

Periodic Orbits in the Restricted Three-body Problem and Arnold's J^+ -invariant

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Abstract—We apply Arnold's theory of generic smooth plane curves to Stark–Zeeman systems. This is a class of Hamiltonian dynamical systems that describes the dynamics of an electron in an external electric and magnetic field, and includes many systems from celestial mechanics. Based on Arnold's J^+ -invariant, we introduce invariants of periodic orbits in planar Stark–Zeeman systems and study their behavior.

Keywords: generic immersions into the plane, Arnold's plane curve invariants, restricted three-body problem

1. INTRODUCTION

By the Whitney–Graustein theorem, the rotation number of an immersed loop in the plane completely classifies such loops up to regular homotopy. During a generic regular homotopy three disasters can occur, namely, triple intersections as well as direct and inverse self-tangencies. Arnold [3] noticed that the theory of immersions in the plane becomes extremely rich if one allows only generic homotopies avoiding the three disasters. He introduced three invariants. The one of interest in this paper is the invariant J^+ , which is invariant under triple intersections and inverse self-tangencies but is sensitive to direct self-tangencies.

Immersions without direct self-tangencies are of geometric interest because their conormal lifts give Legendrian embeddings in the unit cotangent bundle of the plane [3]. They are also natural from a physical point of view: orbits of a particle on the plane moving in a conservative force field cannot have direct self-tangencies, whereas inverse self-tangencies can occur if the force is allowed to be velocity-dependent (like the Coriolis force or the Lorentz force).

To analyze this phenomenon we consider conservative force fields on the plane generated by a vector potential and a scalar potential, which are smooth except for a $1/r$ singularity at the origin for the scalar potential. Such systems describe the motion of an electron in the presence of a proton and additional electric and magnetic fields, which is why we call them *Stark–Zeeman systems*. On the other hand, Newton's law of gravitation is mathematically equivalent to Coulomb's law and the Coriolis force is modelled in the same way as the Lorentz force. Therefore, Stark–Zeeman systems also describe many interesting cases of the restricted three-body problem in celestial mechanics, in which a satellite experiences the gravitational forces of two primary masses (which might be the earth and the moon) as well as the centrifugal force and the Coriolis force.

In a generic homotopy of periodic orbits of Stark–Zeeman systems, apart from triple intersections and inverse self-tangencies, the following two events can occur. First, the orbit might have velocity zero for some isolated moment in time. If a family of periodic orbits moves through this event, an exterior loop is created or destroyed as shown in Fig. 2. This phenomenon was

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first observed by Hill in his discovery of the “moon of maximal lunarity” and will be recalled in Section 2. Secondly, the orbit might pass through the origin, corresponding to a collision of the satellite with one of the primaries in the three-body problem. If this occurs in a family, a loop around the origin is created or destroyed as shown in Fig. 4. See also Fig. 1 for a family of periodic orbits in the restricted 3-body problem in which both events occur. We refer to such homotopies as *Stark–Zeeman homotopies*. The task we address is the now following.

Task: *Find invariants under Stark–Zeeman homotopies.*

We introduce two such invariants both based on Arnold's J^+ -invariant. The first invariant \mathcal{J}_1 is a combination of Arnold's J^+ -invariant with the winding number around the origin. For the second invariant \mathcal{J}_2 we Levi-Civita regularize collisions and then take the J^+ -invariant of the regularized orbit. Our first result is

Theorem A. *\mathcal{J}_1 and \mathcal{J}_2 are invariant under Stark–Zeeman homotopies.*

We then ask how independent the two invariants are and prove

Theorem B. *If the winding number of the loop around the origin is odd, the two invariants determine each other, while in the case of an even winding number they are completely independent.*

This paper is organized as follows. In Section 2 we recall the role of families of orbits in the search for periodic orbits in celestial mechanics. In Section 3 we introduce the class of Stark–Zeeman systems and describe prominent examples (which include several integrable systems) as well as their Levi-Civita and Moser regularizations. In Section 4 we introduce the notion of a Stark–Zeeman homotopy. After recalling the definition of Arnold's J^+ -invariant, we introduce the invariants \mathcal{J}_1 and \mathcal{J}_2 and prove Theorems A and B.

2. FAMILIES OF PERIODIC ORBITS IN THE RESTRICTED THREE-BODY PROBLEM

According to Poincaré's famous dictum, “periodic orbits are the only tool to enter an otherwise impenetrable stronghold” [19]. Therefore the search for periodic orbits in the restricted 3-body problem has a long history. Everybody having some experience with the search for these objects knows that one does not find them by trial and error. Instead, one usually looks at families of periodic orbits. The family parameter can be, for example, the energy or the mass ratio of the two primaries. One starts the family with an obvious periodic orbit. This can, for example, be a critical point, i. e., a Lagrange point, or a periodic orbit in a completely integrable system. For instance, as the energy goes to minus infinity, the restricted 3-body problem just becomes the Kepler problem, for which on each negative energy level one has a circular periodic orbit. As this orbit can be traversed forwards and backwards, it gives rise to two periodic orbits in the restricted 3-body problem, the retrograde and the direct one. Strictly speaking, there are actually four of these orbits since one has direct and retrograde orbits around each of the primaries. Starting with these orbits for very low energy one then carries out a numerical homotopy by increasing the energy to obtain a family of periodic orbits.

The pioneer of this approach was Hill [13]. In his theory of the moon he considered a limiting case of the restricted 3-body problem in which one of the primaries (the sun) becomes infinitely much heavier than the second primary but in turn is infinitely far away. Hill's major interest was the direct periodic orbit, since this orbit provides a good model for the orbit of the moon around the earth. In the actual restricted 3-body problem, pioneers of this method were Darwin [7, 8] and Moulton [17]. In particular, Darwin considered the case where one of the primaries is ten times heavier than the other. An extremely detailed study over many years was carried out by Strömngren and his school at the observatory of Copenhagen [21]. The case considered by Strömngren is where the two primaries have equal mass. The current notation of these families is as well due to the school of Strömngren. In this notation the family of the retrograde orbit is referred to as the f -family, the family of the direct one as the g -family, where the families a through e start at the Lagrange points. For more information we refer to the book by Szebehely [22].

A different approach to Hill's direct periodic orbit was found by Lyapunov [15]. In Hill's lunar problem three forces are acting: the gravitational force of the earth, the Coriolis force, and a third force which one can interpret as a tidal force. This tidal force arises in the limit process in which

the gravitational force of the infinitely heavy but infinitely far away sun and the centrifugal force centered in the infinitely far away sun cancel each other to first order. While the gravitational force of the earth and the Coriolis force are rotationally invariant, the tidal force breaks the rotational symmetry. Lyapunov symmetrizes the tidal force and then considers a homotopy between the tidal force and its symmetrization. During the homotopy he fixes the period so that he obtains a one-parameter family. For the symmetrized tidal force there exist circular periodic orbits. The homotopy leads to a family of periodic orbits which interpolates between a circular periodic orbit in the symmetrized problem and the direct periodic orbit in Hill's lunar problem, see also the paper by Ryabov [20].

Poincaré [19] found another method to obtain periodic orbits in the restricted 3-body problem, namely, by fixing energy but varying the mass ratio of the two primaries. As the mass ratio goes to zero, one obtains the rotating Kepler problem, i.e., the Kepler problem in rotating coordinates. This problem is again completely integrable and its periodic orbits are well known, so that they can be used as the starting points for a family. There are two kinds of periodic orbits in the rotating Kepler problem: circular ones and honestly elliptical ones. Poincaré referred to periodic orbits in the restricted three-body problem obtained from the circular ones as periodic orbits of the first kind, and to the latter ones as periodic orbits of the second kind. Deep interest in the periodic orbits of the second kind was sparked during the Apollo program through the work of Arenstorf [1, 2]. For more recent developments about this approach we refer to the books by Bruno [5] and Hénon [11].

In a family of periodic orbits some disasters can occur. The first disaster is that the velocity of a periodic orbit can vanish at some moment. As the acceleration at this moment need not vanish, the periodic orbit need not be constant. However, it is no longer immersed but has a cusp where its velocity vanishes. This phenomenon was observed by Hill [13] in his “moon of maximal lunarity”. Here “lunarity” refers to the period of the orbit, which in the case of the moon is about the length of a month. Hill used the period as the family parameter which is changing with changing energy. He observed that for some length of the month which is more than half a year, the velocity of the direct periodic orbit can vanish at some moment. Hill thought that the family of the direct periodic orbit might stop at this moment and wrote

Whether this class of satellites is properly to be prolonged beyond this moon, can only be decided by further employment of mechanical quadratures. But it is at least certain that the orbits, if they do exist, do not intersect the line of quadratures, and that the moons describing them would make oscillations to and for, never departing as much as 90 degrees from the point of conjunction or opposition.

This reasoning was criticized by Adams and Poincaré [4, Chapter 2.3.9]. In particular, Poincaré explained [19, Chapter III] that after increasing the period further the cusps should give rise to little loops. Numerically, the phenomenon predicted by Poincaré was verified by Matukuma [16] and Hénon [10].

The second disaster which can occur along a family are collisions. However, two-body collisions can be regularized and the family can be continued beyond the collision. In view of these disasters, we see that neither the rotation number nor the winding number around the primary are preserved in a family of periodic orbits. Nevertheless, in this paper we construct invariants for families of periodic orbits which are not affected by these disasters.

To conclude this section, we include a family of periodic orbits in the restricted three-body problem, found by numerical integration and a simple shooting argument, that illustrate three of the disasters, see Fig. 1.

3. PLANAR STARK–ZEEMAN SYSTEMS

3.1. Stark–Zeeman Systems

A *Stark–Zeeman system* describes the dynamics of an electron attracted by a proton subject to an exterior electric and magnetic field. We suppose that the proton is at the origin, and consider

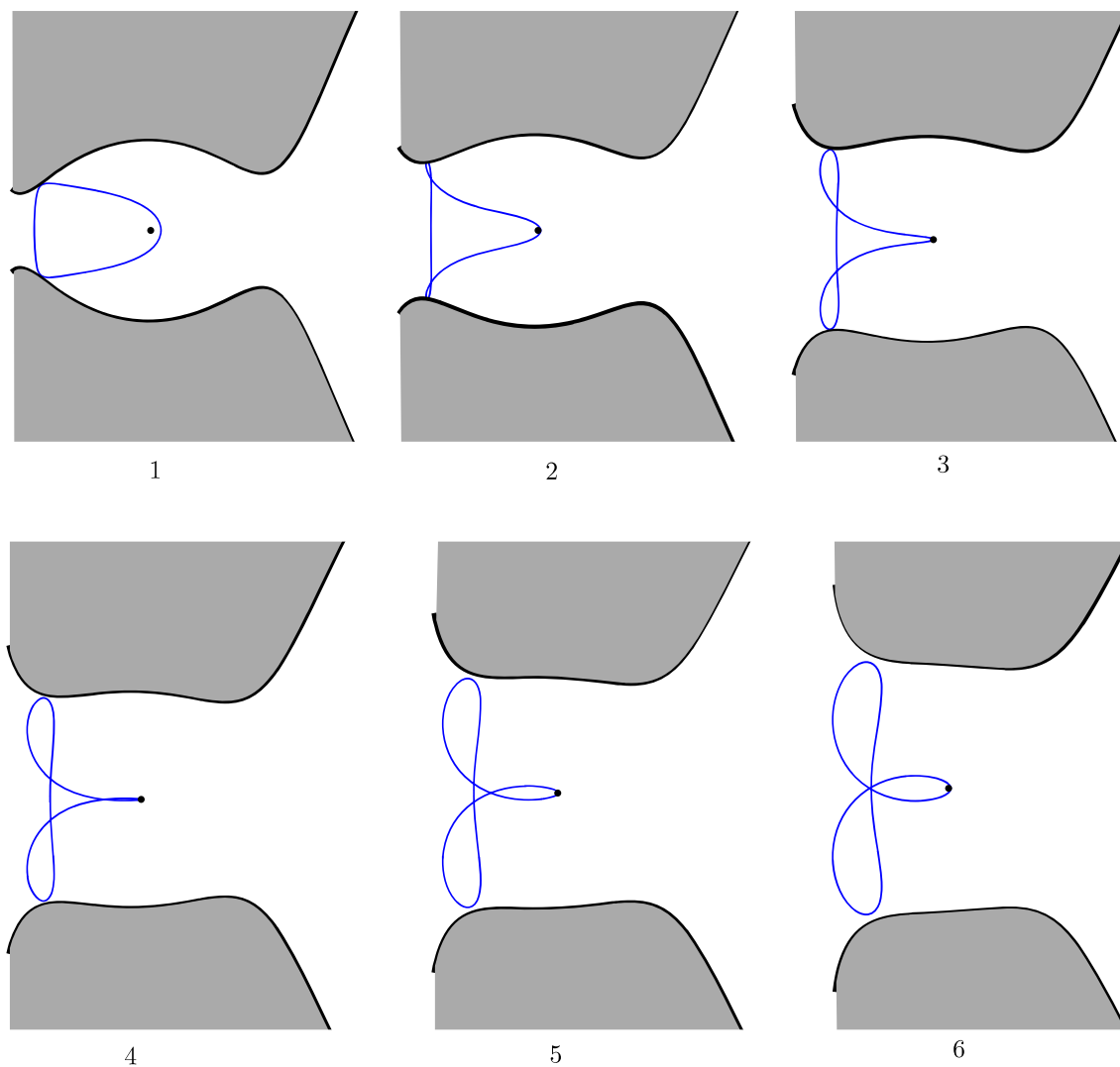


Fig. 1. Some members of a 1-parameter family of periodic orbits in the restricted 3-body problem ($\mu = 0.99$) with Jacobi energy c ranging from 1.575 to 1.55. When projected to the q -plane, the resulting family of immersions goes through several disasters. Specifically, between 1 and 2 event (I_∞) occurs: an exterior loop forms. Between 3 and 4 event (I_0) occurs: an interior loop forms. Finally, between 5 and 6 the vertical arc goes through a triple point. This is event (III). These are three of the four possible disasters that can happen in a Stark–Zeeman homotopy, see Definition 1.

the spatial, so 3-dimensional, Coulomb potential¹⁾ for a particle in \mathbb{R}^n

$$V_0: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}, \quad q \mapsto -\frac{1}{|q|}.$$

The exterior electric field is described by a smooth potential $V_1: U_0 \rightarrow \mathbb{R}$, where $U_0 \subset \mathbb{R}^n$ is an open subset containing the origin. We abbreviate $U := U_0 \setminus \{0\}$. The full potential is then the sum

$$V = V_0 + V_1: U \rightarrow \mathbb{R}.$$

¹⁾The n -dimensional Coulomb potential has the form $-1/|q|^{n-2}$ for $n > 2$, and $\log |q|$ for $n = 2$. However, the spatial potential has both special mathematical (for example, for regularization) and physical properties (for example, physical space has 3 dimensions), which are essential for our purposes. This is also done traditionally. Indeed, in the planar Kepler problem, one still considers the spatial potential.

