FIRST-PASSAGE TIMES OVER MOVING BOUNDARIES FOR ASYMPTOTICALLY STABLE WALKS[*]

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Abstract. Let $\{S_n,\, n \geqslant 1\}$ be a random walk with independent and identically distributed increments, and let $\{g_n,\, n \geqslant 1\}$ be a sequence of real numbers. Let T_g denote the first time when S_n leaves (g_n, ∞). Assume that the random walk is oscillating and asymptotically stable, that is, there exists a sequence $\{c_n,\, n \geqslant 1\}$ such that S_n/c_n converges to a stable law. In this paper we determine the tail behavior of T_g for all oscillating asymptotically stable walks and all boundary sequences satisfying $g_n = o(c_n)$. Furthermore, we prove that the rescaled random walk conditioned to stay above the boundary up to time n converges, as $n \to \infty$, towards the stable meander.

Key words. random walk, stable distribution, first-passage time, overshoot, moving boundary

1. Introduction and main results. Consider a classical random walk

$$S_0 = 0, \qquad S_n = X_1 + \cdots + X_n, \quad n \geqslant 1,$$

where $X, X_1, X_2, \ldots$ are independent and identically distributed (i.i.d.) random variables. For a real-valued sequence $\{g_n\}$ let

$$(1) \qquad T_g := \min\{n \geqslant 1 \colon S_n \leqslant g_n\}$$

be the first crossing time of the moving boundary $\{g_n\}$ by $\{S_n\}$. The aim of this paper is to study the asymptotics of $\mathbf{P}(T_g > n)$ as n goes to infinity.

An important particular case of this problem is the case of a constant boundary $g_n \equiv -x$ for some x. In this case $T_g \equiv \tau_x$, where

$$\tau_x := \min\{n \geqslant 1 \colon S_n \leqslant -x\}.$$

For constant boundaries the following result (see [9]) is available: if

$$(2) \qquad \mathbf{P}(S_n > 0) \to \rho \in (0, 1),$$

then, for every fixed $x \geqslant 0$,

$$(3) \qquad \mathbf{P}(\tau_x > n) \sim V(x) n^{\rho-1} L(n),$$

where $V(x)$ denotes the renewal function corresponding to the weak descending ladder height process and $L(n)$ is a slowly varying function. (Here and in what follows all unspecified limits are taken as $n \to \infty$.)

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Greenwood and Novikov [11, Theorem 1] have shown that if the sequence $\{g_n\}$ is decreasing and concave, then

$$(4) \qquad \frac{\mathbf{P}(T_g > n)}{\mathbf{P}(\tau_0 > n)} \to R_g \in (0, \infty].$$

If, in addition, $\mathbf{E}|g_{\tau_0}|$ is finite, then $R_g < \infty$. This result has been generalized by Wachtel and Denisov [6]: if $\{g_n\}$ decreases and $\{V(-g_n)\}$ is subadditive, then (4) holds and R_g is finite for random walks satisfying $\mathbf{E}V(-g_{\tau_0}) < \infty$.

If $g_n \geqslant 0$ is increasing, then, according to Proposition 1 in [6],

$$(5) \qquad \frac{\mathbf{P}(T_g > n)}{\mathbf{P}(\tau_0 > n)} \to L_g \in [0, 1].$$

Moreover, if $\mathbf{E}X = 0$ and $\mathbf{E}X^2 < \infty$, then $L_g > 0$ if and only if $\mathbf{E}g_{\tau_0} < \infty$. An alternative version of this result was obtained in [11]: it was assumed there that $\mathbf{E}X = 0$ and $\mathbf{E}e^{-\lambda X} < \infty$ for some $\lambda > 0$.

Relation (3) implies that $\mathbf{E}|g_{\tau_0}| < \infty$ provided that $|g_n| = O(n^\gamma)$ with some $\gamma < 1 - \rho$. Since the asymptotic behavior of the renewal function V cannot be expressed in terms of ρ only, it is not clear how to use the condition $\mathbf{E}V(|g_{\tau_0}|) < \infty$. The trivial bound $V(x) \leqslant Cx$ reduces $\mathbf{E}V(|g_{\tau_0}|) < \infty$ to $\mathbf{E}|g_{\tau_0}| < \infty$. In order to have more accurate information on V we need to impose further restrictions on the distribution of X.

In the present paper, we consider the class of asymptotically stable random walks. Let

$$\mathcal{A} := \{0 < \alpha < 1; \ |\beta| < 1\} \cup \{1 < \alpha < 2; \ |\beta| \leqslant 1\}$$
$$\cup \{\alpha = 1, \ \beta = 0\} \cup \{\alpha = 2, \ \beta = 0\}$$

be a subset in $\mathbf{R}^2$. For $(\alpha, \beta) \in \mathcal{A}$ and a random variable X, write $X \in \mathcal{D}(\alpha, \beta)$ if the distribution of X belongs to the domain of attraction of a stable law with a characteristic function

$$(6) \qquad G_{\alpha,\beta}(t) := \exp\left\{-c|t|^\alpha \left(1 - i\beta \frac{t}{|t|} \tan \frac{\pi\alpha}{2}\right)\right\}, \qquad c > 0,$$

and, in addition, $\mathbf{E}X = 0$ if this moment exists. Let $\{c_n\}$ be a sequence of positive numbers specified by the relation

$$(7) \qquad c_n := \inf\{u \geqslant 0 \colon \mu(u) \leqslant n^{-1}\}, \qquad n \geqslant 1,$$

where

$$\mu(u) := \frac{1}{u^2} \int_{-u}^{u} x^2 \, \mathbf{P}(X \in dx).$$

It is known (see, for instance, [10, Chap. XVII, section 5]) that for every $X \in \mathcal{D}(\alpha, \beta)$ the function $\mu(u)$ is regularly varying with index $(-\alpha)$. This implies that c_n is regularly varying with index α^{-1}; i.e., there exists a function $l_1(x)$ slowly varying at infinity such that

$$(8) \qquad c_n = n^{1/\alpha} l_1(n).$$

In addition, the scaled sequence $\{S_n/c_n,\ n \geqslant 1\}$ converges in distribution to the stable law given by (6). In this case we say that S_n is an *asymptotically stable random walk.* For every $X \in \mathcal{D}(\alpha, \beta)$, there is an explicit formula for ρ,

$$
(9) \qquad \rho = \begin{cases} \dfrac{1}{2}, & \alpha = 1, \\[2ex] \dfrac{1}{2} + \dfrac{1}{\pi\alpha}\tan^{-1}\left(\beta\tan\dfrac{\pi\alpha}{2}\right) & \text{otherwise.} \end{cases}
$$

If $X \in \mathcal{D}(\alpha, \beta)$, then the function $V(x)$ is regularly varying with index $\alpha(1-\rho)$. Moreover, according to Lemma 13 in [13],

$$
(10) \qquad \lim_{n\to\infty} V(c_n)\mathbf{P}(\tau_0 > n) =: A \in (0, \infty).
$$

By Corollary 1 in [6], if S_n is asymptotically stable, then the finiteness of $\mathbf{E}V(|g_{\tau_0}|)$ is equivalent to

$$
\sum_{n=1}^{\infty} \frac{V(|g_n|)}{nV(c_n)} < \infty.
$$

Using the fact that $V(x)$ is a regularly varying function of index $\alpha(1-\rho)$, we see that $\mathbf{E}V(|g_{\tau_0}|)$ is finite if $|g_n| = O(c_n/\ln^a n)$ with some $a > 1/\alpha(1-\rho)$.

If $\{g_n\}$ is decreasing but $V(-g_n)$ is not subadditive, then we cannot apply Theorem 1 of [6]. But in Theorem 2 of [6] it is shown that (4) with finite R_g remains valid for boundaries satisfying

$$
(11) \qquad \sum_{n=1}^{\infty} \frac{V(|g_n|)}{nV(c_n/\ln n)} < \infty.
$$

Moreover, it is proved in [6] that if $\{g_n\}$ increases and satisfies (11), then the constant L_g in (5) is strictly positive. We note also that (11) is fulfilled if, for example, $g_n = O(c_n/\ln^{1+a} n)$ with some $a > 1/\alpha(1-\rho)$. A logarithmic version of this result has been given by Aurzada and Kramm [2]. More precisely, they proved that

$$
\mathbf{P}(T_g > n) = n^{\rho-1+o(1)}
$$

for any boundary satisfying $g_n = O(n^\gamma)$ with some $\gamma < 1/\alpha$.

In the present paper, we derive the asymptotics of $\mathbf{P}(T_g > n)$ for all boundaries $g_n = o(c_n)$. Since c_n is the scaling sequence for the random walk S_n, it is natural to expect that the behavior of $\mathbf{P}(T_g > n)$ is quite similar to that of $\mathbf{P}(\tau_0 > n)$. The following result confirms this conjecture.

