Quantum aspects of doubly deformed CFTs

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Abstract

We study quantum aspects of the recently constructed doubly λ-deformed σ-models representing the effective action of two WZW models interacting via current bilinears. We show that although the exact beta-functions and current anomalous dimensions are identical to those of the λ-deformed models, this is not true for the anomalous dimensions of generic primary field operators in accordance with the fact that the two models differ drastically. Our proofs involve CFT arguments, as well as effective σ-model action and gravity calculations.
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1. Introduction and conclusions

Recently, a new class of integrable σ-model theories based on current algebra theories for a general semisimple group G was constructed in [1]. This was achieved by utilizing a left–right asymmetric gauging of two independent WZW models both at the same positive integer

level k, combined with two independent principal chiral models (PCM). The resulting theories are characterized by k and by two generic matrices λ_1 and λ_2. This class of theories was shown to be integrable for certain mono- or multi-parameter choices for λ_1 and λ_2 [1]. A slight modification of the construction provides integrability for deformations corresponding to coset G/H exact CFTs for cases in which G/H is a symmetric space. The above construction, although different, is reminiscent of that for single λ-deformations [2].

The aim of this work is to study the quantum properties of the aforementioned models. More precisely, we will derive the all-loop, exact in λ_1 and λ_2, but leading in the large k-expansion, β-functions of the theory, as well as the exact anomalous dimensions of the currents and of the bilinear operators that deform the CFT. This will be achieved by using different independent methods which have their own advantages separately and in addition they complement each other conceptually. In the first one we employ CFT techniques to show that the correlation functions involving exclusively currents can be mapped to correlation functions of the single λ-deformed models calculated in [3,4]. We demonstrate that this can be done only for current correlators and not for generic correlation functions that involve affine primary operators. In the second method, we employ the all-loop effective action of these doubly-deformed CFTs obtained in [1] (provided in (1.1) below) by considering the Lagrangian of the quantum fluctuations around a classical constant background solution along the lines of [5] and for the case of isotropic coupling matrices, i.e. proportional to the identity. This method will provide the β-functions for the two cases of principal interest, the isotropic double deformations corresponding to a general semisimple group G and the double deformations corresponding to the symmetric coset space G/H exact CFTs. In fact, these are actually the two cases for which this method is applicable. The results obtained are completely consistent with those of the first method. The gravitational background corresponding to the doubly λ-deformed is quite complicated. However, according to the findings of the first two methods we may compute the known β-functions for the usual λ-deformed models by setting one of the matrices to zero identically. Indeed, we set $\lambda_2 = 0$ and use the β-function of the resulting gravity background which in fact is quite simple. Then we determine the running of the remaining coupling λ_1 under the renormalization group (RG), in the general case where λ_1 is an arbitrary matrix. For isotropic deformations the result is in complete agreement with that obtained from the two previous methods.

We summarize the main results of the present paper: At the level of current operators the correlation functions of our model factorize and can be obtained from two copies each of which corresponds to a usual λ-deformed model, one with coupling matrix λ_1 and the other with coupling λ_2. In particular, this implies the remarkable fact that the running of each of the couplings λ_1 and λ_2 is the same as in the case of the single λ-deformations computed in [6] and [7] (see (2.5) and (4.20), respectively). This is despite the fact that the doubly λ-deformed model is fundamentally different from the sum of two copies of single λ-deformations. Furthermore, the anomalous dimensions of the currents and composite operators of the doubly deformed model (1.1) are given by (2.6) and (2.7), respectively, in the large k-limit as computed in [3]. Finally we calculate correlation functions involving affine primary operators by using CFT methods similar to the ones used in [8,9]. In this case the correlation functions will depend non-trivially on both λ_1 and λ_2 since they have non-vanishing transformations with the left, as well as with the right currents.

Before proceeding, let us briefly review the models under consideration. The action defining them depends on two group elements $g_i \in G$, $i = 1, 2$ and is given by the deformation of the sum of two WZW models $S_k(g_1)$ and $S_k(g_2)$ [1]

$$S_{k,\lambda_1,\lambda_2}(g_1, g_2) = S_k(g_1) + S_k(g_2)$$
$$+ \frac{k}{\pi} \int \mathrm{d}^2\sigma \left(J_{1+} \quad J_{2+} \right) \begin{pmatrix} \Lambda_{21}\lambda_1 D_2^T \lambda_2 & \Lambda_{21}\lambda_1 \\ \Lambda_{12}\lambda_2 & \Lambda_{12}\lambda_2 D_1^T \lambda_1 \end{pmatrix} \begin{pmatrix} J_{1-} \\ J_{2-} \end{pmatrix},$$
$$(1.1)$$

where the WZW action $S_k(g)$ for a group element $g \in G$ is given by

$$S_k(g) = \frac{k}{2\pi} \int \mathrm{d}^2\sigma \, \mathrm{Tr}(\partial_+ g^{-1} \partial_- g) + \frac{k}{12\pi} \int \mathrm{Tr}(g^{-1}\mathrm{d}g)^3 \qquad (1.2)$$

and

$$\Lambda_{12} = (\mathbb{I} - \lambda_2 D_1^T \lambda_1 D_2^T)^{-1}, \qquad \Lambda_{21} = (\mathbb{I} - \lambda_1 D_2^T \lambda_2 D_1^T)^{-1}. \qquad (1.3)$$

The matrices D_{ab} and the currents $J_\pm^a$ are given by

$$J_+^a = -i\,\mathrm{Tr}(t^a \partial_+ g g^{-1}), \qquad J_-^a = -i\,\mathrm{Tr}(t^a g^{-1}\partial_- g), \qquad D_{ab} = \mathrm{Tr}(t_a g t_b g^{-1}), \qquad (1.4)$$

where t^a's are Hermitean matrices with $[t_a, t_b] = if_{abc}t_c$, here the structure constants f_{abc} are taken to be real.[1] When a current or the matrix D has an index 1 or 2 this means that one should use the corresponding group element in its definition.

The above action has the following remarkable duality-type symmetry [1][2]

$$k \mapsto -k, \quad \lambda_1 \mapsto \lambda_1^{-1}, \quad \lambda_2 \mapsto \lambda_2^{-1}, \quad g_1 \mapsto g_2^{-1}, \quad g_2 \mapsto g_1^{-1}. \qquad (1.5)$$

A similar non-perturbative duality is also present in the case of the λ-deformed action of [2] as discovered in [6,7] and predicted before using path integral arguments in [11].

For small values of the entries of the matrices λ_i's the action (1.1) becomes

$$S_{k,\lambda_1,\lambda_2}(g_1, g_2) = S_k(g_1) + S_k(g_2) + \frac{k}{\pi} \int \mathrm{d}^2\sigma \left((\lambda_1)_{ab} J_{1+}^a J_{2-}^b + (\lambda_2)_{ab} J_{2+}^a J_{1-}^b \right) + \cdots, \qquad (1.6)$$

thus representing a current-current deformation of the original WZW actions. Notice, however, that unlike the λ-deformed action [2] the currents building the bilinear interactions belong to different WZW models. The action (1.1) is said to be the effective all-loop in λ_1 and λ_2 action corresponding to (1.6) defined as a model on its own.

2. Exact β-functions, current & primary fields correlators

In this section, we will calculate the exact expressions for the running of the couplings λ_1 and λ_2, as well as for the anomalous dimensions of the currents $J_{1\pm}^a$, $J_{2\pm}^a$ and current bilinears $J_{1+}^a J_{2-}^b$ and $J_{2+}^a J_{1-}^b$. We will also derive the exact scaling dimensions of the affine primary operators of the model under consideration.

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[1] The structure constants f_{abc} are taken be real for gravity computations and imaginary for CFT ones, appearing in Secs. 1, 3, 4 and 2, respectively.

[2] The action is also invariant under the generalized parity transformation:

$$\sigma^\pm \mapsto \sigma^\mp, \quad g_1 \mapsto g_1^{-1}, \quad g_2 \mapsto g_2^{-1}, \quad \lambda_1 \mapsto \lambda_2^T, \quad \lambda_2 \mapsto \lambda_1^T,$$

similar to the λ-deformed action of [2] found in [10].

2.1. Exact β-functions & current correlators

Consider all the correlation functions involving current operators or composite current operators and split them into two sets. We will work in the Euclidean regime and denote the Euclidean versions of J^a_{i+} and J^a_{i-}, by J^a_i and $\bar{J}^a_i$, respectively. In the first set we assemble J^a_1, $\bar{J}^a_2$ and all the composite operators built from these two and we do the same for J^a_2, $\bar{J}^a_1$ and all of their composite operator, that is

$$\mathcal{O} = \{J^a_1, \ \bar{J}^a_2, \ J^a_1 \bar{J}^b_2, \ \cdots\}, \qquad \tilde{\mathcal{O}} = \{J^a_2, \ \bar{J}^a_1, \ J^a_2 \bar{J}^b_1, \ \cdots\}. \tag{2.1}$$

