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Martingale approach to subexponential asymptotics for random walks*

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Abstract

Consider the random walk $S_n = \xi_1 + \dots + \xi_n$ with independent and identically distributed increments and negative mean $\mathbf{E}\xi = -m < 0$. Let $M = \sup_{0 \leq i} S_i$ be the supremum of the random walk. In this note we present derivation of asymptotics for $\mathbf{P}(M > x)$, $x \rightarrow \infty$ for long-tailed distributions. This derivation is based on the martingale arguments and does not require any prior knowledge of the theory of long-tailed distributions. In addition the same approach allows to obtain asymptotics for $\mathbf{P}(M_\tau > x)$, where $M_\tau = \max_{0 \leq i < \tau} S_i$ and $\tau = \min\{n \geq 1 : S_n \leq 0\}$.

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1 Introduction, statement of results and discussion

Let ξ, ξ_1, ξ_2, \dots be independent random variables with a common distribution function F and negative mean, i.e., $\mathbf{E}\xi = -a < 0$. Let S_n denote the random walk with the increments ξ_k , that is,

$$S_0 = 0, \quad S_n = \xi_1 + \xi_2 + \dots + \xi_n, \quad n \geq 1.$$

It follows from the assumption $\mathbf{E}\xi < 0$ that the total maximum $M := \sup_{n \geq 0} S_n$ is finite almost surely. The asymptotic behaviour of $\mathbf{P}(M > x)$ has been considered by many authors. The first results are due to Cramer and Lundberg: if there exists $h_0 > 0$ such that $\mathbf{E}e^{h_0\xi} = 1$ and $\mathbf{E}\xi e^{h_0\xi} < \infty$ then

$$\mathbf{P}(M > x) \sim c_0 e^{-h_0 x} \quad \text{as } x \rightarrow \infty \tag{1.1}$$

for some $c_0 \in (0, 1)$ and, furthermore,

$$\mathbf{P}(M > x) \leq e^{-h_0 x} \quad \text{for all } x > 0. \tag{1.2}$$

The proof of these statements is based on the following observation: The assumption $\mathbf{E}e^{h_0\xi} = 1$ implies that the sequence $e^{h_0 S_n}$ is a martingale. Applying the Doob inequality

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we obtain immediately (1.2). The same martingale property allows one to make an exponential change of measure, which is used in the proof of (1.1).

If the distribution of ξ is long-tailed, i.e., $\mathbf{E}e^{h\xi} = \infty$ for all $h > 0$, then one can investigate $\mathbf{P}(M > x)$ under some additional regularity restrictions on the tail function $\bar{F}(x) := 1 - F(x)$. One of the most popular regularity assumption is the so-called subexponentiality of the distribution tails.

Definition 1.1. *The distribution function F on \mathbf{R}_+ is called subexponential if*

$$\int_0^x \bar{F}(x-y)dF(y) \sim 2\bar{F}(x) \quad \text{as } x \rightarrow \infty.$$

The following result is known in the literature as Veraverbecke’s theorem: Let F_I be defined by the tail $\bar{F}_I(x) := \min(1, \int_x^\infty \bar{F}(y)dy)$, $x > 0$. If F_I is subexponential then

$$\mathbf{P}(M > x) \sim \frac{1}{a} \bar{F}_I(x) \quad \text{as } x \rightarrow \infty. \tag{1.3}$$

We next turn to the maximum of the positive excursion of the random walk. Let

$$\tau := \inf\{n \geq 1 : S_n \leq 0\}$$

and

$$M_\tau := \max_{0 \leq n < \tau} S_n.$$

If the Cramer-Lundberg condition holds then one can derive the asymptotics for the quantity $\mathbf{P}(M_\tau > x)$ from that for the total maximum M . This way has been suggested first by Iglehart [10]. Namely, it follows from the Markov property that

$$\mathbf{P}(M > x) = \mathbf{P}(M_\tau > x) + \int_{-\infty}^0 \mathbf{P}(M > x-y)\mathbf{P}(S_\tau \in dy, M_\tau \leq x).$$

