Infinite Time Optimal Control and Periodicity*

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Abstract. For smooth nonlinear systems

$$\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{r} u_i(t) X_i(x(t)),$$

the infinite time optimal control problems: maximize

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T g(x(t), u(t))\, dt \qquad \text{(average yield criterion)}$$

or

$$\lim_{T\to\infty} \int_0^T e^{-\delta t} g(x(t), u(t))\, dt \qquad \text{(discounted criterion)}$$

are considered, where the initial value $x(0)$ may be free or restricted. We study the existence of optimal periodic solutions for the above problems: if approximately optimal solutions have a limit point in the interior of some control set, then there exist approximately optimal periodic solutions. This result is applied to the growth of linear control semigroups and to a three-dimensional predator–prey harvesting model.

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1. Introduction

Consider a control system described by

$$\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{r} u_i(t) X_i(x(t)), \qquad t \geq 0, \tag{1.1}$$

where $u(t) := (u_i(t)) \in U \subset R^r$ and $x(t) \in R^d$. We are interested in optimal control problems where either an average criterion

$$\limsup_{T\to\infty} \frac{1}{T} \int_0^T g(x(t), u(t))\, dt \tag{1.2}$$

or a discounted criterion

$$\lim_{T\to\infty} \int_0^T e^{-\delta t} g(x(t), u(t))\, dt \tag{1.3}$$

have to be maximized, the initial value $x(0) = x_0$ may be free or restricted.

The results for the problem on the infinite time interval $R_+^0 = [0, \infty)$ are rather scarce (see [11] or [4]). Hence a common practice in applications has been to restrict attention to constant controls u yielding a steady state x_u of (1.1)—then, in a second step, one tries to reach the optimal equilibrium. A more sophisticated approach allows for trajectories and controls of some common period $\tau > 0$. Up to a certain degree, this takes into account the dynamics of the system and often leads to substantial improvements, notably in chemical and aerospace engineering. The relations between the steady state and the periodic optimization problems (for criterion (1.2)) have been studied to some extent (see [7] for a recent presentation).

In this paper we address the problem from the following point of view: under which conditions does the general problem on R_+^0 have a periodic solution? As simple counterexamples show, this reduction is not always possible. Even for bounded trajectories, and under periodic forcing, the solutions of (1.1) may show a quite complicated behavior. Note that methods of Fourier expansion are not helpful, since (1.1) is nonlinear, the control values are restricted and an approximation on the unbounded time interval $[0, \infty)$ is needed. Instead, finite time controllability properties and control sets (see [9]) are crucial.

In Section 2 we recall the notion and some properties of control sets, and prove a finite time controllability property. In Section 3 we define limit sets for trajectories of (1.1) corresponding to a control $u(t)$. We show that every limit set has nonvoid intersection with some control set. Furthermore, a discussion of the one-dimensional case (with an arbitrary number of controls) is given. Section 4 presents the main result of this paper: periodic solutions are approximately optimal for the infinite time problem, if approximately optimal solutions have a limit point in the interior of some control set. In Sections 5 and 6 this result is applied to the growth of linear control semigroups, motivated by a stability problem from linear, parameter-excited stochastic systems, and to a three-dimensional predator–prey harvesting model.

2. Control Sets and Finite Time Controllability

The main result of this section is concerned with finite time controllability of nonlinear control systems in certain subsets of the state space, called control sets (compare [1]). We first have to introduce some notation:

Consider the control system

$$\dot{x} = X_0(x) + \sum_{i=1}^{r} u_i X_i(x) \tag{2.1}$$

on a paracompact, connected, finite-dimensional C^∞ manifold M (of dimension m) with a Riemannian structure ρ. The vector fields $X_0, \ldots, X_r$ are assumed to be C^∞ and the set of admissible controls $\mathcal{U}$ is $\mathcal{U} = \{u: \mathbf{R}_+^0 \to U \subset \mathbf{R}^r \text{ measurable}\}$. By $\varphi(t, x, u)$ we denote the solution of (2.1) at time t with initial value $x \in M$ under the control action $u \in \mathcal{U}$ and, since we are dealing with infinite time problems, it is assumed that all solutions exist for all $t \geq 0$. (This is always true, if M is compact, or in a compact C-invariant set $K \subset M$, i.e., $\varphi(t, x, u) \in K$ for all $t \geq 0$ and all $x \in K$, $u \in \mathcal{U}$.)

Recall the following definitions from geometric control theory: $\mathcal{O}^+(x, t) = \{y \in M, \text{ there exists } u \in U \text{ such that } y = \varphi(t, x, u)\}$ is the positive orbit of $x \in M$ at time t, $\mathcal{O}^+_{\leq T}(x) = \bigcup_{0 \leq t \leq T} \mathcal{O}^+(x, t)$, $\mathcal{O}^+(x) = \mathcal{O}^+_{<\infty}(x)$, and similarly $\mathcal{O}^-(x, t) = \{y \in M, \text{ there exists } u \in \mathcal{U} \text{ such that } x = \varphi(t, y, u)\}$, etc.

A set $D \subset M$ is called a control set of (2.1), if $D \subset \overline{\mathcal{O}^+(x)}$ for all $x \in D$ and D is maximal with respect to this property. (One-point sets $\{x\} \subset M$ are considered as control sets only if they are rest points of (2.1), i.e., if $X_i(x) = 0$ for all $i = 0, \ldots, r$.) A control set C is called invariant, if $\bar{C} = \overline{\mathcal{O}^+(x)}$ for all $x \in C$, all other control sets are called variant. Note that because of maximality, control sets are either disjoint or identical (and always path connected).

In this set-up M can be decomposed uniquely as

$$M = A \cup B \cup C, \tag{2.2}$$

where A is the set of points outside control sets, B contains the points in some variant control set, and C contains those points that are elements of invariant control sets (see, e.g., [1]).

Note, furthermore, that for these systems the closures of the (positive or negative) orbits at time t coincide for measurable and for piecewise constant controls, and also for $U \subset \mathbf{R}^r$ and the convex hull co $U \subset \mathbf{R}^r$. Hence the (variant and invariant) control sets for the different classes of admissible controls agree.

We now introduce an assumption under which we prove our main results:

For all control sets D_α and all $x \in D_\alpha$ it holds: there exists a time $T > 0$ (depending on x) such that for all open neighborhoods $U(x)$ we have int $\mathcal{O}^+_{\leq T}(x) \cap U(x) \neq \emptyset$ and int $\mathcal{O}^-_{\leq T} \cap U(x) \neq \emptyset$. (2.3)

Condition (2.3) is, e.g., satisfied in the following set-up, which is standard in geometric control theory: let $\mathscr{L} = \mathscr{L}\mathscr{A}(X_0, \ldots, X_r)$ be the Lie algebra generated by the vector fields $X_0, \ldots, X_r$ and $\Delta_{\mathscr{L}}$ the distribution corresponding to $\mathscr{L}$ in the tangent bundle TM. If $\mathscr{L}$ has the maximal integral manifolds property (see [14]), then (2.1) lives on one of these maximal integral manifolds. Hence we can restrict ourselves without loss of generality to the case $\Delta_{\mathscr{L}}(x) = T_x M$ for all $x \in M$, where $T_x M$ is the tangent space to M at x. This implies local accessibility of the control system (2.1) and hence (2.3) is satisfied for all $x \in M$, all $T > 0$.