Our aim is to evaluate correlation functions involving an arbitrary number of $\mathcal{O}$ and $\tilde{\mathcal{O}}$, namely

$$\langle \prod_{i=1}^n \mathcal{O}_i(z_i) \prod_{j=1}^m \tilde{\mathcal{O}}_j(z_j) \rangle = \frac{1}{\mathcal{Z}} \langle \prod_{i=1}^n \mathcal{O}_i(z_i) \prod_{j=1}^m \tilde{\mathcal{O}}_j(z_j) e^{-\frac{1}{\pi} \int d^2 z \, ((\lambda_1)_{ab} J^a_1 \bar{J}^b_2 + (\lambda_2)_{ab} J^a_2 \bar{J}^b_1)} \rangle_0 ,$$

$$\tag{2.2}$$

where the interaction is the leading term in the small λ expansion given in (1.6). The symbol $\langle \dots \rangle_0$ in the right hand side of (2.2) denotes the average performed over the currents with the CFT action being $S_k(g_1) + S_k(g_2)$. The crucial observation is that the particular form of the interaction vertices in (1.6) leads to a factorization of the correlation function (2.2) since the operators $\mathcal{O}$ can be contracted only with the currents coming from the expansion of $e^{-\frac{1}{\pi} \int d^2 z \, (\lambda_1)_{ab} J^a_1 \bar{J}^b_2}$ and similarly the operators $\tilde{\mathcal{O}}$ can be contracted only with the currents coming from the expansion of the exponential $e^{-\frac{1}{\pi} \int d^2 z \, (\lambda_2)_{ab} J^a_2 \bar{J}^b_1}$. In conclusion (2.2) can be written as follows

$$\langle \prod_{i=1}^n \mathcal{O}_i(z_i) \prod_{j=1}^m \tilde{\mathcal{O}}_j(z_j) \rangle = \langle \prod_{i=1}^n \mathcal{O}_i(z_i) \rangle \cdot \langle \prod_{j=1}^m \tilde{\mathcal{O}}_j(z_j) \rangle , \tag{2.3}$$

where

$$\langle \prod_{i=1}^n \mathcal{O}_i(z_i) \rangle = \frac{1}{\mathcal{Z}_1} \langle \prod_{i=1}^n \mathcal{O}_i(z_i) e^{-\frac{1}{\pi} \int d^2 z \, (\lambda_1)_{ab} J^a_1 \bar{J}^b_2} \rangle_0 ,$$

$$\langle \prod_{j=1}^m \tilde{\mathcal{O}}_j(z_j) \rangle = \frac{1}{\mathcal{Z}_2} \langle \prod_{j=1}^m \tilde{\mathcal{O}}_j(z_j) e^{-\frac{1}{\pi} \int d^2 z \, (\lambda_2)_{ab} J^a_2 \bar{J}^b_1} \rangle_0 . \tag{2.4}$$

Thus we see that at the level of current operators the correlation functions of our model can be obtained from two copies each of which corresponds to a usual λ-deformed model, one with coupling $(\lambda_1)_{ab}$ and the other with coupling $(\lambda_2)_{ab}$. The above factorization of correlators is only true when restricted to correlation functions involving exclusively currents. When one or more affine primary operators are involved in the correlation function then such a factorization will no longer be true (except for primaries transforming trivially under the left or the right current algebras). This will be shown in the next subsection. This is in accordance with the fact that the σ-model action (1.1) entangles the group elements g_1 and g_2 non-trivially in such a way that its action can not be written as a sum of two λ-deformed models, with coupling matrices λ_1 and λ_2, respectively.

The aforementioned factorization of (2.3) implies that the β-functions for the couplings λ_1 and λ_2 are the same as in the single λ-deformed theory since the correlation functions from which they are derived involve only currents and as such they take the form of two copies of

λ-deformed models. The above assertion is valid for all values of the deformation matrices λ_1 and λ_2 and to all order in the level k. In particular, for the case of isotropic couplings these read [7,12,13]

$$\beta_i := \frac{\mathrm{d}\lambda_i}{\mathrm{d}t} = -\frac{c_G \lambda_i^2}{2k(1+\lambda_i)^2}, \quad t := \ln \mu^2, \quad i = 1, 2 \,, \tag{2.5}$$

to leading order in the $1/k$-expansion and exactly in the deformation parameters. c_G is the second Casimir of the adjoint representation defined by the structure constants of the group through $f_{acd} f_{bcd} = -c_G \delta_{ab}$, here the structure constants are imaginary (see footnote 1). Similarly, the pairs of currents $(J_1^a, \bar{J}_2^a)$ and $(J_2^a, \bar{J}_1^a)$ acquire anomalous dimensions that depend only on λ_1 and λ_2, respectively. For the isotropic case their exact in λ_1 and λ_2 large k expressions are given by [3,4]

$$\gamma^{(J_{1+})} = \gamma^{(J_{2-})} = \frac{c_G \lambda_1^2}{k(1-\lambda_1)(1+\lambda_1)^3} \,, \quad \gamma^{(J_{2+})} = \gamma^{(J_{1-})} = \frac{c_G \lambda_2^2}{k(1-\lambda_2)(1+\lambda_2)^3} \,, \tag{2.6}$$

which are both positive. Furthermore, for the composite operators deforming the sum of the two CFTs we have from [3] that

$$\gamma^{(J_{1+}^a J_{2-}^a)} = -\frac{2c_G}{k} \frac{\lambda_1(1-\lambda_1(1-\lambda_1))}{(1-\lambda_1)(1+\lambda_1)^3} \,, \quad \gamma^{(J_{2+}^a J_{1-}^a)} = -\frac{2c_G}{k} \frac{\lambda_2(1-\lambda_2(1-\lambda_2))}{(1-\lambda_2)(1+\lambda_2)^3} \,, \tag{2.7}$$

which are both negative.

The above considerations can be easily extended to the left–right asymmetric cases where the levels of the four currents algebras involved are different. This is done using the corresponding results for the left–right asymmetric λ-deformations in [8].

2.2. Exact dimensions of primary operators

In this subsection, we calculate the anomalous dimensions of affine primary operators along the lines of [4,8]. We will verify the expectation that in this case the dimensions, as well as all correlators will depend on both couplings λ_1 and λ_2 and will not just be what was found for the single λ-deformations in [4,8]. The reason is that generic primary fields are sensitive under transformations from both the left and the right current algebras. Such results might help in finding out the fate of these theories under the RG flow towards the IR, especially in the case of unequal levels.

The CFT we are studying contains two kinds of affine primary fields belonging to the original WZW models and transforming under the corresponding left and right current algebras. From this point we focus on one of them, i.e. $\Phi_{i,i'}(z, \bar{z})$ which under the action of the currents J_1^a and $\bar{J}_1^a$ transforms in the irreducible representations R and R', with the corresponding matrices being t_a and $\tilde{t}_a$. Hence, we have for the indices labeling them that $i = 1, 2, \ldots, \dim R$ and $i' = 1, 2, \ldots, \dim R'$. In addition these primaries are inert under the action of the currents belonging to the second WZW model J_2^a and $\bar{J}_2^a$. The relevant OPE equations are

$$J_1^a(z)\Phi_{i,i'}(w,\bar{w}) = -\frac{1}{\sqrt{k}}\frac{(t_a)_i{}^j\,\Phi_{j,i'}(w,\bar{w})}{z-w}\,,$$

$$\bar{J}_1^a(\bar{z})\Phi_{i,i'}(w,\bar{w}) = \frac{1}{\sqrt{k}}\frac{(\tilde{t}_a)^{j'}{}_{i'}\,\Phi_{i,j'}(w,\bar{w})}{\bar{z}-\bar{w}}\,, \tag{2.8}$$

$$J_2^a(z)\Phi_{i,i'}(w,\bar{w}) = \text{regular}\,, \qquad \bar{J}_2^a(\bar{z})\Phi_{i,i'}(w,\bar{w}) = \text{regular}\,.$$

Our conventions for the transformation matrices are $[t_a,t_b]=f_{abc}t_c$ and $[\tilde{t}_a,\tilde{t}_b]=f_{abc}\tilde{t}_c$, i.e. here f_{abc} are taken to be imaginary. At the conformal point, these affine primary fields are also Virasoro primaries with holomorphic and antiholomorphic dimensions given by [14]

$$\Delta_R = \frac{c_R}{2k+c_G}\,, \qquad \bar{\Delta}_{R'} = \frac{c_{R'}}{2k+c_G}\,, \tag{2.9}$$

where c_R, $c_{R'}$ and c_G are the quadratic Casimir operators in the representations R, R' and in the adjoint representation. For the latter $(t_a)_{bc}=-f_{abc}$. They are defined as

$$(t_at_a)_i{}^j = c_R\delta_i{}^j\,, \qquad (\tilde{t}_a\tilde{t}_a)_{i'}{}^{j'} = c_{R'}\delta_{i'}{}^{j'}\,, \qquad f_{acd}f_{bcd} = -c_G\delta_{ab}\,. \tag{2.10}$$

Finally, the current OPEs are given in our conventions by

$$J_1^a(z)J_1^a(z) = \frac{\delta_{ab}}{(z-w)^2} + \frac{f_{abc}}{\sqrt{k}}\frac{J_1^c(w)}{(z-w)}\,, \tag{2.11}$$

and similarly for the others.

Next we proceed with the calculation of the exact dimensions of the primary fields. A typical term in the perturbative expansion of the two-point function of primary fields will schematically have the form $\lambda_1^n\lambda_2^m\langle\Phi_{i,i'}^{(1)}(x_1)\,(J_1^a\bar{J}_2^a)^n\,(J_2^b\bar{J}_1^b)^m\,\Phi_{j,j'}^{(2)}(x_2)\rangle$, where $\Phi^{(1)}$ denotes an affine primary operator while $\Phi^{(2)}$ denotes its complex conjugate. The field $\Phi^{(1)}$ transforms as in (2.8) whereas $\Phi^{(2)}$ transforms similarly but with t_a and $\tilde{t}_a$ replaced by $-t_a^*$ and $-\tilde{t}_a^*$, respectively. We will first argue that to order $1/k$ in the perturbative expansion that we are interested in, there will be no terms with both $n\neq 0$ and $m\neq 0$, that is, mixed terms of the two couplings λ_1 and λ_2 never appear.

