The topology of rationally and polynomially convex domains

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Abstract We give in this article necessary and sufficient conditions on the topology of a compact domain with smooth boundary in $\mathbb{C}^n$, $n \geq 3$, to be isotopic to a rationally or polynomially convex domain.

1 Introduction

1.1 Polynomial, rational and holomorphic convexity

Recall the following complex analytic notions of convexity for domains in $\mathbb{C}^n$. For a compact set $K \subset \mathbb{C}^n$, one defines its *polynomial hull* as

$$\widehat{K}_{\mathcal{P}} := \left\{ z \in \mathbb{C}^n \;\middle|\; |P(z)| \leq \max_{u \in K} |P(u)| \text{ for all complex polynomials } P \text{ on } \mathbb{C}^n \right\},$$

and its *rational hull* as

$$\widehat{K}_{\mathcal{R}} := \Big\{ z \in \mathbb{C}^n \;\Big|\; |R(z)| \leq \max_{u \in K} |R(u)| \text{ for all rational functions } R = \frac{P}{Q},\; Q|_K \neq 0 \Big\}.$$

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Given an open set $U \supset K$, the *holomorphic hull of K in U* is defined as

$$\widehat{K}^U_{\mathcal{H}} := \left\{ z \in U \;\middle|\; |f(z)| \leq \max_{u \in K} |f(u)| \text{ for all holomorphic functions } f \text{ on } U \right\}.$$

A compact set $K \subset \mathbb{C}^n$ is called *rationally* (resp. *polynomially*) *convex* if $\widehat{K}_{\mathcal{R}} = K$ (resp. $\widehat{K}_{\mathcal{P}} = K$). An open set $U \subset \mathbb{C}^n$ is called *holomorphically convex* if $\widehat{K}^U_{\mathcal{H}}$ is compact for all compact sets $K \subset U$. A compact set $K \subset \mathbb{C}^n$ is called *holomorphically convex* if it is the intersection of its holomorphically convex open neighborhoods. We have

$$\text{polynomially convex} \Longrightarrow \text{rationally convex} \Longrightarrow \text{holomorphically convex}.$$

The first implication is obvious, while the second one follows from the fact that by definition a rationally convex compact set K is an intersection of *bounded rational polyhedra* $\{|R_i| < c_1, i = 1, \ldots, N\}$, where the R_i are rational functions, and any bounded rational polyhedron is clearly holomorphically convex.

Given a real valued function $\phi : U \to \mathbb{R}$ on an open subset $U \subset \mathbb{C}^n$, we denote by $d^{\mathbb{C}}\phi := d\phi \circ i$ its differential twisted by multiplication with $i = \sqrt{-1}$ on $\mathbb{C}^n$, and we set $\omega_\phi := -dd^{\mathbb{C}}\phi = 2i\partial\overline{\partial}\phi$. A function ϕ is called i it -convex if $\omega_\phi(v, iv) > 0$ for all $v \neq 0$. A cooriented hypersurface $\Sigma \subset \mathbb{C}^n$ (of real codimension 1) is called *i-convex* if there exists an i-convex function ϕ defined on some neighborhood of Σ such that $\Sigma = \{\phi = c\}$, and Σ is cooriented by a vector field v satisfying $d\phi(v) > 0$.

Remark 1.1 Traditionally i-convexity for functions is called *strict plurisubharmonicity*, and i-convexity for cooriented hypersurfaces *strict pseudoconvexity*. We prefer to use the term i-convexity for smooth functions and hypersurfaces (or more generally J-convexity in a manifold V with a complex or even an almost complex structure J) not only because it is shorter, but also because it explicitly shows the dependence on the ambient complex structure. Note that we are also "downgrading" the traditional terminology from the theory of functions of several complex variables: i-convexity corresponds to *strict* plurisubharmonicity or pseudoconvexity, while for non-strict plurisubharmonicity or pseudoconvexity we will use the term *weak i-convexity*.

In this paper, by a *domain* we will always mean a *compact* manifold W with smooth boundary ∂W, and by a *domain in* $\mathbb{C}^n$ an embedded domain $W \subset \mathbb{C}^n$ of real dimension $2n$. In particular, a domain $W \subset \mathbb{C}^n$ in our terminology is always a *closed subse*t. According to a theorem of Levi [15], any holomorphically convex domain $W \subset \mathbb{C}^n$ has *weakly* i-convex boundary ∂W. The converse statement that the interior of any domain in $\mathbb{C}^n$ with weakly i-convex boundary is holomorphically convex is known as the *Levi problem*.

It was proved by Oka [19], Bremermann [3], and Norguet [18]. In a more general context of domains in Stein manifolds, the Levi problem was resolved by Grauert [11] in the i-convex case, and by Docquier and Grauert in the weakly i-convex one [5].

We call a domain $W \subset \mathbb{C}^n$ *i-convex* if its boundary is i-convex. Note that any weakly i-convex domain in $\mathbb{C}^n$ can be C^∞-approximated by a slightly smaller i-convex one.

1.2 Topology of rationally and polynomially convex domains

We call a function $\phi : W \to \mathbb{R}$ on a domain W *defining* if ∂W is a regular level set of ϕ and $\phi|_{\partial W} = \max_W \phi$. Any i-convex domain $W \subset \mathbb{C}^n$ admits a defining i-convex function, so in particular it admits a defining Morse function without critical points of index $> n$ (see e.g. [4]). It follows that any holomorphically, rationally or polynomially convex domain has the same property. It was shown in [7] (see Theorem 1.3.6 there and also [4, Theorem 8.19]) that for $n \geq 3$, any domain in $\mathbb{C}^n$ with such a Morse function is smoothly isotopic to an i-convex one.

The first result of this paper states that, for $n \geq 3$, there are no additional constraints on the topology of rationally convex domains.

Theorem 1.2 *A compact domain $W \subset \mathbb{C}^n$, $n \geq 3$, is smoothly isotopic to a rationally convex domain if and only if it admits a defining Morse function without critical points of index $> n$.*

Our second result gives necessary and sufficient constraints on the topology of polynomially convex domains.

Theorem 1.3 *A compact domain $W \subset \mathbb{C}^n$, $n \geq 3$, is smoothly isotopic to a polynomially convex domain if and only if it satisfies the following topological condition:*

(T) W admits a defining Morse function without critical points of index $> n$, and $H_n(W; G) = 0$ for every abelian group G.

The "only if" part is well known and due to Andreotti and Narasimhan [1], see also [10] or Remark 3.3 below. Note that, in view of the universal coefficient theorem, condition (T) is equivalent to the condition

(T') W admits a defining Morse function without critical points of index $> n$, $H_n(W) = 0$, and $H_{n-1}(W)$ has no torsion.

Theorems 1.2 and 1.3 are consequences of the more precise Theorem 1.7 below. Further analysis of condition (T) yields

Proposition 1.4 *(a) If W is simply connected, then condition (T) is equivalent to the existence of a defining Morse function without critical points of index $\geq n$.*

(b) For any $n \geq 3$ there exists a (non-simply connected) domain W satisfying condition (T) with $\pi_n(W, \partial W) \neq 0$. In particular, W does not admit a defining function without critical points of index $\geq n$.

This paper was motivated by the questions raised by S. Nemirovski whether every polynomially convex domain in $\mathbb{C}^n$ is *subcritical*, i.e., it admits a defining i-convex Morse function without critical points of index $\geq n$, and whether there are any additional constraints on the topology of rationally convex domains besides the fact that they admit defining Morse functions without critical points of index $> n$. Theorem 1.2 provides a complete answer to the latter question in the case $n \geq 3$, while Theorem 1.3 together with Proposition 1.4 (b) show that the answer to the former question is in general negative. In the simply connected case, Proposition 1.4 (a) provides a defining Morse function without critical points of index $\geq n$, but we do not know whether there exists such a function which is i-convex. See also the discussion after Theorem 1.7 below.

1.3 Symplectic topology of rationally and polynomially convex domains

One may ask which *deformation classes* of Stein domains can be realized as polynomially or rationally convex domains in $\mathbb{C}^n$. Here by a *Stein domain* we mean a domain W with an integrable complex structure J (making W a complex manifold with boundary) which admits a defining J-convex function $\phi : W \to \mathbb{R}$. Two Stein domains (W, J) an (W', J') are called *deformation equivalent* if there exists a diffeomorphism $f : W \to W'$ such that f^*J' is Stein homotopic to J; see [4]. We call a diffeomorphism $f : W \to W'$ with this property a *deformation equivalence*.

Recall from [4,8] that a *Weinstein domain structure* on a domain W is a triple (ω, X, ϕ) consisting of a symplectic form ω on W, a defining Morse function $\phi : W \to \mathbb{R}$, and a vector field X on W which is Liouville for ω (i.e. $L_X\omega = \omega$) and gradient-like for ϕ. A *Weinstein homotopy* on a domain W is a smooth 1-parameter family (ω_t, X_t, ϕ_t), $t \in [0, 1]$, of triples satisfying all the conditions on Weinstein domain structures, except that we allow the family ϕ_t to have birth-death critical points.

Any defining J-convex Morse function ϕ on a Stein domain (W, J) induces a Weinstein structure $\mathfrak{W}(W, J, \phi) := (\omega_\phi, X_\phi, \phi)$ on W where $\omega_\phi = -dd^{\mathbb{C}}\phi$, and $X_\phi = \nabla_\phi \phi$ is the gradient of ϕ with respect to the Kähler metric $\omega_\phi(\cdot, J\cdot)$; see [4]. Since the space of defining J-convex functions is contractible, different choices of ϕ lead to homotopic Weinstein structures. Hence any Stein domain (W, J) has a canonically associated homotopy class $\mathfrak{W}(W, J)$ of Weinstein structures on W. It is shown in [4] that two Stein domains (W, J_0) and (W, J_1) are Stein homotopic if and only if $\mathfrak{W}(W, J_0) = \mathfrak{W}(W, J_1)$.

