Vortices on the cylinder

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Abstract

We apply the finite dimensional approximation techniques of Furuta, Kronheimer, and Manolescu to give a new proof of a result of Jaffe and Taubes.

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1 Introduction

The vortex equations are the absolute minima of the Yang-Mills-Higgs functional. For a unitary line bundle L over a Riemann surface let $\mathfrak{A}$ be the space of unitary connections of L and $\Omega(L)$ be the space of smooth sections of L then the Yang-Mills-Higgs functional $YMH\colon \mathfrak{A} \times \Omega(L) \to \mathbb{R}$ is defined as

$$YMH(A,v) = \int_\Sigma \big(|F_A|^2 + |\nabla_A v|^2 + \frac{1}{4}(1-|v|^2)^2\big) dvol.$$

Its absolute minima

$$\begin{aligned} \bar\partial_A v &= 0 \\ *F_A &= \frac{1}{2}\big(|v|^2 - 1\big) \end{aligned} \tag{1}$$

are the vortex equations. The Yang-Mills-Higgs functional is invariant under the action of the gauge group $\mathcal{G} = C^\infty(\Sigma, S^1)$ and hence so are the vortex equations.

For the case $\Sigma = \mathbb{C}$ the moduli spaces for the vortex equations were completely described by Jaffe and Taubes, see [17]. For solutions (A, v) of the vortex equations on $\mathbb{C}$ satisfying an appropriate decay condition at infinity, it turns out that the vortex number

$$N = \frac{1}{2\pi}\int_{\mathbb{C}} F_A$$

is an integer. Jaffe and Taubes proved that the moduli space of vortices with vortex number N modulo gauge is given by the N-fold symmetric product

$$\{(1) : \text{vortex number} = N\}/\mathcal{G} \cong S^N\mathbb{C} \cong \mathbb{C}^N.$$

The case for compact Riemann surfaces Σ was studied by Bradlow and Garcia-Prada, see [1, 10]. In the compact case the vortex number is

$$N = \langle c_1(L), [\Sigma]\rangle$$

and the moduli space was determined by Bradlow and Garcia-Prada

$$\{(1) : \text{vortex number} = N\}/\mathcal{G} \cong \begin{cases} S^N\Sigma & N < vol(\Sigma)/4\pi \\ \emptyset & N > vol(\Sigma)/4\pi. \end{cases}$$

If $N = vol(\Sigma)/4\pi$ then solutions of (1) necessarily satisfy $v \equiv 0$. In this article we consider the case where $\Sigma = \mathcal{Z}$ is the cylinder. We prove

Theorem A *The moduli space N-vortices on the cylinder modulo gauge is $S^N\mathcal{Z}$.*

We do not claim originality for this theorem since the methods of Jaffe and Taubes for the complex plane could also be used to determine the vortices on the cylinder. However, we will present in this paper a new approach for proving existence of PDE's by using the finite dimensional approximation techniques of

Furuta, Kronheimer, and Manolescu [9, 20, 23].
The idea of this new method is the following. Solutions of the vortex equations on the cylinder can be interpreted as flow lines of an action functional $\mathcal{A}$ defined on an infinite dimensional space $\mathscr{L}$. We consider a finite dimensional approximation $L \subset \mathscr{L}$ and homotop the flow lines of $\mathcal{A}$ to the flow lines of $\mathcal{A}|_L$. Since L is finite dimensional the flow lines of the restricted action functional are solutions of an ODE. This enables us to translate the question of existence of a PDE to the question of existence of an ODE.
However, to prove existence of finite energy Morse flow lines on a noncompact manifold is still a hard task. To do that we will take advantage of the fact that the restriction of our action functional to the finite dimensional approximation has the form of a Lagrange multiplier functional. It is well known from basic calculus that critical points of a function under a constraint can be found by considering the Lagrange multiplier functional. However, the Morse flow lines of the Lagrange multiplier functional and the Morse flow lines of the function restricted to the constraint are in general quite different. We will develop a theory which shows how they can be homotoped to each other. This theory allows us to translate the question of existence of Morse flow lines on a noncompact manifold to the question of existence of Morse flow lines on a compact manifold.

This paper is organized as follows. In Section 2 we prove Theorem A using finite dimensional approximation modulo the theory of Morse functions with Lagrange multipliers. In Section 3 we discuss further examples were our methods could be applied. In the Appendix we discuss the Theory of Morse functions with Lagrange multipliers.

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2 Proof of Theorem A

2.1 The gradient equation

The standard circle action on $\mathbb{C}$ given by

$$z \mapsto e^{i\theta} z, \quad e^{i\theta} \in S^1$$

is Hamiltonian with respect to the standard symplectic structure $\omega = dx \wedge dy$ on $\mathbb{C}$. A moment map for the action is given by

$$\mu(z) = -\frac{i}{2}|z|^2 + \frac{i}{2} \in i\mathbb{R} = \mathrm{Lie}(S^1).$$

We consider the loop space

$$\mathscr{L} = C^\infty(S^1, \mathbb{C} \times i\mathbb{R})$$

and define the action functional $\mathcal{A}\colon \mathscr{L} \to \mathbb{R}$ by

$$\mathcal{A}(v, \eta) = \int_0^1 \lambda(v)(\partial_t v) + \int_0^1 \langle \mu(v), \eta \rangle dt \tag{2}$$

where $\lambda = ydx$ is the Liouville one-form on $\mathbb{C}$ satisfying $d\lambda = -\omega$. The first term in (2) is just Floer's action functional on the loop space $C^\infty(S^1, \mathbb{C})$,

$$\mathcal{A}_{fl}(v) = \int_0^1 \lambda(v)(\partial_t v)$$

where the second term may be thought of as a Lagrange multiplier to the constraint $\mu^{-1}(0)$.

The gauge group

$$\mathcal{H} = C^\infty(S^1, S^1)$$

acts on $(v, \eta) \in \mathscr{L}$ by

$$h_*(v, \eta) = (hv, \eta - h^{-1}\partial_t h), \quad h \in \mathcal{H}.$$

The differential of the action functional $d\mathcal{A}$ and the L^2-metric g_{L^2} on $\mathscr{L}$ are invariant under the gauge action and hence also the gradient flow lines of $\nabla_{g_{L^2}}\mathcal{A}$ which are solutions $(v, \eta) \in C^\infty(\mathbb{R} \times S^1, \mathbb{C} \times i\mathbb{R})$ of the following PDE

$$\begin{aligned} \partial_s v + i\partial_t v + i\eta v &= 0 \\ \partial_s \eta + \mu(v) &= 0. \end{aligned} \tag{3}$$

Solutions of (3) are solutions of (1) in radial gauge. One has a natural bijection

$$\{(1)\}/\mathcal{G} \cong \{(3)\}/\mathcal{H}$$

by setting $A = \eta dt + \zeta ds$ and using a gauge transformation $g \in \mathcal{G}$ such that $g_*\zeta = 0$.

Remark 2.1 *There is a straightforward generalization of the action functional (2) to general Hamiltonian group actions on symplectic manifolds. The gradient flow lines of these action functionals are the symplectic vortex equations [2, 3, 7] which reduce in the special case of a circle action on 1-dimensional complex space to the classical vortex equations.*

Since the Marsden-Weinstein quotient $\mu^{-1}(0)/S^1$ is just a point, the critical manifold of $\mathcal{A}$ is homeomorphic to the gauge group $\mathcal{H}$. In particular,

$$\pi_0(\mathrm{crit}(\mathcal{A})) \cong \pi_0(\mathcal{H}) \cong \pi_1(S^1) \cong \mathbb{Z}$$

which enables us to recover the vortex number in this setting. The vortex number of a solution of (3) is proportional to the energy via

$$N = \frac{1}{\pi} \lim_{s\to\infty} \Big(\mathcal{A}\big((v,\eta)(s,\cdot)\big) - \mathcal{A}\big((v,\eta)(-s,\cdot)\big) \Big) = \frac{1}{\pi} E(v,\eta)$$

We denote by $\mathfrak{V}^N$ the moduli space of N-vortices on the cylinder, i.e. gradient flow lines of $\nabla_{g_{L_2}} \mathcal{A}$ modulo gauge which converge at the ends to connected components of $\mathrm{crit}(\mathcal{A})$ of difference $N \in \mathbb{Z}$. The following theorems follow from the results in [2].

Theorem 2.2 (Regularity) *The moduli spaces $\mathfrak{V}^N$ are smooth manifolds of dimension $2N$.*

Theorem 2.3 (Compactness moduli breaking) *Let (v^ν, η^ν) be a sequence of N-vortices. Then there exists a subsequence ν_j, a sequence of gauge transformations $h_j \in \mathcal{H}$, N_i-vortices (v_i, η_i) for $1 \leq i \leq \ell$, and sequences of real numbers S_i^j such that the timeshifted vortices converge uniformly in the C^∞_{loc}-topology*

$$(h_j)_*(v^{\nu_j}, \eta^{\nu_j})(\cdot, \cdot + S_i^j) \longrightarrow_{j\to\infty} (v_i, \eta_i)$$

different timeshifts diverge

$$\lim_{j\to\infty} |S_i^j - S_{i'}^j| = \infty, \quad i \neq i'$$

and the total vortex number is preserved

$$\sum_{i=1}^{\ell} N_i = N.$$

2.2 Finite dimensional approximation

The gauge group decomposes

$$\mathcal{H} = \mathcal{H}_0 \oplus S^1 \oplus \mathbb{Z}.$$

where the infinite dimensional contractible group $\mathcal{H}_0$ is given by

$$\mathcal{H}_0 = \left\{ g = \exp(\xi) \in \mathcal{H} : \xi \in C^\infty(S^1, i\mathbb{R}), \int_0^1 \xi dt = 0 \right\}.$$

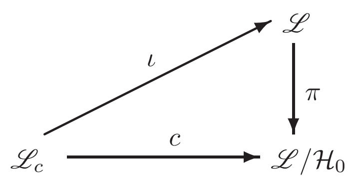
Following Manolescu [23] we get rid of $\mathcal{H}_0$ by projecting the gradient equations to the Coulomb section in $\mathscr{L}$. To see how this works, observe that $\mathcal{H}_0$ acts freely on $\mathscr{L}$ and since our gauge group is abelian we can put each $\eta \in C^\infty(S^1, i\mathbb{R})$ into global Coulomb gauge on the circle, namely there exists a unique $h_\eta \in \mathcal{H}_0$ such that

$$0 = d^*\big((h_\eta)_*\eta\big) = -\partial_t\big((h_\eta)_*\eta\big).$$

Hence we may think of

$$\mathscr{L}_c = C^\infty(S^1, \mathbb{C}) \times i\mathbb{R}$$

as a section in the principal $\mathcal{H}_0$-bundle $\mathscr{L}$, or more precisely, we have a commutative diagram



where ι denotes the canonical inclusion and c denotes the bijection which is induced by Coulomb gauge.

