Symplectic Tate homology

Peter Albers, Kai Cieliebak and Urs Frauenfelder

ABSTRACT

For a Liouville domain W satisfying $c_1(W) = 0$, we propose in this note two versions of symplectic Tate homology, $\underrightarrow{H}\,\underleftarrow{T}(W)$ and $\underleftarrow{H}\,\underrightarrow{T}(W)$, which are related by a canonical map $\kappa\colon \underrightarrow{H}\,\underleftarrow{T}(W) \to \underleftarrow{H}\,\underrightarrow{T}(W)$. Our geometric approach to Tate homology uses the moduli space of finite energy gradient flow lines of the Rabinowitz action functional for a circle in the complex plane as a classifying space for S^1-equivariant Tate homology. For rational coefficients the symplectic Tate homology $\underleftarrow{H}\,\underrightarrow{T}(W;\mathbb{Q})$ has the fixed point property and is therefore isomorphic to $H(W;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[u, u^{-1}]$, where $\mathbb{Q}[u, u^{-1}]$ is the ring of Laurent polynomials over the rationals. Using a deep theorem of Goodwillie, we construct examples of Liouville domains where the canonical map κ is not surjective and examples where it is not injective.

1. Introduction

Equivariant Tate homology was introduced by Swan [40] in the topological context of a finite group G acting on a space X; see the book by Brown [9]. Later this was generalized to actions of compact Lie groups by Adem, Cohen, and Dwyer [4] and Greenlees and May [29]; see also Tene [41] for another construction using stratifolds. If $G = S^1$ and X is a finite CW complex, then S^1-equivariant Tate homology with rational coefficients has the fixed point property

$$\widehat{H}_*^{S^1}(X;\mathbb{Q}) \cong \widehat{H}_*^{S^1}(X^{S^1};\mathbb{Q}),$$

where X^{S^1} is the fixed point set. For infinite-dimensional spaces this property fails in general, as was observed by Goodwillie in [28]. Jones and Petrack [33] introduced a variant of S^1-equivariant Tate homology which retains the fixed point property; see also Cencelj [10].

In this paper, we construct analogues of both versions of Tate homology for symplectic homology, together with a canonical map between the two versions. Our geometric approach twists symplectic homology with Rabinowitz Floer homology on the complex plane. There is partial overlap between our work and the work of Zhao [47], who found independently a more algebraic approach to construct the two symplectic Tate homologies.

Let us briefly recall the main definition of symplectic homology; see [12, 22, 39, 43] for details. We restrict to the case of a Liouville domain, that is, a compact manifold W with boundary equipped with a 1-form λ such that $d\lambda$ is symplectic and $\lambda|_{\partial W}$ is a positive contact form. Its completion is the Liouville manifold $V = W \cup_{\partial W} ([1, \infty) \times \partial W)$ with the 1-form λ_V that equals λ on W and $r\lambda$ on $[1, \infty) \times \partial W$. The symplectic homology $SH_*(W)$ of the Liouville domain W (we will usually omit λ from the notation) is by definition the Floer homology with $\mathbb{Z}$-coefficients of any Hamiltonian on V that grows quadratically in r at infinity. It is $\mathbb{Z}$-graded by Conley–Zehnder indices if the first Chern class $c_1(W)$ (with respect to any compatible almost complex structure) vanishes, and it is invariant under Liouville isomorphisms of the completions (that is, diffeomorphisms $f : V \to V'$ such that $f^*\lambda' - \lambda$ is exact and compactly

supported). An S^1-equivariant version $SH_*^{S^1}(W)$ of symplectic homology with respect to the circle action on the loop space of V has been defined in [6, 42]. It is a graded module over the polynomial ring $\mathbb{Z}[u^{-1}]$, where u is a formal variable of degree 2. Moreover, it has a localization with respect to the endomorphism u^{-1}; see Subsection 2.4. Now we can state our main result.

THEOREM 1.1. (a) *To every Liouville domain W with $c_1(W) = 0$ one can naturally associate two versions of symplectic Tate homology: the Jones–Petrack version $\underrightarrow{\underleftarrow{H\,T}}(W)$ and the Goodwillie version $\underleftarrow{\underrightarrow{H\,T}}(W)$. They are graded modules over the graded ring $\mathbb{Z}[u, u^{-1}]$ of Laurent polynomials with coefficients in $\mathbb{Z}$ in a variable u of degree 2, and they are invariant under Liouville isomorphisms of the completions.*

(b) *There exists a natural map of $\mathbb{Z}[u, u^{-1}]$-modules*

$$\kappa\colon \underrightarrow{\underleftarrow{H\,T}}(W) \longrightarrow \underleftarrow{\underrightarrow{H\,T}}(W).$$

(c) *With rational coefficients, the Jones–Petrack version $\underrightarrow{\underleftarrow{H\,T}}$ has the fixed point property*

$$\underrightarrow{\underleftarrow{H\,T}}_*(W; \mathbb{Q}) \cong H_{*+n}(W, \partial W; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[u, u^{-1}].$$

(d) *With coefficients in a field $\mathfrak{k}$, the Goodwillie version $\underleftarrow{\underrightarrow{H\,T}}(W; \mathfrak{k})$ agrees with the localization of the S^1-equivariant symplectic homology $SH^{S^1}(W; \mathfrak{k})$.*

REMARK. The hypothesis $c_1(W) = 0$ provides us with well-defined integer gradings and allows us to work over the ring $\mathbb{Z}[u, u^{-1}]$. We expect that Theorem 1.1 continues to hold without this hypothesis if we drop the integer grading and replace $\mathbb{Z}[u, u^{-1}]$ by a suitable Novikov completion.

Let us discuss some examples of symplectic Tate homology groups.

(1) As an immediate consequence of Theorem 1.1(d), for a Liouville domain with vanishing equivariant symplectic homology we have

$$\underleftarrow{\underrightarrow{H\,T}}(W; \mathfrak{k}) = 0.$$

For example, this applies to $\mathbb{C}^n$, or more generally to subcritical Stein domains [7, 11]. More generally, suppose that a Liouville domain $\widetilde{W}$ is obtained from a Liouville domain W by attaching subcritical handles. Then the restriction map induces an isomorphism $SH^{S^1}(\widetilde{W}; \mathfrak{k}) \cong SH^{S^1}(W; \mathfrak{k})$ (this follows from [11]; see also [8, 16]). Hence the Goodwillie versions of symplectic Tate homology satisfy

$$\underleftarrow{\underrightarrow{H\,T}}(\widetilde{W}; \mathfrak{k}) \cong \underleftarrow{\underrightarrow{H\,T}}(W; \mathfrak{k}).$$

Theorem 1.1(c) shows that this fails for the Jones–Petrack version because the singular homology groups $H_{*+n}(\widetilde{W}, \partial\widetilde{W}; \mathbb{Q})$ and $H_{*+n}(W, \partial W; \mathbb{Q})$ are in general different.

(2) The fixed point property for the Jones–Petrack version $\underrightarrow{\underleftarrow{H\,T}}$ may fail if the coefficients are not $\mathbb{Q}$. For example, with integer coefficients we have (see Proposition 5.1)

$$\underrightarrow{\underleftarrow{H\,T}}(\mathbb{C}^n) \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[u, u^{-1}].$$

(3) The Goodwillie version $\underleftarrow{\underrightarrow{H\,T}}(W; \mathbb{Q})$ does in general not have the fixed point property. This observation is due to Goodwillie [28]. For a closed Riemannian manifold (N, g), denote by D^*N the unit disk cotangent bundle of N with its canonical structure as a Liouville domain. Goodwillie's theorem implies the following theorem.

THEOREM 1.2. *Assume that N is a closed, simply connected, spin manifold. Then $\underleftarrow{\underrightarrow{H\,T}}(D^*N; \mathbb{Q}) = \mathbb{Q}[u, u^{-1}]$.*

Proof. The equivariant symplectic homology of D^*N equals the equivariant homology $H^{S^1}(LN)$ of the free loop space of N; see [**1**, **37**, **42**]. Here the spin assumption is needed to make coherent orientations work out correctly [**2**, **34**]. By Theorem 1.1(d), we conclude that $\underleftarrow{H}\,\underrightarrow{T}(D^*N)$ equals the localization of $H^{S^1}(LN)$. By Goodwillie's theorem [**28**], the localization of $H^{S^1}(LN)$ with rational coefficients only depends on the fundamental group. Since N is simply connected, we conclude that $\underleftarrow{H}\,\underrightarrow{T}(D^*N;\mathbb{Q})$ equals the localized equivariant loop space homology of a point and the result follows. $\qquad\square$

REMARK. For symplectic homology with twisted coefficients as in [**2**], Theorem 1.2 continues to hold without the spin hypothesis on N.

(4) The map κ in Theorem 1.1(b) need in general neither be injective nor surjective; see Example 5.5. If κ has non-trivial kernel, then there exist infinite chains of periodic orbits connected by Floer cylinders with action going to infinity; see Section 5 for examples of this phenomenon.

The main objective of the current paper is to lay the foundations for symplectic Tate homology, establish its algebraic properties, and compute it for some examples. However, our interest in this construction is motivated by embedding questions and dynamical applications. For example, Viterbo observed in [**42**, p. 1020] that Goodwillie's theorem implies that the unit disk cotangent bundle D^*N of a simply connected closed manifold satisfies the 'strong equivariant algebraic Weinstein conjecture'; namely, the fundamental class of N becomes zero in localized equivariant symplectic homology of D^*N with rational coefficients. As a consequence, every closed contact-type hypersurface in D^*N carries a closed characteristic. Another consequence should be finiteness of the Hofer–Zehnder capacity of D^*N; see [**31**] for such an argument in a slightly different context. Further potential applications concern the construction of finite energy planes and establishing uniruledness of Liouville domains in the sense of McLean [**35**].

2. *Algebraic preliminaries*

In this section, we collect some algebraic facts, most of which can be found in [**14**]. For the applications, we have in mind we have to work in the category of graded $\mathbb{Z}$-modules. For simplicity, we will skip the reference to the grading.

2.1. *Direct and inverse limits*

A *direct system* is a tuple (G,π), where $G = \{G_a\}_{a\in\mathbb{R}}$ is a family of $\mathbb{Z}$-modules indexed by the real numbers, and $\pi = \{\pi_{a_2,a_1}\}_{a_1\leqslant a_2}$ is a family of homomorphisms

$$\pi_{a_2,a_1}\colon G_{a_1} \longrightarrow G_{a_2}$$

satisfying

$$\pi_{a,a} = \mathrm{id}|_{G_a}, \quad \pi_{a_3,a_1} = \pi_{a_3,a_2} \circ \pi_{a_2,a_1}, \quad a_1 \leqslant a_2 \leqslant a_3.$$

We denote its *inverse limit* as $a \to -\infty$ by

$$\underleftarrow{\lim} G = \varprojlim_a G_a.$$

For $a \in \mathbb{R}$, we have canonical maps

$$\pi_a\colon \varprojlim G \longrightarrow G_a$$

satisfying

$$\pi_{a_2} = \pi_{a_2,a_1} \circ \pi_{a_1}, \quad a_1 \leqslant a_2.$$

In order to make the notation more easily adaptable to our later purposes, we denote the direct system by (G, ι) with homomorphisms

$$\iota^{b_2,b_1} \colon G^{b_1} \longrightarrow G^{b_2}$$

when we talk about direct limits. We denote its *direct limit* as $b \to \infty$ by

$$\varinjlim G = \varinjlim_b G^b.$$

For $b \in \mathbb{R}$, we have canonical maps

$$\iota^b \colon G^b \longrightarrow \varinjlim G$$

satisfying

$$\iota^{b_1} = \iota^{b_2} \circ \iota^{b_2,b_1}, \quad b_1 \leqslant b_2.$$

Direct and inverse limits do not necessarily commute. To describe this, we consider a double indexed family of $\mathbb{Z}$-modules G_a^b with $a, b \in \mathbb{R}$. We suppose that, for every $b \in B$ and every $a_1 \leqslant a_2 \in A$, there exists a homomorphism

$$\pi_{a_2,a_1}^b \colon G_{a_1}^b \longrightarrow G_{a_2}^b$$

and, for every $a \in A$ and $b_1 \leqslant b_2 \in B$, there exists a homomorphism

$$\iota_a^{b_2,b_1} \colon G_a^{b_1} \longrightarrow G_a^{b_2}$$

such that the following holds. For any fixed $b \in B$ and any fixed $a \in A$ the tuples (G^b, π^b) and (G_a, ι_a) are direct systems. Moreover, for every $a_1 \leqslant a_2$ and $b_1 \leqslant b_2$ the square

$$\begin{array}{ccc}
G_{a_1}^{b_1} & \xrightarrow{\pi_{a_2,a_1}^{b_1}} & G_{a_2}^{b_1} \\
{\scriptstyle \iota_{a_1}^{b_1,b_2}} \downarrow & & \downarrow {\scriptstyle \iota_{a_2}^{b_2,b_1}} \\
G_{a_1}^{b_2} & \xrightarrow[\pi_{a_2,a_1}^{b_2}]{} & G_{a_2}^{b_2}
\end{array} \tag{1}$$

is commutative. We refer to the triple (G, π, ι) as a *bidirect system*. We denote the direct (respectively, inverse) limits for fixed a (respectively, b) by

$$\varinjlim G_a = \varinjlim_b G_a^b, \quad \varprojlim G^b = \varprojlim_a G_a^b.$$

Due to the commutation relation between π and ι, for $a_1 \leqslant a_2 \in A$ and $b_1 \leqslant b_2 \in B$ we obtain induced maps

$$\pi_{a_2,a_1} \colon \varinjlim G_{a_1} \longrightarrow \varinjlim G_{a_2}, \quad \iota^{b_2,b_1} \colon \varprojlim G^{b_1} \longrightarrow \varprojlim G^{b_2}$$

that make $(\varinjlim G, \pi)$ and $(\varprojlim G, \iota)$ direct systems.

PROPOSITION 2.1 (see [**14**]). *For a bidirect system (G, π, ι) there exists, for every $b \in B$, a unique homomorphism*

$$\kappa^b \colon \varprojlim G^b \longrightarrow \varprojlim \varinjlim G$$

and a unique homomorphism

$$\kappa \colon \varinjlim \varprojlim G \longrightarrow \varprojlim \varinjlim G.$$

such that, for each $a \in A$ and $b \in B$, the following diagram commutes:

$$
\begin{array}{ccccc}
G_a^b & \xleftarrow{\pi_a^b} & \varprojlim G^b & \xrightarrow{\iota^b} & \varinjlim\varprojlim G \\
\downarrow{\iota_a^b} & & \vdots\,\exists!\,\kappa^b & & \exists!\,\kappa \\
\varinjlim G_a & \xleftarrow{\pi_a} & \varprojlim\varinjlim G & &
\end{array}
$$

In general, the map κ is neither injective nor surjective; we will see examples of this in Section 5. Conditions under which the canonical homomorphism κ is an isomorphism were obtained by Eckmann and Hilton [**18**] and by Frei and Macdonald [**26**].

2.2. Bidirect systems of chain complexes

A *bidirect system of chain complexes* is a quadruple (C, p, i, ∂), where (C, p, i) is a bidirect system which in addition is endowed, for each $a \in A$ and $b \in B$, with a boundary operator

$$\partial_a^b \colon C_a^b \longrightarrow C_a^b,$$

which commutes with i and p in the sense that

$$p_{a_2,a_1}^b \circ \partial_{a_1}^b = \partial_{a_2}^b \circ p_{a_2,a_1}^b \tag{2}$$

and

$$i_a^{b_2,b_1} \circ \partial_a^{b_1} = \partial_a^{b_2} \circ i_a^{b_2,b_1}. \tag{3}$$

If

$$HC_a^b = \frac{\ker \partial_a^b}{\operatorname{im} \partial_a^b}$$

are the homology groups, and Hp_{a_2,a_1}^b and $Hi_a^{b_2,b_1}$ are the induced maps on homology, then the triple (HC, Hp, Hi) is a bidirect system. As in the previous subsection, we let

$$\kappa \colon \varinjlim\varprojlim HC \longrightarrow \varprojlim\varinjlim HC$$

be the canonical homomorphism on homology level. We refer to

$$k \colon \varinjlim\varprojlim C \longrightarrow \varprojlim\varinjlim C$$

as the canonical homomorphism on chain level. Since ∂ commutes with i and p, we obtain an induced map

$$Hk \colon H(\varinjlim\varprojlim C) \longrightarrow H(\varprojlim\varinjlim C).$$

PROPOSITION 2.2. (a) *Let (C, p, i, ∂) be a bidirect system of chain complexes of $\mathbb{Z}$-modules such that all the maps p_{a_2,a_1}^b are surjective. Then there is a canonical diagram*

$$
\begin{array}{ccc}
H(\varinjlim\varprojlim C) & \xrightarrow{\rho} & \varinjlim\varprojlim HC \\
\downarrow{Hk} & & \downarrow{\kappa} \\
H(\varprojlim\varinjlim C) & \xrightarrow{\sigma} & \varprojlim\varinjlim HC
\end{array}
\tag{4}
$$

with σ surjective.

(b) *If instead of $\mathbb{Z}$-modules the C_a^b are graded vector spaces that are finite-dimensional in each degree, then the diagram commutes and ρ is an isomorphism.*

Proof.   This follows by combining several results in [**14**]. However, as everything there is stated in the category of vector spaces, we need to check which parts carry over to $\mathbb{Z}$-modules. Let us indicate the required modifications, using the notation from [**14**].

Theorem 3.7 from [**14**] is proved in [**46**] for $\mathbb{Z}$-modules. Using this, it follows that our bidirect system (C, p, i, ∂) satisfies the 'tameness' condition in [**14**], except that the map ν^b is only surjective rather than an isomorphism. Moreover, the argument from the end of proof of Theorem A still works to show that ν is surjective. With this, the discussion after Definition 3.3 still yields the desired diagram and surjectivity of the map σ. If instead of $\mathbb{Z}$-modules the C_a^b are graded vector spaces that are finite-dimensional in each degree, then our bidirect system (C, p, i, ∂) is actually 'tame', and Proposition 3.4 from [**14**] implies that the diagram commutes and ρ is an isomorphism.   □

REMARK 2.3.   (a) In our applications the spaces C_a^b appear as sublevel sets (for b) and quotients (for a) with respect to two real filtrations on a chain complex (C, ∂). In [**14**], we considered the special case that the two filtrations are equal and showed that then the map Hk in (4) is an isomorphism. We will see below that this fails in the case of two different filtrations. This failure can be traced to the failure of the commutative square (1) to be exact in the sense of [**26**].