A *Weinstein manifold structure* on an open manifold V is a triple (ω, X, ϕ) consisting of a symplectic form ω on V, an exhausting Morse function $\phi : W \to \mathbb{R}$, and a vector field X on V which is Liouville for ω, gradient-like for ϕ, and complete (i.e., its flow exists for all times). As in the Stein domain case, one can associate to a Stein manifold (V, J) and an exhausting J-convex function $\phi : V \to \mathbb{R}$ the Weinstein structure $\mathfrak{W}(V, J, \phi) := (\omega_\phi = -dd^{\mathbb{C}}\phi, X_\phi = \nabla_\phi \phi, \phi)$ provided that X_ϕ is complete, which can always be achieved by composing ϕ with a sufficiently convex function $\mathbb{R} \to \mathbb{R}$ (see [4, Section 11.5]). By the *standard Weinstein structure* on $\mathbb{C}^n$ we mean the structure $\mathfrak{W}_{\mathrm{st}} = (\omega_{\mathrm{st}}, X_{\mathrm{st}}, \phi_{\mathrm{st}}) := \mathfrak{W}(\mathbb{C}^n, i, \frac{|z|^2}{4})$.

Theorem 1.5 *Let (W, J) be a Stein domain, $\mathfrak{W}(W, J)$ the associated homotopy class of Weinstein domain structures on W, and $f : W \hookrightarrow \mathbb{C}^n$ a smooth embedding. Then f is isotopic to a deformation equivalence onto*

(a) An i-convex domain if and only if the induced complex structure f^*i is homotopic to J through almost complex structures;
(b) A rationally convex domain if and only if in addition to (a) there is a representative $(\omega, X, \phi) \in \mathfrak{W}(W, J)$ such that f is isotopic to a symplectic embedding $\widetilde{f} : (W, \omega) \hookrightarrow (\mathbb{C}^n, \omega_{\mathrm{st}})$;
(c) A polynomially convex domain if and only if in addition to (a) and (b) the push-forward Weinstein structure $(\omega_{\mathrm{st}}, \widetilde{f}_*X, \phi \circ \widetilde{f}^{-1})$ extends to a Weinstein structure $(\widetilde{\omega}, \widetilde{X}, \widetilde{\phi}) \in \mathfrak{W}(\mathbb{C}^n, i)$ on the whole $\mathbb{C}^n$.

Remark 1.6 The proof of Theorem 1.5 shows that the Weinstein structure $(\widetilde{\omega}, \widetilde{X}, \widetilde{\phi})$ in (c) can be chosen with symplectic form $\widetilde{\omega} = \omega_{\mathrm{st}}$, standard at infinity, and homotopic to the standard Weinstein structure via a homotopy fixed at infinity and with fixed symplectic form.

A class of Stein domains where conditions (b) and (c) hold are the *flexible* Stein domains defined in [4]; we refer the reader to there for their definition and properties. Let us just mention that every subcritical Stein domain is flexible, and every domain which admits a Stein structure also admits a flexible one which is unique (in a given homotopy class of almost complex structures) up to Stein homotopy.

For flexible Stein domains, Theorem 1.5 implies the following refinement of Theorems 1.2 and 1.3.

Theorem 1.7 *Let (W, J) be a* flexible *Stein domain of complex dimension $n \geq 3$, and $f : W \hookrightarrow \mathbb{C}^n$ a smooth embedding such that f^*i is homotopic to J through almost complex structures. Then (W, J) is deformation equivalent to a rationally convex domain in $\mathbb{C}^n$. More precisely, f is smoothly isotopic to* *an embedding* $g : W \hookrightarrow \mathbb{C}^n$ *such that* $g(W) \subset \mathbb{C}^n$ *is rationally convex, and* g^*i *is Stein homotopic to* J.

If in addition $H_n(W; G) = 0$ *for every abelian group* G*, then* $g(W)$ *can be made polynomially convex.*

Theorems 1.2 and 1.3 follow directly from Theorem 1.7 and Theorems 13.1 and 13.5 in [4] which assert that, given a defining Morse function $\phi : W \to \mathbb{R}$ on a domain W of dimension $2n \geq 6$ without critical points of index $> n$, any almost complex structure J on W is homotopic to an integrable complex structure $\widetilde{J}$ for which the function ϕ (after composition with a convex increasing function $\mathbb{R} \to \mathbb{R}$) is $\widetilde{J}$-convex and such that the Stein structure $(\widetilde{J}, \phi)$ is flexible.

We conjecture that polynomial convexity in complex dimension ≥ 3 implies flexibility. Note that, by [4, Theorem 15.11], this conjecture would imply that if a polynomially convex domain $W \subset \mathbb{C}^n$, $n \geq 3$, admits a defining Morse function without critical points of index $\geq n$, then it is subcritical.

For rational convexity, flexibility is not necessary. This follows, for example, from another corollary of Theorem 1.5 which we now describe. Let D^*L denote the unit cotangent disc bundle of a closed n-dimensional manifold L (with respect to some Riemannian metric on L). By a theorem of Grauert [11], D^*L carries a Stein structure J_{Grauert} which admits a defining J_{Grauert}-convex function ϕ_{Grauert} whose Morse-Bott critical point locus is the zero section L. The Stein domain $(D^*L, J_{\mathrm{Grauert}})$ is called a *Grauert tube* of L, and any two Grauert tubes of L are Stein homotopic through Grauert tubes. Note that the corresponding Weinstein structure $\mathfrak{W}(D^*L, J_{\mathrm{Grauert}}, \phi_{\mathrm{Grauert}})$ has the zero section L as a Lagrangian submanifold.

Corollary 1.8 *A Grauert tube* $(D^*L, J_{\mathrm{Grauert}})$ *of a closed* n*-dimensional manifold* L

(a) Is deformation equivalent to an i*-convex domain in* $\mathbb{C}^n$ *if and only if* L *admits a totally real embedding into* $(\mathbb{C}^n, i)$*;*
(b) Is deformation equivalent to a rationally convex domain in $\mathbb{C}^n$ *if and only if* L *admits a Lagrangian embedding into* $(\mathbb{C}^n, \omega_{\mathrm{st}})$*;*
(c) Is not deformation equivalent to a polynomially convex domain in $\mathbb{C}^n$.

Note that part (c) is an immediate corollary of the vanishing of $H_n(W; G)$ for polynomially convex domains, which was already stated in Theorem 1.3. The "if" in part (b) was proved by Duval and Sibony in [6].

Statement (b) shows that the question whether D^*L is deformation equivalent to a rationally convex domain in $\mathbb{C}^n$ depends on the topology on L in a subtle way. By a theorem of Gromov, the answer is negative for manifolds with $H^1(L; \mathbb{R}) = 0$. For example, since S^3 admits a totally real embedding into $\mathbb{C}^3$ but no Lagrangian one, D^*S^3 is deformation equivalent to an i-convex domain in $\mathbb{C}^3$, but not to a rationally convex one.

According to [2], any flexible Weinstein domain has vanishing symplectic homology. On the other hand, by a result of several authors (see [4, Section 17.1] for references), $(D^*L, J_{\mathrm{Grauert}})$ has nonvanishing symplectic homology and is therefore not flexible. Thus Grauert tubes of Lagrangian submanifolds of $\mathbb{C}^n$ provide examples of rationally convex domains that are not flexible. However, we do not know any example of a rationally convex domain W in $\mathbb{C}^n$ with $H^1(\partial W; \mathbb{R}) = 0$ which is not flexible.

1.4 Isotopy through i-convex domains

One can ask when an i-convex domain $W \subset \mathbb{C}^n$ is isotopic to a polynomially or rationally convex domain *via an isotopy through i-convex domains*. (Recall that in our terminology an i-convex domain in $\mathbb{C}^n$ is a compact domain with smooth strictly pseudoconvex boundary.) The answer is provided by

Theorem 1.9 *Let $f_t : W \hookrightarrow \mathbb{C}^n$, $t \in [0, 1]$, be an isotopy of smooth embeddings of a domain W such that $f_0(W)$ and $f_1(W)$ are i-convex. Then the path f_t is homotopic with fixed end points in the space of embeddings to a path of embeddings $\widetilde{f}_t : W \hookrightarrow \mathbb{C}^n$ onto i-convex domains $\widetilde{f}_t(W)$ if and only if there exists a Stein homotopy (W, J_t) such that $J_0 = f_0^* i$ and $J_1 = f_1^* i$, and the paths J_t and $f_t^* i$ are homotopic with fixed end points in the space of almost complex structures on W.*

Proof The "only if" follows simply by setting $J_t := \widetilde{f}_t^* i$.

For the "if", pick a generic family of defining J_t-convex functions $\phi_t : W \to \mathbb{R}$ and consider the Weinstein homotopy $\mathfrak{W}_t := \mathfrak{W}(W, J_t, \phi_t)$. By an ambient version of [4, Theorem 15.2], after composing the ϕ_t with convex increasing function $\mathbb{R} \to \mathbb{R}$, there exists a 2-parametric family of embeddings $h_{s,t} : W \hookrightarrow W$, $s, t \in [0, 1]$, such that

- $h_{0,t} = h_{s,0} = h_{s,1} = \mathrm{Id}$;
- The functions ϕ_t are $h_{1,t}^* f_t^* i$-convex;
- The paths of Weinstein structures $\mathfrak{W}_t$ and $\mathfrak{W}(h_{1,t}^* f_t^* i, \phi_t)$ are homotopic with fixed end points and with fixed functions ϕ_t.