The L^2-metric g_{L^2} on $\mathscr{L}$ induces two natural metrics on $\mathscr{L}_c$

$$g_0 = \iota^* g_{L^2}, \quad g_1 = c^*[g_{L^2}]$$

where $[g_{L^2}]$ denotes the quotient metric of the L^2-metric on $\mathscr{L}/\mathcal{H}_0$.

Abbreviate

$$\mathcal{A}_c = \mathcal{A}|_{\mathscr{L}_c}.$$

Then $\mathcal{H}_0$-gauge equivalence classes of flow lines of $\nabla_{g_{L^2}}\mathcal{A}$ are in natural one-to-one correspondence with flow lines of $\nabla_{g_1}\mathcal{A}_c$ by projection. The importance of g_0 lies in the fact that flow lines of $\nabla_{g_0}\mathcal{A}_c$ are contained in natural finite dimensional subspaces of the infinite dimensional space $\mathscr{L}_c$. For integers $\mu \leq \nu$ consider the Fourierapproximations

$$L_\mu^\nu = \{z = \sum_{j=\mu}^{\nu} z_j e^{2\pi i j} : z_j \in \mathbb{C}\}$$

of the loop space $C^\infty(S^1, \mathbb{C})$. The metric g_0 is just the product of the L^2-metric on $C^\infty(S^1, \mathbb{C})$ and the metric induced from the inner product on $i\mathbb{R}$ and hence

$$\nabla_{g_0}\mathcal{A}_c(z, \eta) \in L_\mu^\nu \times i\mathbb{R} \subset \mathscr{L}_c, \quad (z, \eta) \in L_\mu^\nu \times i\mathbb{R}.$$

It follows that flow lines of $\mathcal{A}_c|_{L_\mu^\nu \times i\mathbb{R}}$ are actually flow lines of $\mathcal{A}_c$. Moreover, critical points of $\mathcal{A}_c$ are tuples

$$(v_0 e^{2\pi i m t}, 2\pi m i), \quad |v_0| = 1,\ m \in \mathbb{Z}$$

and hence for every pair of critical points of $\mathcal{A}_c$ there exists a finite dimensional approximation as above which contains both of them.

The following proposition shows that for each finite energy flow line of $\mathcal{A}_c$ one can find a finite dimensional approximation such that the flow line is entirely contained in it. For a set of flow lines which converge at both ends to the same critical points there can be found a finite dimensional approximation which contains the whole set simultanuously. However note, that since there are infinitely many critical points of $\mathcal{A}_c$ there is no finite dimensional approximation in which all finite energy flow lines lie simultanuously.

Proposition 2.4 *Assume that* $(v,\eta) \in C^\infty(\mathbb{R}\times S^1,\mathbb{C})\times C^\infty(\mathbb{R},i\mathbb{R})$ *is a gradient flow line of* $\nabla_{g_0}\mathcal{A}_c$ *such that*

$$\lim_{s\to\pm\infty}(v,\eta)(s,t) = (v_\pm e^{2\pi i m_\pm t}, 2\pi m_\pm i), \quad |v_\pm| = 1,\ m_\pm \in \mathbb{Z}$$

where the limit is uniformly with respect to the C^∞*-topology. Then* $(v,\eta)(s,\cdot)$ *is contained in* $L^{-m_+}_{-m_-} \times i\mathbb{R}$.

Proof: Abbreviate $\bar{\mu}(v) \in C^\infty(\mathbb{R},i\mathbb{R})$ by

$$\bar{\mu}(v)(s) = \int_0^1 \mu(v(s,t))dt, \quad s \in \mathbb{R}.$$

A gradient flow line of $\nabla_{g_0}\mathcal{A}_c$ is a solution of the following PDE

$$\begin{aligned} \partial_s v + i\partial_t v + i\eta v &= 0 \\ \partial_s \eta + \bar{\mu}(v) &= 0. \end{aligned} \tag{4}$$

Plugging in the Fourierexpansion

$$v(s,t) = \sum_{m=-\infty}^{\infty} v_j(s)e^{2\pi i j t}$$

into the first equation of (4) we obtain for each Fouriercoefficient the ODE

$$\partial_s v_m(s) + \big(i\eta(s) - 2\pi m\big)v_m(s) = 0. \tag{5}$$

Using (5), $\lim_{s\to\pm\infty}\partial_s v_m(s) = 0$, and the asymptotic behaviour of $\eta(s)$, we conclude that v_m vanishes identically unless m is contained in $\{-m_-,\dots,-m_+\}$. This proves the proposition. □

In order to homotop the PDE (3) to an ODE it remains in view of the proposition above to find a homotopy between g_0 and g_1. This homotopy has to be compact, i.e. the moduli spaces of finite energy flow lines should be compact modulo breaking and modulo the remaining action of the noncompact group $\mathbb{Z}$. Moreover, we require the homotopy to be equivariant with respect to the following torus action. There is the circle action on the target manifold $\mathbb{C}$ and there is a further circle action on the domain S^1 given by rotating the circle. The two actions commute on $\mathscr{L}_c$ and lead to an action of the two torus $T^2 = S^1 \times S^1$ on $\mathscr{L}_c$. Note that the action functional $\mathcal{A}_c$, and the metrics g_0 and g_1 are T^2-invariant.

Theorem 2.5 *There exists a continuous family of T^2-invariant metrics g_r for $r \in [0,1]$ on $\mathscr{L}_c$ with the following property. Assume that for $\nu \in \mathbb{N}$ there exists a sequence of flow lines (v^ν, η^ν) of $\nabla_{g_{r^\nu}} \mathcal{A}_c$ for $r^\nu \in [0,1]$ whose energy is uniformly bounded, i.e. there exists a constant $c > 0$ such that for all ν it holds*

$$E(v^\nu, \eta^\nu) \leq c.$$

Then there exists a subsequence ν_j, a sequence of gauge transformations $h_j \in \mathbb{Z}$, flow lines (v_i, η_i) for $1 \leq i \leq \ell$ of $\nabla_{g_{r_\infty}} \mathcal{A}_c$ for $r_\infty \in [0,1]$, and sequences of real numbers S_i^j such that the timeshifted vortices converge uniformly in the C^∞_{loc}-topology

$$(h_j)_*(v^{\nu_j}, \eta^{\nu_j})(\cdot, \cdot + S_i^j) \longrightarrow_{j \to \infty} (v_i, \eta_i)$$

different timeshifts diverge

$$\lim_{j \to \infty} |S_i^j - S_{i'}^j| = \infty, \quad i \neq i'$$

and the total energy is preserved

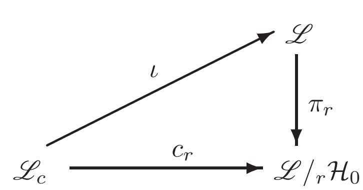
$$\lim_{j \to \infty} E(v^{\nu_j}, \eta^{\nu_j}) = \sum_{i=1}^{\ell} E(v_i, \eta_i)$$

Proof: We first construct a T^2-invariant homotopy between g_0 and g_1. In order to do that, observe that the geometric reason that g_0 and g_1 are different lies in the fact that the infinitesimal gauge action of $\mathcal{H}_0$ is not orthogonal to the Coulomb section $\mathscr{L}_c$ with respect to the L^2-metric on $\mathscr{L}$. To construct the homotopy we consider a family of $\mathcal{H}_0$ actions on $\mathscr{L}$ such that $\mathscr{L}_c$ is a section for the whole family of actions but that for the final action the Coulomb section gets orthogonal to the infinitesimal gauge action.

Taking advantage of the contractibility of the gauge group $\mathcal{H}_0$ we define for $r \in [0,1]$ and $h = \exp(\xi) \in \mathcal{H}_0$ the h_{*_r} action on $(v, \eta) \in \mathscr{L}$ by

$$h_{*_r}(v, \eta) = (\exp(r\xi)v, \eta - h^{-1}\partial_t h).$$

The deformed actions of $\mathcal{H}_0$ are still free on $\mathscr{L}$ and $\mathscr{L}_c$ is a simultanuous section for the whole family of actions. For each $r \in [0,1]$ we have a commutative diagram



where $\mathscr{L}/_r\mathcal{H}_0$ denotes the quotient of $\mathscr{L}$ under the r-action of $\mathcal{H}_0$, π_r denotes the according canonical projection, and c_r refers to the Coulomb gauge of the r-action. The L^2-metric on $\mathscr{L}$ is simultanuously $\mathcal{H}_0$ invariant for the whole family of actions and hence induces for every $r \in [0,1]$ a quotient metric $[g_{L^2}]_r$ on $\mathscr{L}/_r\mathcal{H}_0$. We define

$$g_r = c_r^*[g_{L^2}]_r.$$

It is easy to check that g_r are T^2-invariant for every $r \in [0,1]$. Moreover, for $r = 0$ the Coulomb section is orthogonal to the 0-action of $\mathcal{H}_0$ and hence g_0 defined in this way agrees with the previous definition of g_0.

