# Linear Control Semigroups Acting on Projective Space[1]

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Linear control semigroups $\mathcal{S} \subset \mathrm{Gl}(d, \mathbb{R})$ are associated with semilinear control systems of the form

$$\dot{x}(t) = A(u(t))\, x(t), \qquad t \in \mathbb{R}, \qquad x(0) = x_0 \in \mathbb{R}^d \setminus \{0\}$$
$$u \in \mathcal{U} := \{u\colon \mathbb{R} \to \mathbb{R}^m\colon \text{measurable, } u(t) \in U \subset \mathbb{R}^m\}$$

where $A\colon \mathbb{R}^m \to \mathrm{gl}(d, \mathbb{R})$ is continuous in some open set containing $U$. The semigroup $\mathcal{S}$ then corresponds to the solutions with piecewise constant controls, i.e.,

$$\mathcal{S} = \{e^{t_n B_n} \cdot \ldots \cdot e^{t_1 B_1}\colon t_j \geqslant 0,\, B_j = A(u_j),\, u_j \in U,\, j = 1, \ldots, n \in \mathbb{N}\}$$

$\mathcal{S}$ acts in a natural way on $\mathbb{R}^d \setminus \{0\}$, on the sphere $\mathbb{S}^{d-1}$, and on the projective space $\mathbb{P}^{d-1}$. Under the assumption that the group generated by $\mathcal{S}$ in $\mathrm{Gl}(d, \mathbb{R})$ acts transitively on $\mathbb{P}^{d-1}$, we analyze the control structure of the action of $\mathcal{S}$ on $\mathbb{P}^{d-1}$: We characterize the sets in $\mathbb{P}^{d-1}$, where the system is controllable (the control sets) using perturbation theory of eigenvalues and (generalized) eigenspaces of the matrices $g \in \mathcal{S}$. For nonlinear control systems on finite-dimensional manifolds $M$, we study the linearization on the tangent bundle $TM$ and the projective bundle $\mathbb{P}M$ via the theory of Morse decompositions, to obtain a characterization of the chain-recurrent components of the control flow on $\mathcal{U} \times \mathbb{P}M$. These components correspond uniquely to the chain control sets on $\mathbb{P}M$, and they induce a subbundle decomposition of $\mathcal{U} \times TM$. These results are used to characterize the chain control sets of $\mathcal{S}$ acting on $\mathbb{P}^{d-1}$ and to compare the control sets and chain control sets.

**KEY WORDS:** semilinear control systems; bilinear control systems; linear control semigroups; control sets; chain control sets; control flows on vector bundles.

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## 1. INTRODUCTION

While the structure of linear Lie groups acting transitively on a sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$, or on $\mathbb{R}^d \backslash \{0\}$, is well understood (see, e.g., Ref. 5), this is not true for linear semigroups acting on these spaces. The problem is of considerable interest, however, in control theory, where transitivity of the systems semigroup translates into complete controllability. The problem of complete controllability of systems given by Lie groups has found various solutions (cf. Refs. 4, 15, 19). In this paper we study linear control semigroups acting on projective spaces (or on spheres) that describe situations, where the system is not completely controllable. We characterize the dynamical and the control behavior of such semigroups.

Linear control semigroups arise in connection with semilinear control systems of the form

$$\begin{aligned} \dot{x}(t) = A(u(t))\, x(t), \quad t \in \mathbb{R}, \quad x(0) = x_0 \in \mathbb{R}^d \backslash \{0\} \\ u \in \mathcal{U} := \{u: \mathbb{R} \to \mathbb{R}^m: \text{measurable}, u(t) \in U \text{ a.e.}\} \end{aligned} \tag{1.1}$$

where $U \subset \mathbb{R}^m$, $A: V \to \mathrm{gl}(d, \mathbb{R})$ is continuous on an open set $V \supset U$. [Here $\mathrm{gl}(d, \mathbb{R})$ denotes the space of real $d \times d$-matrices.] We will assume that (1.1) has for every $x_0 \neq 0$ and every $u \in \mathcal{U}$, a unique trajectory $\varphi\ (\cdot, x_0, u)$ on $\mathbb{R}$.

Associated with (1.1) is the control semigroup

$$\mathcal{S} = \{e^{t_n B_n} \cdot \ldots \cdot e^{t_1 B_1}: t_j \geqslant 0, B_j = A(u_j), u_j \in U,$$
$$j = 1, \ldots, n \in \mathbb{N}\} \subset \mathrm{Gl}(d, \mathbb{R})$$

corresponding to solutions of (1.1) with piecewise constant $u \in \mathcal{U}$. Note that $\mathcal{S}$ acts in a natural way on $\mathbb{R}^d \backslash \{0\}$, on the sphere $\mathbb{S}^{d-1}$, and on the projective space $\mathbb{P}^{d-1}$. If $\mathcal{S}$ acts transitively on one of these spaces $M$, i.e., if for all $x$, $y \in M$ there exists $g \in \mathcal{S}$ with $y = gx$, then the system (1.1) is called completely controllable on $M$. Conditions for controllability of bilinear systems $\dot{x} = A_0 x + \sum u_i A_i x$ with unbounded control range, i.e., $u_i(t) \in \mathbb{R}$, on the state space $\mathbb{R}^d \backslash \{0\}$ can be found, e.g., in Ref. 6 or 21. If the matrices $A_0, \ldots, A_m$ are skew symmetric, then the system lives on the sphere with radius $|x_0|$, and Ref. 7 gives criteria for controllability in this case.

Here we are interested in the action of $\mathcal{S}$ on the projective space $\mathbb{P}^{d-1}$. This action is given by the following control system: Denote $s = (x/|x|) \in \mathbb{S}^{d-1}$, and define $h(s, u) := [A(u) - s^T A(u)\, s \cdot \mathrm{id}]\, s$. Then $h(s, u)$ can be considered as vector field on $\mathbb{P}^{d-1}$, and the projection of (1.1) onto $\mathbb{P}^{d-1}$ reads

$$\dot{s}(t) = h(s(t), u(t)), \qquad t \in \mathbb{R}, \qquad s(0) = s_0 \in \mathbb{P}^{d-1} \tag{1.2}$$

For $u \in \mathscr{U}$ denote by $\varphi(\cdot, x_0, u)$ the solution of (1.1) and by $s(\cdot, s_0, u)$ the corresponding solution of (1.2) with $s_0 = x_0/|x_0|$. Then $s(t, s_0, u) = |\varphi(t, x_0, u)|^{-1} \varphi(t, x_0, u)$ for all $t \in \mathbb{R}$, and hence (1.2) describes the action of $\mathscr{S}$ on $\mathbb{P}^{d-1}$.

In this paper we analyze the controllability properties of (1.2) under the assumption that the Lie group $\mathscr{G}$, generated by $\mathscr{S}$ in $\mathrm{Gl}(d, \mathbb{R})$, acts transitively on $\mathbb{P}^{d-1}$. The point of view here is the same as in Ref. 9, where the notions of control sets (see Definition 2.3 below) and chain control sets (see Definition 4.6 below) were related to dynamical systems properties of control systems, such as topological mixing or chain recurrence. We will completely describe the $k$ control sets, $1 \leqslant k \leqslant d$; their interior consists of eigenspaces of the matrices in int $\mathscr{S}$, and these sets can be ordered linearly according to the reachability structure of (1.2); see Theorem 3.10.

Under additional assumptions ($U$ compact, $A(u)$ affine in $u$) we can describe also all chain control sets of the system (1.2). This is based on the connection, derived in Ref. 9, between chain control sets and the connected components of the chain recurrent set of an associated dynamical system (here: a linear flow) on $\mathscr{U} \times \mathbb{P}^{d-1}$. Thus we can apply Conley's theory of (finest) Morse decompositions in order to describe the chain control sets. There are $l$ chain control sets with $1 \leqslant l \leqslant k \leqslant d$ (see Theorem 4.9). Theorem 5.6 describes their relation to the main control sets. According to the results in Ref. 9, the main control sets correspond to the maximal topologically transitive components of the associated dynamical system. Hence our results on control sets bear some resemblance to Smale's decomposition of the nonwandering set of Axiom-A flows (cf. Remark 5.3).

The consequences of these results for the Lyapunov exponents, and hence for the stability properties of the semilinear control system (1.1), will be studied in a forthcoming paper (10). Some results pertaining to the extremal control sets and the extremal Lyapunov exponents are given in Refs. 8 and 11. Applications to robustness analysis of linear systems are contained in Ref. 12. Our results about control sets of (1.2) on $\mathbb{P}^{d-1}$ apply to directional control of linear systems in the following way:

Given the linear control system in $\mathbb{R}^d$ with input in $\mathbb{R}^m$,

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t), \qquad y \in \mathbb{R}^n \tag{1.3}$$

We say that (1.3) is directionally controllable from $x_0 \in \mathbb{R}^d \setminus \{0\}$ to $x_1 \in \mathbb{R}^d \setminus \{0\}$, if there exists an admissible control $u(\cdot)$, and $\alpha \in \mathbb{R} \setminus \{0\}$ such that $\varphi(t, x_0, u) = \alpha x_1$, for some $t > 0$. The problem is to find a (time-varying) output feedback $u(t) = F(t)\, y(t)$, such that the closed loop system

$$\dot{x}(t) = Ax(t) + BF(t)\, Cx(t) \tag{1.4}$$

is directionally controllable, where, due to systems constraints, the feedback satisfies $F(t) \in \Omega \subset M(m, n; \mathbb{R})$, the real $m \times n$ matrices. The closed loop system (1.4) is of the type (1.1), and (1.3) is directionally controllable from $x_0$ to $x_1$, via restricted output feedback, if we have for the projected points $s_i = x_i/|x_i|$, $i = 0, 1$, that $s_1 \in \mathcal{S} s_0$. In particular, $x_0$ is directionally controllable to $x_1$, and $x_1$ directionally controllable to $x_0$ iff $s_0$ and $s_1$ are in the (interior) of the same control set. The ordering between control sets (see Theorem 3.10) tells us which directions can be reached from all $x_0 \neq 0$. On the other hand, directions in the minimal control set can be reached only from within this set. Of course, (1.3) is completely directionally controllable via restricted, time-varying output feedback iff the corresponding system (1.2) has exactly one control set; compare Remark 3.16.

The structure of this paper is as follows: In Section 2 we collect preliminary results on the action of linear control semigroups on $\mathbb{P}^{d-1}$, on control sets, and on the behavior of eigenvalues and eigenspaces under continuous perturbations. Section 3 contains the construction of the main control sets of (1.2) on $\mathbb{P}^{d-1}$. In Section 4 we study nonlinear control systems on finite dimensional manifolds $M$ and their linearizations on the tangent bundle $TM$ from a dynamical systems point of view. In particular, we use the theory of Morse decompositions for the induced system on the projective bundle $\mathbb{P}M$ to characterize the chain control sets. Section 5 contains our results on chain control sets for bilinear systems, and the comparison of these sets with the main control sets.

**Notation.** The projective space $\mathbb{P}^{d-1}$ in $\mathbb{R}^d$ is denoted $\mathbb{P}$ throughout this paper (the dimension being clear from the context), and for $A \subset \mathbb{R}^d$, $\mathbb{P}A$ denotes the set of elements in $\mathbb{P}$ obtained by projecting the nonzero elements of $A$ onto $\mathbb{P}$.

## 2. PRELIMINARIES

In this section we introduce some notation and collect preliminary results. In particular, control sets are defined and the behavior of eigenvalues and eigenspaces under continuous perturbations is studied.

Using piecewise constant controls, one can associate with system (1.1) the systems group and semigroup, respectively:

$$\mathcal{G} := \left\{ \begin{array}{c} \exp(t_n B_n) \ldots \exp(t_1 B_1) \colon t_j \in \mathbb{R},\ B_j = A(u_j), \\ u_j \in U,\ j = 1, \ldots, n,\ n \in \mathbb{N} \end{array} \right\} \subset \mathrm{Gl}(d, \mathbb{R}) \tag{2.1}$$

$$\mathcal{S} := \left\{ \begin{array}{c} \exp(t_n B_n) \ldots \exp(t_1 B_1) \colon t_j \geqslant 0,\ B_j = A(u_j), \\ u_j \in U,\ j = 1, \ldots, n,\ n \in \mathbb{N} \end{array} \right\} \subset \mathrm{Gl}(d, \mathbb{R}). \tag{2.2}$$

Then $\mathscr{G}$ is a Lie group, since it is a path-connected subgroup of $\mathrm{Gl}(d, \mathbb{R})$ (cf., e.g., Ref. 20, pp. 275). The sets of group and semigroup elements at time $t$, i.e., with $\sum |t_i| = t$, are denoted by $\mathscr{G}_t$ and $\mathscr{S}_t$, respectively. Note that $\mathscr{G}$ acts naturally on $\mathbb{R}^d \setminus \{0\}$, and on the projective space $\mathbb{P} := \mathbb{P}^{d-1}$ via $g(x) = \mathbb{P}(gx)$, where $\mathbb{P}: \mathbb{R}^d \setminus \{0\} \to \mathbb{P}^{d-1}$ is the natural projection. We do not distinguish in our notation between these two actions, because it is always clear from the context, which one is referred to.