The magnetic field is modeled by a closed 2-form

$$\sigma_B = \frac{1}{2} \sum_{i,j=1}^n B_{ij}(q) dq_i \wedge dq_j \in \Omega^2(U_0), \quad B_{ij} = -B_{ji}.$$

It induces the twisted symplectic form

$$\omega_B = \sum_{i=1}^n dp_i \wedge dq_i + \pi^* \sigma_B$$

on the cotangent bundle T^*U , where $\pi: T^*U \rightarrow U$ is the footpoint projection. We consider the Hamiltonian

$$H_V: T^*U \rightarrow \mathbb{R}, \quad (q, p) \mapsto \frac{1}{2}|p|^2 + V(q),$$

where the norm is taken with respect to the standard metric on \mathbb{R}^n . The Hamiltonian vector field X_V^B is implicitly defined by the condition

$$-dH_V = \omega_B(X_V^B, \cdot).$$

Explicitly, it is given by the formula

$$X_V^B = \sum_i p_i \frac{\partial}{\partial q_i} + \sum_i \left(\sum_j B_{ij}(q) p_j - \frac{\partial V(q)}{\partial q_i} \right) \frac{\partial}{\partial p_i}.$$

In particular, in terms of the matrix $B = (B_{ij})$ Hamilton's equation for X_V^B reads

$$\begin{cases} \dot{q} = p, \\ \dot{p} = B(q)p - \nabla V(q), \end{cases}$$

or interpreted as a second-order ODE,

$$\ddot{q} = B(q)\dot{q} - \nabla V(q). \quad (3.1)$$

If the summands on the right-hand side are interpreted as forces, then the potential leads to a force which is only position-dependent but independent of the velocity, while the magnetic field gives rise to a force depending linearly on the velocity. For $c \in \mathbb{R}$ we abbreviate the energy hypersurface of energy c by

$$\Sigma_c := H_V^{-1}(c).$$

Then the *Hill's region* is defined as

$$\mathfrak{K}_c := \pi(\Sigma_c) = \{q \in U \mid V(q) \leq c\}.$$

In view of the singularity of V at the origin, the Hill's region has a unique connected component

$$\mathfrak{K}_c^b \subset \mathfrak{K}_c$$

which contains 0 in its closure. We denote by

$$c_1 = c_1(B, V) \in \mathbb{R} \cup \{\infty\}$$

the supremum over all real numbers c_1 such for all $c < c_1$ the following holds: c is a regular value of V (and therefore as well of H_V), and the component \mathfrak{K}_c^b is diffeomorphic to a closed disk from which an interior point is removed (note that this always holds for c sufficiently small). For example, if $V = V_0$ is the Coulomb potential, then $c_1 = 0$ is referred to as the "escape threshold" in the physics literature, see, for instance, [9]. If V_1 is defined on the whole space \mathbb{R}^n and goes to $-\infty$ as $|q|$ goes to infinity, then c_1 is the first critical value of V (respectively H_V).

In this paper we will restrict to the case that σ_B is exact (this is automatic in the planar case $n = 2$), so $\sigma_B = dA$ for a 1-form

$$A = \sum_j A_j(q) dq_j \in \Omega^1(U_0), \quad B_{ij} = \frac{\partial A_j}{\partial q_i} - \frac{\partial A_i}{\partial q_j}.$$

Then we can untwist the symplectic form by the map

$$\Phi_A: T^*U \rightarrow T^*U, \quad (q, p) \mapsto (q, p - A(q)).$$

Indeed, Φ_A pulls back the twisted symplectic form to the standard symplectic form

$$\Phi_A^* \omega_B = \omega_0 = \sum_i dp_i \wedge dq_i$$

and the Hamiltonian H_V to

$$H_V^A: T^*U \rightarrow \mathbb{R}, \quad (q, p) \mapsto \frac{1}{2}|p - A(q)|^2 + V(q),$$

so Hamilton's equations of H_V with respect to the twisted symplectic form ω_B transform into Hamilton's equations of the twisted Hamiltonian H_V^A with respect to the untwisted form ω_0 . Explicitly, the Hamiltonian vector field X_V^A of the Hamiltonian with respect to the untwisted symplectic form ω_0 equals

$$X_V^A = \sum_i (p_i - A_i(q)) \frac{\partial}{\partial q_i} + \sum_i \left(\sum_j \frac{\partial A_j(q)}{\partial q_i} (p_j - A_j(q)) - \frac{\partial V(q)}{\partial q_i} \right) \frac{\partial}{\partial p_i},$$

hence Hamilton's equation for X_V^A reads

$$\begin{cases} \dot{q}_i = p_i - A_i(q), \\ \dot{p}_i = \sum_j \frac{\partial A_j(q)}{\partial q_i} (p_j - A_j(q)) - \frac{\partial V(q)}{\partial q_i}, \end{cases}$$

which gives rise to the same second-order ODE (3.1).

The planar case. A *planar Stark-Zeeman system* is a Stark-Zeeman system with $n = 2$ degrees of freedom. In this case the magnetic field can be written as

$$\sigma_B = B(q) dq_2 \wedge dq_1$$

for a smooth function $B : U_0 \rightarrow \mathbb{R}$ (so our convention is such that positive B corresponds to a magnetic field pointing in the negative q_3 -direction). The corresponding 2×2 -matrix is given by $B(q)J$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is the matrix of complex multiplication by i if \mathbb{R}^2 is identified with \mathbb{C} . Thus, Newton's equation (3.1) reads

$$\ddot{q} = BJ\dot{q} - \nabla V(q), \tag{3.2}$$

or written out in components,

$$\begin{cases} \ddot{q}_1 = -B\dot{q}_2 - \frac{\partial V}{\partial q_1} \\ \ddot{q}_2 = B\dot{q}_1 - \frac{\partial V}{\partial q_2}. \end{cases}$$

3.2. Examples of Planar Stark–Zeeman Systems

Planar Stark–Zeeman systems include many systems of physical relevance in classical mechanics as well as in quantum mechanics. We consider here only the classical systems. In all the examples the magnetic field B will be constant. The integrability of some of these systems is discussed in Section 3.3 below.

Kepler problem. The case $B = V_1 = 0$ corresponds to the Kepler problem of two bodies moving under their mutual gravitation, or equivalently, a (classical) proton and electron moving under their mutual electric attraction.

Stark problem. The case $B = 0$, $V_1(q) = -Eq_1$ corresponds to the 2-dimensional Stark problem of an electron moving in the field of a proton and a constant electric field of strength E in the q_1 -direction. This system is integrable.

Zeeman problem. The case $V_1 = 0$ and constant $B \neq 0$ correspond to the 2-dimensional Zeeman problem of an electron in the plane moving in the field of a proton and a constant magnetic field perpendicular to the plane.

Diamagnetic Kepler problem. Consider the 3-dimensional Zeeman problem of an electron in \mathbb{R}^3 with coordinates $\mathbf{r} = (x, y, z)$ moving in the field of a proton at the origin and a constant magnetic field $B\partial_z$ in the z -direction. This field has the vector potential $A = \frac{B}{2}(x\partial_y - y\partial_x)$, so the twisted Hamiltonian is

$$\begin{aligned} H_V^A &= \frac{1}{2}|p - A|^2 - \frac{1}{|\mathbf{r}|} \\ &= \frac{|p|^2}{2} - \frac{B}{2}(xp_y - yp_x) + \frac{B^2}{8}(x^2 + y^2) - \frac{1}{|\mathbf{r}|} \\ &= \frac{|p|^2}{2} - \frac{B}{2}(xp_y - yp_x) + \frac{B^2}{8}(x^2 + y^2) - \frac{1}{|\mathbf{r}|} \\ &= \frac{1}{2}\left(p_\rho^2 + p_z^2 + \frac{p_\theta^2}{\rho^2} - Bp_\theta\right) + \frac{B^2}{8}\rho^2 - \frac{1}{\sqrt{\rho^2 + z^2}}, \end{aligned}$$

where (ρ, θ, z) are cylindrical coordinates and $p_\theta = xp_y - yp_x$ is the z -component of angular momentum. Note that p_θ is a conserved quantity. Restricting to the case $p_\theta = 0$ and setting $q_1 = \rho$ and $q_2 = z$, we obtain a planar Stark–Zeeman system with a vanishing magnetic field and $V_1 = \frac{B^2}{8}q_1^2$, i. e., with twisted Hamiltonian

$$H_V^A = \frac{1}{2}|p|^2 - \frac{1}{|q|} + \frac{B^2}{8}q_1^2.$$

While the preceding systems were variations of the classical hydrogen atom, the following systems will describe variations of the 2-body problem in celestial mechanics.

Rotating Kepler problem. Consider a coordinate system (q_1, q_2) that is rotating with angular velocity ω in a counterclockwise direction with respect to the fixed coordinate system (q'_1, q'_2) . Thus, in complex notation $q = q_1 + iq_2$ and $q' = q'_1 + iq'_2$ and their time derivatives are related by

$$\begin{aligned} q &= e^{i\omega t}q', & \dot{q} &= e^{i\omega t}(\dot{q}' + i\omega q'), \\ \ddot{q} &= e^{i\omega t}(\ddot{q}' + 2i\omega\dot{q}' - \omega^2q') = e^{i\omega t}F_t(e^{-i\omega t}q) + 2i\omega\dot{q} + \omega^2q, \end{aligned}$$

where F_t is the (possibly time-dependent) force in Newton's law $\ddot{q}' = F_t(q')$. In the Kepler problem, $F(q) = -q/|q|^3 = e^{i\omega t}F_t(e^{-i\omega t}q)$ and Newton's law in rotating coordinates reads

$$\ddot{q} = -\frac{q}{|q|^3} + 2i\omega\dot{q} + \omega^2q,$$

which is a planar Stark–Zeeman system with magnetic field 2ω (generating the Coriolis force) and potential $V_1 = -\frac{1}{2}\omega|q|^2$ (generating the centrifugal force).