The gradient flow lines of $\mathcal{A}_c$ with respect to the metric g_r are solutions of the following problem

$$\begin{aligned} \partial_s v + \xi_v v + i\partial_t v + i\eta v &= 0 \\ \partial_s \eta + \bar{\mu}(v) &= 0 \end{aligned} \tag{6}$$

where $\xi_v \in C^\infty(\mathbb{R}\times S^1, i\mathbb{R})$ which is determined for every $s \in \mathbb{R}$ by the conditions

$$\partial_t \xi_v(s,\cdot) = r^2\big(\mu(v(s,\cdot)) - \bar{\mu}(v)(s)\big), \qquad \int_0^1 \xi_v(s,t)dt = 0.$$

The main difficulty for proving the compactness statement in Theorem 2.5 is to obtain a uniform L^∞-estimate independent of $r \in [0,1]$ for all finite energy solutions of (6). This provides the following lemma.

Lemma 2.6 *Let $(v,\eta) \in C^\infty(S^1\times\mathbb{R},\mathbb{C})\times C^\infty(\mathbb{R},i\mathbb{R})$ be a finite energy solution of (6) for $r \in [0,1]$. Then there exists a constant $c < \infty$ independent of r such that $||v||_\infty < c$*

Proof: Define $u(s) = \frac{1}{2}\int_0^1 |v(s,t)|^2 dt$ for $s \in \mathbb{R}$.

Step 1: *$u(s) \leq 1/2$ for every $s \in \mathbb{R}$.*

Using a computation similar to the one in the proof of [3, Proposition 3.5] we estimate

$$\begin{aligned} \partial_s^2 u &= \int_0^1 \big(|\partial_s v + \xi_v v|^2 + |\partial_t v + \eta v|^2\big)dt + \\ &\quad 2\int_0^1 \langle \mu(v), (1-r^2)\bar{\mu}(v) + r^2\mu(v) - i/2\rangle dt \\ &\geq 2\langle \bar{\mu}(v), \bar{\mu}(v) - i/2\rangle \\ &= 2\langle u, u + 1/2\rangle \\ &\geq 2u(u - 1/2). \end{aligned}$$

Hence if $u(s_0) > 1/2$ for $s_0 \in \mathbb{R}$, then u cannot have a local maximum at s_0. However the finite energy assumption implies that $\lim_{s\to\pm\infty} u(s) = 1/2$ which proves Step 1.

Step 2: *There exists a constant* c_1 *and a gauge transformation* $h \in \mathbb{Z}$ *such that* $||\partial_t(h_* v)||_\infty \leq c_1 ||h_* v||_\infty^2 = c_1 ||v||_\infty^2$.

Fix some integer $n > 3$ and consider the finite cylinder $\mathcal{Z}_n = S^1 \times [-n, n]$. It follows from Step 1 that

$$||v||_{L^2(\mathcal{Z}_n)} = \mathcal{O}(1). \tag{7}$$

After a gauge transformation we may assume without loss of generality that

$$||\eta(0)|| = \mathcal{O}(1).$$

Using the second equation in (6) and Step 1 we conclude that

$$||\eta||_{L^\infty(\mathcal{Z}_n)} = \mathcal{O}(1). \tag{8}$$

The definition of ξ_v together with Step 1 implies that

$$||\xi_v||_{L^\infty(\mathcal{Z}_n)} = \mathcal{O}(1). \tag{9}$$

Combining (7),(8), and (9) and using the first equation in (6) we conclude that

$$||\bar{\partial} v||_{L^2(\mathcal{Z}_n)} = \mathcal{O}(1)$$

from which we deduce using (7) and elliptic regularity for the Cauchy-Riemann operator

$$||v||_{W^{1,2}(\mathcal{Z}_{n-1})} = \mathcal{O}(1). \tag{10}$$

It follows from Sobolev's embedding theorem that for every $p < \infty$ we have

$$||v||_{L^p(\mathcal{Z}_{n-1})} = \mathcal{O}_p(1) \tag{11}$$

from which we deduce analogously as before

$$||v||_{W^{1,p}(\mathcal{Z}_{n-2})} = \mathcal{O}_p(1). \tag{12}$$

Using (12) and the definition of ξ_v we conclude

$$||\xi_v||_{W^{1,p}(\mathcal{Z}_{n-2})} = \mathcal{O}_p(||v||_\infty). \tag{13}$$

The Laplacian of v satisfies the equation

$$\begin{aligned} \Delta v \quad = \quad & i(\partial_t \xi_v)v + i\xi_v(\partial_t v) - (\partial_s \xi_v)v - \xi_v(\partial_s v) \\ & -i(\partial_s \eta)v - i\eta(\partial_s v) - \eta(\partial_t v). \end{aligned} \tag{14}$$

Using (13) and (12) we conclude from (14) that

$$||\Delta v||_{L^p(\mathcal{Z}_{n-2})} = \mathcal{O}_p(||v||_\infty^2)$$

from which we conclude by elliptic regularity for the Laplace operator and (11)

$$||v||_{W^{2,p}(\mathcal{Z}_{n-3})} = \mathcal{O}_p(||v||^2_\infty). \tag{15}$$

Step 2 follows now from (15) and the Sobolev embedding theorem.

Step 3: *Proof of the lemma.*

Abbreviate $v_s = v(s, \cdot)$ and let $||v_s||_p$ be the L^p-norm on the circle. It follows from Step 1 and Step 2 that there exist constants c_0 and c_1 such that

$$||v_s||_2 \leq c_0, \quad ||\partial_t v_s||_\infty \leq c_1 ||v_s||^2_\infty. \tag{16}$$

We may assume without loss of generality that

$$|v_s(0)| = ||v_s||_\infty.$$

We then estimate for $t \in S^1 = \mathbb{R}/\mathbb{Z}$ using the second inequality in (16)

$$|v_s(t)| \geq ||v_s||_\infty - c_1 ||v_s||^2_\infty |t|. \tag{17}$$

Hence

$$\begin{aligned} ||v_s||_2 &\geq \left(2 \int_0^{1/(c_1||v_s||_\infty)} c_1^2 ||v_s||^4_\infty t^2 dt \right)^{1/2} \\ &= \sqrt{\frac{2}{3c_1}} ||v_s||^{1/2}_\infty \end{aligned}$$

from which we deduce using the second inequality in (16)

$$||v_s||_\infty \leq \frac{3c_1}{2} \cdot c_0^2.$$

This proves the lemma. □

Proof of Theorem 2.5 continued: It follows from the previous lemma that for gradient flow lines of $\nabla_{g_r} \mathcal{A}_c$ the first factor v remains in the compact 1-ball around 0 in the complex plane. Compactness modulo breaking can now be deduced from the results in [2]. However note, that their arguments simplify in our case. Since our gauge group is abelian we only need an easy version of Uhlenbeck's compactness theorem. Moreover, the bubbling analysis can be avoided for the standard symplectic structure on $\mathbb{C}$ by using the elliptic estimate

$$||v||_{W^{1,2}([-N,N]\times S^1)} \leq c_N \big(||\bar{\partial} v||_{L^2([-N-1,N+1]\times S^1)} + ||v||_{L^2([-N-1,N+1]\times S^1)} \big)$$

for every $N \in \mathbb{N}$ and a constant $c_N > 0$. □

2.3 The maps of Jaffe and Taubes

In [17] Jaffe and Taubes defined a map from the moduli space of N-vortices on the complex plane to the N-fold symmetric product of the complex plane and showed that it is bijective. In this subsection we define the analogon of their map for the gradient flow lines of $\nabla_{g_r}\mathcal{A}_c$ for all $r \in [0,1]$. We prove that for $r=1$ the map is bijective. This proves Theorem A in the introduction. As a biproduct we will obtain the proof of the compactness statement in Theorem 2.5.

Denote by $\mathfrak{V}_r^N$ the moduli space of N-vortices with respect to the metric g_r. Then $\mathfrak{V}_1^N = \mathfrak{V}^N$ the moduli space introduced before. It is useful to write the map of Jaffe and Taubes from $\mathfrak{V}_r^N$ to the N-fold symmetric product of the cylinder as the composite of two maps. Denote by $\mathfrak{W}_r^N$ the space of distributions w on the cylinder $\mathcal{Z}$ for which there exists N not necessarily distinct points $z_j \in \mathcal{Z}$ such that w is smooth outside of $\bigcup_{j=1}^N \{z_j\}$ on $\mathcal{Z}$ and satisfies the following integro Kazdan-Warner type problem with singularities and prescribed asymptotic behaviour

$$\begin{aligned} -\Delta w + r^2 e^w + (1-r^2)\int_0^1 e^w dt - 1 &= -4\pi \sum_{j=1}^N \delta(z-z_j) \\ \lim_{s\to\pm\infty} w(s,t) &= 0 \end{aligned} \tag{18}$$

where the limit is uniform with respect to the t-variable. Define the map $\mathfrak{T}_r^N : \mathfrak{V}_r^N \to \mathfrak{W}_r^N$ by

$$\mathfrak{T}_r^N(v,\eta) = \ln|v|^2$$

and the map $\mathfrak{J}_r^N : \mathfrak{W}_r^N \to S^N\mathcal{Z}$ by

$$\mathfrak{J}_r^N(w) = [z_1, \ldots, z_N].$$

Note that the composition $\mathfrak{J}_r^N \circ \mathfrak{T}_r^N$ maps a pair (v,η) to the zeros of v counted with multiplicity. For simplicity of notation we will often drop the index 1, i.e. $\mathfrak{J}^N$ means $\mathfrak{J}_1^N$, etc. We prove the following two theorems.

Theorem 2.7 *For every $r \in [0,1]$ and every $N \in \mathbb{N}$ the map $\mathfrak{T}_r^N$ is bijective.*

Theorem 2.8 *For every $N \in \mathbb{N}$ the map $\mathfrak{J}^N = \mathfrak{J}_1^N$ is bijective.*

As an easy corollary of the above two theorems we get Theorem A from the introduction.

Proof of Theorem A: By Theorem 2.7 and Theorem 2.8 the map $\mathfrak{J}^N \circ \mathfrak{T}^N$ gives a bijection for every $N \in \mathbb{N}$ between the moduli space of N-vortices on the cylinder and the N-fold symmetric product of the cylinder. □

Remark 2.9 *For $r=1$ the problem (18) simplifies to the following Kazdan-Warner type problem with singularities (see [19])*

$$\begin{aligned} -\Delta w + e^w - 1 &= -4\pi \sum_{j=1}^N \delta(z-z_j) \\ \lim_{s\to\pm\infty} w(s,t) &= 0. \end{aligned} \tag{19}$$

The bijectivity in Theorem 2.8 means that the above problem has a unique solution. The hard part is to prove existence of a solution. Our existence proof is based on finite dimensional approximation.