The positive (and negative, respectively) orbit of an element $x \in \mathbb{P}$ up to time $t \geqslant 0$ is given by

$$\begin{aligned} \mathcal{O}^+_{\leqslant t}(x) &= \{ y \in \mathbb{P}: \text{there is } g \in \mathscr{S}_{\leqslant t} \text{ with } y = gx \} \\ \mathcal{O}^-_{\leqslant t}(x) &= \{ y \in \mathbb{P}: \text{there is } g \in \mathscr{S}_{\leqslant t} \text{ with } x = gy \} \end{aligned} \tag{2.3}$$

$\mathcal{O}^+(x)$ (and $\mathcal{O}^-(x)$) are the orbits $\mathcal{O}^+_{<\infty}(x)$ (and $\mathcal{O}^-_{<\infty}(x)$), i.e., with respect to the entire semigroup $\mathscr{S}$. Under the assumptions above, $\mathrm{cl}\, \mathcal{O}^+_{<t}(x)$ coincides with the closure of the set of points in $\mathbb{P}$ attainable from $x$ in time $\leqslant t$ with controls $u \in \mathscr{U}$; this follows from the fact that such trajectories can (uniformly on bounded intervals) be approximated by trajectories corresponding to piecewise constant controls.

The following hypothesis (H) will frequently be imposed. Let

$$\mathscr{L} = \mathscr{L}A\{h(\cdot, u),\ u \in U\}$$

denote the Lie algebra generated by the vectorfields $h(\cdot, u)$, $u \in U$, on $\mathbb{P}$, and let $\Delta_{\mathscr{L}}$ denote the corresponding distribution in the tangent bundle $T\mathbb{P}$; we assume that

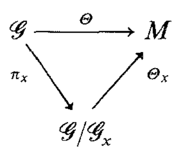
$$\Delta_{\mathscr{L}}(x) = T_x \mathbb{P} \qquad \text{for all} \quad x \in \mathbb{P} \tag{H}$$

This hypothesis guarantees that system (1.2) is locally accessible on $\mathbb{P}$, i.e., for all $x \in \mathbb{P}$ and all neighborhoods $V$ of $x$ we have $\mathrm{int}\, \mathcal{O}^+_{\leqslant t}(x) \cap V \neq \varnothing$ for all $t > 0$, and similarly for the negative orbit (cf. Proposition 2.1 below). Furthermore, (H) implies that the systems group $\mathscr{G}$ acts transitively on $\mathbb{P}$, i.e., for all $x \in \mathbb{P}$ one has $\{gx: g \in \mathscr{G}\} = \mathbb{P}$.

We recall the following facts from the theory of Lie groups (cf., e.g., Ref. 16): Let $\mathscr{G}$ be a Lie group acting transitively on a manifold $M$ and fix $x \in M$. Let $\mathscr{G}_x$ be the isotropy group of $x \in M$, i.e.,

$$\mathscr{G}_x = \{ g \in \mathscr{G}: gx = x \}$$

Then the following diagram is commutative



with $\Theta(g) = gx$; $\pi_x$ is the natural projection and $\Theta_x$ the induced map. Furthermore, $\Theta$ is open and continuous, $\pi_x$ is an open and continuous projection, and $\Theta_x$ is a diffeomorphism.

***2.1. Proposition.*** *Let hypothesis* $(H)$ *be satisfied.*

*(i)* *For all* $t > 0$ *and all* $x \in \mathbb{P}$, $\operatorname{cl}\operatorname{int} \mathcal{O}^+_{\leqslant t}(x) = \operatorname{cl} \mathcal{O}^+_{\leqslant t}(x)$.

*(ii)* *For all* $t > 0$, $\operatorname{cl}\operatorname{int} \mathcal{S}_{\leqslant t} = \operatorname{cl} \mathcal{S}_{\leqslant t}$, *where the interior is taken w.r.t. the topology of the Lie group* $\mathcal{G}$.

*(iii)* *If* $g \in \operatorname{int} \mathcal{S}_{\leqslant t}$, *then* $gx \in \operatorname{int} \mathcal{O}^+_{\leqslant t}(x)$ *and* $x \in \operatorname{int} \mathcal{O}^-_{\leqslant t}(gx)$ *for all* $x \in \mathbb{P}$.

*(iv)* *If* $g_i \in \operatorname{int} \mathcal{S}_{\leqslant t_i}$, *for* $i = 0, 1$, *then there is a continuous path in* $\mathcal{S}_{\leqslant t_0 + t_1}$ *connecting* $g_0$ *and* $g_1$.

**Proof.** For (ii) see, e.g., Ref. 19 (Lemma 6.1); then (i) and (iii) are easy consequences of (ii) and the preceding remarks. In order to see (iv), represent $g_0$ and $g_1$ as in (2.2):

$$g_i = \exp(s^i_{n_i} B^i_{n_i}) \dots \exp(s^i_1 B^i_1), \qquad i = 0, 1$$

with $s^i_j > 0$, $\sum_{j=1}^{n_i} s^i_j = t_i$, $n_i \in \mathbb{N}$, $i = 0, 1$.

Define a continuous path $g\colon [0, t_1] \to \mathcal{S}$: if $t \in [\sum_{j=0}^{k} s^1_j, \sum_{j=0}^{k+1} s^1_j)$ for some $k \in \{0, 1, \dots, n_1 - 1\}$, set $g(t) = \exp(s^1_t B^1_k) \dots \exp(s^1_1 B^1_1)\, g_0$, where $s^1_t = t - \sum_{j=0}^{k} s^1_j$ and $s^1_0 = 0$. Then $g(0) = g_0 \in \operatorname{int} \mathcal{S}_{\leqslant t_0}$, $g(t_1) = g_1 g_0 \in \operatorname{int} \mathcal{S}_{\leqslant t_0 + t_1}$, and hence $g(t) \in \operatorname{int} \mathcal{S}_{\leqslant t_0 + t_1}$ for all $t \in [0, t_1]$. Similarly one can connect $g_1 g_0$ continuously with $g_1$. Taken together, there exists a continuous path from $g_0$ to $g_1$ in $\operatorname{int} \mathcal{S}_{\leqslant t_0 + t_1}$. ■

We obtain the following approximation result.

***2.2. Proposition.*** *Assume that* $(H)$ *is satisfied and let* $g \in \mathcal{S}_t$, $t > 0$. *Then there are* $t_n \searrow t$ *and* $g_n \in \mathcal{S}_t \cap \operatorname{int} \mathcal{S}_{\leqslant t_n}$ *such that* $g_n \to g$. *In particular, for every* $g \in \mathcal{S}_t$ *there are* $g_n \in \mathcal{S}_t \cap \operatorname{int} \mathcal{S}_{\leqslant t+1}$ *with* $g_n \to g$.

**Proof.** The second assertion is obvious from the first one. By the preceding proposition, there are $h_n \in \operatorname{int} \mathcal{S}_{\leqslant \tau_n}$ with $\tau_n \searrow 0$. A look at the construction, e.g., in Ref. 18 (p. 69), shows that we may assume that $h_n \to \mathrm{id}$ for $n \to \infty$. Then $h_n \in \mathcal{S}_{\sigma_n}$ for some $\sigma_n$ with $0 < \sigma_n \leqslant \tau_n$. Since $t > 0$, it is also clear that there are $h'_n \in \mathcal{S}_{t - \sigma_n}$ with $h'_n \to g$. Thus by continuity for $n$ large enough,

$$g_n := h_n \cdot h'_n \in \mathcal{S}_t \cap \operatorname{int} \mathcal{S}_{\leqslant t_n}$$

with $t_n := \tau_n + t - \sigma_n \searrow t$ and $g_n \to g$. ■

Next we introduce the notion of control sets (cf. Ref. 9, Remark 3.2).

**2.3. Definition.** A set $D \subset \mathbb{P}$ is called a control set of the control system (1.2) if

(i) $D \subset \operatorname{cl} \mathcal{O}^+(x)$ for every $x \in D$;

(ii) for every $x \in D$ there is $u \in \mathcal{U}$ such that the corresponding trajectory of (1.2) satisfies $\varphi(t, x, u) \in D$ for all $t \in \mathbb{R}$;

(iii) $D$ is maximal (with respect to set inclusion) with the properties (i) and (ii).

Of particular interest are control sets with nonvoid interior, since in their interior exact controllability in finite time holds (cf. Ref. 11, Proposition 2.3).

***2.4. Proposition.*** *Assume that (H) is satisfied and let D be a control set with nonvoid interior for (1.2). Then for all $x \in D$, all $y \in \operatorname{int} D$ there is $g \in \mathcal{S}$ with $y = gx$.*

In Section 3 we describe all control sets with nonvoid interior of (1.2), called the main control sets.

Next we cite a result on the behavior of eigenvalues and generalized eigenspaces under continuous perturbations. Let spec $A$ denote the set of different eigenvalues of a bounded linear operator $A \in \mathrm{gl}(d, \mathbb{R})$. Then the following holds (see, e.g., Ref. 3, Chap. II.8).

***2.5. Proposition.*** *Fix $A_0 \in \mathrm{gl}(d, \mathbb{R})$.*

*(i) If N is an open set of complex numbers with $N \supset \operatorname{spec} A_0 = \{\lambda_1, ..., \lambda_r\}$, then $N \supset \operatorname{spec} A$ for all $A \in \mathrm{gl}(d, \mathbb{R})$ with $\|A - A_0\| \leqslant \alpha$ for some $\alpha > 0$, where $\|\cdot\|$ is an arbitrary norm on the space of bounded linear operators on $\mathbb{R}^d$.*

*(ii) For $\lambda_0 \in \operatorname{spec} A_0$, choose $\varepsilon > 0$ such that the dotted disk $0 < |\lambda - \lambda_0| \leqslant \varepsilon$ contains no eigenvalue of $A_0$. Then the Riesz projection*

$$P(A) = \frac{1}{2\pi i} \int_{|\lambda - \lambda_0| = \varepsilon} (\lambda - A)^{-1} \, d\lambda$$

*is, in a certain neighborhood of $A_0$, a continuous function of A. The range of $P(A_0)$ is the generalized eigenspace of $\lambda_0$ and for A close to $A_0$,*

$$n(\lambda_0) = \sum_{\lambda \in \operatorname{spec} A \cap \{\lambda : |\lambda - \lambda_0| < \varepsilon\}} n(\lambda) \tag{2.4}$$

*where $n(\lambda_0)$ and $n(\lambda)$ denote the algebraic multiplicities of the eigenvalues $\lambda_0$ of $A_0$ and of $\lambda$ of A, respectively.*

In the following, we consider for eigenvalues $\lambda$ the *real* generalized eigenspace $E(\lambda)$, i.e., if $\lambda \in \mathbb{C}\backslash\mathbb{R}$ is an eigenvalue of a real matrix $A$, then $E(\lambda)$ denotes the subspace of $\mathbb{R}^d$ obtained by adding the real and the imaginary parts of the complex generalized eigenspace of $\lambda$ (cf., e.g., Ref. 14).

***2.6. Proposition.*** *Assume that (H) is satisfied and let $g \in \mathcal{S}_t$, $t > 0$. Then for every $\varepsilon > 0$ there is $g' \in \mathcal{S}_t \cap \operatorname{int} \mathcal{S}_{\leqslant t+1}$ with $\|g - g'\| < \varepsilon$ such that for all $\lambda \in$ spec $g$, there is $\lambda' \in \operatorname{spec} g'$ with $|\lambda - \lambda'| < \varepsilon$ and the generalized eigenspaces $E(\lambda)$ and $E(\lambda')$ satisfy*

$$\inf\{d(x, x') : x \in \mathbb{P}E(\lambda),\, x' \in \mathbb{P}E(\lambda')\} < \varepsilon$$

*here d denotes some Riemannian metric on $\mathbb{P}$.*

**Proof.** By Proposition 2.2, there are $g_n \in \mathcal{S}_t \cap \operatorname{int} \mathcal{S}_{\leqslant t+1}$ with $g_n \to g$. Now the assertion follows from Proposition 2.5. ■

The metric $d$ referred to in Proposition 2.6 and the norm $\|\cdot\|$ in $\mathrm{gl}(d, \mathbb{R})$ will be kept fixed in the sequel, but all results hold for arbitrary Riemannian metrics on $\mathbb{P}$ and arbitrary norms in $\mathrm{gl}(d, \mathbb{R})$

## 3. CONSTRUCTION OF THE MAIN CONTROL SETS

In this section we analyze the behavior in $\mathbb{P}$ of trajectories corresponding to increasingly more general control functions. Theorem 3.10, the main result of this section, characterizes the main control sets, i.e., the control sets with nonvoid interior.

We start by analyzing the behavior of trajectories corresponding to constant controls in the generalized eigenspaces. Consider for $A \in \mathrm{gl}(d, \mathbb{R})$ the equation

$$\dot{x}(t) = Ax(t) \tag{3.1}$$

and its projection onto $\mathbb{P}$

$$\dot{s}(t) = h(A, s(t)) \tag{3.1'}$$

with $h(A, s) = [A - s^T As \cdot \mathrm{id}]\, s$.