Restricted 3-body problem. The planar circular restricted 3-body problem describes the motion $q(t)$ of a massless body in the plane in the (time-dependent) gravitational field generated

by two massive bodies (say, the sun and the earth of masses M_S and M_E) moving on circular orbits around their center of mass. We put the center of mass at the origin and write the motion of the sun and earth using complex notation as

$$q_S(t) = e^{-i\omega t}q_S, \quad q_E(t) = e^{-i\omega t}q_E$$

with fixed $q_S, q_E \in \mathbb{C}$. Newton's law for their Keplerian motion yields the relations

$$\omega^2 = \frac{M_S}{|q_E||q_S - q_E|^2} = \frac{M_E}{|q_S||q_S - q_E|^2}.$$

In particular, $M_S/M_E = |q_E|/|q_S|$. So we can uniquely write

$$M_E = \mu M, \quad M_S = (1 - \mu)M, \quad q_E = -R(1 - \mu), \quad q_S = R\mu$$

for the total mass $M = M_S + M_E$ and parameters $\mu, R > 0$, where we have chosen q_E on the negative real axis and q_S on the positive real axis. The relation for ω then becomes

$$\omega^2 = \frac{M}{R^3}, \tag{3.3}$$

which just expresses Kepler's 3rd law. The motions of the massive bodies read

$$q_E(t) = -e^{-i\omega t}R(1 - \mu), \quad q_S(t) = e^{-i\omega t}R\mu$$

and generate the gravitational force

$$F_t(q) = -\frac{\mu M(q - q_E(t))}{|q - q_E(t)|^3} - \frac{(1 - \mu)M(q - q_S(t))}{|q - q_S(t)|^3}$$

on the massless body. We switch to rotating coordinates as in the rotating Kepler problem. The gravitational force then becomes time-independent,

$$e^{i\omega t}F_t(e^{-i\omega t}q) = -\frac{\mu M(q + R(1 - \mu))}{|q + R(1 - \mu)|^3} - \frac{(1 - \mu)M(q - R\mu)}{|q - R\mu|^3},$$

and adding in the Coriolis and centrifugal forces, we obtain Newton's equation in rotating coordinates,

$$\ddot{q} = 2\omega i\dot{q} + \omega^2 q - \frac{\mu M(q + R(1 - \mu))}{|q + R(1 - \mu)|^3} - \frac{(1 - \mu)M(q - R\mu)}{|q - R\mu|^3}.$$

This equation is generated by the magnetic field $B = 2\omega$ and the potential

$$V(q) = -\frac{1}{2}\omega^2|q|^2 - \frac{\mu M}{|q + R(1 - \mu)|} - \frac{(1 - \mu)M}{|q - R\mu|},$$

where ω is given by (3.3). Normalizing $M = R = \omega = 1$ this yields the familiar description of the planar circular restricted 3-body problem. This becomes a Stark-Zeeman system if we shift the origin to one of the massive bodies, say the sun. Replacing q by $q - R(1 - \mu)$ then yields the equivalent potential

$$\tilde{V}(q) = V(q - R(1 - \mu)) = -\frac{1}{2}\omega^2|q - R(1 - \mu)|^2 - \frac{\mu M}{|q|} - \frac{(1 - \mu)M}{|q - R|}.$$

Hill's lunar problem. Hill's lunar problem is the limiting case of the planar circular restricted 3-body problem as $M, R \rightarrow \infty$ and $\mu \rightarrow 0$ subject to the restrictions

$$\frac{M}{R^3} = \omega^2 \rightarrow \text{constant} \neq 0, \quad \mu M \rightarrow 1. \tag{3.4}$$

Using the Taylor expansion

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + O(x^3),$$

we expand the gravitational potential of the sun in \tilde{V} to second order in $1/R$:

$$\begin{aligned} \frac{(1-\mu)M}{|q-R|} &= \frac{(1-\mu)M}{R} \frac{1}{\sqrt{1-\frac{2q_1}{R} + \frac{q^2}{R^2}}} \\ &= \frac{(1-\mu)M}{R} \left(1 - \frac{|q|^2}{2R^2} + \frac{q_1}{R} + \frac{3}{2} \frac{q_1^2}{R^2} + O\left(\frac{1}{R^3}\right) \right). \end{aligned}$$

The conditions (3.4) are chosen such that the terms with q_i and $|q|^2$ cancel with the corresponding terms in the centrifugal potential

$$\frac{1}{2}\omega^2|q-R(1-\mu)|^2 = \frac{M}{2R^3} \left(|q|^2 - 2R(1-\mu)q_1 + R^2(1-\mu)^2 \right),$$

so dropping constant terms we obtain in the limit the potential

$$V_{\text{Hill}}(q) = -\frac{1}{|q|} - \frac{3}{2}\omega^2q_1^2.$$

Together with the magnetic field of strength $B = 2\omega$ this describes Hill’s lunar problem. By constant rescaling in time we can normalize ω to 1.

Frozen Hill’s problem. Dropping the magnetic field in Hill’s problem gives rise to the “frozen Hill’s problem”. The frozen Hill’s problem can be interpreted as a diamagnetic Kepler problem for an imaginary magnetic field B .

Frozen Hill’s problem with centrifugal force. Surprisingly, adding a suitable centrifugal term to the frozen Hill’s problem leads to an integrable system described by the Hamiltonian

$$H_V^A = \frac{1}{2}|p|^2 - \frac{1}{|q|} - \frac{3}{2}q_1^2 - \frac{1}{2}|q|^2.$$

Euler problem. Dropping the magnetic field and the centrifugal potential from the restricted 3-body problem leads to the Euler problem with Hamiltonian

$$H_V^A = \frac{1}{2}|p|^2 - \frac{\mu M}{|q+R(1-\mu)|} - \frac{(1-\mu)M}{|q-R\mu|}.$$

It describes the motion of an electron in the electric field of two protons held at fixed position. This system is integrable.

Lagrange problem. Adding an additional elastic force with center between the two masses to the potential of the Euler problem leads to another integrable system first considered by Lagrange. Namely, for any $a \in \mathbb{R}$ the Hamiltonian

$$H_V^A = \frac{1}{2}|p|^2 - \frac{\mu M}{|q+\frac{1}{2}R|} - \frac{(1-\mu)M}{|q-\frac{1}{2}R|} + a|q|^2$$

is integrable. If we think of this force as a centrifugal force, then in the case where the two masses are equal this corresponds to the restricted 3-body problem with Coriolis force removed. Note that in this case the Lagrange problem and the restricted 3-body problem have the same potential and therefore the same Hill’s regions.

3.3. Integrability

In this section we explain how the integrability of various systems in the preceding section can be derived in a unified manner from results in [12, 14]. Consider a mechanical Hamiltonian, i. e., a Hamiltonian just consisting of kinetic and potential energy,

$$H(q, p) = \frac{1}{2}p^2 + V(q),$$

where $V: U \rightarrow \mathbb{R}$ is the potential for $U \subset \mathbb{R}^2$ an open subset. Abbreviate $r = |q|$. The following two well-known theorems give some conditions on the potential V which ensure that the Hamiltonian H admits an integral which is quadratic in the momenta. The following two theorems can be found, for example, in [14, Section 48] or [12].

Theorem 1. Assume that the potential V can be written in the form

$$V = \frac{f(r + q_1) + g(r - q_1)}{r}$$

for smooth functions $f, g: \mathbb{R} \rightarrow \mathbb{R}$. Then the function

$$I = (q_1 p_2 - q_2 p_1) p_2 + \frac{(r + q_1)g(r - q_1) - (r - q_1)f(r + q_1)}{r}$$

is an integral for the Hamiltonian H , i. e., H and I Poisson commute.

There are other conditions on the potential V which guarantee an integral quadratic in the momenta. For $c > 0$ we abbreviate

$$r_1 = \sqrt{(q_1 + c)^2 + q_2^2}, \quad r_2 = \sqrt{(q_1 - c)^2 + q_2^2}$$

and further set

$$u = \frac{1}{2}(r_1 + r_2), \quad v = \frac{1}{2}(r_1 - r_2).$$

Theorem 2. Assume that the potential V can be written in the form

$$V = \frac{f(u) - g(v)}{u^2 - v^2}$$

for smooth functions $f, g: \mathbb{R} \rightarrow \mathbb{R}$. Then the function

$$I = (q_1 p_2 - q_2 p_1)^2 + c^2 p_1^2 + 2 \frac{v^2 f(u) - u^2 g(v)}{u^2 - v^2}$$

is an integral for the Hamiltonian H .

By an elementary but tedious computation one can check directly that under the assumptions of Theorem 1 and Theorem 2 one has $\{H, I\} = 0$. The reader might wonder how people could have come up with these conditions and found the integral I . We refer to [12] for an enlightening discussion concerning these questions. Alternatively Hamilton–Jacobi theory leads to a proof of the two theorems above as explained in [14].

Using the above two theorems we can now readily check the integrability of several of the examples of Stark–Zeeman systems discussed in Section 3.2.

Stark problem. Plug $f(s) = -\frac{1}{2} - \frac{E}{4}s^2$ and $g(s) = -\frac{1}{2} + \frac{E}{4}s^2$ into Theorem 1 to obtain $V = -\frac{1}{r} - Eq_1$.

Frozen Hill's problem with centrifugal force. Plug $f(s) = -\frac{1}{2} - \frac{1}{4}s^3$ and $g(s) = -\frac{1}{2} + s^3$ into Theorem 1 to obtain $V = -\frac{1}{r} - \frac{3}{2}q_1^2 - \frac{1}{2}r^2$.

Euler problem. For $c = \frac{R}{2}$ plug $f(s) = -Ms$ and $g(s) = (1 - 2\mu)Ms$ into Theorem 2 to obtain $V = -\frac{\mu M}{r_1} - \frac{(1-\mu)M}{r_2}$, which is the Euler problem up to a translation of the position coordinates.

Lagrange problem. Concerning the Lagrange problem we compute

$$\frac{u^4 - v^4}{u^2 - v^2} = u^2 + v^2 = \frac{1}{4}(r_1^2 + r_2^2) = \frac{1}{2}(r^2 + c^2).$$

In view of Theorem 2, this means that we can add to the Euler problem an elastic force with center in the middle of the two masses and still keep the system integrable.

3.4. Levi-Civita Regularization of a Planar Stark–Zeeman System

We consider a planar Stark–Zeeman system

$$H = H_V^A: T^*U \rightarrow \mathbb{R}, \quad (q, p) \mapsto \frac{1}{2}|p - A(q)|^2 + V(q)$$

where $U = U_0 \setminus \{0\}$ for $U_0 \subset \mathbb{C}$ an open subset containing the origin,

$$V = V_0 + V_1: U \rightarrow \mathbb{R}$$

where V_0 is the Coulomb potential and V_1 is a smooth function on U extending smoothly to U_0 , and

$$A \in \Omega^1(U_0)$$

such that $dA = \sigma_B \in \Omega^2(U)$ is the magnetic field.

Abbreviate $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. The *Levi-Civita map* is the transformation

$$\mathcal{L}: T^*\mathbb{C}^* \rightarrow T^*\mathbb{C}^*$$

given by

$$\mathcal{L}(v, u) = (v^2, u/2\bar{v}), \quad (v, u) \in \mathbb{C}^* \times \mathbb{C} = T^*\mathbb{C}^*.$$

It is the cotangent lift of the complex squaring map

$$L: \mathbb{C} \rightarrow \mathbb{C}, \quad v \mapsto v^2.$$

In particular, the Levi-Civita map pulls back the canonical symplectic structure of $T^*\mathbb{C}$ to itself, i. e., with $(q, p) = (v^2, u/2\bar{v})$ we have

$$\mathcal{L}^*(dq_1 \wedge dp_1 + dq_2 \wedge dp_2) = dv_1 \wedge du_1 + dv_2 \wedge du_2.$$

Set

$$\mathcal{U} := L^{-1}(U) \subset \mathbb{C}^*.$$

For $c \in \mathbb{R}$ we introduce the Hamiltonian

$$\mathcal{H}_c: T^*\mathcal{U} \rightarrow \mathbb{R}, \quad (v, u) \mapsto 4|v|^2(H \circ \mathcal{L}(v, u) - c).$$

Since the Levi-Civita map is a symplectomorphism, it maps up to reparametrization flow lines of the Hamiltonian flow of \mathcal{H}_c of energy zero to flow lines of the Hamiltonian flow of H of energy c .

Our first goal is to understand the structure of the Hamiltonian \mathcal{H}_c . For that purpose we introduce some notation. First observe that 0 lies in the closure of \mathcal{U} and we abbreviate

$$\mathcal{U}_0 = \mathcal{U} \cup \{0\}.$$

Note that \mathcal{U}_0 is an open subset of \mathbb{C} .

Using the canonical identification of the tangent and cotangent bundle of \mathbb{C} we can think of the one-form A as well as a vector field $A: U_0 \rightarrow \mathbb{C}$ and we introduce the one-form

$$\mathcal{A} \in \Omega^1(\mathcal{U}_0)$$

as the vector field

$$\mathcal{A}: \mathcal{U}_0 \rightarrow \mathbb{C}, \quad v \mapsto 4\bar{v}A(v^2).$$

We further introduce the potential

$$\mathcal{V}_c: \mathcal{U}_0 \rightarrow \mathbb{R}, \quad v \mapsto -4c|v|^2 + 4|v|^2V_1(v^2) - 4,$$

where we recall that $V_1 = V - V_0$, where V_0 is the Coulomb potential.

With these preparations we compute for $(v, u) \in T^*\mathcal{U}$

$$\begin{aligned} \mathcal{H}_c(v, u) &= \frac{4|v|^2}{2} \left| \frac{u}{2\bar{v}} - A(v^2) \right|^2 + 4|v|^2V(v^2) - 4c|v|^2 \\ &= \frac{1}{2} |u - 4\bar{v}A(v^2)|^2 - 4c|v|^2 + 4|v|^2V_1(v^2) - 4 \\ &= \frac{1}{2} |u - \mathcal{A}(v)|^2 + \mathcal{V}_c(v). \end{aligned} \tag{3.5}$$

We first observe that, although we just defined \mathcal{H}_c on $T^*\mathcal{U}$, both maps \mathcal{A} and \mathcal{V}_c are actually defined smoothly on \mathcal{U}_0 , so that we can smoothly extend \mathcal{H}_c to $T^*\mathcal{U}_0$. In the following we think by abuse of notation of

$$\mathcal{H}_c: T^*\mathcal{U}_0 \rightarrow \mathbb{R}$$

as the smooth map given by formula (3.5) and refer to it as the *Levi-Civita regularized Hamiltonian*. The fiber over the origin has the following physical significance: it corresponds to collisions of the particle with the charge at the origin. The Hamiltonian \mathcal{H}_c regularizes these collisions after time reparametrization, so that the regularized orbits just pass through the origin.