Consider the case when n is an odd number. We firstly choose to apply the Ward identity for one of the $\bar{J}_2^a$ currents. This current can not be contracted with one of the external fields but only with another $\bar{J}_2^a$. Once such contraction gives another $\bar{J}_2^a$ current via the non-Abelian part delivering another factor of $1/\sqrt{k}$. The remaining $\bar{J}_2^a$ currents, even in number in total, should then be contracted among themselves only through the Abelian term of their OPE since in the opposite case the resulting expression will be of order higher or equal to $1/k^{3/2}$ in the large k-expansion and such terms are subleading. The next current for which we choose to apply the Ward identity is one of the J_1^a. This can be contracted either with another current of the same species through the non-Abelian term of the OPE or with one of the external fields. In both cases, we have already saturated the $1/k$ factor of the two-point function and as a result the currents associated with the second interaction term $(J_2^b\bar{J}_1^b)^m$ should be contracted only among themselves making the corresponding diagram disconnected.

Now we turn to the case when n is an even number and as before we apply first the Ward identity for one of the currents $\bar{J}_2^a$ currents through the Abelian or the non-Abelian part of their OPE. When this contraction is Abelian, at least one of the J_1^a currents should be contracted with one of the external fields, otherwise all currents associated with the first interaction vertex $(J_1^a\bar{J}_2^a)^n$ will have been contracted among themselves giving a disconnected diagram. Thus, the contraction of

the J_1^a current with one of the external fields will leave us with an odd number of J_1^a currents which means that another J_1^a current should be contracted with one of the external fields, hence saturating the factor $1/k$ to which we perform computations. Hence, the currents associated with the second interaction term $(J_2^b \bar{J}_1^b)^m$ have to be contracted only among themselves resulting to a disconnected diagram. In the second case, when two of the $\bar{J}_2^a$ currents are contracted through the non-Abelian part of their OPE, we are left with an odd number of $\bar{J}_2^a$ currents which means that another non-Abelian contraction among two of the remaining $\bar{J}_2^a$ currents is necessary, hence saturating the $1/k$ overall factor. Then as before, the currents associated with the second interaction vertices $(J_2^b \bar{J}_1^b)^m$ should only be contracted among themselves giving a disconnected diagram.

In conclusion, we have shown that the perturbative expansion can never produce terms containing mixed factors of λ_1 and λ_2. This implies that the anomalous dimension of the affine primary fields will take the generic form

$$\gamma_{R,R'} = \frac{f_1(\lambda_1)c_R + f_2(\lambda_1)c_{R'}}{k(1-\lambda_1)(1+\lambda_1)^3} + \frac{h_1(\lambda_2)c_R + h_2(\lambda_2)c_{R'}}{k(1-\lambda_2)(1+\lambda_2)^3} \,, \tag{2.12}$$

where the pole structure of the dimensions in the above equation is dictated by the fact that each of the β-functions of the model, β_{λ_1} and β_{λ_2}, is given by the same expression as in the single λ-deformed σ-model. Then one can use the Callan–Symanzik equation in a similar manner to that in [3] in order to pin down the form of the anomalous dimensions. The unknown polynomials f_1, f_2, h_1 and h_2 appearing in (2.12) can be determined by exploiting the symmetry of the action (1.5), as well as the results from low order perturbation theory presented in Appendix A. From (A.18) we have that

$$\gamma_{R,R'} = \frac{c_R}{k}\left(1 + \lambda_1^2(1+\lambda_1^2)\right) + \frac{c_{R'}}{k}\lambda_2^2(1+\lambda_2^2) + \mathcal{O}(\lambda^5/k) \,. \tag{2.13}$$

The symmetry of the model (1.5) when combined with the obvious $\mathbb{Z}_2$ symmetry exchanging (g_1, λ_1) with (g_2, λ_2) gives the following constraint for the anomalous dimensions

$$\gamma_{R',R}(k, \lambda_1, \lambda_2) = \gamma_{R,R'}(-k, \lambda_2^{-1}, \lambda_1^{-1}) \,. \tag{2.14}$$

The exchange of the representations R and R' is related to the fact that under the combined symmetry mentioned above, the inversion of the group elements, i.e. $g_i \mapsto g_i^{-1}$, $i = 1, 2$, results into the interchange of the representations R and R', for details see [4]. Then (2.14) implies that the functions f_1, f_2 and h_1, h_2 obey the following relations

$$f_1(\lambda_1) = h_2(\lambda_1^{-1})\lambda_1^4 \,, \qquad f_2(\lambda_1) = h_1(\lambda_1^{-1})\lambda_1^4 \,. \tag{2.15}$$

Hence, the unknown polynomials are of order λ_1^4, at most. We fix them by in addition requiring agreement with the perturbative result (2.13). In fact, one needs the perturbative result only up to order λ_1^2 and λ_2^2. In this way, we get $f_2 = h_1 = 0$ and $f_1(\lambda) = (1+\lambda)^2$ and $h_2(\lambda) = \lambda^2(1+\lambda)^2$. Therefore the exact anomalous dimension is

$$\boxed{\gamma_{R,R'} = \frac{c_R}{k}\frac{1}{1-\lambda_1^2} + \frac{c_{R'}}{k}\frac{\lambda_2^2}{1-\lambda_2^2}} \,. \tag{2.16}$$

This reproduces correctly the perturbative result (2.13) up to order λ_1^4 and λ_2^4 and serves as a non-trivial check of the all-loop expression (2.16).[3] We see that, unlike the case with the current's anomalous dimensions, it depends on both deformation parameters λ_1 and λ_2. In addition, comparing with the anomalous dimensions of primary fields for the λ-deformed models [4] a main difference is the absence of a term proportional to the eigenvalues of the matrix $t_a \otimes t_a^*$. Such a matrix does not appear here. The reason lies on the fact that in the λ-deformed models the deformation is driven by current bilinears of the same WZW model, whereas here by current bilinears of different WZW models.

3. Isotropic couplings: RG flows from the effective action

In this section, we will employ the all-loop effective action of (1.1) in order to determine the β-functions for the double isotropic deformation of two WZW models for a group G as well as for two coset CFTs for which G/H is a symmetric space.

3.1. Group space

We first consider the case of two isotropic couplings for a group G so that $(\lambda_i)_{ab} = \lambda_i \, \delta_{ab}$, $i = 1, 2$. To compute the β-functions we need to specify a classical background solution and compute the quantum fluctuations around it. Of course self-consistency requires that the result is background independent. The discussion goes along the lines of [5]. The equations of motion for our models can be cast in the form [1]

$$\partial_\pm I_\mp^i = \mp \frac{1}{2} [I_+^i, I_-^i], \qquad i = 1, 2, \tag{3.1}$$

with

$$I_\pm^i = -\frac{2}{1 + \lambda_i} A_\pm^i, \qquad i = 1, 2. \tag{3.2}$$

Consider group elements of the form[4]

$$g_i = e^{\sigma^\alpha \Theta_\alpha^i}, \qquad i = 1, 2, \tag{3.3}$$

where the Θ_α^i's are arbitrary constant commuting elements of the Lie algebra $\mathfrak{g}$ of G. The gauge fields evaluated at the above classical configuration follow by inserting the classical solutions in the expressions for the gauge fields obtained in eqs. (2.9) and (2.10) of [1] (with $A_\pm \mapsto A_\pm^1$ and $B_\pm \mapsto A_\pm^2$ to conform with the notation in the present paper)

$$A_+^1 = \frac{\lambda_1}{1 - \lambda_1 \lambda_2} \left(\Theta_+^1 + \lambda_2 \, \Theta_+^2 \right), \qquad A_-^1 = -\frac{\lambda_1}{1 - \lambda_1 \lambda_2} \left(\Theta_-^2 + \lambda_2 \, \Theta_-^1 \right),$$
$$A_+^2 = \frac{\lambda_2}{1 - \lambda_1 \lambda_2} \left(\Theta_+^2 + \lambda_1 \, \Theta_+^1 \right), \qquad A_-^2 = -\frac{\lambda_2}{1 - \lambda_1 \lambda_2} \left(\Theta_-^1 + \lambda_1 \, \Theta_-^2 \right). \tag{3.4}$$

Therefore the $I_\pm^i$, for $i = 1, 2$ become constant commuting matrices which we denote by $I_{0,\pm}^i$ so that the equations of motion are indeed satisfied. The Lagrangian evaluated on the background fields reads

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$$\mathcal{L}^{(0)} = -\frac{k}{2\pi(1 - \lambda_1\lambda_2)} \begin{pmatrix} \Theta^1_+ & \Theta^2_+ \end{pmatrix} \begin{pmatrix} 1 + \lambda_1\lambda_2 & 2\lambda_1 \\ 2\lambda_2 & 1 + \lambda_1\lambda_2 \end{pmatrix} \begin{pmatrix} \Theta^1_- \\ \Theta^2_- \end{pmatrix} . \tag{3.5}$$

To compute the one-loop β-function we expand (3.1) around the classical solution (3.3) and we derive the operator acting on the fluctuations. We let $I^i_\pm = I^i_{0,\pm} + \delta I^i_\pm$ and we linearize the equations of motion (3.1). After some rearrangements we get that

$$\mathcal{D}^i \begin{pmatrix} \delta I^i_+ \\ \delta I^i_- \end{pmatrix} = 0 , \qquad i = 1, 2 , \tag{3.6}$$

where the matrix differential operators acting on the fluctuations are given by

$$\mathcal{D}^i = \begin{pmatrix} \partial_- - \frac{1}{2}\tilde{I}^i_- & \frac{1}{2}\tilde{I}^i_+ \\ \frac{1}{2}\tilde{I}^i_- & \partial_+ - \frac{1}{2}\tilde{I}^i_+ \end{pmatrix} , \quad i = 1, 2 , \tag{3.7}$$

with

$$\left(\tilde{I}^i_\pm\right)_{ab} = i f_{abc} I^i_{0,\pm c} , \quad i = 1, 2 . \tag{3.8}$$