***3.1. Lemma.*** *Consider Eq. (3.1).*

*(i) If A is a single Jordan block corresponding to a real eigenvalue $\lambda$*

$$A = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & \lambda & 1 \end{pmatrix}$$

*then every solution of (3.1′) converges for* $t \to \pm\infty$ *to* $\mathbb{P}(1\ 0\ \dots\ 0)^T =: \mathbb{P}E_0$.

(ii) *If* $A$ *is a single (real) Jordan block corresponding to a pair of complex eigenvalues* $\lambda, \bar{\lambda} \in \mathbb{C}\backslash\mathbb{R}$, $\lambda = a + ib$, $a, b \in \mathbb{R}$

$$A = \begin{bmatrix} a & -b & \vdots & 1 & 0 & \vdots & & & \\ b & a & \vdots & 0 & 1 & \vdots & & \mathbf{0} & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & & & \\ & & \vdots & a & -b & \vdots & & & \\ & & \vdots & b & a & \vdots & & & \\ & & & & & \ddots & & & \\ & & & & & & \ddots & \vdots & 1 & 0 \\ & & & & & & \ddots & \vdots & 0 & 1 \\ & & \mathbf{0} & & & & & \vdots & \cdots & \cdots \\ & & & & & & & \vdots & a & -b \\ & & & & & & & \vdots & b & a \end{bmatrix}$$

then every solution of (3.1′) converges for $t \to \pm\infty$ to the projected linear space $\mathbb{P}E_0$ corresponding to

$$E_0 := \operatorname{span}\{(1, 0, 0,\dots, 0)^T, (0, 1, 0,\dots, 0)^T\}$$

more precisely for every solution $\varphi(\cdot)$ of (3.1′), and every $\varepsilon > 0$, there is $T_1 > 0$ such that for all $|t| > T_1$

$$d(\varphi(t), \mathbb{P}E_0) < \varepsilon$$

**Proof.** Consider first part (ii). The corresponding fundamental solution $\Psi$ of (3.1) with $\Psi(0) = \text{id}$ has the form (cf., e.g., Ref. 17, p. 84).

$$\Psi = e^{at} \begin{bmatrix} \cos bt & -\sin bt & \vdots & t\cos bt & -t\sin bt & \vdots & \cdots & \vdots & \frac{t^{k-1}}{(k-1)!}\cos bt & \frac{-t^{k-1}}{(k-1)!}\sin bt \\ \sin bt & \cos bt & \vdots & t\sin bt & t\cos bt & \vdots & \cdots & \vdots & \frac{t^{k-1}}{(k-1)!}\sin bt & \frac{t^{k-1}}{(k-1)!}\cos bt \\ \cdots & \cdots & \vdots & \cdots & \cdots & \vdots & & \vdots & \cdots & \cdots \\ & & & & & & \ddots & & \vdots & \vdots \\ & & & & & & & \vdots & \cdots & \cdots \\ & & & & & & & \vdots & \cos bt & -\sin bt \\ & & & & & & & \vdots & \sin bt & \cos bt \end{bmatrix}$$

For large $|t|$, the first two components become dominant. Hence (ii) follows. Note that $\Psi(t)$ has the eigenvalues $e^{(a \pm ib)t}$. The proof of part (i) follows similarly. ■

Using Lemma 3.1 one easily verifies the following result for general $A \in \mathrm{gl}(d, \mathbb{R})$.

***3.2. Proposition.*** *Let* $\lambda_{i_1}, \ldots, \lambda_{i_j} \in \operatorname{spec} A$ *with* $\operatorname{Re} \lambda_{i_1} = \cdots = \operatorname{Re} \lambda_{i_j}$. *Then for every* $\varepsilon > 0$ *there is* $T(\varepsilon) > 0$ *such that for all* $T > T(\varepsilon)$ *and all* $x, y \in \mathbb{P}(E(\lambda_{i_1}) \oplus \cdots \oplus E(\lambda_{i_j}))$, *there are* $k \in \mathbb{N}$ *and* $x_1, \ldots, x_k \in \mathbb{P}$ *such that* $x_1 = x$, $x_k = y$, *and*

$$d(\varphi(T, x_i), x_{i+1}) < \varepsilon \qquad \text{for} \qquad i = 1, \ldots, k-1 \tag{3.2}$$

*where* $\varphi(\cdot, x)$ *is the trajectory of (3.1′) corresponding to the initial value* $\varphi(0, x) = x$.

**3.3. Remark.** As is well-known (Refs. 13 and 25) $\mathbb{P}(E(\lambda_{i_1}) \oplus \cdots \oplus E(\lambda_{i_j}))$ is a connected component of the chain recurrent set (hence chain transitive) of system (3.1′), provided that there are no other eigenvalues of $A$ with equal real part. The property indicated above is a slight strengthening of chain transitivity, since the time $T$ in (3.2) is fixed in advance, depending only on $\varepsilon$. Note that $k$ also depends on $T$.

Next we analyze the behavior of the trajectories corresponding to the system semigroup $\mathcal{S}_t$. Recall that elements $g$ of the systems semigroup correspond to piecewise constant periodic control functions in the following way: Every

$$g = \exp(A(u_n)\, t_n) \ldots \exp(A(u_1)\, t_1) \in \mathcal{S}_t \tag{3.3}$$

with $u_i \in U$, $t_i > 0$, $i = 1, \ldots, n$, $\sum_i t_i = t$, corresponds to $u_g \in \mathcal{U}$ defined by

$$u_g(\tau) = u_i \qquad \text{for} \qquad \tau \in [t_1 + \cdots + t_i, t_1 + \cdots + t_i + t_{i+1}) \tag{3.4}$$

extended $t$-periodically to $\mathbb{R}$. Conversely, every piecewise constant $t$-periodic control function $u$ defines an element $g_u$ of $\mathcal{S}_t$.

Combining Proposition 3.2 with Floquet theory, one obtains a result corresponding to Proposition 3.2 for elements $g$ of $\mathcal{S}_t$.

***3.4. Proposition.*** *Let* $g \in \mathcal{S}_t$ *be given by (3.3) and suppose* $\lambda_{i_1}, \ldots, \lambda_{i_j} \in \operatorname{spec} g$ *with* $|\lambda_{i_1}| = |\lambda_{i_2}| = \cdots = |\lambda_{i_j}|$. *Then for every* $\varepsilon > 0$, *there is* $N(\varepsilon) \in \mathbb{N}$ *such that for all* $x, y \in \mathbb{P}(E(\lambda_{i_1}) \oplus \cdots \oplus E(\lambda_{i_j}))$ *and all* $n \geq N(\varepsilon)$, *there are* $k \in \mathbb{N}$ *and* $x_1, \ldots, x_k \in \mathbb{P}$ *with* $x_1 = x$, $x_k = y$, *and*

$$d(g^{2n} x_i, x_{i+1}) < \varepsilon \qquad \text{for} \qquad i = 1, \ldots, k-1$$

**Proof.** Consider the differential equation with $t$-periodic coefficients

$$\dot{x}(\tau) = A(u_g(\tau))\, x(\tau), \qquad \tau \in \mathbb{R} \tag{3.5}$$

with $u_g$ given by (3.4). By Floquet theory (see, e.g., Ref. 24, p. 455), there exists a $2t$-periodic coordinate transformation $P(t)$ with $P(0) = \text{id}$, transforming (3.5) into an equation

$$\dot{x}(t) = Bx(t)$$

of the form (3.1). Note that the doubling of the period comes from the requirement that $B$ must be a real matrix. Choosing $T$ as a multiple of $2t$ in Proposition 3.2, one obtains the assertion, since $\lambda \in \text{spec}\, B$ (modulo $2\pi i$) iff $e^{\lambda t} \in \text{spec}\, g$. ■

The following two crucial lemmas connect the chain transitivity property indicated in Proposition 3.4 to control sets. The first one states a certain *uniform interior reachability condition.*

***3.5. Lemma.*** *Assume (H) and let* $g \in \text{int}\, \mathcal{S}_{\leqslant t}$, $t > 0$. *Then there is* $\varepsilon > 0$ *such that for all* $x \in \mathbb{P}$,

$$B_\varepsilon(g^2 x) \subset \text{int}\, \mathcal{O}^+_{\leqslant 2t}(x) \qquad \textit{and} \qquad B_\varepsilon(x) \subset \text{int}\, \mathcal{O}^-_{\leqslant 2t}(g^2 x)$$

*where* $B_\varepsilon(z) := \{y \in \mathbb{P}: d(y, z) \leqslant \varepsilon\}$.

**Proof.** Since $g \in \text{int}\, \mathcal{S}_{\leqslant t}$, Proposition 2.1 implies that for $x \in \mathbb{P}$ there are neighborhoods $V_1$ of $x$ and $V_2$ of $g^2 x$ such that

$$V_1 \subset \text{int}\, \mathcal{O}^-_{\leqslant t}(gx) \qquad \text{and} \qquad V_2 \subset \text{int}\, \mathcal{O}^+_{\leqslant t}(gx)$$

Hence there is $\delta = \delta(x) > 0$ such that $B_\delta(x) \subset V_1$ and $B_\delta(g^2 x) \subset V_2$. Denote

$$g^{-2}[B_{\delta/2}(g^2 x)] \cap B_\delta(x) := V_x$$

Then for all $y \in V_x$ we obtain $g^2 y \in B_{\delta/2}(g^2 x)$, and with $V_x \subset \text{int}\, \mathcal{O}^-_{\leqslant t}(gx)$, this implies for all $y \in V_x$: $B_{\delta/2}(g^2 y) \subset \text{int}\, \mathcal{O}^+_{\leqslant 2t}(y)$. Now a compactness argument over $x \in \mathbb{P}$ completes the proof of the first assertion; the second one follows similarly. ■

***3.6. Lemma.*** *Assume (H) and let* $x$, $y \in \mathbb{P}$ *and* $g \in \text{int}\, \mathcal{S}_{\leqslant t}$, $t > 0$. *Suppose that for every* $\varepsilon > 0$ *there are* $n \in \mathbb{N}$, $n \geqslant 2$, $k \in \mathbb{N}$ *and* $x_1, ..., x_k \in \mathbb{P}$ *with* $x_1 = x$, $x_k = y$ *such that*

$$d(g^n x_i, x_{i+1}) < \varepsilon \qquad \textit{for all} \qquad i = 1, ..., k-1 \tag{3.6}$$

*Then* $y \in \text{int}\, \mathcal{O}^+_{\leqslant T}(x)$ *for some* $T > 0$.

**Proof.** Choose $\varepsilon > 0$ as the uniform constant determined by Lemma 3.5 and take $n \geqslant 2$, $k \in \mathbb{N}$ and $x_1, ..., x_k$ with the properties indicated above. Since $n \geqslant 2$

$$g^n x_i = g^2 g^{n-2} x_i \qquad \text{with} \qquad n-2 \geqslant 0$$

Hence, by choice of $\varepsilon$

$$\begin{aligned} x_{i+1} \in B_\varepsilon(g^n x_i) &\subset \operatorname{int} \mathcal{O}^+_{\leqslant 2t}(g^{n-2} x_i) \\ &\subset \operatorname{int} \mathcal{O}^+_{\leqslant T_i}(x_i) \qquad \text{for some} \qquad T_i > 0 \end{aligned}$$

Thus $x_n = y \in \operatorname{int} \mathcal{O}^+_{\leqslant T}(x)$ for some $T > 0$. ■

Combining Lemma 3.6 with Proposition 3.4, we obtain the following result.

***3.7. Proposition.*** *Assume (H); let $g \in \operatorname{int} \mathcal{S}_{\leqslant t}$ for some $t > 0$, and suppose that $\lambda_{i_1}, ..., \lambda_{i_j} \in \operatorname{spec} g$ are such that $|\lambda_{i_1}| = \cdots = |\lambda_{i_j}|$. Then there exists a control set $D$ such that the corresponding generalized eigenspaces satisfy*

$$\mathbb{P}(E(\lambda_{i_1}) \oplus \cdots \oplus E(\lambda_{i_j})) \subset \operatorname{int} D$$

**Proof.** By Proposition 3.4 for all $x, y \in \mathbb{P}(E(\lambda_{i_1}) \oplus \cdots \oplus E(\lambda_{i_j}))$ and every $\varepsilon > 0$, there are $n \geqslant 2$, $k \in \mathbb{N}$, and $x_1, ..., x_k \in \mathbb{P}$ with $x_1 = x$, $x_k = y$, and

$$d(g^n x_i, x_{i+1}) < \varepsilon$$

Hence Lemma 3.6 implies $y \in \operatorname{int} \mathcal{O}^+_{\leqslant T}(x)$ for some $T > 0$; similarly, $x \in \operatorname{int} \mathcal{O}^+_{\leqslant T'}(y)$. Thus $x, y \in D$ for some control set $D$.