Another interesting observation is that the Levi-Civita regularized Hamiltonian is still a magnetic Hamiltonian, consisting of a twisted kinetic energy term and a potential term. In contrast to H , the potential in \mathcal{H}_c has no singularity at the origin.

Abbreviate the energy hypersurface of the Levi-Civita regularized Hamiltonian by

$$\mathcal{S}_c = \mathcal{H}_c^{-1}(0).$$

As for the nonregularized Hamiltonian we define the *Hill's region* for the regularized Hamiltonian as

$$\mathfrak{K}_c^{\text{reg}} = \pi(\mathcal{S}_c) = \{v \in \mathbb{C} : \mathcal{V}_c(v) \leq 0\} \subset \mathcal{U}_0,$$

where $\pi: T^*\mathcal{U}_0 \rightarrow \mathcal{U}_0$ is the footpoint projection. Note that the regularized Hill's region is related to the nonregularized Hill's region by

$$\mathfrak{K}_c^{\text{reg}} = L^{-1}(\mathfrak{K}_c) \cup \{0\}.$$

Recall that $c_1 \in \mathbb{R} \cup \{\infty\}$ denotes the first critical value respectively the escape threshold. If $c < c_1$, the Hill's region contains a unique bounded component $\mathfrak{K}_c^b \subset \mathfrak{K}_c$ which is characterized by the property that the origin lies in its closure. Moreover, \mathfrak{K}_c^b is diffeomorphic to a pointed closed disk. We define

$$\mathfrak{K}_c^{b,\text{reg}} \subset \mathfrak{K}_c^{\text{reg}}$$

as the connected component of $\mathfrak{K}_c^{\text{reg}}$ which contains the origin. If $c < c_1$, the two components are related to each other by

$$\mathfrak{K}_c^{b,\text{reg}} = L^{-1}(\mathfrak{K}_c^b) \cup \{0\}.$$

In particular, $\mathfrak{K}_c^{b,\text{reg}}$ is diffeomorphic to a closed disk.

We denote by

$$\mathcal{S}_c^b = \{(v, u) \in \mathcal{S}_c : v \in \mathfrak{K}_c^{b,\text{reg}}\}$$

the connected component of the energy hypersurface \mathcal{S}_c^b which projects to $\mathfrak{K}_c^{b,\text{reg}}$. Note that for each point v in the interior of the Hill's region $\mathfrak{K}_c^{b,\text{reg}}$ the fiber in \mathcal{S}_c^b lying over v is a circle, where at the zero velocity curves (i. e., the boundary of $\mathfrak{K}_c^{b,\text{reg}}$) the fiber collapses to a point. We deduce that topologically \mathcal{S}_c^b is a 3-dimensional sphere.

We summarize the preceding discussion as follows.

Proposition 1. *The Levi-Civita regularized Hamiltonian at energy $c < c_1$ is a nonsingular magnetic Hamiltonian $\mathcal{H}_c: T^*\mathcal{U}_0 \rightarrow \mathbb{R}$. The bounded component $\mathfrak{K}_c^{b,\text{reg}}$ of its Hill's region is a closed disk whose preimage $\mathcal{S}_c^b \subset \mathcal{H}_c^{-1}(0)$ is a 3-sphere. \square*

3.5. Moser Regularization of a Stark–Zeeman System

Let us consider a Stark–Zeeman system in any dimension n . Even if we restrict our attention for an energy value $c < c_1$ to the connected component of the energy hypersurface above the bounded component \mathfrak{K}_c^b of the Hill's region,

$$\Sigma_c^b := \{(q, p) \in \Sigma_c \mid q \in \mathfrak{K}_c^b\},$$

we are facing noncompactness issues because the electron might collide with the proton. However, two-body collisions can always be regularized as we now recall. For any regular energy value c of H_V we introduce the Hamiltonian $K_c: T^*U \rightarrow \mathbb{R}$ given by

$$\begin{aligned} K_c(q, p) &:= |q|(H_V^A(q, p) - c) \\ &= \frac{|q|}{2} |p - A(q)|^2 - 1 + |q|V_1(q) - c|q| \\ &= \frac{|q|}{2} |p|^2 + |q| \left(\langle p, A(q) \rangle + \frac{1}{2} |A(q)|^2 + V_1(q) - c \right) - 1. \end{aligned}$$

Since $q \neq 0$ on U , the level sets satisfy

$$K_c^{-1}(0) = (H_V^A)^{-1}(c) = \Phi_A^{-1}(\Sigma_c). \quad (3.6)$$

Note that $|p| \rightarrow \infty$ as $q \rightarrow 0$ in this level set. To study the behavior as p approaches infinity, we pull back K_c under the simple symplectic map

$$\sigma: T^*\mathbb{R}^n \rightarrow T^*\mathbb{R}^n, \quad (q, p) \mapsto (-p, q).$$

Note that this map is from the physical point of view rather revolutionary because it interchanges the roles of base and fiber, so positions become momenta and vice versa. We will keep the original notation (q, p) and just remember that the p_i are now interpreted as positions and the q_i as negative momenta.

To study the behavior as $|p| \rightarrow \infty$, we view \mathbb{R}^n as a chart of the sphere $S_r^n \subset \mathbb{R}^{n+1}$ of radius $r > 0$ via stereographic projection

$$\psi_r^N: \mathbb{R}^n \xrightarrow{\cong} S_r^n \setminus \{N\}, \quad p \mapsto (x, x_{n+1}),$$

where $N = (0, \dots, 0, r)$ is the north pole and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. By elementary geometry, the map ψ_r^N is given by

$$x = \frac{2r^2 p}{|p|^2 + r^2}, \quad x_{n+1} = r \frac{|p|^2 - r^2}{|p|^2 + r^2}. \quad (3.7)$$

To see whether the Hamiltonian K_c smoothly extends to the fiber at infinity of T^*S^n , we consider the transition map

$$\psi_r := (\psi_r^S)^{-1} \circ \psi_r^N: \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{0\}, \quad p \mapsto \frac{r^2 p}{|p|^2}$$

between the stereographic projections from the north and south poles. The Jacobi matrix $D\psi_r(p)$ has entries

$$D\psi_r(p)_{ij} = \frac{r^2}{|p|^4} \begin{cases} -2p_i p_j & i \neq j, \\ |p|^2 - 2p_i^2 & i = j. \end{cases}$$

Since this matrix is symmetric and $\psi_r \circ \psi_r = \text{id}$, its inverse transpose computes to be (using $|\psi_r(p)| = r^2/|p|$)

$$(D\psi_r(p)^{-1})_{ij}^T = D\psi_r(\phi_r(p))_{ij} = \frac{1}{r^2} \begin{cases} -2p_i p_j & i \neq j, \\ |p|^2 - 2p_i^2 & i = j. \end{cases}$$

Therefore, the induced symplectomorphism

$$\Psi_r: T^*(\mathbb{R}^n \setminus \{0\}) \rightarrow T^*(\mathbb{R}^n \setminus \{0\}), \quad (p, q) \mapsto \left(\psi_r(p), (D\psi_r(p)^{-1})_{ij}^T \cdot q \right) =: (\tilde{p}, \tilde{q})$$

is given by

$$\tilde{p} = \frac{r^2 p}{|p|^2}, \quad \tilde{q} = \frac{|p|^2 q - 2\langle p, q \rangle p}{r^2}.$$

In particular, $|\tilde{q}| = |p|^2|q|/r^2$, and using this we obtain for $(p, q) \in \Psi_r^{-1}(T^*U)$

$$K_c \circ \Psi_r(p, q) = \frac{r^2}{2}|q| + |q|\langle p, A(\tilde{q}) \rangle + \frac{|p|^2|q|}{r^2} \left(\frac{|A(\tilde{q})|^2}{2} + V_1(\tilde{q}) - c \right) - 1.$$

We see that this extends continuously over $p = 0$ via

$$K_c \circ \Psi_r(0, q) = \frac{r^2}{2}|q| - 1.$$

In particular, on the level set $K_c^{-1}(0)$ we have $|q| \rightarrow 2/r^2$ as $p \rightarrow 0$, so the extension of $K_c \circ \Psi_r$ is smooth near this level set. Hence the hypersurface

$$\Phi_r^{-1}(K_c^{-1}(0)) = \Psi_r^{-1} \circ \Phi_A^{-1}(\Sigma_c) \subset T^*(\mathbb{R}^n \setminus \{0\})$$

can be extended smoothly over $p = 0$ by adding the sphere $\{p = 0, |q| = 2/r^2\}$. Now a straightforward computation using (3.7) shows that the pullback of the round metric g_r on the sphere $S_r^n \subset \mathbb{R}^{n+1}$ of radius r under the stereographic projection $\psi_r^N : \mathbb{R}^n \rightarrow S_r^n \setminus \{N\}$ is given at the point $p \in \mathbb{R}^n$ by

$$((\psi_r^N)^* g_r)_p = \frac{4r^4}{(|p|^2 + r^2)^2} \langle \cdot, \cdot \rangle.$$

So the sphere $\{p = 0, |q| = 2/r^2\} \subset T_0^* \mathbb{R}^n$ is the preimage under Ψ_r^N of the sphere in $T_N^* S_r^n$ of radius $1/r^2$ with respect to the dual of the round metric g_r . We summarize this discussion in

Proposition 2. *For any regular value c of H_V and any $r > 0$ the closure of the hypersurface $\Psi_r^N \circ \Phi_A^{-1}(\Sigma_c) \subset T^* S_r^n$ is a smooth hypersurface $S_{c,r} \subset T^* S_r^n$ which intersects the fiber over the north pole in the sphere of radius $1/r^2$ with respect to the dual of the round metric g_r on S_r^n . \square*

Example 1. An illustrative example is the case where there is neither an exterior electric nor magnetic field, i. e., just the case of an electron attracted by a proton which corresponds to the Kepler problem. Then

$$H(q, p) = \frac{1}{2}|p|^2 - \frac{1}{|q|}, \quad K_c = |q| \left(\frac{1}{2}|p|^2 - c \right) - 1.$$

Suppose $c < 0$ and set $r := \sqrt{-2c}$. Then the energy hypersurface

$$\Sigma_c = K_c^{-1}(0) = \left\{ (p, q) \mid \frac{|p|^2 + r^2}{2r^2} |q| = \frac{1}{r^2} \right\}$$

corresponds under stereographic projection to the radius $1/r^2$ sphere bundle in $T^* S_r^n$ (where we interpret p as position and q as momentum in $T^* S_r^n$). So the regularized energy hypersurface $\mathcal{S}_{c,r}$ is the radius $1/r^2$ sphere bundle in $T^* S_r^n$ and the regularized Hamiltonian flow just becomes (up to time reparametrization) the geodesic flow for the round metric on the sphere S_r^n of radius $r = \sqrt{-2c}$. In particular, for $c = -1/2$ we get the geodesic flow on the unit sphere.

The preceding example was the main reason for allowing for an arbitrary radius $r > 0$ in our discussion. From now on we will restrict to the case $r = 1$ and drop r from the notation.

In general, suppose that $c < c_1$ and consider the connected component Σ_c^b of the energy hypersurface lying above the bounded part \mathcal{R}_c^b of the Hill's region which contains the origin in its closure. This component is compact up to collisions of the electron with the proton, which after regularization correspond to points in the cotangent sphere over the north pole in S^n . Hence the corresponding connected component

$$\mathcal{S}_c^b \subset \mathcal{S}_c$$

of the regularized hypersurface $\mathcal{S}_c \subset T^*S^n$ is compact. We can homotop \mathcal{S}_c^b through regular compact energy hypersurfaces to Example 1 by first decreasing c to a very negative value and then switching off the external electric and magnetic field. In particular, the energy hypersurfaces in such a homotopy remain diffeomorphic so that we proved

Corollary 1. *If $c < c_1$, then the bounded component \mathcal{S}_c^b of the Moser regularized energy hypersurface is diffeomorphic to the unit cotangent bundle S^*S^n . \square*

Remark 1. The Levi-Civita regularized energy hypersurface is the preimage of the Moser regularized energy hypersurface under a suitable 2–1 cover $S^3 \rightarrow \mathbb{R}P^3 \cong S^*S^2$.

4. TOPOLOGY OF PERIODIC ORBITS

4.1. Periodic Orbits in Planar Stark–Zeeman Systems

Local structure of orbits. Consider an orbit $q : \mathbb{R} \rightarrow \mathbb{R}^2$ of a planar Stark–Zeeman system of energy $c < c_1$. We allow q to pass through the origin, where it is understood to be the projection of a regularized collision orbit. Let us describe the structure of q near a point $q_0 = q(t_0)$. We distinguish 3 cases.

Case 1: $q_0 \neq 0$ lies in the interior of Hill’s region \mathfrak{K}_c . Then $\dot{q}(t_0) \neq 0$, so q is an immersion near t_0 .

Case 2: $q_0 \neq 0$ lies on the boundary of Hill’s region \mathfrak{K}_c , i. e., the velocity $\dot{q}(t_0)$ vanishes.