The one-loop effective Lagrangian in momentum space, after Wick rotating to Euclidean space and integrating out the fluctuations appearing in a Gaussian path integral, reads[5]

$$-\mathcal{L}^{\text{eff}}_{\text{E}} = \mathcal{L}^{(0)} + \int^\mu \frac{d^2 p}{(2\pi)^2} \ln \det \begin{pmatrix} \widehat{\mathcal{D}}^1 & \mathbf{0} \\ \mathbf{0} & \widehat{\mathcal{D}}^2 \end{pmatrix}^{-1/2} , \tag{3.9}$$

where

$$\widehat{\mathcal{D}}^i = \begin{pmatrix} p_- - \frac{1}{2}\tilde{I}^i_- & \frac{1}{2}\tilde{I}^i_+ \\ \frac{1}{2}\tilde{I}^i_- & p_+ - \frac{1}{2}\tilde{I}^i_+ \end{pmatrix} , \quad i = 1, 2 , \quad p_\pm = \frac{1}{2}(p_1 \pm i\, p_2) . \tag{3.10}$$

Next we evaluate the determinant in (3.9)

$$\ln \det \begin{pmatrix} \widehat{\mathcal{D}}^1 & \mathbf{0} \\ \mathbf{0} & \widehat{\mathcal{D}}^2 \end{pmatrix} = \ln \det \widehat{\mathcal{D}}^1 + \ln \det \widehat{\mathcal{D}}^2 , \tag{3.11}$$

where

$$\ln \det \widehat{\mathcal{D}}^i = \ln \det C + \text{Tr} \ln \left(\mathbb{I}_2 + C^{-1} E_i \right) ,$$
$$C = \begin{pmatrix} p_- & 0 \\ 0 & p_+ \end{pmatrix} , \quad E_i = \frac{1}{2} \begin{pmatrix} -\tilde{I}^i_- & \tilde{I}^i_+ \\ \tilde{I}^i_- & -\tilde{I}^i_+ \end{pmatrix} , \quad i = 1, 2 . \tag{3.12}$$

To proceed we expand the field dependent term

$$\text{Tr} \ln \left(\mathbb{I}_2 + C^{-1} E_i \right) = -\frac{\text{Tr}\left(\tilde{I}^i_+ p_- + \tilde{I}^i_- p_+\right)^2}{2p^4} + \mathcal{O}\left(\frac{p^4_\pm}{p^8}\right) , \quad i = 1, 2 . \tag{3.13}$$

The logarithmically divergent term in (3.9) comes only from the explicitly depicted term above. After performing the momentum integration we get that

---

[5] The analytic continuation $(\tau, \sigma) \mapsto (i\sigma_1, \sigma_2)$ and so $(p_0, p_1) \mapsto (ip_1, p_2)$.

$$
\begin{aligned}
\mathcal{L}_{\mathrm{E}}^{\mathrm{eff}} &= -\mathcal{L}^{(0)} - \frac{1}{4\pi}\mathrm{Tr}\left(\tilde{I}_+^1\tilde{I}_-^1 + \tilde{I}_+^2\tilde{I}_-^2\right)\ln\mu \\
&= -\mathcal{L}^{(0)} - \frac{c_G}{4\pi}\left(I_{0,+a}^1 I_{0,-a}^1 + I_{0,+a}^2 I_{0,-a}^2\right)\ln\mu\,,
\end{aligned}
\tag{3.14}
$$

where we have used (3.8) to obtain the second line above. The one-loop β-function is derived by demanding that the effective action (3.14), after inserting (3.2), (3.4) and (3.5) is independent of the cutoff scale μ. After some algebraic manipulations we obtain the same result as that from the field theory calculation (2.5). This agreement is non-trivial evidence that (1.1) is indeed the all-loop effective action of the linearized model described by (1.6).

3.2. Symmetric space

Consider now the case of a deformation of two coset CFTs corresponding to symmetric spaces. For convenience we spit the group index into a part belonging to the subgroup H and the rest corresponding to the coset. We denote by Latin letters the subgroup indices and by Greek letters those of the coset. Using this notation the coupling matrices have non-vanishing elements [1]

$$
(\lambda_i)_{ab} = \delta_{ab}\,, \quad (\lambda_i)_{\alpha\beta} = \lambda_i\,\delta_{\alpha\beta}\,, \quad i = 1,2\,.
\tag{3.15}
$$

We also split the fields in the subgroup and coset components as

$$
I_\pm^i = I_\pm^{h|i} + I_\pm^{g/h|i}\,, \qquad i = 1,2\,.
\tag{3.16}
$$

The equations of motion are of the form [1]

$$
\begin{aligned}
&\partial_+ I_-^{h|i} - \partial_- I_+^{h|i} + [I_+^{h|i}, I_-^{h|i}] + [I_+^{g/h|i}, I_-^{g/h|i}] = 0\,, \\
&\partial_\pm I_\mp^{g/h|i} = [I_\mp^{g/h|i}, I_\pm^{h|i}]\,, \qquad i = 1,2\,,
\end{aligned}
\tag{3.17}
$$

with

$$
I_\pm^{h|i} = -A_\pm^{h|i}\,, \quad I_\pm^{g/h|i} = -\frac{A_\pm^{g/h|i}}{\sqrt{\lambda_i}}\,, \quad i = 1,2
\tag{3.18}
$$

and we have used the fact that $f_{\alpha\beta\gamma} = 0$ for symmetric spaces. We fix the residual gauge by enforcing the covariant gauge fixing condition

$$
\partial_+ I_-^{h|i} + \partial_- I_+^{h|i} = 0\,, \qquad i = 1,2\,.
\tag{3.19}
$$

We will comment on other gauge choices at the end of this section. As in the group case, to derive the β-function we need to identify the proper background classical solution. We make the same choice as in (3.3) but now Θ_α^i, $i = 1,2$ are arbitrary constant commuting elements of $\mathfrak{g}/\mathfrak{h}$. The Lagrangian $\mathcal{L}^{(0)}$ evaluated on the background fields has the same form (3.5). In addition the gauge fields in the coset $A_\pm^{g/h|1}$ and $A_\pm^{g/h|2}$ take the form of (3.4) whereas those in the subgroup $A_\pm^{h|1} = A_\pm^{h|2} = 0$.

Varying the equations of motion (3.17) and the gauge fixing condition (3.19) yields

$$
\mathcal{D}^i \begin{pmatrix} \delta I_+^{g/h|i} \\ \delta I_-^{g/h|i} \\ \delta I_+^{h|i} \\ \delta I_-^{h|i} \end{pmatrix} = 0\,, \qquad i = 1,2\,,
\tag{3.20}
$$

where

$$
\mathcal{D}^i = \begin{pmatrix} \partial_- & 0 & 0 & \widetilde{I}_+^{g/h|i} \\ 0 & \partial_+ & \widetilde{I}_-^{g/h|i} & 0 \\ \widetilde{I}_-^{g/h|i} & -\widetilde{I}_+^{g/h|i} & -\partial_- & \partial_+ \\ 0 & 0 & \partial_- & \partial_+ \end{pmatrix} , \qquad i = 1, 2 , \tag{3.21}
$$

with

$$
\left(\widetilde{I}_\pm^{g/h|i} \right)_{\alpha b} = i f_{\alpha b \gamma} \, I_{\pm\gamma}^{g/h|i} , \qquad i = 1, 2 . \tag{3.22}
$$

The one-loop effective Lagrangian in momentum space, after Wick rotating to Euclidean space takes the form (3.9) where now

$$
\widehat{\mathcal{D}}^i = \begin{pmatrix} p_- & 0 & 0 & \widetilde{I}_+^{g/h|i} \\ 0 & p_+ & \widetilde{I}_-^{g/h|i} & 0 \\ \widetilde{I}_-^{g/h|i} & -\widetilde{I}_+^{g/h|i} & -p_- & p_+ \\ 0 & 0 & p_- & p_+ \end{pmatrix} , \qquad i = 1, 2 . \tag{3.23}
$$

Working along the lines with the group case we get

$$
\begin{aligned}
\mathcal{L}_{\mathrm{E}}^{\mathrm{eff}} &= -\mathcal{L}^{(0)} - \frac{1}{\pi} \mathrm{Tr} \left(\widetilde{I}_+^{g/h|1} \widetilde{I}_-^{g/h|1} + \widetilde{I}_+^{g/h|2} \widetilde{I}_-^{g/h|2} \right) \ln \mu , \\
&= -\mathcal{L}^{(0)} - \frac{c_G}{\pi} \left(I_{+\alpha}^{g/h|1} I_{-\alpha}^{g/h|1} + I_{+\alpha}^{g/h|2} I_{-\alpha}^{g/h|2} \right) \ln \mu ,
\end{aligned} \tag{3.24}
$$

due to (3.22) and the fact that for symmetric spaces $f_{\alpha\beta\gamma} = 0$.

As before the one-loop β function can be derived by demanding that the effective action (3.24) is independent of the cutoff scale μ. The result is

$$
\boxed{ \beta_i = -\frac{c_G \lambda_i}{2k} \quad i = 1, 2 } , \tag{3.25}
$$

for arbitrary constant Θ_α^i's. This result is identical to that obtained for the λ-deformed $SU(2)/U(1)$ coset using gravity in [7] and generalized for λ-deformations for arbitrary G/H symmetric coset CFTs in [5]. We end this section by noting that a different gauge choice than (3.19) (necessarily involving only $A_\pm^{h|i}$) would have resulted in different (43) and (44) elements in (3.21) and (3.23). It turns out that this does not affect the logarithmic behavior in (3.24). Hence (3.25) is unchanged as it should be.

