

AN ITERATED GRAPH CONSTRUCTION AND PERIODIC ORBITS OF HAMILTONIAN DELAY EQUATIONS

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ABSTRACT. According to the Arnold conjectures and Floer's proofs, there are non-trivial lower bounds for the number of periodic solutions of Hamiltonian differential equations on a closed symplectic manifold whose symplectic form vanishes on spheres. We use an iterated graph construction and Lagrangian Floer homology to show that these lower bounds also hold for certain Hamiltonian delay equations.

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1. INTRODUCTION AND MAIN RESULTS

Delay equations. An ordinary differential equation (ODE) on a manifold M is a problem of the form

$$\dot{x}(t) = X_t(x(t))$$

where X_t is a time-dependent vector field on M . A delay differential equation (DDE) on M is a problem of the form

$$\dot{x}(t) = \mathcal{X}(x_{\leq t}) \tag{1}$$

where $x_{\leq t} = \{x(s) : s \leq t\}$ is the “past before time t ” of the curve x and where \mathcal{X} associates with each curve $x_{\leq t}$, $t \in \mathbb{R}$, a vector in $T_{x(t)}M$. In a DDE, the velocity vector $\dot{x}(t)$ therefore does not only depend on the instantaneous position $x(t)$, but also on positions of x in the past. Note that if x is a periodic orbit, then $x_{\leq t}$ is the whole orbit x . Delay equations form a

Date: February 23, 2018.

Key words and phrases. Delay equations, Hamiltonian systems, Floer homology.

2000 Mathematics Subject Classification. Primary 34K13, Secondary 58E05, 58F05.

vast topic; we refer to [13] for the general theory and to [7] for a wealth of DDEs arising in applications.

The simplest delay equations on \mathbb{R}^d are of the form

$$\dot{x}(t) = \sum_{j=1}^N X_t^j(x(t - \tau_j)) \quad (2)$$

where X_t^j are time-dependent vector fields on \mathbb{R}^d and $0 \leq \tau_1 < \dots < \tau_N$ are the delay times. On a general manifold M , one has to ensure that both sides of (2) live in the same tangent space, that is, the vector fields X_t^j are of the form

$$\dot{x}(t) = \sum_{j=1}^N f_t^j(x(t - \tau_j)) Y_t^j(x(t)) \quad (3)$$

for time-dependent functions f_t^j and vector fields Y_t^j on M .

Assume now that (M, ω) is a closed symplectic manifold such that the cohomology class $[\omega]$ vanishes on spherical homology classes: $[\omega]|_{\pi_2(M)} = 0$. Also assume that X_H is the Hamiltonian vector field of a function $H: M \times S^1 \rightarrow \mathbb{R}$, where $S^1 = \mathbb{R}/\mathbb{Z}$. Then by the proofs of Arnold's celebrated conjectures (see for instance [18, Chapter 11]) the number of 1-periodic solutions of the Hamiltonian ODE $\dot{x}(t) = X_H(x(t))$ is at least $\text{cuplength}(M) + 1$, and at least $\dim H_*(M; \mathbb{Z}_2)$ generically (namely if the graph of ϕ_H^1 in $M \times M$ intersects the diagonal transversally). Here, $\text{cuplength}(M)$ is the maximal length of a non-vanishing product of elements of positive degree in the cohomology ring $H^*(M; \mathbb{Z}_2)$, and $\dim H^*(M; \mathbb{Z}_2)$ is the sum of the Betti numbers of M with respect to \mathbb{Z}_2 coefficients.

Question 1.1. *Do the Arnold conjectures generalize to delay equations?*

To make the question meaningful, one should first decide what a Hamiltonian delay equation should be. In [2], we give one possible answer to the sub-question what a periodic solution of a Hamiltonian delay equation should be, by characterising such solutions variationally. In this note we prove the Arnold conjectures for a special class of Hamiltonian delay equations by using classical tools from the theory of J -holomorphic curves and Floer homology, and by an iterated graph construction. The delay equations we can deal with here in particular have only a finite number of delay times. We believe that the Arnold conjectures hold true for a far more general class of Hamiltonian delay equations, cf. [2]. For this one should construct a non-local Floer homology, in which the gradient flow lines will not anymore be solutions of an elliptic PDE on a finite dimensional symplectic manifold, but an ODE on an infinite dimensional scale manifold. For a detailed description of this kind of equations we refer to [1], where a first step of this construction (namely compactness of the space of solutions of non-local Floer equations) was taken. The results of this paper, that rely only on

classical Floer theory, indicate that such a more general Floer theory, that would imply Theorem 1.2 below as a special case, does exist.

Delay equations by means of an iterated graph construction. In accordance with A. Weinstein’s dictum from [25] that “everything is a Lagrangian submanifold”, many results on periodic orbits of Hamiltonian systems on a symplectic manifold (M, ω) can be deduced from results on Lagrangian intersections. Indeed, for every Hamiltonian function $H: M \times S^1 \rightarrow \mathbb{R}$ the intersections of the diagonal Δ in $(M \times M, \omega \oplus -\omega)$ with the graph $\{(\phi_H^1(z), z)\}$ of the Hamiltonian diffeomorphism ϕ_H^1 are in bijection with the 1-periodic orbits of the Hamiltonian flow of H .

To give this correspondence a more precise dynamical interpretation, we consider the loop space $\mathcal{P}_0 = W^{1,2}(S^1, M)$ and the path space $\mathcal{P}_1 = \{v \in W^{1,2}([0, 1], M \times M) \mid v(0), v(1) \in \Delta\}$. The map

$$\psi: \mathcal{P}_0 \rightarrow \mathcal{P}_1, \quad \psi(v)(t) = (v(t), v(0))$$

is an embedding. Given $H: M \times S^1 \rightarrow \mathbb{R}$, define $\tilde{H}: M \times M \times S^1 \rightarrow \mathbb{R}$ by

$$\tilde{H}_t(z_1, z_2) = H_t(z_1).$$

Then the flow-lines of \tilde{H} are of the form $(\phi_H^t(z_1), z_2)$, and so the Hamiltonian chords of \tilde{H} that start and end at time 1 on the diagonal correspond to 1-periodic orbits of H on M .

One may now wonder what more general Hamiltonian functions on $M \times M$ can tell us about dynamical systems on M . The flow lines of such more general Hamiltonians may, however, not be in the image of ψ , and thus cannot be pulled back to M by ψ . Following [5, §5.2] we therefore consider another map $\Psi: \mathcal{P}_0 \rightarrow \mathcal{P}_1$, given by

$$\Psi(v)(t) = \left(v\left(\frac{t}{2}\right), v\left(1 - \frac{t}{2}\right) \right), \quad t \in [0, 1],$$

which in contrast to ψ is a diffeomorphism, with inverse

$$\Psi^{-1}(w)(t) = \begin{cases} w_1(2t), & t \in [0, \frac{1}{2}], \\ w_2(2 - 2t), & t \in [\frac{1}{2}, 1]. \end{cases}$$

Under the “change of variables” Ψ , the Hamiltonian loops of $H: M \times S^1 \rightarrow \mathbb{R}$ correspond to Hamiltonian chords of

$$\tilde{H}: M \times M \times S^1 \rightarrow \mathbb{R}, \quad \tilde{H}_t(z_1, z_2) = \frac{1}{2}H_{\frac{t}{2}}(z_1) + \frac{1}{2}H_{1-\frac{t}{2}}(z_2).$$

For other functions $K: M \times M \times S^1 \rightarrow \mathbb{R}$, however, Hamiltonian chords are pulled back to periodic orbits on M that solve a *delay equation*. For instance, taking K in product form, $K_t(z_1, z_2) = \frac{1}{2}F_t(z_1)G_t(z_2)$, Hamiltonian chords of K are pulled back to 1-periodic orbits v solving

$$\dot{v}(t) = \begin{cases} G_{2t}(v(1-t))X_{F_{2t}}(v(t)), & t \in [0, \frac{1}{2}], \\ F_{2-2t}(v(1-t))X_{G_{2-2t}}(v(t)), & t \in [\frac{1}{2}, 1]. \end{cases}$$

While the term $v(1-t)$ may look a bit awkward at a first glance, since time is running backwards, along 1-periodic orbits the above equation is an honest delay equation of the form (1), because along such orbits we have $x(t+k) = x(t)$ for all $k \in \mathbb{Z}$ and so there is no difference between the future and the past.

To obtain periodic solutions of more general Hamiltonian delay equations on M , and in particular such of the classical form (3), we iterate the above graph construction: Set $(M_1, \omega_1) = (M \times M, \omega \oplus -\omega)$ and define the symplectic manifold

$$(M_2, \omega_2) := (M_1 \times M_1, \omega_1 \oplus -\omega_1) = (M^4, \omega \oplus -\omega \oplus -\omega \oplus \omega),$$

and in (M_2, ω_2) take two Lagrangian submanifolds, namely the product of the diagonal in M_1 with itself:

$$\Delta_2^0 := \Delta \times \Delta \subset M_2,$$

and the diagonal

$$\Delta_2^1 := \{(z, z) : z \in M_1\} \subset M_2.$$

Then the same formula as for Ψ above yields a diffeomorphism Ψ_1 from \mathcal{P}_1 to the space

$$\mathcal{P}_2 = \{v \in W^{1,2}([0, 1], M_2) : v(0) \in \Delta_2^0, v(1) \in \Delta_2^1\}.$$

Pulling back Hamiltonian chords of, say,

$$K_t(z) = F_t^1(z_1) F_t^4(z_4) + F_t^2(z_2) F_t^3(z_3)$$

on M_2 by $\Psi^1 = \Psi_1 \circ \Psi$ we then get 1-periodic orbits on M that solve a delay equation with delay time $\frac{1}{2}$.