The proof of this ambient version is essentially identical to the proof of Theorem 15.2 given in [4], replacing the Parametric Stein Existence Theorem 13.6 by a 1-parametric version of the Ambient Stein Existence Theorem 13.4. Now the isotopy $\widetilde{f}_t := f_t \circ h_{1,t} : W \hookrightarrow \mathbb{C}^n$ has the required properties. □

As a corollary of Theorem 1.9, Theorem 1.7, and [4, Theorem 15.14] we get

Corollary 1.10 *(a) Any two* flexible *i-convex domains in $\mathbb{C}^n$, $n \geq 3$, that are smoothly isotopic are isotopic through i-convex domains.*

(b) *Every* flexible *i-convex domain $W \subset \mathbb{C}^n$, $n \geq 3$, is isotopic through i-convex domains to a rationally convex domain.*
(c) *Every* flexible *i-convex domain $W \subset \mathbb{C}^n$, $n \geq 3$, satisfying $H_n(W; G) = 0$ for every abelian group G is isotopic through i-convex domains to a polynomially convex domain.*

Without the flexibility hypothesis, Corollary 1.10(a) becomes false:

Corollary 1.11 *For every $n \geq 3$ there exist i-convex (even rationally convex) domains in $\mathbb{C}^n$ that are smoothly isotopic, but not isotopic through i-convex domains.*

Proof Let L be a closed Lagrangian submanifold of $(\mathbb{C}^n, \omega_{\mathrm{st}})$. By Corollary 1.8(b), the Grauert tube $(D^*L, J_{\mathrm{Grauert}})$ is deformation equivalent to a rationally convex domain $W_0 \subset \mathbb{C}^n$. On the other hand, by Theorems 13.1 and 13.5 in [4], D^*L carries a flexible Stein structure J_{flex}, and by Theorem 1.7, $(D^*L, J_{\mathrm{flex}})$ is deformation equivalent to a rationally convex domain $W_1 \subset \mathbb{C}^n$. Since $(D^*L, J_{\mathrm{Grauert}})$ and $(D^*L, J_{\mathrm{flex}})$ are not deformation equivalent, W_0 and W_1 are not isotopic through i-convex domains. □

1.5 Generalizations to other Stein manifolds

The notions of rational and polynomial convexity generalize in a straightforward way from $\mathbb{C}^n$ to a general Stein manifold (V, J). Let us denote by $\mathcal{O} := \mathcal{O}(V, J)$ the algebra of holomorphic functions on (V, J), and by $\mathcal{M} := \mathcal{M}(V, J)$ its field of fractions, i.e., the algebra of meromorphic functions on V. For a compact set $K \subset V$, one defines its *$\mathcal{O}$-hull* as

$$\widehat{K}_{\mathcal{O}} := \left\{ z \in V \;\middle|\; |f(z)| \leq \max_{u \in K} |f(u)| \text{ for all functions } f \in \mathcal{O} \right\},$$

and its *$\mathcal{M}$-hull* as

$$\widehat{K}_{\mathcal{M}} := \left\{ z \in V \;\middle|\; |R(z)| \leq \max_{u \in K} |R(u)| \text{ for all functions } R = \frac{f}{g} \in \mathcal{M},\ g|_K \neq 0 \right\}.$$

A compact set is called *$\mathcal{O}$-convex* (resp. *$\mathcal{M}$-convex*) if $\widehat{K}_{\mathcal{O}} = K$ (resp. $\widehat{K}_{\mathcal{M}} = K$).

Given a proper holomorphic embedding $(V, J) \hookrightarrow (\mathbb{C}^N, i)$, a compact subset $K \subset V$ is $\mathcal{O}$-(resp. $\mathcal{M}$-)convex if and only its image in $\mathbb{C}^N$ is polynomially (resp. rationally) convex. This follows from the standard corollary of Cartan's Theorem B (see e.g. [4, Corollary 5.37]) that any holomorphic (resp. meromorphic) function on $V \subset \mathbb{C}^N$ is the restriction of a holomorphic (resp. meromorphic) function on $\mathbb{C}^N$, together with the fact that any holomorphic (resp. meromorphic) function on $\mathbb{C}^N$ can be approximated uniformly on compact sets by polynomials (resp. rational functions). In particular, for $(V, J) = (\mathbb{C}^n, i)$ the notions of $\mathcal{O}$- and $\mathcal{M}$-convexity reduce to polynomial and rational convexity, respectively.

Most of the results of this paper have analogues in this more general situation. The proofs are essentially identical, and we do not discuss them in this paper. In particular, Theorems 1.2 and 1.3 generalize to

Theorem 1.12 *Let (V, J) be a Stein manifold of complex dimension $n \geq 3$ and $W \subset V$ be a compact domain.*

(a) W is smoothly isotopic to an $\mathcal{M}$-convex domain if and only if it admits a defining Morse function without critical points of index $> n$.
(b) W is smoothly isotopic to an $\mathcal{O}$-convex domain if and only if it satisfies, in addition, the following topological condition:
   ($\mathrm{T_V}$) *The inclusion homomorphism $H_n(W; G) \to H_n(V; G)$ is injective for every abelian group G.*

2 Topological preliminaries

In this section we deal with the topological parts of our results. We begin with the proof of Proposition 1.4. The example in part (b) is an adaptation of Example 4.35 from Hatcher's book [13].

Proof of Proposition 1.4 (a) Clearly, the existence of a defining Morse function without critical points of index $\geq n$ implies condition (T). Conversely, suppose that $\phi : W \to \mathbb{R}$ is a defining Morse function without critical points of index $> n$. If W is simply connected and $H_n(W) = 0$, then Smale's theorem on the existence of Morse functions with the minimal number of critical points [20, Theorem 6.1] allows us to cancel all index n critical points against index $n-1$ critical points to obtain a defining Morse function without critical points of index $\geq n$

(b) Fix $n \geq 3$. Let W_0 be the boundary connected sum of $S^{n-1} \times D^{n+1}$ and $S^1 \times D^{2n-1}$, i.e., the domain obtained by connecting $S^{n-1} \times D^{n+1}$ and $S^1 \times D^{2n-1}$ by a 1-handle. Note that W_0 is homotopy equivalent to $S^1 \vee S^{n-1}$, so by [13] it has $\pi_{n-1}(W_0) \cong \mathbb{Z}[t, t^{-1}]$, the group ring of $\pi_1(W_0) = \mathbb{Z}$. Let $f : S^{n-1} \to W_0$ be a smooth map representing $[f] = 2t - 1 \in \pi_{n-1}(W_0)$. For dimension reasons, we can choose f to be an embedding $f : S^{n-1} \hookrightarrow \partial W_0$. This embedding has trivial normal bundle. (To see this, note that the normal bundle of f can be described by gluing two copies of $D^{n-1} \times \mathbb{R}^n$ via a map $g : S^{n-2} \to O(n)$. Since W can be embedded into $\mathbb{R}^{2n}$, the normal bundle is stably trivial, so the homotopy class $[g] \in \pi_{n-2}O(n)$ maps to zero under the stabilization map $\pi_{n-2}O(n) \to \pi_{n-2}O$. As the stabilization map is an isomorphism for $n \geq 3$, this shows that $[g] = 0$ and thus the normal bundle is trivial.)

Let W be the manifold obtained from W_0 by attaching an n-handle along the embedding f, using any framing. Since W is built using handles of index $0, 1, n-1, n$, it carries a defining Morse function without critical points of index $> n$. Moreover, the domain W is homotopy equivalent to the space X in [13, Example 4.35], where it is shown that $H_i(X) = 0$ for all $i \geq 2$. In particular, $H_n(W; G) = 0$ for any coefficient group G, so W satisfies condition (T). Alternatively, this also follows from the observation that attaching to W a two-handle to kill its fundamental group yields a ball.

It remains to show that $\pi_n(W, \partial W) \neq 0$. For this, we consider the universal cover $\widetilde{W}$ of W. For subsets $A \subset W$, we denote by $\widetilde{A}$ their preimage in $\widetilde{W}$. The group ring $R = \mathbb{Z}[t, t^{-1}]$ of the fundamental group $\pi_1(W) = \mathbb{Z}$ acts (by deck transformations) on the singular chain complex $C_*(\widetilde{W})$, so the homology groups $H_i(\widetilde{W})$ are R-modules. The same applies to relative homology groups $H_i(\widetilde{A}, \widetilde{B})$ for subsets $B \subset A \subset W$. Set $W_1 := W \setminus \operatorname{Int} W_0$ and consider the following commuting diagram:

$$
\begin{array}{ccccccc}
\pi_{n+1}(W, W_1) & \longrightarrow & \pi_n(W_1, \partial W) & \longrightarrow & \pi_n(W, \partial W) & \longrightarrow & \pi_n(W, W_1) \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
H_{n+1}(\widetilde{W}, \widetilde{W}_1) & \longrightarrow & H_n(\widetilde{W}_1, \partial\widetilde{W}) & \longrightarrow & H_n(\widetilde{W}, \partial\widetilde{W}) & \longrightarrow & H_n(\widetilde{W}, \widetilde{W}_1) \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
R & \xrightarrow{\phi} & R & \longrightarrow & R/\operatorname{im}(\phi) & \longrightarrow & 0\,.
\end{array}
$$

Here the first two rows are parts of the long exact sequences of the triple $(\widetilde{W}, \widetilde{W}_1, \partial\widetilde{W})$ (where we identify $\pi_i(\widetilde{W}, \widetilde{W}_1) \cong \pi_i(W, W_1)$ etc.), and the top vertical maps are Hurewicz isomorphisms on the universal covers (which are simply connected). For example, W_1 is obtained from ∂W by attaching an n-cell e^n, so the pair $(\widetilde{W}_1, \partial\widetilde{W})$ is $(n-1)$-connected and the Hurewicz map $\pi_n(W_1, \partial W) \cong \pi_n(\widetilde{W}_1, \partial\widetilde{W}) \to H_n(\widetilde{W}_1, \partial\widetilde{W})$ is an isomorphism. Moreover, $H_n(\widetilde{W}_1, \partial\widetilde{W}) \cong R$ is generated as an R-module by $[e^n]$. Similarly, W is obtained from W_1 by attaching an $(n+1)$-cell e^{n+1} plus higher dimensional cells, so the pair $(\widetilde{W}, \widetilde{W}_1)$ is n-connected, in particular $\pi_n(W, W_1) \cong H_n(\widetilde{W}, \widetilde{W}_1) = 0$, and the Hurewicz map $\pi_{n+1}(W, W_1) \cong \pi_{n+1}(\widetilde{W}, \widetilde{W}_1) \to H_{n+1}(\widetilde{W}, \widetilde{W}_1)$ is an isomorphism. Moreover, $H_{n+1}(\widetilde{W}, \widetilde{W}_1) \cong R$ is generated as an R-module by $[e^{n+1}]$. By the five-lemma, the remaining Hurewicz map $\pi_n(W, \partial W) \cong \pi_n(\widetilde{W}, \partial\widetilde{W}) \to H_n(\widetilde{W}, \partial\widetilde{W})$ is an isomorphism as well. Note that the homotopy groups in the diagram are also R-modules and the Hurewicz maps are R-module homomorphisms.