4. A simple action and the β-function from gravity

In this section, we consider the special case where $\lambda_2 = 0$. The other matrix λ_1, renamed as λ, will be kept general. We will use the expressions for the running of the σ-model couplings given in terms of the metric and antisymmetric tensor fields of [15–17] in order to determine the running of the couplings λ_{ab}. The resulting expression is in complete agreement with the CFT we have provided and will coincide with the general result for the λ-deformed backgrounds computed in [6]. However, in the case at hand the computation will be significantly easier since the corresponding action extremely simplifies and reads

$$S_{k,\lambda}(g_1, g_2) = S_k(g_1) + S_k(g_2) + \frac{k}{\pi} \int \mathrm{d}^2\sigma\, \lambda_{ab} J_{1+}^a J_{2-}^b \,, \tag{4.1}$$

which we emphasize is not an approximation for small entries λ_{ab}, but it is just obtained from (1.1) as described above.

4.1. Computation of the β-function

To proceed with the computation we first read off the line element

$$\mathrm{d}s^2 = R^a R^a + L^{\hat{a}} L^{\hat{a}} + 2\lambda_{ab} R^a L^{\hat{b}} \,, \tag{4.2}$$

where

$$R^a = -i\,\mathrm{Tr}(t^a \mathrm{d}g_1 g_1^{-1})\,, \qquad L^{\hat{a}} = -i\,\mathrm{Tr}(t^a g_2^{-1} \mathrm{d}g_2)\,,$$
$$\mathrm{d}L^a = \frac{1}{2} f_{abc} L^b \wedge L^c\,, \quad \mathrm{d}R^a = -\frac{1}{2} f_{abc} R^b \wedge R^c\,. \tag{4.3}$$

Hence, the unhatted and hatted indices denote the Maurer–Cartan forms of g_1 and g_2 respectively. By introducing the vielbeins

$$\mathrm{e}^a = R^a\,, \quad \mathrm{e}^{\hat{a}} = L^{\hat{a}} + \lambda_{ba} R^b\,, \tag{4.4}$$

as well as the double index notation $A = (a, \hat{a})$ the line element can be written as

$$\mathrm{d}s^2 = \tilde{g}_{ab}\mathrm{e}^a\mathrm{e}^b + \mathrm{e}^{\hat{a}}\mathrm{e}^{\hat{a}} = G_{AB}\,\mathrm{e}^A\mathrm{e}^B\,, \tag{4.5}$$

where $\tilde{g}_{ab} = (\mathbb{I} - \lambda\lambda^T)_{ab}$ and for later use we also define $g_{ab} = (\mathbb{I} - \lambda^T\lambda)_{ab}$. We will also need the two-form which is given by

$$B = B_0 + \lambda_{ab} R^a \wedge L^{\hat{b}}\,, \tag{4.6}$$

where B_0 is the two-form corresponding to the two WZW models with

$$H_0 = \mathrm{d}B_0 = -\frac{1}{6} f_{abc}\left(R^a \wedge R^b \wedge R^c + L^{\hat{a}} \wedge L^{\hat{b}} \wedge L^{\hat{c}}\right)\,. \tag{4.7}$$

Note that we have not included an overall factors of $\frac{k}{2\pi}$ in the definitions of $\mathrm{d}s^2$ and B. Also, the sign of H_0 is in accordance with that of the WZ term in (1.2).

The tangent metric G_{AB} is constant and so the spin connection ω_{AB} is antisymmetric. A practical way to compute it is by first defining the quantities $C^A{}_{BC} = -C^A{}_{CB}$ from

$$\mathrm{d}\mathrm{e}^A = \frac{1}{2} C^A{}_{BC}\,\mathrm{e}^B \wedge \mathrm{e}^C\,, \quad C_{ABC} = G_{AD} C^D{}_{BC}\,. \tag{4.8}$$

Then, simply

$$\omega_{AB} = \omega_{AB|C}\mathrm{e}^C = \frac{1}{2}\left(C_{ABC} - C_{CAB} + C_{BCA}\right)\mathrm{e}^C\,, \tag{4.9}$$

which also defines the useful, in explicit computations, quantity $\omega_{AB|C}$. Employing the above and (4.4) we find that

$$\omega_{ab} = -\frac{1}{2}\left(\tilde{g}_{ad} f_{dbc} - \tilde{g}_{cd} f_{dab} + \tilde{g}_{bd} f_{dca}\right)\mathrm{e}^c + \frac{1}{2}\left(\lambda_{dc} f_{dab} - \lambda_{ad}\lambda_{be} f_{cde}\right)\mathrm{e}^{\hat{c}}\,,$$
$$\omega_{\hat{a}b} = \frac{1}{2}\left(f_{ade}\lambda_{bd}\lambda_{ce} - \lambda_{da} f_{dbc}\right)\mathrm{e}^c\,, \tag{4.10}$$
$$\omega_{\hat{a}\hat{b}} = -f_{abd}\lambda_{cd}\,\mathrm{e}^c + \frac{1}{2} f_{abc}\,\mathrm{e}^{\hat{c}}\,.$$

Next we evaluate the field strength of the two-form

$$H = dB = \left(-\frac{1}{6} f_{abc} - \frac{1}{3} \lambda_{ad} \lambda_{be} \lambda_{cf} f_{def} + \frac{1}{2} f_{abd} \left(\lambda \lambda^T \right)_{cd} \right) e^a \wedge e^b \wedge e^c$$
$$+ \frac{1}{2} \left(f_{ade} \lambda_{bd} \lambda_{ce} - \lambda_{da} f_{dbc} \right) e^{\hat{a}} \wedge e^b \wedge e^c - \frac{1}{6} f_{abc} e^{\hat{a}} \wedge e^{\hat{b}} \wedge e^{\hat{c}} . \tag{4.11}$$

It will be convenient to use spin connections which include the torsion, defined as

$$\omega_{AB}^{\pm} = \omega_{AB} \pm \frac{1}{2} H_{ABC} e^C = \omega_{AB|C}^{\pm} e^C . \tag{4.12}$$

The torsion-full spin connections are found to be

$$\omega_{ab}^+ = \left(-f_{abc} - \lambda_{ad} \lambda_{be} \lambda_{cf} f_{def} + \left(\lambda \lambda^T \right)_{ad} f_{dbc} + \left(\lambda \lambda^T \right)_{bd} f_{adc} \right) e^c ,$$
$$\omega_{\hat{a}b}^+ = \left(f_{ade} \lambda_{bd} \lambda_{ce} - \lambda_{da} f_{dbc} \right) e^c , \tag{4.13}$$
$$\omega_{\hat{a}\hat{b}}^+ = -f_{abd} \lambda_{cd} e^c$$

and

$$\omega_{ab}^- = \left(\lambda_{ad} \lambda_{be} \lambda_{cf} f_{def} - \left(\lambda \lambda^T \right)_{cd} f_{abd} \right) e^c + \left(\lambda_{dc} f_{dab} - \lambda_{ad} \lambda_{be} f_{dec} \right) e^{\hat{c}} ,$$
$$\omega_{\hat{a}b}^- = 0 , \tag{4.14}$$
$$\omega_{\hat{a}\hat{b}}^- = -f_{abd} \lambda_{cd} e^c + f_{abc} e^{\hat{c}} .$$

Finally, we compute the torsion-full Ricci tensor by a useful rewriting

$$R_{AB}^{\pm} = \partial_C \omega^{\pm C}{}_{A|B} - \omega_{AC|D}^{\pm} \omega_B^{\mp D|C} - \nabla_B^{\pm} \omega^{\pm C}{}_{A|C} , \tag{4.15}$$

where $\partial_A = e_A{}^M \partial_M$. The last term corresponds to a diffeomorphism associated with

$$\omega^{\pm C}{}_{A|C} = \left(\partial_M e_N^C - \partial_N e_M^C \right) e_A^M e_C^N , \tag{4.16}$$

which shows that it is a vector in target space. The one-loop RG flow equations read

$$\frac{d}{dt} (G_{MN} + B_{MN}) = R_{MN}^- + \nabla_N^+ \xi_M , \tag{4.17}$$

where the second term corresponds to diffeomorphisms along ξ^M. The above may be rewritten in the tangent frame $e^A = e_M^A dX^M$ as

$$\frac{d}{dt} (G_{MN} + B_{MN}) = \left(R_{AB}^- + \nabla_B^- \xi_A \right) e_M^A e_N^B . \tag{4.18}$$

To proceed we evaluate the left hand side as

$$\frac{d}{dt} (G_{MN} + B_{MN}) = \frac{d\lambda_{ab}}{dt} R_M^a L_N^{\hat{b}} = \frac{d\lambda_{ab}}{dt} e_M^a (e_N^{\hat{b}} - \lambda_{cb} e_N^c) , \tag{4.19}$$

where we have used (4.4). In addition we use the freedom to perform diffeomorphisms in order to absorb in the expression for R_{AB}^- the term involving $\omega^{-C}{}_{A|C}$, by choosing $\xi_A = \omega^{-C}{}_{A|C}$. Then, employing the latter, as well as (4.13)–(4.15) and (4.18) we find that

$$\boxed{ \frac{d\lambda_{ab}}{dt} = \frac{1}{2k} \mathcal{N}(\lambda)_{ac}{}^d \mathcal{N}(\lambda^T)_{bd}{}^c } , \tag{4.20}$$

with

$$\mathcal{N}(\lambda)_{ab}{}^c = \left(\lambda_{ae}\lambda_{bd}f_{edf} - f_{abe}\lambda_{ef}\right)g^{fc},\tag{4.21}$$

where we have reinstalled the overall integer k, which, it does not flow. The expression (4.20) is identical to that found in [6]. For the isotropic coupling case it reduces to (2.5), identical to the one of [7].