Thus,

$$\mathbf{P}(M_\tau > x) = \mathbf{P}(M > x) \left(1 - \int_{-\infty}^0 \frac{\mathbf{P}(M > x-y)}{\mathbf{P}(M > x)} \mathbf{P}(S_\tau \in dy, M_\tau \leq x) \right).$$

Noting that (1.1) yields

$$\lim_{x \rightarrow \infty} \frac{\mathbf{P}(M > x-y)}{\mathbf{P}(M > x)} = e^{h_0 y} \quad \text{for every } y < 0,$$

and applying the dominated convergence, we obtain

$$\int_{-\infty}^0 \frac{\mathbf{P}(M > x-y)}{\mathbf{P}(M > x)} \mathbf{P}(S_\tau \in dy, M_\tau \leq x) \sim \mathbf{E}e^{h_0 S_\tau}.$$

As a result we get

$$\mathbf{P}(M_\tau > x) \sim (1 - \mathbf{E}e^{h_0 S_\tau}) \mathbf{P}(M > x) \sim (1 - \mathbf{E}e^{h_0 S_\tau}) c_0 e^{-h_0 x}. \tag{1.4}$$

It turns out that Iglehart’s approach can not be applied to heavy-tailed random walks without further restrictions on the distribution of ξ . Here one has to assume that F is *strong subexponential*. This class of distribution functions was introduced by Klüppelberg [11].

Definition 1.2. *The distribution function F on \mathbf{R} belongs to the class \mathcal{S}^* if*

$$\int_0^x \bar{F}(x-y)\bar{F}(y)dy \sim 2a_+\bar{F}(x) \quad \text{as } x \rightarrow \infty, \tag{1.5}$$

where $a_+ = \int_0^\infty \bar{F}(y)dy \in (0, \infty)$.

Denisov [3] adopted Iglehart’s reduction from M_τ to M to the class of strong subexponential distributions: If $F \in \mathcal{S}^*$ then

$$\mathbf{P}(M_\tau > x) \sim \mathbf{E}\tau\bar{F}(x), \quad x \rightarrow \infty. \tag{1.6}$$

The asymptotics (1.6) were found first by Asmussen [1] for $F \in \mathcal{S}^*$ and by Heath, Resnick and Samorodnitsky [9] for regularly varying F . An extension of this result to the general stopping time can be found in Foss and Zachary [8], and in Foss, Palmowsky and Zachary [7]. These extensions rely on (1.6).

The *main purpose* of the present note is to give alternative proofs of (1.3) and (1.6) using martingale techniques.

In order to state our main result we introduce some notation. For any $y > 0$ let

$$\mu_y := \min\{n \geq 0 : S_n > y\}.$$

The latter stopping time is naturally connected with the supremum since

$$\mathbf{P}(M > x) = \mathbf{P}(\mu_x < \infty).$$

Let

$$\bar{F}_s(x) := \int_x^\infty \bar{F}(u)du$$

and

$$G_c(x) = \begin{cases} \bar{F}_s(x), & \text{if } x \geq 0 \\ c, & \text{if } x < 0 \end{cases}. \tag{1.7}$$

Define also

$$\hat{G}_c(x) := \min\{G_c(x), c\}. \tag{1.8}$$

Theorem 1.3. *Assume that F is long-tailed. For any $\varepsilon > 0$ there exists $R > 0$ such that the stopped sequence*

$$\hat{G}_{a+\varepsilon}(x - S_{n \wedge \mu_{x-R}}) \quad \text{is a submartingale.} \tag{1.9}$$

Assume in addition that $F \in \mathcal{S}^$. For any $\varepsilon > 0$ there exists $R > 0$ such that the stopped sequence*

$$\hat{G}_{a-\varepsilon}(x - S_{n \wedge \mu_{x-R}}) \quad \text{is a supermartingale.} \tag{1.10}$$

Having constructed super- and submartingale we can obtain subexponential asymptotics for M and M_τ by applying the optional stopping theorem.

