n dx_i \prod_{1 \leq i < l \leq n} \frac{\Gamma_h(\tau \pm x_i \pm x_l)}{\Gamma_h(\pm x_i \pm x_l)} \prod_{i=1}^n \frac{\prod_{a=1}^6 \Gamma_h(\mu_a \pm x_i)}{\Gamma_h(\pm 2x_i)} \\ &= \prod_{j=0}^{n-1} \Gamma_h((j+1)\tau) \prod_{a < b} \Gamma_h(j\tau + \mu_a + \mu_b). \end{aligned} \tag{5.1}$$

This identity holds provided that the complex parameters  $\tau$  and  $\mu_a$  satisfy the balancy condition

$$2(n-1)\tau + \sum_{a=1}^6 \mu_a = 2\omega. \tag{5.2}$$

The l.h.s. of equation (5.1) corresponds to an  $SP(2n)$  theory with an antisymmetric and six fundamentals. Furthermore, it is an interesting observation that (5.1) can be obtained in a physical way. This follows from the circle reduction of a limiting case of “rank changing” dualities for symplectic gauge theories with eight fundamentals and an antisymmetric, recently discovered in [33]. This observation deserves further studies and it may be shed some light on the four dimensional origin of the dualities with monopole superpotentials having higher powers discovered in [18].

Here we restrict to the case  $n = 1$ . In this case the measure factor simplifies leaving an  $SU(2)$  theory. We further choose the mass parameters as

$$\mu_i = m_i + m_B + m_A, \quad \mu_{i+3} = n_i - m_B + m_A \quad \text{for } i = 1, 2, 3 \tag{5.3}$$

with the further constraints  $\sum_i m_i = \sum_i n_i = 0$ . This fixes  $m_A = \frac{\omega}{3}$ , that will be crucial in the following. So far we are just reassembling the real masses, with a parameterization compatible with a global  $SU(3)_L \times SU(3)_R \times U(1)_B$  symmetry. We can modify the  $SU(2)$  gauge symmetry to  $U(2)$  by gauging the global  $U(1)_B$  symmetry. This gauging corresponds to integrate both sides of equation (5.1) over  $\int dm_B$ . On the l.h.s. we also re-define the integration variables as  $m_B + x = \sigma_1$  and  $m_B - x = \sigma_2$ , while we denote  $m_B = \sigma$  on the r.h.s. . Thus we arrive at the following identity

$$\int \prod_{i=1}^2 d\sigma_i \frac{\prod_{a=1}^3 \Gamma_h(m_a + m_A + \sigma_i; n_a + m_A - \sigma_i)}{\Gamma_h(\pm(\sigma_1 - \sigma_2))} \tag{5.4}$$

$$= \prod_{a,b} \Gamma_h(m_a + n_b + 2m_A) \int d\sigma \prod_{a=1}^3 \Gamma_h(2m_A - m_a + \sigma; 2m_A - n_a - \sigma).$$

This relation looks similar to the one expected for an Aharony duality between a  $U(2)$  and a  $U(1)$  theory with three flavors, but without electric monopoles acting as singlets and constraining the chiral ring of the dual phase. Moreover, differently from the ordinary Aharony duality, here the dual quarks have real mass  $2m_A$  instead of  $-m_A$ .

In order to study the field theory properties of the duality underlining the identity (5.4), we make use of the constraint  $m_A = \omega/3$ . This signals the fact that the real part of  $m_A$  vanishes, while its imaginary part, corresponding to the trial R-charge, is fixed. This is also the exact R-charge, because of the non-abelian nature of the other global symmetries. In this case, the duality map implies that the electric quarks have  $R$ -charge  $\Delta_Q$  while the magnetic quarks have  $R$ -charge  $\Delta_q = 2\Delta_Q$ . It is different from the expected one in the ordinary Aharony duality,  $1 - \Delta_Q$ . However, the balancing condition gives  $\Delta_Q = \frac{1}{3}$ , compatible with  $\Delta_Q = 1 - \Delta_q$ .

We conclude the analysis by checking that, when the real masses and the R-charges are constrained by condition (5.2), the relation (5.4) corresponds to the new duality conjectured

in [18], involving monopole superpotentials with quadratic powers. This duality has been formulated for 3d  $\mathcal{N} = 2$  U( $n$ ) SQCD with  $f$  flavors and superpotential

$$W = T_+^2 + T_-^2. \tag{5.5}$$

The dual theory corresponds to 3d  $\mathcal{N} = 2$  U( $f - n$ ) SQCD with  $f$  dual flavors and

$$W = Mq\tilde{q} + t_+^2 + t_-^2. \tag{5.6}$$

The presence of a monopole superpotential imposes that

$$\Delta_{T_\pm} = f(1 - \Delta_Q) - n + 1 = 1, \quad \Delta_{t_\pm} = f(1 - \Delta_q) - \tilde{n} + 1 = 1. \tag{5.7}$$

In our analysis we have studied the case with  $n = 2$  and  $f = 3$ . In this case the duality of [18] implies  $\tilde{n} = 1$ . Plugging these values in (5.7) we find

$$\Delta_Q = \frac{1}{3}, \quad \Delta_q = \frac{2}{3} \tag{5.8}$$

that corresponds to the values obtained above from the analysis of the partition functions.