Lemma 2.1. *Assume* (2.3), *then the following assertions hold*:

(i) *Every invariant control set C has nonvoid interior, and a control set D with* $\operatorname{int} D \neq \emptyset$ *is invariant iff it is closed.*
(ii) *For a control set D with* $\operatorname{int} D \neq \emptyset$ *we have* $\mathcal{O}^+(x) \supset \operatorname{int} D$ *for all* $x \in D$.

Proof. (i) Follows from Lemma 2.1 of [9].

(ii) For each $x \in D$ we have by definition that $\mathcal{O}^+(x)$ is dense in D. For $y \in \operatorname{int} D$ it follows from (2.3) that $\operatorname{int} \mathcal{O}^-_{\leq T}(y) \cap \operatorname{int} D \neq \emptyset$, hence $\operatorname{int} \mathcal{O}^-_{\leq T}(y) \cap \mathcal{O}^+(x) \neq \emptyset$. Therefore there exists a control u such that, for some $t \in \mathbf{R}^+$, $\varphi(t, x, u) = y$. □

However, variant control sets $B \subset M$ need not have nonvoid interior:

Example 2.2. In $\mathbf{R}^2$ consider the control system

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} -u & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \qquad \text{with} \quad U = [0, a] \subset \mathbf{R}.$$

The eigenvalues of the systems matrix are $\lambda_{1,2} = -u/2 \pm \sqrt{\frac{1}{4}u^2 - 1}$. Hence for $u \equiv 0$ the system moves on circles centered at 0 and thus each of these circles is contained in a control set. For $u > 0$ we have $\operatorname{Re} \lambda < 0$, and so the circles are the control sets, each of them being variant, closed, and with void interior. The point $\{0\}$ is the only invariant control set. Of course the system satisfies the above Lie algebra condition on the manifold $\mathbf{R}^2 \backslash \{0\}$.

The next proposition is concerned with the crucial step for the existence of periodic controls: finite time controllability. We define the following first hitting time map

$$h: M \times M \to \mathbf{R}^0_+ \cup \{\infty\}, \qquad h(x, y) = \inf_{u \in \mathcal{U}} \{t \geq 0, \varphi(t, x, u) = y\}. \tag{2.4}$$

Note that in general h is not upper semicontinuous.

Proposition 2.3. *Let $D \subset M$ be a control set and assume* (2.3). *Let $K_1 \subset D$, and $K_2 \subset \operatorname{int} D$ be compact sets, then there exists a time $T = T(K_1, K_2) < \infty$ such that $h(x, y) \leq T$ for all $x \in K_1$, $y \in K_2$, with h defined in* (2.4).

Proof. (i) For $x \in K_1$, $y \in K_2$ we show that there is an open neighborhood $U(x)$ such that $h(z, y) \leq t_x < \infty$ for all $z \in U(x)$. By (2.3) there is $T < \infty$ and $y_1 \in \operatorname{int} D \cap \mathcal{O}^-_{\leq T}(y)$, let $U(y_1)$ be an open neighborhood of y_1 contained in $\operatorname{int} D \cap \mathcal{O}^-_{\leq T}(y)$. For $x \in K_1$ there exists a control $u \in \mathcal{U}$ and a time $t_1 < \infty$ such that $\varphi(t_1, x, u) = y_1$ by Lemma 2.1. The solutions of (2.1) depend continuously on the initial value, hence there is an open neighborhood $U(x)$ with $\varphi(t_1, z, u) \in U(y_1)$ for all $z \in U(x)$. Putting this together yields $U(x) \subset \mathcal{O}^-_{\leq t_1 + T}(y)$, hence $h(z, y) \leq t_1 + T$ for all $z \in U(x)$.

(ii) For $x \in K_1, y \in K_2$ we show that $h(x, z) \leq t_y < \infty$ for all z in some open neighborhood of y: let $x_1 \in \operatorname{int} D$ and $u_1 \in \mathcal{U}$, $t_1 < \infty$ such that $\varphi(t_1, x, u_1) = x_1$ by Lemma 2.1. By (2.3) there is $T < \infty$ and $y_1 \in \operatorname{int} D \cap \mathcal{O}^+_{\leq T}(x_1)$, let $U(y_1)$ be an open neighborhood of y_1 contained in int $D \cap \mathcal{O}^+_{\leq T}(x_1)$. Again from Lemma 2.1 there is $u_2 \in \mathcal{U}$, $t_2 < \infty$ with $\varphi(t_1, y, u_2) = y$. The solution of (2.1) under the control action u_2 defines a semigroup of homeomorphisms on M, thus at time t_2 the open set $U(y_1)$ is mapped onto an open neighborhood $U(y)$, i.e., $U(y) \subset \mathcal{O}^+_{\leq t_1+T+t_2}(x)$. This means that $h(x, z) \leq t_1 + T + t_2$ for all $z \in U(y)$.

(iii) A standard compactness argument now finishes the proof of this result. □

Finally we mention a result for existence and uniqueness of invariant control sets, which is useful for the example in Section 6.

Proposition 2.4. *Suppose that* (2.3) *is satisfied and that $K \subset M$ is a C-invariant compact set of* (2.1). *Then*:

(i) *There exists an invariant control set $C \subset K$ (which is compact with* int $C \neq \emptyset$).

(ii) *Denote by $S = \{x_u \in M, X_0(x_u) + \sum u_i X_i(x_u) = 0$ for some $(u_i)_{i=1,\dots,r} \in U\}$ the steady states of* (2.1) *for constant controls u, and assume that all $x \in S \cap K$ are asymptotically stable*:

(a) *For each $x \in S \cap K$ there is a control set $D \subset K$ with $x \in D$.*

(b) *If for each $u \in U$ there is exactly one corresponding $x_u \in S \cap K$, and all x_u are asymptotically stable, globally in K, then the invariant control set C is unique in K and $C = \overline{\mathcal{O}^+(x)}$ for any $x \in S \cap C$.*

Proof. (i) See Lemma 2.2 of [9].

(ii)(a) Follows from the definition of control sets. (b) Is an easy consequence of asymptotic stability, globally in K and of the construction in the proof of Lemma 2.2 of [9]. □

Note however that even (ii)(b) does not exclude the existence of additional variant control sets in K.

3. Limit Sets and Control Sets

A second important aspect of control sets is stressed by the fact that they describe the possible limit points of the trajectories of a control system (2.1) in compact sets. These limit sets will be crucial for the periodicity principle in Section 4.

Let $K \subset M$ denote a compact, C-invariant set and let U be compact and convex.