Corollary 1.4. *For any long-tailed distribution function F with negative mean,*

$$\liminf_{x \rightarrow \infty} \frac{\mathbf{P}(M > x)}{\bar{F}_s(x)} \geq \frac{1}{a}. \tag{1.11}$$

Assume in addition that $F \in \mathcal{S}^$. Then,*

$$\mathbf{P}(M > x) \sim \frac{1}{a}\bar{F}_s(x), \quad x \rightarrow \infty.$$

To the best of our knowledge, all existing in the literature proofs of the Veraverbecke theorem are based on representations via geometric sums. More precisely, $\mathbf{P}(M > x)$ can be estimated by $\sum_{n=1}^{\infty} (1-p)p^n \mathbf{P}(Y_1 + Y_2 + \dots + Y_n > x)$, where $p \in (0, 1)$ and Y_i are independent identically distributed random variables with $\mathbf{P}(Y_1 > x) \sim \overline{F_s}(x)$. In order to obtain (1.3) from that geometric sum one uses the following two properties of subexponential distributions:

- (a) $\mathbf{P}((Y_1 + Y_2 + \dots + Y_n > x)) \sim n\mathbf{P}(Y_1 > x)$ for every fixed k ,
- (b) For every $\varepsilon > 0$ there exists $C(\varepsilon) < \infty$ such that

$$\mathbf{P}((Y_1 + Y_2 + \dots + Y_n > x)) \leq C(\varepsilon)(1 + \varepsilon)^n \mathbf{P}(Y_1 > x).$$

A recent elegant proof based on (a) and (b) can be found in [13]. Our proof does not use any property of F besides (1.5).

Unfortunately, our method does not allow us to derive (1.3) for the whole class of subexponential distributions. The condition $F_I \in \mathcal{S}$ and $F \in \mathcal{S}^*$ are close but do not coincide, see Section 6 in [4]. But we can apply the same construction to M_τ and, as it was shown in [8], the strong subexponentiality is necessary and sufficient for asymptotics (1.6) to hold.

Corollary 1.5. *Let $F \in \mathcal{S}^*$. Then*

$$\mathbf{P}(M_\tau > x) \sim \mathbf{E}\tau \overline{F}(x). \tag{1.12}$$

It is worth mentioning that, in contrast to all previous proofs, our approach to (1.12) is direct, i.e., it does not use any knowledge on the asymptotic behaviour of M .

One of the important advantages of the martingale approach is the possibility to obtain non-asymptotic inequalities for $\mathbf{P}(M > x)$ and $\mathbf{P}(M_\tau > x)$. For example, it follows from (1.9) that for every $\varepsilon > 0$ there exists $R > 0$ such that (see the proof of Corollary 1.4)

$$\mathbf{P}(M > x) \geq \frac{\overline{F_s}(x + R)}{a + \varepsilon}, \quad x > 0. \tag{1.13}$$

Using a supermartingale property of $G_{a-\varepsilon}$ we obtain the following upper bound

$$\mathbf{P}(M > x) \leq \frac{\overline{F_s}(x - R')}{a - \varepsilon}, \quad x > R'. \tag{1.14}$$

Of course, in order to apply these inequalities, one has to know how to compute R and R' for given values of ε . And we believe that one can do it rather easy for certain subclasses of \mathcal{S}^* , e.g., for regularly varying or Weibull tails.

Foss, Korshunov and Zachary [6] have shown that the inequality

$$\mathbf{P}(M > x) \geq \frac{\overline{F_s}(x)}{a + \overline{F_s}(x)}, \quad x > 0$$

holds without any restriction on the distribution function F , see Theorem 5.1 in [6]. This bound is better than (1.13). It's proof is based on the fact, that the distribution of M is the stationary distribution of the Lindley recursion $W_{n+1} = (W_n + \xi_{n+1})^+$. This property of M can be written as follows: Let ξ' a copy of ξ , which is independent of M . Then $\mathcal{L}(M) = \mathcal{L}((M + \xi'))$. This can be seen as a martingale property: Define $\pi(x) := \mathbf{P}(M > x)$. Then the sequence $\pi(x - S_{n \wedge \mu_x})$ is a martingale.