## 5.2 A limiting case of Aharony duality

As the last example of the hyperbolic identities with application to the 3d  $\mathcal{N} = 2$  dualities we discuss the identity corresponds to *Theorem 5.6.6* of [23] as follow

$$\begin{aligned} & \frac{\Gamma_h(\tau)^n}{2^n n!} \int \prod_{i=1}^n dx_i \prod_{1 \leq i < l \leq n} \frac{\Gamma_h(\tau \pm x_i \pm x_l)}{\Gamma_h(\pm x_i \pm x_l)} \prod_{i=1}^n \frac{\prod_{a=1}^4 \Gamma_h(\mu_a \pm x_i)}{\Gamma_h(\pm 2x_i)} \\ &= \prod_{j=0}^{n-1} \frac{\Gamma_h((j+1)\tau)}{\Gamma_h((2n-2-j)\tau + \sum_r \mu_r)} \prod_{a < b} \Gamma_h(j\tau + \mu_a + \mu_b). \end{aligned} \tag{5.9}$$

Same as previous example, we fix  $n = 1$  and then we observe that this identity can be obtained as a limiting case of equation (5.1). This corresponds to a real mass flow on the field theory side.

In addition, we parameterize the real masses  $\mu$  as

$$\mu_a = m_a + m_B + m_A, \quad \mu_{a+2} = n_a - m_B + m_A \quad \text{for } a = 1, 2 \tag{5.10}$$

with the further constraints  $\sum_i m_i = \sum_i n_i = 0$ . So far we are just reassembling the real masses, with a parameterization compatible with a global  $SU(2)_L \times SU(2)_R \times U(1)_B \times U(1)_A$  symmetry. The  $SU(2)$  gauge symmetry becomes  $U(2)$  provided that we gauge the  $U(1)_B$  symmetry and add the FI term. This is done by integrating both sides of the identity by

$$\int dm_B e^{4i\pi\lambda m_B}, \tag{5.11}$$

where the normalization on the FI is arbitrarily chosen for the future purposes.

Redefine the integration variables as  $m_B + x = \sigma_1$  and  $m_B - x = \sigma_2$  we get

$$\begin{aligned} & \int \prod_{i=1}^2 d\sigma_i e^{2\pi i \lambda x_i} \prod_{a=1}^2 \Gamma_h(m_a + m_A + \sigma_i) \Gamma_h(n_a + m_A - \sigma_i) \Gamma_h^{-1}(\pm(\sigma_1 - \sigma_2)) \\ &= \frac{\prod_{a,b} \Gamma_h(m_a + n_b + 2m_A)}{\Gamma_h(4m_A)} \int d\sigma e^{2i\pi\lambda\sigma} \Gamma_h(2m_A \pm \sigma). \end{aligned} \tag{5.12}$$

The integral on the r.h.s. corresponds to the partition function of U(1) with two flavors, dual to the XYZ model [20]. In this case we can use the following identity

$$\int d\sigma e^{2i\pi\lambda\sigma} \Gamma_h(2m_A \pm \sigma) = \Gamma_h(4m_A) \Gamma_h\left(\pm \frac{\lambda}{2} - 2m_A\right), \tag{5.13}$$

and substituting it into (5.12) and get the following equality

$$\begin{aligned} & \int \frac{\prod_{i=1}^2 d\sigma_i e^{2\pi i \lambda x_i} \prod_{a=1}^2 \Gamma_h(m_a + m_A + \sigma_i) \Gamma_h(n_a + m_A - \sigma_i)}{\Gamma_h(\pm(\sigma_1 - \sigma_2))} \\ &= \prod_{a,b} \Gamma_h(m_a + n_b + 2m_A) \Gamma_h\left(\pm \frac{\lambda}{2} - 2m_A\right). \end{aligned} \tag{5.14}$$

This equality corresponds to the limiting case of Aharony duality for the U(2) model with two flavors [19]. Indeed in this case the expected dual has the following superpotential

$$W = v_+ v_- \det M = v_+ v_- (M_{11} M_{22} - M_{12} M_{21}). \tag{5.15}$$

We can observe that the constraints imposed from this superpotential are exactly encoded in (5.14). The FI term  $\lambda$  corresponds to the real mass for the monopoles  $v_{\pm}$  and the  $SU(2)_L \times SU(2)_R \times U(1)_A$  global charges reproduce the monopole and the mesons contributions.

## 6 Conclusions

In this paper we studied 3d  $\mathcal{N} = 2$  theories arising from the reduction of 4d  $\mathcal{N} = 1$  Lagrangian theories, conjectured to enhance to  $(A_1, A_{2n-1})$  AD theories. We provided a check of the IR duality relating the model reduced to 3d and its mirror dual, matching the three sphere partition functions. This corroborates the duality claimed among these models in [5, 6]. This check has been possible thanks to an integral identity, listed in [23], in terms of hyperbolic hypergeometric integrals. Meanwhile, we did not find an analogous relation for the  $(A_1, D_4)$  case. We studied also other identities, associated to  $SP(2n)$  gauge theories with fundamentals and an antisymmetric, showing that in the  $n = 1$  case they reduce to known 3d  $\mathcal{N} = 2$  dualities.

We believe that two main aspects of our analysis require a deeper analysis and may lead to interesting results. The first aspects regards the  $(A_1, D_4)$  case. In section 4 we have obtained a dual description of the Lagrangian reduction of the  $\mathcal{N} = 1$  theory that enhances to AD. This 3d duality can be helpful in proving the abelianization and it deserves further

investigations. Another interesting connection that emerged in the analysis regards the relation between the identity (5.1) and the dualities proposed in [33], based on the results of [34]. We have seen how the relation can be interpreted, in a simplified case in absence of antisymmetric matter, in terms of the dualities of [18] involving quadratic powers in the monopole superpotential. It may be interesting to develop a more general analysis for the 3d dualities obtained from the  $S^1$  reduction of [33] and further real mass and Higgs flow. We hope to report on progress in this direction in the next future.

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