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A Γ-structure on Lagrangian Grassmannians

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Abstract. For n odd the Lagrangian Grassmannian of $\mathbb{R}^{2n}$ is a Γ-manifold.

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1. Introduction and statement of the result

We denote by $(\mathbb{R}^{2n}, \omega)$ the standard symplectic vector space. The (unoriented) Lagrangian Grassmannian $\mathscr{L}$ is the space of all Lagrangian subspaces of $\mathbb{R}^{2n}$. It is a homogeneous space

$$\mathscr{L} \cong \mathrm{U}(n)/\mathrm{O}(n),$$

see [AG01], [MS98]. Every Lagrangian subspace can be identified with the fixed point set of a linear orthogonal anti-symplectic involution. Using this identification, we define a smooth map

$$\Theta\colon \mathscr{L} \times \mathscr{L} \to \mathscr{L}$$

by

$$(R, S) \mapsto RSR,$$

which we think of as a product. On every space there are products such as constant maps and projections to one factor. In [Hop41] Hopf introduced the notion of Γ-manifolds which rules these trivial products out. The purpose of this paper is to prove that the above product gives the Lagrangian Grassmannian $\mathscr{L}$ the structure of a Γ-manifold for n odd.

Definition 1.1. A closed, connected, orientable manifold M carries the structure of a Γ-manifold if there exists a map

$$\Theta\colon M \times M \to M$$

such that the maps

$$x \mapsto \Theta(x, y_0) \quad \text{and} \quad y \mapsto \Theta(x_0, y)$$

have non-zero mapping degree for one and thus all pairs $(x_0, y_0) \in M \times M$.

It is well known that $\mathscr{L}$ is orientable if and only if n odd, see [Fuk68]. The main result of this article is the following theorem.

Theorem 1.2. *If n is odd, then $(\mathscr{L}, \Theta)$ is a Γ-manifold.*

Using Hopf's theorem ([Hop41], Satz 1), we get a new proof of the following corollary due to Fuks [Fuk68].

Corollary 1.3 ([Fuk68]). *For n odd, the rational cohomology ring of $\mathscr{L}$ is an exterior algebra on generators of odd degree.*

Remark 1.4. The cohomology ring of the oriented and unoriented Lagrangian Grassmannian was computed by Borel and Fuks for all n, see [Bor53a], [Bor53b], [Fuk68]. A nice summary of these results can be found in Chapter 22 of the book by Vassilyev [Vas88].

The above situation fits into the following general framework. It is well known that $\mathscr{L}$ embeds into $\mathrm{U}(n)$ as the set $\mathrm{U}(n) \cap \mathrm{Sym}(n)$, i.e. the symmetric unitary matrices. Indeed the image of a Lagrangian subspace $\Lambda \subset \mathbb{C}^n$ is the symmetric unitary matrix $A_\Lambda := uu^t \in \mathrm{U}(n) \cap \mathrm{Sym}(n)$ where $a \in \mathrm{U}(n)$ maps $\mathbb{R}^n$ onto Λ. The unique orthogonal anti-symplectic involution with fixed point set Λ is then the map $A_\Lambda \circ \tau$ where $\tau \colon \mathbb{C}^n \to \mathbb{C}^n$ is complex conjugation. Thus, the Lagrangian Grassmannian $\mathscr{L}$ can be interpreted as the fixed point set of the involutive anti-isomorphism $A \mapsto A^T$ of $\mathrm{U}(n)$. On any Lie group G we can define a new product: $(g, h) \mapsto gh^{-1}g$. If $I \colon G \to G$ is an involutive anti-isomorphism then this new product restricts to a product on the fixed point set $\mathrm{Fix}(I)$. This is precisely the situation for the Lagrangian Grassmannian, namely the map Θ corresponds under the embedding of $\mathscr{L}$ into $\mathrm{U}(n)$ to $(g, h) \mapsto gh^{-1}g$.

For general Lie groups this new product does not always give rise to a Γ-structure for various reasons. For example, if we take $G = \mathrm{O}(n)$ resp. $G = \mathrm{U}(n)$ and $I(A) := A^{-1}$, then $\mathrm{Fix}(I)$ can be identified with $\bigcup_k G(k, n)$, the union of all real resp. complex Grassmannians, which is not connected. Another example is $G = \mathrm{SU}(n)$ with $I =$ transposition. Then for $n = 2$ we can identify $\mathrm{Fix}(I) \cong S^2$. But by Hopf's theorem ([Hop41], Satz 1) S^2 is not a Γ-manifold.