Definition 3.1. For a trajectory $\varphi(\cdot, x, u)$ of (2.1) with $x \in K$ define $\hat{\omega}(x, u) = \bigcap_{n \in \mathbf{N}} \operatorname{cl}\{(\varphi(t, x, u), u(t + \cdot)),\ t \geq n\} \subset K \times L_{2,\mathrm{loc}}(\mathbf{R}^0_+, \mathbf{R}^r)$ where "cl" denotes the closure taken with respect to the given topology on M and the weak topology on compact intervals in $L_{2,\mathrm{loc}}(\mathbf{R}_+^0, \mathbf{R}^r)$. The set $\omega(x, u) = \{y \in M$, there exists a sequence $t_k \to \infty$ with $\varphi(t_k, x, u) = y\}$ is called the *limit set* of $\varphi(\cdot, x, u)$. Observe that

$$\omega(x, u) = \{y \in M, \text{there exists } v \in \mathcal{U} \text{ such that } (y, v) \in \hat{\omega}(x, u)\}.$$

Definition 3.2. A nonempty set $L \subset \omega(x, u)$ is called *positively invariant* if $\varphi(\cdot, y, v) \subset L$ for all $y \in L$, all $v \in \mathcal{U}$ with $(y, v) \in \hat{\omega}(x, u)$.

Remark 3.3. An important feature of the definitions above is that they refer to a single control function u, and not to all admissible controls $u \in \mathcal{U}$. In particular the definition of positive invariance requires invariance only with respect to control functions v that are obtained as (weak) limit points of the "tails" $u(t + \cdot)$. This is in contrast to the notion introduced by Roxin [12]. The definition above turns out to be appropriate for the study of the limiting behavior of special, here optimal, solutions. (Roxin also introduces "holding sets," which resemble the idea of control sets, where exact instead of approximate controllability is required.)

Lemma 3.4. *The set $\omega(x, u)$ is nonempty, compact, connected, and positively invariant.*

Proof. (i) The sets $\mathrm{cl}\{(\varphi(t, x, u), u(t + \cdot)), t \geq n\}$ are nonempty and compact for each $n \in \mathbf{N}$. Finitely many of these sets have a nonvoid intersection, hence $\hat{\omega}(x, u) \neq \emptyset$ and therefore $\omega(x, u)$ is nonempty, compact, and connected (compare, e.g., Theorems 3.01 and 3.09 in Chapter V of [10]).

(ii) For positive invariance it suffices to show that for $x_n \to x_0$, $u_n \to v$ we have $\varphi_n(t) := \varphi(t, x_n, u_n) \to \varphi_0(t) =: \varphi(t, x_0, v)$ for all $t \geq 0$. This follows for globally Lipschitz vector fields in $\mathbf{R}^m$ from the following estimate (even uniformly on compact intervals):

$$\begin{aligned}
|\varphi_n(t) - \varphi_0(t)| &\leq |x_n - x_0| + \left| \int_0^t \left\{ X_0(\varphi_n(t)) + \sum_{i=1}^r u_{n,i}(t) X_i(\varphi_n(t)) - X_0(\varphi_0(t)) \right.\right. \\
&\qquad \left.\left. + \sum_{i=1}^r v_i(t) X_i(\varphi_0(t)) \right\} dt \right| \\
&\leq |x_n - x_0| + \left| \int_0^t \left\{ X_0(\varphi_n(t)) - X_0(\varphi_0(t)) \right.\right. \\
&\qquad \left.\left. + \sum_{i=1}^r u_{n,i}(t)[X_i(\varphi_m(t)) - X_i(\varphi_0(t))] \right\} dt \right| \\
&\quad + \left| \int_0^t \left\{ \sum_{i=1}^r [u_{n,i}(t) - v_i(t)] X_i(\varphi_0(t)) \right\} dt \right|.
\end{aligned}$$

The first and the third summand converge by assumption; the second one is bounded from above by $c_1 \int_0^t |\varphi_n(t) - \varphi_0(t)|\, dt$, where c_1 is a Lipschitz constant for the vector fields $X_0, \ldots, X_r$. Now Gronwall's inequality implies uniform convergence on compact time intervals.

To prove this result for system (2.1) in the compact C-invariant set K, select a finite covering (U_i, ψ_i) of coordinate charts for K. Then $x_0 \in U_i$ for some i, and the above estimate holds in $\psi_i(U_i)$ until the first exit time t_1 of $\psi(\varphi(\cdot, x_0, v))$ from $\psi_i(U_i)$. Therefore $\rho(\varphi_n(t), \varphi_0(t)) \to 0$ as $n \to \infty$ uniformly in the time intervals $[0, t_1 - \varepsilon]$ for any $\varepsilon > 0$; here ρ denotes the Riemannian distance on M. Since K is compact, for any $T > 0$ the trajectory $\varphi_0(t)$ in $[0, T]$ lies in any U_i at most finitely many times, and hence patching in local coordinates finishes the proof. □

Lemma 3.5.

(i) *Each compact, positively invariant set* L *of* $\varphi(\cdot, x, u)$ *contains a minimal, positively invariant set* $\tilde{L}$.

(ii) *Each trajectory* $\{\varphi(t, y, v), (y, v) \in \hat{\omega}(x, u), t \geq 0\}$ *in a minimal, positively invariant set* $\tilde{L}$ *is dense in* $\tilde{L}$.

Proof. These results are proved exactly as in the classical case, see, e.g., Theorems 7.02 and 7.06 in Chapter V of [10]. □

Proposition 3.6. *Let* $\varphi(\cdot, x, u)$ *be a trajectory of* (2.1) *in* K *and* $\omega(x, u)$ *its limit set. Then there exists a control set* D *of* (2.1) *such that* $\omega(x, u) \cap D \neq \emptyset$.

Proof. By Lemmas 3.4 and 3.5 the limit set $\omega(x, u)$ contains a point y in a minimal, positively invariant set $\tilde{L}$. $\tilde{L}$ in turn contains a dense trajectory $\varphi(\cdot, y, v)$ and thus y lies in some control set D. □

While Proposition 3.6 analyzes the relation between a limit set of a trajectory and control sets, the following result describes the behavior of trajectories, which actually hit a control set.

Proposition 3.7.

(i) *If* $\varphi(\cdot, x, u)$ *has a limit point in the interior of some control set* D, *then* $\varphi(t, x, u) \in D$ *for all* $t > t_0$, where $t_0 = \inf\{t \geq 0, \varphi(t, x, u) \in D\}$.

(ii) *For invariant control sets* C *we even have: if there exists* $t_0 \in \mathbf{R}_+^0$ *with*

 (a) $\varphi(t_0, x, u) \in C$, *then* $\varphi(t, x, u) \in C$ *for all* $t \geq t_0$,

 (b) $\varphi(t_0, x, u) \in \operatorname{int} C$, *then* $\varphi(t, x, u) \in \operatorname{int} C$ *for all* $t \geq t_0$.

