# A NOTE ON MEAN CURVATURE, MASLOV CLASS AND SYMPLECTIC AREA OF LAGRANGIAN IMMERSIONS

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**In this note we prove a simple relation between the mean curvature form, symplectic area, and the Maslov class of a Lagrangian immersion in a Kähler-Einstein manifold. An immediate consequence is that in Kähler-Einstein manifolds with positive scalar curvature, minimal Lagrangian immersions are monotone.**

## 1. Introduction

Let $(M, \omega)$ be a Kähler-Einstein manifold whose Ricci curvature is a multiple of the metric by a real number $\lambda$. In particular, the Kähler form $\omega$ and the first Chern class $c_1(M)$ are related by $c_1(M) = \frac{\lambda[\omega]}{2\pi}$ (see Section 3). Let $L$ be an immersed Lagrangian submanifold of $M$. Let $H$ be the trace of the second fundamental form of $L$ (the mean curvature vector field of $L$). Thus $H$ is a section of the normal bundle to $L$ in $M$ and we have a corresponding 1-form $\sigma_L := i_H \omega$ on $L$, called the *mean curvature form* of $L$. Consider a smooth map $F : \Sigma \to M$ from a compact oriented surface $\Sigma$ to $M$ whose (possibly empty) boundary $\partial F := F(\partial\Sigma)$ is contained in $L$. Let $\mu(F)$ be the Maslov class of $F$ (see Section 2) and $\omega(F) := \int_\Sigma F^*\omega$ its symplectic area. The goal of this note is to prove the following simple relation between these quantities:

$$\lambda\omega(F) - \pi\mu(F) = \sigma_L(\partial F). \tag{1}$$

This relation was given in [**Mor**] for $\mathbb{C}^n$ and in [**Ars**] for Calabi-Yau manifolds. Dazord [**Daz**] showed that the differential of the mean curvature form is the Ricci form restricted to $L$, so in the Kähler-Einstein case $\sigma_L$ is closed (see Section 3). Y.-G. Oh [**Oh2**] investigated the symplectic area in the case that the mean curvature form is exact.

Lagrangian submanifolds for which $\mu(F) = a\,\omega(F)$ on all disks $F$, for some $a > 0$, are called *monotone* in the symplectic geometry literature, cf. [**Oh1**].

An immediate consequence of (1) is that in Kähler-Einstein manifolds with positive scalar curvature (i.e. $\lambda > 0$), minimal (i.e. $\sigma_L \equiv 0$) Lagrangian immersions are monotone.

In view of the exact sequence in cohomology (with real coefficients)

$$H^1(M) \longrightarrow H^1(L) \stackrel{\delta}{\longrightarrow} H^2(M,L) \longrightarrow H^2(M),$$

formula (1) can be rephrased as

$$\lambda[\omega] - \pi\mu = \delta\sigma_L \in H^2(M,L).$$

Note that the class $\lambda[\omega]-\pi\mu$ is equivariant under symplectomorphisms of $M$. It follows that if the map $H^1(M) \to H^1(L)$ is trivial, then the cohomology class of the mean curvature form $\sigma_L$ is equivariant under symplectomorphisms of $M$. This generalizes Oh's observation [**Oh2**] that the cohomology class is invariant under Hamiltonian deformations.

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## 2. Maslov class

We first recall a definition of the Maslov index that is suitable for our purposes. Let $V$ be a Hermitian vector space of complex dimension $n$. Let $\Lambda^{(n,0)}V$ be the (one-dimensional) space of holomorphic $(n,0)$-forms on $V$ and set

$$K^2(V) := \Lambda^{(n,0)}V \otimes \Lambda^{(n,0)}V.$$

Let $L$ be a Lagrangian subspace of $V$. We can associate to $L$ an element $\kappa(L)$ in $\Lambda^{(n,0)}V$ of unit length which restricts to a real volume form on $L$. This element is unique up to sign and therefore defines a unique element of unit length