(b) The map ρ in Proposition 2.2 is in general not an isomorphism for finitely generated $\mathbb{Z}$-modules. A counterexample is given in Appendix A of [**14**]. This failure can be traced to the failure of the Mittag-Leffler condition due to the existence of infinite chains of subrings such as $\mathbb{Z} \supset 2\mathbb{Z} \supset 4\mathbb{Z} \supset \cdots$.

REMARK 2.4.   Suppose that we are given a short exact sequence $0 \to A \to B \to C \to 0$ of bidirect systems of chain complexes of graded vector spaces as in Proposition 2.2(b). Since direct limits preserve exactness, and inverse limits preserve exactness for finite-dimensional vector spaces [**19**], we can apply either first the inverse limit, then the direct limit and then homology, or first homology, then the inverse limit and then the direct limit to obtain long exact sequences fitting in the following commuting diagram with the map ρ from (4):

$$\cdots \longrightarrow H(\varinjlim\varprojlim A) \longrightarrow H(\varinjlim\varprojlim B) \longrightarrow H(\varinjlim\varprojlim C) \longrightarrow \cdots \qquad (5)$$
$$\cong \Big\downarrow \rho_A \qquad\qquad \cong \Big\downarrow \rho_B \qquad\qquad \cong \Big\downarrow \rho_C$$
$$\cdots \longrightarrow \varinjlim\varprojlim HA \longrightarrow \varinjlim\varprojlim HB \longrightarrow \varinjlim\varprojlim HC \longrightarrow \cdots$$

2.3. *Module structure over Laurent polynomials*

In our applications, we will have, for all a, b additional isomorphisms,

$$u_a^b : C_a^b \xrightarrow{\cong} C_{a+1}^b$$

commuting with all the other maps in the obvious sense. By functoriality of direct and inverse limits and homology, the u_a^b induce isomorphisms on the four groups in (4) commuting with the maps in this diagram. Denoting these isomorphisms simply by u, we thus have the following corollary.

COROLLARY 2.5.   *Additional isomorphisms u_a^b as above make the four groups in (4) modules over the ring $\mathbb{Z}[u, u^{-1}]$ of Laurent polynomials in u and the maps in this diagram module homomorphisms.*

REMARK 2.6.   (a) In our applications the u_a^b will shift gradings by 2.

(b) For a field $\mathfrak{k}$, the ring $\mathfrak{k}[u, u^{-1}]$ is a principal ideal domain. To see this, let I be an ideal in $\mathfrak{k}[u, u^{-1}]$. Then $I \cap \mathfrak{k}[u]$ is an ideal in the principal ideal domain $\mathfrak{k}[u]$ and thus generated by one polynomial p. Now, for any $q \in I$ we have $u^n q \in I \cap \mathfrak{k}[u]$ for some large integer n, so $u^n q = rp$ for some $r \in \mathfrak{k}[u]$ and thus $q = u^{-n} rp$.

(c) By [**32**, Subsection 3.9, Exercise 3], modules over a principal ideal domain have a well-defined rank satisfying the rank formula for a submodule $N \subset M$:

$$\operatorname{rank} M = \operatorname{rank} N + \operatorname{rank} M/N.$$

2.4. Localization

In this subsection, we work in the category of vector spaces. Assume that V is a vector space over a field $\mathfrak{k}$ and $T \colon V \to V$ is a linear map. We define $V^T \subset \prod_{i \in \mathbb{N}} V$ to be the subvector space consisting of $(v_i)_{i \in \mathbb{N}}$ satisfying $T v_{i+1} = v_i$. If one thinks of the pair (V, T) as a direct system with $V^k = V$ for $k \in \mathbb{N}$ and $T^k = T \colon V^k \to V^{k-1}$ for $k \geqslant 2$, then we can think of V^T as the inverse limit

$$V^T = \varprojlim_k V^k. \tag{6}$$

(Note that here the notation differs from Subsection 2.1 in that k tends to $+\infty$ rather than $-\infty$.) The map T induces a linear map $\overline{T} \colon V^T \to V^T$ given by

$$\overline{T}(v_i)_{i \in \mathbb{N}} = (T v_i)_{i \in \mathbb{N}}.$$

Although T was not assumed to be invertible, the map $\overline{T}$ is invertible with inverse given by

$$\overline{T}^{-1} \colon V^T \longrightarrow V^T, \quad (v_i)_{i \in \mathbb{N}} \longmapsto (v_{i+1})_{i \in \mathbb{N}}.$$

In particular, V^T becomes a module over the ring of Laurent polynomials $\mathfrak{k}[u, u^{-1}]$, where u acts via $\overline{T}$ on V^T. We call the $\mathfrak{k}[u, u^{-1}]$-module V^T the *localization* of V with respect to T.

There is a natural map

$$P = P_T \colon V^T \longrightarrow V, \quad (v_i)_{i \in \mathbb{N}} \longmapsto v_1.$$

Note that P interchanges the maps $\overline{T}$ and T

$$TP = P\overline{T}.$$

LEMMA 2.7. *If T is an isomorphism, then P is an isomorphism as well.*

Proof. We first show that P is injective. To see this, suppose that $(v_i)_{i \in \mathbb{N}}$ is in the kernel of P. This means that $v_1 = 0$. By induction on the formula $v_i = T v_{i+1}$ and using the injectivity of T, we conclude that $v_i = 0$ for every $i \in \mathbb{N}$. This shows that P is injective. To see that P is surjective, let $v \in V$. Since T is an isomorphism, the element $(T^{-i+1} v)_{i \in \mathbb{N}}$ exists in V^T. But $P(T^{-i+1} v)_{i \in \mathbb{N}} = v$. This proves surjectivity of P, and hence the lemma. $\square$

In particular, Lemma 2.7 implies that if T is an isomorphism, then the pair (V, T) is naturally identified with the pair $(V^T, \overline{T})$ via the map P.

LEMMA 2.8. *Assume that V is finite-dimensional. Then P is injective.*

Proof. Assume that $v = (v_i)_{i \in \mathbb{N}} \in V^T$ lies in the kernel of P, that is, $v_1 = 0$. We suppose by contradiction that $v \neq 0$. Hence there exists $i \in \mathbb{N}$ satisfying $v_i \neq 0$. Let $i_0 > 1$ be the minimal positive integer with this property. Hence, for every $i \geqslant i_0$ the vector v_i has the property that $T^j v_i \neq 0$ for $0 \leqslant j \leqslant i - i_0$ but $T^{i-i_0+1} v_i = 0$. This implies that the vectors v_i for $i \geqslant i_0$ are

linearly independent. But this contradicts the assumption that V is finite-dimensional and the lemma is proved. □

EXAMPLE 2.9.   In infinite dimensions, Lemma 2.8 is far from true. Consider, for example, a vector space V of infinite countable dimension with basis $\{e_i\}_{i \in \mathbb{N}}$. Let $T \colon V \to V$ be the shift operator

$$Te_i := \begin{cases} e_{i-1} & i > 1, \\ 0 & i = 1. \end{cases}$$

A basis for V^T is given by $\overline{e}_j$, $j \in \mathbb{Z}$, defined by

$$(\overline{e}_j)_i := \begin{cases} e_{i+j} & i + j \geqslant 1, \\ 0 & i + j \leqslant 0. \end{cases}$$

The induced linear operator becomes the shift operator

$$\overline{T}\overline{e}_j = \overline{e}_{j-1}, \quad j \in \mathbb{Z}.$$

The kernel of P is spanned by the vectors $\overline{e}_j$ for $j < 0$, so P is not injective.

On V^T we define a descending filtration as follows. If $v = (v_i)_{i \in \mathbb{N}}$, then we set

$$|v| := \min\{i \in \mathbb{N} : v_i \neq 0\}$$

with the convention that the minimum of the empty set equals ∞, that is, $|0| = \infty$. For $k \in \mathbb{N}$, we set

$$Z_k^T := \{v \in V^T \mid |v| > k\} = \{v \in V^T \mid v_1 = \cdots = v_k = 0\}.$$

Note that Z_k^T is $\overline{T}$-invariant. Therefore, if

$$V_k^T := V^T / Z_k^T$$

is the quotient vector space, then $\overline{T}$ induces a map

$$T_k \colon V_k^T \longrightarrow V_k^T.$$

There is a well-defined map

$$Q_k \colon V_k^T \longrightarrow V$$

which is given for $[v] = [(v_i)_{i \in \mathbb{N}}] \in V_k^T$ by

$$Q_k[v] := v_k.$$

Note that Q_k is injective and satisfies

$$TQ_k = Q_k T_k.$$

We next describe its image. For this purpose set

$$V_T := \bigcap_{j \in \mathbb{N}} T^j V \subset V.$$

Observe that V_T is a T-invariant subspace of V on which T acts surjectively. In fact, V_T is the largest T-invariant subspace of V on which T acts surjectively. In particular, $V_T = V$ if and only if T is surjective. Moreover,

$$\operatorname{im} Q_k = V_T.$$

It is worth to mention that the image is independent of $k \in \mathbb{N}$.

Since $Z_k^T \subset Z_{k-1}^T$, we get projection maps

$$\pi_k \colon V_k^T \longrightarrow V_{k-1}^T$$

and hence a direct system of vector spaces (V^T, π). If $v = [(v_i)_{i \in \mathbb{N}}] \in V_k^T$, then we compute

$$Q_{k-1}\pi_k[v] = v_{k-1} = Tv_k = TQ_k[v],$$

so the following diagram commutes:

$$
\begin{array}{ccc}
V_k^T & \xrightarrow{\ \pi_k\ } & V_{k-1}^T \\
{\scriptstyle \cong}\Big\downarrow {\scriptstyle Q_k} & & {\scriptstyle \cong}\Big\downarrow {\scriptstyle Q_{k-1}} \\
V_T & \xrightarrow{\ T\ } & V_T
\end{array}
$$

We denote by $V_T^T := (V_T)^T$ the localization of V_T with respect to the map $T|_{V_T}$, defined by the inverse limit (6). So the preceding commuting diagram yields the following lemma.

LEMMA 2.10. *The maps Q_k give rise to an isomorphism*

$$Q : \varprojlim V_k^T \longrightarrow V_T^T$$

which interchanges $\varprojlim T_k$ and $\overline{T}$. In particular, the spaces $\varprojlim V_k^T$ and V_T^T are isomorphic as $\mathfrak{k}[u, u^{-1}]$-modules.

We further have

$$V_T^T = V^T. \tag{7}$$

Indeed, $V_T^T \subset V^T$ is clear. On the other hand, if $v = (v_i)_{i \in \mathbb{N}} \in V^T$, then the condition $Tv_{i+1} = v_i$ implies that $v_i \in \bigcap_{j=1}^{\infty} T^j V = V_T$ for every $i \in \mathbb{N}$ and therefore $v \in V_T^T$. This proves (7). We point out that if T is surjective, then (7) is actually trivial because in this case we already have $V_T = V$.

As a consequence of Lemma 2.10 and equation (7), we get the following corollary.

COROLLARY 2.11. *The spaces $\varprojlim V_k^T$ and V^T are isomorphic as $\mathfrak{k}[u, u^{-1}]$-modules.*

Next, we will carry over the preceding discussion from vector spaces to chain complexes.

DEFINITION 2.12. A *Tate triple* (V, T, ∂) consists of a vector space V, a linear map $T : V \to V$, and a boundary operator $\partial : V \to V$ commuting with T.

Let (V, T, ∂) be a Tate triple. We define a boundary operator $\overline{\partial}$ on V^T by setting, for $(v_i)_{i \in \mathbb{N}} \in V^T$,

$$\overline{\partial}(v_i)_{i \in \mathbb{N}} := (\partial v_i)_{i \in \mathbb{N}}.$$

Note that $\overline{\partial}$ commutes with $\overline{T}$ and, for every $k \in \mathbb{N}$, the subspace V_k^T is invariant under $\overline{\partial}$. Therefore, $\overline{\partial}$ induces boundary operators

$$\partial_k : V_k^T \longrightarrow V_k^T$$

which commute with the maps T_k. Note further that V_T is invariant under ∂ and the maps Q_k identify the triples (V_k^T, T_k, ∂_k) and (V_T, T, ∂). In particular, the map Q_k induces an isomorphism on homology

$$HQ_k : H(V_k^T, \partial_k) \longrightarrow H(V_T, \partial)$$

which interchanges the induced maps

$$HT_k : H(V_k^T, \partial_k) \longrightarrow H(V_k^T, \partial_k), \quad HT : H(V_T, \partial) \longrightarrow H(V_T, \partial).$$

Finally, we observe that the maps $\pi_k \colon V_k^T \to V_{k-1}^T$ interchange the boundary operators ∂_k and ∂_{k-1}, so that we obtain a direct system $(H(V_k^T, \partial_k), H\pi_k)$. So we have proved the following generalization of Lemma 2.10, where $H(V_T, \partial)^{HT}$ denotes the localization of $H(V_T, \partial)$ with respect to the map HT.

LEMMA 2.13.   *Assume that (V, T, ∂) is a Tate triple. Then the maps HQ_k give rise to an isomorphism*

$$HQ \colon \varprojlim H(V_k^T, \partial_k) \longrightarrow H(V_T, \partial)^{HT}$$

which interchanges $\varprojlim HT_k$ and $\overline{HT}$. In particular, the spaces $\varprojlim H(V_k^T, \partial_k)$ and $H(V_T, \partial)^{HT}$ are isomorphic as $\mathfrak{k}[u, u^{-1}]$-modules.

Recall that if T is surjective, then $V_T = V$. Therefore, we obtain the following corollary.

COROLLARY 2.14.   *Assume that (V, T, ∂) is a Tate triple and $T \colon V \to V$ is surjective. Then the spaces $\varprojlim H(V_k^T, \partial_k)$ and $H(V, \partial)^{HT}$ are isomorphic as $\mathfrak{k}[u, u^{-1}]$-modules.*

Without the surjectivity assumption the assertion of Corollary 2.14 can fail, despite the fact that $V_T^T = V^T$ by (7). The following example describes such a scenario.

EXAMPLE 2.15.   Let V be a vector space with basis vectors $\{e_{i,j}, f_{i,j} \mid i, j \in \mathbb{N}, j \geqslant i\}$. Define T and ∂ on basis vectors by

$$Te_{i,j} := \begin{cases} e_{i-1,j} & i \geqslant 2, \\ 0 & i = 1, \end{cases} \quad Tf_{i,j} := \begin{cases} f_{i-1,j} & i \geqslant 2, \\ 0 & i = 1, \end{cases}$$

and

$$\partial f_{i,j} := e_{i,j} + e_{i,j+1}, \quad \partial e_{i,j} := 0.$$

Note that ∂ is a boundary operator that commutes with T, so that (V, T, ∂) is a Tate triple. However, the map T is not surjective. We claim that

$$V_T = \{0\}. \tag{8}$$

To see this, we first decompose V as follows. Define subspaces

$$E := \langle e_{i,j} \mid i, j \in \mathbb{N}, j \geqslant i \rangle \subset V, \quad F := \langle f_{i,j} \mid i, j \in \mathbb{N}, j \geqslant i \rangle \subset V.$$

Note that

$$V = E \oplus F$$

and both subspaces are T-invariant. We therefore have

$$V_T = E_T \oplus F_T.$$

Moreover, observe that the linear map $\Phi \colon E \to F$ which is given on basis vectors by $\Phi(e_{i,j}) = f_{i,j}$ is a T-equivariant isomorphism between E and F. Therefore, E_T is isomorphic to F_T and we are left with showing that $E_T = \{0\}$. Indeed, for every $k \in \mathbb{N}$ we have

$$T^k E = \langle e_{i,j} \mid i, j \in \mathbb{N}, j \geqslant i + k \rangle$$

and therefore

$$E_T = \bigcup_{k \in \mathbb{N}} T^k E = \{0\}.$$

This completes the proof of (8). In view of Lemma 2.13, we deduce from (8) that

$$\varprojlim H(V_k^T, \partial_k) = \{0\}.$$

To compute $H(V, \partial)^{HT}$, we first describe $H(V, \partial)$. Note that, for every $i \in \mathbb{N}$, the vector $e_{i,i}$ gives rise to a non-trivial homology class

$$\epsilon_i := [e_{i,i}],$$

which coincides with $[e_{i,j}]$ for every $j \geqslant i$. The homology becomes

$$H(V, \partial) = \langle \epsilon_i \mid i \in \mathbb{N} \rangle,$$

and the induced map becomes the shift operator

$$HT\epsilon_i = \begin{cases} \epsilon_{i-1} & i > 1, \\ 0 & i = 1. \end{cases}$$

We conclude that $H(V, \partial)^{HT}$ is isomorphic to the free $\mathfrak{k}[u, u^{-1}]$-module on one generator. In particular, $\varprojlim H(V_k^T, \partial_k)$ and $H(V, \partial)^{HT}$ are not isomorphic.

REMARK. We say that a Tate triple (V, T, ∂) is *graded of degree d* if the vector space V is additionally graded, $T \colon V \to V$ is a map of degree $\deg(T) = d$, and the boundary operator is a map of degree $\deg(\partial) = -1$. We can define a grading on V^T as well by setting for a non-zero $v = (v_i)_{i \in \mathbb{N}}$

$$\deg(v) = \deg(v_i) + id,$$

for an arbitrary $i \in \mathbb{N}$ satisfying $v_i \neq 0$. Note that this well-defined, that is, independent of the choice of i. In this setup, Lemma 2.13 is true in the graded sense, where the ring of Laurent polynomials $\mathfrak{k}[u, u^{-1}]$ also has to be graded with $\deg(u) = d$.