Proof. (i) Let $t_1 > t_0$ such that $\varphi(t_1, x, u) \in D$, and $t_2 > t_1$ any time with $\varphi(t_2, x, u) \in \operatorname{int} D$. Denote $y = \varphi(t, x, u)$ for some $t \in [t_1, t_2]$. Then y is reachable from $\varphi(t_1, x, u)$, hence approximately reachable from any $z \in D$ by continuous dependence of the solutions of (2.1) on initial values. On the other hand, $\varphi(t_2, x, u)$ is reachable from y, hence any $z \in D$ is approximately reachable from y by Lemma 2.1(ii). This implies (i).

(ii) Follows from the C-invariance of $\bar{C}$ and int C for invariant control sets (compare, e.g., [2]). □

The proof of our main result, Theorem 4.2, relies on the assumption

$$\omega(x, u) \cap \text{int } D \neq \emptyset \qquad \text{for some control set} \quad D \subset K. \tag{3.1}$$

By Proposition 3.6, in a compact, C-invariant set there is always a control set D with $\omega(x, u) \cap D \neq \emptyset$. There is however no general criterion to ensure that $\text{int } D \neq \emptyset$ or that (3.1) holds. Sections 5 and 6 indicate how to verify (3.1) for specific systems. The next result clarifies the situation for one-dimensional manifolds M.

Proposition 3.8. *Consider system* (2.1) *on a one-dimensional manifold M, and assume that for all $v \in U$ there are at most finitely many $y \in M$ with $X_0(y) + \sum v_i X_i(y) = 0$. Then either*

(i) *there exists a control set with $\omega(x, u) \cap \text{int } D \neq \emptyset$ or*
(ii) *$\omega(x, u)$ consists of a single equilibrium corresponding to some $v \in \partial U$.*

Proof. Lemma 3.4 implies that either (i) $\omega(x, u)$ is an interval $[x_1, x_2]$ (or an arc $[x_1, x_2]$, if $M \simeq S^1$), or (ii) $\omega(x, u)$ consists of a single equilibrium. In case (i), let $y \in \text{int}[x_1, x_2]$. Then there are $t_2 > t_1 > t_0 \geq 0$ with $y = \varphi(t_0, x, u) = \varphi(t_2, x, u)$, and $y \neq \varphi(t_1, x, u) \in \text{int}[x_1, x_2]$. Hence y lies in some control set D with $\omega(x, u) \cap \text{int } D \neq \emptyset$. Now consider case (ii), i.e., $\omega(x, u) = \{y\}$, y is in some control set D, and

$$X_0(y) + \sum_{i=1}^{r} v_i^0 X_i(y) = 0 \qquad \text{for some} \quad v^0 \in U. \tag{3.2}$$

It follows from the characterization of control sets for $M = \mathbf{R}^1$ (see, e.g., Theorem 3.2 of [1]) that either $\{y\} = D$, some invariant control set, i.e., (3.2), holds for all $v \in U$, or D is an interval between two points $y_1 < y_2$. In the latter case assume that $\{y\} \cap D = \omega(x, u) \cap D = \{y_1\}$ (the other case with y_2 can be treated similarly). Then D is closed at $y = y_1$ and the characterization referred to above shows that

$$X_0(y) + \sum_{i=1}^{r} v_i X_i(y) \geq 0 \qquad \text{for all} \quad v \in U,$$

and

$$= 0 \qquad \text{for at least one value in } U.$$

Now assume that $v^0 \in \text{int } U$, v^0 from (3.2). Then there is $v^1 \in U$ with $X_0(y) + \sum_{i=1}^{r} v_i^1 X_i(y) < 0$. (Note that $X_i(y) = 0$ for all $i = 1, \ldots, r$ is impossible, because then (3.2) yields $X_0(y) = 0$, which is ruled out by Assumption (2.3) and the fact that $y \in D$.) This contradiction shows that in this case $v^0 \in \partial U$. □

4. A Periodicity Principle

In this section we prove that a class of infinite time optimal control problems has periodic solutions with performance arbitrarily close to the optimal. We consider the following situations: given the system (2.1) $\dot{x}=X_0(x)+\sum_{i=1}^r u_i X_i(x)$ on M with $U\subset \mathbf{R}^r$ convex, compact and the performance criterion

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t g(x,u)\,dt=\kappa(x_0,u) \qquad \text{(infinite time average yield criterion)} \tag{4.1}$$

or

$$\int_0^\infty e^{-\delta t}g(x,u)\,dt=\eta(x_0,u) \qquad \text{(infinite time discounted criterion),} \tag{4.2}$$

where we assume that g is continuous in both variables. Denote

$$\kappa=\sup_{u\in\mathcal{U}}\sup_{x_0\in M}\kappa(x_0,u)$$

and assume that κ is finite. (If x_0 is fixed, we write $\kappa(x_0):=\sup_{u\in\mathcal{U}}\kappa(x_0,u)$.) Then we say that (4.1) has *ε-optimal periodic solutions* if for each $\varepsilon>0$ there exists a periodic solution (x_p,u_p) (depending on ε) such that $\kappa(x_p,u_p)>\kappa-\varepsilon$, and similarly for Problem (4.2) and η. Note that we do not assume that there exists an optimal trajectory with performance equal to κ, or η.

We first analyze the infinite time average yield problem (4.1) where we have to assume that κ is the supremum over all *bounded* trajectories $\varphi(\cdot,x,u)\subset M$, see Example 4.4 at the end of this section for a counterexample in the unbounded case. For bounded trajectories we know from Proposition 3.6 that the limit points $\omega(x,u)$ intersect some control set D of (2.1). If $\omega(x,u)$ is a one-point set, we have the following preliminary result:

Proposition 4.1. *Suppose that g is affine in u. If for $\varepsilon>0$ there exists a trajectory (x,u) with $\kappa(x,u)>\kappa-\varepsilon$ and $\omega(x,u)=\{y\}\subset M$, then there exists a stationary solution (x_s,u_s) with $\kappa(x_s,u_s)>\kappa-\varepsilon$.*

Proof. If $\omega(x,u)=\{y\}$, then there exists a constant control $\tilde{u}$ such that $\varphi(t,y,\tilde{u})=y$ for all $t\in\mathbf{R}^+$ by Lemma 3.4. Since $\varphi(t,x,u)\to\varphi(t,y,\tilde{u})$ and by continuity of g we see that $\kappa(y,\tilde{u})>\kappa-\varepsilon$. □

The main result of this paper is:

Theorem 4.2. *If for $\varepsilon>0$ there exists a bounded trajectory (x,u) with $\kappa(x,u)>\kappa-\varepsilon$ and $\omega(x,u)\cap\operatorname{int}D\neq\emptyset$ for some control set D, then there exists a periodic solution (x_p,u_p) with $\kappa(x_p,u_p)>\kappa-\varepsilon$.*

Proof. Let (x,u) be a bounded trajectory with $\kappa(x,u)>\kappa-\varepsilon$ and let $y=\varphi(t_0,x,u)\in\operatorname{int}D$. The trajectory $\varphi(\cdot,y,u(t_0+\cdot))$ has the same limit points as $\varphi(\cdot,x,u)$, hence in particular an accumulation point $y_0\in\operatorname{int}D$. By Proposition 3.7, $\varphi(t,y,u)\in D$ for all $t\geq 0$ and by assumption $\varphi(\cdot,y,u)\subset K$ for some compact set $K \subset D$. Hence by Proposition 2.3 there exists a $T > 0$ such that $h(z, y) \leq T$ for all $z \in K$.