Lemma 1. *Suppose that $\dot{q}(t_0) = 0$ and $B(q_0) \neq 0$. Then the orbit q has a semi-cubical cusp at $t = t_0$. Moreover, in a 1-parameter family q^s of nearby orbits of the same energy with $q^0 = q$ in which the component of $\dot{q}^s(t_0)$ normal to $\nabla V(q^s(t_0))$ changes sign, an exterior loop shrinks to a cusp and disappears (or the reverse) as shown in Fig. 2.*

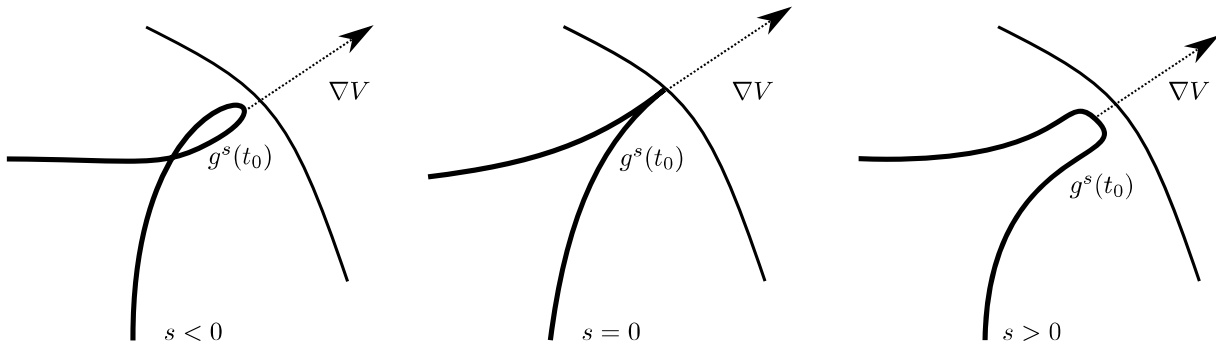


Fig. 2. Passing through a cusp at the zero velocity curve (depicted in black).

Proof. Let us take $t_0 = 0$ and consider an orbit $q = q^s$ in a 1-parameter family near q^0 . Denote by $\dot{q}_0, \ddot{q}_0, \ddot{\ddot{q}}_0$ its derivatives at time $t = 0$. From Newton’s equation $\ddot{q} = iB(q)\dot{q} - \nabla V(q)$ and its time derivative we obtain with $B_0 = B(q_0)$:

$$\begin{aligned} \ddot{q}_0 &= iB_0\dot{q}_0 - \nabla V(q_0), \\ \ddot{\ddot{q}}_0 &= i\nabla_{\dot{q}_0}B\dot{q}_0 + iB_0\ddot{q}_0 - D^2V(q_0)\dot{q}_0 \\ &= -iB_0\nabla V(q_0) - (B_0^2 + D^2V(q_0) - i\nabla_{\dot{q}_0}B)\dot{q}_0. \end{aligned}$$

From the nonvanishing of $\nabla V(q_0)$ and B_0 we conclude that for small \dot{q}_0 the vectors \ddot{q}_0 and $\ddot{\ddot{q}}_0$ are nonzero and roughly orthogonal. Picking complex coordinates in which $q_0 = 0$, $\ddot{q}_0 = 2$ and $\ddot{\ddot{q}}_0 = 6i$ and treating the initial velocity $\dot{q}_0 = a + ib$ as a complex parameter, the Taylor expansion of q to 3rd order becomes

$$\begin{aligned} q(t) &= q_0 + \dot{q}_0t + \frac{1}{2}\ddot{q}_0t^2 + \frac{1}{6}\ddot{\ddot{q}}_0t^3 + O(t^4) \\ &= (at + t^2) + i(bt + t^3) + O(t^4). \end{aligned}$$

Neglecting the terms of order 4 or higher, a short computation shows that for parameters satisfying $3a^2 + 4b < 0$ the curve q has a loop which vanishes as the parameters cross the discriminant curve $\{3a^2 + 4b = 0\}$, see Fig. 3. Note that a 1-parameter family in which the component b of $q^s(t_0)$ normal to $\nabla V(q^s(t_0))$ changes sign corresponds to a crossing of the discriminant curve at the origin. Since this picture persists under perturbations of order 4 or higher, the lemma is proved. \square

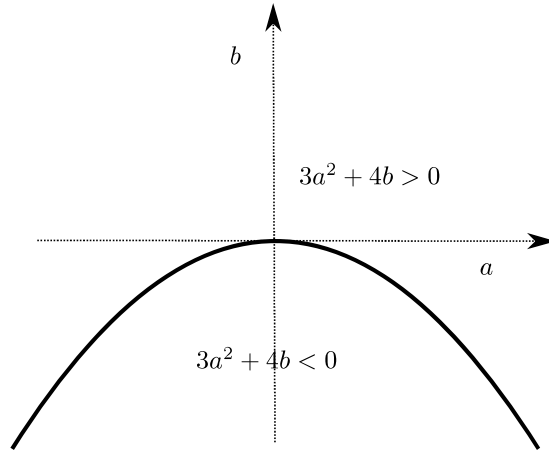


Fig. 3. Crossing the discriminant curve.

Case 3: $q_0 = 0$ is a collision.

Lemma 2. Suppose that $q_0 = 0$ and $B(q_0) \neq 0$. Then the orbit q has a cusp at $t = t_0$. Moreover, in a 1-parameter family q^s of nearby orbits of the same energy with $q^0 = q$, a loop around 0 shrinks to a cusp and becomes a curve going around 0 the other way (or the reverse) as shown at the bottom of Fig. 4.

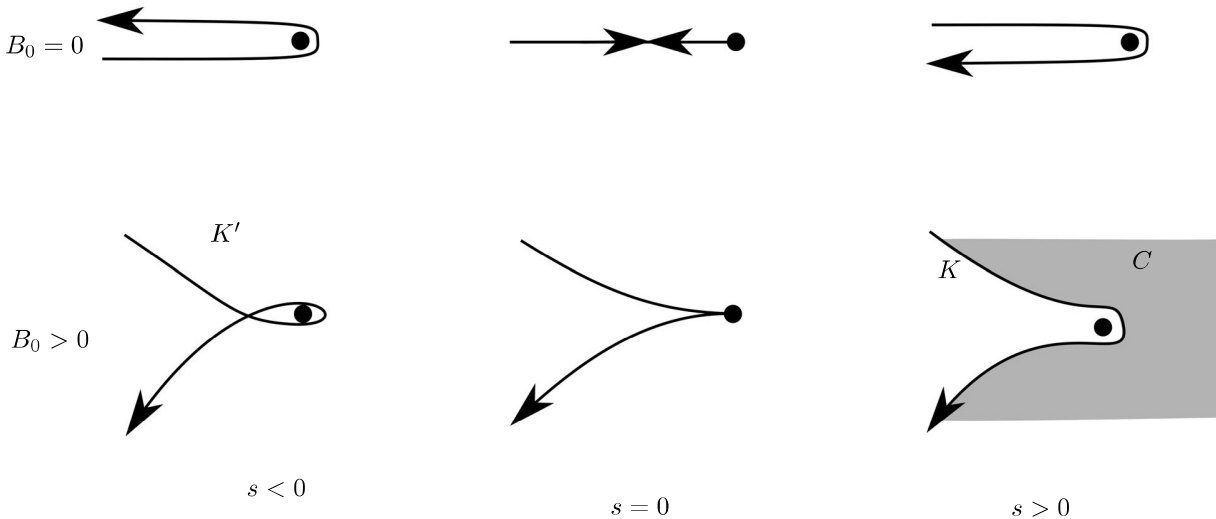


Fig. 4. Passing through a cusp at the origin (depicted in black).

Proof. Consider first the situation with $B_0 = B(q_0) = 0$. Then a collision orbit q bounces back from the origin, and a 1-parameter family q^s with $q^0 = q$ changes the direction in which it encircles the origin as shown at the top of Fig. 4. If we now switch on a magnetic field with $B_0 > 0$, then the

collision orbit curves everywhere to the left and thus becomes a cusp at the origin. Moreover, the 1-parameter family q^s now looks as at the bottom of Fig. 4. (This holds because away from the origin q^s is close to the cusp q^0 for small s , whereas near the origin the Newtonian force dominates the magnetic field and thus q^s looks like in the case $B_0 = 0$.) \square

Remark 2. Performing a time change $\tau = t^{1/3}$ and considering the Taylor expansion of $q(\tau)$ to fourth order near $\tau = 0$, one finds that the cusp of q at the origin is *quintical*, i. e., equivalent to $\tau \mapsto (\tau^2, \tau^5)$ in suitable coordinates.

Stark–Zeeman homotopies. Suppose now that $q : \mathbb{R}/T\mathbb{Z}$ is a simple (i. e., not multiply covered) periodic orbit of energy c . Since the position and velocity at any time t_0 uniquely determine the orbit, it follows that q cannot have any *direct self-tangency*, i. e., a self-tangency at which the two velocities agree. By contrast, *inverse self-tangencies* at which the two velocities are opposite to each other can occur. Generically, q will be a *generic immersion*, i. e., an immersion with only transverse double points, avoiding the boundary of Hill’s region as well as the origin. In view of this discussion and the previous two lemmas, the following definition captures all events that can happen in a generic 1-parameter family of simple periodic orbits of (varying) planar Stark–Zeeman systems.

Definition 1. A Stark–Zeeman homotopy is a 1-parameter family $(K^s)_{s \in [0,1]}$ of closed curves in \mathbb{C} which are generic immersions in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ except for the following events at finitely many $s \in (0, 1)$:

- (I_0) birth or death of interior loops through cusps at the origin;
- (I_∞) birth or death of exterior loops through cusps at infinity;
- (II^+) crossings through inverse self-tangencies;
- (III) crossings through triple points.

These special events, where a 1-parameter family K^s fails to a generic immersion are illustrated in Figs. 5, 6, 7 and 8.

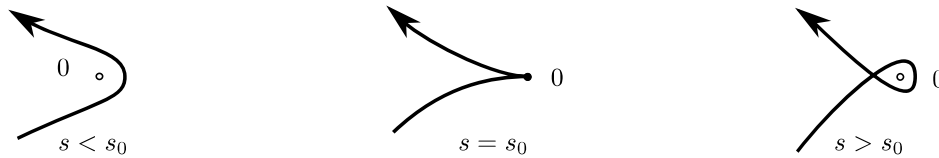


Fig. 5. Event (I_0): development of interior loop through a cusp.

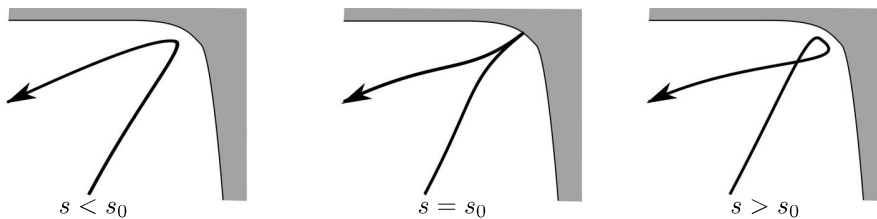


Fig. 6. Event (I_∞): development of exterior loop through a cusp.

Conversely, the following proposition shows that any Stark–Zeeman homotopy can be realized as a family of periodic orbits of Stark–Zeeman systems.

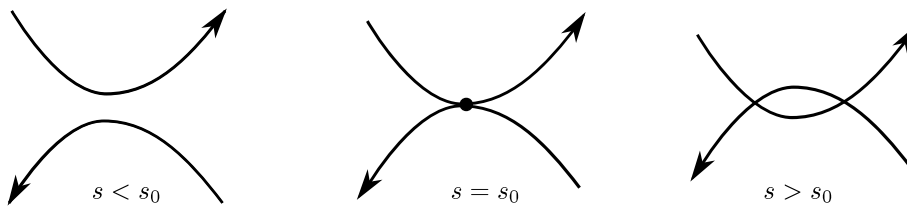


Fig. 7. Event (II^+) , crossing through inverse self-tangency.

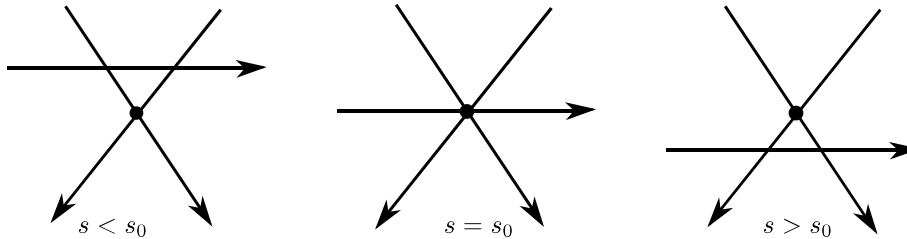


Fig. 8. Event (III) , an ordinary triple point in a 1-parameter family.

Proposition 3. A 1-parameter family $(K^s)_{s \in [0,1]}$ of closed curves in \mathbb{C}^* that are not multiply covered is a Stark–Zeeman homotopy if and only if there exists a family of diffeomorphisms $F^s : \mathbb{C}^* \rightarrow \mathbb{C}^*$ such that the (suitably parametrized) curves $F^s(K^s)$ are simple periodic orbits (possibly with collisions) in a generic family of Stark–Zeeman systems.