THEOREM 1. *Assume that $X \in \mathcal{D}(\alpha, \beta)$. If $g_n = o(c_n)$ and $\mathbf{P}(T_g > n) > 0$ for all $n \geqslant 1$, then*

$$
(12) \qquad \frac{\mathbf{P}(T_g > n)}{\mathbf{P}(\tau_0 > n)} \sim U_g(n),
$$

where U_g is a positive slowly varying function with values

$$
0 < U_g(n) = \mathbf{E}[V(S_n - g_n); T_g > n], \qquad n \geqslant 1.
$$

If $\mathbf{E}X = 0$ and $\mathbf{E}X^2 < \infty$, then (12) is a special case of Theorem 2 from our previous paper [5], where random walks with independent but not necessarily identical distributed increments were considered.

Theorem 1 states that the tail of T_g is a regularly varying tail with index $\rho - 1$ for any boundary $g_n = o(c_n)$. We now turn to the following question: for which boundaries are the sequences $\mathbf{P}(T_g > n)$ and $\mathbf{P}(\tau_0 > n)$ asymptotically equivalent? In other words, we want to find conditions which guarantee that $U_g(n)$ is bounded away from 0 and from ∞.

THEOREM 2. *Assume that $X \in \mathcal{D}(\alpha, \beta)$ and that, as $x \to \infty$,*

$$(13) \qquad V(x+1) - V(x) = O\left(\frac{V(x)}{x}\right).$$

(a) *If*

$$(14) \qquad \sum_{n=1}^{\infty} \frac{\max_{k \leqslant n} |g_k|}{n c_n} < \infty,$$

then there exist positive constants U_* and U^* such that

$$(15) \qquad U_* \leqslant U_g(n) \leqslant U^* \quad \text{for all } n \geqslant 1.$$

(b) *Moreover, if the sequence $\{g_n\}$ is monotone and (14) holds, then*

$$(16) \qquad \lim_{n \to \infty} U_g(n) =: U_g(\infty) \in (0, \infty).$$

Our condition (13) is a bit weaker than the strong renewal theorem for ladder heights. It is well known from renewal theory that the strong renewal theorem and (13) hold for all walks satisfying $\alpha(1 - \rho) < 1/2$. But if $\alpha(1 - \rho) \geqslant 1/2$, then (13) may fail; see Example 4 in [14]. We refer the reader to a recent paper by Caravenna and Doney [4] for necessary and sufficient conditions for the strong renewal theorem.

Mogul'skii and Pecherskii [12] have shown that if the boundary sequence satisfies the condition $g_{n+k} \leqslant g_n + g_k$, then there exists a sequence of events $\{E_n\}$ such that

$$(17) \qquad E_n \subseteq \{S_n > g_n\} \quad \text{for every} \quad n \geqslant 1$$

and

$$(18) \qquad \sum_{n=0}^{\infty} z^n \mathbf{P}(T_g > n) = \exp\left\{\sum_{n=1}^{\infty} \frac{z^n}{n} \mathbf{P}(E_n)\right\}.$$

This relation is a generalization of the classical factorization identity for the stopping time τ_0. Unfortunately, the events E_n have very complicated structure in the case of moving boundaries, and there is no hope of deriving the tail asymptotics for T_g from (18). But (17) allows one to obtain upper bounds for $\mathbf{P}(T_g > n)$. It has been shown in Remark 2 in [6] that

$$\mathbf{P}(T_g > n) \leqslant q_n$$

with q_n defined by

$$(19) \qquad \sum_{n=0}^{\infty} z^n q_n = \left(\sum_{n=0}^{\infty} z^n \mathbf{P}(\tau_0 > n)\right) \exp\left\{\sum_{n=1}^{\infty} \frac{z^n}{n} \Delta_n\right\},$$

where $\Delta_n := \mathbf{P}(S_n > g_n) - \mathbf{P}(S_n > 0)$. Using the standard estimate for the concentration function of S_n, one gets

$$|\Delta_n| \leqslant C \frac{|g_n| + 1}{c_n}.$$

From this bound and (19) we infer that if $|g_n|/(nc_n)$ is summable, then

$$\mathbf{P}(T_g > n) \leqslant q_n \leqslant C\mathbf{P}(\tau_0 > n).$$

It is worth mentioning that the condition (14) is quite close to the summability of the sequence $\{|g_n|/(nc_n)\}$.

If the boundary sequence is strictly positive, $g_n \to \infty$ and $g_n = o(c_n)$, then, by the local limit theorem for S_n,

$$\Delta_n \sim -f_{\alpha,\beta}(0) \frac{g_n}{c_n},$$

where $f_{\alpha,\beta}(x)$ is the density function of the stable distribution given by (6). If we additionally assume that $g_n/(nc_n)$ is not summable, then, by (19),

$$\mathbf{P}(T_g > n) = o\big(\mathbf{P}(\tau_0 > n)\big).$$

This indicates that condition (14) is very close to the optimal one, and it cannot be relaxed in the case of monotone increasing boundaries.

We now turn to the conditional limit theorem. Define the rescaled process

$$(20) \qquad s_n(t) = \frac{S_{[nt]}}{c_n}, \qquad t \in [0, 1].$$

It has been shown by Doney [7] that if $X \in \mathcal{D}(\alpha, \beta)$, then, for every fixed x, s_n conditioned on $\{\tau_x > n\}$ converges weakly on $D[0, 1]$ towards a process $M_{\alpha,\beta}$. This limiting process is usually called the *stable Lévy meander*. Our further result shows that this convergence remains valid for all moving boundaries satisfying $g_n = o(c_n)$.

THEOREM 3. *Assume that the conditions of Theorem* 1 *hold. Then the distribution of s_n conditioned on $\{T_g > n\}$ converges weakly on $D[0, 1]$ towards $M_{\alpha,\beta}$.*

For random walks with zero mean and finite variance we have convergence towards the Brownian meander. In [5] we proved that this convergence holds even for random walks with nonidentically distributed increments satisfying the classical Lindeberg condition. But for random walks with infinite variance the statement of Theorem 3 is new.

The conditional limit theorem allows one to complement Theorem 2 by the following statement: if $g_n = o(c_n)$ is monotone decreasing and $|g_n|/nc_n$ is not summable, then

$$(21) \qquad \lim_{n \to \infty} U_g(n) = \infty.$$

(We prove (21) at the end of the paper.)

Recall that we have shown after Theorem 2 that if g_n is increasing and $g_n/(nc_n)$ is not summable, then $\lim_{n\to\infty} U_g(n) = 0$. This implies that the conditions on the boundary in Theorem 2(b) are optimal.

Our approach to moving boundaries is based on the following universality idea. The condition $g_n = o(c_n)$ means that the boundary reduces to the constant zero boundary after rescaling of the random walk by c_n. Therefore, it is natural to expect that the asymptotic behavior of $\mathbf{P}(T_g > n)$ is similar to that of $\mathbf{P}(\tau_0 > n)$. This is an adaptation of the universality methodology suggested in our recent paper [5], where the first-passage problems for random walks belonging to the domain of attraction of the Brownian motion were considered. It is worth mentioning that in the present paper we use a different type of universality: we fix the distribution of the random walk and look for a possible widest class of boundary functions with the same type of tail behavior for the corresponding first-passage time.

2. Some results from fluctuation theory. In this section we collect some known facts about first-passage problems with constant boundaries. We start with the following result on exit times.

LEMMA 4. *Let S_n be an asymptotically stable random walk. Then, for every $\delta_n \downarrow 0$ there exists $\varepsilon_n \downarrow 0$ such that*

$$(22) \qquad \sup_{x \in [0, \delta_n c_n]} \left| \frac{\mathbf{P}(\tau_x > n)}{V(x)\mathbf{P}(\tau_0 > n)} - 1 \right| \leqslant \varepsilon_n.$$

In addition, the following estimate is valid for all $x \geqslant 0$:

$$(23) \qquad \mathbf{P}(\tau_x > n) \leqslant C_0 V(\min\{x, c_n\})\mathbf{P}(\tau_0 > n).$$

The first assertion (22) is Corollary 3 in [8]; (23) was proved in Lemma 2.1 of [1]. Let τ^+ denote the first ascending ladder epoch, i.e.,

$$\tau^+ := \min\{n \geqslant 1 \colon S_n > 0\}.$$

Let $H(x)$ denote the renewal function of strict ascending ladder epochs. Then, similarly to (10),

$$(24) \qquad \lim_{n \to \infty} H(c_n)\mathbf{P}(\tau^+ > n) =: A^+ \in (0, \infty).$$

Define also $\tau_x^+ := \min\{n \geqslant 1 \colon S_n > x\}$. Then, similarly to (23),

$$(25) \qquad \mathbf{P}(\tau_x^+ > n) \leqslant C_0 H(\min\{x, c_n\})\mathbf{P}(\tau^+ > n), \qquad x \geqslant 0.$$

Combining (10) and (24), and using the well-known relation

$$\mathbf{P}(\tau^+ > n)\mathbf{P}(\tau_0 > n) \sim n^{-1},$$

we conclude that

$$(26) \qquad \lim_{n \to \infty} \frac{V(c_n)H(c_n)}{n} \in (0, \infty).$$

LEMMA 5. *Let f be a continuous functional on $D[0,1]$, and let $\delta_n \to 0$. Then, for the sequence of processes s_n defined in (20),*

$$\sup_{x \leqslant \delta_n c_n} \left| \mathbf{E}[f(s_n) \mid \tau_x > n] - \mathbf{E}f(M_{\alpha,\beta}) \right| \to 0.$$

Proof. Let x_n be a sequence satisfying $x_n \leqslant \delta_n c_n$. Caravenna and Chaumont have shown in [3] that the Doob transform of s_n converges to a stable process conditioned to stay positive at all times. Performing the inverse change of measure, one can easily obtain the convergence

$$\mathbf{E}[f(s_n) \mid \tau_{x_n} > n] \to \mathbf{E}f(M_{\alpha,\beta}).$$

The desired uniformity follows from the standard contradiction argument. Lemma 4 is proved.