3. S^1-equivariant Tate homology

3.1. Rabinowitz Floer homology on $\mathbb{C}$

To define S^1-equivariant Tate homology via the Borel construction, we look for a space with a free S^1-action and an invariant Morse function whose Morse homology vanishes in all degrees. Such a space can be described as follows. Consider the Hilbert space

$$\mathscr{L}_{\mathbb{C}} := \left\{ z = (z_k)_{k \in \mathbb{Z}} \mid z_k \in \mathbb{C}, \ \sum_{k \in \mathbb{Z}} (1 + k^2)|z_k|^2 < \infty \right\}$$

with the Morse function $\mathcal{A} : \mathscr{L}_{\mathbb{C}} \to \mathbb{R}$,

$$\mathcal{A}(z) := \pi \sum_{k \in \mathbb{Z}} k|z_k|^2.$$

(The reason for the normalization constant π will become clear in a moment.) The action of the circle $S^1 = \mathbb{R}/\mathbb{Z}$ on $\mathscr{L}_{\mathbb{C}}$ via

$$(\tau \cdot z)_k := e^{2\pi i \tau} z_k$$

leaves $\mathcal{A}$ invariant and is free on the infinite-dimensional sphere

$$S_\infty^\infty := \left\{ z \in \mathscr{L}_{\mathbb{C}} \mid \|z\|^2 := \sum_{k \in \mathbb{Z}} |z_k|^2 = 1 \right\} \subset \mathscr{L}_{\mathbb{C}}.$$

The function $\mathcal{A}$ descends to a Morse function on the quotient $\mathbb{C}P_\infty^\infty = S_\infty^\infty/S^1$ whose critical points $[z^{(\ell)}]$, $\ell \in \mathbb{Z}$, are given by

$$|z_\ell^{(\ell)}| = 1, \quad z_k^{(\ell)} = 0 \quad \text{for } k \neq \ell.$$

Note that the critical points have infinite index and coindex and critical values $\mathcal{A}(z^{(\ell)}) = \pi\ell$. The space $\mathscr{L}_{\mathbb{C}}$ also carries a natural $\mathbb{Z}$-action given by the shifts

$$(n * z)_k := z_{k-n}.$$

This action preserves S^∞_∞, commutes with the S^1-action, and satisfies

$$\mathcal{A}(n * z) = \mathcal{A}(z) + n\pi\|z\|^2.$$

In particular, $\mathbb{Z}$ acts on the critical points on S^∞_∞ by

$$n * z^{(\ell)} = z^{(\ell+n)}.$$

Identifying elements of $\mathscr{L}_{\mathbb{C}}$ with Fourier series

$$z(t) = \sum_{k \in \mathbb{Z}} z_k \, e^{2\pi i k t},$$

we see that $\mathscr{L}_{\mathbb{C}}$ corresponds to the Sobolev space $W^{1,2}(S^1, \mathbb{C})$ and $\|\ \|$ to the L^2-norm

$$\|z\|^2 = \int_0^1 |z(t)|^2 \, dt.$$

The function $\mathcal{A}$ is the classical symplectic action

$$\mathcal{A}(z) = \int_0^1 z^* \lambda_{\mathbb{C}} = -\frac{1}{2} \int_0^1 \mathrm{Im}(z\dot{\bar{z}}) \, dt, \quad \lambda_{\mathbb{C}} = \frac{1}{2}(x\, dy - y\, dx),$$

and the actions of S^1 and $\mathbb{Z}$ are given by

$$\tau \cdot z(t) = e^{2\pi i \tau} z(t), \quad n * z(t) = e^{2\pi i n t} z(t).$$

Note that the circle action on $\mathscr{L}_{\mathbb{C}}$ is induced by the circle action on the target $\mathbb{C}$ of the loop space and *not* on the domain S^1. The reason is that this action is free on the sphere S^∞_∞, while the action coming from circle action on the domain is not.

The restriction of $\mathcal{A}$ to the sphere S^∞_∞ can be conveniently described in terms of the Rabinowitz action functional for the unit circle in $\mathbb{C}$,

$$\mathcal{A}^\mu \colon \mathscr{L}_{\mathbb{C}} \times \mathbb{R} \longrightarrow \mathbb{R},$$

$$\mathcal{A}^\mu(z, \eta) := \int z^* \lambda_{\mathbb{C}} - \eta \int \mu(z) \, dt.$$

Here $\eta \in \mathbb{R}$ is a Lagrange multiplier and

$$\mu \colon \mathbb{C} \longrightarrow \mathbb{R}, \quad z \longmapsto \pi(|z|^2 - 1)$$

is the moment map for the standard circle action $(t, z) \mapsto e^{2\pi i t} z$ on $\mathbb{C}$. The critical points of $\mathcal{A}^\mu$ appear in critical circles obtained by applying the circle action to the pairs

$$w^{(\ell)} := (z^{(\ell)}, \ell), \quad z^{(\ell)}(t) = e^{2\pi i \ell t}.$$

They have actions $\mathcal{A}^\mu(z^{(\ell)}, \ell) = \pi\ell$. The Rabinowitz action functional is invariant under the circle action $\tau \cdot (z, \eta) = (\tau \cdot z, \eta)$, and with respect to the $\mathbb{Z}$-action

$$n * (z, \eta) := (n * z, \eta + n)$$

it satisfies

$$\mathcal{A}^\mu(n * z, \eta + n) = \mathcal{A}^\mu(z, \eta) + \pi n.$$

The gradient of $\mathcal{A}^\mu$ with respect to the L^2-metric on $\mathscr{L}_{\mathbb{C}}$ and the standard metric on $\mathbb{R}$ is given by

$$\nabla \mathcal{A}^\mu(z, \eta) = (-i\dot{z} - 2\pi\eta z, -\pi(\|z\|^2 - 1)).$$

Thus (positive) gradient flow lines of $\mathcal{A}^\mu$ are maps $(z, \eta): \mathbb{R} \to \mathscr{L}_\mathbb{C} \times \mathbb{R}$ whose Fourier coefficients satisfy the following system of ordinary differential equations, where $'$ denotes the derivative with respect to $s \in \mathbb{R}$:

$$\begin{cases} z_k' = 2\pi(k - \eta)z_k, \\ \eta' = -\pi(\|z\|^2 - 1). \end{cases} \tag{9}$$

Note that the subspaces where some of the z_k are zero are invariant under the gradient flow. In particular, gradient flow lines connecting two critical points never pass through the constant loop $z \equiv 0$, so they remain in the region $\mathscr{L}_\mathbb{C}^* := \mathscr{L}_\mathbb{C} \setminus \{0\}$ where the S^1-action is free and can be projected to $S_\infty^\infty \times \mathbb{R}$ by normalization. Since both the action functional and the metric are S^1-invariant, gradient flow lines descend to the quotient by S^1. Moreover, since the $\mathbb{Z}$-action preserves the metric and changes the action functional only by additive constants, we get a $\mathbb{Z}$-action on gradient flow lines which we will describe next.

Note first that, for integers $m \leqslant n$, the intersection of the stable manifold (with respect to the negative gradient flow) of $[w^{(m)}]$ and the unstable manifold of $[w^{(n)}]$ projects onto the $2(n - m)$-dimensional projective subspace

$$\mathbb{C}P_m^n := \{[z] \mid z_k = 0 \text{ for all } k < m \text{ and for all } k > n\} \subset \mathbb{C}P_\infty^\infty.$$

Moreover, according to [**24**, Proposition A.2], the indices of a Lagrange multiplier functional differ from the indices of the function restricted to the constraint hypersurface only by a global shift. Hence the critical points $[w^{(m)}]$ and $[w^{(n)}]$ have finite index difference $2(n - m)$. Note that the $\mathbb{C}P_m^n$ are closed submanifolds of $\mathbb{C}P_\infty^\infty$ satisfying

$$\mathbb{C}P_m^n \cap \mathbb{C}P_{m'}^{n'} = \mathbb{C}P_{\max(m,m')}^{\min(n,n')},$$

and the action by $\ell \in \mathbb{Z}$ maps $\mathbb{C}P_m^n$ onto $\mathbb{C}P_{m+\ell}^{n+\ell}$.

The *equivariant Rabinowitz Floer complex on $\mathbb{C}$* (with $\mathbb{Z}$-coefficients) is the Floer chain complex of the functional $\mathcal{A}^\mu$ on the quotient space $(\mathscr{L}_\mathbb{C}^* \times \mathbb{R})/S^1$: Its chain group FC^{S^1} is the free $\mathbb{Z}$-module generated by the critical points $[w^{(n)}]$, $n \in \mathbb{Z}$, and its boundary operator ∂^{S^1} counts negative gradient lines between critical points of index difference one. Let us write the group ring of $\mathbb{Z}$ as the ring

$$\Lambda := \mathbb{Z}[u, u^{-1}], \quad |u| = 2$$

of Laurent polynomials in a formal variable u of degree 2. The $\mathbb{Z}$-action gives an isomorphism between Λ and the chain group FC^{S^1} by identifying u^n with the critical point $w^{(n)}$ for $n \in \mathbb{Z}$. Since the critical points $[w^{(n)}]$ and $[w^{(m)}]$ on the quotient $S_\infty^\infty/S^1 = \mathbb{C}P_\infty^\infty$ have index difference $2(n - m)$, this isomorphisms fixes gradings of the critical points compatible with the relative gradings. Since all index differences are even, the boundary operator vanishes and the equivariant Rabinowitz Floer homology equals Λ.

On the other hand, since the unit circle in $\mathbb{C}$ is displaceable by a Hamiltonian diffeomorphism, it follows from [**13**] that the non-equivariant Rabinowitz Floer homology vanishes. So we have shown the following lemma.

LEMMA 3.1. *The non-equivariant Floer homology $FH(A^\mu)$ vanishes. The equivariant Floer homology $FH^{S^1}(\mathcal{A}^\mu)$ equals $\Lambda = \mathbb{Z}[u, u^{-1}]$, the ring of Laurent polynomials in a formal variable u of degree 2.*

Let us describe the non-equivariant Rabinowitz Floer complex more explicitly. We fix a Morse function on the critical circle corresponding to $w^{(0)}$ with two critical points, the maximum $w_+^{(0)}$ and the minimum $w_-^{(0)}$. Via the $\mathbb{Z}$-action, we obtain Morse functions on the critical circles corresponding to $w^{(n)}$ with maximum $w_+^{(n)} = u^n w_+^{(0)}$ and minimum $w_+^{(n)} = u^n w_+^{(0)}$. Their

indices are

$$|w_+^{(n)}| = 2n + 1, \quad |w_-^{(n)}| = 2n,$$

and vanishing of non-equivariant Rabinowitz Floer homology implies that

$$\partial w_+^{(n)} = 0, \quad \partial w_-^{(n)} = \pm w_+^{(n-1)}.$$

Since the boundary operator decreases the action, we have, for each $b \in \mathbb{R}$, a subcomplex $FC^b \subset FC$ generated by critical points of action $\leqslant b$. For $a \leqslant b$, we denote by FC_a^b the quotient of FC^b by the subcomplex generated by critical points of action $< a$, and we set $FC_a := FC_a^\infty$. We denote the corresponding homology groups by FH^b, FH_a^b, and FH_a. The same definitions apply in the equivariant case. Then the explicit description of the chain complexes above yields the filtered Rabinowitz Floer homology groups for integers $m \leqslant n$:

$$FH_m^n(\mathcal{A}^\mu) = \mathbb{Z}w_+^{(n)} \oplus \mathbb{Z}w_-^{(m)},$$
$$FH^n(\mathcal{A}^\mu) = \mathbb{Z}w_+^{(n)}, \quad FH_m(\mathcal{A}^\mu) = \mathbb{Z}w_-^{(m)},$$
$$FH_m^{S^1,n}(\mathcal{A}^\mu) = u^m\mathbb{Z}[u]/u^{n+1}\mathbb{Z}[u],$$
$$FH^{S^1,n}(\mathcal{A}^\mu) = u^n\mathbb{Z}[u^{-1}], \quad FH_m^{S^1}(\mathcal{A}^\mu) = u^m\mathbb{Z}[u].$$

3.2. S^1-equivariant Tate homology

In this section, we give a Morse theoretic argument for Borel's localization theorem for circle actions; cf. [**5**]. Consider a closed manifold M with a circle action. Choose on M a non-negative S^1-invariant Morse–Bott function $f : M \to \mathbb{R}$ with the following properties:

(i) $f(x) = 0$ if and only if $x \in \mathrm{Fix}(S^1)$;
(ii) the components of the critical manifold of positive action consist of Bott non-degenerate critical circles on which the action is locally free and thus defines a finite cover $S^1 \to S^1$.

The existence of such a function follows from the results of Wasserman in [**44**].

We now define the S^1-equivariant Tate complex of (M, f) by a Morse theoretic version of the Borel construction using the Rabinowitz Floer complex on $\mathbb{C}$. Set $\mathscr{L}_\mathbb{C}^* := \mathscr{L}_\mathbb{C} \setminus \{0\}$ and let

$$\widehat{M} := (M \times \mathscr{L}_\mathbb{C}^* \times \mathbb{R})/S^1$$

be the quotient by the (free) diagonal circle action on $M \times \mathscr{L}_\mathbb{C}$. Since f and $\mathcal{A}^\mu$ are S^1-invariant, they descend to functions

$$\widehat{f}, \widehat{\mathcal{A}}^\mu : \widehat{M} \longrightarrow \mathbb{R}$$

that are invariant and Morse–Bott with respect to the anti-diagonal circle action. We define the S^1-equivariant Tate complex $\widehat{C}^{S^1}(M, f)$ of (M, f) as the (non-equivariant) Morse chain complex of $\widehat{f} + \widehat{\mathcal{A}}^\mu : \widehat{M} \to \mathbb{R}$. Its homology is the S^1-equivariant Tate homology $\widehat{H}^{S^1}(M, f)$.

More precisely, we pick a generic family g of metrics on the fibres of the fibration $M \to (M \times S_\infty^\infty)/S^1 \to \mathbb{C}P_\infty^\infty$ which is invariant under the $\mathbb{Z}$-action on $(M \times S_\infty^\infty)/S^1$ given by $n * [x, z] = [x, n * z]$. Note that such g can be constructed inductively over the strata $\mathbb{C}P_m^n$ of increasing dimensions. (If g is already constructed over the $\mathbb{C}P_m^n$ with $n - m < k$ in a $\mathbb{Z}$-invariant way, then we extend it arbitrarily to $\mathbb{C}P_0^k$ and then by $\mathbb{Z}$-invariance to all $\mathbb{C}P_m^n$ with $n - m = k$; this is possible because g is already defined on the intersections $\mathbb{C}P_m^n \cap \mathbb{C}P_{m+\ell}^{n+\ell} = \mathbb{C}P_{m+\ell}^n$ for all $\ell \geqslant 0$.) Consider the pullback diagram

$$
\begin{array}{ccccc}
M & \longrightarrow & (M \times \mathscr{L}_\mathbb{C}^* \times \mathbb{R})/S^1 & \longrightarrow & (\mathscr{L}_\mathbb{C}^* \times \mathbb{R})/S^1 \\
& & \downarrow & & \downarrow \\
M & \longrightarrow & (M \times S_\infty^\infty)/S^1 & \longrightarrow & \mathbb{C}P_\infty^\infty,
\end{array}
$$

where the vertical maps are induced by the normalization map $\mathscr{L}_\mathbb{C}^* \to S_\infty^\infty$, $z \mapsto z/\|z\|$. Thus g induces a family of metrics on fibres of the fibration $M \to \widehat{M} \to (\mathscr{L}_\mathbb{C}^* \times \mathbb{R})/S^1$, which combines with the pullback of the L^2-metric on $\mathscr{L}_\mathbb{C}$ and the standard metric on $\mathbb{R}$ (with respect to some choice of horizontal subspaces) to a metric on $\widehat{M}$. The boundary operator of the Morse chain complex of $\widehat{f} + \widehat{\mathcal{A}}^\mu$ counts negative gradient flow lines with respect to such a metric. Here are some properties of this construction.

(0) Since $\widehat{f} + \widehat{\mathcal{A}}^\mu$ is still invariant under the anti-diagonal S^1-action on $\widehat{M}$, its critical points appear in Bott non-degenerate families of the following types:

(i) $C \times \{[w^{(\ell)}]\}$ for a component $C \subset M$ of the fixed point set;
(ii) $(\gamma \times S^1 \cdot w^{(\ell)})/S^1$ for a critical circle $\gamma \subset M$ of f outside the fixed point set.

To define the Morse chain complex, we pick additional Morse functions on all these critical components and count cascades of negative gradient flow lines.

(1) The $\mathbb{Z}$-action on the Rabinowitz Floer chain complex induces a $\mathbb{Z}$-action on the Tate chain complex that gives Tate homology the structure of a module over $\Lambda = \mathbb{Z}[u, u^{-1}]$.

(2) Under the Tate boundary operator, both $\widehat{f}$ and $\widehat{\mathcal{A}}^\mu$ are non-increasing. Since gradient flow lines of $\widehat{f} + \widehat{\mathcal{A}}^\mu$ project onto gradient flow lines of $\mathcal{A}^\mu$, the index of critical points of $\mathcal{A}^\mu$ is also non-increasing. However, the index of critical points of f may increase due to the fact that not every metric on M in the generic family g has to be generic.

(3) The filtration by values of $\widehat{f}$ yields a spectral sequence. Since M is a closed manifold, the Morse–Bott function is bounded from above and below, and hence the spectral sequence is bounded. Therefore, it converges to Tate homology; see [**46**, Theorem 5.5.1]. Its first page is the direct sum of contributions from the critical components of f, with boundary operator given by the Rabinowitz Floer boundary operator. Let us compute the corresponding 'local homologies'.

If $p \in M$ is a fixed point of the S^1-action (and thus a critical point of f), then the local homology is just the equivariant Rabinowitz Floer homology $\Lambda = \mathbb{Z}[u, u^{-1}]$, shifted by the index of p. In particular, if the circle action on M is trivial, then $\widehat{H}^{S^1}(M, f) = H(M; \Lambda)$ is just the homology of M with coefficients in Λ.