Remark 3. The restricted three-body problem comes with an antisymplectic involution $(q_1, q_2; p_1, p_2) \mapsto (q_1, -q_2, -p_1, p_2)$ making this problem very nongeneric. For example, if a cusp develops in the upper half-plane, then this cusp has a paired cusp in the lower half-plane. When this occurs with an event of type I_0 , one obtains a double cusp through 0.

The proof of this proposition uses the following lemma. The domain of each curve in the lemma is an open interval.

Lemma 3. (a) Let $\gamma_1, \gamma_2, \gamma_3$ be three embedded curves through the origin 0 in \mathbb{R}^2 whose derivatives at 0 are pairwise linearly independent. Let $\tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\gamma}_3$ be another such triple such that $\tilde{\gamma}_i$ is tangent to γ_i at 0 for $i = 1, 2, 3$. Then there exist neighborhoods U, U' and a diffeomorphism $F : U \rightarrow U'$ with $DF(0) = id$ such that $F(\gamma_i)$ is a reparametrization of $\tilde{\gamma}_i$ for $i = 1, 2, 3$.

(b) Let $\gamma_1, \gamma_2, \tilde{\gamma}_1, \tilde{\gamma}_2$ be four embedded curves through the origin 0 in \mathbb{R}^2 all having the same nonzero derivative at 0. Suppose that the corresponding curvatures at 0 satisfy $\kappa_2 - \kappa_1 = \tilde{\kappa}_2 - \tilde{\kappa}_1 \neq 0$. Then there exist neighborhoods U, U' of 0 and a diffeomorphism $F : U \rightarrow U'$ with $DF(0) = id$ such that $F(\gamma_i)$ is a reparametrization of $\tilde{\gamma}_i$ for $i = 1, 2$.

Proof. (a) By transitivity, it suffices to prove this for the case that the γ_i are straight lines. Moreover, it suffices to consider the case $\gamma_i = \tilde{\gamma}_i$ for $i = 1, 2$ (because we can then arrange this condition by applying the result twice with some arbitrary auxiliary lines). We pick linear coordinates in which γ_1 and $\tilde{\gamma}_1$ both map to the x -axis, γ_2 and $\tilde{\gamma}_2$ both to the y -axis, and γ_3 to the line $\{y = ax\}$ for some $a \neq 0$. Then for a sufficiently small neighborhood U' of the origin, the image of the curve $\tilde{\gamma}_3$ is the graph $\{y = g(x)\}$ of a smooth function g with $g(0) = 0$ and $g'(0) = a \neq 0$. By Taylor's formula we can write $g(x) = axh(x)$ for a unique smooth function h with $h(0) = 1$. By choosing a sufficiently small neighborhood U as domain we obtain a diffeomorphism $F(x, y) := (x, yh(x)) = (X, Y)$ that satisfies $DF(0) = id$ and preserves the x - and y -axes. Moreover, it sends the graph $y = ax$ to the graph $Y = yh(x) = axh(x) = g(x) = g(X)$ and thus the image of γ_i to the image of $\tilde{\gamma}_i$ for $i = 1, 2, 3$.

(b) Arguing as in (a), we see that it suffices to prove this for the case that $\gamma_1 = \tilde{\gamma}_1$ is the x -axis, γ_2 is the graph of the function $y = \frac{1}{2}ax^2$, and $\tilde{\gamma}_2$ is the graph of a function $y = g(x)$ with

$g(0) = g'(0) = 0$ and $g''(0) = a \neq 0$. By Taylor's formula we can write $g(x) = \frac{1}{2}ax^2h(x)$ for a unique smooth function h with $h(0) = 1$. Choosing neighborhoods as before, we find a diffeomorphism $F(x, y) := (x, yh(x)) = (X, Y)$ that satisfies $DF(0) = \text{id}$ and preserves the x - and y -axes. Moreover, it sends the graph $y = \frac{1}{2}ax^2$ to the graph $Y = yh(x) = \frac{1}{2}ax^2h(x) = g(x) = g(X)$ and thus the image of γ_i to the image of $\tilde{\gamma}_i$ for $i = 1, 2$. \square

Proof (of Proposition 3). As explained above, simple periodic orbits of Stark–Zeeman systems are immersions without direct self-tangencies and with cusps at the zero velocity curves and at the origin. Hence Lemma 1 and Lemma 2 imply that a generic 1-parameter family of such orbits is a Stark–Zeeman homotopy, and the latter condition is preserved by composition with diffeomorphisms of the plane.

Conversely, suppose that $(K^s)_{s \in [0,1]}$ is a Stark–Zeeman homotopy. We will construct a smooth family of Stark–Zeeman systems such that the K^s agree (suitably parametrized) with periodic solutions of energy 0 of the ODE

$$\ddot{q} = B^s(q)i\dot{q} - \nabla V^s(q). \quad (4.1)$$

We first pick a smooth family of potentials V^s with a Coulomb singularity at the origin and without critical points such that the curves K^s lie in the interior of the Hill's region $\mathfrak{R}^s = \{-\infty < V^s \leq 0\}$, except for their cusps which lie on the boundary $\{0\} \cup \{V^s = 0\}$. We add preliminary magnetic fields B_0^s near the cusps and apply diffeomorphisms there to make K^s agree with zero energy solutions of (4.1) near the cusps. We cut off B_0^s and extend this function by zero outside neighborhoods of the cusps. Next, we use the neighborhoods and diffeomorphisms provided by Lemma 3 near all double and triple points to map K^s to solutions of (4.1) near double and triple points. These diffeomorphisms can be chosen smoothly in s , agreeing with the given ones near the cusps, and can be extended to a smooth family of diffeomorphisms F^s . We parametrize the curves $F^s(K^s)$ near the cusps and double/triple points as solutions of (4.1), and extend these parametrizations to parametrizations $q^s : [0, T^s] \rightarrow \mathbb{C}^*$ of $F^s(K^s)$ by the energy requirement $\frac{1}{2}|\dot{q}^s|^2 + V^s(q^s) = 0$. Note that $\dot{q}^s \neq 0$ away from the cusps. Taking a time derivative of the energy condition, we find $0 = \langle \dot{q}^s, \ddot{q}^s + \nabla V^s(q^s) \rangle$, so $\ddot{q}^s + \nabla V^s(q^s)$ is a multiple of $i\dot{q}^s$ away from the cusps. Since q^s is injective away from the double and triple points, we thus find unique well-defined functions $B^s(q)$ along $F^s(K^s)$ such that $\ddot{q}^s + \nabla V^s(q^s) = iB^s(q^s)\dot{q}^s$, i.e., q^s is a zero energy solution of (4.1). We arbitrarily extend B^s away from K^s in a way that smoothly depends on s and Proposition 3 is proved. \square

4.2. Topology of Plane Curves

In this subsection we recall some facts from [3] about immersed loops in the plane, which will be applied to Stark–Zeeman systems in the next subsection. In particular, we recall the definition of Arnold's J^+ -invariant and derive some of its properties.

By an *immersion* we will mean an immersion $q : S^1 \rightarrow \mathbb{C}$ of the circle into the complex plane, considered up to orientation preserving reparametrization. Note that q is canonically oriented (by \dot{q}) and cooriented (by $i\dot{q}$). By a slight abuse of notation we will identify an immersion with its image $K \subset \mathbb{C}$. A generic immersion has only transverse double points. During a generic homotopy of immersions the following three events occur: *triple points* (as in Fig. 8), *direct self-tangencies* (where the coorientations agree, see Fig. 9), and *inverse self-tangencies* (where the coorientations do not agree, see Fig. 7). The discriminant hypersurfaces corresponding to these events are naturally cooriented, and Arnold defines three invariants (St , J^+ , J^-) of generic immersions in terms of their changes under positive crossings of the discriminant hypersurfaces. They are independent of the orientation and coorientation and additive under connected sums. Of these, the invariant J^+ is relevant for us. It increases by 2 under positive crossings through direct self-tangencies (where the number of double points increases) and remains unchanged under crossings through triple points and inverse self-tangencies. It is normalized to 0 on the figure eight K_0 and to $2 - 2|j|$ on the circle K_j with $|j| - 1$ interior loops and rotation number (winding number of the tangent vector) $j \neq 0$; see Fig. 10 for pictures of the *standard curves* K_j .

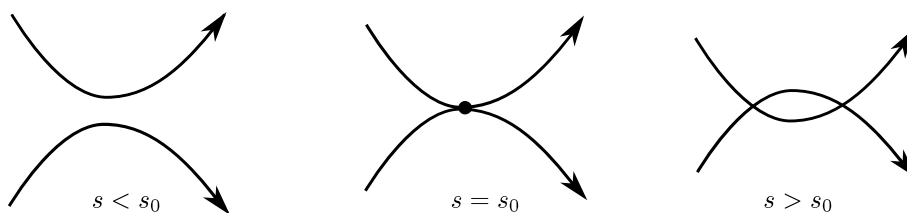


Fig. 9. A 1-parameter family of immersions going through a direct self-tangency.

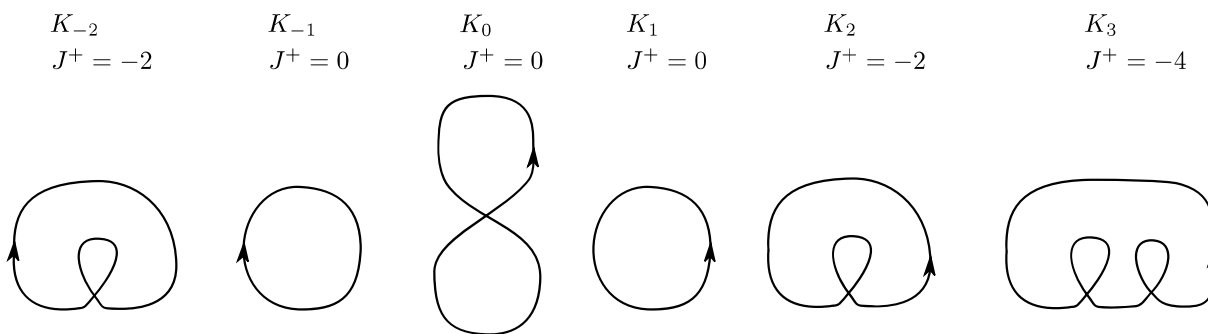


Fig. 10. The standard curves and their J^+ .

Lemma 4. For an immersion $K \subset \mathbb{C}$, consider a connected component C of $\mathbb{C} \setminus K$ and a boundary arc A of C . Orient K such that A inherits the boundary orientation from C and denote by $w(K, C) \in \mathbb{Z}$ the winding number of K (with this orientation) around a point in C . Then the invariant J^+ changes by $-2w(K, C)$ under addition of a small loop in C to the arc A , see Fig. 11.

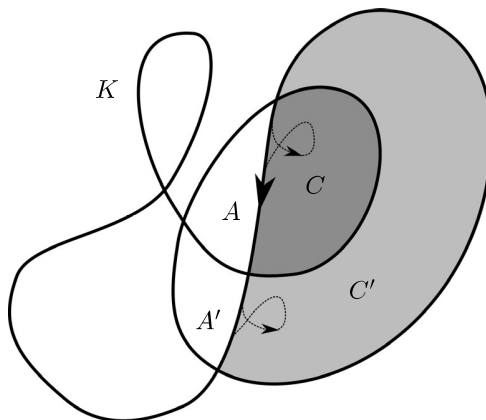


Fig. 11. Adding a loop to an arc A of K ; here $w(K, C) = 2$ and $w(K, C') = 1$.

Proof. For K, C, A as in the lemma, let \tilde{K} be the immersion obtained by adding a small loop in C to the arc A of K and set

$$I(K, C, A) := J^+(\tilde{K}) - J^+(K) + 2w(K, C).$$

Consider a component C' of $\mathbb{C} \setminus K$ adjacent to C with boundary arc A' adjacent to A and winding number $w(K, C') = w(K, C) \pm 1$ as in Fig. 11. Let \tilde{K}' be the immersion obtained by adding a small loop in C' to the arc A' of K . Figure 11 shows that $J^+(\tilde{K}') - J^+(\tilde{K}) = \mp 2$, and therefore $I(K, C, A) = I(K, C', A')$. Thus, $I(K, C, A)$ is independent of C and A and we can simply denote

it by $I(K)$ (note that at this point $I(K)$ may still depend on the orientation of K). It follows that $I(K)$ is invariant under crossings through triple points or (direct or inverse) self-tangencies (because we can always attach the new loop along an arc that is not involved in the crossing, so that $J^+(\tilde{K}) - J^+(K)$ and $w(K, C)$ both do not change under the crossing). Hence the invariant I is uniquely determined by its value on the standard curves K_j , which is computed using Fig. 12 as follows. For $j > 0$ we take the component C with $w(K_j, C) = 1$; then $\tilde{K}_j = K_{j+1}$ and thus

$$I(K_j) = J^+(K_{j+1}) - J^+(K_j) + 2w(K_j, C) = -2j - (2 - 2j) + 2 = 0.$$

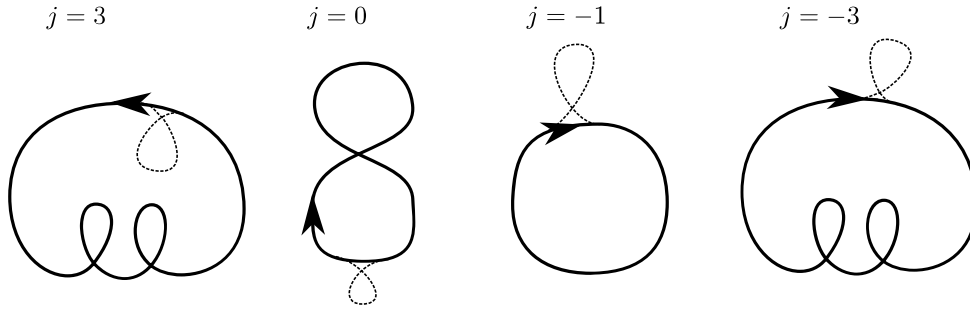


Fig. 12. Adding a loop to the standard curves.