To compute the map ϕ recall that, by construction of the attaching map f above, the boundary map $\psi : R \cong H_n(\widetilde{W}_1, \partial\widetilde{W}_0) \to H_{n-1}(\widetilde{W}_0) \cong R$ is multiplication by $(2t-1)$. By the duality lemma in [16, §10], the boundary map $\phi : R \cong H_{n+1}(\widetilde{W}, \widetilde{W}_1) \cong H_{n+1}(\widetilde{W}_0, \partial\widetilde{W}_0) \to H_n(\widetilde{W}_1, \partial\widetilde{W}) \cong R$ is dual to ψ in the sense that ϕ is multiplication by $(-1)^n(2t^{-1}-1)$. Thus the image of ϕ is the ideal $\langle 2t^{-1}-1\rangle$ generated by $2t^{-1}-1$, and the above diagram shows $\pi_n(W, \partial W) \cong \mathbb{Z}[t, t^{-1}]/\langle 2t^{-1}-1\rangle \neq 0$. □

The following lemma will be used in the proof of Theorem 1.3.

Lemma 2.1 *Let $W \subset \mathbb{C}^n$, $n \geq 3$, be a domain satisfying condition (T) in Theorem 1.3, i.e., W admits a defining Morse function $\psi : W \to \mathbb{R}$ without critical points of index $> n$, and $H_n(W; G) = 0$ for every abelian group G. Then the function ψ extends to a Morse function $\widetilde{\psi} : \mathbb{C}^n \to \mathbb{R}$ without critical points of index $> n$ which equals $\widetilde{\psi}(z) = |z|^2$ outside a compact set.*

Proof Note that condition (T) implies $H_i(W; G) = 0$ for all $i \geq n$ and any abelian group G. Let us pick any gradient-like vector field X for ψ.

Any map $g : S^{k-1} \to W$ from a sphere of dimension $k-1 \leq n-1$ is generically an embedding which does not meet any stable manifold of the vector field X. Moreover, generically no trajectory of X is tangent to $g(S^{k-1})$ or intersects it in more than one point. Hence, using the flow of X, g is isotopic to an embedding $f : S^{k-1} \hookrightarrow \partial W$. We claim that f is contractible in $V := \mathbb{C}^n \setminus \mathrm{Int}\, W$. For $k < n$ this is true since $\mathbb{C}^n$ is obtained from V by attaching handles of indices $\geq n$, and thus $\pi_i(V) = \pi_i(\mathbb{C}^n) = 0$ for all $i \leq n-2$. For $k = n$ it follows from

$$\pi_{n-1}(V) \cong H_{n-1}(V) \cong H_{n-1}(S^{2n} \setminus W) \cong H^n(W) \cong 0,$$

where the first isomorphism follows from the Hurewicz theorem and $(n-2)$-connectivity of V, the third one from Alexander duality (see [13, Theorem 3.44]), and the last one from condition (T).

So f extends to an embedding $F : D^k \hookrightarrow \mathbb{C}^n \setminus \mathrm{Int}\, W$ transversely attaching the k-disk to ∂W along its boundary. Let $W' \subset \mathbb{C}^n$ be a tubular neighborhood of $W \cup F(D^k)$ in $\mathbb{C}^n$ with smooth boundary, so W' is obtained from W by attaching a k-handle along f (with some framing). Then:

(i) $H_i(W'; G) \cong H_i(W; G)$ for all $i > k$ and for every G;
(ii) $\pi_i(W') \cong \pi_i(W)$ for all $i < k-1$;
(iii) $\pi_{k-1}(W')$ equals $\pi_{k-1}(W)$ modulo the subgroup generated by $[f]$;
(iv) $\pi_k(W')$ equals $\pi_k(W)$ if $[f] \in \pi_{k-1}(W)$ is non-torsion, and $\pi_k(W) \oplus \mathbb{Z}$ if $[f]$ is torsion; the same holds for H_k in place of π_k.

For property (iv), consider the long exact sequence

$$0 = \pi_{k+1}(W', W) \to \pi_k(W) \hookrightarrow \pi_k(W') \xrightarrow{\phi} \pi_k(W', W) \xrightarrow{\psi} \pi_{k-1}(W).$$

Let $[F]$ be the generator of $\pi_k(W', W) \cong \mathbb{Z}$ that is mapped to $[f]$ under ψ. If $[f]$ is non-torsion, then $\operatorname{im}\phi = \ker\psi = \{0\}$ and thus $\pi_k(W) \cong \ker\phi = \pi_k(W')$. If $[f]$ is torsion, say $m[f] = 0$ for some $m \in \mathbb{N}$, then $\operatorname{im}\phi = \ker\psi \cong \mathbb{Z}$ generated by $m[F]$. In the resulting exact sequence $\pi_k(W) \hookrightarrow \pi_k(W') \xrightarrow{\phi} \mathbb{Z} = \langle m[F]\rangle \to 0$ the map ϕ has a right inverse (gluing the k-disk mF to a k-disk in W filling mf), so it follows that $\pi_k(W') \cong \pi_k(W) \oplus \mathbb{Z}$. The proof for H_k in place of π_k is analogous.

Hence we can successively attach handles of indices $1, 2, \dots, n-1$ to obtain a domain $W' \subset \mathbb{C}^n$ containing W with

- $H_i(W'; G) = 0$ for all $i \geq n$ and for every G;
- $\pi_i(W') = 0$ for all $i < n-1$.

By the Hurewicz theorem, $H_i(W'; \mathbb{Z}) = 0$ for all $i < n-1$ and $H_{n-1}(W'; \mathbb{Z}) \cong \pi_{n-1}(W')$. Moreover, by the universal coefficient theorem, $H_n(W'; G) = 0$ for every G implies that $H_{n-1}(W'; \mathbb{Z})$ is torsion free. Hence we can attach n-handles to W' along spheres representing a basis of $\pi_{n-1}(W')$ to obtain a domain $\widetilde{W} \subset \mathbb{C}^n$ containing W with

- $H_i(\widetilde{W}; \mathbb{Z}) = 0$ for all $i > n$;
- $H_i(\widetilde{W}; \mathbb{Z}) \cong \pi_i(\widetilde{W}) = 0$ for all $i \leq n-1$;
- $H_n(\widetilde{W}; \mathbb{Z}) \cong H_n(W'; Z) = 0$.

Here the last condition follows from property (iv) above aplied to H_n. So $\widetilde{W}$ is simply connected and $H_i(\widetilde{W}) = 0$ for all $i > 0$. Since $\dim \widetilde{W} \geq 6$, the h-cobordism theorem [20] implies that $\widetilde{W}$ is diffeomorphic to the closed ball B^{2n}.

Now recall that $\widetilde{W}$ is obtained from W by attaching handles of indices $\leq n$. So we can extend the given defining Morse function $\psi : W \to \mathbb{R}$ to a Morse function $\widetilde{\psi} : \widetilde{W} \to \mathbb{R}$ without critical points of index $> n$ such that $\widetilde{W} = \{\widetilde{\psi} \leq 1\}$. Since $\widetilde{W}$ is diffeomorphic to B^{2n}, the embedding $\widetilde{W} \hookrightarrow \mathbb{C}^n$ is isotopic to the standard embedding $B^{2n} \hookrightarrow \mathbb{C}^n$, so we can extend $\widetilde{\psi}$ to an exhausting Morse function on $\mathbb{C}^n$ (still denoted by $\widetilde{\psi}$) without critical points outside $\widetilde{W}$, and equal to $\widetilde{\psi}(z) = |z|^2$ at infinity. □

3 Complex analytic preliminaries

3.1 Criteria for rational and polynomial convexity

The proofs of Theorems 1.2 and 1.3 are based on the following characterizations of rational and polynomial convexity. Consider the following condition on a J-convex domain W in a complex manifold (X, J):

(R) There exists a J-convex function $\phi : W \to \mathbb{R}$ such that $W = \{\phi \leq 0\}$, and the form $-dd^{\mathbb{C}}\phi$ on W extends to a Kähler form ω on the whole X.

The following criterion for rational convexity was proved by Nemirovski [17] as a corollary of a result of Duval and Sibony [6, Theorem 1.1].

Criterion 3.1 *An i-convex domain $W \subset \mathbb{C}^n$ is rationally convex if and only if it satisfies condition* (R).

Proof The "if" is the first proposition in [17]. The following proof of the "only if" was pointed out to us by Nemirovski. Let $W \subset \mathbb{C}^n$ be a rationally convex domain. Let $\phi : U \to \mathbb{R}$ be an i-convex function on a bounded open neighborhood U of W such that $W = \{\phi \leq 0\}$. Pick a cutoff function $\rho : \mathbb{C}^n \to [0, 1]$ which equals 0 outside U and 1 on a smaller open neighbourhood $V \subset U$ of W. Let $B \subset \mathbb{C}^n$ be a closed ball around the origin containing U, and let $\psi : \mathbb{C}^n \to \mathbb{R}$ be such that $-dd^{\mathbb{C}}\psi$ vanishes on B and is strictly positive outside B. By [6, Theorem 2.1], for every $z \notin W$ there exists a nonnegative closed $(1, 1)$-form ω_z which vanishes on W and is strictly positive on an open neighbourhood V_z of z. Finitely many such neighborhoods $V_{z_1}, \dots, V_{z_N}$ cover the compact set $B \setminus V$. Then for sufficiently large constants $c_i > 0$,

$$\omega := -dd^{\mathbb{C}}(\rho\phi) - dd^{\mathbb{C}}\psi + \sum_{i=1}^{N} c_i \omega_{z_i}$$

is a Kähler form with $\omega|_W = -dd^{\mathbb{C}}\phi$. □

We will also need the following classical criterion for polynomial convexity which goes back to Oka's paper [19] (see also [21, Theorem 1.3.8]).