Proof of Theorem 2.7: Note that for $r \in [0,1]$ the action functional $\mathcal{A}^r : \mathscr{L} \to \mathbb{R}$ defined by

$$\mathcal{A}^r(v,\eta) = \int_0^1 \lambda(v)\partial_t v + \int_0^1 \langle r\mu(v(t)) + (1-r)\bar{\mu}(v), \eta(t)\rangle dt$$

is invariant under the r-action of $\mathcal{H}_0$ on $\mathscr{L}$ and

$$\mathcal{A}^r|_{\mathscr{L}_c} = \mathcal{A}_c.$$

It follows that the gradient flow of $\mathcal{A}^r$ with respect to the L^2-metric on $\mathscr{L}$ are in natural one-to-one correspondence with gradient flow lines of $\mathcal{A}_c$ with respect to the g_r-metric on $\mathscr{L}_c$ by projection on the Coulomb section. Note that projection on the Coulomb section does not change the value of $\ln|v|^2$ and hence we are left with showing the equivalence of flow line of $\nabla_{g_{L^2}}\mathcal{A}^r$ and solutions of the problem (18). Using the notation $\bar{\eta} = \int_0^1 \eta(t)dt$ for $\eta \in C^\infty(S^1, i\mathbb{R})$ gradient flow lines $(v,\eta) \in C^\infty(\mathcal{Z}, \mathbb{C} \times i\mathbb{R})$ of $\mathcal{A}^r$ with respect to the L^2-metric solve

$$\begin{aligned} \partial_s v + i\partial_t v + ir\eta v + i(1-r)\bar{\eta}v &= 0 \\ \partial_s \eta + r\mu(v) + (1-r)\bar{\mu}(v) &= 0. \end{aligned} \tag{20}$$

It is now an easy exercise to show that $\mathfrak{T}_r^N$ is well-defined, i.e. $\ln|v|^2$ of solutions of (20) are solutions of the integro type Kazdan-Warner type problem with singularities, and that for each solution of (18) there exists a unique gauge equivalence class of N-vortices satisfying (20), for details see [17]. $\square$

We finally embark on the prove of Theorem 2.8. We first prove a lemma.

Lemma 2.10 *For every $N \in \mathbb{N}$ the map $\mathfrak{J}^N$ is injective and its image is open and closed in $S^N\mathcal{Z}$.*

Proof: Assume that w and w' are two solution of the problem (19) for the same N-tuple of singularities $[z_1, \dots, z_N]$. Then its difference $w - w'$ is asymptotically zero and $\Delta(w - w') \geq e^{w'}(w - w')$. Hence $w = w'$.
The map $\mathfrak{J}^N \circ \mathfrak{T}^N$ is a continuous, one-to-one map between manifolds of the same dimension. Hence it is open by the Invariance of domain theorem, see for example [13, Corollary 18.9].
To show that the image of $\mathfrak{J}^N$ is closed, assume that $w_\nu \in \mathfrak{W}^N$ is a sequence such that $z_\nu = \mathfrak{J}^N(w_\nu)$ converges to $z \in S^N\mathcal{Z}$ as ν goes to infinity. For the sequence of flow lines $v_\nu = (\mathfrak{T}^N)^{-1}(w_\nu)$ there exists a subsequence v_{ν_j} which converges to a broken flow line. But since z_ν converges, the limit broken flow line is actually unbroken. Hence v_{ν_j} converges to $v \in \mathfrak{V}^N$ and

$$z = \mathfrak{J}^N(\mathfrak{T}^N(v)).$$

Hence the image of $\mathfrak{I}^N$ is closed. $\square$

The main work lies in the following existence statement for 1-vortices.

Theorem 2.11 *$\mathfrak{V}^1_r$ is not empty for every $r \in [0,1]$.*

Proof: Consider the following cylinder action on the gradient flow lines

$$(v,\eta)(s,t) \mapsto (v,\eta)(s+\sigma, t+\tau), \quad (\sigma,\tau) \in \mathcal{Z}.$$

1-vortices cannot break and hence $\mathfrak{V}^1_r/\mathcal{Z}$ is compact by Theorem 2.5 for every $r \in [0,1]$. By Proposition 2.4 we can identify the flow lines of $\nabla_{g_0}\mathcal{A}_c$ describing 1-vortices with Morse-flow lines on the finite dimensional approximation L^1_0. The finite dimensional approximation L^1_0 can be identified with $\mathbb{C}^2$ via its Fourierbasis and the T^2-action is given by

$$(e^{i\theta_1}, e^{i\theta_2})(z_1,z_2) \mapsto (e^{i\theta_1}z_1, e^{i(\theta_1+\theta_2)}z_2).$$

Denote by $\mu_{L^1_0}$ the moment map of the circle action of the first factor in $S^1 \times S^1 = T^2$ given by

$$\mu_{L^1_0}(z_1,z_2) = -\frac{i}{2}\big(|z_1|^2+|z_2|^2\big)+\frac{i}{2}.$$

The restriction of the action functional

$$A = \mathcal{A}_c|_{L^1_0\times i\mathbb{R}} = \mathcal{A}|_{L^1_0\times i\mathbb{R}} \in C^\infty(L^1_0 \times i\mathbb{R})$$

is given by

$$A(v,\eta) = \mathcal{A}_{fl}(v) + \langle \mu_{L^1_0}(v), \eta\rangle.$$

The moduli space $\mathfrak{V}^1_0/\mathcal{Z}$ can now be identied by Proposition 2.4 with the space of finite energy Morse flow lines of A modulo $T^2 \times \mathbb{R}$ where the group $\mathbb{R}$ acts by reparametrisation of flow lines.
The space $L^1_0 \times i\mathbb{R}$ is finite dimensional but still noncompact. Using the results of Appendix A we can homotop our Morse flow lines further to Morse flow lines on a compact manifold. In order to do that note that the function A on $L^1_0 \times i\mathbb{R}$ is the Lagrange multiplier functional of

$$H = \mathcal{A}_{fl}|_{\mu^{-1}_{L^1_0}(0)} \in C^\infty(\mu^{-1}_{L^1_0}(0)).$$

Hence finite energy Morse flow lines of A can be homotoped inside a compact subset of $L^1_0 \times i\mathbb{R}$ to Morse flow lines of H. The manifold $\mu^{-1}_{L^1_0}(0)$ is the three sphere S^3, the circle action of the first factor in $T^2 = S^1 \times S^1$ is the Hopf fibration $S^3 \to S^2$ and the circle action of the second factor in T^2 acts by rotation on the two-sphere S^2. The function H induces on S^2 the height function. In particular, the action of $T^2 \times \mathbb{R}$ on the Morse flow lines of H is free and the quotient consists of exactly one point.
The upshot of our construction is that we can homotop the moduli space $\mathfrak{V}^1_r/\mathcal{Z}$

by a compact homotopy to a point. During this homotopy the Fredholm index is unchanged by Proposition A.2. We are now in position to show that $\mathfrak{V}_r^1$ is nonempty for every $r \in [0,1]$. Assume the contrary. Then we apply the abstract perturbation theory of [8, 21, 22, 24, 25] to our compact homotopy. Actually, since we do not have to compactify our moduli spaces by broken flow lines containing bubble trees, the more elementary theory of [4] is already sufficient. What we obtain is a compact branched manifold containing just one boundary point of weight one. But such an object does not exist. Hence $\mathfrak{V}_r^1$ is nonempty for every $r \in [0,1]$. □

Proof of Theorem 2.8: It follows from Floer's gluing construction and Theorem 2.11 that $\mathfrak{V}^N$ is not empty for every $N \in \mathbb{N}$. Hence $\mathfrak{J}^N(\mathfrak{W}^N) = \mathfrak{J}^N \circ \mathfrak{T}^N(\mathfrak{V}^N)$ is not empty in $S^N\mathcal{Z}$. Since the image im $\mathfrak{J}^N$ is open and closed by Lemma 2.10 and $S^N\mathcal{Z}$ is connected it follows that $\mathfrak{J}^N$ is surjective. Since again by Lemma 2.10 $\mathfrak{J}^N$ is injective the theorem follows. □

3 Further directions

3.1 The symplectic vortex equations and Givental's toric map spaces

Instead of the circle action on $\mathbb{C}$ we can study more generally linear torus actions on a complex vector space. Assume that for $k \leq n$ the torus $T^k = \{e^{iv} : v \in \mathbb{R}^k\}$ acts on the complex vector space $\mathbb{C}^n$ via the action

$$\rho(e^{iv})z = e^{iAv}z, \quad z \in \mathbb{C}^n, \ v \in \mathbb{R}^k$$

for some $(n \times k)$-matrix A with integer entries. We endow the Lie algebra of the torus

$$\mathrm{Lie}(T^k) = \mathfrak{t}^k = i\mathbb{R}^k$$

with its standard inner product. The action of the torus on $\mathbb{C}^n$ is Hamiltonian with respect to the standard symplectic structure $\omega = \sum_{i=1}^n dx_i \wedge dy_i$. Denoting by A^T the transposed matrix of A a moment map $\mu\colon \mathbb{C}^n \to \mathfrak{t}^k$ is given by

$$\mu(z) = -iA^T w, \quad w = \frac{1}{2}\begin{pmatrix} |z_1|^2 \\ \vdots \\ |z_n|^2 \end{pmatrix}, \tag{21}$$

i.e.

$$d\langle \mu, \xi\rangle = \iota_{X_\xi}\omega, \quad \xi \in \mathfrak{t}^k$$

for the vector field X_ξ on $\mathbb{C}^n$ given by the infinitesimal action

$$X_\xi(z) = \dot{\rho}(\xi)(z), \quad z \in \mathbb{C}^n.$$

We assume the following hypothesis,

(H) The moment map μ is proper and T^k acts freely on $\mu^{-1}(\tau)$.