Iterating the graph construction, we obtain for every $n \geq 1$ a diffeomorphism Ψ^n from \mathcal{P}_0 to the space \mathcal{P}_n of paths in $M_n := M^{2^n}$ that end on the diagonal Δ_n^1 of $M_n = M_{n-1} \times M_{n-1}$ and start on $\Delta_n^0 := \Delta_{n-1}^0 \times \Delta_{n-1}^1$. From this diffeomorphism we get 1-periodic orbits on M that solve delay equations with time delays $\frac{1}{2^n}, \dots, \frac{2^n-1}{2^n}$, and pre-composing Ψ, Ψ_1, \dots with a diffeomorphism of the circle or the interval, we get delay equations with various other delay times, see Sections 4 and 5.

The existence of many Hamiltonian chords on \mathcal{P}_n is guaranteed by a recent cuplength estimate of Albers–Hein [3] and by Pozniak’s theorem on clean Lagrangian intersections [19]. From this we obtain

Theorem 1.2. *Let (M, ω) be a closed symplectic manifold with $[\omega]|_{\pi_2(M)} = 0$, and fix a function $K: M_n \times S^1 \rightarrow \mathbb{R}$. Let $\Psi^n: \mathcal{P}_0 \rightarrow \mathcal{P}_n$ be the diffeomorphism obtained by the iterated graph construction, or any of its variants induced by pre-composing with diffeomorphisms of the circle or the interval. Then the number of contractible 1-periodic orbits on M that solve the Hamiltonian delay equation induced by K via pullback under Ψ^n is*

- (i) at least $\text{cuplength}(M) + 1$;
- (ii) at least $\dim H_*(M; \mathbb{Z}_2)$ if $\phi_K^1(\Delta_n^0)$ intersects Δ_n^1 transversally.

Note that the transversality assumption on K in (ii) is C^∞ -generic in the space of functions $M_n \times S^1 \rightarrow \mathbb{R}$.

Comparison with previous results. We conclude this introduction with comparing our method and result with previous ones.

The search for periodic solutions of (non-linear) DDEs, that has been an important topic in the study of DDEs since the 1960s, is much harder than for ODEs: Most results are for very special classes of autonomous DDEs on \mathbb{R} , and just one or two periodic orbits are found. Existence results for periodic orbits of DDEs on \mathbb{R}^d , or even on manifolds, are even scarcer, see [4], [13, §11], [24] and the references therein. The methods used are bifurcation and fixed point theorems, and index methods. In addition to these tools, there are a few tricks that for some DDEs convert the problem of finding periodic solutions to the problem of finding certain solutions of related ODEs. One is the “chain trickery” going back to [8], see also [6]. Another one is the Kaplan–Yorke method from [15]. Our graph construction adds one more trick to this lists.

While the search for periodic orbits of *Hamiltonian* ODEs is a topic with a rich tradition and many profound results, much less is known on the existence of periodic orbits of Hamiltonian DDEs. A Hamiltonian DDE on \mathbb{R}^{2n} is, for instance, of the form (2) with $X_t^j = J\nabla H_t^j$, where J is the usual complex structure on $\mathbb{R}^{2n} \cong \mathbb{C}^n$. For the special case

$$\dot{x}(t) = J(\nabla G_t(x(t - \tau)) + \nabla G_t(x(t - 2\tau)) + \cdots + \nabla G_t(x(t - N\tau)))$$

with G_t of periodic τ and meeting suitable growth conditions, Liu [17] proved the existence of one $N\tau$ -periodic solution, and, under a generic assumption on this solution, of two or even three $N\tau$ -periodic solutions. The main idea of his proof, that can be traced back to the Kaplan–Yorke method, was one inspiration for our graph trick.

In summary, both our methods and results are different to previous ones: We use the *graph trick* and *Floer homology* to prove a *multiplicity* result for periodic solutions of certain *Hamiltonian* delay equations on *manifolds*.

The paper is organized as follows. In the next section we describe the iterated graph construction in detail. In Section 3 we investigate how the action functionals of classical mechanics on \mathcal{P}_0 and \mathcal{P}_n relate under Ψ^n . In Sections 4 and 5 we give many examples of Hamiltonian delay equations for periodic orbits that are obtained by pulling back Hamiltonian chords on \mathcal{P}_n by Ψ^n and its variants. In Section 6 we prove Theorem 1.2, and in the last section we discuss improvements of Theorem 1.2 in various directions.

Acknowledgment. FS cordially thanks Augsburg University for its warm hospitality in the autumn of 2017. We are grateful to Felix Schmäscke for his explanations on orienting the space of Floer strips. PA is supported by DFG CRC/TRR 191, UF is supported by DFG FR/2637/2-1, and FS supported by SNF grant 200020-144432/1

2. THE ITERATED GRAPH CONSTRUCTION

Assume that (M, ω) is a symplectic manifold. Abbreviate by $S^1 = \mathbb{R}/\mathbb{Z}$ the circle and let

$$\mathcal{P}_0 = W^{1,2}(S^1, M)$$

be the space of free loops on M of Sobolev class $W^{1,2}$. The product manifold

$$(M_1, \omega_1) := (M \times M, \omega \oplus -\omega)$$

is again a symplectic manifold, and the diagonal

$$\Delta_1 = \{(z, z) : z \in M\} \subset M_1$$

is a Lagrangian submanifold canonically diffeomorphic to M via the map $(z, z) \mapsto z$. Abbreviate by

$$\mathcal{P}_1 = \{v \in W^{1,2}([0, 1], M_1) : v(0), v(1) \in \Delta_1\}$$

the space of paths of class $W^{1,2}$ in $M \times M$ that start and end at the diagonal. Note that the boundary condition makes sense since the elements of $W^{1,2}([0, 1], M_1)$ are continuous.

The Hilbert manifolds \mathcal{P}_0 and \mathcal{P}_1 are diffeomorphic via the diffeomorphism $\Psi: \mathcal{P}_0 \rightarrow \mathcal{P}_1$ given by

$$\Psi(v)(t) = \left(v\left(\frac{t}{2}\right), v\left(1 - \frac{t}{2}\right)\right), \quad t \in [0, 1]. \quad (4)$$

Indeed, since $v \in \mathcal{P}_0$ is a continuous loop it holds that $v(0) = v(1)$, so that

$$\Psi(v)(0) = (v(0), v(1)) = (v(0), v(0)) \in \Delta_1.$$

Moreover, $\Psi(v)(1) = (v(\frac{1}{2}), v(\frac{1}{2})) \in \Delta_1$, whence $\Psi(v)$ satisfies the required boundary conditions, i.e., $\Psi(v) \in \mathcal{P}_1$. The inverse $\Phi: \mathcal{P}_1 \rightarrow \mathcal{P}_0$ of Ψ can be explicitly written down: For $w = (w_1, w_2) \in \mathcal{P}_1$ it is given by

$$\Phi(w)(t) = \begin{cases} w_1(2t), & t \in [0, \frac{1}{2}], \\ w_2(2 - 2t), & t \in [\frac{1}{2}, 1]. \end{cases} \quad (5)$$

Since $w \in \mathcal{P}_1$ it holds that $w_1(0) = w_2(0)$ and $w_1(1) = w_2(1)$, which guarantees that $\Phi(w)(0) = \Phi(w)(1)$ and that $\Phi(w)$ is continuous at $t = \frac{1}{2}$, so that $\Phi(w)$ belongs to \mathcal{P}_0 . Further, $\Phi \circ \Psi = \text{id}_{\mathcal{P}_0}$ and $\Psi \circ \Phi = \text{id}_{\mathcal{P}_1}$. We note that it is important that we work with loops and paths of class $W^{1,2}$. For spaces $W^{k,2}$ of higher regularity, Ψ would not be surjective and Φ would not take values in the space of $W^{k,2}$ -loops.

The next iteration step proceeds as follows. Define the symplectic manifold

$$(M_2, \omega_2) := (M_1 \times M_1, \omega_1 \oplus -\omega_1) = (M^4, \omega \oplus -\omega \oplus -\omega \oplus \omega).$$

In (M_2, ω_2) we introduce two Lagrangian submanifolds, namely the product of the diagonal in M_1 with itself:

$$\Delta_2^0 := \Delta_1 \times \Delta_1 \subset M_2,$$

and the diagonal

$$\Delta_2^1 := \{(z, z) : z \in M_1\} \subset M_2.$$

Note that both Lagrangians Δ_2^0 and Δ_2^1 are canonically diffeomorphic to $M_1 = M \times M$. Their intersection

$$\Delta_2^0 \cap \Delta_2^1 = \{(z, z, z, z) : z \in M\} \subset M_2 = M^4$$

is the total diagonal in the fourfold product M^4 and is therefore canonically diffeomorphic to M . We introduce

$$\mathcal{P}_2 = \{v \in W^{1,2}([0, 1], M_2) : v(0) \in \Delta_2^0, v(1) \in \Delta_2^1\},$$

the space of paths in M_2 of Sobolev class $W^{1,2}$ which start at Δ_2^0 and end in Δ_2^1 . This Hilbert manifold is diffeomorphic to \mathcal{P}_1 and therefore also to \mathcal{P}_0 . A diffeomorphism $\Psi_1: \mathcal{P}_1 \rightarrow \mathcal{P}_2$ is given by the same formula as for Ψ , namely

$$\Psi_1(v)(t) = \left(v\left(\frac{t}{2}\right), v\left(1 - \frac{t}{2}\right)\right), \quad t \in [0, 1].$$

The inverse $\Phi_1: \mathcal{P}_2 \rightarrow \mathcal{P}_1$ of Ψ_1 is given for $w = (w_1, w_2) \in \mathcal{P}_2$ by the same formula as for Φ , namely

$$\Phi_1(w)(t) = \begin{cases} w_1(2t), & t \in [0, \frac{1}{2}], \\ w_2(2 - 2t), & t \in [\frac{1}{2}, 1]. \end{cases}$$