By assumption there exists t_1 (arbitrarily large) such that $(1/t_1)\int_0^{t_1} g(\varphi(t, y, u(t_0+t)), u)\,dt > \kappa - 2\varepsilon$ and

$$\begin{aligned}
\frac{1}{t_1+T}\int_0^{t_1+T} & g(\varphi(t, y, u(t_0+t)), u)\,dt \\
&= \frac{1}{t_1+T}\int_0^{t_1} g(\varphi(t, y, u(t_0+t)), u)\,dt \\
&\quad + \frac{1}{t_1+T}\int_{t_1}^{t_1+T} g(\varphi(t, y, u(t_0+t)), u)\,dt \\
&\geq \frac{1}{t_1+T}\int_0^{t_1} g(\varphi(t, y, u(t_0+t)), u)\,dt + \frac{1}{t_1+T} T \cdot \min_{(z,u)\in K\times U} g(z, u) \\
&\geq \kappa - 3\varepsilon - \varepsilon \qquad \text{for } t_1 \text{ sufficiently large.}
\end{aligned}$$

Define a periodic control

$$u_p = \begin{cases} u(t_0+t) & \text{for} \quad t \in [0, t_1], \\ u_c(t) & \text{for} \quad t \in (t_1, t_1+t_2], \end{cases}$$

where u_c steers $\varphi(t_1, y, u)$ to y in time $t_2 \leq T$, and continue u_p periodically for $t > t_1 + t_2$. Then $\varphi(\cdot, y, u_p)$ is periodic and in each period we have $(1/(t_1+t_2))\int_0^{t_1+t_2} g(\varphi(t, y, u(t_0+t)), u)\,dt \geq \kappa - 4\varepsilon$. □

For invariant control sets C we can strengthen this result to:

Corollary 4.3. *If for $\varepsilon > 0$ there exists a solution (x, u) with $\kappa(x, u) > \kappa - \varepsilon$ and a time $t_0 \geq 0$ such that $\varphi(t_0, x_0, u) \in \operatorname{int} C$, then there exists a periodic solution (x_p, u_p) with $\kappa(x_p, u_p) > \kappa - \varepsilon$.*

Proof. Same as before with Proposition 3.7(ii). □

Note that under Condition (2.3) any invariant control set has nonvoid interior. Proposition 4.1 and Theorem 4.2 guarantee ε-optimal periodic solutions for the infinite time average yield problem, except for the case where the set of limit points has more than one element and does not intersect the interior of some control set.

If the ε-optimal trajectories are unbounded, we cannot expect ε-optimal periodic solutions for (4.1), since the optimal value κ does not depend on the behavior on finite time intervals.

Example 4.4. Let $M = (0, \infty)$ and consider the problem

$$\dot{x} = 3 - 2u, \qquad u \in U = [1, 2],$$

$$g(x, u) = 5 - 4u.$$

The unique control set $C = M = (0, \infty)$ is invariant and we may start at any (interior) point $x_0 \in C$. The solutions are $\varphi(t, x_0, u) = (3-2u)t + x_0$, $\kappa(x_0, u) = 5 - 4u$ and $\kappa = 1$ is obtained for $u \equiv 1$, where the corresponding solution is

$\varphi(t, x_0, 1) = t + x_0$. Any periodic solution has to use controls in $(\frac{3}{2}, 2]$, where $g(x, u) < -1$, and hence any periodic solution has performance ≤ 0.

For the discounted problem (4.2) we cannot expect ε-optimal periodic solutions, if we start outside some control set. Our result in this case is:

Theorem 4.5. *If for $\varepsilon > 0$ there exists a pair (x, u) with $\eta(x, u) > \eta - \varepsilon$ and*

(i) *if $\omega(x, u) \cap \operatorname{int} D \neq \emptyset$ for some control set D, or*
(ii) *if there exists $t_0 \geq 0$ such that $\varphi(t_0, x, u) \in \operatorname{int} C$ for some invariant control set C,*

then there exists a solution (x_p, u_p) with $\eta(x_p, u_p) > \eta - \varepsilon$ and (x_p, u_p) is periodic after the time of the first entrance into int D *(or* int C, *respectively).*

Proof. Use the same ideas as in the proof of Theorem 4.2 and Corollary 4.3. □

Remark 4.6. (a) If Problems (4.1) or (4.2) have a solution that actually realizes κ (or η), then in general there need not be a periodic control having performance κ (or η).

(b) Consider the discounted problem (4.2) for $\delta \to 0$: if we start outside some control set D, then the initial part of the trajectory until the entrance into int D will carry less and less weight, as the period of the solution constructed in Theorem 4.5 grows. In this sense the solutions of Problem (4.2) converge toward those of (4.1) for $\delta \to 0$ (compare [8]).

(c) Theorem 4.5 remains valid, if instead of $e^{-\delta t}$ we use any discount function $\varphi(t)$ which decreases monotonically to 0 as $t \to \infty$ and such that η is still finite.

(d) Let us mention once again that outside control sets, system (2.1) cannot have any solution which is periodic in the state variable $x \in M$, since any periodic solution lies in some control set.

5. Growth of Linear Control Semigroups

Consider a linear, parameter-controlled system

$$\dot{x} = ux \quad \text{in } \mathbf{R}^d \tag{5.1}$$

with $u \in N \subset \mathrm{gl}(d, \mathbf{R})$ compact. According to the remarks after (2.2) we can restrict ourselves to piecewise constant controls and we define the systems group $\mathcal{G}$ and the system semigroup $\mathcal{S}$, describing the orbits of (5.1) by

$$\mathcal{G} = \{e^{t_r A_r} \times \cdots \times e^{t_1 A_1},\ A_i \in N,\ t_i \in \mathbf{R},\ 1 \leq i \leq r \in \mathbf{N}\},$$

$$\mathcal{S} = \{e^{t_r A_r} \times \cdots \times e^{t_1 A_1},\ A_i \in N,\ t_i \in \mathbf{R}^+,\ 1 \leq i \leq r \in \mathbf{N}\}.$$

$\mathcal{G}$ is a connected Lie subgroup of $\mathrm{Gl}(d, \mathbf{R})$ with Lie algebra $g = \mathcal{LA}(N)$, the Lie algebra generated by N in $\mathrm{gl}(d, \mathbf{R})$. Let $\mathcal{G}_t$ and $\mathcal{S}_t$ denote the subset of the (semi-) group with $\sum_{i=1}^{r} t_i \leq t$.