By Lemma 3.5 there is $\varepsilon > 0$ such that $B_\varepsilon(x) \subset \operatorname{int} \mathcal{O}^-_{\leqslant 2t}(g^2 x)$. By restricting $\varepsilon$, if necessary, we have also $B_\varepsilon(x) \subset \operatorname{int} \mathcal{O}^+_{\leqslant T'}(y)$. Note that $z := g^2 x \in \mathbb{P}(E(\lambda_{i_1}) \oplus \cdots \oplus E(\lambda_{i_j}))$. Then $y \in \operatorname{int} \mathcal{O}^+_{\leqslant T''}(z)$ and therefore $B_\varepsilon(x) \subset D$. Since $x \in \mathbb{P}(E(\lambda_{i_1}) \oplus \cdots \oplus E(\lambda_{i_j}))$ was arbitrary, this finishes the proof. ■

The next proposition shows that the interior of every control set (provided it is nonvoid) consists of eigenspaces of elements $g \in \operatorname{int} \mathcal{S}$.

***3.8. Proposition.*** *Assume (H) and let $D$ be a control set with nonvoid interior. Then for every $x \in \operatorname{int} D$ there are $t > 0$, $g \in \mathcal{S}_t \cap \operatorname{int} \mathcal{S}_{\leqslant t+1}$, and $\lambda \in \operatorname{spec} g \cap \mathbb{R}$ with $x \in \mathbb{P}E(\lambda) \subset \operatorname{int} D$.*

**Proof.** By the remarks preceding Proposition 2.1, we have for $x \in \operatorname{int} D$ that $V := \{h \in \mathcal{G} : hx \in \operatorname{int} D\}$ is open in $\mathcal{G}$. Since $x \in \operatorname{int} D$ there is for some $\tau > 0$, a $h \in \mathcal{S}_\tau \cap V$. By Proposition 2.2, there are $h_n \in \mathcal{S}_\tau \cap \operatorname{int} \mathcal{S}_{\leqslant \tau + 1}$ with $h_n \to h$. Since $V$ is open, we can find $n \in \mathbb{N}$ such that $h_n \in V$. Fixing this $h_n$ we can, by exact controllability in $\operatorname{int} D$ (Proposition 2.4), find $g \in \mathcal{S}_\sigma$ such that $g h_n x = x$. Clearly $g h_n \in \mathcal{S}_{\sigma + \tau} \cap \operatorname{int} \mathcal{S}_{\leqslant \sigma + \tau + 1}$. This proves that $x$ is in some eigenspace of some element of $\mathcal{S}_t \cap \operatorname{int} \mathcal{S}_{\leqslant t+1}$ with $t := \sigma + \tau$. ∎

**3.9. Remark.** For $d = 2$, a corresponding result is true for all $x \in D$, $D$ any control set on $\mathbb{P}^1$: If $x \in \partial D$, then there exists $u \in U$ such that $g_u x = x$, see Ref. 1. For $d > 2$, $\partial D$ will in general not consist only of projected eigenspaces of elements $g \in \mathcal{S}$.

Next we formulate the main result of this section.

***3.10. Theorem.*** *Assume that the semilinear control system (1.2) in $\mathbb{P}$ satisfies hypothesis (H). Then it has the following properties.*

(i) *There are $k$ control sets $D_i$ with nonvoid interior in $\mathbb{P}$ and $1 \leqslant k \leqslant d$; we call these control sets the main control sets of the system.*

(ii) *The main control sets are linearly ordered, where the order is defined by*

$$D_i < D_j \quad \textit{iff there exist} \quad x_i \in D_i,\ x_j \in D_j,\ t \geqslant 0,$$
$$\textit{and} \quad g \in \mathcal{S}_t \quad \textit{with} \quad g x_i = x_j$$

*We enumerate the control sets such that $D_1 < D_2 < \cdots < D_k$.*

(iii) *For every $t > 0$ and every $g \in \operatorname{int} \mathcal{S}_{\leqslant t}$ and every $\lambda \in \operatorname{spec} g$, there is a main control set $D_i$ such that the generalized eigenspace $E(\lambda)$ satisfies*

$$\mathbb{P}(E(\lambda)) \subset \operatorname{int} D_i$$

*the interior of the main control sets consists exactly of those elements $x \in \mathbb{P}$ which are eigenvectors for a (real) eigenvalue of some $g \in \mathcal{S}_t \cap \operatorname{int} \mathcal{S}_{\leqslant t+1}$ for some $t > 0$.*

(iv) *For every $g \in \mathcal{S}$ and every $\lambda \in \operatorname{spec} g$ there is some main control set $D_i$ with $\mathbb{P}E(\lambda) \cap \operatorname{cl} D_i \neq \varnothing$; for every main control set $D_i$ and every $g \in \mathcal{S}$ there is $\lambda \in \operatorname{spec} g$ with $\mathbb{P}E(\lambda) \cap \operatorname{cl} D_i \neq \varnothing$.*

(v) *The control set $C := D_k$ is closed and invariant and $C = \bigcap_{x \in \mathbb{P}} \operatorname{cl} \mathcal{O}^+(x)$; the control set $C^* := D_1$ is open and $\operatorname{cl} C^* = \bigcap_{x \in \mathbb{P}} \operatorname{cl} \mathcal{O}^-(x)$; all other main control sets are neither open nor closed.*

**Proof.** (i) By Proposition 3.7 there exists a control set with nonvoid interior. Hence $k \geqslant 1$. Next we show that the number $k$ of control sets $D$ with nonvoid interior is less or equal $d$. For every such $D$ there exist by Proposition 3.8 for some $t > 0$ an element $g \in \mathscr{S}_t \cap \operatorname{int} \mathscr{S}_{\leqslant t+1}$ and $\lambda \in \operatorname{spec} g$ such that $\mathbb{P}E(\lambda) \subset \operatorname{int} D$. Clearly, it suffices to prove that there is $h \in \mathscr{S}_T \cap \operatorname{int} \mathscr{S}_{\leqslant T+1}$ for some $T > 0$, such that for all control sets $D$ with nonvoid interior there is $\lambda \in \operatorname{spec} h$ with $\mathbb{P}E(\lambda) \subset \operatorname{int} D$.

Let $g_j \in \mathscr{S}_{t_j} \cap \operatorname{int} \mathscr{S}_{\leqslant t_j+1}$, $t_j > 0$, $j = 0, 1$ and let $\lambda \in \operatorname{spec} g_0$ be such that the corresponding eigenspace satisfies $\mathbb{P}E(\lambda) \subset \operatorname{int} D$ for some control set $D$ with nonvoid interior.

By Proposition 2.1 there is a continuous path $g(\sigma)$, $\sigma \in [0, 1]$, in $\operatorname{int} \mathscr{S}_{\leqslant t_0 + t_1 + 2}$ such that $g(0) = g_0$ and $g(1) = g_1$. Thus, by Proposition 2.5 there is $\sigma_0 > 0$ such that for all $\sigma \in [0, \sigma_0)$ there is an eigenvalue $\lambda(\sigma)$ of $g(\sigma)$ with corresponding generalized eigenspace $E(\lambda(\sigma))$ satisfying $\mathbb{P}E(\lambda(\sigma)) \cap \operatorname{int} D \neq \varnothing$. Proposition 3.7 and maximality of control sets imply $\mathbb{P}E(\lambda(\sigma)) \subset \operatorname{int} D$.

Now define

$$\bar{\sigma} := \sup \left\{ \begin{array}{c} \sigma_0 \colon \text{for all } \sigma \in [0, \sigma_0) \text{ there is a generalized eigenspace} \\ E(\lambda(\sigma)) \text{ of } g(\sigma) \text{ with } \mathbb{P}E(\lambda(\sigma)) \subset \operatorname{int} D \end{array} \right\}$$

Consider $\sigma_n \nearrow \bar{\sigma}$ and corresponding eigenvalues $\lambda(\sigma_n)$ with $E(\lambda(\sigma_n)) \subset \operatorname{int} D$. Since the spectral radius of $g(\sigma)$ is bounded, we may assume that $\lambda(\sigma_n)$ converges to some eigenvalue $\lambda(\bar{\sigma})$ of $g(\bar{\sigma})$. By Proposition 3.7, $E(\lambda(\bar{\sigma}))$ is contained in the interior of some control set $\tilde{D}$. By Proposition 2.5, $\mathbb{P}E(\lambda(\sigma_n)) \cap \tilde{D} \neq \varnothing$ for $n$ large enough. Hence $\tilde{D} = D$, and there is a neighborhood $N(\bar{\sigma})$ of $\bar{\sigma}$ such that for all $\tau \in N(\bar{\sigma})$, there is a generalized eigenspace of $g(\tau)$ in $\operatorname{int} D$. Thus $\bar{\sigma} = 1$.

We have found that $g_1$ has an eigenvalue $\lambda(1)$ such that the corresponding generalized eigenspace satisfies $\mathbb{P}E(\lambda(1)) \subset \operatorname{int} D$. This shows that $g_1$ has a generalized eigenspace in each control set with nonvoid interior and hence the number $k$ of main control sets satisfies $k \leqslant d$.

Actually, we have proved more than statement (i), namely, for each $g \in \operatorname{int} \mathscr{S}_{\leqslant t}$, $t > 0$, and each main control set $D_i$ there exists $\lambda \in \operatorname{spec} g$ such that $\mathbb{P}E(\lambda) \subset \operatorname{int} D_i$.

Turning to assertion (ii), note that in the proof of (i) we have constructed $g \in \operatorname{int} \mathscr{S}_{\leqslant t}$ such that each main control set contains a generalized eigenspace of $g$ in its interior. Some of these generalized eigenspaces may lie in the same main control set; certainly this is true for all generalized eigenspaces corresponding to eigenvalues of equal absolute value by Proposition 3.7. For the other generalized eigenspaces, one obtains a linear ordering according to the absolute values of the eigenvalues. This induces a linear ordering between the main control sets of the form

$$D_i \overset{\wedge}{<} D_j \text{ iff there exist } x_i \in \operatorname{int} D_i,\ x_j \in \operatorname{int} D_j,\ t > 0, \\ \text{and } g \in \mathcal{S}_t \cap \operatorname{int} \mathcal{S}_{\leqslant t+1} \text{ with } gx_i = x_j$$

That this defines the same order as the one in the statement of (ii), can be seen as follows: Recall that for each $x_j \in D_j$ there exists $\tau > 0$ and $h \in \operatorname{int} \mathcal{S}_{\leqslant \tau}$ such that $y := hx_j \in \operatorname{int} D_j$, hence if $D_i < D_j$, then there are $x_i \in D_i$, $y \in \operatorname{int} D_j$, $\sigma > 0$, and $g \in \operatorname{int} \mathcal{S}_{\leqslant \sigma}$ with $y = gx_i$. Since $g$ is a diffeomorphism, there exists a neighborhood $V$ of $x_i$ such that $gV \subset \operatorname{int} D_j$; in particular, there is $x \in \operatorname{int} D_i$ with $y = gx$. Now by maximality of the control sets, the order does not depend on the choice of $g$, which proves (ii).

(iii) follows directly from Propositions 3.7 and 3.8.

(iv) is a consequence of the proof of (i) and the approximation result in Proposition 2.6.

It remains to prove (v), by Ref. 2 (Lemma 3.1) there is for every $x \in \mathbb{P}$ an invariant control set $C_x \subset \operatorname{cl} \mathcal{O}^+(x)$. By Ref. 2 (Remark 3.2), each $C_x$ is compact with nonvoid interior. Now the linear ordering from (ii) between main control sets implies that all $C_x$ coincide proving the assertion for $C := C_x = D_k$. By Ref. 2 (Remark 3.1), $C$ is the only closed control set. Then the proof of the other assertions in (v) is immediate from Proposition 3.20 below. ■

**3.11. Remark.** The proof of (v) above greatly simplifies the proof of Ref. 2 (Theorem 3.1), which states uniqueness of the invariant control set.

Next we study the relation between main control sets and generalized eigenspaces more closely.

***3.12. Lemma.*** *Assume (H) and let D be a main control set. Then for every* $g \in \operatorname{int} \mathcal{S}_{\leqslant t}$, $t > 0$,

$$\mathbb{P}\left(\bigoplus_\lambda E(\lambda)\right) \subset \operatorname{int} D$$

*where the sum is taken over all* $\lambda \in \operatorname{spec} g$ *such that the corresponding generalized eigenspace* $E(\lambda)$ *satisfies* $\mathbb{P}E(\lambda) \subset \operatorname{int} D$.