Using (1.10), one gets for all $x > R'$ the inequality

$$\mathbf{P}(M_\tau > x) \leq \frac{\overline{F_s}(x - R') - \mathbf{E}\overline{F_s}(x - R' - S_\tau)}{a - \varepsilon}.$$

And an upper estimate for the difference in the nominator is easy to get:

$$\overline{F}_s(x - R') - \mathbf{E}\overline{F}_s(x - R' - S_\tau) = \mathbf{E} \left[\int_{x-R'}^{x-R'-S_\tau} \overline{F}(z) dz \right] \leq \overline{F}(x - R') \mathbf{E}[-S_\tau].$$

Applying the Wald identity, we obtain

$$\mathbf{P}(M_\tau > x) \leq \frac{a}{a - \varepsilon} \mathbf{E}\tau \overline{F}(x - R'). \tag{1.15}$$

A lower bound is not as obvious. Here we can conclude from (1.9) that

$$\mathbf{P}(M_\tau > x) \geq \frac{\overline{F}_s(x + R) - \mathbf{E}\overline{F}_s(x + R - S_\tau)}{a + \varepsilon}. \tag{1.16}$$

Thus one needs an appropriate estimate for the difference in the nominator.

Martingale approach has been used also by Kugler and Wachtel [12] in deriving upper bounds for $\mathbf{P}(M > x)$ and $\mathbf{P}(M_{\tau_z} > x)$, where $\tau_z := \min\{k : S_n \leq -z\}$ under the assumption that some power moments of ξ are finite. Their strategy is completely different: They truncate the summands ξ_i in order to construct an exponential supermartingale for the random walk with truncated increments.

2 Proofs.

2.1 Proof of Theorem 1.3.

Fix $\varepsilon > 0$. To prove the submartingale property we need to show that

$$\mathbf{E}\widehat{G}_{a+\varepsilon}(x - y - \xi) \geq \widehat{G}_{a+\varepsilon}(x - y) \tag{2.1}$$

for all $y \leq x - R$.

Put, for brevity, $t := x - y \geq R$. By the definition (1.7),

$$\begin{aligned} \mathbf{E}\widehat{G}_{a+\varepsilon}(t - \xi) &= (a + \varepsilon)\mathbf{P}(\xi > t - r_c) + \int_{-\infty}^{t-r_c} F(dz)\overline{F}_s(t - z) \\ &= (a + \varepsilon)\overline{F}(t - r_c) + \left(\int_0^{t-r_c} + \int_{-\infty}^0 \right) F(dz)\overline{F}_s(t - z), \end{aligned}$$

where $r_c := \min\{x \geq 0 : \overline{F}_s(x) \leq c\}$. Integrating the first integral by parts, we obtain

$$\int_0^{t-r_c} F(dz)\overline{F}_s(t - z) = \overline{F}(0)\overline{F}_s(t - r_c) - \overline{F}(t - r_c)\overline{F}_s(0) + \int_0^{t-r_c} dz\overline{F}(z)\overline{F}(t - z).$$

Integrating the second integral by parts, we obtain

$$\int_{-\infty}^0 F(dz)\overline{F}_s(t - z) = F(0)\overline{F}_s(t) - \int_{-\infty}^0 dz\overline{F}(t - z)F(z).$$

Combining the above inequalities, we get

$$\begin{aligned} \mathbf{E}\widehat{G}_{a+\varepsilon}(t - \xi) &= (a + \varepsilon)\overline{F}(t - r_c) - \overline{F}(t - r_c)\overline{F}_s(0) + \overline{F}(0)\overline{F}_s(t - r_c) \\ &\quad + \int_0^{t-r_c} dz\overline{F}(z)\overline{F}(t - z) + F(0)\overline{F}_s(t) - \int_{-\infty}^0 dz\overline{F}(t - z)F(z). \end{aligned} \tag{2.2}$$