4.2. Some properties of the action

According to our general discussion when we consider the action (4.1) the currents J_{2+}^a and J_{1-}^a acquire no anomalous dimension. That implies that the action (4.1) should have on-shell chiral and anti-chiral currents. The equations of motion corresponding to the variation of the two group elements g_1 and g_2 can be readily derived. They are most easily obtained from eqs. (3.6) and (3.7) of [1] after using eqs. (2.9) and (2.10) of the same work, setting $\lambda_2 = 0$ and renaming λ_1 by λ. Following this approach we obtain

$$\begin{aligned}\lambda\partial_+ J_{2-} + \partial_- J_{1+} &= i[J_{1+}, \lambda J_{2-}],\\ \partial_+ J_{2-} + \lambda^T \partial_- J_{1+} &= i[\lambda^T J_{1+}, J_{2-}]\end{aligned}\tag{4.22}$$

and

$$\begin{aligned}\partial_- \mathcal{J}_+ &= 0, & \mathcal{J}_+ &= J_{2+} + D_2\lambda^T J_{1+},\\ \partial_+ \mathcal{J}_- &= 0, & \mathcal{J}_- &= J_{1-} + D_1^T \lambda J_{2-}.\end{aligned}\tag{4.23}$$

The first (second) of (4.22) is equivalent to the second (first) of (4.23). To prove that we used the identities $(D^T\partial_- D)^{ab} = f^{ab}{}_c\mathcal{J}_-^c$ and $(\partial_+ DD^T)^{ab} = f^{ab}{}_c\mathcal{J}_+^c$. The above chiral and anti-chiral conserved currents $\mathcal{J}_\pm$ are deformations of J_{2+} and J_{1-}, to which they reduce for $\lambda = 0$. This is consistent with their vanishing anomalous dimensions. The equations (4.23) for the action (4.1) were derived before in [18].

Acknowledgements

Appendix A. Perturbative 2-point functions for primary fields

In this appendix we will compute the $\langle\Phi_{i,i'}^{(1)}\Phi_{j,j'}^{(2)}\rangle$ correlator up to four-loop $\mathcal{O}(\lambda^4)$ and to leading order in the large k expansion. We concentrate to the case where these fields transform non-trivially under the J_1^a and $\bar{J}_1^a$ as in (2.8).

One-loop $\mathcal{O}(\lambda)$: It is easily seen that the corresponding contribution is zero, i.e.

$$\langle\Phi_{i,i'}^{(1)}(x_1,\bar{x}_1)\Phi_{j,j'}^{(2)}(x_2,\bar{x}_2)\rangle_\lambda^{(1)} = 0.\tag{A.1}$$

Two-loop $\mathcal{O}(\lambda^2)$: To this order we find that

$$\langle\Phi_{i,i'}^{(1)}(x_1,\bar{x}_1)\Phi_{j,j'}^{(2)}(x_2,\bar{x}_2)\rangle_\lambda^{(2)} = \frac{1}{2!\pi^2}\int d^2 z_{12}\left(\lambda_1^2 A_2 + \lambda_2^2 B_2\right),\tag{A.2}$$

where

$$A_2 = \langle \Phi^{(1)}_{i,i'}(x_1, \bar{x}_1) J_1^{a_1}(z_1) \bar{J}_2^{a_1}(\bar{z}_1) J_1^{a_2}(z_2) \bar{J}_2^{a_2}(\bar{z}_2) \Phi^{(2)}_{j,j'}(x_2, \bar{x}_2) \rangle,$$

$$B_2 = \langle \Phi^{(1)}_{i,i'}(x_1, \bar{x}_1) J_2^{a_1}(z_1) \bar{J}_1^{a_1}(\bar{z}_1) J_2^{a_2}(z_2) \bar{J}_1^{a_2}(\bar{z}_2) \Phi^{(2)}_{j,j'}(x_2, \bar{x}_2) \rangle. \tag{A.3}$$

We note that the mixed terms proportional to $\lambda_1 \lambda_2$ do not contribute, as explained in the main text. After further contractions with $J_1^{a_1}$, followed by contractions of $\bar{J}_2^{a_1}$ with $\bar{J}_2^{a_2}$ we end up with

$$A_2 = \frac{c_R}{k} \frac{(\mathbb{I}_R \otimes \mathbb{I}_{R'})_{ii',jj'}}{x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}} \left(\frac{1}{\bar{z}_{12}^2 (z_1 - x_1)(z_2 - x_1)} - \frac{1}{\bar{z}_{12}^2 (z_1 - x_1)(z_2 - x_2)} \right.$$
$$\left. - \frac{1}{\bar{z}_{12}^2 (z_1 - x_2)(z_2 - x_1)} + \frac{1}{\bar{z}_{12}^2 (z_1 - x_2)(z_2 - x_2)} \right). \tag{A.4}$$

The expression for B_2 is found by performing a parity transformation in A_2 to be

$$B_2 = \frac{c_{R'}}{k} \frac{(\mathbb{I}_R \otimes \mathbb{I}_{R'})_{ii',jj'}}{x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}} \left(\frac{1}{z_{12}^2 (\bar{z}_1 - \bar{x}_1)(\bar{z}_2 - \bar{x}_1)} - \frac{1}{z_{12}^2 (\bar{z}_1 - \bar{x}_1)(\bar{z}_2 - \bar{x}_2)} \right.$$
$$\left. - \frac{1}{z_{12}^2 (\bar{z}_1 - \bar{x}_2)(\bar{z}_2 - \bar{x}_1)} + \frac{1}{z_{12}^2 (\bar{z}_1 - \bar{x}_2)(\bar{z}_2 - \bar{x}_2)} \right). \tag{A.5}$$

Then we perform the double integration in (A.2), choosing the order of integration from left to right. Using the symmetry under $x_1 \leftrightarrow x_2$ we compute the final result for the two-point function of primary fields up to order $\mathcal{O}(\lambda^2/k)$

$$\langle \Phi^{(1)}_{i,i'}(x_1, \bar{x}_1) \Phi^{(2)}_{j,j'}(x_2, \bar{x}_2) \rangle^{(2)}_\lambda = \frac{1}{k} \left(c_R \lambda_1^2 + c_{R'} \lambda_2^2 \right) \frac{(\mathbb{I}_R \otimes \mathbb{I}_{R'})_{ii',jj'}}{x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}} \ln \frac{\varepsilon^2}{|x_{12}|^2}. \tag{A.6}$$

All necessary integrals have been encountered before in similar computations in various works, for example in [4].

Three-loop $\mathcal{O}(\lambda^3)$: To this order, we have that

$$\langle \Phi^{(1)}_{i,i'}(x_1, \bar{x}_1) \Phi^{(2)}_{j,j'}(x_2, \bar{x}_2) \rangle^{(3)}_\lambda = -\frac{1}{3! \pi^3} \int d^2 z_{123} (\lambda_1^3 A_3 + \lambda_2^3 B_3), \tag{A.7}$$

where

$$A_3 = \langle \Phi^{(1)}_{i,i'}(x_1, \bar{x}_1) J_1^{a_1}(z_1) \bar{J}_2^{a_1}(\bar{z}_1) J_1^{a_2}(z_2) \bar{J}_2^{a_2}(\bar{z}_2) J_1^{a_3}(z_3) \bar{J}_2^{a_3}(\bar{z}_3) \Phi^{(2)}_{j,j'}(x_2, \bar{x}_2) \rangle,$$

$$B_3 = \langle \Phi^{(1)}_{i,i'}(x_1, \bar{x}_1) J_2^{a_1}(z_1) \bar{J}_1^{a_1}(\bar{z}_1) J_2^{a_2}(z_2) \bar{J}_1^{a_2}(\bar{z}_2) J_2^{a_3}(z_3) \bar{J}_1^{a_3}(\bar{z}_3) \Phi^{(2)}_{j,j'}(x_2, \bar{x}_2) \rangle. \tag{A.8}$$

As explained in the main text, terms with mixed factors of λ_1 and λ_2 do not occur. Furthermore, under a parity transformation, mapping $J_i \leftrightarrow \bar{J}_i$ and $z_i \leftrightarrow \bar{z}_i$, $c_R \leftrightarrow c_{R'}$, the contributions of A_3 and B_3 must be related. We immediately see that

$$A_3 = \frac{1}{\sqrt{k}} \frac{f_{a_1 a_2 a_3}}{\bar{z}_{12} \bar{z}_{13} \bar{z}_{23}} \langle \Phi^{(1)}_{i,i'}(x_1, \bar{x}_1) J_1^{a_1}(z_1) J_1^{a_2}(z_2) J_1^{a_3}(z_3) \Phi^{(2)}_{j,j'}(x_2, \bar{x}_2) \rangle.$$

Subsequently, we apply the Ward identity for the current J_1^a. When contracted with another internal current via the Abelian term the result is zero due to the overall factor $f_{a_1 a_2 a_3}$. When the contraction is via the non-Abelian part then there is no contribution to order $1/k$. When J^{a_1} is contracted with an external Φ one saturates the order $1/k$. Further contraction gives rise either to pieces corresponding to disconnected diagrams or to subleading terms. Similar considerations for B_3 also apply. Hence, all contributions of $\mathcal{O}(\lambda^3)$ are zero and we get that

$$\langle \Phi_{i,i'}^{(1)}(x_1, \bar{x}_1) \Phi_{j,j'}^{(2)}(x_2, \bar{x}_2) \rangle_\lambda^{(3)} = 0 \, . \tag{A.9}$$