3. Proof of Theorem 1.

3.1. Preliminary estimates. Define

$$G_n := \max_{k \leqslant n} |g_k|, \qquad Z_n := S_n - g_n,$$

and

$$Q_{k,n}(y) := \mathbf{P}\Big(y + \min_{k \leqslant j \leqslant n} (Z_j - Z_k) > 0\Big).$$

LEMMA 6. *Fix some sequence $\delta_n \downarrow 0$ such that $\delta_n c_n$ increases. Then, for all $y \geqslant 0$,*

$$\max_{k \leqslant n/2} \left| \frac{Q_{k,n}(y)}{\mathbf{P}(\tau_0 > n - k)} - V(y) \right|$$

(27)
$$\leqslant \overline{\varepsilon}_n V(y) + 2(1 + C_0 + \overline{\varepsilon}_1)V(G_n) + 2C_0 V(y)\mathbf{I}\{y > \overline{\delta}_n c_n - 2G_n\},$$

where

$$\overline{\varepsilon}_n := \max_{k \in [n/2,n]} \varepsilon_k, \quad \overline{\delta}_n := \frac{\min_{k \in [n/2,n]} \delta_k c_k}{c_n},$$

and ε_n is taken from (22).

Proof. It is immediate from the definition of $Q_{k,n}$ that

$$\mathbf{P}\Big(y - 2G_n + \min_{j \leqslant n-k} S_k > 0\Big) \leqslant Q_{k,n}(y) \leqslant \mathbf{P}\Big(y + 2G_n + \min_{j \leqslant n-k} S_k > 0\Big).$$

If $y + 2G_n \leqslant \overline{\delta}_n c_n$, then $y + 2G_n \leqslant \delta_{n-k} c_{n-k}$ for all $k \leqslant n/2$. Therefore, by (22),

$$\mathbf{P}\Big(y + 2G_n + \min_{j \leqslant n-k} S_k > 0\Big) \leqslant (1 + \overline{\varepsilon}_n)V(y + 2G_n)\mathbf{P}(\tau_0 > n - k)$$

for every $y \leqslant \overline{\delta}_n c_n - 2G_n$. Using now the subadditivity of V, we obtain

$$\frac{\mathbf{P}(y + 2G_n + \min_{j \leqslant n-k} S_k > 0)}{\mathbf{P}(\tau_0 > n - k)} \leqslant (1 + \overline{\varepsilon}_n)V(y) + 2(1 + \overline{\varepsilon}_n)V(G_n),$$
$$y \leqslant \overline{\delta}_n c_n - 2G_n.$$

If $y > \overline{\delta}_n c_n - 2G_n$, then, using (23) and the subadditivity of V, we have

$$\frac{\mathbf{P}(y + 2G_n + \min_{j \leqslant n-k} S_k > 0)}{\mathbf{P}(\tau_0 > n - k)} \leqslant C_0 V(y) + 2C_0 V(G_n), \qquad y > \overline{\delta}_n c_n - 2G_n.$$

As a result we have

$$\frac{Q_{k,n}(y)}{\mathbf{P}(\tau_0 > n - k)} \leqslant (1 + \overline{\varepsilon}_n)V(y) + 2(1 + C_0 + \overline{\varepsilon}_n)V(G_n)$$

(28)
$$+ C_0 V(y)\mathbf{I}\{y > \overline{\delta}_n c_n - 2G_n\}.$$

If $y \leqslant \overline{\delta}_n c_n - 2G_n$, then it follows from (22) that

$$\mathbf{P}\Big(y - 2G_n + \min_{j \leqslant n-k} S_k > 0\Big) \geqslant (1 - \overline{\varepsilon}_n)V(y - 2G_n)\mathbf{P}(\tau_0 > n - k).$$

Therefore, due to the subadditivity of V,

$$\frac{Q_{k,n}(y)}{\mathbf{P}(\tau_0 > n - k)} \geqslant \frac{\mathbf{P}(y - 2G_n + \min_{j \leqslant n-k} S_k > 0)}{\mathbf{P}(\tau_0 > n - k)}$$
$$\geqslant (1 - \overline{\varepsilon}_n)V(y) - 2V(G_n) - V(y)\mathbf{I}\{y > \overline{\delta}_n c_n - 2G_n\}.$$

Combining this with (28), we obtain (27). Lemma 6 is proved.

Define

$$Z_n^* := V(Z_n)\mathbf{I}\{T_g > n\}.$$

LEMMA 7. *For every stopping time ν,*

$$|\mathbf{E}Z_{\nu \wedge n}^* - \mathbf{E}Z_n^*| \leqslant 2V(G_n)\mathbf{P}(T_g > \nu \wedge n), \qquad n \geqslant 1.$$

Proof. By the Markov property at time $\nu \wedge n$,

$$\mathbf{E}Z_n^* = \mathbf{E}[V(S_n - g_n); T_g > n]$$
$$= \sum_{k=1}^{n} \int_0^\infty \mathbf{P}(Z_k \in dz; T_g > k, \nu \wedge n = k)$$
$$\times \mathbf{E}\Big[V(z + Z_n - Z_k); z + \min_{k \leqslant j \leqslant n} Z_j - Z_k > 0\Big].$$

Then, we have the estimates from above,

$$\mathbf{E}Z_n^* \leqslant \sum_{k=1}^{n} \int_0^\infty \mathbf{P}(Z_k \in dz; T_g > k, \nu \wedge n = k)$$
$$\times \mathbf{E}\Big[V(z + 2G_n + S_{n-k}); z + 2G_n + \min_{j \leqslant n-k} S_j > 0\Big],$$

and from below,

$$\mathbf{E}Z_n^* = \mathbf{E}[V(S_n - g_n); T_g > n]$$
$$\geqslant \sum_{k=1}^{n} \int_0^\infty \mathbf{P}(Z_k \in dz; T_g > k, \nu \wedge n = k)$$
$$\times \mathbf{E}\Big[V(z - 2G_n + S_{n-k}); z - 2G_n + \min_{j \leqslant n-k} S_j > 0\Big].$$

Then, using the harmonicity and the subadditivity of $V(x)$, we get

$$\mathbf{E}Z_n^* \leqslant \mathbf{E}[V(Z_{\nu \wedge n} + 2G_n); T_g > \nu \wedge n] \leqslant \mathbf{E}Z_{\nu \wedge n}^* + 2V(G_n)\mathbf{P}(T_g > \nu \wedge n)$$

and

$$\mathbf{E}Z_n^* \geqslant \mathbf{E}[V(Z_{\nu \wedge n} - 2G_n); T_g > \nu \wedge n] \geqslant \mathbf{E}Z_{\nu \wedge n}^* - 2V(G_n)\mathbf{P}(T_g > \nu \wedge n).$$

Thus, the proof is complete.

Define the stopping times

(29) $$\nu(h) := \min\{k \geqslant 1 \colon Z_k \geqslant h\} \quad \text{and} \quad \nu_n := \nu(c_n) \wedge n.$$

LEMMA 8. *There exist constants C_1 and C_2 such that*

(30) $$\frac{\mathbf{P}(T_g > n)}{\mathbf{P}(\tau_0 > n)} \leqslant C_1 \mathbf{E}Z_n^*$$

and

(31) $$\frac{\mathbf{P}(T_g > \nu_n)}{\mathbf{P}(\tau_0 > n)} \leqslant C_2 \mathbf{E}Z_n^*$$

for all $n \geqslant 1$.