Consider now a critical circle γ of f on which the circle action is an n-fold covering, $n \in \mathbb{N}$. Then $(\gamma \times S_\infty^\infty)/S^1$ is the infinite-dimensional lens space $S_\infty^\infty/\mathbb{Z}_n$ obtained as the quotient by the stabilizer subgroup $\mathbb{Z}_n \subset S^1$. Picking a Morse function with two critical points on each critical fibre of the degree n circle bundle $S_\infty^\infty/\mathbb{Z}_n \to \mathbb{C}P^\infty$ gives us generators $w_\pm^\ell$ of indices $|w_-^\ell| = 2\ell$ and $|w_+^\ell| = 2\ell + 1$ for $\ell \in \mathbb{Z}$. To compute the boundary operator on this complex, we need to distinguish two cases. Let us call γ *good* if the tangent bundle to the unstable manifold of f along γ is orientable, and *bad* otherwise (the latter can only happen for n even; see the following proof).

LEMMA 3.2. *If γ is good, then the boundary maps on the first page are given by*

$$\cdots \xrightarrow{\cdot n} w_+^\ell \xrightarrow{\ 0\ } w_-^\ell \xrightarrow{\cdot n} w_+^{\ell-1} \xrightarrow{\ 0\ } w_-^{\ell-1} \xrightarrow{\cdot n} \cdots$$

If γ is bad, then the boundary maps on the first page are given by

$$\cdots \xrightarrow{\ 0\ } w_+^\ell \xrightarrow{\cdot 2} w_-^\ell \xrightarrow{\ 0\ } w_+^{\ell-1} \xrightarrow{\cdot 2} w_-^{\ell-1} \xrightarrow{\ 0\ } \cdots.$$

REMARK. We actually only determine the coefficients $\pm n, \pm 2$ up to signs. They can be arranged to be positive by replacing some generators by their negatives if necessary. The same remark applies to subsequent computations.

Proof. Pick a point x on γ and denote by $\Phi_x : T_x M \to T_x M$ the linearization of the S^1-action at time $1/n$. Since Φ_x preserves the tangent space E_x^- to the unstable manifold at x, the

unstable bundle $E^- \to \gamma$ along γ is isomorphic to the bundle $[0,1] \times E_x^-/(0,v) \sim (1, \Phi_x \cdot v)$. So E^- is non-orientable (that is, γ is bad) if and only if $\det(\Phi_x|_{E_x^-}) = -1$. Now $\Phi_x^n = \mathbb{1}$ implies $(-1)^n = 1$, which is only possible if n is even. Consider the Morse complex of a Morse function with two critical points on γ with local coefficients in E^- (or rather its orientation bundle). The two gradient trajectories from the maximum to the minimum occur in the boundary operator with opposite signs if γ is good, and with the same sign if γ is bad. Hence the (non-equivariant) Morse homology of γ with local coefficients in E^- equals

$$H_*(S^1; E^-) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z} & * = 1 \end{cases} \quad \text{if } \gamma \text{ is good,}$$

$$H_*(S^1; E^-) = \begin{cases} \mathbb{Z}_2 & * = 0 \\ \{0\} & * = 1 \end{cases} \quad \text{if } \gamma \text{ is bad.}$$

To compute the equivariant homology, let us identify $\gamma \cong S^1 = \mathbb{R}/\mathbb{Z}$ with the S^1-action $(\tau, t) \mapsto t + n\tau$. The associated critical manifold $(\gamma \times S_\infty^\infty)/S^1$ is diffeomorphic to the doubly infinite lens space $L_n := (S_\infty^\infty)/\mathbb{Z}_n$ via the map sending $[t, z]$ to $[e^{-2\pi i t/n}z]$, with inverse map $[z] \mapsto [0, z]$. Under this diffeomorphism the loop $t \mapsto [t, z] = [0, e^{-2\pi i t/n}z]$, $t \in [0, 1]$ (with $z \in S_\infty^\infty$ fixed), corresponds to a generator of $\pi_1(L_n) = \mathbb{Z}_n$. Thus the pullback bundle $\pi^*E^- \to L_n$ under the projection $[t, z] \mapsto t$ is orientable over the generator of $\mathbb{Z}_n$ if and only if E^- is orientable. The Morse–Bott function $h(z) = \sum_{k \in \mathbb{Z}} k|z_k|^2$ on L_n has critical circles $w^\ell = \{z_j = 0 \text{ for } j \neq \ell\}$ of index 2ℓ for each $\ell \in \mathbb{Z}$. Perturbing h by Morse functions with two critical points on each critical circle, we obtain generators w_+^ℓ of index $2\ell + 1$ and w_-^ℓ of index 2ℓ. For each ℓ there are two gradient lines from w_+^ℓ to w_-^ℓ, and n gradient lines from $w_-^{\ell+1}$ to w_+^ℓ.

For $n = 1$ the lens space is the sphere and its homology is the Rabinowitz Floer homology of $\mathcal{A}^\mu$, which vanishes by [13]. For $n \geqslant 2$, we again distinguish two cases.

If γ is good, then the bundle $\pi^*E^- \to L_n$ is orientable and the homology of L_n with coefficients in π^*E^- is just the ordinary homology of the doubly infinite lens space, which by the computation in [30, Example 2.43] is given by

$$H_*(L_n; \pi^*E^-) = \begin{cases} \{0\} & * \text{ even} \\ \mathbb{Z}_n & * \text{ odd} \end{cases} \quad \text{if } \gamma \text{ is good.}$$

If γ is bad, then the bundle $\pi^*E^- \to L_n$ is non-orientable over each circle w^ℓ (which represents a generator of $\pi_1(L_n)$), so the Morse boundary operator ∂ maps w_+^ℓ to $2w_-^\ell$. The relation $\partial \circ \partial = 0$ then enforces $\partial w_-^\ell = 0$, so the homology is given by

$$H_*(L_n; \pi^*E^-) = \begin{cases} \mathbb{Z}_2 & * \text{ even} \\ \{0\} & * \text{ odd} \end{cases} \quad \text{if } \gamma \text{ is bad.}$$

Since the chain complex has only one generator in each degree, in order to yield these homology groups the boundary operator must have (up to signs) the multiplicities given in the lemma. $\square$

REMARK. The homology of the lens space L_n with local coefficients in π^*E^- in the preceding proof can also be obtained more algebraically as follows. The vanishing of Rabinowitz Floer homology of $\mathcal{A}^\mu$ implies that the local chain complex is a complete resolution of the group $\mathbb{Z}_n$ in the sense of [9]. If the periodic orbit γ is good, then with integer coefficients the local Tate homology is given by the Tate homology of the group $\mathbb{Z}_n$,

$$\widehat{H}_*(\mathbb{Z}_n; \mathbb{Z}) = \begin{cases} \{0\} & * \text{ even,} \\ \mathbb{Z}_n & * \text{ odd.} \end{cases}$$

If γ is bad, then n is necessarily even and we have to consider the twisted $\mathbb{ZZ}_n$-module $\widehat{\mathbb{Z}}$ where the generator of $\mathbb{Z}_n$ acts via -1. In this case, the local Tate homology is given by

$$\widehat{H}_*(\mathbb{Z}_n; \widehat{\mathbb{Z}}) = \begin{cases} \mathbb{Z}_2 & * \text{ even}, \\ \{0\} & * \text{ odd}. \end{cases}$$

COROLLARY 3.3. *The contribution of a critical circle γ of covering number n to the homology of the first page is equivalent to the following*

   (i) *$\{0\}$ if $n = 1$;*
   (ii) *$\mathbb{Z}_n[u, u^{-1}]$ shifted by $|\gamma| + 1$ if $n \geqslant 2$ and γ is good;*
   (iii) *$\mathbb{Z}_2[u, u^{-1}]$ shifted by $|\gamma|$ if γ is bad.*

In particular, if the circle action on M is free, then $\widehat{H}^{S^1}(M, f) = 0$.

With $\mathbb{Q}$-coefficients, the contribution of each critical circle γ to the homology of the first page vanishes, so only the contributions from the fixed points remain. The boundary operator on the second page counts gradient flow lines of f connecting Morse critical points. After this, the spectral sequence collapses. Indeed, since the Morse–Bott function f is constant 0 on the fixed points, different components of the fixed point set cannot interact via gradient flow lines. We have thus derived Borel's localization theorem (see [5]).

COROLLARY 3.4. *The Tate homology with $\mathbb{Q}$-coefficients equals the homology of the fixed point set M_{S^1} with coefficients in $\Lambda_{\mathbb{Q}} := \mathbb{Q}[u, u^{-1}]$:*

$$\widehat{H}^{S^1}(M, f; \mathbb{Q}) \cong H(M_{S^1}; \Lambda_{\mathbb{Q}}) \cong H(M_{S^1}) \otimes_{\mathbb{Q}} \mathbb{Q}[u, u^{-1}].$$

In particular, if the circle action on M has no fixed points, then $\widehat{H}^{S^1}(M, f; \mathbb{Q}) = 0$.

(4) Using the filtration by values of the Rabinowitz action functional $\widehat{\mathcal{A}}^\mu$ (not of $\widehat{f} + \widehat{\mathcal{A}}^\mu!$), we define filtered Tate homology groups $\widehat{H}_a^b(M, f)$, $\widehat{H}_a(M, f)$, and $\widehat{H}^b(M, f)$ with the obvious notation as above. Note that, for integers $m \leqslant n$, multiplication by u^m defines canonical isomorphisms (of degree $2m$)

$$\widehat{H}_0^{n-m}(M, f) \cong \widehat{H}_m^n(M, f), \quad \widehat{H}_0(M, f) \cong \widehat{H}_m(M, f).$$

By definition, $\widehat{H}_0^n(M, f)$ equals the Morse homology of the manifold $(M \times S^{2n+1})/S^1$. Since the homology functor and the direct limit functor commute, it follows that

$$\widehat{H}_0(M, f) = \varinjlim_{n \to \infty} \widehat{H}_0^n(M, f) = \varinjlim_{n \to \infty} H(M \times S^{2n+1})/S^1) = H^{S^1}(M)$$

equals the S^1-equivariant homology of M. Here $H^{S^1}(M) = H(M_{S^1})$ is defined via the Borel construction; see below. So we have shown the following proposition.

PROPOSITION 3.5. *For every $a \in \mathbb{R}$, the filtered Tate homology $\widehat{H}_a(M, f)$ is canonically isomorphic to the S^1-equivariant homology $H^{S^1}(M)$ with degrees shifted by $2[a]$.*

Recall the Borel construction $X_G = X \times_G EG$ for a G-space X. It fits into the diagram

$$\begin{array}{ccc} X \times EG & \longrightarrow & EG \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ X_G & \xrightarrow{\ p\ } & BG \end{array} \tag{10}$$

where the vertical maps are principal G-bundles and the horizontal maps are induced by projection onto the second factor. Pullback under p yields a ring homomorphism $p^* : H^*(BG) \to H^*(X_G)$, which makes equivariant cohomology a module over $H^*(BG)$ via the cup product. The image p^*e of a class $e \in H^*(BG)$ is the characteristic class of the bundle π induced by e, and cap product with these classes makes equivariant homology $H_*(X_G)$ a module over $H^*(BG)$ as well.

For $G = S^1$ and X being a manifold, the class p^*e corresponding to the generator $e \in H^2(BS^1)$ is the Euler class of π, and cap product with this class is realized by intersection with the codimension 2 submanifold of X_{S^1} given by a complex codimension 1 linear subspace of $BS^1 = \mathbb{C}P_0^\infty$.

COROLLARY 3.6. *The isomorphism in Proposition 3.5 is compatible with the module structures over the polynomial ring $\mathbb{Z}[u^{-1}]$, where u^{-1} acts on $H^{S^1}(M)$ by cap product with the Euler class $e \in H^2(BS^1)$.*

Proof. Define a complex codimension 1 linear subspace H of $\mathbb{C}P_0^\infty$ by the equation $\sum_{k=0}^\infty a_k z_k = 0$ on Fourier coefficients, for an ℓ^2 sequence (a_k) with $a_k \neq 0$ for all k. Gradient flow lines of $\mathcal{A}^\mu$ connecting two critical points $(z^{(\ell+1)}, \ell+1)$ and $(z^{(\ell)}, \ell)$ project onto the 2-sphere $\{z_k = 0$ for all $k \neq \ell, \ell+1\}$ in $\mathbb{C}P_0^\infty$, which intersects H in the single point $\{z_k = 0$ for all $k \neq \ell, \ell+1$, $a_\ell z_\ell + a_{\ell+1} z_{\ell+1} = 0\}$. Thus cap product with the Euler class maps $(z^{(\ell+1)}, \ell+1)$ to $(z^{(\ell)}, \ell)$, which is also what the map u^{-1} does. $\square$

4. Symplectic Tate homology

In this section, we carry over the construction of Subsection 3.2 to define symplectic Tate homology, and we prove Theorem 1.1 from Section 1.

4.1. *Definition and basic properties*

Let (V, λ_V) be the completion of a Liouville domain (W, λ) satisfying $c_1(W) = 0$. Let $H \in C^\infty(V, \mathbb{R})$ be an autonomous Hamiltonian growing quadratically at infinity. Denote by $\mathscr{L}_V := C^\infty(S^1, V)$ the free loop space of V. Since the Hamiltonian is autonomous, the action functional of classical mechanics

$$\mathcal{A}_H : \mathscr{L}_V \longrightarrow \mathbb{R}$$

defined by

$$\mathcal{A}_H(v) := \int v^* \lambda_V - \int H(v)\, dt$$

is invariant under the circle action of $\mathscr{L}_V$ induced by rotating the domain. We assume in the following that it is Morse–Bott, which we can always achieve by perturbing H slightly.

Recall from Subsection 3.1 the definition of the Rabinowitz action functional for the unit circle in $\mathbb{C}$,

$$\mathcal{A}^\mu : \mathscr{L}_\mathbb{C} \times \mathbb{R} \longrightarrow \mathbb{R}, \quad \mathcal{A}^\mu(z, \eta) = \int z^* \lambda_\mathbb{C} - \eta \int \mu(z)\, dt.$$

The circle acts on the product space $\mathscr{L}_V \times \mathscr{L}_\mathbb{C} \times \mathbb{R}$ diagonally. Since both $\mathcal{A}_H$ and $\mathcal{A}^\mu$ are S^1-invariant, the functional

$$\mathcal{T}_H : (\mathscr{L}_V \times \mathscr{L}_\mathbb{C} \times \mathbb{R})/S^1 \longrightarrow \mathbb{R}$$

given by

$$\mathcal{T}_H([v, z, \eta]) = \mathcal{A}_H(v) + \mathcal{A}^\mu(z, \eta)$$

is well-defined. Since $\mathcal{A}_H$ was assumed to be Morse–Bott, $\mathcal{T}_H$ is Morse–Bott as well. Choose an auxiliary Morse function f on the critical manifold of $\mathcal{T}_H$. On the set $\mathrm{Crit}(f)$ of critical points of the auxiliary Morse function f we define a double filtration by

$$\mathrm{Crit}_a^b(f) := \{c \in \mathrm{Crit}(f) \mid \mathcal{A}^\mu(c) \geqslant a,\ \mathcal{A}_H(c) \leqslant b\}, \quad a, b \in \mathbb{R}.$$

Since $c_1(W) = 0$, elements in $\mathrm{Crit}(f)$ are naturally graded by the sum of the Conley–Zehnder indices of the periodic orbits for $\mathcal{A}_H$ and $\mathcal{A}^\mu$ and the Morse index of the critical points of f. (Here the Conley–Zehnder indices of periodic orbits of $\mathcal{A}_H$ depend on a choice of trivializations of TV along base loops in each free homotopy class.) It is important to observe that, for all $a, b \in \mathbb{R}$, the set $\mathrm{Crit}_a^b(f)$ is finite in each fixed degree. We omit the reference to the degree in our notation. The *symplectic Tate chain groups* (with $\mathbb{Z}$-coefficients) are defined as the free $\mathbb{Z}$-modules generated by critical points of our auxiliary Morse function,

$$CT_a^b := \mathrm{Crit}_a^b(f) \otimes \mathbb{Z}.$$

The boundary operator

$$\partial_a^b \colon CT_a^b \longrightarrow CT_a^b$$

is defined by counting gradient flow lines with cascades for the pair $(\mathcal{T}_H, f)$ as in [6] or [23, Appendix A]. This definition depends on the choice of a suitable family of almost complex structures, which we defer to Subsection 4.4. We denote its homology groups by

$$HT_a^b := \frac{\ker \partial_a^b}{\mathrm{im}\, \partial_a^b}.$$

Since both action functionals $\mathcal{A}_H$ and $\mathcal{A}^\mu$ are non-increasing along gradient flow lines with cascades, the inclusion and projection maps

$$\iota_a^{b_2, b_1} \colon CT_a^{b_1} \longrightarrow CT_a^{b_2}, \quad b_1 \leqslant b_2, \quad \pi_{a_2, a_1}^b \colon CT_{a_1}^b \longrightarrow CT_{a_2}^b, \quad a_1 \leqslant a_2$$

together with the boundary operators ∂_a^b form a bidirect system of chain complexes as in Section 2. In particular, the induced maps on homology

$$H\iota_a^{b_2, b_1} \colon HT_a^{b_1} \longrightarrow HT_a^{b_2}, \quad b_1 \leqslant b_2, \quad H\pi_{a_2, a_1}^b \colon HT_{a_1}^b \longrightarrow HT_{a_2}^b, \quad a_1 \leqslant a_2$$

form a bidirect system of graded vector spaces. As in Subsection 2.2, we define four versions of *symplectic Tate homology* by

$$\overrightarrow{H}\overleftarrow{T}(W) := H(\varinjlim \varprojlim CT), \quad \underrightarrow{H}\,\underleftarrow{T}(W) := \varinjlim \varprojlim HT \quad \text{(Jones–Petrack version)},$$

$$\overleftarrow{H}\overrightarrow{T}(W) := H(\varprojlim \varinjlim CT), \quad \underleftarrow{H}\,\underrightarrow{T}(W) := \varprojlim \varinjlim HT \quad \text{(Goodwillie version)}.$$

Thus superscript (respectively, subscript) arrows indicate that the direct/inverse limits are applied before (respectively, after) taking homology. As the notation suggests, the symplectic Tate homology groups are independent of H, the auxiliary Morse function f, as well as the metrics used in order to define gradient flow lines with cascades. This can be shown by standard continuation arguments in Morse homology as in [38]. By construction, they are invariant under Liouville isomorphisms of the completions, hence in particular under deformation equivalence of the Liouville domains (see also [39]).