Four-loop $\mathcal{O}(\lambda^4)$: To this order the result has several terms. However, after excluding, according to our general discussion, mixed in λ_1 and λ_2 terms, we have that

$$\langle \Phi_{i,i'}^{(1)}(x_1, \bar{x}_1) \Phi_{j,j'}^{(2)}(x_2, \bar{x}_2) \rangle_\lambda^{(4)} = \frac{1}{4! \pi^4} \int d^2 z_{1234} \left(\lambda_1^4 A_4 + \lambda_2^4 B_4 \right), \tag{A.10}$$

where

$$A_4 = \langle \Phi_{i,i'}^{(1)} J_1^{a_1}(z_1) \bar{J}_2^{a_1}(\bar{z}_1) J_1^{a_2}(z_2) \bar{J}_2^{a_2}(\bar{z}_2) J_1^{a_3}(z_3) \bar{J}_2^{a_3}(\bar{z}_3) J_1^{a_4}(z_4) \bar{J}_2^{a_4}(\bar{z}_4) \Phi_{j,j'}^{(2)} \rangle \tag{A.11}$$

and B_4 follows by the parity transformation in A_4. To compute A_4 term, we use the Ward identity for $J_1^{a_1}$ obtaining that

$$
\begin{aligned}
A_4 = & -\frac{1}{\sqrt{k}} \frac{(t_{a_1}^{(1)})_i{}^l}{z_1 - x_1} \langle \Phi_{l,i'}^{(1)} \bar{J}_2^{a_1}(\bar{z}_1) J_1^{a_2}(z_2) \bar{J}_2^{a_2}(\bar{z}_2) J_1^{a_3}(z_3) \bar{J}_2^{a_3}(\bar{z}_3) J_1^{a_4}(z_4) \bar{J}_2^{a_4}(\bar{z}_4) \Phi_{j,j'}^{(2)} \rangle \\
& -\frac{1}{\sqrt{k}} \frac{(t_{a_1}^{(2)})_j{}^l}{z_1 - x_2} \langle \Phi_{i,i'}^{(1)} \bar{J}_2^{a_1}(\bar{z}_1) J_1^{a_2}(z_2) \bar{J}_2^{a_2}(\bar{z}_2) J_1^{a_3}(z_3) \bar{J}_2^{a_3}(\bar{z}_3) J_1^{a_4}(z_4) \bar{J}_2^{a_4}(\bar{z}_4) \Phi_{l,j'}^{(2)} \rangle \\
& +\frac{\delta_{a_1 a_2}}{z_{12}^2} \langle \Phi_{i,i'}^{(1)} \bar{J}_2^{a_1}(\bar{z}_1) \bar{J}_2^{a_2}(\bar{z}_2) J_1^{a_3}(z_3) \bar{J}_2^{a_3}(\bar{z}_3) J_1^{a_4}(z_4) \bar{J}_2^{a_4}(\bar{z}_4) \Phi_{j,j'}^{(2)} \rangle \\
& +\frac{1}{\sqrt{k}} \frac{f_{a_1 a_2 c}}{z_{12}} \langle \Phi_{i,i'}^{(1)} \bar{J}_2^{a_1}(\bar{z}_1) J_1^c(z_2) \bar{J}_2^{a_2}(\bar{z}_2) J_1^{a_3}(z_3) \bar{J}_2^{a_3}(\bar{z}_3) J_1^{a_4}(z_4) \bar{J}_2^{a_4}(\bar{z}_4) \Phi_{j,j'}^{(2)} \rangle \\
& +\frac{\delta_{a_1 a_3}}{z_{13}^2} \langle \Phi_{i,i'}^{(1)} \bar{J}_2^{a_1}(\bar{z}_1) J_1^{a_2}(z_2) \bar{J}_2^{a_2}(\bar{z}_2) \bar{J}_2^{a_3}(\bar{z}_3) J_1^{a_4}(z_4) \bar{J}_2^{a_4}(\bar{z}_4) \Phi_{j,j'}^{(2)} \rangle \\
& +\frac{1}{\sqrt{k}} \frac{f_{a_1 a_3 c}}{z_{13}} \langle \Phi_{i,i'}^{(1)} \bar{J}_2^{a_1}(\bar{z}_1) J_1^{a_2}(z_2) \bar{J}_2^{a_2}(\bar{z}_2) J_1^c(z_3) \bar{J}_2^{a_3}(\bar{z}_3) J_1^{a_4}(z_4) \bar{J}_2^{a_4}(\bar{z}_4) \Phi_{j,j'}^{(2)} \rangle \\
& +\frac{\delta_{a_1 a_4}}{z_{14}^2} \langle \Phi_{i,i'}^{(1)} \bar{J}_2^{a_1}(\bar{z}_1) J_1^{a_2}(z_2) \bar{J}_2^{a_2}(\bar{z}_2) J_1^{a_3}(z_3) \bar{J}_2^{a_3}(\bar{z}_3) \bar{J}_2^{a_4}(\bar{z}_4) \Phi_{j,j'}^{(2)} \rangle \\
& +\frac{1}{\sqrt{k}} \frac{f_{a_1 a_4 c}}{z_{14}} \langle \Phi_{i,i'}^{(1)} \bar{J}_2^{a_1}(\bar{z}_1) J_1^{a_2}(z_2) \bar{J}_2^{a_2}(\bar{z}_2) J_1^{a_3}(z_3) \bar{J}_2^{a_3}(\bar{z}_3) J_1^c(z_4) \bar{J}_2^{a_4}(\bar{z}_4) \Phi_{j,j'}^{(2)} \rangle \, .
\end{aligned}
$$

$$\tag{A.12}$$

The first two lines of (A.12) correspond to a contraction of $J_1^{a_1}$ with the external fields and the rest with its contraction, Abelian and non-Abelian, with J^{a_2}, J^{a_3} and J^{a_4}.

The first line of (A.12) after further contraction with $J_1^{a_2}$ and subsequently with $J_1^{a_3}$ and $J_1^{a_4}$ gives up to order $1/k$ the expression

$$
\frac{1}{k}\left(\frac{(t_{a_1})_i{}^l(t_{a_2})_l{}^m}{(z_1-x_1)(z_2-x_1)z_{34}^2}\langle\Phi_{m,i'}^{(1)}\bar{J}_2^{a_1}(\bar{z}_1)\bar{J}_2^{a_2}(\bar{z}_2)\bar{J}_2^{a_3}(\bar{z}_3)\bar{J}_2^{a_3}(\bar{z}_4)\Phi_{j,j'}^{(2)}\rangle\right.
$$

$$
-\frac{(t_{a_1})_i{}^l(t_{a_2}^*)_j{}^m}{(z_1-x_1)(z_2-x_2)z_{34}^2}\langle\Phi_{l,i'}^{(1)}\bar{J}_2^{a_1}(\bar{z}_1)\bar{J}_2^{a_2}(\bar{z}_2)\bar{J}_2^{a_3}(\bar{z}_3)\bar{J}_2^{a_3}(\bar{z}_4)\Phi_{m,j'}^{(2)}\rangle
$$

$$
+\frac{(t_{a_1})_i{}^l(t_{a_4})_l{}^m}{(z_1-x_1)(z_4-x_1)z_{23}^2}\langle\Phi_{m,i'}^{(1)}\bar{J}_2^{a_1}(\bar{z}_1)\bar{J}_2^{a_2}(\bar{z}_2)\bar{J}_2^{a_2}(\bar{z}_3)\bar{J}_2^{a_4}(\bar{z}_4)\Phi_{j,j'}^{(2)}\rangle
$$

$$
-\frac{(t_{a_1})_i{}^l(t_{a_4}^*)_j{}^m}{(z_1-x_1)(z_4-x_2)z_{23}^2}\langle\Phi_{l,i'}^{(1)}\bar{J}_2^{a_1}(\bar{z}_1)\bar{J}_2^{a_2}(\bar{z}_2)\bar{J}_2^{a_2}(\bar{z}_3)\bar{J}_2^{a_4}(\bar{z}_4)\Phi_{m,j'}^{(2)}\rangle
$$

$$
-\frac{(t_{a_1})_i{}^l f_{a_2a_3a_4}}{(z_1-x_1)z_{23}z_{34}^2}\langle\Phi_{l,i'}^{(1)}\bar{J}_2^{a_1}(\bar{z}_1)\bar{J}_2^{a_2}(\bar{z}_2)\bar{J}_2^{a_3}(\bar{z}_3)\bar{J}_2^{a_4}(\bar{z}_4)\Phi_{j,j'}^{(2)}\rangle \tag{A.13}
$$

$$
+\frac{(t_{a_1})_i{}^l(t_{a_3})_l{}^m}{(z_1-x_1)(z_3-x_1)z_{24}^2}\langle\Phi_{m,i'}^{(1)}\bar{J}_2^{a_1}(\bar{z}_1)\bar{J}_2^{a_2}(\bar{z}_2)\bar{J}_2^{a_3}(\bar{z}_3)\bar{J}_2^{a_2}(\bar{z}_4)\Phi_{j,j'}^{(2)}\rangle
$$

$$
-\frac{(t_{a_1})_i{}^l(t_{a_3}^*)_j{}^m}{(z_1-x_1)(z_3-x_2)z_{24}^2}\langle\Phi_{l,i'}^{(1)}\bar{J}_2^{a_1}(\bar{z}_1)\bar{J}_2^{a_2}(\bar{z}_2)\bar{J}_2^{a_3}(\bar{z}_3)\bar{J}_2^{a_2}(\bar{z}_4)\Phi_{m,j'}^{(2)}\rangle
$$

$$
\left.-\frac{(t_{a_1})_i{}^l f_{a_2a_4a_3}}{(z_1-x_1)z_{24}z_{34}^2}\langle\Phi_{l,i'}^{(1)}\bar{J}_2^{a_1}(\bar{z}_1)\bar{J}_2^{a_2}(\bar{z}_2)\bar{J}_2^{a_3}(\bar{z}_3)\bar{J}_2^{a_4}(\bar{z}_4)\Phi_{j,j'}^{(2)}\rangle\right),
$$