**Proof.** Take $0 \neq x \in \bigoplus_\lambda E(\lambda)$ as above, and consider a representation of $x$ as

$$x = x_1 + \cdots + x_l$$

where the $x_j$ lie in the direct sum of the generalized eigenspaces corresponding to eigenvalues of equal absolute value. By Proposition 3.7, $g^m x_j \in \operatorname{int} D$ for all $m \in \mathbb{Z}$. Now, for $m \to \infty$, $g^m x$ is attracted by the eigenspaces corresponding to eigenvalues with maximal absolute value, and for $m \to -\infty$, $g^m x$ tends toward those eigenspaces corresponding to eigenvalues with minimal absolute value. Hence $x \in \operatorname{int} D$. ■

For $g \in \mathscr{S}$ and $D_i$ a main control set define, with $u_g$ as in (3.4)

$$D_i(u_g) = \{x \in \mathbb{P}: \varphi(t, x, u_g) \in \operatorname{cl} D_i \text{ for all } t \in \mathbb{R}\} \tag{3.7}$$

***3.13. Theorem.*** *Assume that (H) holds and let $g \in \operatorname{int} \mathscr{S}_{\leqslant t}$, $t > 0$, and $1 \leqslant i \leqslant k$. Then*

$$D_i(u_g) = \mathbb{P}\left(\bigoplus_{\lambda} E(\lambda)\right)$$

*where the sum is taken over all $\lambda \in \operatorname{spec} g$ with $\mathbb{P}E(\lambda) \subset \operatorname{int} D_i$.*

**Proof.** By Lemma 3.12

$$\mathbb{P}\left(\bigoplus_{\lambda} E(\lambda)\right) \subset D_i(u_g), \qquad \lambda \text{ as above}$$

Suppose that there is $x \in D_i(u_g) \backslash \mathbb{P}(\bigoplus E(\lambda))$. Since

$$\mathbb{R}^d = \bigoplus_{\lambda \in \operatorname{spec} g} E(\lambda)$$

$x$ can be represented as

$$x = \sum_{j=1}^{k} x_j$$

with $x_j \in \bigoplus E(\lambda)$, where the sum is taken over all $\lambda$ such that $\mathbb{P}(\bigoplus E(\lambda)) \subset \operatorname{int} D_j$, $j = 1,\ldots, k$. Suppose that there is $j > i$ with $x_j \neq 0$. Take $j$ maximal with this property. Then

$$\varphi(t, x, u_g) \to \mathbb{P}\left(\bigoplus_{\lambda} E(\lambda)\right)$$

where the sum is taken over all $\lambda$ with $\mathbb{P}(E(\lambda)) \subset D_j$. Then we have $\mathbb{P}(\bigoplus E(\lambda)) \subset \operatorname{int} D_j$, which contradicts $\varphi(\cdot, x, u_g) \subset \operatorname{cl} D_i$. Now assume that $x = \sum_{j=1}^{l} x_j$ with $l < i$. Then we arrive at a contradiction in a similar way. Thus $D_i(u_g) \subset \mathbb{P}(\bigoplus E(\lambda))$, with $\lambda$ as above. ■

**3.14. Remark.** The result above shows that for all $g \in \operatorname{int} \mathcal{S}_{\leqslant t}$, $t > 0$,

$$\mathbb{R}^d = \bigoplus_{i=1}^{k} \left\{ \bigoplus_{\lambda} E(\lambda): \begin{array}{l} \text{where the sum is taken over all} \\ \lambda \in \operatorname{spec} g \text{ with } \mathbb{P}E(\lambda) \subset \operatorname{cl} D_i \end{array} \right\}$$
$$= \bigoplus_{i=1}^{k} \mathbb{P}^{-1}[D_i(u_g)]$$

The first decomposition does not remain valid for general $g \in \mathcal{S}_t$, since it may happen that $\mathbb{P}E(\lambda) \cap D_i \neq \varnothing$ and $\mathbb{P}E(\lambda) \cap D_j \neq \varnothing$ for $i \neq j$, $\lambda \in \operatorname{spec} g$. The second decomposition is not valid for general $g \in \mathcal{S}_t$, since here the $D_i(u_g)$ may not be linear objects, cf. Example 5.8 for both cases.

Nevertheless, the following result is valid.

***3.15. Corollary.*** *Suppose that (1.1) satisfies Hypothesis (H), and let $g \in \mathcal{S}$. Then there exists a basis $\{x_1, ..., x_d\}$ of $\mathbb{R}^d$ such that for every $x_j$ there are $\lambda \in \operatorname{spec} g$ and a main control set $D_i$ with*

$$\mathbb{P}x_j \in \operatorname{cl} D_i \cap E(\lambda) \tag{3.8}$$

**Proof.** By Theorem 3.13, the assertion is true for $g \in \operatorname{int} \mathcal{S}_{\leqslant t}$, $t > 0$. Now approximate $g \in \mathcal{S}_t$ by $g_n \in \mathcal{S}_t \cap \operatorname{int} \mathcal{S}_{\leqslant t+1}$ according to Proposition 2.6. Then one finds for every $n \in \mathbb{N}$ a basis $\{x_1^n, ..., x_d^n\}$ with

$$\mathbb{P}x_j^n \in \mathbb{P}E(\lambda_\iota) \subset \operatorname{int} D_i \tag{3.9}$$

with $\lambda_\iota \in \operatorname{spec} g_n$, $\iota = \iota(j, n)$, $D_i$ some main control set. For $n \to \infty$, one obtains by Proposition 2.5 $\{x_1, ..., x_d\}$ with (3.8). Since (3.9) holds, formula (2.4) and Theorem 3.13 show that the $x_j^n$ may be chosen such that the $\{x_1, ..., x_d\}$ form a basis of $\mathbb{R}^d$. ■

**3.16. Remark.** According to Theorems 3.10 and 3.13 we can define for each $g \in \operatorname{int} \mathcal{S}_{\leqslant t}$, $t > 0$, the multiplicity of a main control set $D$:

$$m(D) := \#\{\lambda: \lambda \text{ is an eigenvalue of } g \text{ with } \mathbb{P}E(\lambda) \subset \operatorname{int} D\}$$

where each $\lambda$ is counted according to its multiplicity. By the proof of Theorem 3.10(i), the number $m(D)$ is independent of $g \in \operatorname{int} \mathcal{S}_{\leqslant t}$ for all $t > 0$. But Example 5.8 shows that for other elements $g \in \mathcal{S}$, the number $m(D)$ may not be the multiplicity of all eigenvalues with eigenspaces intersecting with $\operatorname{int} D$; compare Remark 3.14. It follows from Theorem 3.13 that $\sum_{i=1}^{k} m(D_i) = d$, and hence system (1.2) is completely controllable iff there exists a control set $D \subset \mathbb{P}$ with $m(D) = d$ or, equivalently, iff there exists a $g \in \operatorname{int} \mathcal{S}_{\leqslant t}$ for some $t > 0$ such that all generalized eigenspaces of $g$ are contained in one main control set.

We now analyze the behavior of the main control sets under time reversal and the fine structure of their boundary. The time-reversed system corresponding to (1.1) has the form

$$\begin{aligned} &\dot{x}^*(t) = -A(u(t))\, x^*(t), \qquad t \in \mathbb{R}, \qquad x(0) = x_0 \in \mathbb{R}^d \\ &u \in \mathcal{U} \end{aligned} \tag{3.10}$$

***3.17. Proposition.*** *Assume that (H) is satisfied. Then the time reversed system (3.10) has the same number k of main control sets as the forward system (1.1). The order among the main control sets is reversed, and for the corresponding main control sets one has*

$$\operatorname{int} D_i^* = \operatorname{int} D_{k+1-i} \qquad \textit{for} \quad i = 1, \ldots, k$$

*where $D^*$ denotes main control sets of (3.10).*

**Proof.** First observe that (H) remains true under time reversal. Furthermore, Proposition 2.4 shows that in the interior of control sets exact controllability holds. This proves the assertion. ■

**3.18. Definition.** Let $D$ be a main control set. Define the following subsets of $\partial D$:

$$\Gamma(D) := \{x \in \partial D\text{: there exist } y \in \operatorname{int} D \text{ and } g \in \mathcal{S} \text{ with } x = gy\}$$

$$\Gamma^*(D) := \{x \in \partial D\text{: there exist } y \in \operatorname{int} D \text{ and } g \in \mathcal{S} \text{ with } y = gx\}$$

$$\tilde{\Gamma}(D) := \{x \in \partial D\text{: } \mathcal{O}^+(x) \cap D = \varnothing \text{ and } \mathcal{O}^-(x) \cap D = \varnothing\}$$

These sets are called the exit, entrance, and tangential boundary, respectively.

By maximality of control sets, the three sets defined above form a decomposition of $\partial D$.

***3.19. Lemma.*** *Let D be a main control set.*

*(i) $\Gamma(D)$ and $\Gamma^*(D)$ are open in $\partial D$, $\tilde{\Gamma}(D)$ is closed in $\partial D$.*

*(ii) $\tilde{\Gamma}(D) \subset \operatorname{cl} \Gamma(D) \cap \operatorname{cl} \Gamma^*(D)$; in particular, $\operatorname{int}_{\partial D} \tilde{\Gamma}(D) = \varnothing$.*

**Proof.**

(i) Since $g \in \mathcal{S}$ is a diffeomorphism on $\mathbb{P}$, $\Gamma(D)$, and $\Gamma^*$ $(D)$ are open in $\partial D$. This implies closedness of $\tilde{\Gamma}(D)$ by the decomposition property.

(ii) Let $x \in \tilde{\Gamma}(D)$ and pick $y \in \operatorname{int} \mathcal{O}^+_{\leqslant t}(x)$. Then there exists $g \in \operatorname{int} \mathcal{S}_{\leqslant t}$ and a neighborhood $V$ of $x$ such that $gV \subset \operatorname{int} O^+_{\leqslant t}(x)$. Since $V$ contains points in $\operatorname{int} D$, and $\mathcal{O}^+(x) \cap D = \varnothing$, we see that $x \in \operatorname{cl} \Gamma(D)$. A similar argument holds for $\mathcal{O}^-_{\leqslant t}(x)$, showing that $x \in \operatorname{cl} \Gamma^*(D)$. ■

The next proposition clarifies the behavior of the boundary under time reversal [and hence completes the proof of Theorem 3.10(v)].

***3.20. Proposition.*** *Let $D_i$ be a main control set. Then for $x \in \partial D_i$ we have*

*(i)* $x \in \Gamma^*(D_i)$ *iff* $x \in D_i$.

*(ii)* $x \in \Gamma(D_i)$ *iff* $x \in D^*_{k+1-i}$.

*(iii)* $x \in \tilde{\Gamma}(D_i)$ *iff* $x \notin D_i \cup D^*_{k+1-i}$.

**Proof.** (i) follows directly from the definition. (ii) is just the fact that $\mathcal{O}^-(x)$ are the forward orbits of the time reversed system. And (iii) follows from (i) and (ii) because of the decomposition property. ■

We note the following invariance properties of main control sets.

***3.21. Proposition.*** *Let $D$ be a main control set.*

*(i) For all $u \in \mathcal{U}$ there exists $x \in D$ with $\varphi(x, u) := \{\varphi(t, x, u) : t \in \mathbb{R}\} \subset \operatorname{cl} D$.*

*(ii) For all $x \in \operatorname{cl} D$ there exists $u \in \mathcal{U}$ with $\varphi(x, u) \subset \operatorname{cl} D$.*

**Proof.**

(i) The assertion is true for all $u_g$, $g \in \operatorname{int} \mathcal{S}_{\leqslant t}$. Hence it follows from density of these controls in $\mathcal{U}$ (cf. Ref. 9).

(ii) The assertion is valid for all $x \in \operatorname{int} D$ and hence by compactness of $\mathcal{U}$ for all $x \in \operatorname{cl} D$. ■

## 4. CHAIN CONTROL SETS AND MORSE DECOMPOSITIONS OF LINEARIZED FLOWS

In this section we first recall some notions and results from the theory of flows on vector bundles, suitable for our purposes. Then we will consider nonlinear control systems on a manifold $M$ and show that the linearized system defines a linear flow on the tangent bundle $TM$ and an induced flow on the projective bundle $\mathbb{P}M$. For these flows we obtain a finest Morse decomposition, which is related to the so-called chain control sets on $\mathbb{P}M$.

Chain control sets were introduced in Ref. 9, where they were used to characterize certain ergodic properties of nonlinear control systems. Here the emphasis is on the linearization of control systems on $TM$, and on $\mathbb{P}M$, if the control system itself is already chain recurrent, i.e., has only one chain control set. In Section 5 this theory is applied to the analysis of linear control semigroups acting on projective spaces $\mathbb{P}^{d-1}$.

For the following notions and results see, e.g., Refs. 13, 23, and 25. Let $M$ be a compact, metric space and $\pi: E \to M$ a vector bundle over $M$, whose fibers are finite-dimensional, real vectorspaces. A continuous map $\psi: \mathbb{R} \times E \to E$ is called a flow, if $\psi(t+s, e) = \psi(t, \psi(s, e))$ and $\psi(0, e) = e$ for all $e \in E$. $\psi$ is a linear flow on the vector bundle $\pi$, if for $e$, $e' \in E$ with $\pi(e) = \pi(e')$, one has $\pi\psi(t, e) = \pi\psi(t, e')$, $\psi(t, e) + \psi(t, e') = \psi(t, e+e')$, $\lambda\psi(t, e) = \psi(t, \lambda e)$ for all $t \in \mathbb{R}$ and $\lambda \in \mathbb{R}$.