This construction can now be iterated as follows. Abbreviate $(M_0, \omega_0) := (M, \omega)$ and for $n \in \mathbb{N} \cup \{0\}$ recursively define

$$(M_{n+1}, \omega_{n+1}) := (M_n \times M_n, \omega_n \oplus -\omega_n).$$

Note that $M_n = M^{2^n}$ is the 2^n -fold product of M with itself. Let

$$\Delta_1^0 := \Delta_1^1 := \Delta_1 \subset M_1$$

be the diagonal in M_1 and define for $n \geq 1$ Lagrangian submanifolds $\Delta_{n+1}^0, \Delta_{n+1}^1$ in M_{n+1} , where the former is defined recursively as

$$\Delta_{n+1}^0 = \Delta_n^0 \times \Delta_n^1 \subset M_{n+1},$$

while the latter

$$\Delta_{n+1}^1 = \{(z, z) : z \in M_n\} \subset M_{n+1}$$

is just the diagonal Δ_{n+1} of M_{n+1} . Hence

$$\Delta_n^0 = \Delta_1 \times \Delta_1 \times \Delta_2 \times \Delta_3 \times \cdots \times \Delta_{n-1} \quad \text{for } n \geq 2. \quad (6)$$

Recall that two submanifolds S_0, S_1 of a manifold X are said to intersect cleanly if $S_0 \cap S_1$ is a submanifold of X and if for every $p \in S_0 \cap S_1$ it holds that

$$T_p(S_0 \cap S_1) = T_p S_0 \cap T_p S_1.$$

Lemma 2.1. (i) For $n \geq 1$ the intersection of the two Lagrangians Δ_n^0 and Δ_n^1 equals the total diagonal in $M_n = M^{2^n}$, i.e.,

$$\Delta_n^0 \cap \Delta_n^1 = \{(z, \dots, z) : z \in M\} =: \Delta_n^{\text{tot}}.$$

In particular, the intersection $\Delta_n^0 \cap \Delta_n^1$ is canonically diffeomorphic to M .

(ii) For $n \geq 1$ the submanifolds Δ_n^0 and Δ_n^1 intersect cleanly along Δ_n^{tot} .

Proof. (i) We argue by induction on n starting with $n = 1$. For $n = 1$ the assertion holds since $\Delta_1^0 \cap \Delta_1^1 = \Delta_1 \cap \Delta_1 = \Delta_1$ is the diagonal in M_1 . The inclusion $\Delta_{n+1}^0 \cap \Delta_{n+1}^1 \supset \Delta_{n+1}^{\text{tot}}$ follows by induction and the definitions. For the reverse inclusion we write an element $z \in \Delta_{n+1}^0 \cap \Delta_{n+1}^1$ as

$$z = (z_1, z_2), \quad z_1, z_2 \in M_n.$$

Since $z \in \Delta_{n+1}^1$, which is just the diagonal of M_{n+1} , we have

$$z_1 = z_2.$$

Since $z \in \Delta_{n+1}^0 = \Delta_n^0 \times \Delta_n^1$ we have

$$z_1 \in \Delta_n^0, \quad z_2 \in \Delta_n^1.$$

Combining these facts we derive

$$z_1 = z_2 \in \Delta_n^0 \cap \Delta_n^1$$

which by the induction hypothesis is Δ_n^{tot} . Using once more $z_1 = z_2$ it follows that $z \in \Delta_{n+1}^{\text{tot}}$.

(ii) For $n = 1$ the assertion is clear since $\Delta_1^0 = \Delta_1^1 = \Delta_1^{\text{tot}}$, and the inclusions

$$T_z(\Delta_n^0 \cap \Delta_n^1) \subset T_z \Delta_n^0 \cap T_z \Delta_n^1$$

for $z \in \Delta_n^0 \cap \Delta_n^1 = \Delta_n^{\text{tot}}$ and $n \geq 1$ are also obvious. For the reverse inclusions fix $z \in \Delta_{n+1}^{\text{tot}}$ and $\zeta \in T_z \Delta_{n+1}^0 \cap T_z \Delta_{n+1}^1$ and write $z = (z_1, z_1) \in M_n \times M_n$ and $\zeta = (\zeta_1, \zeta_2) \in T_{z_1} M_n \oplus T_{z_1} M_n$. Since $(\zeta_1, \zeta_2) \in T_z \Delta_{n+1}^1$ and Δ_{n+1}^1 is the diagonal in M_{n+1} , we have

$$\zeta_1 = \zeta_2.$$

Since $(\zeta_1, \zeta_2) \in T_z \Delta_{n+1}^0$ and $\Delta_{n+1}^0 = \Delta_n^0 \times \Delta_n^1$, we have

$$\zeta_1 \in T_{z_1} \Delta_n^0, \quad \zeta_2 \in T_{z_1} \Delta_n^1.$$

Together with the induction hypothesis it follows that

$$\zeta_1 = \zeta_2 \in T_{z_1} \Delta_n^0 \cap T_{z_1} \Delta_n^1 = T_{z_1} \Delta_n^{\text{tot}}.$$

This implies that $\zeta \in T_z \Delta_{n+1}^{\text{tot}}$. \square

For $n \geq 1$ we define the path space

$$\mathcal{P}_n = \{v \in W^{1,2}([0, 1], M_n) : v(0) \in \Delta_n^0, v(1) \in \Delta_n^1\}.$$

For every $n \in \mathbb{N} \cup \{0\}$ the same formula as for $\Psi = \Psi_0$ defines a diffeomorphism between Hilbert manifolds

$$\Psi_n : \mathcal{P}_n \rightarrow \mathcal{P}_{n+1}.$$

In particular, the following lemma holds.

Lemma 2.2. *The Hilbert manifolds \mathcal{P}_n for $n \in \mathbb{N} \cup \{0\}$ are all diffeomorphic to each other.*

The following topological lemma will imply that the solution spaces of Floer's equation relevant for the proof of Theorem 1.2 are C_{loc}^∞ -compact.

Lemma 2.3. *Assume that $[\omega]|_{\pi_2(M)} = 0$. Then $[\omega_n]|_{\pi_2(M_n, \Delta_n^i)} = 0$ for $i = 0, 1$ and for every $n \geq 1$.*

Proof. Since $\pi_2(M_n) = \oplus_{2^n} \pi_2(M)$, we have $[\omega_n]|_{\pi_2(M_n)} = 0$ for all $n \geq 1$. The lemma now follows from the following two claims and by induction.

Claim 1. *Let (X, ω) be a symplectic manifold with $[\omega]|_{\pi_2(X)} = 0$, and let Δ be the diagonal in $X \times X$. Then $[\omega \oplus -\omega]|_{\pi_2(X \times X, \Delta)} = 0$.*

Indeed, given $u = (u_0, u_1): (\mathbb{D}, S^1) \rightarrow (X \times X, \Delta)$ define $v: S^2 = \mathbb{C}\mathbb{P}^1 \rightarrow X$ by $v(z) = u_0(z)$ if $|z| \leq 1$ and $v(z) = u_1(1/\bar{z})$ if $|z| \geq 1$. Then $\int_{\mathbb{D}} u^*(\omega \oplus -\omega) = \int_{\mathbb{D}} u_0^* \omega - \int_{\mathbb{D}} u_1^* \omega = \int_{S^2} v^* \omega = 0$.

Claim 2. *For $i = 0, 1$ let $L_i \subset (X_i, \omega_i)$ be Lagrangian submanifolds of symplectic manifolds such that $[\omega_i]|_{\pi_2(X_i, L_i)} = 0$. Then*

$$[\omega_0 \oplus \omega_1]|_{\pi_2(X_0 \times X_1, L_0 \times L_1)} = 0.$$

Indeed, given $u: (\mathbb{D}, S^1) \rightarrow (X_0 \times X_1, L_0 \times L_1)$ write $u = (u_0, u_1)$ with $u_i: (\mathbb{D}, S^1) \rightarrow (X_i, L_i)$. Then $\int_{\mathbb{D}} u^*(\omega_0 \oplus \omega_1) = \int_{\mathbb{D}} u_0^* \omega_0 + \int_{\mathbb{D}} u_1^* \omega_1 = 0 + 0$. \square

3. THE ACTION FUNCTIONAL OF CLASSICAL MECHANICS

In the following discussion of the action functional of classical mechanics we assume that (M, ω) satisfies $[\omega]|_{\pi_2(M)} = 0$.

Denote by $\mathcal{P}_0^c \subset \mathcal{P}_0$ the connected component of *contractible* loops in the free loop space \mathcal{P}_0 of M . Pick a time-dependent Hamiltonian on M which depends periodically on time,

$$H \in C^\infty(M \times S^1, \mathbb{R}),$$

and for $t \in S^1$ abbreviate $H_t = H(\cdot, t) \in C^\infty(M, \mathbb{R})$. Given a contractible loop $v \in \mathcal{P}_0^c$ we can find a filling disk \bar{v} of v , namely a map $\bar{v} \in C^\infty([0, 1], \mathcal{P}_0^c)$ such that

$$\bar{v}(1) = v \tag{7}$$

and such that $\bar{v}(0)$ is a constant loop in \mathcal{P}_0^c . If $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ is the unit disk in the complex plane, we can think of a filling disk as a map

$$\bar{v}: \mathbb{D} \rightarrow M$$

by setting $\bar{v}(re^{2\pi it}) = \bar{v}(r)(t)$ for $re^{2\pi it} \in \mathbb{D}$. In this interpretation (7) becomes

$$\bar{v}(e^{2\pi it}) = v(t), \quad t \in S^1.$$

The action functional of classical mechanics $\mathcal{A}_H: \mathcal{P}_0^c \rightarrow \mathbb{R}$ is now defined by

$$\mathcal{A}_H(v) = - \int_{\mathbb{D}} \bar{v}^* \omega - \int_0^1 H_t(v(t)) dt$$

where \bar{v} is a filling disk for v . Due to the assumption that $[\omega]|_{\pi_2(M)} = 0$, the value $\mathcal{A}_H(v)$ depends only on v but not on the choice of the filling disk \bar{v} .