It follows from (H) that the Marsden-Weinstein quotient

$$\mathbb{C}^n//T^k = \mu^{-1}(\tau)/T^k$$

is a compact symplectic manifold of dimension

$$\dim(\mathbb{C}^n//T^k) = 2(n-k),$$

where the symplectic structure is induced from the standard symplectic structure on $\mathbb{C}^n$.
Let $\mathscr{L}$ be the loop space

$$\mathscr{L} := C^\infty(S^1, \mathbb{C}^n \times \mathfrak{t}^k).$$

The gauge group

$$\mathcal{H} = C^\infty(S^1, T^k)$$

acts on $\mathscr{L}$ by

$$h_*(v,\eta) = (\rho(h)v, \eta - h^{-1}\partial_t h), \quad h \in \mathcal{H},\ (v,\eta) \in \mathscr{L}.$$

Recall Floer's action functional $\mathcal{A}_{fl} \colon C^\infty(S^1, \mathbb{C}^n) \to \mathbb{R}$ given by

$$\mathcal{A}_{fl}(v) = \int_0^1 \lambda(v)(\partial_t v)$$

where λ denotes the Liouville 1-form

$$\lambda = \sum_{i=1}^n y_i dx_i, \quad d\lambda = -\omega.$$

The Moment action functional $\mathcal{A} \colon \mathscr{L} \to \mathbb{R}$ is defined by

$$\mathcal{A}(v,\eta) := \mathcal{A}_{fl}(v) + \int_0^1 \langle \mu(v(t)) - \tau, \eta(t) \rangle dt.$$

Again one may think of η in the second integral as a Lagrange multiplier. In particular, the critical points of $\mathcal{A}$ are the critical points of Floer's action on the constraint $\mu^{-1}(\tau)$.
The gradient flow lines of $\mathcal{A}$ with respect to the L^2-metric g_{L^2} on $\mathscr{L}$ are solutions $(v,\eta) \in C^\infty(\mathbb{R} \times S^1, \mathbb{C}^n \times \mathfrak{t}^k)$ of

$$\begin{aligned} \partial_s v + i\partial_t v + i\dot{\rho}(\eta)v &= 0 \\ \partial_s \eta + \mu(v) &= \tau. \end{aligned}$$

These are examples of the symplectic vortex equations on the cylinder in temporal gauge.

In this setting the symplectic vortex equations can be homotoped again via a Floer homotopy compact up to breaking of flow lines to Morse problems on finite dimensional compact manifolds. If one considers a finite dimensional Fourierapproximation L of the loop space $C^\infty(S^1, \mathbb{C}^n)$ then the T^k-action on $\mathbb{C}^n$ induces a T^k-action on L by coefficientwise multiplication. This action is again Hamiltonian with moment map μ_L normalized such that $\mu_L(0) = 0$. The finite dimensional compact manifolds we end up with are

$$G_L = \mu_L^{-1}(\tau)/T^k$$

and the Morse function is again Floer's action functional restricted to G_L. The spaces G_L are known as Givental's toric map spaces. They were introduced by Givental in [12] and studied by different authors in [11, 15, 28].

3.2 Warped product metrics and Chern-Simons Vortices

If we consider the same action functional $\mathcal{A}\colon \mathscr{L} \to \mathbb{R}$ as for the vortex equations but instead of the standard flat L^2-metric a warped product metric g_γ for a smooth function $\gamma\colon [0,\infty) \to (0,\infty)$ given by

$$g_\gamma(v,\eta)\big((\hat{v}_1,\hat{\eta}_1),(\hat{v}_2,\hat{\eta}_2)\big) = \int_0^1 \langle \hat{v}_1, \hat{v}_2\rangle dt + \int_0^1 \gamma(|v|)^2 \langle \hat{\eta}_1, \hat{\eta}_2\rangle dt$$

for $(v,\eta) \in \mathscr{L}$ and $(\hat{v}_1,\hat{\eta}_1),(\hat{v}_2,\hat{\eta}_2) \in T_{(v,\eta)}\mathscr{L}$ we obtain the following gradient equations for $(v,\eta) \in C^\infty(\mathbb{R} \times S^1, \mathbb{C} \times i\mathbb{R})$

$$\begin{aligned} \partial_s v + i\partial_t v + i\eta v &= 0 \\ \partial_s \eta + \frac{1}{\gamma(|v|)^2}\mu(v) &= 0. \end{aligned}$$

In particular, if we choose

$$\gamma(r) = \frac{1}{r}$$

we obtain

$$\begin{aligned} \partial_s v + i\partial_t v + i\eta v &= 0 \\ \partial_s \eta + |v|^2 \mu(v) &= 0. \end{aligned}$$

These are the selfduality equations for the Chern-Simons vortices on the cylinder discovered by Hong-Kim-Pac and Jackiw-Weinberg, see [14, 16]. We refer the reader to the excellent textbook of Y. Yang [29] for a detailed treatment of this equation. This textbook may also serve as a guide to the corresponding literature.

Note that for this choice of γ the metric γ_v becomes singular if v goes to zero. One may think of this as a continuum of "critical points at infinity" for each $(0,\eta)$ where η is a smooth loop in the Lie algebra $i\mathbb{R}$. In particular, the action functional $\mathcal{A}$ takes on the set of "critical points at infinity" every value in $\mathbb{R}$. The energy of flow lines which converge at one end to a "critical point at infinity" can therefore be any value in $\mathbb{R}$, in contrast to the classical vortex equations where the energy of finite energy flow lines was quantized. Such solutions are called in the physics literature "nontopological solutions". We again refer to the textbook of Y. Yang [29] and the literature cited therein for a detailed treatment of nontopological solutions. For compact Riemann surfaces existence of "nontopological solutions" was proved by Tarantello and Ding-Jost-Li-Peng-Wang [5, 27]. In this case the "nontopological solutions" are characterized by the property that v converges to 0 under the adiabatic limit obtained by letting the Chern-Simons coupling parameter tend to zero.

On the finite dimensional approximations the Chern-Simons vortices are flow lines of a Lagrange multiplier functional with respect to a warped product metric. "Critical points at infinity" are responsible for the failure of the Palais-Smale condition discussed in the appendix. So the study of flow lines of Lagrange multiplier functionals with respect to a warped product metric is a finite dimensional analogon of the Abelian Chern-Simons-Higgs theory and should lead to a deeper understanding of the phenomenons occuring in this theory.

A Morse functions with Lagrange multipliers

Assume that M is a finite dimensional manifold and V is a finite dimensional real vector space. It is well known from basic calculus that critical points of a smooth function $f \in C^\infty(M)$ satisfying a constraint given by the zero set of a smooth function $h \in C^\infty(M, V)$ can be found by considering the Lagrange multiplier functional $F \in C^\infty(M \times V^*)$, where V^* is the dual vector space of V, given by

$$F(x, v^*) = f(x) + v^*(h(x)).$$

If 0 is a regular value of h then there is a natural one-to-one correspondence between critical points of F and critical points of $f|_{h^{-1}(0)}$. However, Morse flow lines of F and Morse flow lines of $f|_{h^{-1}(0)}$ may be quite different. Even if $h^{-1}(0)$ is compact it is not a priori clear that the moduli spaces of flow lines of F are compact modulo breaking since F is neither bounded from above nor below. However, we will show that if h is locally proper around 0, then F satisfies the Palais-Smale condition from which we can deduce that flow lines of F remain in a compact subset of the noncompact manifold $M \times V^*$.

A first possibility to homotop Morse flow lines of F to Morse flow lines of $f|_{h^{-1}(0)}$ would be the adiabatic limit method. For a fixed Riemannian metric g_M on M and a fixed Riemannian metric g_{V^*} on V^*, induced from a Euclidean scalar product on V we consider the family of metrics g_ϵ on $M \times V^*$ for $\epsilon \in (0, 1]$

$$g_\epsilon = g_M \oplus \epsilon^2 g_{V^*}.$$

If ϵ goes to zero, then the gradient flow lines of F with respect to the metric g_ϵ converge to gradient flow lines of $f|_{h^{-1}(0)}$ with respect to the metric $g_M|_{h^{-1}(0)}$.

If a generalized implicit function theorem as in [6] can be established at $\epsilon = 0$, then this would lead to a homotopy compact modulo breaking between the two moduli spaces of gradient flow lines.

In this section we will pursue another approach. We will consider a homotopy of f and the Riemannian metric g_M such that $f|_{h^{-1}(0)}$ is unchanged during the homotopy but the normal derivatives of $\nabla_{g_M} f$ at $h^{-1}(0)$ are homotoped to zero. Since $f|_{h^{-1}(0)}$ is fixed the critical points of the Lagrange multiplier functional can be canonically identified with the set of critical points of $f|_{h^{-1}(0)}$ during the whole homotopy. If the normal derivatives of $\nabla_{g_M} f$ at $h^{-1}(0)$ vanish then the moduli space of Morse flow lines of $f|_{h^{-1}(0)}$ is canonically contained in the moduli space of flow lines of F. We will prove that for special choices of f and g_M there are no other flow lines of F. The main idea is to choose g_M in such a way that a tubular neighbourhood of $h^{-1}(0)$ in M becomes very huge and then prove that finite energy flow lines have to remain in this tubular neighbourhood.

It is natural to formulate our main theorem in the language of Morse-Bott functions. In order to fix notation we recall briefly its definition. A function F on a finite dimensional manifold M is called Morse-Bott if the critical set is a submanifold of M and for each $x \in \mathrm{crit}(F)$ we have

$$T_x \mathrm{crit}(F) = \ker H_F(x)$$

where $H_F(x)$ is the Hessian of F at x. It is well known that the Morse-Bott condition implies that flow lines which remain in a compact set of M converge at both ends exponentially fast to critical points of F. For a Riemannian metric g on M we denote by $\mathcal{M}(F, g)$ the moduli space of finite energy flow lines of $\nabla_g F$.

The main theorem of this section can now be statet in the following way.