Let $\{U_\alpha, \alpha \in I\}$ be a finite covering of $M$ with associated homeomorphisms

$$\varphi_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times V$$

defining the vector bundle structure on $E$; then the zero section of $E$ is defined by $Z := \{e \in E: \varphi_\alpha(e) = (\pi(e), 0)$ whenever $\pi(e) \in U_\alpha\}$. Note that the zero section $Z$ is homeomorphic to $M$. The linear flow $\Psi$ on $E$ induces flows on the base space $M$, the zero section $Z$, and the projective bundle $\mathbb{P}E$.

Now let $\Theta: \mathbb{R} \times S \to S$ be a (continuous) flow on a compact metric space $S$. The limit sets of $Y \subset S$ are $\omega(Y) := \bigcap_{t \geq 0} \mathrm{cl}\{\Theta(s, Y), s \geq t\}$ and $\omega^*(Y) := \bigcap_{t \leq 0} \mathrm{cl}\{\Theta(s, Y), s \leq t\}$. A set $K \subset S$ is an isolated invariant set, if there exists a compact neighborhood $N$ of $K$, such that $\{\Theta(t, x): t \in \mathbb{R}\} \subset N$ implies $x \in K$.

A compact invariant set $A$ is said to be an attractor, if it admits a neighborhood $N$ such that $\omega(N) = A$. In this case the set $A^* = \{x \in S: \omega(x) \cap A = \varnothing\}$ is its complementary repeller, i.e., there exists a neighborhood $N^*$ of $A^*$ with $\omega^*(N^*) = A^*$, and $\omega^*(x) \subset A^*$, $\omega(x) \subset A$ for all $x \notin A \cup A^*$; see, e.g., Refs. 13 and 22. The pair $(A, A^*)$ is called an attractor-repeller pair.

**4.1. Definition.** For $\varepsilon > 0$ and $T > 0$ an $(\varepsilon, T)$-chain from $x \in S$ to $y \in S$ consists of a sequence $x_0, \ldots, x_n$ in $S$ and a sequence $t_0, \ldots, t_{k-1}$ in $\mathbb{R}$ such that $t_j \geq T$, $x_0 = x$, $x_k = y$, and $d(\Theta(t_j, x_j), x_{j+1}) \leq \varepsilon$ for $j = 0, \ldots, k-1$, where $d(\cdot, \cdot)$ is the metric on $S$. For $X \subset S$ define

$$\Omega(X) := \{y \in S: \text{for all } \varepsilon > 0, \text{ all } T > 0 \text{ there exists } x \in X \text{ and an } (\varepsilon, T)\text{-chain from } x \text{ to } y\}$$

$$\Omega^*(X) := \{y \in S: \text{for all } \varepsilon > 0, \text{ all } T > 0 \text{ there exists } x \in X \text{ and an } (\varepsilon, T)\text{-chain from } y \text{ to } x\}$$

By Ref. 13 (II 4.1.C),

$$\Omega(X) = \bigcap \{A\colon A \text{ attractor}, \omega(X) \subset A\}$$

$$\Omega^*(X) = \bigcap \{A^*\colon A^* \text{ repeller}, \omega^*(X) \subset A^*\}$$

The dynamical system on $S$ is called chain recurrent, if $x \in \Omega(x)$ for all $x \in S$ or, equivalently, $S = A \cup A^*$ for every attractor-repeller pair. It is called chain transitive, if $y \in \Omega(x)$ for all $x, y \in S$ or, equivalently, if $A = S$ and $A = \varnothing$ are the only attractors in $S$. Note that the flow on $S$ is chain transitive iff it is chain recurrent and $S$ is connected.

By Ref. 13(II 6.2) the chain recurrent set $\mathcal{CR}$, i.e., the maximal chain recurrent subset of $S$, is given by

$$\mathcal{CR} = \bigcap \{A \cup A^*\colon A \text{ is an attractor}\}.$$

***4.2. Proposition.*** *Let $A$ be an attractor in $\mathbb{P}E$ for a linear flow $\Psi$ on a vector bundle $E$ with base space $M$ and suppose that the induced flow on $M$ is chain transitive. Then $\{e \in E\colon e \notin Z \Rightarrow \mathbb{P}e \in A\}$ is a subbundle of $E$.*

**Proof.** It suffices to show (cf. (Ref. 23, Lemma A2) that $\mathbb{P}^{-1} A$ is a closed subset of $E$ which intersects each fiber in a linear subspace $F_p$, $p \in M$, and $\dim F_p = \dim F_q$ for all $p, q \in M$. Closedness is clear by definition of $A$; by Ref. 23, (Corollary 2.11), for all $p \in M$

$$A_p := \{e \in E_p\colon e \notin Z \Rightarrow \mathbb{P}e \in A\}$$

is a linear subspace of $E_p$ with

$$\dim A_q \geqslant \dim A_p \qquad \text{for all} \qquad q \in \Omega(p)$$

Hence chain transitivity implies that $\dim A_p$ is constant for $p \in M$. ■

A *Morse set* is the intersection of an attractor and a repeller. A *Morse decomposition* of a flow $\Theta$ on $S$ is given by the following: Let

$$\varnothing = A_0 \subset A_1 \subset \cdots \subset A_n = S$$

be an increasing sequence of attractors and define

$$\mathcal{M}_j := A_j \cap A^*_{j-1}, \qquad j = 1,..., n$$

Then $\{\mathcal{M}_1,..., \mathcal{M}_n\}$ is a Morse decomposition of $(S, \Theta)$. Morse decompositions have the following properties; see Ref. 13, (II.7.1 and 2).

**4.3. *Proposition.*** *Let $\{\mathcal{M}_1,..., \mathcal{M}_n\}$ be a Morse decomposition.*

(i) $\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_n = \bigcap_{i=1}^n (A_i \cup A_i^*)$, *hence* $\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_n \supset \mathscr{CR}$.

(ii) *For all* $x \in S$ *there are* $i$, $j$ *with* $i \leqslant j$ *such that* $\omega(x) \subset \mathcal{M}_i$, $\omega^*(x) \subset \mathcal{M}_j$; *if* $i = j$, *then* $x \in \mathcal{M}_i = \mathcal{M}_j$.

(iii) *Assume that* $\mathcal{M}'_1,..., \mathcal{M}'_m$ *is a collection of disjoint invariant sets in* $S$. *Suppose that for all* $x \in S$ *there are* $i$, $j$ *with* $i \leqslant j$ *such that* $\omega(x) \subset \mathcal{M}'_i$, $\omega^*(x) \subset \mathcal{M}'_j$, *and, if* $i = j$, *then* $x \in \mathcal{M}'_i = \mathcal{M}'_j$. *Then* $\{M'_1,..., \mathcal{M}'_m\}$ *is a Morse decomposition.*

(iv) *Let also* $\{\mathcal{M}'_1,..., \mathcal{M}'_m\}$ *be a Morse decomposition, and define* $\mathcal{M}_{ij} = \mathcal{M}_i \cap \mathcal{M}'_j$, $i = 1,..., n$, $j = 1,..., m$. *Any ordering of the* $\mathcal{M}_{ij}$ *with the property that, if* $\mathcal{M}_{ij}$ *comes before* $\mathcal{M}_{kl}$, *then either* $i < k$ *or* $j < l$, *is a Morse decomposition (of course, several of the* $\mathcal{M}_{ij}$ *may be empty).*

**4.4. Definition.** Let $\{\mathcal{M}_1,..., \mathcal{M}_n\}$ and $\{\mathcal{M}'_1,..., \mathcal{M}'_m\}$ be Morse decompositions. Then the first one is finer than the second one if for every $\mathcal{M}_i$ there is $\mathcal{M}'_j$ with $\mathcal{M}_i \subset M'_j$. A finest Morse decomposition is a Morse decomposition which is finer than every other Morse decomposition.

**4.5. *Theorem.*** *Consider a linear flow* $\Psi$ *on a vector bundle* $E$ *with base space* $M$ *and suppose that the induced flow on* $M$ *is chain transitive. Then there exists a (unique) finest Morse decomposition* $\{\mathcal{M}_1,..., \mathcal{M}_l\}$ *of* $\mathbb{P}\Psi$, *the induced flow on the projective bundle* $\mathbb{P}E$, *and* $l \leqslant d := \dim E_p$, $p \in M$; *every* $\mathcal{M}_i$ *defines a subbundle of* $E$ *via*

$$\mathcal{V}_i = \{e : e \notin Z \Rightarrow \mathbb{P}e \in \mathcal{M}_i\}$$

*and the following decomposition into a Whitney sum holds:*

$$E = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_l$$

**Proof.** Note first that there always is a Morse decomposition of $\mathbb{P}\Psi$: Define $A_0 = \emptyset$, $A_1 = \mathbb{P}E$ and $\mathcal{M}_1 = A_1 \cap A_0^*$; then a Morse decomposition is given by $\{\mathcal{M}_1\}$. Next we claim that for every Morse decomposition $\{\mathcal{M}_1,..., \mathcal{M}_n\}$ corresponding to an attractor sequence $\emptyset = A_0 \subset A_1 \subset \cdots \subset A_n = \mathbb{P}E$, all $\mathbb{P}^{-1}\mathcal{M}_i = \mathbb{P}\mathcal{M}_i = \mathbb{P}^{-1}A_i \cap \mathbb{P}^{-1}A_{i-1}^*$ define subbundles. For $n = 1$, this is obviously true. So we assume that the assertion is true for $n - 1$ and prove it for $n$. Clearly $\mathcal{M}_1 = A_1$ is an attractor, and hence by Proposition 4.2 $A_1$ and $A_1^*$ are subbundles. We show that the assertion is true for $\{A_1, A_1^*\}$ and that $\{\mathcal{M}_2,..., \mathcal{M}_n\}$ is a Morse decomposition of $A_1^*$. The second claim follows from $\mathcal{M}_i \subset A_1^*$ for $i \geqslant 2$ and Proposition 4.3(iii).

For the proof of the first claim denote $A := A_1$, choose $p \in M$, and assume that the corresponding fibres of $A$ and $A^*$ have dimensions $r$ and $s$, respectively. Since $A$ and $A^*$ are disjoint, it suffices to prove that $r+s \geqslant \dim E =: k$.

Fix $p \in M$ and consider a subspace $F_p^{k-s}$ of $E_p$ complementary to $A_p^*$. By the definition of $A^*$, $\omega(\mathbb{P}F_p^{k-s}) \subset A$. Since $A$ is an attractor, also $\Omega(\mathbb{P}F_p^{k-s}) \subset A$. Now chain recurrence and Ref. 23, (Corollary 2.11) show that for each $q \in M$, the set $\Omega(F_q^{k-s})$ meets $\mathbb{P}E_q$ in a subspace of dimension at least $k-s$. Therefore $r \geqslant k-s$.

Now let $\{\mathcal{M}_1,..., \mathcal{M}_n\}$ and $\{\bar{\mathcal{M}}_1,..., \bar{\mathcal{M}}_m\}$ be two Morse decompositions corresponding to the attractor sequences $\varnothing = A_0 \subset \cdots \subset A_n = \mathbb{P}E$ and $\varnothing = \bar{A}_0 \subset \cdots \subset \bar{A}_m = \mathbb{P}E$. By Proposition 4.2, all $\mathbb{P}^{-1}A_i$, $\mathbb{P}^{-1}\bar{A}_j$ are subbundles of $E$; hence by a dimension argument $n, m \leqslant d$.

By the result proven above $\mathbb{P}^{-1}\mathcal{M}_i = \mathbb{P}^{-1}A_i \cap \mathbb{P}^{-1}A_{i-1}^*$ and $\mathbb{P}^{-1}\bar{\mathcal{M}}_j = \mathbb{P}^{-1}\bar{A}_j \cap \mathbb{P}^{-1}\bar{A}_{j-1}^*$ are subbundles. By Proposition 4.3(iv), $\mathcal{M}_{ij} = \mathcal{M}_i \cap \bar{\mathcal{M}}_j$ defines a Morse decomposition. This shows that a refinement of Morse decompositions of $(\mathbb{P}E, \mathbb{P}\Psi)$ leads to finer subbundle decompositions of $E$. Therefore, again by a dimension argument, there exists a (unique) finest Morse decomposition $\{\mathcal{M}_1,..., \mathcal{M}_l\}$ of $\mathbb{P}\Psi$, with $l \leqslant d$. ∎

We now return to control systems and consider the following class of nonlinear systems on a para-compact $d$-dimensional manifold $M$, with constrained control:

$$\begin{aligned} &\dot{x}(t) := X_0(x(t)) + \sum_{i=1}^{m} u_i(t)\, X_i(x(t)), \quad t \in \mathbb{R}, \quad x(0) = x_0 \in M \\ &u := (u_i) \in \mathcal{U} := \{u \colon \mathbb{R} \to U \colon \text{measurable}\} \end{aligned} \tag{4.1}$$

where $U \subset \mathbb{R}^m$ is compact and convex, and $X_0,..., X_m$ are $C^\infty$ vectorfields on $M$. We assume that for all $u \in \mathcal{U}$ and $x_0 \in M$, Eq. (4.1) has a unique solution $\varphi(\cdot, x_0, u)$, defined on $\mathbb{R}$. Some aspects of control systems as dynamical systems were treated in Ref. 9. We recall some definitions and facts from this paper.