Recall that for every $n \geq 1$ there is a diffeomorphism

$$\Psi^n := \Psi_{n-1} \circ \Psi_{n-2} \circ \cdots \circ \Psi_0: \mathcal{P}_0 \rightarrow \mathcal{P}_n.$$

Abbreviate

$$\mathcal{P}_n^c := \Psi^n(\mathcal{P}_0^c) \subset \mathcal{P}_n$$

the connected component of \mathcal{P}_n which is the image of the contractible loops. Geometrically, the elements of \mathcal{P}_n^c are the paths in M_n from Δ_n^0 to Δ_n^1 which are homotopic through such paths to a constant path in the total diagonal $\Delta_n^{\text{tot}} = \Delta_n^0 \cap \Delta_n^1$.

We next discuss the pushforward of the action functional \mathcal{A}_H under the diffeomorphism Ψ^n , i.e.,

$$\Psi_*^n \mathcal{A}_H = \mathcal{A}_H \circ (\Psi^n)^{-1}: \mathcal{P}_n^c \rightarrow \mathbb{R}.$$

We first consider the unperturbed case \mathcal{A}_0 , where the Hamiltonian is zero, which is just (minus) the area functional. For $v \in \mathcal{P}_0^c$ let \bar{v} be a filling disk. The pushforward

$$\bar{w} := \Psi^n \circ \bar{v} \in C^\infty([0, 1], \mathcal{P}_n^c)$$

can be geometrically interpreted as a *filling half disk* of the path in M_n

$$w := \Psi^n \circ v \in \mathcal{P}_n^c$$

that starts at Δ_n^0 and ends at Δ_n^1 , see Figure 1. Indeed, if $\mathbb{D}_+ := \{z \in \mathbb{D} : \text{Im}(z) \geq 0\}$ denotes the unit half disk lying in the upper half plane, we can think of \bar{w} as a map

$$\bar{w}: \mathbb{D}_+ \rightarrow M_n$$

by setting $\bar{w}(re^{\pi it}) = \bar{w}(r)(t)$ for $re^{\pi it} \in \mathbb{D}_+$. Then \bar{w} satisfies the boundary conditions

$$\bar{w}(r) \in \Delta_n^0, \quad r \in [0, 1], \quad \bar{w}(-r) \in \Delta_n^1, \quad r \in [0, 1], \quad \bar{w}(e^{\pi it}) = w(t), \quad t \in [0, 1].$$

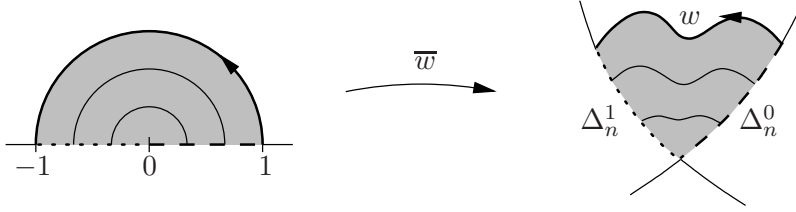
In particular, it holds that $\bar{w}(0) \in \Delta_n^0 \cap \Delta_n^1$. If we now apply the pushforward of the area functional \mathcal{A}_0 to $w \in \mathcal{P}_n^c$ we just obtain

$$(\Psi_*^n \mathcal{A}_0)(w) = - \int_{\mathbb{D}_+} \bar{w}^* \omega_n$$

where \bar{w} is a filling half disk of w , i.e., the pushforward of the area functional is the area functional again. For instance, for $n = 1$ and with $\bar{v}(r, t) =$

$$\bar{v}(re^{2\pi it}) \text{ and } \bar{w}(re^{\pi it}) = (\bar{w}_1(r, t), \bar{w}_2(r, t)) = \left(\bar{v}\left(r, \frac{t}{2}\right), \bar{v}\left(r, 1 - \frac{t}{2}\right)\right),$$

$$\begin{aligned} \int_{\mathbb{D}_+} \bar{w}^*(\omega \oplus -\omega) &= \int_{\mathbb{D}_+} \bar{w}_1^* \omega - \int_{\mathbb{D}_+} \bar{w}_2^* \omega \\ &= \int_{\mathbb{D}_+} \bar{v}^* \omega + \int_{\mathbb{D}_-} \bar{v}^* \omega \\ &= \int_{\mathbb{D}} \bar{v}^* \omega = -\mathcal{A}_0(v) = -(\Psi_*^1 \mathcal{A}_0)(w). \end{aligned}$$


 FIGURE 1. A filling half disk \bar{w}

We next discuss the pushforward of the Hamiltonian perturbation term

$$\mathcal{H}: \mathcal{P}_0^c \rightarrow \mathbb{R}, \quad v \mapsto \int_0^1 H_t(v(t)) dt.$$

For $n \geq 1$ define the time-dependent Hamiltonian

$$H^n \in C^\infty(M_n \times [0, 1], \mathbb{R})$$

for $z = (z_1, z_2, \dots, z_{2^n}) \in M_n = M^{2^n}$ and $t \in [0, 1]$ by the formula

$$H^n(z, t) = \frac{1}{2^n} \sum_{j=1}^{2^n} H(z_j, \tau_j^n(t)).$$

where $\tau_j^n(t)$ is recursively defined by $\tau_1^0(t) = t$ and

$$\tau_j^{n+1}(t) = \begin{cases} \frac{1}{2} \tau_j^n(t) & \text{if } 1 \leq j \leq 2^n, \\ 1 - \frac{1}{2} \tau_{j-2^n}^n(t) & \text{if } 2^n + 1 \leq j \leq 2^{n+1}. \end{cases}$$

Hence

$$\begin{aligned}
\tau_1^1(t) &= \frac{t}{2} & \tau_1^2(t) &= \frac{t}{4} & \tau_1^3(t) &= \frac{t}{8} \\
\tau_2^1(t) &= 1 - \frac{t}{2} & \tau_2^2(t) &= \frac{1}{2} - \frac{t}{4} & \tau_2^3(t) &= \frac{1}{4} - \frac{t}{8} \\
& & \tau_3^2(t) &= 1 - \frac{t}{4} & \tau_3^3(t) &= \frac{1}{2} - \frac{t}{8} \\
& & \tau_4^2(t) &= \frac{1}{2} + \frac{t}{4} & \tau_4^3(t) &= \frac{1}{4} + \frac{t}{8} \\
& & & & \tau_5^3(t) &= 1 - \frac{t}{8} \\
& & & & \tau_6^3(t) &= \frac{3}{4} + \frac{t}{8} \\
& & & & \tau_7^3(t) &= \frac{1}{2} + \frac{t}{8} \\
& & & & \tau_8^3(t) &= \frac{3}{4} - \frac{t}{8}.
\end{aligned}$$

Abbreviating again $H_t^n = H^n(\cdot, t) \in C^\infty(M_n, \mathbb{R})$ for $t \in [0, 1]$, the pushforward of \mathcal{H} at $w \in \mathcal{P}_n^c$ becomes

$$\Psi_*^n \mathcal{H}(w) = \int_0^1 H_t^n(w(t)) dt.$$

For instance, for $n = 1$ we find using (5),

$$\begin{aligned}
\Psi_*^1 \mathcal{H}(w) &= \mathcal{H}(\Phi(w)) \\
&= \int_0^{\frac{1}{2}} H_t(w_1(2t)) dt + \int_{\frac{1}{2}}^1 H_t(w_2(2-2t)) dt \\
&= \frac{1}{2} \int_0^1 H_{\frac{t}{2}}(w_1(t)) dt + \frac{1}{2} \int_0^1 H_{1-\frac{t}{2}}(w_2(t)) dt \\
&= \int_0^1 H_t^1(w(t)) dt.
\end{aligned}$$

In particular, the pushforward of the Hamiltonian perturbation term is again a Hamiltonian perturbation term, however of a rather specific form. Summarising we have shown that the pushforward of the action functional of classical mechanics at a point $w \in \mathcal{P}_n^c$ reads

$$\Psi_*^n \mathcal{A}_H(w) = - \int_{\mathbb{D}_+} \bar{w}^* \omega_n - \int_0^1 H_t^n(w(t)) dt \quad (8)$$

for a filling half disk \bar{w} of w .

Coming back to the unperturbed functionals, we have

Lemma 3.1. *The area functional $\Psi_*^n \mathcal{A}_0: \mathcal{P}_n^c \rightarrow \mathbb{R}$ is Morse–Bott with critical manifold the constant paths in Δ_n^{tot} , for every $n \geq 1$.*

First proof. The area functional $\mathcal{A}_0: \mathcal{P}_0^c \rightarrow \mathbb{R}$, $v \mapsto - \int_{\mathbb{D}} \bar{v}^* \omega$ is Morse–Bott with critical manifold the constant loops in M . Indeed, the differential of \mathcal{A}_0 at a loop $x \in \mathcal{P}_0^c$ is given by

$$d\mathcal{A}_0(x)v = - \int_{S^1} \omega(x(t)) (\dot{x}(t), v(t)) dt$$

for $v \in T_x \mathcal{P}_0^c = \{v \in W^{1,2}(S^1, TM) \mid v(t) \in T_{x(t)}M\}$. The non-degeneracy of ω thus implies that $\text{Crit } \mathcal{A}_0$ is the set of constant loops in M , that we identify with M . We must show that at each $x \in M$ the kernel of the Hessian $d^2 \mathcal{A}_0: T_x \mathcal{P}_0^c \times T_x \mathcal{P}_0^c \rightarrow \mathbb{R}$ is just $T_x M$. We compute

$$d^2 \mathcal{A}_0(x)(v, w) = - \int_{S^1} \omega(x(t)) (\dot{v}(t), w(t)) dt.$$

Invoking the non-degeneracy of ω again we find that $v \in \ker d^2 \mathcal{A}_0(x)$ if and only if $\dot{v}(t) \equiv 0$, that is, $v \in T_x M$ is constant.