Theorem A.1 *Let M be a finite dimensional manifold, Γ be a Lie group acting on M, and let $(V, \langle\,,\,\rangle)$ be a finite dimensional Euclidean vector space. Assume that g_M is a Γ-invariant, geodesically complete Riemannian metric on M, $h \in C^\infty(M, V)$ and $f \in C^\infty(M)$ are Γ-invariant functions satisfying the following conditions.*

- *0 is a regular value of h,*
- *h is locally proper around 0, i.e. there exists an open neighbourhood V_0 of 0 in V such that $h^{-1}(\mathrm{cl}(V_0))$ is compact,*
- *the restriction of f to the compact manifold $h^{-1}(0)$ is Morse-Bott.*

Denote by V^* the dual vector space of V and let Γ act on $M \times V^*$ by $\gamma(x, v^*) = (\gamma x, v^*)$ for $\gamma \in \Gamma$ and $(x, v^*) \in M \times V^*$. Then there exists a smooth family of Γ-invariant Morse-Bott functions $F_r \in C^\infty(M \times V^*)$ for $r \in [0, 1]$ and a smooth family of Γ-invariant Riemannian metrics g_r on $M \times V^*$ satisfying

$$F_0(x, v^*) = f(x) + v^*(h(x)), \quad g_0 = g_M \oplus g_{V^*}$$

where g_{V^*} is the metric on V^* induced from the scalar product $\langle\,,\,\rangle$ on V, such that the following conditions are satisfied.

(i) *The inclusion $\iota\colon h^{-1}(0) \to M \times V^*, x \mapsto (x,0)$ induces a bijection*

$$\iota_*\colon \mathcal{M}(f|_{h^{-1}(0)}, g_M|_{h^{-1}(0)}) \to \mathcal{M}(F_1, g_1)$$

defined by

$$\iota_* y(s) = \iota(y(s)), \quad y \in \mathcal{M}(f|_{h^{-1}(0)}, g_M|_{h^{-1}(0)}),\ s \in \mathbb{R}.$$

(ii) *For $r \in [0,1]$ there exists a smooth family of diffeomorphism $\phi_r\colon \mathrm{crit}F_0 \to \mathrm{crit}F_r \subset M \times V^*$.*

(iii) *There exists a compact set $K \subset M \times V^*$ such that*

$$\left\{ y(\sigma) : y \in \bigcup_{r\in[0,1]} \mathcal{M}(F_r, g_r),\ \sigma \in \mathbb{R} \right\} \subset K.$$

Proof: We prove the theorem in seven steps.

Step 1 (Neighbourhood of the constraint): *There exists an open neighbourhood V_1 of 0 in V, a Γ-invariant open neighbourhood U of $h^{-1}(0)$ in M, and a Γ-equivariant diffeomorphism*

$$\phi\colon h^{-1}(0) \times V_1 \to U$$

where Γ acts on $h^{-1}(0) \times V_1$ by $\gamma(x,v) = (\gamma x, v)$ for $\gamma \in \Gamma$ and $(x,v) \in h^{-1}(0) \times V_1$ such that

$$h(\phi(x,v)) = v, \quad (x,v) \in h^{-1}(0) \times V_1. \tag{22}$$

Since 0 is a regular value of h there exists an open neighbourhood U_0 of $h^{-1}(0)$ such that $dh(y)$ is surjectiv for every $y \in U_0$. For $v \in V$ define the vector field ξ_v on U_0 by the conditions

$$dh(y)\xi_v(y) = v, \quad \xi_v(y) \in \mathrm{ker}dh(y)^\perp, \quad y \in U_0$$

where $\mathrm{ker}dh(y)^\perp$ denotes the orthogonal complement of the kernel of $dh(y)$ with respect to the metric g_M. Since $h^{-1}(0)$ is compact there exists an open neighbourhood V_1 of 0 in V such that for each $v \in V_1$ and for each $x \in h^{-1}(0)$ there exists a unique solution $y_{x,v} \in C^\infty([0,1], U)$ of the problem

$$y_{x,v}(0) = x, \quad \partial_t y_{x,v}(t) = \xi_v(y_{x,v}(t)), \quad t \in [0,1],$$

and the map

$$\phi(x,v) = y_{x,v}(1), \quad (x,v) \in h^{-1}(0) \times V_1$$

is a diffeomorphism. Set $U = \phi(h^{-1}(0) \times V_1)$. Since h and g_M are Γ-invariant it follows that $\gamma(y_{x,v}) = y_{\gamma x,v}$ for $\gamma \in \Gamma$ and $(x,v) \in h^{-1}(0) \times V_1$. Hence U is Γ-invariant and $\phi\colon h^{-1}(0) \times V_1 \to U$ is a Γ-equivariant diffeomorphism. Moreover, we compute

$$\begin{aligned} h(\phi(x,v)) &= h(y_{x,v}(1)) \\ &= h(y_{x,v}(0)) + \int_0^1 \frac{d}{dt} h(y_{x,v}(t))dt \\ &= h(x) + \int_0^1 dh(y_{x,v}(t))\partial_t y_{x,v}(t)dt \\ &= \int_0^1 dh(y_{x,v}(t))\xi_v(y_{x,v}(t))dt \\ &= \int_0^1 vdt \\ &= v. \end{aligned}$$

This proves (22) and hence Step 1.

Step 2: *We construct the homotopies.*

In this step we construct a Γ-invariant function $f_1 \in C^\infty(M)$ and a Γ-invariant metric $g_{M,1}$ on M. We then set for $r \in [0,1]$

$$f_r = (1-r)f + rf_1, \quad g_{M,r} = (1-r)g_M + rg_{M,1}$$

and define the homotopy of functions $F_r \in C^\infty(M \times V^*)$ by

$$F_r(x, v^*) = f_r(x) + v^*(h(x)), \quad (x, v^*) \in M \times V^*$$

and the homotopy of metrics g_r on $M \times V^*$ by

$$g_r = g_{M,r} \oplus g_{V^*}.$$

Choose a small number $\delta > 0$ such that the open δ-ball $B_\delta = \{v \in V : ||v|| < \delta\}$ is contained in the neighbourhood V_1 of 0 in V constructed in Step 1. Moreover, since h is locally proper at 0, we may assume that

$$h(x) > \delta \Rightarrow x \notin \phi(h^{-1}(0) \times B_\delta). \tag{23}$$

Choose further a cutoff function $\hat{\beta} \in C^\infty([0,\delta),[0,1])$ such that $\hat{\beta}|_{[0,\delta/2]} = 1$ and $\hat{\beta}|_{[3\delta/4,\delta)} = 0$. Denote by $\pi_1\colon h^{-1}(0) \times B_\delta \to h^{-1}(0)$ and by $\pi_2\colon h^{-1}(0) \times B_\delta \to B_\delta$ the projection to the first, respectively the second, factor. We will use the following Γ-invariant cutoff function on M given by

$$\beta(x) = \begin{cases} \beta\big(|\pi_2(\phi^{-1}(x))|\big) & x \in \phi(h^{-1}(0) \times B_\delta) \\ 0 & x \notin \phi(h^{-1}(0) \times B_\delta). \end{cases}$$

Define the function $f_1 \in C^\infty(M)$ by

$$f_1(x) = \begin{cases} \beta(x) f\big(\pi_1(\phi^{-1}(x))\big) + \big(1-\beta(x)\big) f(x) & x \in \phi(h^{-1}(0) \times B_\delta) \\ f(x) & x \notin \phi(h^{-1}(0) \times B_\delta). \end{cases}$$

Set

$$C := \max_{x \in h^{-1}(0)} \{f(x)\} - \min_{x \in h^{-1}(0)} \{f(x)\} \tag{24}$$

and choose a constant

$$\kappa > \frac{16C}{\delta^2}. \tag{25}$$

Let g_κ be the product metric on $h^{-1}(0) \times B_\delta$

$$g_\kappa = g_M|_{h^{-1}(0)} \oplus \kappa^2 g_{B_\delta}$$

where g_{B_δ} is the standard euclidean metric on the ball $B_\delta \subset V$. We are now able to define the metric $g_{M,1}$ on M by the formula

$$g_{M,1}(x) = \begin{cases} \beta(x)(\phi_* g_\kappa)(x) + \big(1-\beta(x)\big) g_M(x) & x \in \phi(h^{-1}(0) \times B_\delta) \\ g_M(x) & x \notin \phi(h^{-1}(0) \times B_\delta). \end{cases}$$

Step 3: *The trace of each finite energy flow line $y \in C^\infty(\mathbb{R}, M \times V^*)$ of $\nabla_{g_1} F_1$ is contained in $\phi(h^{-1}(0) \times B_{\delta/2}) \times V^*$.*

First note that if $x \in M \setminus \phi(h^{-1}(0) \times B_{\delta/4})$ and $v^* \in V^*$ then it follows from (22) and (23) that

$$||\nabla_{g_1} F_1(x, v^*)|| \geq ||h(x)|| \geq \frac{\delta}{4}. \tag{26}$$

Observe further that the energy of a finite energy flow line is bounded from above by the constant C introduced in (24), i.e.

$$\int_{-\infty}^{\infty} ||\nabla_{g_1} F_1(y(s))||_{g_1}^2 ds \leq C. \tag{27}$$

Now assume by contradiction that there exists $\sigma \in \mathbb{R}$ such that

$$y(\sigma) \notin \phi(h^{-1}(0) \times B_{\delta/2}) \times V^*. \tag{28}$$

Denote by $\tau(\sigma) > \sigma$ the real number

$$\tau(\sigma) := \min\{s \in \mathbb{R} : y(s) \notin \phi(h^{-1}(0) \times B_{\delta/4}) \times V^*\}.$$

Note that $\tau(\sigma)$ is finite, since the energy of the flow line y is assumed to be finite and the critical points of F_1 lie in $h^{-1}(0) \times V^*$. Denoting by $\mathrm{dist}_{g_1}(\cdot,\cdot)$

the distance with respect to the metric g_1 we estimate using (25), (26), and (27)

$$\begin{aligned} C &< \frac{\delta^2\kappa}{16} \\ &\leq \frac{\delta}{4}\cdot \mathrm{dist}_{g_1}\big(y(\sigma), y(\tau(\sigma))\big) \\ &\leq \frac{\delta}{4}\int_\sigma^{\tau(\sigma)} ||\partial_s y(s)||_{g_1} ds \\ &= \frac{\delta}{4}\int_\sigma^{\tau(\sigma)} ||\nabla_{g_1} F_1(y(s))|| ds \\ &\leq \int_\sigma^{\tau(\sigma)} ||\nabla_{g_1} F_1(y(s))||^2 ds \\ &\leq \int_{-\infty}^{\infty} ||\nabla_{g_1} F_1(y(s))||^2 ds \\ &\leq C. \end{aligned}$$

This contradiction shows that (28) cannot hold which proves Step 3.