It is clear that

$$\int_{-\infty}^0 dz\overline{F}(t - z)F(z) \leq \overline{F}(t) \int_{-\infty}^0 dzF(z) = a_- \overline{F}(t).$$

Further,

$$\overline{F}(0)\overline{F}_s(t - r_c) + F(0)\overline{F}_s(t) = \overline{F}_s(t) + \overline{F}(0) \int_{t-r_c}^t \overline{F}(z)dz$$

and

$$\int_0^{t-r_c} dz\overline{F}(z)\overline{F}(t - z) = \int_0^t dz\overline{F}(z)\overline{F}(t - z) - \int_{t-r_c}^t dz\overline{F}(z)\overline{F}(t - z).$$

Now, put $a_+ := \overline{F}_s(0)$, $a_- := \int_{-\infty}^0 dzF(z)$ and note that $a = a_- - a_+$. Consequently,

$$\begin{aligned} \mathbf{E}\widehat{G}_{a+\varepsilon}(t - \xi) &\geq \overline{F}_s(t) + (a + \varepsilon)\overline{F}(t - r_c) - a_+\overline{F}(t - r_c) - a_-\overline{F}(t) \\ &\quad + 2 \int_0^{t/2} dz\overline{F}(z)\overline{F}(t - z) + \int_{t-r_c}^t \overline{F}(z) (\overline{F}(0) - \overline{F}(t - z)) dz \\ &\geq \overline{F}_s(t) + (-2a_+ + \varepsilon)\overline{F}(t - r_c) + 2\overline{F}(t) \int_0^{t/2} dz\overline{F}(z). \end{aligned}$$

Now, taking R_1 sufficiently large, we can ensure that

$$2 \int_0^{t/2} \overline{F}(z)dz \geq 2a_+ - \frac{\varepsilon}{2} \quad \text{for all } t \geq R_1.$$

Furthermore, we can choose R_2 so large that

$$\frac{\overline{F}(t - r_c) - \overline{F}(t)}{\overline{F}(t)} \leq \frac{\varepsilon}{4a_+}.$$

As a result, for $t > \max\{R_1, R_2\}$ we have

$$\mathbf{E}\widehat{G}_{a+\varepsilon}(t - \xi) \geq \overline{F}_s(t)$$

This proves (1.9).

To prove the supermartingale property it sufficient to show that

$$\mathbf{E}G_{a-\varepsilon}(x - y - \xi) \leq G_{a-\varepsilon}(x - y) \tag{2.3}$$

for all $y \leq x - R$. Using (2.2) with $r_c = 0$, we obtain

$$\begin{aligned} \mathbf{E}G_{a-\varepsilon}(t - \xi) &= G_{a-\varepsilon}(t) + (a - \varepsilon - a_+)\overline{F}(t) \\ &\quad + \int_0^t dz\overline{F}(z)\overline{F}(t - z) - \int_{-\infty}^0 dz\overline{F}(t - z)F(z). \end{aligned}$$

According to the definition of \mathcal{S}^* there exists R_1 such that

$$\int_0^t dz\overline{F}(z)\overline{F}(t - z) \leq (2a_+ + \varepsilon/2)\overline{F}(t)$$

for all $t \geq R_1$. Furthermore, since F is long-tailed, we have

$$\lim_{t \rightarrow \infty} \frac{1}{\overline{F}(t)} \int_{-\infty}^0 dz\overline{F}(t - z)F(z) = \int_{-\infty}^0 dzF(z) = a_-.$$

Therefore, there exists R_2 such that

$$\int_{-\infty}^0 dz\overline{F}(t - z)F(z) \geq (a_- + \varepsilon/2)\overline{F}(t), \quad t \geq R_2.$$

This immediately implies (1.10) with $R = \max\{R_1, R_2\}$.