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2. Proof of Theorem 1.2

We recall that the (unoriented) Lagrangian Grassmannian $\mathscr{L}$ is the homogeneous space

$$\mathscr{L} \cong \mathrm{U}(n)/\mathrm{O}(n),$$

see [AG01], [MS98]. Since n is odd, $\mathscr{L}$ is a closed connected orientable manifold [Fuk68]. The space $\mathscr{L}$ is naturally identified with the space of linear orthogonal anti-symplectic involutions of $\mathbb{R}^{2n}$ with the standard symplectic structure. Using this identification, we define the map

$$\Theta\colon \mathscr{L} \times \mathscr{L} \to \mathscr{L}$$

by $(R, S) \mapsto RSR$. In order to prove Theorem 1.2, it suffices to show for one choice of basepoint R_0 that the mapping degrees of

$$S \mapsto \Theta(R_0, S) \quad \text{and} \quad S \mapsto \Theta(S, R_0)$$

are non-zero. Since

$$S \mapsto \Theta(R_0, S) = R_0 S R_0 \mapsto \Theta(R_0, R_0 S R_0) = R_0 R_0 S R_0 R_0 = S,$$

the first map is an involution and therefore has mapping degree ± 1. The non-trivial case is to compute the mapping degree of

$$\Theta_0(S) := \Theta(S, R_0) = S R_0 S.$$

Theorem 1.2 follows immediately from the following proposition.

Proposition 2.1. *The mapping degree of Θ_0 equals*

$$\deg \Theta_0 = 2^{m+1}$$

where $n = 2m + 1$.

Proof. Identify $\mathbb{R}^{2n} = \mathbb{C}^n$ in the standard way. Denote by $\tau\colon \mathbb{C}^n \to \mathbb{C}^n$ the map given by complex conjugation of all coordinates simultaneously. It is a standard fact, see for instance [MS98], that an orthogonal symplectic map $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ corresponds to a unitary map $\mathbb{C}^n \to \mathbb{C}^n$. It follows that an orthogonal anti-symplectic map

$R\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ can be written as the composition $A \circ \tau\colon \mathbb{C}^n \to \mathbb{C}^n$ for A a unitary linear map. The condition $R^2 = \mathrm{Id}$ then translates to $A\bar{A} = \mathrm{Id}$. So, we identify

$$\mathscr{L} = \{A \in \mathrm{U}(n) \mid A\bar{A} = \mathrm{Id}\}.$$

Under this identification, the map Θ is given by

$$\Theta(A, B) = A\bar{B}A.$$

Let B_0 be the unitary matrix corresponding to R_0. Then the map Θ_0 is given by

$$\Theta_0(A) = \Theta(A, B_0) = A\bar{B}_0A.$$

In the following, we take $B_0 = B$, the diagonal matrix with entries $b_{jk} = e^{i\theta_j}\delta_{jk}$ where

$$0 < \theta_j < 2\pi, \quad \theta_1 < \theta_2 < \cdots < \theta_n.$$

For this choice of B_0, we show that Id is a regular value of Θ_0 and compute the signed cardinality of $\Theta_0^{-1}(\mathrm{Id})$.

Indeed, if $\Theta_0(A) = \mathrm{Id}$, then $A\bar{B}A = \mathrm{Id}$, and therefore $\bar{A}B = A$. Throughout this paper, we do *not* use the Einstein summation convention. Letting a_{jk} denote the matrix entries of A, we have

$$\bar{a}_{jk}e^{i\theta_k} = a_{jk}.$$

Write $a_{jk} = r_{jk}e^{i\psi_{jk}}$, where $r_{jk} \in \mathbb{R}$ and $0 \le \psi_{jk} < \pi$. So,

$$e^{i2\psi_{jk}} = a_{jk}/\bar{a}_{jk} = e^{i\theta_k},$$

and therefore $\psi_{jk} = \theta_k/2$. Writing the unitary condition for A in terms of r_{jk} and ψ_{jk}, we have

$$\delta_{jl} = \sum_k a_{jk}\bar{a}_{lk} = \sum_k r_{jk}r_{lk}e^{i(\psi_{jk}-\psi_{lk})} = \sum_k r_{jk}r_{lk}.$$

Thus r_{jk} is an orthogonal matrix. Furthermore, the condition $A\bar{A} = \mathrm{Id}$ translates to

$$\delta_{jl} = \sum_k a_{jk}\bar{a}_{kl} = \sum_k r_{jk}r_{kl}e^{i(\theta_k-\theta_l)/2}.$$