Proof. According to Lemma 24 in [5],

$$\mathbf{P}(S_n \geqslant x \mid T_g > n) \geqslant \mathbf{P}(S_n \geqslant x), \qquad x \in \mathbf{R}.$$

This implies that

(32) $$\frac{\mathbf{E}Z_n^*}{\mathbf{P}(T_g > n)} = \mathbf{E}[V(Z_n) \mid T_g > n] \geqslant \mathbf{E}V(Z_n).$$

Since S_n is asymptotically stable and $V(x)$ is regularly varying of index $\alpha(1 - \rho)$,

$$\mathbf{E}V(Z_n) = \mathbf{E}V(S_n - g_n) \sim V(c_n)\mathbf{E}[Y^{\alpha(1-\rho)}; Y > 0],$$

where Y is distributed according to the stable law from (6).

Combining this with (32), we obtain

$$\liminf_{n \to \infty} \frac{\mathbf{E}Z_n^*}{V(c_n)\mathbf{P}(T_g > n)} \geqslant \mathbf{E}[Y^{\alpha(1-\rho)}; Y > 0] > 0.$$

Using now (10), we get (30).

In order to prove (31) we note that

$$\begin{aligned}
\mathbf{P}(T_g > \nu_n) &= \mathbf{P}(T_g > \nu_n, Z_{\nu_n} < c_n) + \mathbf{P}(T_g > \nu_n, Z_{\nu_n} \geqslant c_n) \\
&\leqslant \mathbf{P}(T_g > n) + \mathbf{P}(Z_{\nu_n}^* \geqslant V(c_n)).
\end{aligned}$$

Applying (30) to the first summand and the Markov inequality to the second summand, we obtain

(33) $$\mathbf{P}(T_g > \nu_n) \leqslant C_1 \mathbf{E}Z_n^* \mathbf{P}(\tau_0 > n) + \frac{\mathbf{E}Z_{\nu_n}^*}{V(c_n)}.$$

By Lemma 7,

$$\frac{\mathbf{E}Z_{\nu_n}^*}{V(c_n)} \leqslant \frac{\mathbf{E}Z_n^*}{V(c_n)} + \frac{2V(G_n)}{V(c_n)}\mathbf{P}(T_g > \nu_n).$$

Substituting this into (33), we have

$$\mathbf{P}(T_g > \nu_n) \leqslant C_1 \mathbf{E}Z_n^* \mathbf{P}(\tau_0 > n) + \frac{\mathbf{E}Z_n^*}{V(c_n)} + \frac{2V(G_n)}{V(c_n)} \mathbf{P}(T_g > \nu_n).$$

Since $G_n = o(c_n)$, $2V(G_n)/V(c_n) < 1/2$ for all n sufficiently large. For such values of n we have

$$\mathbf{P}(T_g > \nu_n) \leqslant 2C_1 \mathbf{E}Z_n^* \mathbf{P}(\tau_0 > n) + 2\frac{\mathbf{E}Z_n^*}{V(c_n)},$$

and (31) follows now from (10). Lemma 8 is proved.

LEMMA 9. *Sequences* $\mathbf{E}Z_n^*$ *and* $\mathbf{E}Z_{\nu_n}^*$ *are slowly varying and, moreover,*

$$\mathbf{E}Z_n^* \sim \mathbf{E}Z_{\nu_n}^*.$$

Proof. Taking $\nu \equiv k < n$ in Lemma 7 and using (30), we obtain

$$|\mathbf{E}Z_k^* - \mathbf{E}Z_n^*| \leqslant 2V(G_n)\mathbf{P}(T_g > k) \leqslant 2C_1 V(G_n)\mathbf{E}Z_k^* \mathbf{P}(\tau_0 > k).$$

Therefore,

$$\max_{k \in [m,n]}\left|\frac{\mathbf{E}Z_n^*}{\mathbf{E}Z_k^*} - 1\right| \leqslant 2C_1 V(G_n)\mathbf{P}(\tau_0 > m).$$

Assumption $G_n = o(c_n)$ and (10) imply that $V(G_n) = o(1/\mathbf{P}(\tau_0 > n))$. Recalling that $\mathbf{P}(\tau_0 > n)$ is regularly varying, we infer that $V(G_n) = o(1/\mathbf{P}(\tau_0 > m(n)))$ if $m(n)/n \to 0$ sufficiently slow. Thus,

$$\max_{k \in [m(n),n]}\left|\frac{\mathbf{E}Z_k^*}{\mathbf{E}Z_n^*} - 1\right| \to 0$$

provided that $m(n)/n$ is bounded from below or goes to zero sufficiently slow. In particular, the sequence $\mathbf{E}Z_n^*$ is slowly varying.

Taking $\nu = \nu_n$ in Lemma 7 and using (31), we have

$$|\mathbf{E}Z_{\nu_n}^* - \mathbf{E}Z_n^*| \leqslant 2V(G_n)\mathbf{P}(T_g > \nu_n) \leqslant 2C_2 V(G_n)\mathbf{E}Z_n^* \mathbf{P}(\tau_0 > k) = o(\mathbf{E}Z_n^*).$$

In other words, $\mathbf{E}Z_{\nu_n}^* \sim \mathbf{E}Z_n^*$. Thus, the proof is finished.

LEMMA 10. *For every sequence* A_n *satisfying* $A_n \gg c_n$, *we have*

$$\mathbf{E}[Z_{\nu_n}^*; Z_{\nu_n} > A_n] = o(\mathbf{E}Z_n^*).$$

Proof. Since V is increasing and subadditive, for all n sufficiently large,

$$\mathbf{E}[Z_{\nu_n}^*; Z_{\nu_n} > A_n]$$
$$= \sum_{j=1}^{n} \int_{g_{j-1}}^{c_n} \mathbf{P}(S_{j-1} \in dy, T_g > j-1)\mathbf{E}[V(y - g_j + X_1); y - g_j + X_1 > A_n]$$
$$\leqslant \sum_{j=1}^{n} \mathbf{P}(T_g > j-1)\mathbf{E}[V(c_n + 2G_n + X_1); c_n + 2G_n + X_1 > A_n]$$
$$\leqslant \sum_{j=1}^{n} \mathbf{P}(T_g > j-1)\left(\mathbf{E}\left[V(X_1); X_1 > \frac{A_n}{2}\right] + 3V(c_n)\mathbf{P}\left(X_1 > \frac{A_n}{2}\right)\right).$$

Combining (30), Lemma 9, and the fact that $\mathbf{P}(\tau_0 > j)$ is regularly varying of index $\rho - 1 \in (-1, 0)$, we get

$$\sum_{j=1}^{n} \mathbf{P}(T_g > j - 1) \leqslant 1 + C_1 \sum_{j=1}^{n-1} \mathbf{E} Z_j^* \mathbf{P}(\tau_0 > j) \leqslant Cn \mathbf{E} Z_n^* \mathbf{P}(\tau_0 > n).$$

Therefore,

$$\frac{\mathbf{E}[Z_{\nu_n}^*; Z_{\nu_n} > A_n]}{\mathbf{E} Z_n^*}$$

$$(34) \qquad \leqslant Cn \mathbf{P}(\tau_0 > n) \left(\mathbf{E}\left[V(X_1); X_1 > \frac{A_n}{2} \right] + 3 V(c_n) \mathbf{P}\left(X_1 > \frac{A_n}{2} \right) \right).$$

The assumption $A_n \gg c_n$ implies that $\mathbf{P}(X_1 > A_n) = o(n^{-1})$. Consequently,

$$(35) \qquad V(c_n) \mathbf{P}\left(X_1 > \frac{A_n}{2} \right) = o\left(\frac{1}{n \mathbf{P}(\tau_0 > n)} \right).$$

Furthermore,

$$\mathbf{E}\left[V(X_1); X_1 > \frac{A_n}{2} \right] = \int_{A_n/2}^{\infty} V(x) \, \mathbf{P}(X_1 \in dx) \leqslant \int_{A_n/2}^{\infty} \frac{V(x)}{x^2} \, \theta(dx),$$

where $\theta(dx) := x^2 \mathbf{P}(|X_1| \in dx)$. If S_n is asymptotically stable, then $\Theta(x) := \theta((0, x))$ is regularly varying of index $2 - \alpha$. Since $V(x)/x^2$ is regularly varying of index $\alpha(1 - \rho) - 2$, we infer that

$$\mathbf{E}\left[V(X_1); X_1 > \frac{A_n}{2} \right] \leqslant C \frac{V(A_n)}{A_n^2} \Theta(A_n) = o\left(\frac{V(c_n)}{c_n^2} \Theta(c_n) \right),$$

where the last step follows from the fact that $(V(x)/x^2)\Theta(x)$ is regularly varying of index $-\alpha \rho < 0$. By the definition of c_n, $c_n^{-2} \Theta(c_n) \sim n^{-1}$. Using (10) once again, we get

$$(36) \qquad \mathbf{E}\left[V(X_1); X_1 > \frac{A_n}{2} \right] = o\left(\frac{1}{n \mathbf{P}(\tau_0 > n)} \right).$$

By combining (34)–(36) we complete the proof.