Note that the spaces C_a^b are finitely generated in each degree and the projections π_{a_2, a_1}^b are surjective. Hence Proposition 2.2 and Corollary 2.5 imply the following theorem.

THEOREM 4.1. *Let W be a Liouville domain with $c_1(W) = 0$. Then there is a canonical diagram*

$$\begin{array}{ccc} \overrightarrow{H}\overleftarrow{T}(W) & \xrightarrow{\ \rho\ } & \underrightarrow{H}\underleftarrow{T}(W) \\ \downarrow{\scriptstyle Hk} & & \downarrow{\scriptstyle \kappa} \\ \overleftarrow{H}\overrightarrow{T}(W) & \xrightarrow{\ \sigma\ } & \underleftarrow{H}\underrightarrow{T}(W) \end{array} \qquad (11)$$

of $\mathbb{Z}[u, u^{-1}]$-module maps with σ surjective. With coefficients in a field, the diagram commutes and ρ is an isomorphism.

This proves parts (a) and (b) of Theorem 1.1.

4.2. *Localization properties*

Next we turn to the proof of Theorem 1.1(c) and (d). For $a \in \mathbb{R}$, we abbreviate

$$HT_a(W) = HT_a := \varinjlim_b HT_a^b.$$

Again, HT_a only depends on W up to Liouville isomorphisms of the completions. The next proposition identifies these groups with the S^1-equivariant symplectic homology $SH^{S^1}(W)$ of the Liouville domain W as defined by Bourgeois and Oancea [7].

PROPOSITION 4.2. *For each $a \in \mathbb{R}$, we have a canonical isomorphism of $\mathbb{Z}[u^{-1}]$-modules*

$$HT_a(W) \cong SH^{S^1}(W).$$

Proof.   Because of the $\mathbb{Z}$-invariance of the chain complex, it is clear that the HT_a for different choices of a are canonically isomorphic. It therefore suffices to show that HT_0 is canonically isomorphic to SH^{S^1}. We filter the action of $\mathcal{A}^\mu$ also from above and denote, for $n \in \mathbb{N}$, the corresponding vector space by $H^n T_0$. Since the homology functor and the direct limit functor commute, we have

$$\varinjlim_{n \to \infty} H^n T_0 = HT_0.$$

For a loop $z \in \mathscr{L}_{\mathbb{C}}$, we consider its Fourier expansion

$$z(t) = \sum_{k=-\infty}^{\infty} z_k \, e^{2\pi i k t}, \quad z_k \in \mathbb{C}.$$

Consider the finite-dimensional subspace $(\mathscr{L}_{\mathbb{C}})_0^n \cong \mathbb{C}^{n+1}$ of the loop space $\mathscr{L}_{\mathbb{C}}$ consisting of all Fourier series with non-vanishing Fourier coefficients only in the range from 0 to n. The gradient flow equation (9) for $\mathcal{A}^\mu$ in terms of the Fourier coefficients shows that the subspace $(\mathscr{L}_{\mathbb{C}})_0^n$ is invariant under the gradient flow of $\mathcal{A}^\mu$. The restriction of $\mathcal{A}^\mu$ to the subspace $(\mathscr{L}_{\mathbb{C}})_0^n$ is given by the Lagrange multiplier function

$$\mathcal{A}^\mu|_{(\mathscr{L}_{\mathbb{C}})_0^n}(z, \eta) = \bar{\mathcal{A}}(z) + \eta \bar{\mu}(z),$$

where

$$\bar{\mathcal{A}}(z) = \pi \sum_{k=0}^{n} k |z_k|^2, \quad \bar{\mu}(z) = \pi \left(\sum_{k=0}^{n} |z_k|^2 - 1 \right).$$

Note that $\bar{\mu}^{-1}(0) = S^{2n-1}$. Now homotope the Morse homology of the Lagrange multiplier functional to the Morse homology of the function $\bar{\mathcal{A}}$ on the constraint $\bar{\mu}^{-1}(0)$ as in [24, Theorem A1] to recover the definition of S^1-equivariant symplectic homology of Bourgeois

and Oancea [**7**, **8**]; namely, one homotopes the metric and the function $\bar{A}$ in a neighbourhood of the constraint $\bar{\mu}^{-1}(0)$ without changing $\bar{A}$ on the constraint itself to make the gradient of $\bar{A}$ tangent to the constraint. Then gradient flow lines of the restriction of $\bar{A}$ to the constraint $\bar{\mu}^{-1}(0)$ give rise to gradient flow lines of the Lagrange multiplier functional. On the other by choosing the metric appropriately in order to make this neighbourhood large, all gradient flow lines of the Lagrange multiplier functional arise in this way. The resulting isomorphism is compatible with the $\mathbb{Z}[u^{-1}]$-module structures, where the action of u^{-1} on $SH^{S^1}(W)$ is induced by the cap product with the Euler class $e \in H^2(BS^1)$ in the definition by Bourgeois and Oancea, which is modelled on the Borel construction. $\qquad\square$

Now we specialize to coefficients in a field $\mathfrak{k}$. The following corollary establishes Theorem 1.1(d).

COROLLARY 4.3. *For any field $\mathfrak{k}$, the Goodwillie version of Tate homology $\underleftarrow{H}\underrightarrow{T}(W; \mathfrak{k})$ coincides with the localization of S^1-equivariant symplectic homology of W.*

Proof. Recall that the map u defines isomorphisms $u : CT^b_a \to CT^{b+\pi}_{a+\pi}$ on the filtered Rabinowitz Floer complex. Consider the vector space $V := CT^\infty_0$ and the linear map

$$T := p_{-1} \circ u^{-1}|_V : V \longrightarrow V,$$

where $p_{-1} := \pi^\infty_{0,-\pi} : CT^\infty_{-\pi} \to CT^\infty_0$ is the canonical projection. Since T commutes with the boundary operator $\partial := \partial^\infty_0$, the triple (V, T, ∂) is a Tate triple as in Definition 2.12. Since $T : V \to V$ is surjective, Corollary 2.14 yields the isomorphism of $\mathfrak{k}[u, u^{-1}]$-modules

$$\varprojlim_{k \to \infty} H(V^T_k, \partial_k) \cong H(V, \partial)^{HT}. \tag{12}$$

In view of Proposition 4.2 and $H(V, \partial) = HT_0(W; \mathfrak{k})$, the right-hand side of (12) is the localization of S^1-equivariant symplectic homology of W. To understand the left-hand side, recall from Subsection 2.4 that V^T_k is the quotient of the space of sequences $(v_i)_{i \in \mathbb{N}}$ of $v_i \in V$ with $Tv_{i+1} = v_i$ by those sequences with $v_k = 0$. We claim that we have a commuting diagram

$$
\begin{array}{ccc}
V^T_k & \xrightarrow{\;\;\pi_k\;\;} & V^T_{k-1} \\
{\scriptstyle\cong}\downarrow{\scriptstyle f_k} & & {\scriptstyle\cong}\downarrow{\scriptstyle f_{k-1}} \\
CT^\infty_{-k\pi} & \xrightarrow{\;\;p_{-k}\;\;} & CT^\infty_{-(k-1)\pi}.
\end{array}
$$

Here the horizontal maps are the canonical projections and the vertical map f_k sends $[(v_i)] \in V^T_k$ to $u^{-k}v_k \in CT^\infty_{-k\pi}$. For injectivity of f_k, note that $0 = u^{-k}v_k \in CT^\infty_{-k\pi}$ implies $0 = v_k \in CT^\infty_0$ and thus $0 = [(v_i)] \in V^T_k$. For surjectivity, a preimage of $x \in CT^\infty_{-k\pi}$ is obtained by setting $v_k := u^k x \in CT^\infty_0$ and extending it to $[(v_i)] \in V^T_k$ using surjectivity of T. Commutativity of the diagram follows from

$$f_{k-1}\pi_k[(v_i)] = u^{-(k-1)}v_{k-1} = u^{-(k-1)}Tv_k = u^{-(k-1)}p_{-1}u^{-1}v_k$$
$$= p_{-k}u^{-(k-1)}u^{-1}v_k = p_{-k}f_k[(v_i)].$$

In view of the commuting diagram, the left-hand side of (12) becomes

$$\varprojlim_{k \to \infty} H(V^T_k, \partial_k) \cong \varprojlim_{a \to -\infty} H(CT^\infty_a, \partial^\infty_a) \cong \varprojlim_{a \to -\infty} \varinjlim_{b \to \infty} HT^b_a \cong \underleftarrow{H}\underrightarrow{T}(W; \mathfrak{k}),$$

where the second isomorphism comes from the fact that the direct limit commutes with the homology functor. This proves the corollary. $\qquad\square$

Finally, we prove that with rational coefficients $\overrightarrow{H}\overleftarrow{T}$ has the fixed point property [**10**, **29**, **33**], which in view of the isomorphism $\overrightarrow{H}\overleftarrow{T} \cong \underrightarrow{H}\overleftarrow{T}$ in Theorem 4.1 establishes Theorem 1.1(c).

PROPOSITION 4.4.   *For any Liouville domain W satisfying $c_1(W) = 0$, the Jones–Petrack version of symplectic Tate homology has the fixed point property*

$$\overrightarrow{H}\overleftarrow{T}(W;\mathbb{Q}) \cong H(W,\partial W;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[u,u^{-1}].$$

Proof.   We consider the spectral sequence associated to the increasing filtration $\mathcal{F}^b := \{\mathcal{A}_H \leqslant b\}$ on the chain complex $\underrightarrow{\lim}\underleftarrow{\lim}CT$; see [**46**, Chapter 5.4]. Since H grows quadratically at infinity, the filtration is bounded from below. Moreover, it is exhaustive in the sense that $\bigcup_{b\geqslant 0} \mathcal{F}^b = \underrightarrow{\lim}\underleftarrow{\lim}CT$. (Note that this fails for the chain complex $\underleftarrow{\lim}\underrightarrow{\lim}CT$.) Therefore, the spectral sequence converges to $\overrightarrow{H}\overleftarrow{T}(W;\mathbb{Q})$.

The spectral sequence depends on the choice of our Hamiltonian. We explain next how to choose the Hamiltonian H on the completion V of the Liouville domain W so that the result can easily be read off from the spectral sequence. After a small perturbation of W we may assume without loss of generality that the Reeb flow on ∂W is non-degenerate. Now choose a smooth function $\beta \in C^\infty([0,\infty),[0,\infty))$ satisfying

$$\beta(r) \begin{cases} = 0 & r \leqslant 1, \\ > 0 & r > 1, \\ = r & r \geqslant 2. \end{cases}$$

Define $h \in C^\infty([0,\infty),[0,\infty))$ by the requirement

$$h(0) = 0, \quad h'(0) = 0, \quad h''(r) = \frac{\beta(r)}{r} \quad \text{for } r > 0.$$

Note that h grows quadratically at infinity. Moreover, since h is strictly convex for $r > 1$, it satisfies

$$h'(r)r - h(r) > 0 \quad \text{for all } r > 1. \tag{13}$$

If $(\partial W \times (0,\infty), d(r\lambda|_{\partial W}))$ is embedded as the symplectization of the Liouville domain W in its completion V, then we define the Hamiltonian $H \in C^\infty(V,\mathbb{R})$ as

$$H(v) = \begin{cases} h(r) & v = (x,r) \in \partial W \times (0,\infty), \\ 0 & v \in W. \end{cases}$$

The critical set of the action functional $\mathcal{A}_H$ associated to H can be identified with W corresponding to the constant loops and a disjoint union of circles, one for each periodic Reeb orbit of the Reeb flow on ∂W. In particular, the action functional $\mathcal{A}_H$ is not quite Morse–Bott in the orthodox sense, since the critical manifold W has a boundary. However, note that the action of $\mathcal{A}_H$ vanishes on W and is positive on the non-constant periodic Reeb orbits in view of (13). In view of this fact, we can define Morse–Bott homology of $\mathcal{A}_H$ as for an honest Morse–Bott functional treated in [**25**]. Namely, choose a Morse function f on $\mathrm{Crit}(\mathcal{A}_H)$ with the property that ∇f points outward at the boundary of W for one (and hence every) Riemannian metric on $\mathrm{Crit}(\mathcal{A}_H)$. Since the action is decreasing along gradient flow lines, no flow line with cascades can contain a Floer cylinder starting or ending on a constant in the boundary of W, so the boundary operator squares to zero.

The Liouville domain W corresponds precisely to the fixed point set of the S^1-action on $\mathrm{Crit}(\mathcal{A}_H)$. We can assume without loss of generality that W is connected (otherwise we look at the symplectic Tate homologies of each connected component of W). Therefore, its local Tate homology (the contribution to the first page of the spectral sequence) with rational coefficients

is $H(W, \partial W; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[u, u^{-1}]$. The reason we have to take homology relative to the boundary of W is that the auxiliary Morse function points outward at the boundary.

On the other hand, by Lemma 4.5, the contribution of each non-constant periodic orbit γ of the Hamiltonian vector field X_H to the first page of the spectral sequence is torsion and thus vanishes with $\mathbb{Q}$-coefficients. This completes the proof of Proposition 4.4. $\qquad\square$

In the proof, we have used the following analogue of Corollary 3.3, where as in [20] a Hamiltonian orbit γ is called *bad* if it is an even multiple of a simple orbit whose linearized return map has an odd number of eigenvalues in the interval $(-1, 0)$, and *good* otherwise.

LEMMA 4.5. *The local symplectic Tate homology (that is, the contribution to the homology of the first page of the spectral sequence above) of a non-constant periodic orbit γ of the Hamiltonian vector field X_H of covering number n equals (with $\mathbb{Z}$-coefficients)*

  (i) $\{0\}$ *if $n = 1$;*
  (ii) $\mathbb{Z}_n[u, u^{-1}]$ *shifted by $|\gamma| + 1$ if $n \geqslant 2$ and γ is good;*
  (iii) $\mathbb{Z}_2[u, u^{-1}]$ *shifted by $|\gamma|$ if γ is bad.*

Proof. The local symplectic Tate homology of γ is computed by a chain complex with the same generators $w_{\pm}^{\ell}$, $\ell \in \mathbb{Z}$, and the same gradient flow lines between them as in the proof of Lemma 3.2. However, the signs with which the gradient flow lines contribute to the boundary operator are now determined by the coherent orientations from [21]. Rather than analysing the coherent orientations, we can deduce the signs from Lemma 3.2 and the following two facts.

(1) The local (non-equivariant) symplectic homology $SH(\gamma; \mathbb{Q})$ of γ with rational coefficients vanishes if and only if γ is bad; this follows from [6, Lemma 4.28].

(2) The local non-equivariant symplectic homology $SH(\gamma; \mathbb{Q})$ of γ with rational coefficients vanishes if and only if the local equivariant symplectic homology $SH^{S^1}(\gamma; \mathbb{Q})$ vanishes; this follows from the Gysin exact sequence in [7] and the fact that both groups live in non-negative degrees.

Now the local symplectic Tate homology of γ is the homology of the lens space $L_n = (S_{\infty}^{\infty})/\mathbb{Z}_n$ with some local system. For n odd (so γ is good) there is no non-trivial group homomorphism $\mathbb{Z}_n \to \mathbb{Z}_2$, so the local system is trivial and the homology is of the first (for $n = 1$) or second type in the statement of Lemma 4.5. Suppose now that n is even, so there exists a unique non-trivial group homomorphism $\mathbb{Z}_n \to \mathbb{Z}_2$ and thus a unique non-trivial local system on L_n. By facts (1) and (2), γ is bad if and only if the local equivariant symplectic homology $SH^{S^1}(\gamma; \mathbb{Q})$ with rational coefficients vanishes. This is the case if and only if the complex agrees with the second complex in Lemma 3.2 (set to zero in negative degrees, so the first complex has rational homology in degree 0), so the local system on L_n is non-trivial and the local symplectic Tate homology is of the third type in the statement of Lemma 4.5. If γ is good, then the local system on L_n is trivial and the local symplectic Tate homology is of the second type in the statement of Lemma 4.5. $\qquad\square$

REMARK 4.6. The construction of symplectic Tate homology shows: Whenever the map κ has non-trivial kernel, then there exist infinite chains of periodic orbits connected by Floer cylinders with $\mathcal{A}_H$ going to infinity. We will see examples of this phenomenon in Section 5.

4.3. *Backwards symplectic homology*

For $a < 0$, let us denote by $CB_a^b \subset CT_a^b$ the subcomplex defined by $\mathcal{A}^{\mu} \leqslant 0$. We call $HB(W) := \overrightarrow{H}\overleftarrow{B}(W)$ the *backwards S^1-equivariant symplectic homology* of W. We get a short exact sequence of bidirect systems of chain complexes $0 \to CB_a^b \to CT_a^b \to CT_0^b \to 0$ (defined for

$a < 0$ and with one filtration trivial on CT_0^b). According to Remark 2.4, with coefficients in a field $\mathfrak{k}$ we obtain long exact sequences fitting in the following commuting diagram with the map ρ from (11):

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & \overrightarrow{H}\overleftarrow{B}(W;\mathfrak{k}) & \longrightarrow & \overrightarrow{H}\overleftarrow{T}(W;\mathfrak{k}) & \longrightarrow & SH^{S^1}(W;\mathfrak{k}) & \longrightarrow \cdots \\
& \cong \Big\downarrow \rho & & \cong \Big\downarrow \rho & & \cong \Big\downarrow \rho & \\
\cdots \longrightarrow & \underrightarrow{H}\underleftarrow{B}(W;\mathfrak{k}) & \longrightarrow & \underrightarrow{H}\underleftarrow{T}(W;\mathfrak{k}) & \longrightarrow & SH^{S^1}(W;\mathfrak{k}) & \longrightarrow \cdots
\end{array}
\tag{14}
$$

The term 'backwards homology' is taken from [41], where an exact sequence analogous to (14) is derived in a different context.

4.4. Notes on transversality

The definition of Tate homology of a smooth manifold (respectively, symplectic Tate homology of a Liouville domain) M depends on the choice of a generic family of metrics (respectively, compatible almost complex structures) on the fibres of the fibration $M \to (M \times S_\infty^\infty)/S^1 \to \mathbb{C}P_\infty^\infty$ which is invariant under the $\mathbb{Z}$-action. Here we explain the construction of such a family of almost complex structures, the case of a family of metrics being similar but simpler.