In particular, taking $j = l$, we obtain

$$1 = \sum_k r_{jk}r_{kj}\cos((\theta_k - \theta_j)/2).$$

Writing $r'_{jk} = \cos((\theta_k - \theta_j)/2)r_{jk}$, we can reformulate the preceding equation in terms of the inner product of the row and column vectors $r'_{j\cdot}$ and $r_{\cdot j}$. Namely,

$$r'_{j\cdot} \cdot r_{\cdot j} = 1. \tag{2.1}$$

On the other hand, since r_{jk} is unitary, $|r_{\cdot j}| = 1$ and

$$|r'_{j\cdot}|^2 = \sum_k r_{jk}^2 \cos^2((\theta_k - \theta_j)/2) \leq \sum_k r_{jk}^2 = |r_{j\cdot}|^2 = 1,$$

with equality only if $r_{jk} = 0$ when $k \neq j$. Applying Cauchy–Schwartz to equation (2.1), we have

$$1 \leq |r'_{j\cdot}||r_{\cdot j}| = |r'_{j\cdot}| \leq 1.$$

Thus equality must hold, and the matrix r_{jk} is diagonal. Moreover, orthogonality implies that $r_{jk} = \pm\delta_{jk}$. Summing up, $A \in \Theta_0^{-1}(\mathrm{Id})$ if and only if we have $A = A^\epsilon$, where

$$\epsilon = (\epsilon_1, \ldots, \epsilon_n), \quad \epsilon_k \in \{0, 1\},$$

and A^ϵ is the matrix with elements

$$a_{jk}^\epsilon = e^{i(\theta_k/2 + \epsilon_k \pi)}.$$

In particular, $\Theta_0^{-1}(\mathrm{Id})$ has unsigned cardinality 2^n.

It remains to show that Id is a regular value and compute the signs. Let $\mathrm{Sym}(n)$ denote the space of real $n \times n$ symmetric matrices. It is easy to see that the tangent space to $\mathscr{L}$ at $A = \mathrm{Id}$ is given by

$$T_{\mathrm{Id}}\mathscr{L} = \{T \in \mathfrak{u}(n) \mid T + \overline{T} = 0\} = \{iQ \mid Q \in \mathrm{Sym}(n)\}.$$

Recall that $\mathrm{U}(n)$ acts on $\mathscr{L}$ by $A \mapsto UA\bar{U}^{-1}$. Thus, if $A = U\bar{U}^{-1}$, we have an isomorphism

$$\kappa_U : T_{\mathrm{Id}}\mathscr{L} \xrightarrow{\sim} T_A\mathscr{L}$$

given by $T \mapsto UT\bar{U}^{-1}$. Since $\mathscr{L}$ is a $\mathrm{U}(n)$ homogeneous space, the isomorphism κ_U preserves orientation. Moreover, for $T \in T_{A^\epsilon}\mathscr{L}$ we have

$$d\Theta_0|_{A^\epsilon}(T) = T\overline{B}A^\epsilon + A^\epsilon\overline{B}T = T\overline{A}^\epsilon + \overline{A}^\epsilon T.$$

If $U^\epsilon \in \mathrm{U}(n)$ satisfies

$$A^\epsilon = U^\epsilon(\bar{U}^\epsilon)^{-1},$$

then A^ϵ is a regular point of Θ_0 if the linear map

$$\alpha^\epsilon = d\Theta_0|_{A^\epsilon} \circ \kappa_U : T_{\mathrm{Id}}\mathscr{L} \to T_{\mathrm{Id}}\mathscr{L}$$

is invertible, and in that case the sign of A^ϵ is $\operatorname{sign}\det(\alpha^\epsilon)$. Explicitly,

$$\begin{aligned}
\alpha^\epsilon(T) &= U^\epsilon T(\bar{U}^\epsilon)^{-1}\overline{A}^\epsilon + \overline{A}^\epsilon U^\epsilon T(\bar{U}^\epsilon)^{-1} \\
&= U^\epsilon T(U^\epsilon)^{-1} + \bar{U}^\epsilon T(\bar{U}^\epsilon)^{-1} \\
&= U^\epsilon T(U^\epsilon)^{-1} - \bar{U}^\epsilon \overline{T}(\bar{U}^\epsilon)^{-1} \\
&= 2i\,\mathrm{Im}(U^\epsilon T(U^\epsilon)^{-1}).
\end{aligned}$$