Step 4: *The trace of each finite energy flow line* $y = (x, v^*) \in C^\infty(\mathbb{R}, M \times V^*)$ *of* $\nabla_{g_1} F_1$ *is contained in* $h^{-1}(0) \times \{0\}$.

It follows from Step 3 that x is contained in the image of ϕ. Denoting

$$(q, w) = \phi^{-1}(x) \in C^\infty(\mathbb{R}, h^{-1}(0) \times B_\delta)$$

we observe that the triple (q, w, v^*) is a flow line of the function

$$F(q, w, v^*) = f(q) + v^*(w)$$

with respect to the metric

$$g_M|_{h^{-1}(0)} \oplus \kappa^2 g_{B_\delta} \oplus g_{V^*}.$$

Denote by $\Lambda\colon V^* \to V$ the isomorphism induced from the euclidean scalar product on V. Flow lines of F are solutions of the following ODE

$$\begin{aligned} \partial_s q + \nabla_{g_M|_{h^{-1}(0)}} f(q) = 0 \\ \partial_s w + \frac{1}{\kappa}\Lambda v^* = 0 \\ \partial_s v^* + \Lambda^{-1} w = 0. \end{aligned} \tag{29}$$

It follows from the two last equations in (29) that there exist $w_0, w_1 \in V$ such that

$$w(s) = w_0 \exp\left(\frac{s}{\sqrt{\kappa}}\right) + w_1 \exp\left(-\frac{s}{\sqrt{\kappa}}\right).$$

Since the energy of the flow line y is assumed to be finite it follows that $w_0 = w_1 = 0$ and hence

$$w(s) = 0, \quad v^*(s) = 0, \quad s \in \mathbb{R}.$$

This proves Step 4.

Step 5 (Uniform Palais-Smale condition): *There exists a geodesically complete Riemannian metric g_{PS} on $M \times V^*$, a compact set $K_0 \subset M \times V^*$ and a constant $\epsilon > 0$ such that for $y \in (M \times V^*) \setminus K_0$ and $r \in [0,1]$*

$$\nabla_{gr} F_r(y) \neq 0, \quad ||\nabla_{g_r} F_r(y)||^2_{g_r} \geq \epsilon ||\nabla_{g_r} F_r(y)||_{g_{PS}} \tag{30}$$

where $||\,||_g$ denotes the norm induced from the metric g.

We choose $g_{PS} = g_M \oplus g_{V^*} = g_0$. Then g_{PS} is geodesically complete by assumption. For $x \in M$ and $r \in [0,1]$ we denote by $dh(x)^{*_r} \colon V^* \to T_x M$ the adjoint of $dh(x)$ with respect to the inner products $g_{M,r}(x)$ on $T_x M$ and $\langle\, , \rangle$ on V^*. With respect to the natural splitting $T_{(x,v^*)}(M \times V*) \cong T_x M \times V^*$ for $(x, v^*) \in M \times V^*$ the gradient of F_r reads

$$\begin{aligned} \nabla_{g_r} F_r(x, v^*) &= \begin{pmatrix} \nabla_{g_{M,r}} f_r(x) + \nabla_{g_{M,r}} (v^* \circ h)(x) \\ h(x) \end{pmatrix} \\ &= \begin{pmatrix} \nabla_{g_{M,r}} f_r(x) + dh^{*_r}(x) v^* \\ h(x) \end{pmatrix}. \end{aligned} \tag{31}$$

Since 0 is a regular value of h and $h^{-1}(\mathrm{cl}(V_0))$ is compact we can find an open neighbourhood V_0' of 0 in V satisfying $V_0' \subset V_0$ such that $dh(x)^{*_r}$ is injectiv for every $x \in \mathrm{cl}(V_0')$ and every $r \in [0,1]$. Set

$$\epsilon' := \min_{v \in V \setminus V_0'} ||v|| > 0.$$

Since the family of injective maps $dh(x)^{*_r}$ depends smoothly on the compact parameter $(x, r) \in h^{-1}(\mathrm{cl}(V_0')) \times [0,1]$ there exists a compact subset $W \in V^*$ such that

$$||\nabla_{g_{M,r}} f_r(x) + dh(x)^{*_r} v^*||_{g_{M,r}} \geq \epsilon', \quad v^* \in V^* \setminus W,\ x \in h^{-1}(\mathrm{cl}(V_0')),\ r \in [0,1]. \tag{32}$$

We set

$$K_0 = h^{-1}(\mathrm{cl}(V_0')) \times W.$$

Then K_0 is compact and we claim that

$$||\nabla_{g_{M,r}} f_r(x) + dh(x)^{*_r} v^*||_{g_{M,r}} + ||h(x)|| \geq \epsilon', \quad y = (x, v^*) \in (M \times V^*) \setminus K_0. \tag{33}$$

To prove the claim we first assume that $x \notin h^{-1}(\mathrm{cl}(V_0'))$. We then estimate

$$||\nabla_{g_{M,r}} f_r(x) + dh(x)^{*_r} v^*||_{g_{M,r}} + ||h(x)|| \geq ||h(x)|| \geq \epsilon'$$

by the definition of ϵ'. Now assume that $x \in h^{-1}(\mathrm{cl}(V_0'))$ but $v^* \notin W$. We estimate in this case using (32)

$$||\nabla_{g_{M,r}} f_r(x) + dh(x)^{*_r} v^*||_{g_{M,r}} + ||h(x)|| \geq ||\nabla_{g_r} f_r(x) + dh(x)^{*_r} v^*||_{g_{M,r}} \geq \epsilon'.$$

This proves (33).
Using (31) and (33) we estimate for $y = (x, v^*) \in (M \times V^*) \setminus K_0$

$$||\nabla_{g_r} F_r(y)||_{g_r} \geq \frac{1}{\sqrt{2}} \Big(||\nabla_{g_{M,r}} f_r(x) + dh(x)^{*_r} v^*||_{g_{M,r}} + ||h(x)|| \Big) \geq \frac{\epsilon'}{\sqrt{2}} > 0$$

which implies the first inequality in (30). To prove the second one we observe that since the metrics $g_{M,r}$ differ from g_M only on a compact subset of M the metrics $g_{M,r}$ are equivalent for every $r \in [0,1]$, i.e. there exists a constant $c \geq 1$ such that

$$\frac{1}{c^2} g_M \leq g_{M,r} \leq c^2 g_M, \quad r \in [0,1]. \tag{34}$$

Using (31), (33), and (34) we estimate for $y = (x, v^*) \in (M \times V^*) \setminus K_0$

$$\begin{aligned} ||\nabla_{g_r} F_r(y)||_{g_r}^2 &= ||\nabla_{g_{M,r}} f_r(x) + dh^{*_r}(x) v^*||_{g_{M,r}}^2 + ||h(x)||^2 \\ &\geq \frac{1}{2} \Big(||\nabla_{g_{M,r}} f_r(x) + dh^{*_r}(x) v^*||_{g_{M,r}} + ||h(x)|| \Big)^2 \\ &\geq \frac{\epsilon'}{2} \Big(\frac{1}{c} ||\nabla_{g_{M,r}} f_r(x) + dh^{*_r}(x) v^*||_{g_M} + ||h(x)|| \Big) \\ &\geq \frac{\epsilon'}{2^{3/2} c} ||\nabla_{g_r} F_r(y)||_{g_{PS}}. \end{aligned}$$

Hence the second inequality in (30) follows with $\epsilon = \epsilon'/2^{3/2}c$. This proves Step 5.

Step 6: *We prove (iii).*

Let $y \in \bigcup_{r \in [0,1]} \mathcal{M}(F_r, g_r)$. Let $K_0 \subset M \times V^*$ be the compact set found in Step 5. We estimate for each $\sigma \in \mathbb{R}$ the distance $\mathrm{dist}_{PS}(y(\sigma), K_0)$ between $y(\sigma)$ and K_0 with respect to the Palais-Smale metric g_{PS} found in Step 5. We abbreviate

$$m := \max_{\substack{x \in K_0, \\ r \in [0,1]}} F_r(x) - \min_{\substack{x \in K_0, \\ r \in [0,1]}} F_r(x).$$

Since the Morse flow line y has finite energy it follows from (30) that for each $\sigma \in \mathbb{R}$ the set $\{s \geq \sigma : y(s) \in K_0\}$ is nonempty. We set

$$\tau(\sigma) = \inf\{s \geq \sigma : y(s) \in K_0\}.$$

Using (30) and the gradient equation we estimate

$$\begin{aligned}
\mathrm{dist}_{PS}(y(\sigma), K_0) &\leq \int_\sigma^{\tau(\sigma)} ||\partial_s y(s)||_{g_{PS}} ds \\
&= \int_\sigma^{\tau(\sigma)} ||\nabla_{g_r} F_r(y(s))||_{g_{PS}} ds \\
&\leq \frac{1}{\epsilon} \int_\sigma^{\tau(\sigma)} ||\nabla_{g_r} F_r(y(s))||_{g_r}^2 ds \\
&\leq \frac{1}{\epsilon} \int_{-\infty}^{\infty} ||\nabla_{g_r} F_r(y(s))||_{g_r}^2 ds \\
&= -\frac{1}{\epsilon} \int_{-\infty}^{\infty} g_r(y(s))(\nabla_{g_r} F_r(y(s)), \partial_s y(s)) ds \\
&= -\frac{1}{\epsilon} \int_{-\infty}^{\infty} dF_r(y(s))\partial_s y(s) ds \\
&= -\frac{1}{\epsilon} \int_{-\infty}^{\infty} \frac{d}{ds} F_r(y(s)) ds \\
&\leq \frac{1}{\epsilon}\Big(\limsup_{s \to -\infty} F_r(y(s)) - \liminf_{s \to \infty} F_r(y(s)) \Big) \\
&\leq \frac{m}{\epsilon}.
\end{aligned}$$

We now set

$$K := \Big\{ y \in M \times V^* : \mathrm{dist}_{PS}(y, K_0) \leq \frac{m}{\epsilon} \Big\}.$$

Since g_{PS} is geodesically complete, the set K is compact. Moreover, the estimate above shows that

$$\Big\{ y(\sigma) : y \in \bigcup_{r \in [0,1]} \mathcal{M}(F_r, g_r),\ \sigma \in \mathbb{R} \Big\} \subset K$$

holds. This proves Step 6.