  Recall from Subsection 3.1 the Hilbert space $\mathscr{L}_\mathbb{C}$ of Fourier series, equipped with the $W^{1,2}$-norm and the $\mathbb{Z}$-action $(n * z)_k = z_{k-n}$.

  LEMMA 4.7. *The $\mathbb{Z}$-action on $\mathscr{L}_\mathbb{C}^* = \mathscr{L}_\mathbb{C} \setminus \{0\}$ is free and proper, so the quotient by this action is a smooth Hilbert manifold.*

  Proof. Freeness holds because an element with $n * z = z$ for some $n \neq 0$ can only be in $W^{1,2}$ if it is zero. Recall that the $\mathbb{Z}$-action is proper if and only if, for all $x, y \in \mathscr{L}_\mathbb{C}^*$, there exist open neighbourhoods U_x, U_y such that $(n * U_x) \cap U_y = \emptyset$ for all but finitely many $n \in \mathbb{Z}$. To show this, consider $x, y \in \mathscr{L}_\mathbb{C}^*$. Pick $\ell \in \mathbb{Z}$ with $y_\ell \neq 0$ and let U_x, U_y be the open ε-balls with respect to the L^2-norm, where $\varepsilon := |y_\ell|/3$. Suppose that $z \in U_x$ and $n * z \in U_y$. Then $|z_{\ell-n} - y_\ell| \leqslant \|n * z - y\| < \varepsilon$, and thus $|z_{\ell-n}| \geqslant 2\varepsilon$. On the other hand, there exists $K \in \mathbb{N}$ such that $|x_k| < \varepsilon$ for all $k \in \mathbb{Z}$ with $|k| > K$. It follows that $|z_k| < 2\varepsilon$ for all $k \in \mathbb{Z}$ with $|k| > K$, which together with the previous estimate yields $|\ell - n| \leqslant K$, which is only satisfied for finitely many $n \in \mathbb{Z}$. The proof in [3] that the quotient of a manifold by a proper free action is again a manifold carries over readily to Hilbert manifolds. $\square$

  Since the free S^1-action on $\mathscr{L}_\mathbb{C}^*$ commutes with the $\mathbb{Z}$-action, we obtain a principal circle bundle

$$
S^1 \longrightarrow \mathscr{L}_\mathbb{C}^*/\mathbb{Z} \longrightarrow \mathscr{L}_\mathbb{C}^*/(S^1 \times \mathbb{Z}).
$$

Given a completed Liouville domain (V, λ_V), let $\mathcal{J}(V, \lambda_V)$ be the space of almost complex structures on V compatible with $d\lambda_V$ that are cylindrical over the collar $[1, \infty) \times \partial V$. Let

$$
\mathcal{J} := C^\infty(\mathscr{L}_\mathbb{C}^*/\mathbb{Z}, \mathcal{J}(V, \lambda_V))
$$

be the space of almost complex structures in $\mathcal{J}(V, \lambda_V)$ parametrized by $\mathscr{L}_\mathbb{C}^*/\mathbb{Z}$. An element $J \in \mathcal{J}$ defines by integration of $d\lambda(\cdot, J\cdot)$ an L^2-inner product on the vertical tangent bundle of the fibre bundle

$$
\mathscr{L}_V \longrightarrow (\mathscr{L}_V \times \mathscr{L}_\mathbb{C}^*)/(S^1 \times \mathbb{Z}) \longrightarrow \mathscr{L}_\mathbb{C}^*/(S^1 \times \mathbb{Z}).
$$

Identifying $\mathscr{L}_\mathbb{C}^*/\mathbb{Z}$ with the total space of the associated bundle $S^1 \to (\mathscr{L}_\mathbb{C}^* \times S^1)/(S^1 \times \mathbb{Z}) \to \mathscr{L}_\mathbb{C}^*/(S^1 \times \mathbb{Z})$, we can think of $J \in \mathcal{J}$ as a family of almost complex structures $J_{z,t}$ parametrized by $(z, t) \in \mathscr{L}_\mathbb{C}^* \times S^1$, invariant under the diagonal S^1-action on (z, t) and the $\mathbb{Z}$-action on z.

Then the V-component of a gradient flow line $[v, z, \eta]$ of the functional $\mathcal{T}_H$ satisfies the Floer equation

$$\partial_s v + J_{z(s),t}(v)(\partial_t v - X_H(v)) = 0.$$

This is the Floer equation on a cylinder with an S^1-dependent almost complex structure. It follows by standard methods in Floer theory (see, for example, [36]) that one an achieve transversality for all moduli spaces by generic choice of J in this class. Compactness modulo breaking follows by combining compactness modulo breaking for the components (z, η), which satisfy the gradient flow equation for $\mathcal{A}^\mu$, with Floer compactness for the component v satisfying the Floer equation above.

Let us point out that the preceding discussion is reminiscent of the one in [15], where transversality for holomorphic curves is established using domain-dependent almost complex structures parametrized by the universal curve over Deligne–Mumford space; see also [8].

We conclude this section with a restriction on contributions to the Tate boundary operator ∂ arising from transversality. Note that generators of the Tate chain complex can be written as $u^k x$, where $k \in \mathbb{Z}$ and x is a either a critical point of H or a critical point of the Morse function f on a non-constant 1-periodic orbit of H. Since the index of critical points of $\mathcal{A}^\mu$ is non-increasing, non-zero matrix elements $\langle x_+, u^\ell x_- \rangle$ can only exist for $\ell \leqslant 0$.

LEMMA 4.8. *By appropriate choice of the Hamiltonian H and a generic $J \in \mathcal{J}$ we can achieve that $\langle \partial x_+, u^\ell x_- \rangle = 0$ in the following cases:*

(i) *$x_\pm$ are critical points of H and $\ell < 0$;*

(ii) *$x_\pm$ are critical points of the Morse function f on distinct non-constant 1-periodic orbits $\gamma_\pm$ of H whose multiplicities k_+, k_- are relatively prime, and either $\ell < 0$, or $\ell = 0$ and $\mathrm{CZ}(\gamma_+) \leqslant \mathrm{CZ}(\gamma_-)$.*

Proof. In both cases, we can achieve transversality for all Floer cylinders in V connecting x_+ and x_- (respectively, γ_+ and γ_-) using a generic z-independent $J \in \mathcal{J}(V, \lambda_V)$. In case (i) this holds because, for H sufficiently C^2-small on W, all Floer cylinders from x_+ to x_- are gradient trajectories; see [36]. In case (ii), it holds because the relative primeness condition ensures that Floer cylinders from γ_+ to γ_- cannot be multiply covered.

For such a z-independent $J \in \mathcal{J}(V, \lambda_V)$, Tate trajectories project onto Floer trajectories in V. Hence, in case (i), $\langle \partial x_+, u^\ell x_- \rangle \neq 0$ implies $\mathrm{CZ}(x_+) \geqslant \mathrm{CZ}(x_-)$. In view of the grading condition $\mathrm{CZ}(x_+) = \mathrm{CZ}(x_-) + 2\ell + 1$, this yields $\ell \geqslant 0$. In case (ii), $\langle \partial x_+, u^\ell x_- \rangle \neq 0$ and $\gamma_+ \neq \gamma_-$ implies $\mathrm{CZ}(\gamma_+) \geqslant \mathrm{CZ}(\gamma_-) + 1$. Since the index $\mathrm{ind}(x_\pm)$ equals $\mathrm{CZ}(\gamma_\pm)$ or $\mathrm{CZ}(\gamma_\pm) + 1$, the grading condition $\mathrm{ind}(x_+) = \mathrm{ind}(x_-) + 2\ell + 1$ yields $\mathrm{CZ}(x_+) \leqslant \mathrm{CZ}(x_-) + 2\ell + 2$ and thus $\ell \geqslant 0$.

This proves vanishing of $\langle \partial x_+, u^\ell x_- \rangle$ in cases (i) and (ii) for a z-independent $J \in \mathcal{J}(V, \lambda_V)$. Now this vanishing persists for a sufficiently small perturbation of J to a generic element in $\mathcal{J}$. $\square$

5. *Computations and examples*

5.1. *$\mathbb{C}^n$ and subcritical Stein manifolds*

The Tate chain complex for $\mathbb{C}^n$ with integer coefficients is shown in Figure 1. Here the orbits are arranged according to their $\mathcal{A}_H$- and $\mathcal{A}^\mu$-actions, where each pair of nearby dots corresponds to the maximum and minimum of a small Morse function on the circle. The red numbers denote the *degree* $|x| := \mathrm{CZ}(x) - n$, that is, the Conley–Zehnder index minus n. For H we use a quadratic Hamiltonian close to $z \mapsto |z|^2$, composed with a strictly convex increasing function $\mathbb{R} \to \mathbb{R}$. Then the generators with $\mathcal{A}^\mu = 0$ along the horizontal axis are given by the minimum w_0^+ of H (of degree 0), and to the generators $w_k^\pm$ corresponding to the maximum
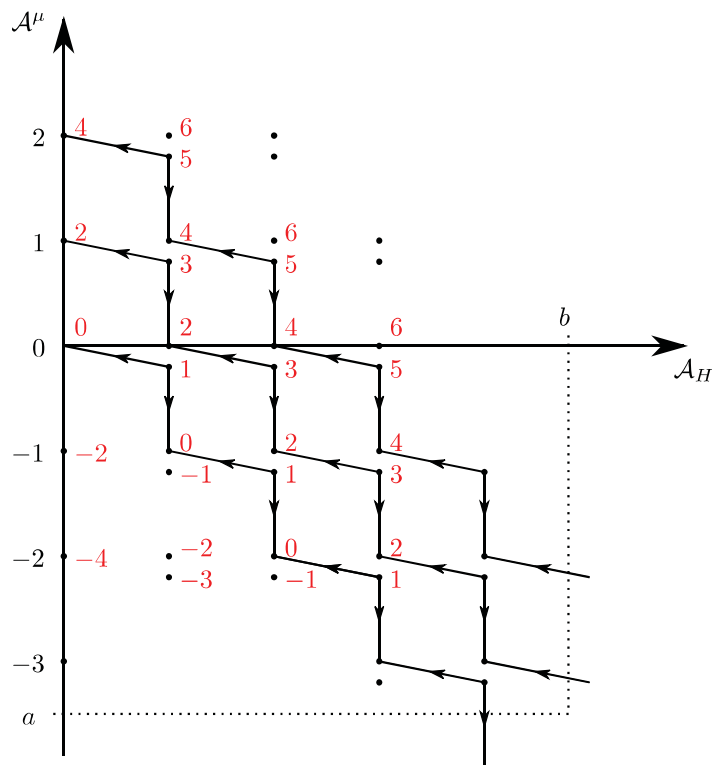
FIGURE 1 (colour online). *The Tate chain complex for $\mathbb{C}^n$.*

and minimum of a small Morse function on the periodic orbit w_k of Conley–Zehnder index $n-1+2k$, of degrees

$$|w_k^+| = 2k, \quad |w_k^-| = 2k - 1.$$

The other generators in the diagram are obtained from these by applying u^k for $k \in \mathbb{Z}$. The horizontal arrows map generators of odd degree onto those of even degree and thus compute, for each fixed value of $\mathcal{A}^\mu$, the symplectic homology of $\mathbb{C}^n$, which vanishes. The vertical arrows map the odd generators w_k^- onto multiples $a_k u^{-1} w_k^+$ of the even generators, so altogether we have

$$\partial w_k^+ = 0, \quad \partial w_{k+1}^- = w_k^+ + a_{k+1} u^{-1} w_{k+1}^+, \quad k = 0, 1, 2, \ldots.$$

Here $a_1, a_2, \ldots$ is a non-decreasing sequence of natural numbers which is given by $a_k = k$ for $n = 1$, while for higher n each natural number is repeated n times. The values a_k correspond to the multiplicities of the closed Reeb orbits on the (slightly perturbed) unit sphere $S^{2n-1} \subset \mathbb{C}^n$ generating the positive part of equivariant symplectic homology.

Note that, for each degree, there is an infinite sequence of generators on which $\mathcal{A}_H$ tends to $+\infty$ and $\mathcal{A}^\mu$ to $-\infty$. To compute the Tate homology groups, it is convenient to introduce the new generators

$$x_k := (-1)^k u^{-k} w_k^+, \quad y_k := (-1)^{k+1} u^{-k+1} w_k^-$$

of degrees $|x_k| = 0$ and $|y_k| = 1$, on which the boundary operator is given by

$$\partial x_k = 0, \quad \partial y_{k+1} = x_k - a_{k+1} x_{k+1}, \quad k = 0, 1, 2, \ldots.$$

Viewing Figure 1 as a graph in the obvious way, we see that it consists of an infinite number of connected components ('staircases'). The component containing x_0 is shown in Figure 2 for $n = 1$ (the case of higher n differs only by the weights on the vertical arrows), and the other
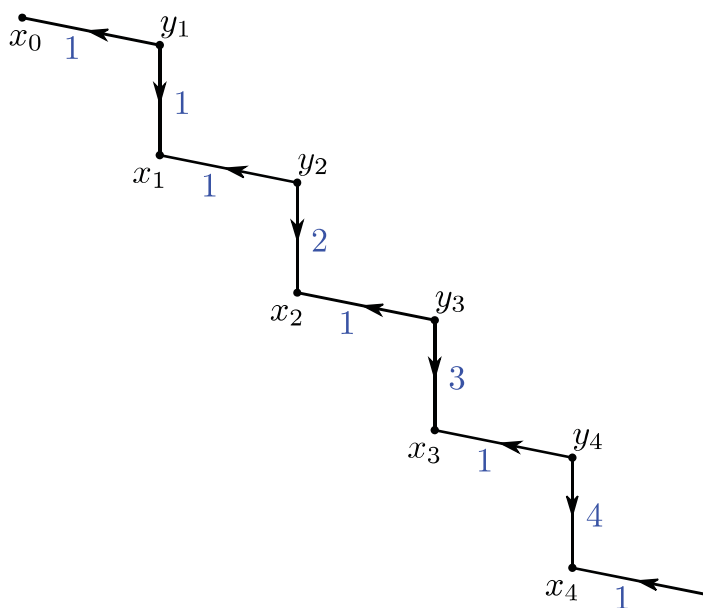
FIGURE 2 (colour online). *A connected component in the Tate chain complex for* $\mathbb{C}$.

components are obtained from this one by applying u^k for $k \in \mathbb{Z}$. Now we can read off the Tate homology groups. The chain complex for $\overrightarrow{H}\overleftarrow{T}$ consists of *finite* sums of generators. All generators $x_0, x_1, \ldots$ of a given even degree are cycles, and x_k is homologous to $a_{k+1}x_{k+1}$ for each $k = 0, 1, 2, \ldots$. So $\overrightarrow{H}\overleftarrow{T} = G \otimes_{\mathbb{Z}} \mathbb{Z}[u, u^{-1}]$, where G is the abelian group described in terms of generators and relations by

$$G = \langle x_k, k \in \mathbb{N} \mid x_k - a_{k+1}x_{k+1} \; k \in \mathbb{N}\rangle.$$

According to Proposition A.1, the group G is isomorphic to $\mathbb{Q}$.

In contrast, the chain complex for $\overleftarrow{H}\overrightarrow{T}$ consists of possibly infinite sums of generators on which $\mathcal{A}^\mu$ tends to $-\infty$. All such sums of a given even degree are cycles, but now each even generator x_k is the boundary of the infinite sum $y_{k+1} + \sum_{i=k+1}^{\infty} a_{k+1} \cdots a_i y_{i+1}$, so $\overleftarrow{H}\overrightarrow{T} = 0$.

In the complex HT_a^b an even generator is a boundary if and only if the staircase that begins there exits the region $\{\mathcal{A}^\mu \geqslant a, \mathcal{A}_H \leqslant b\}$ at the bottom, that is, through the line $\{\mathcal{A}^\mu = a\}$. It follows that HT_a^b is generated by the elements of even degree along the right vertical edge of the region $\{\mathcal{A}^\mu \geqslant a, \mathcal{A}_H \leqslant b\}$, that is, whose $\mathcal{A}_H$-value is just below b. Note that their degrees are bounded below by a fixed multiple of $a + b$; see Figure 1. Since in the direct limit $b \to \infty$ these degrees go to ∞, we conclude that $\varinjlim HT_a = 0$ for all a, and hence $\overrightarrow{H}\,\overrightarrow{T} = 0$.

The inverse limit $\varprojlim HT^k$ as $a \to -\infty$ for $b = k \in \mathbb{N}$ is the free $\mathbb{Z}[u, u^{-1}]$-module on one generator x_k of degree 0, and the map $\varprojlim HT^k \to \varprojlim HT^{k+1}$ sends x_k onto $a_{k+1}x_{k+1}$. Thus $\varprojlim HT^b = \mathbb{Z}[u, u^{-1}]$ for all b and $\overrightarrow{H}\,\overleftarrow{T} = G \otimes_{\mathbb{Z}} \mathbb{Z}[u, u^{-1}]$, where $G \cong \mathbb{Q}$ is the group above. So we have shown the following proposition.

PROPOSITION 5.1. *For the Liouville domain* $\mathbb{C}^n$ *and integer coefficients, the diagram* (11) *becomes (with* $\mathbb{Q}$ *viewed as an abelian group)*

$$
\begin{array}{ccc}
\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[u, u^{-1}] & \xrightarrow{\;\rho\;} & \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[u, u^{-1}] \\
\Big\downarrow{\scriptstyle Hk} & & \Big\downarrow{\scriptstyle \kappa} \\
0 & \xrightarrow{\;\sigma\;} & 0
\end{array}
\qquad (15)
$$

With rational coefficients, the diagram for $\mathbb{C}^n$ becomes (with Q as a field)

$$
\begin{array}{ccc}
\mathbb{Q}[u,u^{-1}] & \xrightarrow[\cong]{\rho} & \mathbb{Q}[u,u^{-1}] \\
\downarrow{\scriptstyle Hk} & & \downarrow{\scriptstyle \kappa} \\
0 & \xrightarrow{\sigma} & 0
\end{array}
\tag{16}
$$

Note that this is consistent with $\overrightarrow{H}\overleftarrow{T} \cong H(\mathbb{C}^n;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[u,u^{-1}] = \mathbb{Q}[u,u^{-1}]$ being the homology of the fixed point set, and $\underleftarrow{H}\,\underrightarrow{T}$ being the localization of equivariant symplectic homology, which vanishes for $\mathbb{C}^n$. These two facts together with the vanishing of equivariant symplectic homology for subcritical Stein manifolds (see, for example, [7]) and the isomorphism property of ρ yield more generally the following proposition.