3.2. Proof of Theorem 1. Let $\{m(n)\}$ be a sequence of natural numbers such that $m(n) \to \infty$ and $m(n) = o(n)$. By the Markov property,

$$\mathbf{P}(T_g > n) = \mathbf{E}[Q_{\nu_{m(n)}, n}(Z_{\nu_{m(n)}}); T_g > \nu_{m(n)}].$$

Applying Lemma 6 and noting that $\mathbf{P}(\tau_0 > n - k) \sim \mathbf{P}(\tau_0 > n)$ uniformly in $k \leqslant m(n)$, we get

$$\frac{\mathbf{P}(T_g > n)}{\mathbf{P}(\tau_0 > n)} = (1 + o(1)) \mathbf{E} Z_{\nu_{m(n)}}^* + O\big(V(G_n) \mathbf{P}(T_g > \nu_{m(n)}) \big)$$

$$(37) \qquad\qquad + O\big(\mathbf{E}[Z_{\nu_{m(n)}}^*; Z_{\nu_{m(n)}} > \overline{\delta}_n c_n - G_n] \big).$$

By (31), $\mathbf{P}(T_g > \nu_{m(n)}) \leqslant C_2 \mathbf{E} Z^*_{m(n)} \mathbf{P}(\tau_0 > m(n))$. From this estimate and from the fact that $\mathbf{P}(\tau_0 > n) V(G_n) \to 0$ we infer that, for every sequence $\{m(n)\}$ such that $m(n)/n \to 0$ sufficiently slow,

$$(38) \qquad V(G_n) \mathbf{P}(T_g > \nu_{m(n)}) = o(\mathbf{E} Z^*_{m(n)}).$$

For every sequence $m(n) = o(n)$, we can choose $\{\delta_n\}$ satisfying $\overline{\delta}_n c_n \gg G_n$ and $\overline{\delta}_n c_n \gg c_{m(n)}$. Then, by Lemma 10,

$$\mathbf{E}[Z^*_{\nu_{m(n)}}; \, Z_{\nu_{m(n)}} > \overline{\delta}_n c_n - G_n] = o(\mathbf{E} Z^*_{m(n)}).$$

Plugging this and (38) into (37), we obtain

$$\frac{\mathbf{P}(T_g > n)}{\mathbf{P}(\tau_0 > n)} = (1 + o(1)) \mathbf{E} Z^*_{\nu_{m(n)}} + o(\mathbf{E} Z^*_{m(n)}).$$

According to Lemma 9,

$$(39) \qquad \mathbf{E} Z^*_{\nu_{m(n)}} \sim \mathbf{E} Z^*_{m(n)} \sim \mathbf{E} Z^*_n$$

provided that $m(n)/n \to 0$ sufficiently slow. Consequently,

$$\frac{\mathbf{P}(T_g > n)}{\mathbf{P}(\tau_0 > n)} \sim \mathbf{E} Z^*_n.$$

Thus, the proof is complete.

4. Proof of Theorem 2.

4.1. Technical preparations.

LEMMA 11. *For any sequence $\{r_n\}$ satisfying $r_n = o(c_n)$ we have*

$$\mathbf{E}[V(S_n + r_n); T_g > n] \sim \mathbf{E} Z^*_n.$$

Proof. By the subadditivity of $V(x)$,

$$|V(x+y) - V(x)| \leqslant V(|y|), \qquad x, y \in \mathbf{R}.$$

Therefore,

$$\begin{aligned} |\mathbf{E}[V(S_n &+ r_n); T_g > n] - \mathbf{E} Z^*_n| \\ &= |\mathbf{E}[V(S_n + r_n); T_g > n] - \mathbf{E}[V(S_n - g_n); T_g > n]| \\ &\leqslant V(|r_n + g_n|) \mathbf{P}(T_g > n). \end{aligned}$$

According to Theorem 1, $\mathbf{P}(T_g > n) \sim \mathbf{E} Z^*_n \mathbf{P}(\tau_0 > n)$. Therefore,

$$\frac{\mathbf{E}[V(S_n + r_n); T_g > n]}{\mathbf{E} Z^*_n} - 1 = O\big(V(|r_n + g_n|) \mathbf{P}(\tau_0 > n)\big).$$

Recalling that $|r_n + g_n| = o(c_n)$ and taking into account (10), we conclude that $V(|r_n + g_n|) \mathbf{P}(\tau_0 > n)$ converges to zero. This completes the proof.

LEMMA 12. *Under the conditions of Theorem* 1,

$$\mathbf{P}\big(S_n \in (x, x+1], T_g > n\big) = O\bigg(\frac{H(\min\{x + G_n, c_n\})}{nc_n}\mathbf{E}Z_n^*\bigg)$$

uniformly in x.

Proof. Set $m = [n/2]$. By the Markov property at time m,

$$\mathbf{P}\big(S_n \in (x, x+1], T_g > n\big)$$
$$\leqslant \int_{g_m}^{\infty} \mathbf{P}(S_m \in dy, T_g > m)\,\mathbf{P}\big(S_{n-m} \in (x - y, x - y + 1], \tau_{y+G_n} > n - m\big).$$

Define $X_k^* = -X_{n-m+1-k}$, $S_k^* = X_1^* + \cdots + X_k^*$ for $k = 1, \ldots, n - m$. Define also $\tau_y^* := \min\{k \geqslant 1\colon S_k^* < -y\}$. Then

$$\mathbf{P}\big(S_{n-m} \in (x - y, x - y + 1], \tau_{y+G_n} > n - m\big)$$
$$\leqslant \mathbf{P}\big(S_{n-m}^* \in [y - x - 1, y - x), \tau_{x+1+G_n}^* > n - m\big).$$

Since S_k^* is also asymptotically stable, one has the following standard bound for the concentration function:

$$\sup_x \mathbf{P}\big(S_n^* \in (x, x+1]\big) \leqslant \frac{C}{c_n}.$$

Using this bound, we infer that

$$\mathbf{P}\big(S_n^* \in (x, x+1], \tau_y^* > n\big)$$
$$\leqslant \int_{-\infty}^{\infty} \mathbf{P}\bigg(S_{n/2}^* \in dz, \tau_y^* > \frac{n}{2}\bigg)\mathbf{P}\big(S_{n/2}^* \in (x - z, x - z + 1]\big)$$
$$\leqslant \frac{C}{c_{n/2}}\mathbf{P}\bigg(\tau_y^* > \frac{n}{2}\bigg).$$

Therefore,

$$\mathbf{P}\big(S_{n-m} \in (x - y, x - y + 1], \tau_{y+G_n} > n - m\big)$$
$$= O\bigg(\frac{\mathbf{P}(\tau_{x+1+G_n}^* > (n - m)/2)}{c_{(n-m)/2}}\bigg) = O\bigg(\frac{\mathbf{P}(\tau_{x+1+G_n}^* > n)}{c_n}\bigg).$$

It is obvious that $\mathbf{P}(\tau_{x+1+G_n}^* > n) = \mathbf{P}(\tau_{x+1+G_n}^+ > n)$. Then, taking into account (25) and (24), we conclude that

$$\mathbf{P}\big(S_n \in (x, x+1], T_g > n\big) = O\bigg(\frac{H(\min\{c_n, x + G_n\})}{c_n H(c_n)}\mathbf{P}(T_g > n)\bigg).$$

Recalling that $\mathbf{P}(T_g > n) = O(\mathbf{E}Z_n^*/V(c_n))$ and using (26), we obtain the desired bound.