2.2 Proof of Corollary 1.4.

Fix $\varepsilon > 0$ and pick R such that

$$Y_n = \widehat{G}_{a+\varepsilon}(x - S_{n \wedge \mu_{x-R}})$$

is a submartingale. Then,

$$\begin{aligned} \overline{F}_s(x) &= \widehat{G}_{a+\varepsilon}(x) = \mathbf{E}Y_0 \leq \mathbf{E}Y_\infty \\ &= \mathbf{E} \left[\widehat{G}_{a+\varepsilon}(x - S_{\mu_{x-R}}), \mu_{x-R} < \infty \right] \\ &\leq (a + \varepsilon) \mathbf{P}(\mu_{x-R} < \infty). \end{aligned}$$

Hence,

$$\mathbf{P}(M > x) = \mathbf{P}(\mu_x < \infty) \geq \frac{1}{a + \varepsilon} \overline{F}_s(x + R).$$

Letting x to infinity we obtain,

$$\liminf_{x \rightarrow \infty} \frac{\mathbf{P}(M > x)}{\overline{F}_s(x)} \geq a + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary the lower bound in (1.11) holds.

To prove the corresponding upper bound fix $\varepsilon > 0$ and pick R such that the process $Y_n = G_{a-\varepsilon}(x - S_{n \wedge \mu_{x-R}})$ is a supermartingale. Then,

$$\begin{aligned} \overline{F}_s(x) &= G_{a-\varepsilon}(x) = \mathbf{E}Y_0 \geq \mathbf{E}Y_\infty \\ &= (a - \varepsilon) \mathbf{P}(\mu_{x-R} < \infty, S_{\mu_{x-R}} > x) \\ &\quad + \mathbf{E} \left[\overline{F}_s(x - S_{\mu_{x-R}}); \mu_{x-R} < \infty, S_{\mu_{x-R}} \in (x - R, x] \right] \\ &\geq (a - \varepsilon) \mathbf{P}(\mu_{x-R} < \infty, S_{\mu_{x-R}} > x) \\ &\quad + \overline{F}_s(R) \mathbf{P}(\mu_{x-R} < \infty, S_{\mu_{x-R}} \in (x - R, x]). \end{aligned} \tag{2.4}$$

Let $r > 0$ be a number which we pick later. Then,

$$\begin{aligned} \mathbf{P}(M > x + r) &\leq \mathbf{P}(\mu_{x-R} < \infty, S_{\mu_{x-R}} > x) \\ &\quad + \mathbf{P}(\mu_{x-R} < \infty, S_{\mu_{x-R}} \in (x - R, x], M > x + r) \\ &\leq \mathbf{P}(\mu_{x-R} < \infty, S_{\mu_{x-R}} > x) \\ &\quad + \mathbf{P}(M > r) \mathbf{P}(\mu_{x-R} < \infty, S_{\mu_{x-R}} \in (x - R, x]), \end{aligned}$$

where we use the strong Markov property. Now pick sufficiently large r such that $\mathbf{P}(M > r) \leq \overline{F}_s(R)/(a - \varepsilon)$. Then,

$$\begin{aligned} \mathbf{P}(M > x + r) &\leq \mathbf{P}(\mu_{x-R} < \infty, S_{\mu_{x-R}} > x) \\ &\quad + \frac{\overline{F}_s(R)}{a - \varepsilon} \mathbf{P}(\mu_{x-R} < \infty, S_{\mu_{x-R}} \in (x - R, x]). \end{aligned}$$

Combining this with (2.4), we get

$$\mathbf{P}(M > x + r) \leq \frac{\overline{F}_s(x)}{a - \varepsilon}.$$

Letting x to infinity we obtain,

$$\limsup_{x \rightarrow \infty} \frac{\mathbf{P}(M > x)}{\overline{F}_s(x)} \leq \frac{1}{a - \varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary the upper bound holds.