Since $\Psi^n: \mathcal{P}_0^c \rightarrow \mathcal{P}_n^c$ is a diffeomorphism, the functional $\Psi_*^n \mathcal{A}_0: \mathcal{P}_n^c \rightarrow \mathbb{R}$ is also Morse–Bott with critical manifold the constant paths $\Psi^n(M) = \Delta_n^{\text{tot}}$.

Second proof. We fix $n \geq 1$ and abbreviate $L_i = \Delta_n^i$ for $i = 0, 1$. The differential of $\Psi_*^n \mathcal{A}_0 =: \mathcal{A}^n: \mathcal{P}_n^c \rightarrow \mathbb{R}$, $v \mapsto - \int_{\mathbb{D}_+} \bar{v}^* \omega_n$ at a path $x \in \mathcal{P}_n^c$ is given by

$$d\mathcal{A}^n(x)v = - \int_0^1 \omega_n(x(t)) (\dot{x}(t), v(t)) dt$$

for v in the tangent space $T_x \mathcal{P}_n^c$ of paths $v \in W^{1,2}([0, 1], TM_n)$ with $v(t) \in T_{x(t)}M_n$ and $v(i) \in T_{x(i)}L_i$ for $i = 0, 1$. The non-degeneracy of ω_n thus implies that $\text{Crit } \mathcal{A}^n$ is the set of constant paths in Δ_n^{tot} , that we identify with Δ_n^{tot} . We must show that at each $x \in \Delta_n^{\text{tot}}$ the kernel of the Hessian $d^2 \mathcal{A}^n: T_x \mathcal{P}_n^c \times T_x \mathcal{P}_n^c \rightarrow \mathbb{R}$ is just $T_x \Delta_n^{\text{tot}}$. For a constant path $x \in \Delta_n^{\text{tot}}$,

$$T_x \mathcal{P}_n^c = \{v \in W^{1,2}([0, 1], TM_n): v(i) \in T_x L_i \text{ for } i = 0, 1\}.$$

We compute

$$d^2 \mathcal{A}^n(x)(v, w) = - \int_0^1 \omega_n(x(t)) (\dot{v}(t), w(t)) dt.$$

Invoking the non-degeneracy of ω_n again we find that $v \in \ker d^2 \mathcal{A}^n(x)$ if and only if $\dot{v}(t) \equiv 0$, that is, $v \in T_x L_0 \cap T_x L_1$ is constant. Finally, since by Lemma 2.1 (ii) the two Lagrangians L_0 and L_1 intersect cleanly along Δ_n^{tot} , we have $T_x L_0 \cap T_x L_1 = T_x(L_0 \cap L_1) = T_x \Delta_n^{\text{tot}}$. \square

4. DELAY EQUATIONS

For $n \geq 1$ we consider a time-dependent Hamiltonian

$$K \in C^\infty(M_n \times [0, 1], \mathbb{R}),$$

abbreviate $K_t = K(\cdot, t) \in C^\infty(M_n, \mathbb{R})$ for $t \in [0, 1]$, and define the action functional $\mathcal{A}_K: \mathcal{P}_n^c \rightarrow \mathbb{R}$ by

$$\mathcal{A}_K(w) = - \int_{\mathbb{D}_+} \bar{w}^* \omega_n - \int_0^1 K_t(w(t)) dt$$

where \bar{w} is a filling half disk for w . Since $\Psi^n: \mathcal{P}_0^c \rightarrow \mathcal{P}_n^c$ is a diffeomorphism, the critical points of the pulled-back functional

$$\mathcal{A}_K \circ \Psi^n: \mathcal{P}_0^c \rightarrow \mathbb{R}$$

are the loops $(\Psi^n)^{-1}w$ where w is a critical point of \mathcal{A}_K , i.e., w is a Hamiltonian chord from Δ_0^n to Δ_1^n . In the special case that $K = H^n$ for a Hamiltonian $H \in C^\infty(M \times S^1, \mathbb{R})$, we have seen in (8) that

$$\mathcal{A}_{H^n} \circ \Psi^n = \mathcal{A}_H.$$

Therefore, in this case $(\Psi^n)^{-1}$ bijectively maps the time 1 H^n -Hamiltonian chords from Δ_0^n to Δ_1^n in M_n to the 1-periodic H -Hamiltonian orbits in M . In this section we shall see that for a general K the critical points of $\mathcal{A}_K \circ \Psi^n$ are 1-periodic solutions of a certain delay equation on M .

First take $n = 1$ and recall that $M_1 = M \times M$. Given a smooth function $K: M \times M \times [0, 1] \rightarrow \mathbb{R}$ define for $z_1, z_2 \in M$ the functions $M \times [0, 1] \rightarrow \mathbb{R}$ by

$$K_t^{z_2}(z_1) := K_t(z_1, z_2), \quad K_t^{z_1}(z_2) := K_t(z_1, z_2).$$

For $z = (z_1, z_2)$ we identify $T_z(M \times M)$ with $T_{z_1}M \times T_{z_2}M$. We can then write the Hamiltonian vector field X_K on $M \times M$ as

$$X_{K_t}(z) = (X_{K_t}^1(z), X_{K_t}^2(z)).$$

Using the definition of Hamiltonian vector fields on (M, ω) and on $(M \times M, \omega \oplus -\omega)$ we obtain

$$X_{K_t}^1(z) = X_{K_t^{z_2}}(z_1), \quad X_{K_t}^2(z) = -X_{K_t^{z_1}}(z_2). \quad (9)$$

Now assume that $w(t) = (w_1(t), w_2(t)) \in \mathcal{P}_1^c$ satisfies $\dot{w}(t) = X_{K_t}(w(t))$. Let $v = \Psi^{-1}w = \Phi w \in \mathcal{P}_0^c$ be the loop in M given by formula (5). Together with (9) for $t \in [0, \frac{1}{2}]$ we compute that

$$\begin{aligned} \dot{v}(t) &= \frac{d}{dt}(w_1(2t)) = 2\dot{w}_1(2t) = 2X_{K_{2t}}^1(w(2t)) \\ &= 2X_{K_{2t}}^1(v(t), v(1-t)) \\ &= 2X_{K_{2t}^{z_2=v(1-t)}}(v(t)). \end{aligned}$$

In the same way for $t \in [\frac{1}{2}, 1]$ we find that

$$\begin{aligned} \dot{v}(t) &= \frac{d}{dt}(w_2(2-2t)) = -2\dot{w}_2(2-2t) = -2X_{K_{2-2t}}^2(w(2-2t)) \\ &= -2X_{K_{2-2t}}^2(v(1-t), v(t)) \\ &= 2X_{K_{2-2t}^{z_1=v(1-t)}}(v(t)). \end{aligned}$$

Altogether, $v \in \mathcal{P}_0^c$ solves the equation

$$\dot{v}(t) = \begin{cases} 2X_{K_{2t}^{z_2=v(1-t)}}(v(t)), & t \in [0, \frac{1}{2}], \\ 2X_{K_{2-2t}^{z_1=v(1-t)}}(v(t)), & t \in [\frac{1}{2}, 1]. \end{cases} \quad (10)$$

Note that the loop v is smooth for $t \notin \{0, \frac{1}{2}\}$, but only continuous at $t = 0$ and $t = \frac{1}{2}$. To see what kind of an equation (10) is we specialize K in several ways:

Example 4.1. Assume first that K splits as a sum: $K_t(z_1, z_2) = F_t(z_1) + G_t(z_2)$. Then (10) becomes

$$\dot{v}(t) = \begin{cases} 2X_{F_{2t}}(v(t)), & t \in [0, \frac{1}{2}], \\ 2X_{G_{2-2t}}(v(t)), & t \in [\frac{1}{2}, 1]. \end{cases} \quad (11)$$

Thus v is a periodic orbit of the Hamiltonian flow defined by the “jumping” Hamiltonian vector field given by the right hand side of (11): v runs through a 2-gon, first under the law given by $2F_{2t}$, then under the law given by $2G_{2-2t}$, then again under the law given by $2F_{2t}$, and so on, see the left drawing in Figure 2. Note that the flow defined by (11) and in particular the periodic orbit v is actually smooth if F_t and G_t are constant in z for t near 0 and 1.

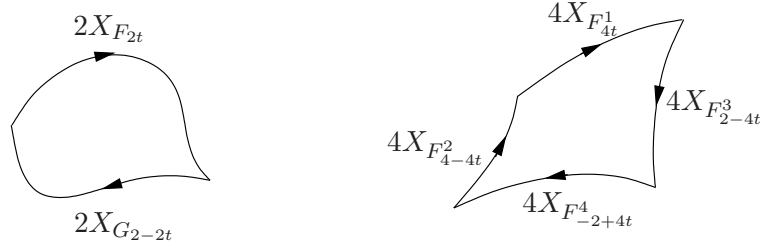


FIGURE 2. A solution of (11) and of (16)

Example 4.2. Assume now that K splits as a product: $K_t(z_1, z_2) = F_t(z_1)G_t(z_2)$. Then

$$X_{K_t}^{z_2}(z_1) = G_t(z_2)X_{F_t}(z_1), \quad X_{K_t}^{z_1}(z_2) = -F_t(z_1)X_{G_t}(z_2),$$

and so v solves

$$\dot{v}(t) = \begin{cases} 2G_{2t}(v(1-t))X_{F_{2t}}(v(t)), & t \in [0, \frac{1}{2}], \\ 2F_{2-2t}(v(1-t))X_{G_{2-2t}}(v(t)), & t \in [\frac{1}{2}, 1]. \end{cases} \quad (12)$$

This is a Hamiltonian *delay equation*: $\dot{v}(t)$ is not just given by a vector field depending on t and $v(t)$ (as it is the case for ordinary differential equations), but $\dot{v}(t)$ also depends on the place the loop v was at another time $1-t$.