Criterion 3.2 *An i-convex domain $W \subset \mathbb{C}^n$ is polynomially convex if and only if there exists an exhausting i-convex function $\phi : \mathbb{C}^n \to \mathbb{R}$ such that $W = \{\phi \leq 0\}$.*

Remark 3.3 This criterion shows that to any polynomially convex domain $W \subset \mathbb{C}^n$ we can attach handles of indices at most n to obtain a ball $B^{2n} \subset \mathbb{C}^n$. From the homology exact sequence of the pair (B^{2n}, W),

$$0 = H_{n+1}(B^{2n}, W; G) \to H_n(W; G) \to H_n(B^{2n}; G) = 0,$$

we conclude that $H_n(W; G) = 0$ for any coefficient group G.

We will also need the following lemma, where $\phi_{\text{st}}(z) := |z|^2/4$ and $\omega_{\text{st}} := -dd^{\mathbb{C}}\phi_{\text{st}}$ denote the standard i-convex function and Kähler form on $\mathbb{C}^n$.

Lemma 3.4 *Let $A \subset \mathbb{C}^n$ be a compact subset and $B \subset \mathbb{C}^n$ a closed ball with $A \subset$* Int *B.*

(a) *For every i-convex function $\phi : \mathbb{C}^n \to \mathbb{R}$ there exist an i-convex function $\widetilde{\phi}$ which equals ϕ on A, and $c\,\phi_{\rm st}$ outside B for some constant $c > 0$.*
(b) *For every Kähler form ω on $(\mathbb{C}^n, i)$ there exist a Kähler form $\widetilde{\omega}$ which equals ω on A, and $c\,\omega_{\rm st}$ outside B for some constant $c > 0$.*

Proof (a) We pick a smooth function $\alpha : \mathbb{R} \to \mathbb{R}$ with the following properties:

- $\alpha' > 0$, $\alpha'' \geq 0$, and $\alpha(x) = cx$ for large x and some constant $c > 0$;
- $\alpha \circ \phi_{\rm st} < \phi$ on A, and $\alpha \circ \phi_{\rm st} > \phi$ near ∂B.

(To find α, we choose a constant b such that $\phi_{\rm st} + b < \phi$ on A, then increase the function $x \mapsto x + b$ steeply near $\phi_{\rm st}^{-1}(\partial B)$ to arrange $\alpha \circ \phi_{\rm st} > \phi$ near ∂B, and finally change it outside B to arrange $\alpha(x) = cx$ near infinity.) Then a smoothing of the function $\max(\alpha \circ \phi_{\rm st}, \phi)$ gives an i-convex function $\widetilde{\phi}$ with the desired properties.

(b) By the d-Poincaré Lemma and the $\bar{\partial}$-Poincaré lemma on $\mathbb{C}^n$ (see e.g. [14, Corollary 1.3.9], and also the proof of Lemma 3.7 below), we can write $\omega = -dd^{\mathbb{C}}\phi$ for some function $\phi : \mathbb{C}^n \to \mathbb{R}$. Then $\widetilde{\omega} := -dd^{\mathbb{C}}\widetilde{\phi}$ for the function $\widetilde{\phi}$ provided by (a) has the desired properties. $\square$

Remark 3.5 For any $c > 0$ there exists a Kähler form ω on $\mathbb{C}^n$ which coincides with $c\omega_{\rm st}$ on a ball B of radius R and with $\omega_{\rm st}$ outside a ball of the radius $2cR$. Hence, in the statement of Lemma 3.4(b) we can set $c = 1$ if we are allowed to choose the ball B sufficiently large.

3.2 Construction of Kähler potentials

The following proposition provides Kähler potentials with prescribed behavior on a Lagrangian submanifold.

Proposition 3.6 *Let L be a real analytic Lagrangian submanifold (possibly noncompact and/or with boundary) in a Kähler manifold (X, J, ω) with real analytic Kähler form ω, and let $\rho : L \to \mathbb{R}$ be any real analytic function. Then there exists a unique real analytic function ϕ on a neighborhood of L satisfying*

$$-dd^{\mathbb{C}}\phi = \omega, \quad \phi|_L = \rho, \quad (d^{\mathbb{C}}\phi)|_L = 0.$$

We denote by $D^n \subset \mathbb{R}^n \subset \mathbb{C}^n$ the closed unit disc, and by $\mathcal{O}p\, D^n$ a sufficiently small (but not specified) open neighborhood of D^n in $\mathbb{C}^n$. Covering an arbitrary manifold L by discs and using uniqueness, Proposition 3.6 is an immediate consequence of the following special case.

Lemma 3.7 *Let ω be real analytic real $(1,1)$-form on $\mathcal{O}p\, D^n$ with $d\omega = 0$ and $\omega|_{D^n} = 0$, and let $\rho : D^n \to \mathbb{R}$ be any real analytic function. Then there exists a unique real analytic function $\phi : \mathcal{O}p\, D^n \to \mathbb{R}$ satisfying*

$$-dd^{\mathbb{C}}\phi = \omega, \quad \phi|_{D^n} = \rho, \quad \frac{\partial \phi}{\partial y_k}\Big|_{D^n} = 0 \textit{ for all } k = 1, \dots, n.$$

The following simple proof was pointed out to us by the referee.

Proof For uniqueness, note that the difference of two solutions is a real analytic function $\phi : \mathcal{O}p\, D^n \to \mathbb{R}$ satisfying

$$-dd^{\mathbb{C}}\phi = 0, \quad \phi|_{D^n} = 0, \quad \frac{\partial \phi}{\partial y_k}\Big|_{D^n} = 0 \text{ for all } k = 1, \dots, n.$$

The last two conditions imply that near a point on D^n we can write

$$\phi(x, y) = \phi_p(x, y) + O(|y|^{p+1}), \quad \phi_p(x, y) = \sum_{|I|=p} a_I(x) y^I$$

for some $p \geq 2$. The first condition yields

$$0 = -dd^{\mathbb{C}}\phi = \sum_{k,\ell=1}^{n} \frac{\partial^2 \phi_p}{\partial y_k \partial y_\ell} dx_k \wedge dy_\ell + O(|y|^{p-1}),$$

which implies $a_I(x) = 0$ for all I. Thus ϕ vanishes to infinite order along D^n, and unique continuation implies $\phi \equiv 0$.

For existence, let ω and ρ as in the lemma be given. By the $\partial\bar{\partial}$-lemma (see e.g. [14]) there exists a real analytic function $\psi : \mathcal{O}p\, D^n \to \mathbb{R}$ with $-dd^{\mathbb{C}}\psi = \omega$. Let $f : \mathcal{O}p\, D^n \to \mathbb{C}$ be the holomorphic extension of $\rho - \psi$. Then the real analytic function $\psi' := \psi + \mathrm{Re}\, f$ satisfies $\psi'|_{D^n} = \rho$ and $-dd^{\mathbb{C}}\psi' = -dd^{\mathbb{C}}\psi = \omega$, where the last property follows from $\partial\bar{\partial} = -\bar{\partial}\partial$ and holomorphicity of f via

$$-dd^{\mathbb{C}}\mathrm{Re}\, f = i\partial\bar{\partial}(f + \bar{f}) = i(\partial\bar{\partial} f - \bar{\partial}\partial \bar{f}) = 0.$$

After renaming ψ' back to ψ, we may thus assume $\psi|_{D^n} = \rho|_{D^n}$ and $-dd^{\mathbb{C}}\psi = \omega$. Since $d(d^{\mathbb{C}}\psi|_{D^n}) = -\omega|_{D^n} = 0$, the Poincaré lemma (see e.g. [12]) yields a real analytic function $\vartheta : D^n \to \mathbb{R}$ with $d^{\mathbb{C}}\psi|_{D^n} = d\vartheta$. Let $g : \mathcal{O}p\, D^n \to \mathbb{C}$ be the holomorphic extension of $i\vartheta$. Then the real analytic function $\phi := \psi + \mathrm{Re}\, g : \mathcal{O}p\, D^n \to \mathbb{R}$ still satisfies $-dd^{\mathbb{C}}\phi = \omega$ (by the argument above) and $\phi|_{D^n} = \rho + \mathrm{Re}\,(i\vartheta) = \rho$. Moreover, holomorphicity of g implies

$$
\begin{aligned}
d^{\mathbb{C}}\phi|_{D^n} &= d^{\mathbb{C}}\psi|_{D^n} + \frac{1}{2}d^{\mathbb{C}}(g+\bar{g})|_{D^n} = d\vartheta + \frac{1}{2}(dg\circ i + d\bar{g}\circ i)|_{D^n} \\
&= d\vartheta + \frac{1}{2}(i\circ dg - i\circ d\bar{g})|_{D^n} = d\vartheta + \frac{1}{2}\big(i\circ(id\vartheta) - i\circ(-id\vartheta)\big) \\
&= d\vartheta - d\vartheta = 0.
\end{aligned}
$$

This concludes the proof of Lemma 3.7, and thus of Proposition 3.6. □

3.3 Attaching isotropic discs to rationally convex domains

Using Proposition 3.6, we now prove that rational convexity persists under suitable attaching of isotropic discs.

Given a J-convex domain W in a complex manifold (V, J), we say that a totally real disc $\Delta \subset V\setminus \mathrm{Int}\, W$ is *J-orthogonally* attached to ∂W along $\partial\Delta$ if $J(T_x\Delta) \subset T_x(\partial W)$ for all $x \in \partial\Delta$. The following result is probably known to specialists, though we could not find it in the literature; S. Nemirovski has informed us that he knew this fact.