PROPOSITION 5.2. *For a Liouville domain W with vanishing equivariant symplectic homology, for example, a subcritical Stein manifold, and rational coefficients the diagram* (11) *becomes*

$$
\begin{array}{ccc}
H(W;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[u,u^{-1}] & \xrightarrow[\cong]{\rho} & H(W;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}[u,u^{-1}] \\
\downarrow{\scriptstyle Hk} & & \downarrow{\scriptstyle \kappa} \\
\overleftarrow{H}\,\overrightarrow{T}(W;\mathbb{Q}) & \xrightarrow{\sigma} & 0
\end{array}
\tag{17}
$$

It would be interesting to compute $\overleftarrow{H}\,\overrightarrow{T}$ in this case.

5.2. *Cotangent bundles*

Let us now restrict to rational coefficients and set

$$
\Lambda_{\mathbb{Q}} := \mathbb{Q}[u,u^{-1}].
$$

PROPOSITION 5.3. *For a cotangent bundle T^*N and rational coefficients, the diagram* (11) *becomes*

$$
\begin{array}{ccc}
H(N;\mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}} & \xrightarrow[\cong]{\rho} & \underrightarrow{H}\,\underleftarrow{T}(T^*N;\mathbb{Q}) \\
\downarrow{\scriptstyle Hk} & & \downarrow{\scriptstyle \kappa} \\
\overleftarrow{H}\,\overrightarrow{T}(T^*N;\mathbb{Q}) & \xrightarrow{\sigma} & \underleftarrow{H}\,\underrightarrow{T}(T^*N;\mathbb{Q})
\end{array}
\tag{18}
$$

where $\underleftarrow{H}\,\underrightarrow{T}(T^*N;\mathbb{Q})$ depends only on $\pi_1(N)$ by Goodwillie's theorem [28]. *In particular, $\underleftarrow{H}\,\underrightarrow{T}(T^*N;\mathbb{Q}) = \Lambda_{\mathbb{Q}}$ if N is simply connected.*

Cotangent bundles of spheres. For $N = S^n$, the above diagram becomes

$$
\begin{array}{ccc}
\Lambda_{\mathbb{Q}} \oplus \Lambda_{\mathbb{Q}} & \xrightarrow[\cong]{\rho} & \Lambda_{\mathbb{Q}} \oplus \Lambda_{\mathbb{Q}} \\
\downarrow{\scriptstyle Hk} & & \downarrow{\scriptstyle \kappa} \\
\overleftarrow{H}\,\overrightarrow{T}(T^*S^n;\mathbb{Q}) & \xrightarrow{\sigma} & \Lambda_{\mathbb{Q}}
\end{array}
\tag{19}
$$

For $N = S^2$, we can compute all the groups explicitly. The Tate chain complex for T^*S^2 is shown in Figure 3, where the red numbers now denote the Conley–Zehnder index. To derive it, we first consider the round metric on S^2. For each $k \in \mathbb{N}$, the k-fold covered unparametrized
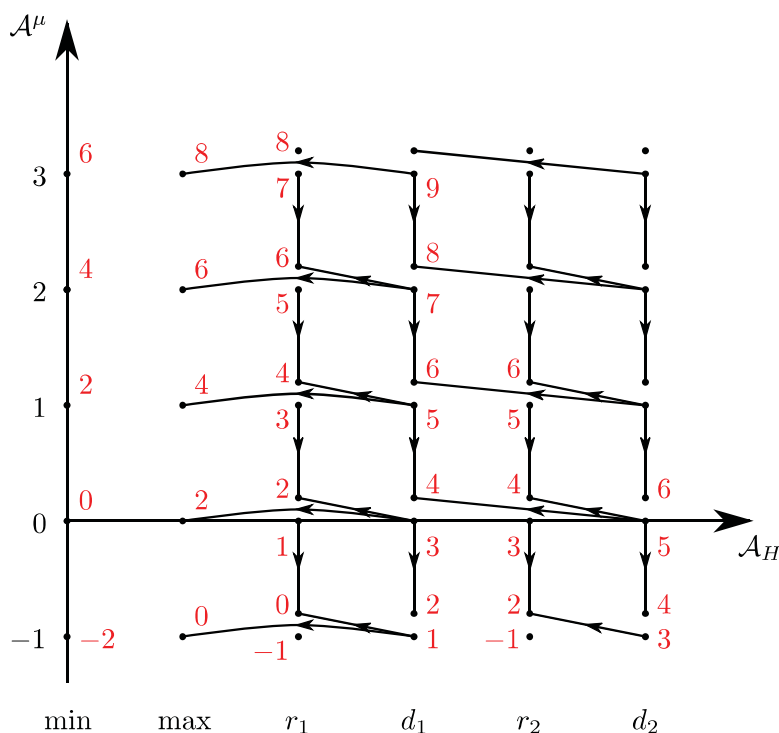
FIGURE 3 (colour online). *The Tate chain complex for T^*S^2.*

oriented closed geodesics have Morse index $2k - 1$ and form an S^2-family. Now we pick a non-reversible Finsler perturbation of the round metric with precisely two simple closed geodesics, the retrograde (shorter) one r_1 and the direct (longer) one d_1. For $k \in \mathbb{N}$, their iterates r_k and d_k correspond to the minimum and maximum of the corresponding S^2-family and thus have Morse index $2k - 1$ and $2k + 1$, respectively. Each of them gives rise to two generators $r_k^{\pm}, d_k^{\pm}$ of the chain complex for the non-equivariant homology of the loop space LS^2, corresponding to the minimum and maximum of a Morse function on the circle, of Conley–Zehnder (= Morse) indices

$$\mathrm{CZ}(r_k^-) = 2k - 1, \quad \mathrm{CZ}(r_k^+) = 2k, \quad \mathrm{CZ}(d_k^-) = 2k + 1, \quad \mathrm{CZ}(d_k^+) = 2k + 2.$$

The remaining two generators on the horizontal axis $\mathcal{A}^{\mu} = 0$ arise from the constant loops corresponding to the minimum and maximum of a Morse function on S^2 and have Conley–Zehnder index 0 and 2, respectively. The other generators in the diagram are obtained from these by applying u^k for $k \in \mathbb{Z}$.

The weights of the boundary operator can be determined as follows. As in the case of $\mathbb{C}^n$, the weights on the vertical arrows correspond to the multiplicities k of the closed geodesics r_k, d_k. The weights on the short horizontal arrows from d_k^- to r_k^+ must be 2 for each k in order for the four generators $r_k^{\pm}, d_k^{\pm}$ to compute the homology of $\mathbb{R}P^3$, the space of parametrized great circles on S^2. The matrix elements $\langle \partial r_k^+, d_{k-1}^- \rangle$ and $\langle \partial r_k^-, u^{-1} d_{k-1}^+ \rangle$ vanish by Lemma 4.8 because the multiplicities $k, (k-1)$ of r_k, d_{k-1} are relative prime. So we have

$$\partial r_k^- = k u^{-1} r_k^+, \quad \partial r_k^+ = 0, \quad \partial d_k^- = 2 r_k^+ + c_k d_{k-1}^+ + k u^{-1} d_k^+, \quad \partial d_k^+ = 0$$

for some coefficients $c_k \in \mathbb{Z}$.

We claim that the c_k *must be even*. To see this, we consider the loop space homology with integer coefficients, which according to [17] is given by the following quotient of an exterior
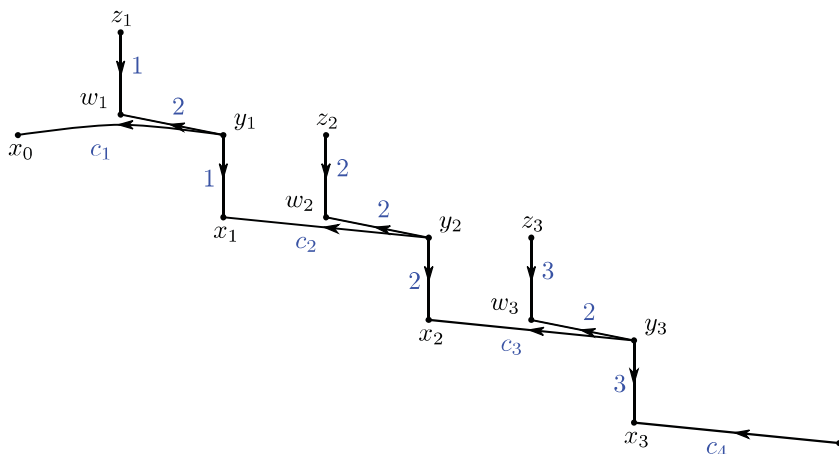
FIGURE 4 (colour online). *A connected component in the Tate chain complex for T^*S^2.*

algebra on three generators:

$$H_{*+2}(LS^2; \mathbb{Z}) \cong \Lambda[x, y, z]/\langle x^2, xz, 2xy \rangle, \quad |x| = -2, \ |y| = 2, \ |z| = -1.$$

From this, we read off that

$$H_k(LS^2; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z} & k \text{ odd}, \\ \mathbb{Z} \oplus \mathbb{Z}_2 & k > 0 \text{ even}. \end{cases}$$

Comparing with Figure 3, we see that, for each $k \geqslant 1$, the subcomplex consisting of d_{k-1}^+, r_k^+, d_k^- with the horizontal boundary operators must compute the homology $\mathbb{Z} \oplus \mathbb{Z}_2$. Since $\partial_{\mathrm{hor}} d_k^- = 2r_k^+ + c_k d_{k-1}^+$, the homology of this subcomplex equals $\mathbb{Z} \oplus \mathbb{Z}_2$ if c_k is even, and $\mathbb{Z}$ if c_k is odd. Thus c_k must be even. In Appendix B, we use the heat flow to show that the first weight c_1 in fact equals 2.

To compute the Tate homology groups, it is convenient to introduce the new generators

$$x_k := u^{-k-1} d_k^+, \quad y_k := u^{-k} d_k^-, \quad z_k := u^{-k+1} r_k^-, \quad w_k := u^{-k} r_k^+$$

of Conley–Zehnder indices $\mathrm{CZ}(x_k) = \mathrm{CZ}(w_k) = 0$ and $\mathrm{CZ}(y_k) = \mathrm{CZ}(z_k) = 1$, on which the boundary operator is given by

$$\partial z_k = w_k, \quad \partial w_k = 0, \quad \partial y_k = 2w_k + c_k x_{k-1} + k x_k, \quad \partial x_k = 0.$$

As in the case of $\mathbb{C}^n$, we see infinitely many connected components (related by the action of u) on which $\mathcal{A}_H \to \infty$ and $\mathcal{A}^\mu \to -\infty$. Figure 4 shows the component containing $x_0 := u^{-1} \max$ together with the weights of the boundary maps.

From Figures 3 and 4, we can now read off the symplectic Tate homology groups with rational coefficients. For this, observe that the rational homology of a component as in Figure 4 does not change if we remove all the generators w_k, z_k. The resulting component looks like the one in Figure 2, just with different weights c_k instead of 1 on the horizontal arrows.

We claim that all the c_k *are non-zero.* Arguing by contradiction, suppose that $c_k = 0$ and $c_i \neq 0$ for $i < k$, for some $k \geqslant 1$. Then the finite connected component containing x_0 in Figure 4 would contribute a direct summand $\Lambda_{\mathbb{Q}}$ to $\underrightarrow{H\,T}(T^*S^2; \mathbb{Q})$ in addition to the one generated by the min, contradicting $\underleftarrow{H\,T}(T^*S^2; \mathbb{Q}) = \Lambda_{\mathbb{Q}}$ from Proposition 5.3.

It follows from the claim that, with rational coefficients, each component has the same contribution to symplectic Tate homology as in the $\mathbb{C}^n$ case. Taking into account the contribution from the min in Figure 3 (which does not interact with anything else), we obtain

the diagram for T^*S^2 with rational coefficients:

$$
\begin{array}{ccc}
\Lambda_{\mathbb{Q}} \oplus \Lambda_{\mathbb{Q}} & \xrightarrow[\cong]{\rho} & \Lambda_{\mathbb{Q}} \oplus \Lambda_{\mathbb{Q}} \\
{\scriptstyle Hk}\downarrow & & \downarrow{\scriptstyle \kappa} \\
\Lambda_{\mathbb{Q}} & \xrightarrow{\sigma} & \Lambda_{\mathbb{Q}}
\end{array}
\tag{20}
$$

Cotangent bundles of tori. For $N = T^n$ with the flat metric the closed geodesics occur in Morse–Bott families, one for each class in $H_1(T^n)$, so there are no gradient trajectories between different components. Since the circle action is locally free in the non-trivial homology classes, their contribution to the Tate homology vanishes with rational coefficients. Thus the only contribution comes from the constants and we obtain the diagram for T^*T^n with rational coefficients:

$$
\begin{array}{ccc}
H(T^n;\mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}} & \xrightarrow[\cong]{\rho} & H(T^n;\mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}} \\
{\scriptstyle Hk}\downarrow & & \downarrow{\scriptstyle \kappa} \\
H(T^n;\mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}} & \xrightarrow{\sigma} & H(T^n;\mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda_{\mathbb{Q}}
\end{array}
\tag{21}
$$

REMARK 5.4. By the same argument, all the groups in the diagram coincide also for manifolds of negative sectional curvature, and for products of such manifolds (probably also for manifolds of non-positive sectional curvature). So, in this class of manifolds the Tate homology of a product is given by the tensor product of the Tate cohomologies via the Künneth formula. One may wonder whether such a result holds more generally.

EXAMPLE 5.5. We can construct a closed 4-manifold with the same fundamental group as T^n as follows. The n-fold connected sum $\#_n(S^1 \times S^3)$ has fundamental group the free group on n generators. Surgery on $\binom{n}{2}$ embedded loops representing all commutators of the generators yields a closed 4-manifold X_n with $\pi_1(X_n) = H_1(X_n) = \mathbb{Z}^n$. By the Mayer–Vietoris sequence, the second homology of X_n equals $\mathbb{Z}^{\binom{n}{2}} \oplus \mathbb{Z}^{\binom{n}{2}}$, where the first $\binom{n}{2}$ generators are represented by the 2-spheres pt $\times S^2$ in the surgery regions $D^2 \times S^2$, and the remaining $\binom{n}{2}$ generators are represented by the 2-tori obtained from the disks $D^2 \times$ pt in the surgery regions under the identifications on the boundary according to the commutators. So the total homology of X_n has dimension $d_n = 2 + 2n + 2\binom{n}{2} = n^2 + n + 2$. Moreover, each 2-sphere intersects the corresponding 2-torus in one point and therefore the intersection form of X_n is even. Since X_n has no 2-torsion in its first homology group, it follows from [**27**, Corollary 5.7.6] that X_n is spin. So we can apply Theorem 1.2 to obtain $\underleftarrow{H}\,\underrightarrow{T}(T^*X_n;\mathbb{Q}) = \underleftarrow{H}\,\underrightarrow{T}(T^*T^n;\mathbb{Q}) = \Lambda_{\mathbb{Q}}^{2^n}$ and we obtain the diagram for T^*X_n with rational coefficients:

$$
\begin{array}{ccc}
\Lambda_{\mathbb{Q}}^{d_n} & \xrightarrow[\cong]{\rho} & \Lambda_{\mathbb{Q}}^{d_n} \\
{\scriptstyle Hk}\downarrow & & \downarrow{\scriptstyle \kappa} \\
\underleftarrow{H}\,\underrightarrow{T}(T^*X_n;\mathbb{Q}) & \xrightarrow{\sigma} & \Lambda_{\mathbb{Q}}^{2^n}
\end{array}
\tag{22}
$$

Since $d_n > 2^n$ for $n \leqslant 4$ and $d_n < 2^n$ for $n \geqslant 6$, this shows that the map κ need neither be injective nor surjective.

Appendix A.   *Some presentations of the group $\mathbb{Q}$*

Let $a = (a_k)_{k \in \mathbb{N}}$ be a sequence of positive integers a_k. To such a sequence we associate a group G_a, presented as

$$G_a = \langle x_k, k \in \mathbb{N} \,|\, x_k - a_k x_{k+1}, k \in \mathbb{N} \rangle.$$

To a sequence $a = (a_k)$ we associate a sequence $a' = (a'_k)$ of prime numbers by removing all the a_k that are equal to 1, and replacing each $a_k > 1$ by the numbers in its prime decomposition, written in non-decreasing order and with repetitions. It is easy to see that $G_a = G_{a'}$. Note that the resulting sequence is finite if only finitely many a_k are bigger than 1, and empty if all the a_k are equal to 1; in these cases $G_a = G_{a'}$ is obviously isomorphic to $\mathbb{Z}$, generated by the last x_k. The following result shows that, for 'generic' sequences, the group G_a is isomorphic to $\mathbb{Q}$.

PROPOSITION A.1.   *Let $a = (a_k)$ be a sequence of positive integers whose associated sequence of primes (a'_k) satisfies*

$$\#\{k \in \mathbb{N} : a'_k = p\} = \infty \text{ for every prime number } p. \tag{A.1}$$

Then the group $G_a = G_{a'}$ is isomorphic to the additive group $\mathbb{Q}$.

Proof.   *Step* 1. By the preceding discussion, we may assume that $a = a'$ is a sequence of prime numbers. Let us first show that if a and b are two sequences of prime numbers satisfying (A.1), then the groups G_a and G_b are isomorphic.