More generally, if K splits as a finite sum of products,

$$K_t(z_1, z_2) = \sum_{j=1}^N F_t^j(z_1)G_t^j(z_2),$$

then v solves

$$\dot{v}(t) = \begin{cases} 2\sum_{j=1}^N G_{2t}^j(v(1-t))X_{F_{2t}^j}(v(t)), & t \in [0, \frac{1}{2}], \\ 2\sum_{j=1}^N F_{2-2t}^j(v(1-t))X_{G_{2-2t}^j}(v(t)), & t \in [\frac{1}{2}, 1]. \end{cases} \quad (13)$$

In the case $n = 1$ the general form of a delay equation we can achieve from pulling back Hamilton's equation for $K: M \times M \times [0, 1] \rightarrow \mathbb{R}$ by Ψ is equation (10). Note that if $M = \mathbb{R}^{2n}$ or a quotient thereof, or if K is supported in a Darboux chart, then (13) approximates (10) arbitrarily well. Indeed, trigonometric polynomials of period, say, 2π are linear combinations of functions of the form

$$F_t(z_1) G_t(z_2) = a(t) (e^{i(k_1, k'_1) \cdot (x_1, y_1)}) (e^{i(k_2, k'_2) \cdot (x_2, y_2)})$$

with $k_1, k'_1, k_2, k'_2 \in \mathbb{Z}$ and $a: [0, 1] \rightarrow \mathbb{R}$ a smooth function, and trigonometric polynomials are C^∞ -dense in the space of all smooth periodic functions.

Recall from the introduction that classically, a delay equation on a manifold is of the form

$$\dot{v}(t) = \sum_{j=1}^N f_t^j(v(t - \tau_j)) Y_t^j(v(t)) \quad (14)$$

for time-dependent functions $f_t^j: M \rightarrow \mathbb{R}$ and vector fields Y_t^j on M . The vector fields in (13) are not of this classical form, since the delay term is not taken at $v(t - \tau_j)$ but at $v(1 - t)$. As we shall see next, we can obtain classical delay equations (14) by iterating the graph construction.

We start with $n = 2$. For $w = (w_1, w_2, w_3, w_4) \in \mathcal{P}_2^c$ we compute, using formula (5) twice, that

$$v(t) = \Phi^2(w)(t) = \begin{cases} w_1(4t), & t \in [0, \frac{1}{4}], \\ w_3(2 - 4t), & t \in [\frac{1}{4}, \frac{1}{2}], \\ w_4(-2 + 4t), & t \in [\frac{1}{2}, \frac{3}{4}], \\ w_2(4 - 4t), & t \in [\frac{3}{4}, 1]. \end{cases} \quad (15)$$

Assume now that w is a Hamiltonian chord, $\dot{w} = X_K(w)$ for a smooth function $K: M_2 \times S^1 \rightarrow \mathbb{R}$. Recall that the symplectic form on M_2 is $\omega \oplus -\omega \oplus -\omega \oplus \omega$.

Example 4.3. Assume that K splits as a sum,

$$K_t(z) = F_t^1(z_1) + F_t^2(z_2) + F_t^3(z_3) + F_t^4(z_4).$$

Then (15) implies

$$\dot{v}(t) = \begin{cases} 4X_{F_{4t}^1}(v(t)), & t \in [0, \frac{1}{4}], \\ 4X_{F_{2-4t}^3}(v(t)), & t \in [\frac{1}{4}, \frac{1}{2}], \\ 4X_{F_{-2+4t}^4}(v(t)), & t \in [\frac{1}{2}, \frac{3}{4}], \\ 4X_{F_{4-4t}^2}(v(t)), & t \in [\frac{3}{4}, 1]. \end{cases} \quad (16)$$

Thus v runs through a 4-gon, solving a different Hamiltonian differential equation along each side, see the right drawing in Figure 2.

Example 4.4. Assume that K splits as a product,

$$K_t(z) = F_t^1(z_1) F_t^2(z_2) F_t^3(z_3) F_t^4(z_4). \quad (17)$$

Then

$$\begin{aligned} X_{K_t}(z) &= F_t^2(z_2) F_t^3(z_3) F_t^4(z_4) X_{F_t^1}(z_1) \\ &\quad - F_t^1(z_1) F_t^3(z_3) F_t^4(z_4) X_{F_t^2}(z_2) \\ &\quad - F_t^1(z_1) F_t^2(z_2) F_t^4(z_4) X_{F_t^3}(z_3) \\ &\quad + F_t^1(z_1) F_t^2(z_2) F_t^3(z_3) X_{F_t^4}(z_4). \end{aligned}$$

To simplify notation, we set given three different numbers $j_1, j_2, j_3 \in \{1, 2, 3, 4\}$,

$$\mathbf{F}_t^{j_1 j_2 j_3}(v(t)) := F_t^{j_1}(v(\frac{1}{2} + t)) \cdot F_t^{j_2}(v(\frac{1}{2} - t)) \cdot F_t^{j_3}(v(1 - t)).$$

With (15) and using that $v(t+1) = v(t)$ along 1-periodic solutions, we then find that v solves the delay equation

$$\frac{1}{4} \dot{v}(t) = \begin{cases} \mathbf{F}_{4t}^{432}(v(t)) X_{F_{4t}^1}(v(t)), & t \in [0, \frac{1}{4}], \\ \mathbf{F}_{2-4t}^{214}(v(t)) X_{F_{2-4t}^3}(v(t)), & t \in [\frac{1}{4}, \frac{1}{2}], \\ \mathbf{F}_{-2+4t}^{123}(v(t)) X_{F_{-2+4t}^4}(v(t)), & t \in [\frac{1}{2}, \frac{3}{4}], \\ \mathbf{F}_{4-4t}^{341}(v(t)) X_{F_{4-4t}^2}(v(t)), & t \in [\frac{3}{4}, 1]. \end{cases}$$

Taking sums of Hamiltonians of the form (17) we get more general delay vector fields. Now take a sum of the form

$$K_t(z) = F_t^1(z_1) F_t^4(z_4) + F_t^2(z_2) F_t^3(z_3).$$

Then v solves the equation

$$\frac{1}{4} \dot{v}(t) = \begin{cases} F_{4t}^4(v(t - \frac{1}{2})) X_{F_{4t}^1}(v(t)), & t \in [0, \frac{1}{4}], \\ F_{2-4t}^2(v(t - \frac{1}{2})) X_{F_{2-4t}^3}(v(t)), & t \in [\frac{1}{4}, \frac{1}{2}], \\ F_{-2+4t}^1(v(t - \frac{1}{2})) X_{F_{-2+4t}^4}(v(t)), & t \in [\frac{1}{2}, \frac{3}{4}], \\ F_{4-4t}^3(v(t - \frac{1}{2})) X_{F_{4-4t}^2}(v(t)), & t \in [\frac{3}{4}, 1]. \end{cases} \quad (18)$$

The four parts of equation (18) have the form (14) of a classical delay equation, although only with one delay, namely $\frac{1}{2}$.

For more than one delay one needs to look at higher values of n . We only work out the case $n = 3$.

Example 4.5. Take $n = 3$ and assume that $K: M_3 \times S^1 \rightarrow \mathbb{R}$ splits as a product:

$$K_t(z) = \prod_{j=1}^8 F_t^j(z_j).$$

Given seven different numbers $j_1, j_2, \dots, j_7 \in \{1, \dots, 8\}$ we set

$$\begin{aligned} \mathbf{F}_t^{j_1 \dots j_7}(v(t)) &:= F_t^{j_1}(v(\frac{1}{4} + t)) \cdot F_t^{j_2}(v(\frac{1}{2} + t)) \cdot F_t^{j_3}(v(\frac{3}{4} + t)) \cdot \\ &F_t^{j_4}(v(\frac{1}{4} - t)) \cdot F_t^{j_5}(v(\frac{1}{2} - t)) \cdot F_t^{j_6}(v(\frac{3}{4} - t)) \cdot F_t^{j_7}(1 - t). \end{aligned}$$

If w solves $\dot{w} = X_K(w)$, then $v(t) = \Phi^3(w)(t)$ solves the delay equation

$$\frac{1}{8}\dot{v}(t) = \begin{cases} \mathbf{F}_{8t}^{7465382}(v(t)) X_{F_{8t}^1}(v(t)), & t \in [0, \frac{1}{8}], \\ \mathbf{F}_{2-8t}^{3821746}(v(t)) X_{F_{2-8t}^5}(v(t)), & t \in [\frac{1}{8}, \frac{2}{8}], \\ \mathbf{F}_{-2+8t}^{4612538}(v(t)) X_{F_{-2+8t}^7}(v(t)), & t \in [\frac{2}{8}, \frac{3}{8}], \\ \mathbf{F}_{4-8t}^{8256174}(v(t)) X_{F_{4-8t}^3}(v(t)), & t \in [\frac{3}{8}, \frac{4}{8}], \\ \mathbf{F}_{-4+8t}^{6178253}(v(t)) X_{F_{-4+8t}^4}(v(t)), & t \in [\frac{4}{8}, \frac{5}{8}], \\ \mathbf{F}_{6-8t}^{2534617}(v(t)) X_{F_{6-8t}^8}(v(t)), & t \in [\frac{5}{8}, \frac{6}{8}], \\ \mathbf{F}_{-6+8t}^{1743825}(v(t)) X_{F_{-6+8t}^6}(v(t)), & t \in [\frac{6}{8}, \frac{7}{8}], \\ \mathbf{F}_{8-8t}^{5387461}(v(t)) X_{F_{8-8t}^2}(v(t)), & t \in [\frac{7}{8}, 1]. \end{cases}$$

If we now take sums of products

$$F_t^1(z_1)F_t^4(z_4)F_t^6(z_6)F_t^7(z_7) \quad \text{and} \quad F_t^2(z_2)F_t^3(z_3)F_t^5(z_5)F_t^8(z_8)$$

we obtain Hamiltonian delay equations with the three different time delays $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$.