For $j = 0$ we take the unbounded component C so that $w(K_0, C) = 0$; then \tilde{K}_0 is obtained from K_1 without crossing through a direct self-tangency and thus

$$I(K_0) = J^+(K_1) - J^+(K_0) + 2w(K_0, C) = 0 - 0 + 0 = 0.$$

For $j = -1$ we take the unbounded component C so that $w(K_{-1}, C) = 0$; then $\tilde{K}_{-1} = K_0$ and thus

$$I(K_{-1}) = J^+(K_0) - J^+(K_{-1}) + 2w(K_0, C) = 0 - 0 + 0 = 0.$$

For $j < -1$ we take the unbounded component C so that $w(K_j, C) = 0$; Fig. 13 shows that \tilde{K}_j is obtained from K_{j+1} by crossing once negatively through a direct double point and thus

$$I(K_j) = J^+(K_{j+1}) - 2 - J^+(K_j) + 2w(K_j, C) = (4 + 2j) - 2 - (2 + 2j) + 0 = 0.$$

This shows that $I(K)$ vanishes identically, which proves the lemma. □

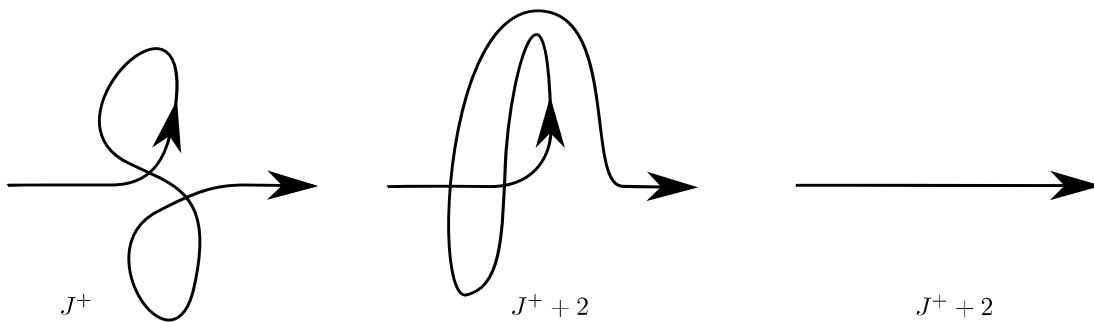


Fig. 13. Cancelling two loops changes J^+ by 2.

In particular, we have the following corollary (which also follows directly from the additivity of J^+ under connected sum and the fact that adding an exterior loop corresponds to connected sum with K_0).

Corollary 2. *The invariant J^+ does not change under addition of an exterior loop, i. e., a loop in the unbounded component.*

4.3. The two Invariants \mathcal{J}_1 and \mathcal{J}_2

For a generic immersion K in \mathbb{C}^* denote by $J^+(K) \in 2\mathbb{Z}$ its J^+ -invariant and by $w_0(K) \in \mathbb{Z}$ its winding number around the origin. We set

$$\mathcal{J}_1(K) := J^+(K) + \frac{w_0(K)^2}{2}.$$

Proposition 4. \mathcal{J}_1 is invariant under Stark–Zeeman homotopies.

Proof. We need to show that \mathcal{J}_1 is invariant under the moves in Definition 1. Under the moves (II^+) and (III) both quantities J^+ and w_0 are invariant. The same is true for the move (I_∞) due to Corollary 2. Therefore, \mathcal{J}_1 is invariant under these three moves. To discuss the invariance of \mathcal{J}_1 under the move (I_0) , first observe that J^+ is invariant under orientation reversion, while w_0 changes sign, so \mathcal{J}_1 is invariant under orientation reversion. Hence we can choose the orientations of the immersions K and K' before and after the move (I_0) as shown at the bottom of Fig. 4. Then their winding numbers are related by $w_0(K') = w_0(K) + 2$. Moreover, K' is obtained from K by adding a small loop in the component C of $\mathbb{C} \setminus K$ shown in the figure (having the arc of K as a positive boundary arc). Since the winding number of K around C is $w_0(K) + 1$, by Lemma 4 the J^+ -invariants are related by

$$J^+(K') = J^+(K) - 2w(K, C) = J^+(K) - 2w_0(K) - 2.$$

Hence we compute

$$\begin{aligned} \mathcal{J}_1(K') &= J^+(K') + \frac{w_0(K')^2}{2} \\ &= J^+(K) - 2w_0(K) - 2 + \frac{(w_0(K) + 2)^2}{2} \\ &= J^+(K) - 2w_0(K) - 2 + \frac{w_0(K)^2 + 4w_0(K) + 4}{2} \\ &= J^+(K) + \frac{w_0(K)^2}{2} \\ &= \mathcal{J}_1(K). \end{aligned}$$

This proves invariance of \mathcal{J}_1 under the move (I_0) and hence the proposition. □

Recall from Section 3.4 the Levi-Civita regularization map

$$L: \mathbb{C}^* \rightarrow \mathbb{C}^*, \quad v \mapsto v^2$$

which regularizes collisions at the origin. Note that the preimage $L^{-1}(K)$ of a generic immersion $K \subset \mathbb{C}^*$ is again a generic immersion in \mathbb{C}^* . We distinguish two cases.

Case 1: $w_0(K)$ is odd. Then the preimage $L^{-1}(K)$ is connected and we set

$$\mathcal{J}_2(K) := J^+(L^{-1}(K)).$$

Note that $L: L^{-1}(K) \rightarrow K$ is a 2–1 covering, in particular, $w_0(L^{-1}(K)) = 2w_0(K)$.

Case 2: $w_0(K)$ is even. Then the preimage $L^{-1}(K)$ is disconnected and we set

$$\mathcal{J}_2(K) := J^+(\tilde{K})$$

for one component \tilde{K} of $L^{-1}(K)$. Since the two components of $L^{-1}(K)$ are related by a 180 degree rotation, this definition does not depend on the choice of \tilde{K} . Note that $L: \tilde{K} \rightarrow K$ is a diffeomorphism, in particular, $w_0(\tilde{K}) = w_0(K)$.

Proposition 5. \mathcal{J}_2 is invariant under Stark–Zeeman homotopies.

Proof. Since the Levi-Civita regularization map regularizes collisions, the quantity \mathcal{J}_2 is invariant under the move (I_0) . By definition of the J^+ -invariant, it is invariant as well under the moves (II^+) and (III) and invariance under the move (I_∞) is the content of Corollary 2. This proves the proposition. \square

Propositions 4 and 5 together prove Theorem A of the Introduction.

4.4. Comparison of the Two Invariants

Next we will discuss the relation between the invariants \mathcal{J}_1 and \mathcal{J}_2 . We begin with the case that $w_0(K)$ is odd.

Proposition 6. *If $w_0(K)$ is odd, then $\mathcal{J}_2(K) = 2\mathcal{J}_1(K) - 1$.*

Proof. On generic immersions $K \subset \mathbb{C}^*$ with $w_0(K)$ odd consider the quantity $I(K) := \mathcal{J}_2(K) - 2\mathcal{J}_1(K)$. Since $L : \mathbb{C}^* \rightarrow \mathbb{C}^*$ is an oriented 2–1 covering, each point p of K has two preimages in $L^{-1}(K)$ near which $L^{-1}(K)$ looks like K near p . This shows that I is invariant under the moves (II^\pm) and (III) . Using these moves and adding exterior loops to adjust the rotation number (which by Corollary 2 does not affect I), we can deform K to any given generic immersion with the same winding number around the origin without changing I . So it remains to compute $I(K^w)$ on one generic immersion K^w of winding number $w_0(K^w) = w$ for each odd integer w . By invariance of I under orientation reversal, it suffices to consider positive w .

For any $w \in \mathbb{N}$ consider the immersion K^w of winding number $w_0(K^w) = w$ as shown at the top left in Fig. 14.

We see that K^w is obtained from K^{w-1} by adding a small loop in its innermost component C , which has winding number $w - 1$ around the origin. In view of Lemma 4, this gives the recursion relation $J^+(K^w) = J^+(K^{w-1}) - 2(w - 1)$. Together with the initial condition $J^+(K^1) = 0$ (which holds because K^1 equals the standard curve K_1), this determines $J^+(K^w)$ to be

$$J^+(K^w) = -2(1 + 2 + 3 + \dots + (w - 1)) = -w(w - 1).$$

Now suppose that $w \in \mathbb{N}$ is odd and consider the immersion $L^{-1}(K^w)$ as depicted at the top right in Fig. 14. Applying a (II^-) move (which does not change J^+) to the two innermost arcs yields the immersion L' shown at the bottom right in Fig. 14. The figure shows that L' is obtained from $L^{-1}(K^{w-2})$ by adding two disjoint small loops in its innermost component C , which has winding number $w - 2$ around the origin. In view of Lemma 4, this gives the recursion relation $J^+(L^{-1}(K^w)) = J^+(L') = J^+(L^{-1}(K^{w-2})) - 4(w - 2)$. Together with the initial condition $J^+(L^{-1}(K^1)) = 0$ (which holds because $L^{-1}(K^1)$ equals the standard curve K_1), this determines $J^+(L^{-1}(K^w))$ to be

$$J^+(L^{-1}(K^w)) = -4(1 + 3 + 5 + \dots + (w - 2)) = -(w - 1)^2.$$

Combining the preceding two displayed equations, it follows that

$$I(K^w) = J^+(L^{-1}(K^w)) - 2(J^+(K^w) + w^2/2) = -(w - 1)^2 + 2w(w - 1) - w^2 = -1.$$

In view of the discussion at the beginning of the proof, this shows that for each generic immersion $K \subset \mathbb{C}^*$ with odd winding number around the origin we have $I(K) = \mathcal{J}_2(K) - 2\mathcal{J}_1(K) = -1$. \square

Satellites. Next we turn to the case that $w_0(K)$ is even. We will generate examples using the following construction. For an oriented generic immersion $K \subset \mathbb{C}^*$ and $n \in \mathbb{N}$, the n -satellite nK is defined as follows, see Fig. 15. We start at some point $p \in K$ and construct a spiral along K slowly moving off linearly to the left of K . After n turns along K , we close off the spiral by a short arc a intersecting it $n - 1$ times. Thus, nK has n^2 double points for each double point of K , plus the $n - 1$ double points on the arc a . The rotation number of nK is n times that of K , and $w_0(nK) = nw_0(K)$. Changing the orientation of K leads to an n -satellite which differs from that of K by a reflection of the plane and thus has the same invariants \mathcal{J}_1 and \mathcal{J}_2 .

Note that nK_{-1} (with K_{-1} enclosing the origin) equals the immersion K^n with $w_0(K^n) = n$ defined above.

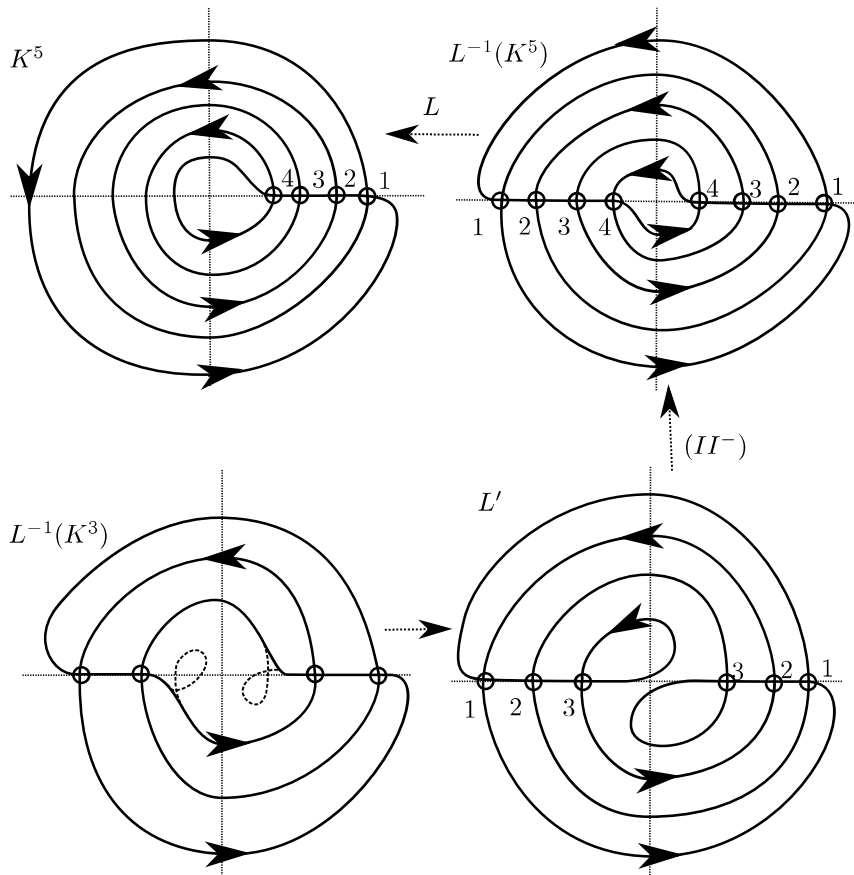


Fig. 14. The curve K^5 and its Levi-Civita lift $L^{-1}(K^5)$.