LEMMA 13. *Assume that the conditions of Theorem* 1 *are valid. Assume, in addition, that* (13) *holds. Then*

$$\mathbf{E}[V(S_n + G_{2n}) - V(S_n + G_n); T_g > n] = O\bigg(\frac{G_{2n}}{c_n}\mathbf{E}Z_n^*\bigg)$$

and

$$\mathbf{E}[V(S_n - G_n) - V(S_n - G_{2n}); T_g > n] = O\bigg(\frac{G_{2n}}{c_n}\mathbf{E}Z_n^*\bigg).$$

Proof. We first note that the subadditivity of V implies the bound

$$\mathbf{E}[V(S_n + G_{2n}) - V(S_n + G_n); S_n < G_{2n}, T_g > n]$$
$$\leqslant V(G_{2n})\mathbf{P}(S_n < G_{2n}, T_g > n).$$

Applying Lemma 12, we then get

$$\mathbf{E}[V(S_n + G_{2n}) - V(S_n + G_n); S_n < G_{2n}, T_g > n]$$
$$= O\left(G_{2n}\frac{H(G_{2n})V(G_{2n})}{nc_n}\mathbf{E}Z_n^*\right).$$

Recalling that $G_n = o(c_n)$ and using (26), we infer that

$$H(G_{2n})V(G_{2n}) = o(n).$$

As a result,

$$(40) \qquad \mathbf{E}[V(S_n + G_{2n}) - V(S_n + G_n); S_n < G_{2n}, T_g > n] = o\left(\frac{G_{2n}}{c_n}\mathbf{E}Z_n^*\right).$$

Furthermore, it follows from (13) that, uniformly for $x \in (G_{2n}, c_n)$,

$$\mathbf{E}\big[V(S_n + G_{2n}) - V(S_n + G_n); S_n \in (x, x+1], T_g > n\big]$$
$$= O\left(G_{2n}\frac{V(x)}{x}\mathbf{P}(S_n \in (x, x+1], T_g > n)\right).$$

Applying now Lemma 12, we conclude that

$$\mathbf{E}\big[V(S_n + G_{2n}) - V(S_n + G_n); S_n \in (x, x+1], T_g > n\big]$$
$$= O\left(G_{2n}\frac{V(x)H(x)}{xnc_n}\mathbf{E}Z_n^*\right).$$

Therefore,

$$\mathbf{E}\big[V(S_n + G_{2n}) - V(S_n + G_n); S_n \in (G_{2n}, c_n], T_g > n\big]$$
$$= O\left(\frac{G_{2n}}{nc_n}\mathbf{E}Z_n^* \sum_{k=[G_{2n}]}^{[c_n]+1} \frac{V(k)H(k)}{k}\right).$$

Recalling that $V(x)H(x)$ is regularly varying with index α and taking into account (26), we arrive at

$$(41) \qquad \mathbf{E}[V(S_n + G_{2n}) - V(S_n + G_n); S_n \in (G_{2n}, c_n], T_g > n] = O\left(\frac{G_{2n}}{c_n}\mathbf{E}Z_n^*\right).$$

Using (13) once again and noting that the function $V(x)/x$ is eventually nonincreasing, we get

$$\mathbf{E}[V(S_n + G_{2n}) - V(S_n + G_n); S_n > c_n, T_g > n]$$
$$= O\left(G_{2n}\frac{V(c_n)}{c_n}\mathbf{P}(T_g > n)\right).$$

By Theorem 1 and (10), $V(c_n)\mathbf{P}(T_g > n) \sim \mathbf{E}Z_n^*$. Consequently,

$$\mathbf{E}[V(S_n + G_{2n}) - V(S_n + G_n); S_n > c_n, T_g > n] = O\left(\frac{G_{2n}}{c_n}\mathbf{E}Z_n^*\right).$$

Combining this with (40) and (41), we complete the proof of the first estimate. The second one can be derived by using the same arguments. For this reason we omit its proof. Lemma 13 is proved.

4.2. Proof of Theorem 2(a). For every $m \in (n, 2n)$, we have

$$\mathbf{E}[V(S_m + G_m); T_g > m]$$
$$= \int_{-G_n}^{\infty} \mathbf{P}(S_n \in dx; T_g > n)\mathbf{E}[V(x + S_{m-n} + G_m); \min_{k \leqslant n-m}(x + S_k - g_{n+k}) > 0]$$
$$\leqslant \int_{-G_n}^{\infty} \mathbf{P}(S_n \in dx; T_g > n)\mathbf{E}[V(x + S_{m-n} + G_{2n}); \tau_{x+G_{2n}} > n - m].$$

Recalling that $V(y + S_k)\mathbf{I}\{\tau_y > k\}$ is a martingale, we obtain

$$\max_{m \in (n,2n]} \mathbf{E}[V(S_m + G_m); T_g > m] \leqslant \mathbf{E}[V(S_n + G_{2n}); T_g > n]$$
$$= \mathbf{E}[V(S_n + G_n); T_g > n] + \mathbf{E}[V(S_n + G_{2n}) - V(S_n + G_n); T_g > n].$$

Applying the first estimate from Lemma 13 and noting that

$$\mathbf{E}Z_n^* = \mathbf{E}[V(S_n - g_n); T_g > n] \leqslant \mathbf{E}[V(S_n + G_n); T_g > n],$$

we infer that, for some constant B and all $n \geqslant 1$,

$$\max_{m \in (n,2n]} \mathbf{E}[V(S_m + G_m); T_g > m] \leqslant \mathbf{E}[V(S_n + G_n); T_g > n]\left(1 + B\frac{G_{2n}}{c_n}\right).$$

Thus, for every $\ell \geqslant 1$,

$$\max_{n \leqslant 2^\ell} \mathbf{E}[V(S_m + G_m); T_g > m] \leqslant \mathbf{E}[V(S_1 + G_1); T_g > 1] \prod_{j=0}^{\ell-1}\left(1 + B\frac{G_{2^{j+1}}}{c_{2^j}}\right).$$

It is obvious that (14) implies that

$$\sum_{j=1}^{\infty} \frac{G_{2^{j+1}}}{c_{2^j}} < \infty.$$

Therefore,

$$\sup_{n \geqslant 1} \mathbf{E}[V(S_n + G_n); T_g > n] < \infty.$$

Recalling that $U_g(n) = \mathbf{E}Z_n^*$ is bounded from above by $\mathbf{E}[V(S_n + G_n); T_g > n]$, we get the upper bound in (15).

The proof of the lower bound in (15) is very similar to that of the upper bound. We first note that

$$\mathbf{E}Z_n^* \geqslant \mathbf{E}[V(S_n - G_n); T_g > n].$$

Furthermore, for every $m \in (n, 2n]$,

$$
\begin{aligned}
\mathbf{E}[V(S_m &- G_m); T_g > m] \\
&\geqslant \int_{G_{2n}}^{\infty} \mathbf{P}(S_n \in dx; T_g > n)\mathbf{E}[V(x + S_{m-n} - G_{2n}); \tau_{x-G_{2n}} > n - m] \\
&= \mathbf{E}[V(S_n - G_{2n}); T_g > n] \\
&= \mathbf{E}[V(S_n - G_n); T_g > n] - \mathbf{E}[V(S_n - G_n) - V(S_n - G_{2n}); T_g > n].
\end{aligned}
$$

Using the second estimate from Lemma 13 and recalling that $\mathbf{E}Z_n^* \sim \mathbf{E}[V(S_n - G_n); T_g > n]$ by Lemma 11, we arrive at the inequality

$$
\min_{m \in (n, 2n]} \mathbf{E}[V(S_m - G_m); T_g > m] \geqslant \mathbf{E}[V(S_n - G_n); T_g > n]\left(1 - B\frac{G_{2n}}{c_n}\right).
$$

Choosing n_0 so that $B(G_{2n}/c_n) < 1/2$ for all $n > n_0$, we then get

$$
\begin{aligned}
\min_{n \leqslant n_0 2^\ell} &\mathbf{E}[V(S_n - G_n); T_g > n] \\
&\geqslant \min_{n \leqslant n_0} \mathbf{E}[V(S_n - G_n); T_g > n] \prod_{j=0}^{\ell-1}\left(1 - B\frac{G_{n_0 2^{j+1}}}{c_{n_0 2^j}}\right).
\end{aligned}
$$

From this bound and (15) we obtain the desired lower bound.

4.3. Proof of Theorem 2(b). If g_n increases, then, in view of Lemma 4 in [6], the sequence $V(S_n - g_n)\mathbf{I}\{T_g > n\}$ is a supermartingale. In particular, the sequence $\mathbf{E}Z_n^*$ decreases and has finite limit. The positivity of the limit follows from (15).

If g_n decreases, then $V(S_n - g_n)\mathbf{I}\{T_g > n\}$ is a submartingale; see Lemma 1 in [6]. This implies that the limit of $\mathbf{E}Z_n^*$ is positive. Its finiteness follows from (15).