The problem is to describe the growth rate, i.e., the Lyapunov exponent of the system (5.1) (which is the growth rate of $\{\mathcal{S}_t, t\geq 0\}$). This can be done using the spectral radius or the operator norm. We define

$$\beta(t)=\frac{1}{t}\log r(\mathcal{S}_t),$$

$$r(\mathcal{S}_t)=\sup_{g\in\mathcal{S}_t} r(g),\quad r(g)\text{ is the spectral radius of } g\in \mathrm{Gl}(d,\mathbf{R}) \tag{5.2}$$

$$\beta=\lim_{t\to\infty}\beta(t)\qquad (=\sup_{t\geq 0}\beta(t)),$$

and

$$\delta(t)=\frac{1}{t}\log\|\mathcal{S}_t\|,$$

$$\|\mathcal{S}_t\|=\sup_{g\in\mathcal{S}_t}\|g\|,\quad \|g\|\text{ is the operator norm of } g\in\mathrm{Gl}(d,\mathbf{R}), \tag{5.3}$$

$$\delta=\lim_{t\to\infty}\delta(t)\qquad (=\inf_{t\geq 0}\delta(t)).$$

Of course $\beta\leq\delta$, and we are looking for conditions that guarantee $\beta=\delta$. The answer to this problem is important for the analysis of the stability of moments of associated stochastic systems and for the description of large deviations in these systems (see [3] for this idea and for the following set-up).

For further investigation we project system (5.1) onto the unit sphere S^{d-1} in $\mathbf{R}^d$, and we obtain for $s(t)=x(t)/|x(t)|$ the equation

$$\dot{s}(t)=h(u,s)=(u-s^*us\cdot Id)s,\qquad s_0=x_0/|x_0|, \tag{5.4}$$

where * denotes the transpose and Id the $d\times d$ identity matrix. Then

$$|x(t,x_0,u)|=|x_0|\exp\left[\int_0^t s(\tau,x_0)^*u(\tau)s(\tau,x_0)\,d\tau\right]$$

$$:=|x_0|\exp\left\{\int_0^t q(u,s)\,d\tau\right\}. \tag{5.5}$$

System (5.1) is linear in x, so we may consider (5.4) on the projective space P^{d-1} instead of S^{d-1}, which we will do from now on. The Lie algebra generated by the vector fields in (5.4) is $\mathcal{L}=\mathcal{LA}(h(u,\cdot),u\in N)$, and our assumption, analogously to the condition mentioned after (2.3), is

$$\Delta_{\mathcal{L}}(s)=T_sP\qquad\text{for all}\quad s\in P. \tag{5.6}$$

The following theorem states that system (5.1) has one Lyapunov exponent:

Theorem 5.1. *Under Assumption* (5.6) *we have* $\beta=\delta$ *for system* (5.1).

In order to prove this theorem we transform (5.2) and (5.3) into optimal control problems involving ε-optimal trajectories. Define

$$\kappa=\sup_{u\in\mathcal{U}}\sup_{s_0\in\mathbf{P}}\limsup_{t\to\infty}\frac{1}{t}\int_0^t q(u,s)\,d\tau,\qquad \mathcal{U}\text{ is the measurable controls with values in co } N,$$

$$\kappa_p = \sup_{u\in\mathcal{U}_p} \sup_{s_0\in\mathbf{P}} \limsup_{t\to\infty} \frac{1}{t}\int_0^t q(u,s)\,d\tau, \qquad \mathcal{U}_p \text{ is the periodic measurable controls with values in co } N.$$

Note that by (5.5) we have

$$\limsup_{t\to\infty}(1/t)\textstyle\int_0^t q(u,s)\,d\tau = \limsup_{t\to\infty}(1/t)\log|x(t,x_0,u)|,$$

the Lyapunov exponent of the solution of (5.1) starting in x_0 with control u.

From Arnold *et al.* [3, Theorem 3.1], we cite the following result:

Lemma 5.2. *Assume* (5.6), *then system* (5.1) *has exactly one invariant control set* C *on* $\mathbf{P}$, C *is compact, and* $\operatorname{int} C \neq \emptyset$.

Lemma 5.3. $\kappa_p = \beta$.

Proof. (i) $\beta \le \kappa_p$: For a given $\varepsilon > 0$ we find a $g = e^{t_r A_r}\times\cdots\times e^{t_1 A_1}\in\mathcal{S}_T$ with $\sum_{i=1}^r t_i = T$ and $(1/T)\log r(g) > \beta - \varepsilon$. Define a piecewise constant control u_0 on $[0, T]$ by $u_0(t) = A_i$ for $t \in [t_1+\cdots+t_{i-1}, t_1+\cdots+t_i)$ and continue T-periodically. Then the periodic system $\dot{x} = u_0 x$ has $g = \Phi(T)$ as the fundamental matrix at time T, its largest characteristic (or Floquet) exponent is therefore $(1/T)\log r(g)$. Hence there exists $s_0 \in \mathbf{P}$ such that for all $t \ge 0$ large enough $|x(t, s_0, u_0)| \ge \exp t\cdot(\beta-\varepsilon)$. Since ε was arbitrary, we see that $\beta \le \kappa_p$.

(ii) $\kappa_p \le \beta$: The argument above also proves that $\beta \le \alpha_p = \sup_{u\in\mathcal{U}_p}\lim_{t\to\infty}(1/t)\log r(\Phi_u(t))$, where $\Phi_u(t)$ is the fundamental matrix of $\dot{x} = ux$ at time t. $\beta \ge \alpha_p$ is clear from the definition, so it remains to show that $\kappa_p \le \alpha_p$. To this end choose $u \in \mathcal{U}_p$ with period T, then $x(t, x_0, u) = \Phi_u(t)s_0 = (P(t)e^{t\Lambda})s_0$, the Floquet decomposition, where $P(t)$ is T-periodic and Λ is a constant matrix with the Floquet exponents. Then

$$\frac{1}{nT}\log|x(nT, s_0, u)| \le \frac{1}{nT}(\log|P(t)| + \log|e^{nT\Lambda}| + \log|s_0|).$$

Thus for $n\to\infty$ we have for all $u \in \mathcal{U}_p$

$$\lim_{t\to\infty}\frac{1}{t}\log|x(t, s_0, u)| \le \lim_{t\to\infty}\frac{1}{t}\log r(\Phi_u(t)), \qquad \text{yielding} \quad \kappa_p \le \alpha_p. \qquad \square$$

Lemma 5.4. $\kappa = \delta$.