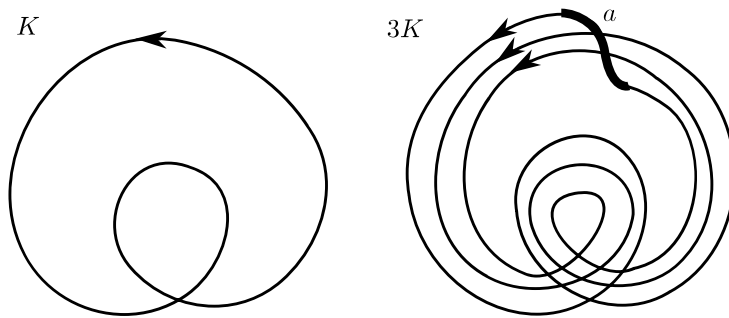


Fig. 15. The 3-satellite $3K$.

Lemma 5. For $j \in \mathbb{N}$, the 2-satellite $2K_j$ of the standard curve K_j with $w_0(K_j) = 1$ has the invariants

$$\mathcal{J}_1(2K_j) = 2\mathcal{J}_2(2K_j) = -8(j - 1).$$

More generally, for any generic immersion $K \subset \mathbb{C}^*$ with $w_0(K) = 1$, the invariants of its 2-satellite $2K$ are related by $\mathcal{J}_1(2K) = 2\mathcal{J}_2(2K)$.

Proof. Consider the double loop in $2K_j$ corresponding to a loop in K_j as shown on the left in Fig. 16. Pulling the interior loop down by a (II^+) move (followed by a (III) and two (II^-) moves) yields an immersion K' with $J^+(K') = J^+(2K_j) + 2$ in which the double loop has been turned into two single loops as shown on the right in Fig. 16. Removing the resulting two

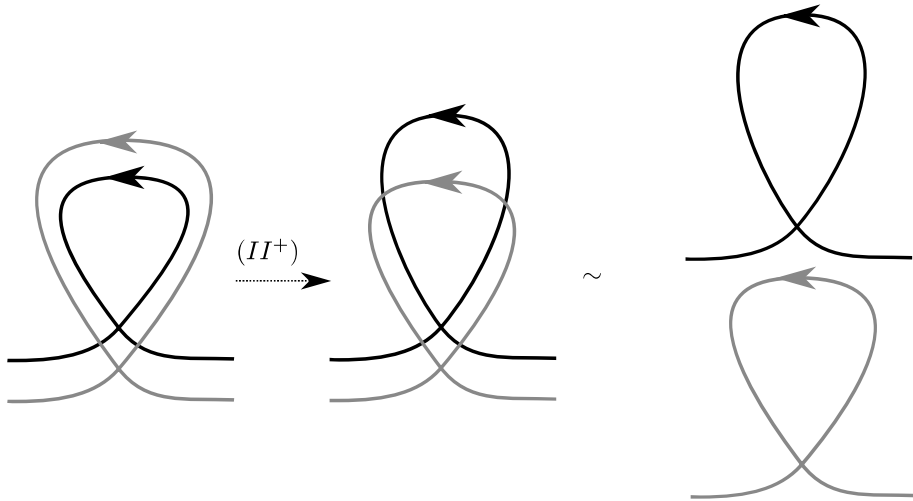


Fig. 16. Resolving a double loop.

loops from K' yields the satellite immersion $2K_{j-1}$. By Lemma 4 the invariants are related by $J^+(K') = J^+(2K_{j-1}) - 2 - 4$, so we get the recursion relation $J^+(2K_j) = J^+(2K_{j-1}) - 8$. Together with the initial condition $J^+(2K_1) = -2$ (since $2K_1 = K_2$), this shows $J^+(2K_j) = -2 - 8(j - 1)$ and therefore $\mathcal{J}_1(2K_j) = J^+(2K_j) + 2^2/2 = -8(j - 1)$.

For \mathcal{J}_2 , note that for each double loop in $2K_j$ the intersection points between the two loops disappear in the lift $\widetilde{2K_j}$ under the Levi-Civita covering. Hence $\widetilde{2K_j}$ has $2(j - 1)$ disjoint loops, so it coincides with the standard immersion K_{2j-1} and we find $\mathcal{J}_2(2K_j) = J^+(K_{2j-1}) = -4(j - 1)$.

Finally, consider the quantity $I(K) := \mathcal{J}_1(2K) - 2\mathcal{J}_2(2K)$ on generic immersions $K \subset \mathbb{C}^*$ with $w_0(K) = 1$. Since every double point in K gives rise to 4 double points in $2K$ and 2 double points in its lift $\widetilde{2K}$ (all of the same type), it follows that I is invariant under the moves (II^\pm) and (III) . Using these moves and adding exterior loops (which by Corollary 2 does not affect I), we can deform any given K to some K_j and conclude $I(K) = I(K_j) = 0$. This proves the lemma. \square

The following proposition shows that for even winding number around the origin the invariants \mathcal{J}_1 and \mathcal{J}_2 are completely independent.

Proposition 7. *On generic immersions $K \subset \mathbb{C}^*$ with $w_0(K)$ even, the pair of invariants $(\mathcal{J}_1(K), \mathcal{J}_2(K))$ attains all values in $2\mathbb{Z} \times 2\mathbb{Z}$.*

Proof. We start with the 2-satellite $2K_j$ of a standard knot K_j with $w_0(K_j) = 1$ and $j \in \mathbb{N}$, which by Lemma 5 has invariants

$$\mathcal{J}_1(2K_j) = -8(j - 1) \quad \text{and} \quad \mathcal{J}_2(2K_j) = -4(j - 1).$$

A (II^+) move pulling one strand of $2K_j$ across the neighboring strand increases $\mathcal{J}_1(2K_j)$ by 2 and leaves $\mathcal{J}_2(2K_j)$ unchanged (because the two new double points do not give rise to double points in the lift $\widetilde{2K_j}$ under the Levi-Civita covering). Performing $k \in \mathbb{N}_0$ such operations, we obtain an immersion $K_{j,k}$ with invariants

$$\mathcal{J}_1(K_{j,k}) = -8(j - 1) + 2k \quad \text{and} \quad \mathcal{J}_2(K_{j,k}) = -4(j - 1).$$

Finally, we take the connected sum $K_{j,k,\ell}$ of $K_{j,k}$ and an immersion L with $w_0(L) = 0$ and $J^+(L) = 2\ell$, for any $\ell \in \mathbb{Z}$. Its lift $\widetilde{K_{j,k,\ell}}$ under the Levi-Civita covering is the connected sum of $\widetilde{K_{j,k}}$ and L , so by additivity of L^+ we get the invariants

$$\mathcal{J}_1(K_{j,k,\ell}) = -8(j - 1) + 2k + 2\ell \quad \text{and} \quad \mathcal{J}_2(K_{j,k,\ell}) = -4(j - 1) + 2\ell. \tag{4.2}$$

By appropriate choices of $j \in \mathbb{N}$, $k \in \mathbb{N}_0$ and $\ell \in \mathbb{Z}$ we can arrange arbitrary values in $2\mathbb{Z} \times 2\mathbb{Z}$ for the pair $(\mathcal{J}_1(K_{j,k,\ell}), \mathcal{J}_2(K_{j,k,\ell}))$. This proves the proposition for winding number 2 around the origin.

For arbitrary even winding number $2w$, let K^w be any generic immersion with $w_0(K^w) = w$. Its 2-satellite $2K^w$ has invariants $(\mathcal{J}_1(2K^w), \mathcal{J}_2(2K^w)) = (2a, 2b)$ for some $a, b \in \mathbb{Z}$ (depending on w). Adding a loop to K^w in a component of winding number 1 corresponds to adding a double loop to $2K^w$, which becomes two disjoint loops in the lift $\widetilde{2K^w}$. Hence this operation decreases $\mathcal{J}_2(2K^w)$ by 4, and thus $\mathcal{J}_1(2K^w)$ by 8 according to Lemma 5. Adding $(j-1)$ such loops thus decreases $\mathcal{J}_1(2K^w)$ by $8(j-1)$ and $\mathcal{J}_2(2K^w)$ by $4(j-1)$. Performing subsequently the other two operations described above with $k \in \mathbb{N}_0$ and $\ell \in \mathbb{Z}$, we can therefore change the original values $(2a, 2b)$ by arbitrary pairs of numbers as in Eq. (4.2), and thus achieve all values in $2\mathbb{Z} \times 2\mathbb{Z}$. \square

Propositions 6 and 7 together prove Theorem B of the Introduction.

4.5. Further Discussion

Smooth knot type. Every periodic orbit γ of energy $c < c_1$ in a planar Stark–Zeeman system describes a smooth knot in the Moser regularized energy hypersurface $\mathcal{S}_c^b \cong \mathbb{R}P^3$. If γ avoids the zero velocity locus and collisions, then it is obtained from its footpoint projection $K \subset \mathbb{R}^2$ by adding to each point on K its velocity vector. Clearly, the knot type of γ is an invariant of K under Stark–Zeeman homotopies.

Alternatively, we can associate to each point on K its (appropriately scaled) normal vector; the resulting knot in $\mathbb{R}P^3$ differs from γ just by a rotation by $-\pi/2$ and thus represents the same knot type. According to [6, Theorem 1.2], every oriented knot type in $\mathbb{R}^2 \times S^1$ is obtained by adding the normal vectors to an immersion in the plane. Since every periodic orbit in $\mathbb{R}P^3$ can be pushed into $\mathbb{R}^2 \times S^1$, the complement of the zero velocity and collision loci, this implies

Corollary 3. *Every oriented knot type in $\mathbb{R}P^3$ is realized by a Moser regularized periodic orbit in some planar Stark–Zeeman system.*

We can further lift the orbit γ from $\mathbb{R}P^3$ to an orbit $\tilde{\gamma}$ in the Levi-Civita regularization S^3 (choosing one of the two components if γ is contractible and traversing γ twice if it is noncontractible). The oriented knot type of $\tilde{\gamma}$ is then also an invariant of the footpoint projection $K \subset \mathbb{R}^2$ under Stark–Zeeman homotopies.

Relation to Legendrian knots. Adding the unit conormal vectors to an immersion $K \subset \mathbb{R}^2$ actually yields a *Legendrian* knot in the unit cotangent bundle $S^*\mathbb{R}^2 = \mathbb{R}^2 \times S^1$ with its canonical contact structure. The invariant $J^+(K)$ can be defined as the Bennequin–Tabachnikov invariant of its Legendrian lift [3], and more refined invariants of immersions in the plane can be obtained from Chekanov type invariants of their Legendrian lifts [18]. Since the moves (I_0) and (I_∞) in Definition 1 do not lift to Legendrian isotopies, general invariants of Legendrian lifts will not be invariant under Stark–Zeeman homotopies. It would be interesting, however, to explore the possibility of defining more refined invariants by suitably modifying Legendrian isotopy invariants.

Higher energy levels. In the planar restricted three-body system, only the bounded components of energy levels below the first critical value c_1 fit our definition of a planar Stark–Zeeman system. For the bounded component of a higher energy level, the potential will have two Coulomb singularities in the Hill's region. The extension of the invariant \mathcal{J}_1 to this situation is straightforward, adding correction terms for both winding numbers around the singularities (and similarly for more than two singularities). For the extension of \mathcal{J}_2 we have infinitely many possibilities due to the different Levi-Civita type regularizations corresponding to the different covers of the Moser regularized energy hypersurface $\mathbb{R}P^3 \# \mathbb{R}P^3$.

Bifurcations. The notion of a Stark–Zeeman homotopy captures the events occurring in a generic 1-parameter family of simple periodic orbits in planar Stark–Zeeman systems. Now such a family generically also undergoes bifurcations, in which an n -fold multiple of the orbits becomes degenerate and gives birth to a new family of simple orbits [1]. The footpoint projection L of one of these new simple orbits is thus a perturbation of the n -fold multiple of the projection K of one of

the original orbits. Based on the above discussion of satellites, we conjecture that $J^+(L)$ is bounded from below by a constant depending only on $J^+(K)$ and n . Such an estimate would be useful for comparing the values of the invariants \mathcal{J}_1 and \mathcal{J}_2 in different planar Stark–Zeeman systems.

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