Writing $T = iQ$, we can think of α^ϵ as the map $\mathrm{Sym}(n) \to \mathrm{Sym}(n)$ given by

$$\alpha^\epsilon(Q) = 2\,\mathrm{Re}(U^\epsilon Q (U^\epsilon)^{-1}).$$

For convenience, we take U^ϵ to be the unitary linear map given by

$$u^\epsilon_{jk} = e^{i(\theta_k/4+\epsilon_k\pi/2)}\delta_{jk}.$$

Then, denoting by q_{jk} the matrix elements of Q, we have

$$\begin{aligned} \alpha^\epsilon(Q)_{jk} &= 2\,\mathrm{Re}\left(e^{i\left((\theta_j-\theta_k)/4+(\epsilon_j-\epsilon_k)\pi/2\right)}\right)q_{jk} \\ &= 2\cos\big((\theta_j-\theta_k)/4+(\epsilon_j-\epsilon_k)\pi/2\big)q_{jk}. \end{aligned}$$

Since Q is a symmetric matrix, it is determined by q_{jk} for $j \le k$. Thus

$$\det(\alpha^\epsilon) = \prod_{j\le k} 2\cos\big((\theta_j-\theta_k)/4+(\epsilon_j-\epsilon_k)\pi/2\big)q_{jk}.$$

We need to show that this determinant does not vanish and compute its sign. For $j = k$, clearly $\cos\big((\theta_j-\theta_k)/4+(\epsilon_j-\epsilon_k)\pi/2\big) = 1$. For $j < k$, by assumption, $0 < \theta_j < \theta_k < 2\pi$, so

$$-\frac{\pi}{2} < \frac{\theta_j-\theta_k}{4} < 0.$$

It follows that for all $j \le k$, we have $\cos\big((\theta_j-\theta_k)/4+(\epsilon_j-\epsilon_k)\pi/2\big) \neq 0$. Therefore $\det(\alpha^\epsilon) \neq 0$ for all ϵ and Id is a regular value. Moreover,

$$\cos\big((\theta_j-\theta_k)/4+(\epsilon_j-\epsilon_k)\pi/2\big) < 0 \quad \text{if and only if} \quad \epsilon_j = 0,\ \epsilon_k = 1.$$

Let Υ_n be the set of all binary sequences $\epsilon = (\epsilon_1,\dots,\epsilon_n)$. For $\epsilon \in \Upsilon_n$ define $\mathrm{sign}(\epsilon)$ to be the number modulo 2 of pairs $j < k$ such that $\epsilon_j = 0$ and $\epsilon_k = 1$. The upshot of the preceding calculations is that

$$\mathrm{sign}\det(\alpha^\epsilon) = \mathrm{sign}(\epsilon),$$

therefore

$$\deg\Theta_0 = \sum_{\epsilon\in\Upsilon_n}(-1)^{\mathrm{sign}(\epsilon)}.$$

A combinatorial argument given below in Lemma 2.2 then implies the theorem. □

Lemma 2.2. For $n = 2m+1$, we have

$$d_n := \sum_{\epsilon\in\Upsilon_n}(-1)^{\mathrm{sign}(\epsilon)} = 2^{m+1}.$$

Proof. Let M_n denote the number of $\epsilon \in \Upsilon_n$ such that $\mathrm{sign}(\epsilon) = 0$. Then

$$d_n = M_n - (2^n - M_n) = 2M_n - 2^n.$$

For $\epsilon \in \Upsilon_n$ denote by $\mathrm{par}(\epsilon)$ the parity of ϵ, or in other words the number modulo 2 of j such that $\epsilon_j = 1$. Let P_n denote the number of $\epsilon \in \Upsilon_n$ such that $\mathrm{sign}(\epsilon)+\mathrm{par}(\epsilon) = 0$. By analyzing what happens when we adjoin either 1 or 0 to the beginning of a sequence $\epsilon \in \Upsilon_{n-1}$, we find that

$$M_n = M_{n-1} + P_{n-1}, \quad P_n = (2^{n-1} - P_{n-1}) + M_{n-1}.$$

Iterating these recursions twice, we obtain

$$M_n = M_{n-2} + P_{n-2} + 2^{n-2} - P_{n-2} + M_{n-2} = 2M_{n-2} + 2^{n-2}.$$

Clearly $M_1 = 2$, so $d_1 = 2$. Using the preceding recursion for M_n, we obtain

$$d_n = 2(2M_{n-2} + 2^{n-2}) - 2^n = 2(2M_{n-2} - 2^{n-2}) = 2d_{n-2}.$$

The lemma follows by induction. □

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