Proposition 3.8 *Let W be a J-convex domain in a complex manifold (V, J). Suppose there exists a defining J-convex function $\phi : W = \{\phi \le c\} \to \mathbb{R}$ such that $-dd^{\mathbb{C}}\phi$ extends to a Kähler form ω on V. Let $\Delta \subset V\setminus \mathrm{Int}\, W$ be a real analytic k-disc, J-orthogonally attached to ∂W along $\partial\Delta$, such that $\omega|_\Delta = 0$.*

Then for every open neighborhood U of $W \cup \Delta$ there exists a J-convex domain $\widetilde{W} \subset U$ with $W \subset \mathrm{Int}\, \widetilde{W}$ and a defining J-convex function $\widetilde{\phi} : \widetilde{W} = \{\widetilde{\phi} \le \widetilde{c}\} \to \mathbb{R}$ such that

- *$\widetilde{\phi}|_W = \phi$, and $\widetilde{\phi}$ has a unique index k critical point in $\widetilde{W}\setminus W$ whose stable manifold is Δ;*
- *$-dd^{\mathbb{C}}\widetilde{\phi}$ extends to a Kähler form $\widetilde{\omega}$ on V which agrees with ω outside U.*

Remark 3.9 The real analyticity assumption on the disc Δ can probably be removed using appropriate results on solutions of the $\bar{\partial}$-equation.

The proof uses the following extension lemma.

Lemma 3.10 *In the situation of Proposition 3.8, there exists an extension of ϕ to a J-convex function ϕ_1 on a slightly larger domain $W_1 = \{\phi_1 \le c_1\} \subset U$ with the following properties:*

(i) *$-dd^{\mathbb{C}}\phi_1$ extends to a Kähler form ω_1 on V which agrees with ω outside U such that $\omega_1|_\Delta = 0$;*
(ii) *$d^{\mathbb{C}}\phi_1|_{\Delta\cap(W_1\setminus W)} = 0$.*

Proof First, we extend the function ϕ to a J-convex function $\widetilde{\phi} : \widetilde{W} \to \mathbb{R}$ on a slightly larger domain such that all level sets of $\widetilde{\phi}$ in $\widetilde{W} \setminus W$ intersect Δ J-orthogonally, or in other words, $d^{\mathbb{C}}\phi|_{\Delta \cap (\widetilde{W} \setminus W)} = 0$.

To find a Kähler form extending $\widetilde{\phi}$, we argue as in the proof of Lemma 3.7. The closed $(1, 1)$-form $\widetilde{\omega} := \omega + dd^{\mathbb{C}}\widetilde{\phi}$ on $\widetilde{W}$ vanishes on W. By the relative d-Poincaré lemma, we find a real 1-form λ on $\widetilde{W}$ with $d\lambda = \widetilde{\omega}$ and $\lambda|_W = 0$. We write $\lambda = \alpha + \bar{\alpha}$ for a $(1, 0)$-form α on $\widetilde{W}$, so that $\alpha|_W = 0$ and

$$\partial\bar{\alpha} + \bar{\partial}\alpha = \widetilde{\omega}, \quad \partial\alpha = 0, \quad \bar{\partial}\bar{\alpha} = 0.$$

For sufficiently small $t \geq 0$ set $\Omega_t := \{\widetilde{\phi} < c + t\} \subset \widetilde{W}$, so that $\Omega_0 = W$. Consider $\varepsilon > 0$ such that $\bar{\Omega}_{3\varepsilon} \subset \widetilde{W}$. By a result of Hörmander and Wermer (see [4, Theorem 8.36 and Remark 8.38]), there exists a smooth solution $\beta_\varepsilon : \Omega_{3\varepsilon} \to \mathbb{C}$ of the equation $\bar{\partial}\beta_\varepsilon = \bar{\alpha}$ which satisfies for each integer $k \geq 0$ an estimate

$$\|\beta_\varepsilon\|_{C^k(\Omega_{2\varepsilon})} \leq C_k \varepsilon^{-n-k} \|\bar{\alpha}\|_{C^k(\Omega_{3\varepsilon})},$$

where $n = \dim_{\mathbb{C}} V$ and the constant C_k depends on k and the diameter of the domain $\widetilde{W}$ but not on ε. The function $\psi_\varepsilon := \operatorname{Im} \beta_\varepsilon : \Omega_{2\varepsilon} \to \mathbb{R}$ satisfies $-dd^{\mathbb{C}}\psi_\varepsilon = \widetilde{\omega}|_{\Omega_{2\varepsilon}}$. Since $\bar{\alpha}|_W = 0$, there exists for every k, N a constant $C_{k,N}$ depending on k and N but not on ε such that $\|\bar{\alpha}\|_{C^k(\Omega_{3\varepsilon})} \leq C_{k,N}\varepsilon^N$. It follows that

$$\|\psi_\varepsilon\|_{C^k(\Omega_{2\varepsilon})} \leq \|\beta\|_{C^k(\Omega_{2\varepsilon})} \leq C'_{k,N}\varepsilon^{N-n-k}$$

with constants $C'_{k,N}$ not depending on ε. Fix a cutoff function $\rho : \mathbb{R} \to [0, 1]$ with $\rho(t) = 1$ for $t \leq 1$ and $\rho(t) = 0$ for $t \geq 2$ and define $\rho_\varepsilon : V \to \mathbb{R}$ by $\rho_\varepsilon(x) := \rho\big(\frac{\phi(x)-c}{\varepsilon}\big)$. Then the closed $(1, 1)$-form

$$\omega_1 := \omega + dd^{\mathbb{C}}(\rho_\varepsilon\psi_\varepsilon)$$

agrees with $\omega + dd^{\mathbb{C}}\psi_\varepsilon = -dd^{\mathbb{C}}\widetilde{\phi}$ on $\bar{\Omega}_\varepsilon$, and with ω outside $\Omega_{2\varepsilon}$. Moreover, for each $v \in TV$ with $|v|^2 = \omega(v, Jv) = 1$, the previous estimates yield

$$\omega_1(v, Jv) \geq \omega(v, Jv) - \|\rho_\varepsilon\psi_\varepsilon\|_{C^2(\Omega_{2\varepsilon})} \geq 1 - C'_N\varepsilon^{N-n-2}$$

with constants C'_N not depending on ε. Choosing $N := n+3$ and ε sufficiently small, we can arrange $\omega_1(v, Jv) \geq 1/2$, so ω_1 is a Kähler form and the restriction ϕ_1 of $\widetilde{\phi}$ to the domain $W_1 := \bar{\Omega}_\varepsilon$ is the desired extension of ϕ. □

Proof of Proposition 3.8 **Step 1.** Let $\phi_1 : W_1 = \{\phi_1 \leq c_1\} \to \mathbb{R}$ and ω_1 be the extensions provided by Lemma 3.10 and rename ϕ_1, ω_1 back to ϕ, ω. We pick a value $c' \in (c, c_1)$ and set $W' := \{\phi \leq c'\}$, $\Delta' := \Delta \cap (V \setminus \mathrm{Int}\, W'\}$. After adding a constant to ϕ, we may assume without loss of generality that $c' = -1$.

For the following, we need some notation from [4]. On $\mathbb{C}^n$ with complex coordinates $z_j = x_j + iy_j$, we introduce the functions

$$r := \sqrt{x_1^2 + \cdots + x_n^2 + y_{k+1}^2 + \cdots + y_n^2}, \qquad R := \sqrt{y_1^2 + \cdot + y_k^2},$$

and for some fixed $a > 1$ the standard i-convex function

$$\psi_{\mathrm{st}}(r, R) := ar^2 - R^2.$$

For $\varepsilon > 0$ we define the subsets

$$\begin{aligned} H_\varepsilon &:= \{R \leq 1 + \varepsilon,\ r \leq \varepsilon\}, \qquad & H_\varepsilon^y &:= H_\varepsilon \cap \{x = 0\}, \\ U_\varepsilon &:= \{1 - \varepsilon \leq R \leq 1 + \varepsilon,\ r \leq \varepsilon\}, \qquad & U_\varepsilon^y &:= U_\varepsilon \cap \{x = 0\}, \end{aligned}$$
$$D^k := \{R \leq 1,\ r = 0\}.$$

Step 2. Using [4, Theorem 5.53], we find a real analytic embedding $f : H_\varepsilon^y \hookrightarrow V$ such that $f(D^k) = \Delta'$, and $\phi \circ f$ agrees with ψ_{st} to first order along ∂D^k. Moreover, by property (ii) in Lemma 3.10 we can arrange that $f^* d^{\mathbb{C}} \phi = 0$ near ∂D^k.

We extend f uniquely to a holomorphic embedding $F : (H_\varepsilon, i) \hookrightarrow (V, J)$ (for some possibly smaller $\varepsilon > 0$) and set $\psi := \phi \circ F : U_\varepsilon \to \mathbb{R}$. Then for any $z \in \partial D^k$ and $v \in T_z H_\varepsilon^y = i\mathbb{R}^n$ we have $d_z \psi(v) = d_z \psi_{\mathrm{st}}(v)$ and $d_z \psi(iv) = (f^* d^{\mathbb{C}} \phi)(v) = 0 = d_z \psi_{\mathrm{st}}(iv)$, so the i-convex functions ψ and ψ_{st} agree to first order at points of ∂D^k. According to [4, Proposition 3.26], there exists an i-convex function $\vartheta : U_\varepsilon \to \mathbb{R}$ which agrees with ψ near ∂U_ε, and with ψ_{st} on U_δ for some $\delta < \varepsilon$. Moreover, according to [4, Remark 3.27 (ii)], the conditions $d^{\mathbb{C}} \psi|_{U_\varepsilon^y} = d^{\mathbb{C}} \psi_{\mathrm{st}}|_{U_\varepsilon^y} = 0$ imply $d^{\mathbb{C}} \vartheta|_{U_\varepsilon^y} = 0$.

Hence, after replacing ϕ by $\vartheta \circ F^{-1}$ on $F(U_\varepsilon)$ and shrinking ε, we may assume that $F^* \phi = \psi_{\mathrm{st}}$ on U_ε.