$$\kappa^2(L) := \kappa(L) \otimes \kappa(L) \in K^2(V).$$

Thus we get a map $\kappa^2$ from the Grassmanian $Gr_{\mathrm{Lag}}(V)$ of Lagrangian planes to the unit circle in $K^2(V)$. This map induces a homomorphism $\kappa^2_*$ of fundamental groups

$$\kappa^2_* : \pi_1(Gr_{\mathrm{Lag}}(V)) \to \mathbb{Z}.$$

To understand the map $\kappa^2_*$, let $L$ be a Lagrangian subspace and let $v_1, \ldots, v_n$ be an orthonormal basis for $L$. For $0 \le t \le 1$ consider the subspace

$$L_t = \mathrm{span}\{v_1, \ldots, v_{n-1}, e^{\pi i t} v_n\}.$$

This loop $\{L_t\}$ is the standard generator of $\pi_1(Gr_{\mathrm{Lag}}(V))$. The induced elements in $\Lambda^{(n,0)}V$ are related by $\kappa(L_t) = \pm e^{-\pi i t}\kappa(L)$, so $\kappa^2(L_t) = e^{-2\pi i t}\kappa^2(L)$ and $\kappa^2_*(\{L_t\}) = -1$. Thus we see that the homomorphism $\kappa^2_*$ is related to the Maslov index $\mu$ (as defined, e.g., in [**AuLa**]) by

$$\kappa^2_* = -\mu : \pi_1(Gr_{\mathrm{Lag}}(V)) \to \mathbb{Z}.$$

Now let $(M, \omega)$ be a symplectic manifold of dimension $2n$. Pick an almost complex structure $J$ on $M$ such that $\omega(\cdot, J\cdot)$ defines a Riemannian metric and let $K(M) := \Lambda^{(n,0)}T^*M$ be the canonical bundle of $M$, i.e., the bundle of $(n, 0)$-forms on $M$. Note that $c_1(K(M)) = -c_1(M)$. Let $K^2(M) := K(M) \otimes K(M)$ be the square of the canonical bundle.

Let $L$ be an immersed Lagrangian submanifold of $M$. For any point $l \in L$ there is an element of unit length $\kappa(l)$ of $K(M)$ over $l$, unique up to sign, which restricts to a real volume form on the tangent space $T_lL$. The squares of these elements give rise to a section of unit length

$$\kappa_L^2 : L \to K^2(M).$$

Note that if $L$ is oriented, then $\kappa_L^2$ is the square of the unit length section $\kappa_L : L \to K(M)$ defined by picking the volume forms $\kappa(l)|_L$ positive with respect to the orientation.

Now let $F : \Sigma \to M$ be a smooth map from a compact oriented surface to $M$ with boundary $\partial F = F(\partial\Sigma)$ on $L$. To define the Maslov class $\mu(F)$, assume first that $\Sigma$ is connected and $\partial\Sigma$ is nonempty. Then $H^2(\Sigma; \mathbb{Z}) = 0$, hence the pullback $F^*K(M)$ to $\Sigma$ is a trivial bundle and we can pick a unit length section $\kappa_F$ of $K(M)$ over $\Sigma$. Now on the boundary $\partial F$ we also have the section $\kappa_L^2$ defined above. We can uniquely write

$$\kappa_L^2 = e^{i\theta}\kappa_F^2$$

for a function $e^{i\theta} : \partial\Sigma \to S^1$ to the unit circle. We define the Maslov class $\mu(F)$ as minus its winding number,

$$\mu(F) := \frac{-1}{2\pi}\int_{\partial F} d\theta.$$

If $\Sigma$ is closed replace some point of $\Sigma$ by a new boundary circle $\partial\Sigma$ which gets mapped under $F$ to a point $x \in M$. Pick a unit length element $\kappa_x$ of $K(M)$ at $x$ and a unit length section $\kappa_F$ of $K(M)$ over $\Sigma$ (which is possible since $\Sigma$ now has nonempty boundary). Now write $\kappa_x^2 = e^{i\theta}\kappa_F^2$ over $\partial\Sigma$ and define $\mu(F) := \frac{-1}{2\pi}\int_{\partial F} d\theta$ as above. For disconnected $\Sigma$ define $\mu(F)$ as the sum over all connected components.