5. Functional convergence.

5.1. Proof of the conditional limit theorem. Fix some sequence $m(n) = o(n)$ such that (39) holds. Let δ_n satisfy the condition

$$
G_n \ll \overline{\delta}_n^2 c_n \ll c_{m(n)} \ll \overline{\delta}_n c_n.
$$

By the Markov property and (23),

$$
\begin{aligned}
\mathbf{P}(T_g &> n, Z_{\nu_{m(n)}} > \overline{\delta}_n c_n) \\
&= \int_{\overline{\delta}_n c_n}^{\infty} \mathbf{P}(Z_{\nu_{m(n)}} \in dz, T_g > \nu_{m(n)})\mathbf{P}\Big(z + \min_{\nu_{m(n)} \leqslant j \leqslant n}(Z_j - Z_{\nu_{m(n)}}) > 0\Big) \\
&\leqslant \int_{\overline{\delta}_n c_n}^{\infty} \mathbf{P}(Z_{\nu_{m(n)}} \in dz, T_g > \nu_{m(n)})\mathbf{P}\Big(z + 2G_n + \min_{j \leqslant n-m(n)} S_j > 0\Big) \\
&\leqslant C_0 \mathbf{P}(\tau_0 > n - m(n))\mathbf{E}[V(Z_{\nu_{m(n)}} + 2G_n); T_g > \nu_{m(n)}, Z_{\nu_{m(n)}} > \overline{\delta}_n c_n].
\end{aligned}
$$

Since $G_n \ll \overline{\delta}_n c_n$ and $m(n) = O(n)$, we have $V(Z_{\nu_{m(n)}} + 2G_n) = O(V(Z_{\nu_{m(n)}}))$ uniformly on the event $\{Z_{\nu_{m(n)}} > \overline{\delta}_n c_n\}$. Consequently,

$$
\mathbf{P}(T_g > n, Z_{\nu_{m(n)}} > \overline{\delta}_n c_n) = O\big(\mathbf{P}(\tau_0 > n)\mathbf{E}[Z_{\nu_{m(n)}}^*; Z_{\nu_{m(n)}} > \overline{\delta}_n c_n]\big).
$$

Now, in view of Lemma 9 and (39),

$$(42) \qquad \mathbf{P}(T_g > n, \, Z_{\nu_{m(n)}} > \overline{\delta}_n c_n) = o\big(\mathbf{P}(\tau_0 > n)\mathbf{E}Z_n^*\big).$$

Using the Markov property and (23) once again, we obtain

$$\mathbf{P}(T_g > n, \, Z_{\nu_{m(n)}} < \overline{\delta}_n^2 c_n)$$
$$= \int_0^{\overline{\delta}_n^2 c_n} \mathbf{P}(Z_{\nu_{m(n)}} \in dz, \, T_g > \nu_{m(n)})\mathbf{P}\Big(z + \min_{\nu_{m(n)} \leqslant j \leqslant n}(Z_j - Z_{\nu_{m(n)}}) > 0\Big)$$
$$\leqslant \int_0^{\overline{\delta}_n^2 c_n} \mathbf{P}(Z_{\nu_{m(n)}} \in dz, \, T_g > \nu_{m(n)})\mathbf{P}\Big(z + 2G_n + \min_{j \leqslant n-m(n)} S_j > 0\Big)$$
$$\leqslant C_0 V(\overline{\delta}_n^2 c_n)\mathbf{P}(\tau_0 > n - m(n))\mathbf{P}(T_g > \nu_{m(n)}).$$

Then, according to (31) and (39),

$$\mathbf{P}(T_g > n, \, Z_{\nu_{m(n)}} < \overline{\delta}_n^2 c_n) = O\big(V(\overline{\delta}_n^2 c_n)\mathbf{P}(\tau_0 > n)\mathbf{E}Z_n^*\mathbf{P}(\tau_0 > m(n))\big).$$

Using the relation $\mathbf{P}(\tau_0 > m(n)) \sim C/V(c_{m(n)})$ and the assumption $c_{m(n)} \gg \overline{\delta}_n^2 c_n$, we get

$$V(\overline{\delta}_n^2 c_n)\mathbf{P}(\tau_0 > m(n)) \to 0.$$

Therefore,

$$(43) \qquad \mathbf{P}(T_g > n, \, Z_{\nu_{m(n)}} < \overline{\delta}_n^2 c_n) = o\big(\mathbf{P}(\tau_0 > n)\mathbf{E}Z_n^*\big).$$

Let f be a uniformly continuous and bounded functional on the space $D[0,1]$. Without loss of generality, we may assume that $0 \leqslant f \leqslant 1$. It follows then from (42), (43), and Theorem 1 that

$$(44) \quad \mathbf{E}[f(s_n); \, T_g > n] = \mathbf{E}\big[f(s_n); \, Z_{\nu_{m(n)}} \in [\overline{\delta}_n^2 c_n, \overline{\delta}_n c_n], \, T_g > n\big] + o(\mathbf{P}(T_g > n)).$$

For every $k \geqslant 0$ and every $y \in \mathbf{R}$, consider the functional $f(k, y; \cdot)$ defined by

$$f(k, y; h) := f\Big(y + \Big(h(t) - h\Big(\frac{k}{n}\Big)\Big)\mathbf{I}\Big\{t \geqslant \frac{k}{n}\Big\}\Big), \qquad h \in D[0,1].$$

It follows from the definition of $\nu_{m(n)}$ that

$$\frac{\max_{k \leqslant \nu_{m(n)}} |S_k - S_{\nu_{m(n)}}|}{c_n} \leqslant \frac{\max_{k \leqslant \nu_{m(n)}} |Z_k - Z_{\nu_{m(n)}}|}{c_n} + \frac{2G_n}{c_n}$$
$$\leqslant \frac{Z_{\nu_{m(n)}}}{c_n} + \frac{3G_n}{c_n} \leqslant \overline{\delta}_n + \frac{3G_n}{c_n}$$

on the event $\{Z_{\nu_{m(n)}} \leqslant \overline{\delta}_n c_n, \, T_g > \nu_{m(n)}\}$. From this bound and the uniform continuity of the functional f we infer that

$$f(s_n) - f\Big(\nu_{m(n)}, \frac{S_{\nu_{m(n)}}}{c_n}; s_n\Big) = o(1) \text{ on the event } \{Z_{\nu_{m(n)}} \leqslant \overline{\delta}_n c_n, \, T_g > \nu_{m(n)}\}.$$

Combining this estimate with (44), we obtain

$$
\mathbf{E}[f(s_n); T_g > n]
$$

$$
(45) \quad = \mathbf{E}\left[f\left(\nu_{m(n)}, \frac{S_{\nu_{m(n)}}}{c_n}; s_n\right); Z_{\nu_{m(n)}} \in [\overline{\delta}_n^2 c_n, \overline{\delta}_n c_n], T_g > n\right] + o\big(\mathbf{P}(T_g > n)\big).
$$

By the Markov property at $\nu_{m(n)}$,

$$
\mathbf{E}\left[f\left(\nu_{m(n)}, \frac{S_{\nu_{m(n)}}}{c_n}, s_n\right); T_g > n, Z_{\nu_{m(n)}} \in [\overline{\delta}_n^2 c_n, \overline{\delta}_n c_n]\right]
$$

$$
= \sum_{k=1}^{m(n)} \int_{\overline{\delta}_n^2 c_n}^{\overline{\delta}_n c_n} \mathbf{P}(Z_k \in dy, \nu_{m(n)} = k, T_g > k)
$$

$$
\times \mathbf{E}\left[f\left(k, \frac{y + g_k}{c_n}; s_n\right); y + \min_{j \in [k,n]}(Z_j - Z_k) > 0\right].
$$

We now note that it suffices to show that, uniformly in $y \in [\overline{\delta}_n^2 c_n, \overline{\delta}_n c_n]$ and $k \leqslant m(n)$,

$$
\mathbf{E}\left[f\left(k, \frac{y + g_k}{c_n}, s_n\right); y + \min_{j \in [k,n]}(Z_j - Z_k) > 0\right]
$$

$$
(46) \qquad = \big(\mathbf{E}f(M_{\alpha,\beta}) + o(1)\big)V(y)\mathbf{P}(\tau_0 > n).
$$

Indeed, this relation implies that

$$
\mathbf{E}\left[f\left(\nu_{m(n)}, \frac{S_{\nu_{m(n)}}}{c_n}, s_n\right); T_g > n, Z_{\nu_{m(n)}} \in [\overline{\delta}_n^2 c_n, \overline{\delta}_n c_n]\right]
$$

$$
= (\mathbf{E}f(M_{\alpha,\beta}) + o(1))\mathbf{P}(\tau_0 > n)
$$

$$
\times \mathbf{E}\big[V(Z_{\nu_{m(n)}}); T_g > \nu_{m(n)}, Z_{\nu_{m(n)}} \in [\overline{\delta}_n^2 c_n, \overline{\delta}_n c_n]\big].
$$

It follows from the assumption $\overline{\delta}_n^2 c_n \ll c_{m(n)}$ and the definition of $\nu_{m(n)}$ that

$$
\mathbf{E}[V(Z_{\nu_m(n)}); T_g > \nu_{m(n)}, Z_{\nu_{m(n)}} < \overline{\delta}_n^2 c_n]
$$

$$
= \mathbf{E}[V(Z_{\nu_m(n)}); T_g > m(n), Z_{m(n)} < \overline{\delta}_n^2 c_n] \leqslant V(\overline{\delta}_n^2 c_n)\mathbf{P}(T_g > m(n)).
$$

Applying now Theorem 1 and recalling that $\mathbf{E}Z_{m(n)}^* \sim \mathbf{E}Z_n^*$, we get

$$
\mathbf{E}[V(Z_{\nu_m(n)}); T_g > \nu_{m(n)}, Z_{\nu_{m(n)}} < \overline{\delta}_n^2 c_n] = O\big(V(\overline{\delta}_n^2 c_n)\mathbf{E}Z_n^*\mathbf{P}(\tau_0 > m(n))\big).
$$