Step 3. After real analytic approximations using [4, Theorem 5.53], we may assume that ϕ and ω are real analytic on $F(H_\varepsilon)$. Then $F^* \omega$ is a real analytic Kähler form on H_ε which vanishes on H_ε^y. By Lemma 3.7, there exists a unique real analytic function $\psi' : H_\varepsilon \to \mathbb{R}$ (for some possibly smaller ε) satisfying

$$-dd^{\mathbb{C}} \psi' = F^* \omega, \quad \psi'|_{H_\varepsilon^y} = \psi_{\mathrm{st}}|_{H_\varepsilon^y}, \quad \left.\frac{\partial \psi'}{\partial y_j}\right|_{H_\varepsilon^y} = 0 \text{ for all } j = 1, \ldots, n.$$

Thus ψ' agrees with ψ_{st} to first order along H_ε^y. Moreover, since $-dd^{\mathbb{C}} \psi_{\mathrm{st}} = F^* \omega$ on U_ε by Step 2, uniqueness of ψ' implies that $\psi' = \psi_{\mathrm{st}}$ on U_ε.

Again by [4, Proposition 3.26], there exists an i-convex function $\vartheta' : H_\varepsilon \to \mathbb{R}$ which agrees with ψ' near ∂H_ε, and with ψ_{st} on $H_\gamma \cup U_\varepsilon$ for some $\gamma < \varepsilon$. Let $\phi' : W' \cup F(H_\varepsilon) \to \mathbb{R}$ be the J-convex function which equals ϕ on W', and $\vartheta' \circ F^{-1}$ on $F(H_\varepsilon)$. By construction, $-dd^{\mathbb{C}}\phi'$ extends by ω to a Kähler form ω' on V which equals ω outside U. Moreover, $F^*\phi' = \psi_{\mathrm{st}}$ on H_γ.

Step 4. According to [4, Corollary 4.4], there exists an i-convex function $\widetilde{\psi} : H_\gamma \to \mathbb{R}$ with the following properties:

- $\widetilde{\psi} = \psi_{\mathrm{st}}$ near ∂H_γ;
- $\widetilde{\psi}$ has a unique index k critical point at the origin whose stable disc is D^k;
- $\widetilde{\psi}|_{D^k} < -1$.

Let $\widetilde{\phi} : W' \cup F(H_\varepsilon) \to \mathbb{R}$ be the J-convex function which equals $\widetilde{\psi} \circ F^{-1}$ on $F(H_\gamma)$ and ϕ' outside, and set $\widetilde{W} := \{\widetilde{\phi} \le -1\}$. By construction, we have $\widetilde{W} \subset U$ and $W \subset \mathrm{Int}\,\widetilde{W}$. Moreover, $-dd^{\mathbb{C}}\widetilde{\phi}|_{\widetilde{W}}$ extends by ω' to a Kähler form $\widetilde{\omega}$ on V which equals ω outside U. This concludes the proof of Proposition 3.8. □

For the proof of Theorem 1.5, we will need the following refinement of Proposition 3.8 in the presence of an ambient Weinstein structure.

Proposition 3.11 *Let (V, J) be a complex manifold equipped with a Weinstein structure (ω, X, ϕ) such that ω is a Kähler form for J. Let $W = \{\phi \le c\} \subset V$ be a J-convex domain such that $\phi|_W$ is defining and J-convex and $\mathfrak{W}(W, J, \phi) = (\omega, X, \phi)$ on W. Let $\Delta \subset V \setminus \mathrm{Int}\,W$ be a real analytic stable disc of an index k critical point p of (V, ω, X, ϕ) on the first critical level above c.*

Then for every open neighborhood U of $W \cup \Delta$ there exists a Weinstein structure $(\widetilde{\omega}, \widetilde{X}, \widetilde{\phi})$ on V and a J-convex domain $\widetilde{W} = \{\widetilde{\phi} \le \widetilde{c}\} \subset U$ such that

- *$W \subset \mathrm{Int}\,\widetilde{W}$ and $(\widetilde{\omega}, \widetilde{X}, \widetilde{\phi}) = (\omega, X, \phi)$ on W;*
- *$\widetilde{\phi}|_{\widetilde{W}}$ is defining and J-convex and $\mathfrak{W}(\widetilde{W}, J, \widetilde{\phi}) = (\widetilde{\omega}, \widetilde{X}, \widetilde{\phi})$ on $\widetilde{W}$;*
- *$\widetilde{\omega}$ is a Kähler form for J;*
- *the Weinstein structures $(\widetilde{\omega}, \widetilde{X}, \widetilde{\phi})$ and (ω, X, ϕ) agree outside U, have the same critical points, and are Weinstein homotopic via a homotopy with fixed critical points and fixed on W as well as outside U.*

Proof As in Steps 1–3 of the proof of Proposition 3.8, we construct a J-convex function $\phi' : U' \to \mathbb{R}$ on a neighborhood $U' \subset U$ of $W \cup \Delta$ with the following properties:

- $\phi' = \phi$ on $W \cup \Delta$;
- $-dd^{\mathbb{C}}\phi' = \omega|_{U'}$;
- p is the unique critical point (of index k) of ϕ' in $U' \setminus W$ with stable disc Δ.

According to [4, Proposition 12.14], there exists a Weinstein structure (ω, X'', ϕ'') on V which agrees with (ω, X, ϕ) outside U and with (ω, X', ϕ') on a neighborhood $U'' \subset U'$ of $W \cup \Delta$ such that the Weinstein structures (ω, X'', ϕ'') and (ω, X, ϕ) have the same critical points and are Weinstein homotopic via a homotopy with fixed critical points, fixed ω, and fixed on W as well as outside U.

Finally, we apply Step 4 of the proof of Proposition 3.8 to modify (ω, X'', ϕ'') inside U'' to obtain the desired Weinstein structure $(\widetilde{\omega}, \widetilde{X}, \widetilde{\phi})$ and J-convex domain $\widetilde{W} = \{\widetilde{\phi} \leq \widetilde{c}\} \subset U$. □

4 Proofs of the main results

Proof of Theorem 1.5 (a) Is a special case of [4, Theorem 13.4].

(b) For the "only if", suppose that (after an isotopy) $f(W) = \widetilde{W} \subset \mathbb{C}^n$ is rationally convex and f^*i is Stein homotopic to J. By Criterion 3.1, there exists a defining i-convex function $\widetilde{\phi} : \widetilde{W} \to \mathbb{R}$ such that $-dd^{\mathbb{C}}\widetilde{\phi}$ extends to a Kähler form $\widetilde{\omega}$ on $\mathbb{C}^n$. After applying Lemma 3.4 and rescaling, we may assume that $\widetilde{\omega} = \omega_{\rm st}$ at infinity. Then Moser's stability theorem yields a family of diffeomorphisms $g_t : \mathbb{C}^n \to \mathbb{C}^n$ with $g_0 = \mathrm{id}$, $g_t = \mathrm{id}$ at infinity, and $g_1^*\omega_{\rm st} = \widetilde{\omega}$. Thus $g_t \circ f$ is an isotopy from f to the symplectic embedding $g_1 \circ f : (W, f^*\widetilde{\omega}) \hookrightarrow (\mathbb{C}^n, \omega_{\rm st})$. Moreover, $f^*\widetilde{\omega}$ is the symplectic form of the Weinstein structure $\mathfrak{W}(W, f^*i, f^*\widetilde{\phi})$, which belongs to the class $\mathfrak{W}(W, J)$ by assumption.

For the "if", suppose that (after an isotopy) $f : (W, \omega) \hookrightarrow (\mathbb{C}^n, \omega_{\rm st})$ is a symplectic embedding for some $(\omega, X, \phi) \in \mathfrak{W}(W, J)$. Applying Proposition 3.11 inductively to sublevel sets of the Weinstein structure $(f_*\omega = \omega_{\rm st}, f_*X, f_*\phi)$ on $\big(f(W), i\big)$, we construct a Weinstein structure $(\widetilde{\omega}, \widetilde{X}, \widetilde{\phi})$ on $f(W)$ and an i-convex domain $\widetilde{W} = \{\widetilde{\phi} \leq \widetilde{c}\} \subset \mathrm{Int}\, f(W)$ such that

(i) $\widetilde{\phi}|_{\widetilde{W}}$ is defining and i-convex and $\mathfrak{W}(\widetilde{W}, i, \widetilde{\phi}) = (\widetilde{\omega}, \widetilde{X}, \widetilde{\phi})$ on $\widetilde{W}$;

(ii) $\widetilde{\omega}$ is a Kähler form for i;

(iii) The Weinstein structures $(\widetilde{\omega}, \widetilde{X}, \widetilde{\phi})$ and $(f_*\omega = \omega_{\rm st}, f_*X, f_*\phi)$ agree near $\partial f(W)$, have the same critical points, and are Weinstein homotopic via a homotopy with fixed critical points and fixed near $\partial f(W)$.

Thus $-dd^{\mathbb{C}}\widetilde{\phi}$ extends via $\widetilde{\omega}$ to a Kähler form on $f(W)$, and from there via $\omega_{\rm st}$ to a Kähler form on $\mathbb{C}^n$, so by Criterion 3.1 the domain $\widetilde{W} \subset \mathbb{C}^n$ is rationally convex. Since $\widetilde{\phi}$ has no critical points in $f(W) \setminus \mathrm{Int}\, \widetilde{W} = \{\widetilde{c} \leq \widetilde{\phi} \leq c\}$, we find a family of diffeomorphisms $f_t : W \to W_t$ onto sublevel sets of $\widetilde{\phi}$ such that $f_0 = f$ and $W_1 = \widetilde{W}$. Hence the pullback Weinstein structure $\mathfrak{W}(W, f_1^*i, f_1^*\widetilde{\phi}) = f_1^*(\widetilde{\omega}, \widetilde{X}, \widetilde{\phi})$ is homotopic to $f^*(\widetilde{\omega}, \widetilde{X}, \widetilde{\phi})$, which by property (iii) is homotopic to $(\omega, X, \phi) \in \mathfrak{W}(W, J)$. By [4, Theorem 15.2], this implies that the Stein structures $f_1^* i$ and J are Stein homotopic.