where we have only kept terms corresponding to connected diagrams. From the second line of (A.12), a similar expression occurs. The contribution from the third line of (A.12), which results from an Abelian contraction among currents, is

$$
\frac{1}{k}\left(\frac{(t_{a_3})_i{}^l(t_{a_4})_l{}^m}{(z_3-x_1)(z_4-x_1)z_{12}^2}\langle\Phi_{m,i'}^{(1)}\bar{J}_2^{a_1}(\bar{z}_1)\bar{J}_2^{a_1}(\bar{z}_2)\bar{J}_2^{a_3}(\bar{z}_3)\bar{J}_2^{a_4}(\bar{z}_4)\Phi_{j,j'}^{(2)}\rangle\right.
$$

$$
-\frac{(t_{a_3})_i{}^l(t_{a_4}^*)_j{}^m}{(z_3-x_1)(z_4-x_2)z_{12}^2}\langle\Phi_{l,i'}^{(1)}\bar{J}_2^{a_1}(\bar{z}_1)\bar{J}_2^{a_1}(\bar{z}_2)\bar{J}_2^{a_3}(\bar{z}_3)\bar{J}_2^{a_4}(\bar{z}_4)\Phi_{m,j'}^{(2)}\rangle
$$

$$
-\frac{(t_{a_3}^*)_j{}^l(t_{a_4})_i{}^m}{(z_3-x_2)(z_4-x_1)z_{12}^2}\langle\Phi_{m,i'}^{(1)}\bar{J}_2^{a_1}(\bar{z}_1)\bar{J}_2^{a_1}(\bar{z}_2)\bar{J}_2^{a_3}(\bar{z}_3)\bar{J}_2^{a_4}(\bar{z}_4)\Phi_{l,j'}^{(2)}\rangle
$$

$$
+\frac{(t_{a_3}^*)_j{}^l(t_{a_4}^*)_l{}^m}{(z_3-x_2)(z_4-x_2)z_{12}^2}\langle\Phi_{i,i'}^{(1)}\bar{J}_2^{a_1}(\bar{z}_1)\bar{J}_2^{a_1}(\bar{z}_2)\bar{J}_2^{a_3}(\bar{z}_3)\bar{J}_2^{a_4}(\bar{z}_4)\Phi_{m,j'}^{(2)}\rangle \tag{A.14}
$$

$$
-\frac{f_{a_3a_4c}(t_{a_c})_i{}^l}{(z_4-x_1)z_{34}z_{12}^2}\langle\Phi_{l,i'}^{(1)}\bar{J}_2^{a_1}(\bar{z}_1)\bar{J}_2^{a_1}(\bar{z}_2)\bar{J}_2^{a_3}(\bar{z}_3)\bar{J}_2^{a_4}(\bar{z}_4)\Phi_{j,j'}^{(2)}\rangle
$$

$$
\left.+\frac{f_{a_3a_4c}(t_{a_c}^*)_j{}^l}{(z_4-x_2)z_{34}z_{12}^2}\langle\Phi_{i,i'}^{(1)}\bar{J}_2^{a_1}(\bar{z}_1)\bar{J}_2^{a_1}(\bar{z}_2)\bar{J}_2^{a_3}(\bar{z}_3)\bar{J}_2^{a_4}(\bar{z}_4)\Phi_{l,j'}^{(2)}\rangle\right).
$$

The rest of the terms arising from Abelian contractions among currents in (A.12) give similar results.

Finally, the fourth line of (A.12), resulting from non-Abelian contraction among currents gives

$$
\begin{aligned}
-\frac{1}{k}\Bigg(& \frac{(t_{a_c})_i{}^l f_{a_1 a_2 c}}{(z_2 - x_1)z_{12}} \frac{\delta_{a_3 a_4}}{z_{34}^2} \langle \Phi^{(1)}_{l,i'} \bar{J}^{a_1}_2(\bar{z}_1) \bar{J}^{a_2}_2(\bar{z}_2) \bar{J}^{a_3}_2(\bar{z}_3) \bar{J}^{a_4}_2(\bar{z}_4) \Phi^{(2)}_{j,j'} \rangle \\
& - \frac{(t^*_{a_c})_j{}^l f_{a_1 a_2 c}}{(z_2 - x_2)z_{12}} \frac{\delta_{a_3 a_4}}{z_{34}^2} \langle \Phi^{(1)}_{i,i'} \bar{J}^{a_1}_2(\bar{z}_1) \bar{J}^{a_2}_2(\bar{z}_2) \bar{J}^{a_3}_2(\bar{z}_3) \bar{J}^{a_4}_2(\bar{z}_4) \Phi^{(2)}_{l,j'} \rangle \\
& + \frac{(t_{a_4})_i{}^l f_{a_1 a_2 a_3}}{(z_4 - x_1)z_{12} z_{23}^2} \langle \Phi^{(1)}_{l,i'} \bar{J}^{a_1}_2(\bar{z}_1) \bar{J}^{a_2}_2(\bar{z}_2) \bar{J}^{a_3}_2(\bar{z}_3) \bar{J}^{a_4}_2(\bar{z}_4) \Phi^{(2)}_{j,j'} \rangle \\
& - \frac{(t^*_{a_4})_j{}^l f_{a_1 a_2 a_3}}{(z_4 - x_2)z_{12} z_{23}^2} \langle \Phi^{(1)}_{i,i'} \bar{J}^{a_1}_2(\bar{z}_1) \bar{J}^{a_2}_2(\bar{z}_2) \bar{J}^{a_3}_2(\bar{z}_3) \bar{J}^{a_4}_2(\bar{z}_4) \Phi^{(2)}_{l,j'} \rangle \\
& + \frac{(t_{a_3})_i{}^l f_{a_1 a_2 a_4}}{(z_3 - x_1)z_{12} z_{24}^2} \langle \Phi^{(1)}_{l,i'} \bar{J}^{a_1}_2(\bar{z}_1) \bar{J}^{a_2}_2(\bar{z}_2) \bar{J}^{a_3}_2(\bar{z}_3) \bar{J}^{a_4}_2(\bar{z}_4) \Phi^{(2)}_{j,j'} \rangle \\
& - \frac{(t^*_{a_3})_j{}^l f_{a_1 a_2 a_4}}{(z_3 - x_2)z_{12} z_{24}^2} \langle \Phi^{(1)}_{i,i'} \bar{J}^{a_1}_2(\bar{z}_1) \bar{J}^{a_2}_2(\bar{z}_2) \bar{J}^{a_3}_2(\bar{z}_3) \bar{J}^{a_4}_2(\bar{z}_4) \Phi^{(2)}_{l,j'} \rangle \Bigg),
\end{aligned}
\tag{A.15}
$$

where we have once again kept only terms corresponding to connected diagrams. The rest of the terms resulting from non-Abelian contraction among currents in (A.12) give similar results.

In the above expressions a 6-point function appears with two external fields and four currents. Since we have already saturated the factor $1/k$, in order to evaluate it we can simply contract the currents among themselves via the Abelian part of their OPE. The result, to $\mathcal{O}(1)$ in which we are interested, is

$$
\begin{aligned}
\langle \Phi^{(1)}_{i,i'} \bar{J}^{a_1}_2(\bar{z}_1) \bar{J}^{a_2}_2(\bar{z}_2) \bar{J}^{a_3}_2(\bar{z}_3) \bar{J}^{a_4}_2(\bar{z}_4) \Phi^{(2)}_{j,j'} \rangle = & \left(\frac{\delta_{a_1 a_2} \delta_{a_3 a_4}}{\bar{z}_{12}^2 \bar{z}_{34}^2} + \frac{\delta_{a_1 a_3} \delta_{a_2 a_4}}{\bar{z}_{13}^2 \bar{z}_{24}^2} + \frac{\delta_{a_1 a_4} \delta_{a_2 a_3}}{\bar{z}_{14}^2 \bar{z}_{23}^2} \right) \\
& \times \frac{(\mathbb{I}_R \otimes \mathbb{I}_{R'})_{ii', jj'}}{x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}}.
\end{aligned}
\tag{A.16}
$$

Using the above in (A.10), by taking into account the symmetry under the exchange of external points $x_1 \leftrightarrow x_2$ and contracting the group indices we get that

$$
\langle \Phi^{(1)}_{i,i'}(x_1, \bar{x}_1) \Phi^{(2)}_{j,j'}(x_2, \bar{x}_2) \rangle^{(4)}_\lambda = \frac{1}{k} \left(c_R \lambda_1^4 + c_{R'} \lambda_2^4 \right) \frac{(\mathbb{I}_R \otimes \mathbb{I}_{R'})_{ii', jj'}}{x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}} \ln \frac{\varepsilon^2}{|x_{12}|^2}.
\tag{A.17}
$$

Adding together the perturbative results of (A.1), (A.6), (A.9) and (A.17), we find the two-point function for primaries up to order $\mathcal{O}(\lambda^4/k)$ to be

$$
\begin{aligned}
\langle \Phi^{(1)}_{i,i'}(x_1, \bar{x}_1) \Phi^{(2)}_{j,j'}(x_2, \bar{x}_2) \rangle_\lambda = & \frac{1}{k} \left(c_R \lambda_1^2 (1 + \lambda_1^2) + c_{R'} \lambda_2^2 (1 + \lambda_2^2) \right) \\
& \times \frac{(\mathbb{I}_R \otimes \mathbb{I}_{R'})_{ii', jj'}}{x_{12}^{2\Delta_R} \bar{x}_{12}^{2\bar{\Delta}_{R'}}} \ln \frac{\varepsilon^2}{|x_{12}|^2} + \mathcal{O}(\lambda^5/k),
\end{aligned}
\tag{A.18}
$$

from which we extract the perturbative expression (2.13) in the main text.

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