More generally, pulling back Hamiltonian chords of suitable products from \mathcal{P}_n^c by Ψ^n we obtain 1-periodic solutions of Hamiltonian delay equations on M with time delays $\frac{j}{2^n}$, $j = 1, \dots, 2^n - 1$. In the next section we explain how to get delay equations with different delay times.

5. GENERALIZATION

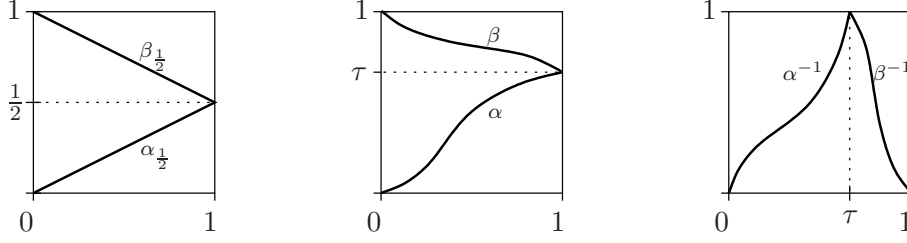
One way to get more general delay equations on M via Hamilton's equation on iterated path spaces is by replacing the diffeomorphism Ψ , Ψ_1, \dots defined in §2 by diffeomorphisms $\Psi \circ \sigma$, $\Psi_1 \circ \sigma_1, \dots$, where σ is a diffeomorphism of the circle S^1 and σ_1, \dots are diffeomorphisms of the interval $[0, 1]$.

Equivalently, we choose $\tau \in (0, 1)$ and smooth functions $\alpha, \beta: [0, 1] \rightarrow \mathbb{R}$ such that $\dot{\alpha}(t) > 0$ and $\dot{\beta}(t) < 0$ and $\alpha(0) = 0$, $\beta(0) = 1$, $\alpha(1) = \beta(1) = \tau$, see the figure below. For $n \geq 0$ define $\Psi_{\alpha\beta}: \mathcal{P}_n \rightarrow \mathcal{P}_{n+1}$ by

$$\Psi_{\alpha\beta}(v)(t) = (v(\alpha(t)), v(\beta(t))), \quad t \in [0, 1].$$

The inverse $\Phi_{\alpha\beta}$ of $\Psi_{\alpha\beta}$ is given by

$$\Phi_{\alpha\beta}(w)(t) = \begin{cases} w_1(\alpha^{-1}(t)), & t \in [0, \tau], \\ w_2(\beta^{-1}(t)), & t \in [\tau, 1]. \end{cases} \quad (19)$$



Example 5.1. For $r \in (0, 1)$ take

$$\begin{aligned} \alpha_r(t) &= rt, & \beta_r(t) &= 1 - (1-r)t, \\ \alpha_r^{-1}(t) &= \frac{t}{r}, & \beta_r^{-1}(t) &= \frac{1-t}{1-r}. \end{aligned}$$

Then $\Psi_{\alpha_r, \beta_r}(v)(t) = (v(rt), v(1 - (1-r)t))$ and

$$\Phi_{\alpha_r, \beta_r}(w)(t) = \begin{cases} w_1\left(\frac{t}{r}\right), & t \in [0, r], \\ w_2\left(\frac{1-t}{1-r}\right), & t \in [r, 1]. \end{cases}$$

For $n = 1$ and a Hamiltonian chord w of K_t as in Examples 4.1 and 4.2, the loop $v = \Phi_{\alpha\beta}(w)$ solves an equation similar to (11) and (12). In (11) the break time is now τ , and in (12) the delayed times are now $\beta \circ \alpha^{-1}(t)$ and $\alpha \circ \beta^{-1}(t)$, which are usually different; for instance $\beta_r \circ \alpha_r^{-1}(t) = 1 - \frac{1-r}{r}t$ and $\alpha_r \circ \beta_r^{-1}(t) = \frac{r}{1-r}(1-t)$ are equal only for $r = \frac{1}{2}$. Note that in contrast to the previous delayed times, that were of the form $\pm t \pm \tau$, in these delayed times t itself is scaled. This happens in many concrete delay equations. For instance, in the scalar non-autonomous linear pantograph equation

$$\dot{x}(t) = a x(\lambda t) + b x(t)$$

with parameter $\lambda \in (0, 1)$ the delay time $\tau(t)$ is given by $t - \tau(t) = \lambda t$, that is, $\tau(t) = (1 - \lambda)t$.

Now take $n = 2$. For $w = (w_1, w_2, w_3, w_4) \in \mathcal{P}_2^c$ we compute, using formula (19) twice, that

$$\Phi_{\alpha\beta}(w)(t) = \begin{cases} (w_1(\alpha^{-1}(t)), w_1(\alpha^{-1}(t))), & t \in [0, \tau], \\ (w_3(\beta^{-1}(t)), w_4(\beta^{-1}(t))), & t \in [\tau, 1]. \end{cases}$$

and

$$(\Phi_{\alpha_1\beta_1} \circ \Phi_{\alpha_2\beta_2})(w)(t) = \begin{cases} w_1(\alpha_2^{-1} \circ \alpha_1^{-1}(t)), & t \in [0, \alpha_1(\tau_2)], \\ w_3(\beta_2^{-1} \circ \alpha_1^{-1}(t)), & t \in [\alpha_1(\tau_2), \tau_1], \\ w_4(\beta_2^{-1} \circ \beta_1^{-1}(t)), & t \in [\tau_1, \beta_1(\tau_2)], \\ w_2(\alpha_2^{-1} \circ \beta_1^{-1}(t)), & t \in [\beta_1(\tau_2), 1]. \end{cases}$$

Abbreviating

$$\begin{aligned} T^1(t) &= (\alpha_1 \circ \alpha_2)^{-1}(t), & T^2(t) &= (\beta_1 \circ \alpha_2)^{-1}(t), \\ T^3(t) &= (\alpha_1 \circ \beta_2)^{-1}(t), & T^4(t) &= (\beta_1 \circ \beta_2)^{-1}(t), \end{aligned}$$

we then find that if w is a Hamiltonian chord, $\dot{w} = X_K(w)$, for a function $K: M_2 \times S^1 \rightarrow \mathbb{R}$ of the form

$$K_t(z) = F_t^1(z_1) F_t^4(z_4) + F_t^2(z_2) F_t^3(z_3),$$

then $v = (\Phi_{\alpha_1\beta_1} \circ \Phi_{\alpha_2\beta_2})(w)$ solves the delay equation

$$\dot{v}(t) = \begin{cases} \dot{T}^1(t) F_{T^1(t)}^4(v((T^4)^{-1} \circ T^1(t))) X_{F_{T^1(t)}^1}(v(t)), & t \in [0, \alpha_1(\tau_2)], \\ -\dot{T}^3(t) F_{T^3(t)}^2(v((T^2)^{-1} \circ T^3(t))) X_{F_{T^3(t)}^3}(v(t)), & t \in [\alpha_1(\tau_2), \tau_1], \\ \dot{T}^4(t) F_{T^4(t)}^1(v((T^1)^{-1} \circ T^4(t))) X_{F_{T^4(t)}^4}(v(t)), & t \in [\tau_1, \beta_1(\tau_2)], \\ -\dot{T}^2(t) F_{T^2(t)}^3(v((T^3)^{-1} \circ T^2(t))) X_{F_{T^2(t)}^2}(v(t)), & t \in [\beta_1(\tau_2), 1]. \end{cases} \quad (20)$$

For instance, with α_r and β_r as in Example 5.1 and $T_{r_1 r_2}^1 = (\alpha_{r_1} \circ \alpha_{r_2})^{-1}(t)$, etc., the four delayed times are

$$\begin{aligned} (T_{r_1 r_2}^4)^{-1} \circ T_{r_1 r_2}^1(t) &= \beta_{r_1} \circ \beta_{r_2} \circ \alpha_{r_2}^{-1} \circ \alpha_{r_1}^{-1}(t) = \frac{(1-r_1)(1-r_2)}{r_1 r_2} t + r_1 \\ (T_{r_1 r_2}^2)^{-1} \circ T_{r_1 r_2}^3(t) &= \beta_{r_1} \circ \alpha_{r_2} \circ \beta_{r_2}^{-1} \circ \alpha_{r_1}^{-1}(t) = \frac{r_2}{r_1} \frac{1-r_1}{1-r_2} (t - r_1) + 1 \\ (T_{r_1 r_2}^1)^{-1} \circ T_{r_1 r_2}^4(t) &= \alpha_{r_1} \circ \alpha_{r_2} \circ \beta_{r_2}^{-1} \circ \beta_{r_1}^{-1}(t) = \frac{r_1 r_2}{(1-r_1)(1-r_2)} (t - r_1) \\ (T_{r_1 r_2}^3)^{-1} \circ T_{r_1 r_2}^2(t) &= \alpha_{r_1} \circ \beta_{r_2} \circ \alpha_{r_2}^{-1} \circ \beta_{r_1}^{-1}(t) = \frac{r_1}{r_2} \frac{1-r_2}{1-r_1} (t - 1) + r_1 \end{aligned}$$

In particular, if $r_1 = r_2 = r$, then with $T_r^1 = T_{rr}^1$, etc.,

$$\begin{aligned} (T_r^4)^{-1} \circ T_r^1(t) &= \left(\frac{1-r}{r}\right)^2 t + r \\ (T_r^2)^{-1} \circ T_r^3(t) &= t + 1 - r \\ (T_r^1)^{-1} \circ T_r^4(t) &= \left(\frac{r}{1-r}\right)^2 (t - r) \\ (T_r^3)^{-1} \circ T_r^2(t) &= t + r - 1 \end{aligned}$$