Using now (10) and the assumption $\overline{\delta}_n^2 c_n \ll c_{m(n)}$, we conclude that

$$
\mathbf{E}[V(Z_{\nu_m(n)}); T_g > \nu_{m(n)}, Z_{\nu_{m(n)}} < \overline{\delta}_n^2 c_n] = o(\mathbf{E}Z_n^*).
$$

We know that $c_{m(n)} \ll \overline{\delta}_n c_n$. Then, by Lemma 10,

$$
\mathbf{E}[V(Z_{\nu_m(n)}); T_g > \nu_{m(n)}, Z_{\nu_{m(n)}} > \overline{\delta}_n c_n] = o(\mathbf{E}Z_n^*).
$$

From the above two relations we infer that

$$
\mathbf{E}\big[V(Z_{\nu_m(n)}); T_g > \nu_{m(n)}, Z_{\nu_{m(n)}} \in [\overline{\delta}_n^2 c_n, \overline{\delta}_n c_n]\big] \sim \mathbf{E}Z_n^*
$$

and, consequently,

$$\mathbf{E}\left[f\left(\nu_{m(n)}, \frac{S_{\nu_{m(n)}}}{B_n}, s_n\right); T_g > n, \, Z_{\nu_{m(n)}} \in [\overline{\delta}_n^2 c_n, \overline{\delta}_n c_n]\right]$$
$$\sim \mathbf{E}f(M_{\alpha,\beta})\mathbf{E}Z_n^*\mathbf{P}(\tau_0 > n) \sim \mathbf{E}f(M_{\alpha,\beta})\mathbf{P}(T_g > n).$$

Plugging this into (45), we have

$$\mathbf{E}[f(s_n); T_g > n] \sim \mathbf{E}f(M_{\alpha,\beta})\mathbf{P}(T_g > n).$$

This implies immediately the desired weak convergence. Thus, it remains to show (46).

We prove (46) by giving bounds for the expectation on the left-hand side in terms of boundary problems with constant boundaries. More precisely,

$$\mathbf{E}\left[f\left(k, \frac{y+g_k}{c_n}, s_n\right); y + \min_{j\in[k,n]}(Z_j - Z_k) > 0\right]$$
$$\leqslant \mathbf{E}\left[f\left(k, \frac{y+g_k}{c_n}, s_n\right); \tau_{y+2G_n} > n\right]$$
$$= \mathbf{E}\left[f\left(k, \frac{y+g_k}{c_n}, s_n\right) \,\Big|\, \tau_{y+2G_n} > n\right]\mathbf{P}(\tau_{y+G_n} > n).$$

Note that $|f(k, S_k/c_n, s_n) - f(s_n)| \to 0$ uniformly over all trajectories s_n with $S_k \leqslant G_n + \overline{\delta}_n c_n$. This convergence is also uniform in $k \leqslant m(n)$. Then, using Lemma 5 and (22), we get

$$\mathbf{E}\left[f\left(k, \frac{y+g_k}{c_n}, s_n\right); y + \min_{j\in[k,n]}(Z_j - Z_k) > 0\right]$$
$$\leqslant V(y + 2G_n)\mathbf{P}(\tau_0 > n)\mathbf{E}f(M_{\alpha,\beta})(1 + o(1)).$$

Noting that $V(y + 2G_n) \sim V(y)$ for $y \in [\overline{\delta}_n^2 c_n, \overline{\delta}_n c_n]$, we obtain the upper bound

$$\mathbf{E}\left[f\left(k, \frac{y+g_k}{c_n}, s_n\right); y + \min_{j\in[k,n]}(Z_j - Z_k) > 0\right]$$
$$\leqslant V(y)\mathbf{P}(\tau_0 > n)\mathbf{E}f(M_{\alpha,\beta})(1 + o(1)).$$

By the same argument,

$$\mathbf{E}\left[f\left(k, \frac{y+g_k}{c_n}, s_n\right); y + \min_{j\in[k,n]}(Z_j - Z_k) > 0\right]$$
$$\geqslant \mathbf{E}\left[f\left(k, \frac{y+g_k}{c_n}, s_n\right); \tau_{y-2G_n} > n\right]$$
$$\geqslant V(y)\mathbf{P}(\tau_0 > n)\mathbf{E}f(M_{\alpha,\beta})(1 + o(1)).$$

These two estimates imply (46). This completes the proof of the functional limit theorem.

5.2. Proof of (21). Since the sequence $\{g_n\}$ is decreasing, the sequence $V(S_n - g_n)\mathbf{I}\{T_g > n\}$ is a submartingale and, in particular, the sequence $\mathbf{E}[V(S_n - g_n); T_g > n]$ is increasing. So, it suffices to show that $\mathbf{E}[V(S_{2^j} - g_{2^j}); T_g > 2^j]$ converges

to ∞. We first note that

$$\mathbf{E}[V(S_{2^{j+1}} - g_{2^{j+1}}); T_g > 2^{j+1}]$$

$$\geqslant \int_{g_{2^j}}^{\infty} \mathbf{P}(S_{2^j} \in dy; T_g > 2^j) \mathbf{E}[V(y + S_{2^j} - g_{2^{j+1}}); \tau_{y-g_{2^j}} > 2^j]$$

$$= \mathbf{E}[V(S_{2^j} - g_{2^j}); T_g > 2^j] + \int_{g_{2^j}}^{\infty} \mathbf{P}(S_{2^j} \in dy; T_g > 2^j)$$

$$\times \mathbf{E}[V(y + S_{2^j} - g_{2^{j+1}}) - V(y + S_{2^j} - g_{2^j}); \tau_{y-g_{2^j}} > 2^j],$$

where we have used the harmonicity of V in the last step. Furthermore, since all terms in the integral are positive, we have

$$\mathbf{E}[V(S_{2^{j+1}} - g_{2^{j+1}}); T_g > 2^{j+1}] - \mathbf{E}[V(S_{2^j} - g_{2^j}); T_g > 2^j]$$

$$\geqslant \int_{c_{2^j}}^{2c_{2^j}} \mathbf{P}(S_{2^j} \in dy; T_g > 2^j)$$

$$\times \mathbf{E}[V(y + S_{2^j} - g_{2^{j+1}}) - V(y + S_{2^j} - g_{2^j}); \tau_{y-g_{2^j}} > 2^j].$$

Since V is a renewal function, there exists a positive constant C such that

$$\liminf_{x \to \infty} \frac{x}{V(x)}(V(x + u) - V(x)) \geqslant Cu$$

for all u large enough. Therefore,

$$\mathbf{E}[V(S_{2^{j+1}} - g_{2^{j+1}}); T_g > 2^{j+1}] - \mathbf{E}[V(S_{2^j} - g_{2^j}); T_g > 2^j]$$

$$\geqslant \int_{c_{2^j}}^{2c_{2^j}} \mathbf{P}(S_{2^j} \in dy; T_g > 2^j)$$

$$\times C'(g_{2^j} - g_{2^{j+1}}) \frac{V(c_{2^j})}{c_{2^j}} \mathbf{P}(S_{2^j} \in [c_{2^j}, 2c_{2^j}], \tau_{y-g_{2^j}} > 2^j).$$

Applying now the standard (nonconditional) limit theorem for S_n and Theorem 3, we obtain

$$\mathbf{E}[V(S_{2^{j+1}} - g_{2^{j+1}}); T_g > 2^{j+1}] - \mathbf{E}[V(S_{2^j} - g_{2^j}); T_g > 2^j]$$

$$\geqslant C''(g_{2^j} - g_{2^{j+1}}) \frac{V(c_{2^j})}{c_{2^j}} \mathbf{P}(T_g > 2^j).$$

Combining Theorem 1 and (10), we have

$$V(c_{2^j}) \mathbf{P}(T_g > 2^j) \sim A \mathbf{E}[V(S_{2^j} - g_{2^j}); T_g > 2^j].$$

Consequently,

$$\mathbf{E}[V(S_{2^{j+1}} - g_{2^{j+1}}); T_g > 2^{j+1}]$$

$$\geqslant \mathbf{E}[V(S_{2^j} - g_{2^j}); T_g > 2^j]\left(1 + C''' \frac{g_{2^j} - g_{2^{j+1}}}{c_{2^j}}\right).$$

Iterating this estimate, we obtain

$$\mathbf{E}[V(S_{2^{j+1}} - g_{2^{j+1}}); T_g > 2^{j+1}]$$

$$\geqslant \mathbf{E}[V(S_1 - g_1); T_g > 1] \prod_{k=0}^{j}\left(1 + c''' \frac{g_{2^k} - g_{2^{k+1}}}{c_{2^k}}\right).$$

It remains to note that the condition $\sum(|g_n|/(nc_n)) = \infty$ implies that the right-hand side in the last display above goes to infinity as $j \to \infty$.

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