2.3 Proof of Corollary 1.5

We start with a lower bound. Fix $\varepsilon > 0$ and pick R such that $Y_n = \widehat{G}_{a+\varepsilon}(x - S_{n \wedge \mu_{x-R}})$ is a submartingale. Then,

$$\begin{aligned} \overline{F}_s(x) &= \widehat{G}_{a+\varepsilon}(x) = \mathbf{E}Y_0 \leq \mathbf{E}Y_\tau \\ &\leq (a + \varepsilon)\mathbf{P}(\mu_{x-R} < \tau) + \mathbf{E}\overline{F}_s(x - S_\tau). \end{aligned}$$

Hence,

$$\mathbf{P}(M_\tau > x + R) = \mathbf{P}(\mu_{x+R} < \tau) \geq \frac{1}{a + \varepsilon} (\overline{F}_s(x) - \mathbf{E}\overline{F}_s(x - S_\tau)).$$

Now

$$\begin{aligned} \overline{F}_s(x) - \mathbf{E}\overline{F}_s(x - S_\tau) &= \int_0^\infty \mathbf{P}(S_\tau \in -dt) (\overline{F}_s(x) - \overline{F}_s(x + t)) \\ &\sim |\mathbf{E}S_\tau| \overline{F}(x), \quad x \rightarrow \infty. \end{aligned} \tag{2.5}$$

By the Wald's identity $|\mathbf{E}S_\tau| = a\mathbf{E}\tau$. Therefore,

$$\liminf_{x \rightarrow \infty} \frac{\mathbf{P}(M_\tau > x)}{\overline{F}(x)} \geq \frac{a\mathbf{E}\tau}{a + \varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary we obtain the lower bound

$$\liminf_{x \rightarrow \infty} \frac{\mathbf{P}(M_\tau > x)}{\overline{F}(x)} \geq \mathbf{E}\tau.$$

To show the upper bound fix $\varepsilon > 0$ and pick R such that $Y_n = G_{a-\varepsilon}(x - S_{n \wedge \mu_{x-R}})$ is a supermartingale. Then,

$$\begin{aligned} \overline{F}_s(x) &= G_{a-\varepsilon}(0) = \mathbf{E}Y_0 \geq \mathbf{E}Y_\tau \\ &= (a - \varepsilon)\mathbf{P}(\mu_{x-R} < \tau, S_{\mu_{x-R}} > x) \\ &\quad + \mathbf{E}[\overline{F}_s(x - S_{\mu_{x-R}}); \mu_{x-R} < \tau, S_{\mu_{x-R}} \in (x - R, x)] \\ &\quad + \mathbf{E}\overline{F}_s(x - S_\tau) \\ &\geq (a - \varepsilon)\mathbf{P}(\mu_{x-R} < \tau, S_{\mu_{x-R}} > x) + \mathbf{E}\overline{F}_s(x - S_\tau) \\ &\quad + \overline{F}_s(R)\mathbf{P}(\mu_{x-R} < \tau, S_{\mu_{x-R}} \in (x - R, x]). \end{aligned}$$

Similarly to the corresponding argument in the proof of Corollary 1.4,

$$\begin{aligned} \mathbf{P}(M_\tau > x + r) &\leq \mathbf{P}(\mu_{x-R} < \tau, S_{\mu_{x-R}} > x) \\ &\quad + \mathbf{P}(\mu_{x-R} < \tau, S_{\mu_{x-R}} \in (x - R, x], M_\tau > x + r) \\ &\leq \mathbf{P}(\mu_{x-R} < \tau, S_{\mu_{x-R}} > x) \\ &\quad + \mathbf{P}(M > r)\mathbf{P}(\mu_{x-R} < \tau, S_{\mu_{x-R}} \in (x - R, x]), \end{aligned}$$

Consequently,

$$\mathbf{P}(M_\tau > x + r) \leq \frac{1}{a + \varepsilon} (\overline{F}_s(x) - \mathbf{E}\overline{F}_s(x - S_\tau)).$$

Now, we can apply (2.5) and obtain

$$\limsup_{x \rightarrow \infty} \frac{\mathbf{P}(M_\tau > x)}{\overline{F}(x)} \leq \frac{a\mathbf{E}\tau}{a - \varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary we obtain the upper bound

$$\limsup_{x \rightarrow \infty} \frac{\mathbf{P}(M_\tau > x)}{\overline{F}(x)} \leq \mathbf{E}\tau.$$

□

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