Proof. From (5.5) we see that $\delta = \lim_{t\to\infty}\sup_{u\in\mathcal{U}}\sup_{s_0\in\mathbf{P}}(1/t)\int_0^t q(u,s)\,d\tau$. $\kappa \le \delta$ is obvious from this formulation, hence it remains to show that $\delta \le \kappa$. For a sequence $t_k \to \infty$ there exist (u^k, s_0^k) with $(1/t_k)\int_0^t q(u^k, s(x_0^k, u^k))\,d\tau > \delta(t_k) - 1/k$. Hence for any compact time interval $[0, T] \subset \mathbf{R}_+^0$ there exists $u^0\colon \mathbf{R}_+^0 \to \text{co } N$, measurable such that $u^k \to u^0$ weakly in $L_2([0, T])$. Take a weak accumulation point u_T^0 of u^k on $[0, T]$ and a diagonal sequence in T, then $u^0(t) \in \text{co } N$ via the Cesaro limit. Let s_0^0 be an accumulation point of s_k^0, then $s^k := s(s_0^k, u^k) \to s(s_0^0, u^0) =: s^0$ uniformly on compact time intervals by the argument from the proof of Lemma 3.4(ii).

Fix $\varepsilon > 0$, for k large enough we have

$$\frac{1}{t_k}\int_0^{t_k} q(u^k, s(s_0^k, u^k))\, d\tau > \delta(t_k) - \varepsilon,$$

$$\left|\frac{1}{t_k}\int_0^{t_k} (s^{k*}u^k s^k - s^{0*}u^0 s^0)\, d\tau\right| < \varepsilon \quad \text{and} \quad |\delta(t_k) - \delta| < \varepsilon.$$

Thus

$$\frac{1}{t_k}\int_0^{t_k} s^{0*}u^0 s^0\, d\tau \geq \frac{1}{t_k}\int_0^{t_k} s^{k*}u^k s^k\, d\tau - \left|\frac{1}{t_k}\int_0^{t_k} (s^{k*}u^k s^k - s^{0*}u^0 s^0)\, d\tau\right|$$
$$\geq \delta(t_k) - 2\varepsilon \geq \delta - 3\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have $\kappa \geq \limsup_{t\to\infty} (1/t)\int_0^t s^{0*}u^0 s^0\, d\tau \geq \delta$. □

Proof of Theorem 5.1. The result follows from $\kappa = \kappa_p$ by using Corollary 4.3, if we can show that initial values for ε-optimal trajectories can be chosen in int C, where C is the unique compact invariant control set from Lemma 5.2. To this end let $A \subset \operatorname{int} C$ be open such that $\bar{A} \subset \operatorname{int} C$. Then A contains a basis of $\mathbf{R}^d$ and for any $u \in \mathcal{U}$ the maximal Lyapunov exponent among the $\limsup_{t\to\infty} (1/t) \log|x(t, \cdot, u)|$ can be obtained by starting in A (see Section 3.12 of [5]). Hence for κ (and κ_p) it suffices to consider initial values in $\bar{A} \subset \operatorname{int} C$. □

6. Management of Interacting Populations

We consider the following three-dimensional model of harvested populations:

$$\begin{aligned}
\dot{x}_1(t) &= q_1 x_1\left(1 - \frac{x_1}{K_1}\right) - \alpha_1 x_1 x_2 x_2 - u x_1 =: g_1(x_1, x_2, x_3, u),\\
\dot{x}_2(t) &= q_2 x_2\left(1 - \frac{x_2}{K_2}\right) + \alpha_2 x_1 x_2 \qquad =: g_2(x_1, x_2),\\
\dot{x}_3(t) &= q_3 x_3\left(1 - \frac{x_3}{K_3}\right) + \alpha_3 x_1 x_3 \qquad =: g_3(x_1, x_3),
\end{aligned} \tag{6.1}$$

with

$$u(t) \in [0, U_{\max}] \subset \mathbf{R},$$

where the constants q_i, α_i, K_i, $U_{\max}$ are positive and $U_{\max} < q_1$. These equations describe the dynamics of a system, where the predators x_2 and x_3 feed on the prey x_1, the prey is subject to harvesting, and the harvesting intensity is considered as the control variable. There is no direct interaction between the predators x_2 and x_3.

Reasonable performance criteria are the average yield

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T x_1(t)u(t)\, dt \tag{6.2}$$

or, taking into account economic considerations,

$$\int_0^\infty e^{-\delta t}(p(u, x_1) - c(x_1))ux_1\, dt, \tag{6.3}$$

where the discount factor δ is positive and p, c are the price and the cost, respectively, of the harvested prey (see [6]).

A simple analysis shows the existence of a compact invariant set of the form $K = \{(x_1, x_2, x_3), 0 \le x_i \le c_i, i = 1, 2, 3\}$. The function g_1 vanishes for

$$x_1 = 0 \quad \text{and} \quad x_1 = K_1\left(1 - \frac{\alpha_1}{q_1} x_2 x_3 - \frac{u}{q_1}\right), \qquad u \in [0, U_{\max}].$$

For $i = 2, 3$ the functions g_i vanish for

$$x_i = 0 \quad \text{and} \quad x_i = K_i\left(1 + \frac{\alpha_i}{q_i} x_1\right).$$

Thus we obtain the following eight sets of steady states:

$$S_1 = \{(0, 0, 0)\},$$

$$S_2 = \{(0, K_2, 0)\},$$

$$S_3 = \{(0, 0, K_3)\},$$

$$S_4 = \{(0, K_2, K_3)\},$$

$$S_5 = \left\{(x_1, 0, 0), x_1 = K_1\left(1 - \frac{u}{q_1}\right), u \in [0, U_{\max}]\right\},$$

$$S_6 = \left\{(x_1, 0, x_3), x_1 = K_1\left(1 - \frac{u}{q_1}\right), x_3 = K_3\left(1 + K_1 \frac{\alpha_3}{q_3}\left(1 - \frac{u}{q_1}\right)\right),\right.$$
$$\left. u \in [0, U_{\max}]\right\},$$

$$S_7 = \left\{(x_1, x_2, 0), x_1 = K_1\left(1 - \frac{u}{q_1}\right), x_2 = K_2\left(1 + K_2 \frac{\alpha_2}{q_2}\left(1 - \frac{u}{q_1}\right)\right),\right.$$
$$\left. u \in [0, U_{\max}]\right\},$$

$$S_8 = \left\{(x_1, x_2, x_3), x_1 = K_1\left(1 - \frac{\alpha_1}{q_1} x_2 x_3 - \frac{u}{q_1}\right), x_2 = K_2\left(1 + \frac{\alpha_2}{q_2} x_1\right),\right.$$
$$\left. x_3 = K_3\left(1 + \frac{\alpha_3}{q_3} x_1\right), u \in [0, U_{\max}]\right\}.$$

The planes P_i given by $x_i = 0$ are invariant for the control system (6.1) and, having empty interior in $\mathbf{R}^3$, they cannot be reached from $\mathbf{R}^3_+$. We therefore concentrate on the three-dimensional situation and assume

$$S_8 \neq \varnothing \quad \text{and} \quad S_8 \subset \mathbf{R}^3_+ =: M \qquad \text{(the state space for (5.1)).} \tag{6.4}$$

It is easy to show that if $S_8 = \varnothing$, then all solutions of the control system (6.1) tend toward one of the planes P_i. Furthermore, $S_8 \neq \varnothing$ can be expressed explicitly in terms of the parameters of (6.1).