We denote the generators of G_a by x_k and the generators of G_b by y_k. Our aim is to construct an isomorphism $h \colon G_a \to G_b$ explicitly. For two integers c and d we write $c|d$ if c divides d. We first define a function $m \colon \mathbb{N} \to \mathbb{N}$ as follows:

$$m(k) = \begin{cases} 1, & k = 1, \\ \min\left\{ \nu \in \mathbb{N} : \prod_{j=1}^{k-1} a_j \,\middle|\, \prod_{j=1}^{\nu-1} b_j \right\}, & k \geqslant 2. \end{cases}$$

Note that, by our assumption (A.1), the function m is well-defined. We next define, for every positive integer k, an integer c_k by

$$c_k = \begin{cases} 1, & k = 1, \\ \dfrac{\prod_{j=1}^{m(k)-1} b_j}{\prod_{j=1}^{k-1} a_j}, & k \geqslant 2. \end{cases}$$

We next define a homomorphism $h \colon G_a \to G_b$ on generators as

$$h(x_k) = c_k y_{m(k)}, \quad k \in \mathbb{N}.$$

It remains to check that h is well-defined, that is, it respects the relations in G_a. For that purpose we compute

$$\begin{aligned} a_k h(x_{k+1}) &= a_k c_{k+1} y_{m(k+1)} \\ &= a_k \frac{\prod_{j=1}^{m(k+1)-1} b_j}{\prod_{j=1}^{k} a_j} y_{m(k+1)} \\ &= \frac{\prod_{j=1}^{m(k)-1} b_j}{\prod_{j=1}^{k-1} a_j} \prod_{j=m(k)}^{m(k+1)-1} b_j y_{m(k+1)} \end{aligned}$$

$$= c_k y_{m(k)}$$
$$= h(x_k).$$

This proves that h is a well-defined homomorphism.

We next check that h is injective. For this purpose, we suppose that $\xi \in G_a$ satisfies

$$h(\xi) = 0.$$

In view of the structure of the group G_a there exists $d \in \mathbb{Z}$ and a generator x_k such that

$$\xi = dx_k.$$

Hence we obtain

$$0 = dh(x_k) = dc_k y_{m(k)}.$$

Since there is no torsion in G_b and $c_k \neq 0$, we conclude that $d = 0$ and consequently

$$\xi = 0.$$

This proves that h is injective.

It remains to check that h is surjective. For this purpose choose $\eta \in G_b$. Since the function m is unbounded, there exist $k \in \mathbb{N}$ and $e \in \mathbb{Z}$ such that

$$\eta = ey_{m(k)}.$$

In view of our assumption (A.1) there exists $\ell > k$ such that

$$c_k \left| \prod_{j=k}^{\ell} a_j \right. .$$

We compute

$$c_k \left(h \left(\frac{e \prod_{j=k}^{\ell} a_j}{c_k} x_{\ell+1} \right) - \eta \right) = eh(x_k) - c_k e y_{m(k)} = 0,$$

and, therefore, since there is no torsion in G_b, we obtain

$$h \left(\frac{e \prod_{j=k}^{\ell} a_j}{c_k} x_{\ell+1} \right) = \eta.$$

This proves that h is surjective. In particular, we have shown that h is a group isomorphism.

Step 2. It remains to prove $G_a \cong \mathbb{Q}$ for some sequence (a_k) whose associated sequence of prime numbers satisfies (A.1). We choose the sequence $a_k = k + 1$, so the group G_a is

$$G_a = \langle x_1, x_2, \ldots \mid x_1 - 2x_2, \ x_2 - 3x_3, \ x_3 - 4x_4, \ldots \rangle.$$

Note that in G_a we have, for all $q < q'$, the identity

$$x_q = \frac{q'!}{q!} x_{q'}.$$

We claim that the required group isomorphism is given by

$$\phi : \mathbb{Q} \longrightarrow G_a, \quad \frac{p}{q} \longmapsto p(q-1)!x_q.$$

Here p, q need not be relatively prime. Let us first check that ϕ is well-defined. For this, we compute, for $r \in \mathbb{N}$,

$$\phi \left(\frac{p}{q} \right) = p(q-1)!x_q = \frac{p(q-1)!(qr)!}{q!} x_{qr} = \frac{p(qr)!}{q} x_{qr}$$
$$= pr(qr-1)!x_{qr} = \phi \left(\frac{pr}{qr} \right).$$

To show that ϕ is a group homomorphism, we compute

$$
\begin{aligned}
\phi\left(\frac{p}{q}\right) + \phi\left(\frac{p'}{q'}\right) &= p(q-1)!x_q + p'(q'-1)!x_{q'} \\
&= \left(p(q-1)!\frac{(qq')!}{q!} + p'(q'-1)!\frac{(qq')!}{q'!}\right)x_{qq'} \\
&= \left(\frac{p}{q} + \frac{p'}{q'}\right)(qq')!x_{qq'} \\
&= (pq' + p'q)(qq'-1)!x_{qq'} \\
&= \phi\left(\frac{p}{q} + \frac{p'}{q'}\right).
\end{aligned}
$$

Injectivity of ϕ holds because $rx_q \neq 0$ in G_a for any $r, q \in \mathbb{N}$, and surjectivity follows from

$$
\phi\left(\frac{1}{q!}\right) = (q!-1)!x_{q!} = (q!-1)!\frac{q!}{(q!)!}x_q = x_q.
$$

This concludes the proof of Proposition A.1. $\square$

Appendix B.   The heat flow on S^2

In view of the results of Salamon and Weber [**37**, **45**], symplectic homology of a cotangent bundle can be understood in terms of the heat flow. In this appendix, we use the heat flow on S^2 to compute the coefficient c_1 in Figure 4 to be $c_1 = 2$.

We consider $S^2 = \{x \in \mathbb{R}^3 \mid |x| = 1\}$ endowed with its standard round metric, that is, the metric induced from the standard metric on $\mathbb{R}^3$. We abbreviate by $\mathscr{L} = C^\infty(\mathbb{R}/\mathbb{Z}, S^2)$ the free loop space. Closed geodesics on S^2 arise as critical points of the energy functional $E \colon \mathscr{L} \to \mathbb{R}$ given for $v \in \mathscr{L}$ by

$$
E(v) = \frac{1}{2}\int_0^1 |\dot{v}|^2 \, dt.
$$

Negative L^2-gradient flow lines of E can be interpreted as solutions $v \in C^\infty(\mathbb{R} \times \mathbb{R}/\mathbb{Z}, S^2)$ of the parabolic PDE

$$
\partial_s v = \nabla_t \partial_t v = \partial_t^2 v + (\partial_t v)^2 v. \tag{B.1}
$$

We consider the geodesic $v_- \in \mathscr{L}$ given by

$$
v_-(t) = (0, \cos(2\pi t), \sin(2\pi t))
$$

and our aim is to determine its unstable manifold $W^u(v_-)$ with respect to the negative L^2-gradient of E. We identify $W^u(v_-)$ with all solutions v of (B.1) satisfying

$$
\lim_{s \to -\infty} v(s, t) = v_-(t) \tag{B.2}
$$

via the map

$$
v \longmapsto v(0, \cdot) \in \mathscr{L}.
$$

For $r \in \mathbb{R}/\mathbb{Z}$ let R_r be the orthogonal 3×3-matrix

$$
R_r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\pi r) & -\sin(2\pi r) \\ 0 & \sin(2\pi r) & \cos(2\pi r) \end{pmatrix}.
$$

Consider the following circle action on $\mathscr{L}$; namely, for $r \in \mathbb{R}/\mathbb{Z}$ and $v \in \mathscr{L}$ define $r_* v \in \mathscr{L}$ by

$$r_* v(t) = R_r v(t - r).$$

Note that v_- is fixed under this circle action. Moreover, both the energy functional E and the L^2-metric are invariant under this circle action and therefore the gradient flow equation (B.1) is invariant as well. Since the Morse index of E at v_- is 1, we conclude that its unstable manifold is one-dimensional. Therefore, each $v \in W^u(v_-)$ has to be fixed under the S^1-action. We conclude that there exist smooth functions x and y depending only on the s-variable such that v can be written as

$$v(s, t) = \begin{pmatrix} x(s) \\ \sqrt{1 - x(s)^2} \cos(2\pi t - y(s)) \\ \sqrt{1 - x(s)^2} \sin(2\pi t - y(s)) \end{pmatrix}.$$

Applying (B.1) to this expression, we arrive at the following system of equations:

$$\left. \begin{array}{r} \partial_s x = 4\pi^2 (1 - x^2) x, \\ \left(\dfrac{x \partial_s x}{\sqrt{1 - x^2}} - 4\pi^2 x^2 \sqrt{1 - x^2} \right) \cos(2\pi t - y) = \sqrt{1 - x^2} \sin(2\pi t - y) \partial_s y, \\ \left(\dfrac{x \partial_s x}{\sqrt{1 - x^2}} - 4\pi^2 x^2 \sqrt{1 - x^2} \right) \sin(2\pi t - y) = -\sqrt{1 - x^2} \cos(2\pi t - y) \partial_s y. \end{array} \right\}$$

Plugging the first equation into the latter two equations, we obtain the following equivalent system:

$$\left. \begin{array}{r} \partial_s x = 4\pi^2 (1 - x^2) x, \\ \partial_s y = 0. \end{array} \right\}$$

In particular, y is constant and in view of the asymptotic behaviour (B.2) we conclude

$$y = 0.$$

The first equation for x is a Bernoulli-type ODE whose explicit solution to the initial condition

$$x(0) = x_0 \in (-1, 1)$$

is given by the expression

$$x(s) = \frac{x_0}{\sqrt{(1 - x_0^2) e^{-8\pi^2 s} + x_0^2}}.$$

Hence v becomes

$$v(s, t) = \begin{pmatrix} \dfrac{x_0}{\sqrt{(1 - x_0^2) e^{-8\pi^2 s} + x_0^2}} \\ \dfrac{(1 - x_0)^2 e^{-8\pi^2 s}}{(1 - x_0^2) e^{-8\pi^2 s} + x_0^2} \cos(2\pi t) \\ \dfrac{(1 - x_0)^2 e^{-8\pi^2 s}}{(1 - x_0^2) e^{-8\pi^2 s} + x_0^2} \sin(2\pi t) \end{pmatrix}.$$

Note that the positive asymptotic $\lim_{s \to \infty} v(s, t)$ of v is given by the constant loop $\pm(1, 0, 0)$, where the sign coincides with the sign of x_0. Using invariance of the gradient flow under the action of the orthogonal group $O(3)$ on S^2, one can now determine explicitly the unstable manifold for any simple closed geodesic on S^2.

It follows that the fixed positive asymptotic $(1, 0, 0)$ is hit by precisely two isolated gradient flow lines. One of these two gradient flow lines has negative asymptotic $t \mapsto (0, \cos(2\pi t), \sin(2\pi t))$ and the other one has negative asymptotics $t \mapsto (0, \cos(-2\pi t), \sin(-2\pi t))$. This shows that the coefficient c_1 in Figure 4 equals 2, so twice the fundamental class of S^2 becomes zero in localized equivariant loop space homology.

References

1. A. ABBONDANDOLO and M. SCHWARZ, 'On the Floer homology of cotangent bundles', *Comm. Pure Appl. Math.* 59 (2006) 254–316.
2. M. ABOUZAID, 'Symplectic cohomology and Viterbo's theorem', Preprint, 2013, arXiv:1312.3354.
3. R. ABRAHAM and J. MARSDEN, *Foundations of mechanics*, 2nd edn (Benjamin/Cummings Publishing Co., Menlo Park, CA, 1978).
4. A. ADEM, R. L. COHEN and W. DWYER, 'Generalized Tate homology, homotopy fixed points and the transfer', *Algebraic topology*, Contemporary Mathematics 96 (American Mathematical Society, Providence, RI, 1989) 1–13.
5. M. ATIYAH and R. BOTT, 'The moment map and equivariant cohomology', *Topology* 23 (1984) 1–28.
6. F. BOURGEOIS and A. OANCEA, 'Symplectic homology, autonomous Hamiltonians, and Morse–Bott moduli spaces', *Duke Math. J.* 146 (2009) 71–174.
7. F. BOURGEOIS and A. OANCEA, 'The Gysin exact sequence for S^1-equivariant symplectic homology', *J. Topol. Anal.* 5 (2013) 361–407.
8. F. BOURGEOIS and A. OANCEA, 'S^1-equivariant symplectic homology and linearized contact homology', Preprint, 2012, arXiv:1212.3731.
9. K. BROWN, *Cohomology of groups*, Graduate Texts in Mathematics 87 (Springer, New York, 1994).
10. M. CENCELJ, 'Jones–Petrack cohomology', *Q. J. Math. Oxford Ser.* (2) 46 (1995) 409–415.
11. K. CIELIEBAK, 'Handle attaching in symplectic homology and the chord conjecture', *J. Eur. Math. Soc. (JEMS)* 4 (2002) 115–142.
12. K. CIELIEBAK, A. FLOER and H. HOFER, 'Symplectic homology II: a general construction', *Math. Z.* 218 (1995) 103–122.
13. K. CIELIEBAK and U. FRAUENFELDER, 'A Floer homology for exact contact embeddings', *Pacific J. Math.* 239 (2009) 251–316.
14. K. CIELIEBAK and U. FRAUENFELDER, 'Morse homology on noncompact manifolds', *J. Korean Math. Soc.* 48 (2011) 749–774.
15. K. CIELIEBAK and K. MOHNKE, 'Symplectic hypersurfaces and transversality in Gromov–Witten theory', *J. Symplectic Geom.* 5 (2007) 281–356.
16. K. CIELIEBAK and A. OANCEA, 'Symplectic homology and the Eilenberg-Steenrod axioms', Preprint, 2015, arXiv:1511.00485.
17. R. COHEN, J. JONES and J. YAN, 'The loop homology algebra of spheres and projective spaces', *Categorical decomposition techniques in algebraic topology* (Isle of Skye, 2001), Progress in Mathematics 215 (Birkhäuser, Basel , 2004) 77–92.
18. B. ECKMANN and P. HILTON, 'Commuting limits with colimits', *J. Algebra* 11 (1969) 116–144.
19. S. EILENBERG and N. STEENROD, *Foundations of algebraic topology* (Princeton University Press, Princeton, NJ, 1952).
20. Y. ELIASHBERG, A. GIVENTAL and H. HOFER, 'Introduction to symplectic field theory', *Geom. Funct. Anal.*, Special Volume, Part II (2000) 560–673.
21. A. FLOER and H. HOFER, 'Coherent orientations for periodic orbit problems in symplectic geometry', *Math. Z.* 212 (1993) 13–38.
22. A. FLOER and H. HOFER, 'Symplectic homology I: open sets in C^n', *Math. Z.* 215 (1994) 37–88.
23. U. FRAUENFELDER, 'The Arnold–Givental conjecture and moment Floer homology', *Int. Math. Res. Not.* 2004 (2004) 2179–2269.
24. U. FRAUENFELDER, 'Vortices on the cylinder', *Int. Math. Res. Not.* (2006). Art. Id 63130.
25. U. FRAUENFELDER and F. SCHLENK, 'Hamiltonian dynamics on convex symplectic manifolds', *Israel J. Math.* 159 (2007) 1–56.
26. A. FREI and J. MACDONALD, 'Limits in categories of relations and limit-colimit commutation', *J. Pure Appl. Algebra* 1 (1971) 179–197.
27. R. GOMPF and A. STIPSICZ, *4-Manifolds and Kirby calculus* (American Mathematical Society, Providence, RI, 1999).
28. T. GOODWILLIE, 'Cyclic homology, derivations and the free loop space', *Topology* 24 (1985) 187–217.
29. J. GREENLEES and J. MAY, 'Generalized Tate cohomology', *Mem. Amer. Math. Soc.* 113 (1995) viii + 178 pp.
30. A. HATCHER, *Algebraic topology* (Cambridge University Press, Cambridge, 2002).
31. K. IRIE, 'Hofer–Zehnder capacity of unit disk cotangent bundles and the loop product', Preprint, 2011, arXiv:1110.2244.
32. N. JACOBSON, *Basic algebra I*, 2nd edn (Dover Publisher, New York, 2009).
33. J. JONES and S. PETRACK, 'The fixed point theorem in equivariant cohomology', *Trans. Amer. Math. Soc.* 322 (1990) 35–49.
34. T. KRAGH, 'Parametrized ring-spectra and the nearby lagrangian conjecture', *Geom. Topol.* 17 (2013) 639–731.
35. M. MCLEAN, 'Symplectic invariance of uniruled affine varieties and log Kodaira dimension', *Duke Math. J.* 163 (2014) 1929–1964.
36. D. SALAMON, 'Lectures on Floer homology', *Symplectic geometry and topology* (Park City, 1997), IAS/Park City Mathematics Series 7 (American Mathematical Society, Providence, RI, 1999) 143–229.
37. D. SALAMON and J. WEBER, 'Floer homology and the heat flow', *Geom. Funct. Anal.* 16 (2006) 1050–1138.
38. M. SCHWARZ, *Morse homology*, Progress in Mathematics 111 (Birkhäuser, Basel, 1993).

39. P. SEIDEL, *A biased view of symplectic cohomology*, Current Developments in Mathematics 2006 (Int. Press, Somerville, MA, 2008) 211–253.

40. R. SWAN, 'A new method in fixed point theory', *Comment. Math. Helv.* 34 (1960) 1–16.

41. H. TENE, 'Some geometric equivariant cohomology theories', Preprint, 2012, arXiv:1210.7923.

42. C. VITERBO, 'Functors and computations in Floer homology with applications. II', Preprint, 1998, Université Paris-Sud no 98-15. http://www.math.ens.fr/∼viterbo/FCFH.II.2003.pdf.

43. C. VITERBO, 'Functors and computations in Floer homology with applications I', *Geom. Funct. Anal.* 9 (1999) 985–1033.

44. A. WASSERMAN, 'Equivariant differential topology', *Topology* 8 (1969) 127–150.

45. J. WEBER, 'Morse homology for the heat flow', *Math. Z.* 275 (2013) 1–54.

46. C. WEIBEL, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics 38 (Cambridge University Press, Cambridge, 1994).

47. J. ZHAO, 'Periodic symplectic cohomologies', Preprint, 2014, arXiv:1405.2084.

Peter Albers
Mathematisches Institut
Westfälische Wilhelms-Universität Münster
Einsteinstrasse 62
D-48149 Münster
Germany

peter.albers@wwu.de

Kai Cieliebak
Mathematisches Institut
Universität Augsburg
Universitätsstrasse 14
D-86159 Augsburg
Germany

kai.cieliebak@math.uni-augsburg.de

Urs Frauenfelder
Department of Mathematics and Research
Institute of Mathematics
Seoul National University
Seoul
South Korea

frauenf@snu.ac.kr