Step 7: *We prove the theorem*

It remains to show that the functions F_r are Morse-Bott. We prove that in Proposition A.2 below. This finishes the proof of the theorem. □

If x is a critical point of a Morse-Bott function, then we define the index $\mathrm{ind}_F(x)$ of F at x as the number of negative eigenvalues of the Hessian of F at x. Note that the Morse-Bott condition implies that the index is constant on each connected component of $\mathrm{crit}(F)$. The following proposition shows that if $f|_{h^{-1}(0)}$ is Morse-Bott, then the Lagrange multiplier functional is also Morse-Bott and its index is independent of the behaviour of f outside of $h^{-1}(0)$.

However note, that the Hessian itself depends also on the derivatives of f in the normal directions of $h^{-1}(0)$.

Proposition A.2 *Let M be a finite dimensional manifold and let $(V, \langle\, ,\, \rangle)$ be a k-dimensional Euclidean vector space. Assume that $f \in C^\infty(M)$, $h \in C^\infty(M, V)$ such that 0 is a regular value of h and $f|_{h^{-1}(0)}$ is Morse-Bott. Then $F \in C^\infty(M \times V^*)$ defined by $F(x, v^*) = f(x) + v^*(h(x))$ for $(x, v^*) \in M \times V^*$ is also Morse-Bott. Moreover, if $\lambda \colon \mathrm{crit}(F) \to \mathrm{crit}(f|_{h^{-1}(0)})$ is the natural bijection given by $(x, v^*) \mapsto x$ for $(x, v^*) \in \mathrm{crit}(F)$, then for the indices the following relation holds*

$$\mathrm{ind}_F(\lambda^{-1}(x)) = \mathrm{ind}_{f|_{h^{-1}(0)}}(x) + k, \quad x \in \mathrm{crit}(f|_{h^{-1}(0)}).$$

Proof: Let $x \in \mathrm{crit}(f|_{h^{-1}(0)})$. We first choose convenient coordinates around x in M. Set $n = \dim(M)$ and chose $\delta_1, \delta_2 > 0$ so small such that there exists a diffeomorphism ϕ from $B^{n-k}_{\delta_1} \times B^k_{\delta_2} = \{v \in \mathbb{R}^{n-k} : ||v|| < \delta_1\} \times \{v \in \mathbb{R}^k : ||v|| < \delta_2\}$ to an open neighbourhood U of x in M such that

$$\phi(0,0) = x, \quad h(\phi(q, w)) = w, \quad q \in B^{n-k}_{\delta_1},\ w \in B^k_{\delta_2}.$$

Choose furthermore an orthonormal basis in V^* to define an isomorphism $\Phi \colon V^* \to \mathbb{R}^k$. Let $\hat{f} \in C^\infty(B^{n-k}_{\delta_1} \times B^k_{\delta_2})$, be given by the pullback of f, i.e.

$$\hat{f} = \phi^* f|_U,$$

and $\hat{F} \in C^\infty(B^{n-k}_{\delta_1} \times B^k_{\delta_2} \times \mathbb{R}^k)$ be given by the pullback of F, i.e.

$$\hat{F} = (\phi \times \Phi)^* F|_{U \times V^*}.$$

Then $\hat{F}$ reads

$$\hat{F}(q, w, v) = \hat{f}(q, v) + \langle v, w \rangle, \quad q \in B^{n-k}_{\delta_1},\ v \in B^k_{\delta_2},\ w \in \mathbb{R}^k.$$

We choose the standard flat metric on $B^{n-k}_{\delta_1} \times B^k_{\delta_2} \times \mathbb{R}^k$ and introduce the $k \times k$-matrix A, the $k \times (n-k)$-matrix B and the $(n-k) \times (n-k)$-matrix H by

$$A_{ij} = \frac{\partial^2 \hat{f}(0,0)}{\partial w_i \partial w_j}, \quad B_{ij} = \frac{\partial^2 \hat{f}(0,0)}{\partial q_i \partial w_j}, \quad H_{ij} = \frac{\partial^2 \hat{f}(0,0)}{\partial q_i \partial q_j}.$$

Denote by $\pi_2 \colon M \times V^* \to V^*$ the projection to the second factor. The Hessian of $\hat{F}$ at $(0, 0, \Phi \circ \pi_2 \circ \lambda^{-1}(x))$ with respect to the standard flat metric is given by

$$H_{\hat{F}}(0, 0, \Phi \circ \pi_2 \circ \lambda^{-1}(x)) = \begin{pmatrix} H & B & 0 \\ B^T & A & \mathrm{id} \\ 0 & \mathrm{id} & 0 \end{pmatrix}$$

We claim that

$$\dim\bigl(\ker H_{\hat{F}}(0, 0, \Phi \circ \pi_2 \circ \lambda^{-1}(x))\bigr) = \dim\bigl(\ker H\bigr). \tag{35}$$

To see that assume that the vector $(\hat{q}, \hat{w}, \hat{v}) \in \mathbb{R}^{n-k} \times \mathbb{R}^k \times \mathbb{R}^k$ lies in the kernel of $H_{\hat{F}}(0, 0, \Phi \circ \pi_2 \circ \lambda^{-1}(x)$. It follows that

$$\begin{cases} H\hat{q} + B\hat{w} = 0 \\ B^T\hat{q} + A\hat{w} + \hat{v} = 0 \\ \hat{w} = 0 \end{cases}$$

which implies that

$$(\hat{q}, \hat{w}, \hat{v}) = (\hat{q}, 0, -B^T\hat{q}), \quad \hat{q} \in \ker(H).$$

Hence (35) follows.
To prove that F is Morse-Bott we denote for $y \in \mathrm{crit}(F)$ by $\dim_y(\mathrm{crit}(F))$ the local dimension at y of the (unconnected) manifold $\mathrm{crit}(F)$ and compute using (35) and the Morse-Bott assumption on $f|_{h^{-1}(0)}$

$$\begin{aligned} \dim\bigl(\ker H_{\hat{F}}(\lambda^{-1}(0)\bigr) &= \dim\bigl(\ker H_{f|_{h^{-1}(0)}}(x)\bigr) \\ &= \dim_x(\mathrm{crit}(f|_{h^{-1}(0)})) \\ &= \dim_{\lambda^{-1}(x)}(\mathrm{crit}(F)). \end{aligned}$$

This proves that F is Morse-Bott.

It remains to compute the index of the Hessian of F. To do that we consider the smooth family of functions $\hat{f}_r \in C^\infty(B^{n-k}_{\delta_1} \times B^k_{\delta_2})$ for $r \in [0, 1]$ defined by

$$\hat{f}_r(q, w) = (1 - r)\hat{f}(q, w) + r\hat{f}(q, 0), \quad q \in B^{n-k}_{\delta_1},\ w \in B^k_{\delta_2}.$$

Then

$$\hat{f}_0 = \hat{f}, \quad \hat{f}_r|_{B^{n-k}_{\delta_1} \times \{0\}} = \hat{f}|_{B^{n-k}_{\delta_1} \times \{0\}},\ r \in [0, 1].$$

We define the smooth family of functions $\hat{F}_r \in C^\infty(B^{n-k}_{\delta_1} \times B^k_{\delta_2} \times \mathbb{R}^k)$ for $r \in [0, 1]$ by

$$\hat{F}_r(q, w, v) = \hat{f}_r(q, w) + \langle v, w\rangle.$$

Define further the smooth family of vectors $v_r \in \mathbb{R}^k$ for $r \in [0, 1]$ by

$$(v_r)_i = -\frac{\partial \hat{f}_r(0, 0)}{\partial w_i}, \quad i \in \{1, \ldots, k\}.$$

The functions $\hat{F}_r$ have critical points at $(0, 0, v_r)$ and it follows from (35) that $\dim\bigl(\ker H_{\hat{F}_r}(0, 0, v_r)\bigr) = \dim\bigl(\ker(H)\bigr)$ does not depend on $r \in [0, 1]$. Since the eigenvalues of a continuous family of matrices are continuous, see [18, Theorem II.5.1] we conclude

$$\mathrm{ind}_{\hat{F}_0}(0, 0, v_0) = \mathrm{ind}_{\hat{F}_1}(0, 0, v_1). \tag{36}$$

The Hessian of $\hat{F}_1$ at $(0, 0, v_1) = (0, 0, 0)$ is given by

$$H_{\hat{F}_1}(0, 0, 0) = \begin{pmatrix} H & 0 & 0 \\ 0 & 0 & \mathrm{id} \\ 0 & \mathrm{id} & 0 \end{pmatrix}$$

from which we deduce

$$\mathrm{ind}_{\hat{F}_1}(0,0,v_1) = \mathrm{ind}_{\hat{f}|_{B^{n-k}_{\delta_1}\times\{0\}}}(0,0) + k. \tag{37}$$

Combining (36) and (37) we compute

$$\begin{aligned} \mathrm{ind}_F(\lambda^{-1}(x)) &= \mathrm{ind}_{\hat{F}_0}(0,0,v_0) \\ &= \mathrm{ind}_{\hat{f}|_{B^{n-k}_{\delta_1}\times\{0\}}}(0,0) + k \\ &= \mathrm{ind}_{f|_{h^{-1}(0)}}(x) + k. \end{aligned}$$

This completes the proof of the proposition. □

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