6.1. *Existence of ε-Optimal Periodic Solutions*

If $\alpha_2 \neq \alpha_3$ or $q_2 \neq q_3$, then condition (2.3) is satisfied in the manifold M (see Section 6.3). Furthermore, there exists an invariant control set $C \subset M$, C is closed with nonvoid interior. The steady states in S_8 are globally asymptotically stable in M (see Section 6.2), thus C is unique and $S_8 \subset C$ by Proposition 2.4. For the criteria (6.2) and (6.3) with initial values in int C we thus infer the existence of ε-optimal periodic controls.

6.2. *Global Asymptotic Stability of S_8 in $\mathbf{R}^3_+$*

Let $x_u \in S_8$, i.e., $g_i(x_u, u) = 0$ for $i = 1, 2, 3$. Using arguments from Sieveking [13, Theorem 4] we construct a Lyapunov function for this system. First note that, for u fixed, we have a predator-prey system, since for $i \neq j$ either

$$\frac{\partial g_i}{\partial x_j}(x, u) = 0 \quad \text{and} \quad \frac{\partial g_j}{\partial x_i}(x, u) = 0 \quad \text{on } \mathbf{R}^3_+$$

or

$$\frac{\partial g_i}{\partial x_j}(x, u) \cdot \frac{\partial g_j}{\partial x_i}(x, u) < 0 \quad \text{on } \mathbf{R}^3_+.$$

To (6.1) we associate an undirected graph with three knots and an edge $i - j$ if $\partial g_i / \partial x_j \neq 0$ and $\partial g_j / \partial x_i \neq 0$. This graph obviously is a tree. Hence (see Lemma 5 of [13]) there are smooth positive functions $k_i(x)$, $x \in \mathbf{R}^3_+$ with

$$k_i(x) \frac{\partial g_i}{\partial x_j} = -k_j(x) \frac{\partial g_j}{\partial x_i} \qquad \text{for} \quad 1 \leq i, j \leq 3.$$

Define

$$H(x, u) = \sum_{i=1}^{3} k_i(x)(x_i - x_{u,i} \log x_i).$$

Then

$$\frac{d}{dt} H(x(t), u) < 0$$

along solutions of (6.1) in $\mathbf{R}^3_+$ and

$$H(x_u, u) = \min_{x \in \mathbf{R}^3_+} H(x, u).$$

Thus x_u is locally asymptotically stable. Furthermore,

$$\lim_{x \to \partial \mathbf{R}^3_+} H(x, u) = +\infty$$

and thus x_u is globally asymptotically stable in $\mathbf{R}^3_+$.

6.3. *Condition (2.3)*

System (6.1) is of the form

$$\dot{x} = X_0(x) + uX_1(x) \qquad \text{on} \quad M = \mathbf{R}^3_+ \quad \text{with}$$

$$X_0 = \begin{pmatrix} q_1x_1\left(1+\dfrac{x_1}{K_1}\right) - \alpha_1x_1x_2x_3 \\ q_2x_2\left(1+\dfrac{x_2}{K_2}\right) + \alpha_2x_1x_2 \\ q_3x_3\left(1-\dfrac{x_3}{K_3}\right) + \alpha_3x_1x_3 \end{pmatrix}, \qquad X_1 = \begin{pmatrix} -x_1 \\ 0 \\ 0 \end{pmatrix}.$$

In a straightforward way we compute the Lie brackets

$$X_2 := [X_0, X_1] = \begin{pmatrix} -\dfrac{q_1}{K_1}x_1^2 \\ \alpha_2x_1x_2 \\ \alpha_3x_1x_3 \end{pmatrix},$$

$$X_3 := [X_0, X_2] = \begin{pmatrix} -\dfrac{q_1^2}{K_1}x_1^2 + \dfrac{q_1\alpha_1}{K_1}x_1^2x_2x_3 + (\alpha_1\alpha_2 + \alpha_1\alpha_3)x_1^2x_2x_3 \\ \alpha_2q_1x_1x_2 - \alpha_1\alpha_2x_1x_2^2x_3 + \dfrac{\alpha_2q_2}{K_2}x_1x_2^2 \\ \alpha_3q_1x_1x_3 - \alpha_1\alpha_3x_1x_2x_3^2 + \dfrac{\alpha_2q_3}{K_3}x_1x_3^2 \end{pmatrix}.$$

The vector fields X_0, X_1, X_2 are linearly dependent for all points (x_1, x_2, x_3) with

$$\alpha_3q_2 - \frac{\alpha_2q_2}{K_2}x_2 = \alpha_2q_3 - \frac{\alpha_2q_3}{K_3}x_3. \tag{6.5}$$

The vector fields X_1, X_2, X_3 are linearly dependent for all points (x_1, x_2, x_3) with

$$K_3q_2x_2 = K_2q_3x_3. \tag{6.6}$$

For $\alpha_2 = \alpha_3$ and $q_2 = q_3$, these equations define the same line in the $x_2 - x_3$ plane, i.e., the same plane $P = \{(c \cdot x_1, x_2 = (K_2/K_3)x_3, d \cdot x_3), c, d, \in \mathbf{R}\}$ in $\mathbf{R}^3$. The vector fields X_0 and X_1 are tangent to P in this case and hence all orbits of (6.1) with initial values in P are contained in P, thus (2.3) cannot be met in this situation.

If however $\alpha_2 = \alpha_3$ and $q_2 \neq q_3$, then (6.5) and (6.6) define parallel lines in the $x_2 - x_3$ plane, hence the vector fields X_i, $i = 0, \ldots, 3$, span the whole tangent space at any point in R^3_+.

If $\alpha_2 \neq \alpha_3$, denote $\beta = (\alpha_3q_2 - \alpha_2q_3)/(q_3\alpha_3 - q_3\alpha_2)$. If $\beta \leq 0$, then the lines defined by (6.5) and (6.6) do not intersect in the $x_2 - x_3$ plane for $x_2 > 0, x_3 > 0$, thus again X_i, $i = 0, \ldots, 3$, span the tangent space at any point in $\mathbf{R}^3_+$.

If $\beta > 0$, then (6.5) and (6.6) define a line

$$\left\{(c \cdot x_1, x_2^0, x_3^0), c \in \mathbf{R}, x_2^0 = \frac{q_3}{q_2}\beta K_2, x_3^0 = \beta K_3\right\} \quad \text{in } \mathbf{R}^3_+.$$

The vector field X_0 is never tangent to this line in $\mathbf{R}^3_+$ hence Condition (2.3) is satisfied in this case.

Summing up, we see that, except for the case $\alpha_2 = \alpha_3$ and $q_2 = q_3$, Assumption (2.3) is always fulfilled.

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