This definition is independent of the choice of $\kappa_F$ and defines a map

$$\mu : H_2(M, L; \mathbb{Z}) \to \mathbb{Z}.$$

To see this, first note that any other unit length section $\kappa_F'$ of $K(M)$ over $F$ is related to $\kappa_F$ by a multiple $e^{i\phi} : \Sigma \to S^1$. So on $F(\partial\Sigma)$ we have $\kappa_L^2 = e^{i\theta'}(\kappa_F')^2$ with $e^{i\theta'} = e^{-2i\phi}e^{i\theta} : \partial\Sigma \to S^1$. By Stokes' theorem, this implies $\int_{\partial F} d\theta' = \int_{\partial F} d\theta$. Next suppose that $F$ and $F'$ have the same boundary $\partial F = \partial F' =: \gamma$ and $[F \cup_\gamma -F'] = 0 \in H_2(M; \mathbb{Z})$. Then the pullback of $K(M)$ to $[F \cup_\gamma -F']$ is a trivial bundle and there is a unit length section $\kappa$ of $K(M)$ over $[F \cup_\gamma -F']$. If we take the restriction of $\kappa$ to $F$ as $\kappa_F$ and the restriction of $\kappa$ to $F'$ as $\kappa_F'$ we get $e^{i\theta} = e^{i\theta'}$, and hence $\mu(F) = \mu(F')$.

In particular, if $[F] = 0 \in H_2(M, L; \mathbb{Z})$ we find an $F' : \Sigma' \to L$ with $\partial F = \partial F' = \gamma$ and $[F \cup_\gamma -F'] = 0 \in H_2(M; \mathbb{Z})$, and thus $\mu(F) = \mu(F') = 0$. This shows that $\mu(F)$ depends only on $[F] \in H_2(M, L; \mathbb{Z})$.

In view of the discussion above, our definition of $\mu$ agrees with the usual definition of the Maslov class, cf. [**AuLa**].

## 3. Proof

Now assume that $(M, \omega)$ is Kähler with complex structure $J$ and Kähler metric $\langle \cdot, \cdot \rangle = \omega(\cdot, J\cdot)$. We denote by $\nabla$ the Levi-Civita connection, as well as the induced connections on $K(M)$ and $K^2(M)$. Let us briefly review the geometry of $K(M)$, following the notations in [**Bes**], pp. 81-82. Any local non-vanishing section $\kappa$ of $K(M)$ over an open subset $U$ of $M$ defines a (complex valued) connection one form $\eta$ on $U$ by $\nabla\kappa = \eta \otimes \kappa$. The curvature of $K(M)$ is defined to be $R_K := -d\eta$; it is a global closed imaginary valued $(1,1)$-form on $M$. By [**Bes**], Prop. 2.45, the Ricci tensor $Ric$ of $M$ is a symmetric bilinear form of type $(1,1)$; the associated 2-form $\rho(\cdot, \cdot) := Ric(J\cdot, \cdot)$ is called the *Ricci form* of $M$. By [**Bes**], Prop. 2.96, the Ricci form satisfies

$$\rho = iR_K.$$

If follows (cf. [**Bes**], Prop. 2.75) that the first Chern class $c_1(M)$ is represented by $\frac{\rho}{2\pi}$. Note that in the Kähler-Einstein case, $\rho = \lambda\omega$.