(c) For the "only if", suppose that (after an isotopy) $f(W) = \widetilde{W} \subset \mathbb{C}^n$ is polynomially convex and $f^* i$ is Stein homotopic to J. By Criterion 3.1, there exists an exhausting i-convex function $\widetilde{\phi} : \mathbb{C}^n \to \mathbb{R}$ such that $\widetilde{W} = \{\widetilde{\phi} \leq 0\}$. After applying Lemma 3.4, we may assume that $\widetilde{\phi} = \phi_{\mathrm{st}}$ at infinity. Then Moser's stability theorem yields a family of diffeomorphisms $g_t : \mathbb{C}^n \to \mathbb{C}^n$ with $g_0 = \mathrm{id}$, $g_t = \mathrm{id}$ at infinity, and $g_1^* \omega_{\mathrm{st}} = \widetilde{\omega} := -dd^{\mathbb{C}} \widetilde{\phi}$. Thus $g_t \circ f$ is an isotopy from f to the symplectic embedding $\widetilde{f} := g_1 \circ f : (W, f^* \widetilde{\omega}) \hookrightarrow (\mathbb{C}^n, \omega_{\mathrm{st}})$. Moreover, $f^* \widetilde{\omega}$ is the symplectic form of the Weinstein structure $\mathfrak{W}(W, f^* i, f^* \widetilde{\phi})$, which belongs to the class $\mathfrak{W}(W, J)$ by assumption. Finally, note that the push-forward Weinstein structure $\widetilde{f}_* \mathfrak{W}(W, f^* i, f^* \widetilde{\phi})$ extends to the Weinstein structure $g_1^* \mathfrak{W}(\mathbb{C}^n, i, \widetilde{\phi})$ on the whole $\mathbb{C}^n$ which is standard at infinity. This Weinstein structure is homotopic via $g_t^* \mathfrak{W}(\mathbb{C}^n, i, \widetilde{\phi})$ to $\mathfrak{W}(\mathbb{C}^n, i, \widetilde{\phi})$, and hence belongs to the class $\mathfrak{W}(\mathbb{C}^n, i)$.

For the "if", suppose that (after an isotopy) $f : (W, \omega) \hookrightarrow (\mathbb{C}^n, \omega_{\mathrm{st}})$ is a symplectic embedding for some $(\omega, X, \phi) \in \mathfrak{W}(W, J)$, and the push-forward Weinstein structure $(f_* \omega = \omega_{\mathrm{st}}, f_* X, f_* \phi)$ extends to a Weinstein structure (ω_1, X_1, ϕ_1) on the whole $\mathbb{C}^n$ which is homotopic to the standard Weinstein structure $(\omega_{\mathrm{st}}, X_0, \phi_0) = \mathfrak{W}(\mathbb{C}^n, i, \frac{|z|^2}{4})$ via a Weinstein homotopy (ω_t, X_t, ϕ_t).

According to [4, Theorem 15.3], after composing the ϕ_t with a convex increasing diffeomorphism $g : \mathbb{R} \to \mathbb{R}$, there exists a family of diffeomorphisms $h_t : \mathbb{C}^n \to \mathbb{C}^n$, $t \in [0, 1]$, with $h_0 = \mathrm{id}$ such that

(i) the functions $\phi_t \circ h_t^{-1} : \mathbb{C}^n \to \mathbb{R}$ are i-convex;
(ii) the paths of Weinstein structures $(\omega_{\mathrm{st}}, X_t, \phi_t)$ and $\mathfrak{W}(\mathbb{C}^n, h_t^* i, \phi_t)$ are homotopic with fixed functions ϕ_t and fixed at $t = 0$.

Thus $f_t := h_t \circ f : W \hookrightarrow \mathbb{C}^n$ is a smooth isotopy from $f_0 = f$ to an embedding f_1 whose image $f_1(W) = h_1\big(f(W)\big)$ is a sublevel set $\{\phi_1 \circ h_1^{-1} \leq c\}$ of the i-convex function $\phi_1 \circ h_1^{-1} : \mathbb{C}^n \to \mathbb{R}$, and hence polynomially convex by Criterion 3.2. Due to property (ii) at $t = 1$, the Weinstein structures $\mathfrak{W}\big(f(W), h_1^* i, \phi_1\big)$ and (ω_1, X_1, ϕ_1) on $f(W) = \{\phi_1 \leq c\}$ are homotopic with fixed function ϕ_1. Therefore, the pullback Weinstein structure $\mathfrak{W}(f_1^* i, \phi_1 \circ f) = f^* \mathfrak{W}\big(f(W), h_1^* i, \phi_1\big)$ is homotopic to $f^*(\omega_1, X_1, \phi_1) = (\omega, X, \phi) \in \mathfrak{W}(W, J)$. By [4, Theorem 15.2], this implies that the Stein structures $f_1^* i$ and J are Stein homotopic. □

Proof of Theorem 1.7 The result is a consequence of Theorem 1.5 and the h-principle for Lagrangian caps in [9]. Let (W, J) be a flexible Stein domain of complex dimension $n \geq 3$, and $f : W \hookrightarrow \mathbb{C}^n$ a smooth embedding such that f^*i is homotopic to J through almost complex structures. Let $(\omega, X, \phi) \in \mathfrak{W}(W, J)$ be an associated flexible Weinstein structure on W. According to [9, Corollary 6.3], the embedding f is isotopic to a symplectic embedding $\widetilde{f} : (W, \omega) \hookrightarrow (\mathbb{C}^n, \omega_{\mathrm{st}})$. Hence, by Theorem 1.5(b), the embedding f is isotopic to a deformation equivalence onto a rationally convex domain.

Now suppose in addition that $H_n(W; G) = 0$ for every abelian group G. Let $\widetilde{\mathfrak{W}} := (\omega_{\mathrm{st}}, \widetilde{X} := \widetilde{f}_*X, \widetilde{\phi} := \phi \circ \widetilde{f}^{-1})$ be the push-forward Weinstein structure on $\widetilde{W} := \widetilde{f}(W) \subset \mathbb{C}^n$. According to Lemma 2.1, the defining function $\widetilde{\phi} : \widetilde{W} \to \mathbb{R}$ extends to a Morse function $\widehat{\phi} : \mathbb{C}^n \to \mathbb{R}$ without critical points of index $> n$ which equals $\widehat{\phi}(z) = |z|^2$ at infinity. By [4, Theorem 13.1], we can extend the Weinstein structure $\widetilde{\mathfrak{W}}$ to a flexible Weinstein structure $\widehat{\mathfrak{W}} = (\widehat{\omega}, \widehat{X}, \widehat{\phi})$ on $\mathbb{C}^n$ such that the forms $\widehat{\omega}$ and ω_{st} are homotopic rel $\widetilde{W}$ as non-degenerate 2-forms. According to [4, Theorem 14.5], the two flexible Weinstein structures $\widehat{\mathfrak{W}}$ and $\mathfrak{W}_{\mathrm{st}}$ are homotopic. Hence, by Theorem 1.5(c), the embedding f is isotopic to a deformation equivalence onto a polynomially convex domain. $\square$

For the proof of Corollary 1.8, we need the following lemma about Weinstein structures.

Lemma 4.1 *Let* $\mathfrak{W}_t = (\omega_t, X_t, \phi_t)$, $t \in [0, 1]$, *be a Weinstein homotopy on* D^*L *with Liouville forms* $\lambda_t = i_{X_t}\omega_t$ *such that* $\mathfrak{W}_0 = \mathfrak{W}(D^*L, J_{\mathrm{Grauert}}, \phi_{\mathrm{Grauert}})$ *is the Weinstein structure associated to a Grauert tube.*

Then there exists an isotopy of submanifolds $L_t \subset D^*L$ *starting with the zero section* L_0 *such that* $\lambda_t|_{L_t}$ *is exact for all* $t \in [0, 1]$.

Proof Set $W := D^*L$ and consider the contact structures $\xi_t := \ker(\lambda_t|_{\partial W})$ on its boundary. By Gray's stability theorem, there exist diffeomorphisms $g_t : \partial W \to \partial W$ with $g_t^*\xi_t = \xi_0$. After extending the g_t to diffeomorphisms of W and replacing $\mathfrak{W}_t$ by their pullbacks, we may hence assume that $\xi_t = \xi$ for all t. Pushing down by the flows of $-X_t$ for suitable times, we then find embeddings $h_t : W \hookrightarrow W$ onto domains with $W_t := h_t(W)$ with ∂W_t transverse to X_t such that $h_t^*\lambda_t = h_0^*\lambda_0$ near ∂W for all t. Note that the zero section $L_0 = h_0(L_0)$ is contained in W_0 and $h_0^*\lambda_0|_{L_0} = \lambda_0|_{L_0} = 0$.

By Moser's stability theorem (in the form stated in [4, Theorem 6.8]), there exist compactly supported diffeomorphisms $f_t : W \to W$ with $f_0 = \mathrm{id}$ such that $f_t^*h_t^*\lambda_t - h_0^*\lambda_0 = d\rho_t$ for compactly supported functions $\rho_t : W \to \mathbb{R}$. Then the manifolds $L_t := h_t \circ f_t(L_0) \subset W_t \subset W$ satisfy

$$(h_t \circ f_t)(\lambda_t|_{L_t}) = (f_t^*h_t^*\lambda_t)|_{L_0} = (h_0^*\lambda_0 + d\rho_t)|_{L_0} = (d\rho_t)|_{L_0},$$

and hence $\lambda_t|_{L_t}$ is exact for all $t \in [0, 1]$. $\square$

Proof of Corollary 1.8 Let L be a closed n-dimensional manifold.

(a) It is well known (see [4, Proposition 2.5]) that every totally real submanifold $L \cong L' \subset \mathbb{C}^n$ has a tubular neighborhood which is a Grauert tube of L. Conversely, suppose that $(D^*L, J_{\text{Grauert}})$ is deformation equivalent to an i-convex domain $W \subset \mathbb{C}^n$. Pick any defining i-convex function $\phi : W \to \mathbb{R}$. By Lemma 4.1, W contains a submanifold diffeomorphic to L which is Lagrangian for $\omega = -dd^{\mathbb{C}}\phi$, hence in particular i-totally real.

(b) Is a special case of Theorem 1.5(b), in view of Weinstein's Lagrangian neighborhood theorem and Lemma 4.1.

(c) Follows immediately from the vanishing of $H_n(W; G)$ for polynomially convex domains stated in Theorem 1.3. □

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