Associated with the control system (4.1) is a control flow, i.e., a dynamical system $\Phi$ on $\mathcal{U} \times M$ defined by

$$\Phi(t, u, x_0) := (u(t + \cdot), \varphi(t, x_0, u)), \qquad t \in \mathbb{R}$$

Here $\mathcal{U} \subset L^\infty(\mathbb{R}, \mathbb{R}^m)$ is considered as a compact, metric space with topology given by the induced weak*-topology of $L^\infty(\mathbb{R}, \mathbb{R}^m) = (L^1(\mathbb{R}, \mathbb{R}^m))^*$.

The control theoretic meaning of chain recurrence is expressed by chain control sets.

**4.6. Definition.** A set $E \subset M$ is called a chain control set of the system (4.1) if

(i) for all $x, y \in E$ and all $\varepsilon, T > 0$, there are $k \in \mathbb{N}$, $x = x_0, ..., x_k = y \in M$, $u_0, ..., u_{k-1} \in \mathcal{U}$ and $t_0, ..., t_{k-1} \geqslant T$ with

$$d(\varphi(t_i, x_i, u_i), x_{i+1}) < \varepsilon \qquad \text{for} \qquad i = 0, ..., k-1 \tag{4.2}$$

(ii) for every $x \in E$ there is $u \in \mathcal{U}$ such that $\varphi(t, x, u) \in E$ for all $t \in \mathbb{R}$, and

(iii) $E$ is maximal with properties (i) and (ii).

Here $d$ denotes some Riemannian metric on $M$ (which will be fixed in the sequel). The following result was proved in Ref. 9.

***4.7. Theorem.***

*(i) Let E be a chain control set for (4.1). Then the set*

$$\mathscr{E} := \{(y, u) \in M \times \mathcal{U} \colon \varphi(t, y, u) \in E \text{ for all } t \in \mathbb{R}\} \tag{4.3}$$

*is a maximal invariant chain transitive set for the induced dynamical system on $\mathcal{U} \times M$.*

*(ii) Let $\mathscr{E}$ be a maximal invariant chain transitive set in $\mathcal{U} \times M$. Then*

$$\pi_M \mathscr{E} := \{y \in M \colon \text{there is } u \in \mathcal{U} \text{ with } (u, y) \in \mathscr{E}\} \tag{4.4}$$

*is a chain control set.*

In this paper, we are interested in linear control flows on vectors bundles, i.e., in linearized control systems. Linearizing system (4.1) with respect to the state variable $x$, we obtain a system defined on the tangent bundle $TM$:

$$\begin{aligned} (\dot{Tx})(t) &= TX_0(Tx) + \sum_{i=1}^{m} u_i(t)\, TX_i(Tx), \qquad t \in \mathbb{R} \\ (Tx)(0) &= (x_0, v_0) \in T_{x_0}M, \text{ the tangent space at } x_0 \in M \\ u &:= (u_i) \in \mathcal{U} \end{aligned} \tag{4.5}$$

where for a vector field $X$ on $M$ its linearization is denoted $TX = (X, DX)$. Locally this means: If $X_j = \sum_{k=1}^{d} \alpha_{kj}(x)(\partial/\partial x_k)$, denote the Jacobian of the coefficient functions by

$$A_j(x) = \left(\frac{\partial \alpha_{kj}(x)}{\partial x_l}\right)$$

Then $TX_j(x, v) = (\alpha_j(x), A_j(x)\, v)$, and (4.5) is a pair of coupled differential equations, given locally by

$$\dot{x}(t) = \alpha_0(x) + \sum_{i=1}^{m} u_i(t)\, \alpha_i(x), \qquad x(0) = x_0$$

$$\dot{v}(t) = A_0(x)\, v + \sum_{i=1}^{m} u_i(t)\, A_i(x)\, v, \qquad v(0) = v_0$$

Note that if $x(t) \equiv x_0 \in M$ is a rest point of each $X_i$, then the linearized equation is a bilinear control system, as a special case of the set up in Section 1.

System (4.5) induces a control system on the projective bundle $\mathbb{P}M$, given by

$$\begin{aligned} (\dot{\mathbb{P}}x)(t) &= \mathbb{P}X_0(\mathbb{P}x) + \sum_{i=1}^{m} u_i(t)\, \mathbb{P}X_i(\mathbb{P}x), \qquad t \in \mathbb{R} \\ (\mathbb{P}x)(0) &= (x_0, s_0) \in \mathbb{P}_{x_0}M, \qquad \text{the projective space at } x_0 \in M \\ u &:= (u_i) \in \mathcal{U} \end{aligned} \tag{4.6}$$

where $\mathbb{P}x$ is the projection of a vectorfield $TX$ on $TM$ onto $\mathbb{P}M$, i.e., the $\mathbb{P}X_j$ read locally $\mathbb{P}X_j(x, s) = (\alpha_j(x), h(A_j(x), s))$ with $h(A_j(x), s) = [A_j(x) - s^T A_j(x)\, s \cdot \mathrm{id}]\, s$. The trajectories of control system (4.5) will be denoted $T\varphi(t, Tx, u)$, $t \in \mathbb{R}$, and those of (4.6) by $\mathbb{P}\varphi(t, \mathbb{P}x, u)$.

Associated with (4.5) is the control flow of the linearized system:

$$T\Phi\colon \mathbb{R} \times \mathcal{U} \times TM \to \mathcal{U} \times TM, \;\; T\Phi(t, u, Tx) = (u(t + \cdot), T\varphi(t, Tx, u))$$

which defines a linear flow on the vector bundle $\mathcal{U} \times TM$ with base space $\mathcal{U} \times M$. Similarly, $\mathbb{P}\Phi$ will denote the control flow on $U \times \mathbb{P}M$.

If the state space $M$ of control system (4.1) is compact, we can apply Theorem 4.5 to the linearized control flow and obtain:

**4.8. Corollary.** *Assume that $M$ is compact and that the control flow $\Phi$ on $\mathcal{U} \times M$ is chain transitive. Then the induced dynamical system $\mathbb{P}\Phi$ on $\mathcal{U} \times \mathbb{P}M$ admits a unique finest Morse decomposition $\{\mathcal{M}_1, ..., \mathcal{M}_l\}$, where $1 \leqslant l \leqslant d = \dim M$. For $i = 1, ..., l$,*

$$\begin{aligned} \mathcal{E}_i := \{&(u, Tx) \in \mathcal{U} \times TM\colon (u, Tx) \notin Z \\ &\Rightarrow (u(t + \cdot), \mathbb{P}\varphi(t, Tx, u)) \in \mathcal{M}_i \textit{ for all } t \in \mathbb{R}\} \end{aligned}$$

*are invariant subbundles of $\mathcal{U} \times TM$ and*

$$\mathcal{U} \times TM = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_l$$

**Proof.** The flow $T\Phi$ is a linear flow on the vector bundle $\mathcal{U} \times TM$, the induced flow on $\mathcal{U} \times M$ is $\Phi$. From Theorem 4.5 we obtain that for $i = 1,..., l$, the sets

$$\mathcal{E}_i = \{(u, Tx) \in \mathcal{U} \times TM: (u, Tx) \notin Z \Rightarrow (u, \mathbb{P}x) \in \mathcal{M}_i\}$$

are subbundles of $\mathcal{U} \times TM$ with $\mathcal{U} \times TM = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_l$. Now note that Morse sets are flow invariant, i.e., here the $\mathcal{M}_i$ are $\mathbb{P}\phi$ invariant, which shows that the $\mathcal{E}_i$ are of the form stated in the formulation of the corollary. ■

The main result of this section is the following characterization of Morse decompositions and chain control sets of the linearized control flow.

***4.9. Theorem.*** *Assume that $M$ is compact and that $M$ is the chain control set of (4.1). Then the induced control system on $\mathbb{P}M$ has $l$ chain control sets $E_1,..., E_l$, with $1 \leqslant l \leqslant d$. The chain control sets are compact, pairwise disjoint, connected, and linearly ordered by*

$$E_i < E_j \quad \textit{if there is} \quad (u, \mathbb{P}x) \in \mathcal{U} \times \mathbb{P}M$$
$$\textit{with} \quad \omega^*(u, \mathbb{P}x) \subset E_i \quad \textit{and} \quad \omega(u, \mathbb{P}x) \subset E_j$$

*We enumerate the chain control sets such that $E_i < E_j$ iff $i < j$.*

*Define the lift of a chain control set $E \subset \mathbb{P}M$ to $\mathcal{U} \times TM$ by*

$$\mathcal{E} := \{(u, Tx) \in \mathcal{U} \times TM: (u, Tx) \notin Z \Rightarrow \mathbb{P}\varphi(t, Tx, u) \in E \textit{ for all } t \in \mathbb{R}\} \tag{4.7}$$

*Then the $\mathcal{E}_i$'s are invariant subbundles of $\mathcal{U} \times TM$ with*

$$\mathcal{U} \times TM = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_l$$

*and $\{\mathbb{P}\mathcal{E}_1,..., \mathbb{P}\mathcal{E}_l\}$ is the (unique) finest Morse decomposition of the flow $(\mathcal{U} \times \mathbb{P}M, \mathbb{P}\phi)$, where $\mathbb{P}\mathcal{E}_i$ is the projection of $\mathcal{E}_i \subset \mathcal{U} \times TM$ onto $\mathcal{U} \times \mathbb{P}M$.*

**Proof.** Since $M$ is a compact chain control set, Theorem 4.7 implies that the corresponding dynamical system $\Phi$ on $\mathcal{U} \times M$ is chain transitive. Hence the assumptions of Corollary 4.8 are met and the induced flow on $\mathcal{U} \times \mathbb{P}M$ admits a unique finest Morse decomposition. This is a decomposition of the chain recurrent set $\mathcal{CR}$ into its connected components, which are chain transitive. Hence by Theorem 4.7 again, these components uniquely correspond to the chain control sets of the induced control system on $\mathbb{P}M$, and their relation is given by (4.7). It remains to show that the chain control sets $E_i$ are compact, pairwise disjoint and connected. But this is Lemma 4.8 in Ref. 9. ■

## 5. CHAIN CONTROL SETS AND CONTROL SETS OF LINEAR CONTROL SEMIGROUPS

In this section we analyze a class of bilinear control systems, for which the results from Sections 3 and 4 are applicable. We characterize the chain control sets on projective spaces, and their relations to the main control sets.

Consider the system

$$\begin{aligned} &\dot{x}(t) = u(t)\, x(t), \qquad t \in \mathbb{R}, \qquad x(0) = x_0 \in \mathbb{R}^d \backslash \{0\} \\ &u \in \mathcal{U} := \{u\colon \mathbb{R} \to \mathrm{gl}(d, \mathbb{R})\colon u \text{ measurable and } u(t) \in U \text{ a.e.}\} \end{aligned} \tag{5.1}$$

where $U \subset \mathrm{gl}(d, \mathbb{R})$ is compact and convex.

Clearly (5.1) is a special case of systems (1.1) and (4.5). Note that, in particular, the bilinear control systems $\dot{x} = A_0 x + \sum u_i A_i x$ with a compact and convex set of control values fit into the framework above.

**5.1. Remark.** Every control system of the form (1.1) with bounded set $\{A(u)\colon u \in U\}$ can be embedded into a control system of the form (5.1): Define

$$\hat{U} := \mathrm{cl}\,\mathrm{co}\{A(u)\colon u \in U\} \subset \mathrm{gl}(d, \mathbb{R})$$

and

$$\hat{\mathcal{U}} := \{\hat{u} \in L^\infty(\mathbb{R}, \mathrm{gl}(d, \mathbb{R}))\colon \hat{u}(t) \in \hat{U} \text{ a.e.}\}$$

Then the system

$$\begin{aligned} &\dot{x}(t) = \hat{u}(t)\, x(t), \qquad x(0) = x_0 \in \mathbb{R}^d \\ &\hat{u} \in \hat{\mathcal{U}} \end{aligned} \tag{5.2}$$

can be viewed as a completion of the original system: In general, trajectories $x_k(\cdot)$ of the original system converging uniformly on bounded intervals will not converge to a trajectory of the original system, but to a trajectory of the system (5.2), which is the corresponding relaxed system (cf. e.g., Ref. 27). It is interesting to note that the numbers of the corresponding main control sets $D_i$ and $\hat{D}_i$ coincide and that int $D_i =$ int $\hat{D}_i$ for $i = 1,..., k$. This follows from exact controllability in the interior of control sets.

The control system (5.1) induces a control system on the projective space $\mathbb{P} := \mathbb{P}^{d-1}$ [cf. Eq. (1.2)], and also, according to the discussion in Section 4, control flows on $\mathcal{U} \times \mathbb{R}^d$ and on $\mathcal{U} \times \mathbb{P}$. From Theorems 3.10 and

4.9 we obtain competing control structures for the system on $\mathbb{P}$: one corresponding to the main control sets $D_1,..., D_k$ [provided (5.1) satisfies the relevant Lie algebra condition] and one corresponding to the chain control sets $E_1,..., E_l$. In order to clarify the relation between these two structures, we first characterize the main control sets via a dynamical systems property.