Now let $L$ be an immersed Lagrangian submanifold of $M$ and let $\kappa_L^2$ be the canonical section of $K^2(M)$ over $L$ as above. The section $\kappa_L^2$ defines a connection 1-form $\eta_L$ for $K^2(M)$ over $L$ by the condition $\nabla\kappa_L^2 = \eta_L \otimes \kappa_L^2$. Since $\kappa_L^2$ has constant length 1, $\eta_L$ is an imaginary valued 1-form on $L$. Let $\sigma_L = i_H\omega$ be the mean curvature form of $L$ as in Section 1. The following fact goes back to [**Oh2**], Prop. 2.2:

$$\eta_L = 2i\sigma_L. \tag{2}$$

Here the factor 2 is due to the fact that $\eta_L$ is a connection 1-form for $K^2(M)$ rather than $K(M)$. In particular, since $d\eta_L = -2R_K = 2i\rho$, this formula implies $d\sigma_L = \rho|_L$, so in the Kähler-Einstein case $\sigma_L$ is closed.

For the convenience of the reader, we recall the proof of formula (2) from [**Gol**] (where, however, the formula is stated with the wrong sign). Pick a point $l \in L$ and let $e_1, \ldots, e_n$ be a local orthonormal frame tangent to $L$. Orient $L$ locally by this frame. Then $\kappa_L(e_1, \ldots, e_n) \equiv 1$, and hence for every local vector field $v$ tangent to $L$,

$$0 = v\big(\kappa_L(e_1, \ldots, e_n)\big) = (\nabla_v\kappa_L)(e_1, \ldots, e_n) + \sum_{j=1}^{n} \kappa_L(e_1, \ldots, \nabla_v e_j, \ldots, e_n).$$

Since the complex structure $J$ is parallel (see [**KoNo**], Ch. IX Thm. 4.3), the $j$-th term in the last sum equals

$$i\langle\nabla_v e_j, Je_j\rangle = i\langle\nabla_{e_j} v, Je_j\rangle = -i\langle v, J\nabla_{e_j} e_j\rangle = i\omega(v, \nabla_{e_j} e_j).$$

Summing over $j$ and inserting $H = \sum_{j=1}^n \nabla_{e_j} e_j$, we find

$$\nabla_v \kappa_L(e_1, \ldots, e_n) = i\sum_{j=1}^n \omega(\nabla_{e_j} e_j, v) = i\omega(H, v) = i\sigma_L(v).$$

Now formula (2) follows from $\nabla\kappa_L^2 = \eta_L \otimes \kappa_L^2$ via

$$\eta_L = 2\nabla\kappa_L(e_1, \ldots, e_n) = 2i\sigma_L.$$

Now let $F : \Sigma \to M$ be a smooth map from a compact oriented surface with boundary on $L$. We will prove the following identity in any Kähler manifold:

$$\rho(F) - \pi\mu(F) = \sigma_L(\partial F). \tag{3}$$

Note that in general the form $\sigma_L$ need not be closed on $L$. It is closed in the Kähler-Einstein case, in which $\rho = \lambda\omega$ and (3) implies formula (1) in the introduction.

To prove identity (3), assume that every connected component of $\Sigma$ has nonempty boundary (closed components are treated similary, see Section 2). Define the section $\kappa_F$ of $K(M)$ over $F$ as in Section 2. Let $\eta_F$ be the connection 1-form along $F$ defined by $\nabla\kappa_F^2 = \eta_F \otimes \kappa_F^2$. By the discussion in the beginning of this section, $d\eta_F = 2iF^*\rho$. Stokes' theorem implies

$$2\rho(F) = \int_{\partial F} -i\eta_F.$$

Recall from Section 2 that along $\partial F$ we have $\kappa_L^2 = e^{i\theta}\kappa_F^2$ for a function $e^{i\theta} : \partial\Sigma \to S^1$, and the Maslov class is given by

$$\mu(F) = \frac{-1}{2\pi}\int_{\partial F} d\theta.$$

The connection 1-forms $\eta_F$ and $\eta_L$ are related by

$$\eta_L = \eta_F + i\,d\theta$$

on $\partial F$. Combining the equations above and formula (2), we find

$$\sigma_L(\partial F) = \int_{\partial F} \frac{-i\eta_L}{2} = \int_{\partial F} \frac{-i\eta_F}{2} + \int_{\partial F} \frac{d\theta}{2} = \rho(F) - \pi\mu(F).$$

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