**5.2. Theorem.** *Assume that system (5.1) satisfies hypothesis (H).*

*(i) The control system induced by (5.1) on $\mathbb{P}$ has exactly $1 \leqslant k \leqslant d$ control sets with nonvoid interior $D_1,..., D_k$. These main control sets are linearly ordered and have the properties indicated in Theorem 3.10.*

*(ii) The dynamical system $\mathbb{P}\Phi$ induced by (5.1) on $\mathcal{U} \times \mathbb{P}$ has exactly $1 \leqslant k \leqslant d$ maximal topologically transitive components $\mathcal{D}_1,..., \mathcal{D}_k$ with* int $\pi_{\mathbb{P}} \mathcal{D}_i \neq \emptyset$, $i = 1,..., k$. *These main topologically transitive components are topologically mixing and*

$$\pi_{\mathbb{P}} \mathcal{D}_i = \text{cl } D_i, \qquad i = 1,..., k$$

Here $\pi_{\mathbb{P}}$ denotes the projection of $\mathcal{U} \times \mathbb{P}$ onto the second component.

**Proof.** Assertion (i) follows from Theorem 3.10. Assertion (ii) is a consequence of (i) and Theorem 3.8 in Ref. 9. ■

**5.3. Remark.** Theorem 5.2 shows that the number of maximal topologically transitive components $\mathcal{D}_i$ with int $\pi_{\mathbb{P}} \mathcal{D}_i \neq \emptyset$ is finite and bounded by $d$. This result may be viewed as an analogue of Smale's decomposition of the set of nonwandering points of Axiom A Flows into finitely many maximal topologically transitive sets (cf. Ref. 26). The situation considered in the theorem above arises from a semigroup (the systems semigroup $\mathcal{S}$) in a Lie group (the systems group $\mathcal{G}$) acting by Hypothesis (H) transitively on $\mathbb{P}$.

**5.4. Remark.** Consider for $i = 1,..., k$ the following subsets of $\mathcal{U} \times \mathbb{R}^d$:

$$\{(u, x) \in \mathcal{U} \times \mathbb{R}^d : x \neq 0 \Rightarrow \mathbb{P}\varphi(t, x, u) \in \text{cl } D_i \text{ for all } t \in \mathbb{R}\}$$

Example 5.8, below, shows that these sets need not be subbundles of $\mathcal{U} \times \mathbb{R}^d$. Hence, in general, the (closures of the) main control sets $D_i$, $i = 1,..., k$, will not coincide with the chain control sets.

Concerning the relation between control sets and chain control sets, one can say in general that several control sets, even with nonvoid interior, can be contained in one chain control set and that chain control sets may contain points, which are in no control set; see, e.g., Examples 4.11 and 4.12 in Ref. 9. [Note that these examples are projections of systems of type (5.1), i.e., they fit into the present framework.] Obviously, every main control set lies in some chain control set. A converse of this property holds as well, as the following result shows.

**5.5. Theorem.** *Suppose that (5.1) satisfies Hypothesis (H). Then every chain control set $E_j$ contains a main control set; in particular, $1 \leqslant l \leqslant k \leqslant d$ and $\operatorname{int} E_j \neq \varnothing$ for all $j = 1, \ldots, l$.*

**Proof.** By Theorem 4.9, for every $u \in \mathscr{U}$ one has the decomposition

$$\mathbb{R}^d = \mathscr{E}_1(u) \oplus \cdots \oplus \mathscr{E}_l(u)$$

with $\mathscr{E}_j(u) = \{x \in \mathbb{R}^d: x \neq 0 \Rightarrow (u, x) \in \mathscr{E}_j\}$. Take $u = u_g$ with $g \in \operatorname{int} \mathscr{S}_{\leqslant t}$, for some $t > 0$. Then one has a corresponding decomposition of $\mathbb{R}^d$ into generalized eigenspaces of $g$ and the sums of generalized eigenspaces corresponding to eigenvalues of equal absolute value lie in some main control set. By the remarks above, every main control set lies in some chain control set. Hence, by a dimension argument, every $\mathscr{E}_j(u_g)$ must have nonvoid intersection with some main control set. Thus every $E_j$ has nonvoid intersection with some main control set, which by maximality implies that it contains some main control set. ∎

The following result gives a very precise description of the relation between the main control sets and the chain control sets, i.e., for the control flow on $\mathscr{U} \times \mathbb{P}$, between the maximal topologically transitive sets whose projection on $\mathbb{P}$ has nonvoid interior and the components of the chain recurrent set. Recall the definition of $D_i(u_g)$ in (3.7),

$$D_i(u_g) := \{x \in \mathbb{P}: \varphi(t, x, u_g) \in \operatorname{cl} D_i \text{ for all } t \in \mathbb{R}\}$$

and the definition of $\mathscr{E}_j$ in (4.7).

**5.6. Theorem.** *Suppose that (5.1) satisfies Hypothesis (H). Then for $j = 1, \ldots, l$*

$$\mathscr{E}_j = \operatorname{cl} \left\{ (u_g, x): \begin{array}{c} g \in \operatorname{int} \mathscr{S}_{\leqslant t} \textit{ for some } t > 0 \quad \textit{and} \\ \mathbb{P}_x \in \bigoplus_i D_i(u_g) \textit{ where the sum is taken} \\ \textit{over all } i \textit{ with } D_i \subset E_j \end{array} \right\} \tag{5.3}$$

*In particular, for every* $u \in \mathcal{U}$ *and every* $j = 1,..., l$, *there are* $x_1,..., x_{i_j} \in \mathbb{R}^d$ *such that*

$$\mathcal{E}_j(u) = \text{span}\{x_1,..., x_{i_j}\}$$

*and every* $\mathbb{P}x_i$ *lies in the closure of some main control set.*

**Proof.** Theorem 3.13 implies that for all $g \in \text{int}\, \mathcal{S}_{\leqslant t}$, $t > 0$, the sets $D_i(u_g)$ are projected linear subspaces and coincide with $\mathbb{P}(\bigoplus_\lambda E(\lambda))$, where the sum is taken over all $\lambda \in \text{spec}\, g$ with $\mathbb{P}E(\lambda) \subset \text{int}\, D_i$ or, equivalently, with $\mathbb{P}E(\lambda) \subset \text{cl}\, D_i$. Clearly, with $\mathcal{E}_j(u)$ as defined in the proof of Theorem 5.5,

$$\mathcal{E}_j(u_g) = \bigoplus \{E(\lambda)\colon \mathbb{P}E(\lambda) \subset E_j\}$$

and each $\mathbb{P}E(\lambda)$ is contained in some $D_i$. Hence $\mathcal{E}_j(u_g) = \bigoplus_i \{E(\lambda)\colon \mathbb{P}E(\lambda) \subset D_i\}$, where the sum is taken over all $i$ with $D_i \subset E_j$. Now observe that

$$\{u_g\colon g \in \text{int}\, \mathcal{S}_{\leqslant t}, \text{ for some } t > 0\}$$

is dense in $\mathcal{U}$. Furthermore, $\mathcal{E}_j$ is closed. Hence $\mathcal{E}_j$ contains the set on the right-hand side of (5.3) and every $u \in \mathcal{U}$ appears as a limit. Now the assertion follows since $\dim \mathcal{E}_j(u)$ is constant over $u \in \mathcal{U}$. ■

**5.7. Remark.** In Remark 3.16 we defined the multiplicity $m(D_i)$ of a main control set $D_i \subset \mathbb{P}$. According to Theorem 5.6, one can define the multiplicity of chain control sets as

$$m(E_j) := \dim \mathcal{E}_j(u)$$

and this number is independent of $u \in \mathcal{U}$. Recall that $m(D_i)$ was independent of $g \in \text{int}\, \mathcal{S}_{\leqslant t}$ for all $t > 0$. In particular, we obtain from the proof of Theorem 5.6,

$$m(E_j) = \sum_i m(D_i)$$

where the sum is taken over all $i$ with $D_i \subset E_j$. Again, we obtain: System (1.2) has exactly one chain control set ($= \mathbb{P}$) iff there exists $g \in \mathcal{S}$ such that all generalized eigenspaces of $g$ are contained in one chain control set $E$, i.e., iff $m(E) = d$.

The following simple two-dimensional example illustrates the relation between control sets and chain control sets in the situation, where the eigenspace structure of $g \in \partial\mathcal{S}$, the boundary of the systems semigroup, is different from that for $g \in \operatorname{int} \mathscr{S}$. This system possesses a single chain control set but two main control sets, which are connected by a continuum of equilibria each of them forming a control set.

**5.8. Example.** Consider the system

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} 1 & u_1(t) \\ u_1(t) & u_2(t) \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \qquad t \in \mathbb{R} \tag{5.4}$$

$$\text{with } u = (u_1, u_2) \in \mathscr{U} := \{u \colon \mathbb{R} \to U \text{ measurable}\}$$

where $U = [0, 1/2] \times [1, 2]$. First note that for constant controls

$$u_1 \equiv \alpha, \qquad u_2 \equiv \beta$$

the eigenvalues $\lambda_{1,2}$ of

$$\begin{pmatrix} 1 & \alpha \\ \alpha & \beta \end{pmatrix}$$

are given by

$$(1 - \lambda)(\beta - \lambda) - \alpha^2 = \lambda^2(1 + \beta)\,\lambda + \beta - \alpha^2 = 0$$

hence

$$\lambda_{1,2} = \frac{1 + \beta}{2} \pm \sqrt{\alpha^2 - \beta + \frac{(1 + \beta)^2}{4}}$$

1. For $\beta = 1$, one has $\lambda_{1,2} = 1 \pm \alpha$ with generalized eigenspaces given by

$$\alpha = 0: \qquad \mathbb{R}^2$$

$$\alpha > 0: \qquad y = \pm x$$

2. For $\beta = 2$, one has

$$\lambda_{1,2} = \tfrac{3}{2} \pm \sqrt{\alpha^2 + \tfrac{1}{4}}$$

The corresponding eigenspaces are given by

$$\alpha = 0: \qquad y = 0 \text{ and } x = 0, \text{ with } \lambda_{1,2} = 1, 2$$

$$\alpha > 0: \qquad y = \left(\frac{1}{2\alpha} \pm \sqrt{1 + \frac{1}{4\alpha^2}}\right) x$$

3. For $\beta \in [1, 2]$ and $\alpha = \frac{1}{2}$ one always has 2 real eigenvalues $\lambda_1 > \lambda_2$, and the corresponding eigenspaces are of the form

$$E_1 = \left\{ c \cdot \begin{pmatrix} x \\ y \end{pmatrix} : c \in \mathbb{R}, \text{ with } y = \gamma x \text{ for some } \gamma > 1 \right\}$$

$$E_2 = \left\{ c \cdot \begin{pmatrix} x \\ y \end{pmatrix} : c \in \mathbb{R}, \text{ with } y = -\delta x \text{ for some } \delta < 1 \right\}$$

In order to determine the main control sets of system (5.4) projected onto $\mathbb{P}$, we can use now Theorem 4.8 in Ref. 1 and obtain (compare Fig. 1) the following.

Parametrize the projective space $\mathbb{P}$ via the angle as $\mathbb{P} = \{\theta: -\pi/2 < \theta \leqslant \pi/2\}$; then $D_1 = (-\pi/4, 0)$ is the open main control set, and $D_2 = [\pi/4, \pi/2]$ is the closed main control set. Note that

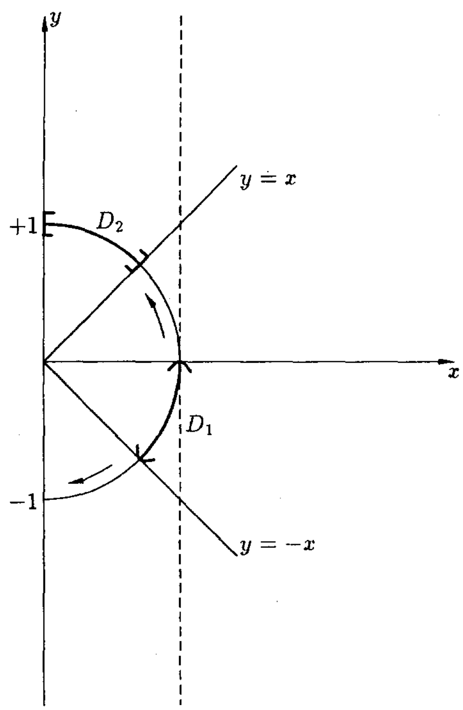



**Fig. 1.** Main control sets for Example 5.8.

$\bar{D}_1 \cap D_2 = \emptyset$, but by considering the case $\alpha = 0, \beta = 1$, one sees that there exists a unique chain control set $E_1 = \mathbb{P}$. The main control sets are connected by a continuum of control sets (with void interior), which are rest points on $\mathbb{P}$ of the diffeomorphism corresponding to $g(t) := \exp t \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} \in \partial \mathcal{S}$.

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