For convenience we now assume that the functions F^j do not depend on time. Taking $F^2 = F^3 = 0$ and $r_1 + r_2 = 1$, i.e., $r_1 r_2 = (1 - r_1)(1 - r_2)$, we then find that v solves the equation

$$r_1 r_2 \dot{v}(t) = \begin{cases} F^4(v(t + r_1)) X_{F^1}(v(t)), & t \in [0, r_1 r_2], \\ 0, & t \in [r_1 r_2, r_1], \\ F^1(v(t - r_1)) X_{F^4}(v(t)), & t \in [r_1, 1 - r_2 + r_1 r_2], \\ 0, & t \in [1 - r_2 + r_1 r_2, 1]; \end{cases} \quad (21)$$

and taking $F^1 = F^4 = 0$ and $r_1 = r_2 = r$ we find that v solves

$$r(1-r) \dot{v}(t) = \begin{cases} 0, & t \in [0, r^2], \\ F^2(v(t - r + 1)) X_{F^3}(v(t)), & t \in [r^2, r], \\ 0, & t \in [r, 1 - r + r^2], \\ F^3(v(t - (1 - r))) X_{F^2}(v(t)), & t \in [1 - r + r^2, 1]. \end{cases} \quad (22)$$

Noting again that $v(t + r_1) = v(t - (1 - r_1))$ and $v(t - r + 1) = v(t - r)$ along 1-periodic solutions, we see that the solutions v of (21) and (22) solve on each of their four segments a delay equation of the classical form (14).

6. PROOF OF THE ARNOLD CONJECTURE FOR SOME DELAY EQUATIONS

Let (M, ω) be a symplectic manifold with $[\omega]|_{\pi_2(M)} = 0$. Fix $n \geq 1$, let Δ_n^0 and Δ_n^1 be the Lagrangian submanifolds of (M_n, ω_n) defined in §2, and as in §3 let \mathcal{P}_n^c be the space of $W^{1,2}$ -paths from Δ_n^0 to Δ_n^1 that are homotopic through such paths to a constant path in the total diagonal $\Delta_n^{\text{tot}} = \Delta_n^0 \cap \Delta_n^1$. Fix a function $K: M_n \times S^1 \rightarrow \mathbb{R}$. Since M is diffeomorphic to Δ_n^{tot} , we must show that the number of solutions of $\dot{x}(t) = X_K(x(t))$ that belong to \mathcal{P}_n^c is

- (i) at least $\text{cuplength}(\Delta_n^{\text{tot}}) + 1$;
- (ii) at least $\dim H_*(\Delta_n^{\text{tot}}; \mathbb{Z}_2)$ if $\phi_K^1(\Delta_n^0)$ intersects Δ_n^1 transversally.

Proof of (ii). By Lemma 2.3, $[\omega_n]$ vanishes on $\pi_2(M_n, \Delta_n^i)$ for $i = 0, 1$, and by Lemmas 2.1 (ii) and 3.1, Δ_n^0 and Δ_n^1 intersect cleanly along Δ_n^{tot} , which is the only critical manifold of the area functional $\Psi_*^n \mathcal{A}_0$. We can thus apply Pozniak’s theorem [19, Theorem 3.4.11] to the “isolating neighbourhood” \mathcal{P}_n^c of Δ_n^{tot} : The Floer homology $\text{HF}_*(\Delta_n^0, \Delta_n^1; \mathbb{Z}_2)$ is well-defined and isomorphic to $H_*(\Delta_n^{\text{tot}}; \mathbb{Z}_2)$. Since

$$\text{HF}_*(\Delta_n^0, \Delta_n^1; \mathbb{Z}_2) \cong \text{HF}_*(\phi_K^1(\Delta_n^0), \Delta_n^1; \mathbb{Z}_2)$$

and since the chain complex of the latter group is generated by the Hamiltonian chords of K from Δ_n^0 to Δ_n^1 , the claim follows. We remark that for the special case $n = 1$ we have $\Delta_1^0 = \Delta_1^1$, so that in this case one can apply Floer’s Lagrangian intersection result from [9] instead of Pozniak’s theorem.

Proof of (i). For the special case $n = 1$, namely when $\Delta_1^0 = \Delta_1^1$, the claim follows from the cup-length estimate for Lagrangian intersections proved independently by Floer [10] and Hofer [14].

In general, the claim readily follows from the general approach to cup-length estimates given by Albers–Hein [3], by taking a tubular neighbourhood U of Δ_n^{tot} in Δ_n^0 , and evaluating “capped Floer strips” with boundary on Δ_n^0 and Δ_n^1 at the boundary points $(jr, 0) \in \Delta_n^0$ for $j = 1, \dots, \text{cuplength}(M)$.

We outline the set-up from [3]. For this we assume that the reader is familiar with Lagrangian Floer homology. On the path space \mathcal{P}_n^c we consider the two functionals

$$\mathbb{A}_0(w) = - \int_{\mathbb{D}_+} \bar{w}^* \omega_n \quad \text{and} \quad \mathbb{A}_K(w) = - \int_{\mathbb{D}_+} \bar{w}^* \omega_n - \int_0^1 K_t(w(t)) dt.$$

Abbreviate $k = \text{cuplength}(M)$, and let β_r , $r \geq 0$, be a smooth family of compactly supported functions with $\beta_0 \equiv 0$ and β_r as in Figure 3 for $r \geq 1$. Fix an ω_n -compatible almost complex structure J on M_n , and for each $r \geq 0$ consider the the Floer equation

$$\partial_s u + J(u) (\partial_t u - X_{\beta_r(s)K(u)}) = 0 \tag{23}$$

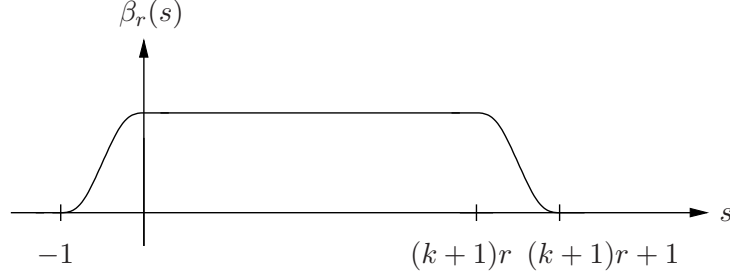


FIGURE 3.

for strips $u: \mathbb{R} \times [0, 1] \rightarrow M_n$ with $u(s, i) \in \Delta_n^i$ for $s \in \mathbb{R}$ and $i = 0, 1$. Denote the space of solutions of (23) with $\lim_{s \rightarrow \pm\infty} u(s) \in \Delta_n^{\text{tot}}$ by \mathcal{M}_r , and write $\mathcal{M}_{[0, R]} = \bigcup_{r \in [0, R]} \mathcal{M}_r$. These spaces are all C^∞ -compact. Indeed, the only obstructions to compactness are breaking and bubbling. From breaking one would get a non-constant J -holomorphic strip u asymptotic for $s \rightarrow \pm\infty$ to the critical manifold of \mathbb{A}_0 . By Lemma 3.1, the only critical component of \mathbb{A}_0 is $\Delta_n^{\text{tot}} = \mathbb{A}_0^{-1}(0)$. Hence the energy of u vanishes, whence u is constant, a contradiction. ‘‘Bubbling’’ means bubbling off of a holomorphic sphere or of a holomorphic disk at the boundary of a sequence of strips. Bubbling of spheres cannot occur since $[\omega_n]|_{\pi_2(M_n)} = 0$, and bubbling of disks cannot occur since it would occur at a point in $\Delta_n^0 \cup \Delta_n^1$, but $[\omega_n]|_{\pi_2(M_n), \Delta_n^i} = 0$ for $i = 0, 1$ by Lemma 2.3.

Now choose Morse functions f_1, \dots, f_k, f_* on Δ_n^{tot} , and extend the f_i to Morse functions \bar{f}_i on M_n that on a tubular neighbourhood U of Δ_n^{tot} in Δ_n^0 are of the form $f_i + q$ where $q: U \rightarrow \mathbb{R}$ is a positive definite quadratic form in normal direction to Δ_n^{tot} . Then $\text{Crit } f_i \subset \text{Crit } \bar{f}_i$.

From here on the argument is exactly as in Sections 4 and 5.2 of [3]: Fix critical points x_*^\pm of f_* and x_j of f_j and Riemannian metrics g_* and g_j on M_n , and denote by $W^u(x_*^-, f_*)$ and $W^s(x_*^+, f_*)$ the unstable manifold of x_*^- and the stable manifold of x_*^+ with respect to the negative gradient flow of g_* , etc. For $R \geq 0$ consider the closed subspace $\mathcal{M}_R(x_1, \dots, x_k, x_*^-, x_*^+)$ of \mathcal{M}_R consisting of those $u \in \mathcal{M}_R$ with

$$u(-\infty) \in W^u(x_*^-, f_*), \quad u(+\infty) \in W^s(x_*^+, f_*), \quad u(jR) \in W^s(x_j, \bar{f}_j)$$

for $j = 1, \dots, k$. The mod 2 count of zero-dimensional components of the spaces $\mathcal{M}_R(x_1, \dots, x_k, x_*^-, x_*^+)$ induces, after identifying Morse (co-)homology with singular (co-)homology, operations

$$\begin{aligned} \Theta_R: H^*(\Delta_n^{\text{tot}}) \otimes \dots \otimes H^*(\Delta_n^{\text{tot}}) \otimes H_*(\Delta_n^{\text{tot}}) &\rightarrow H_*(\Delta_n^{\text{tot}}) \\ a_1 \otimes \dots \otimes a_k \otimes b &\mapsto (a_1 \cup \dots \cup a_k) \cap b. \end{aligned}$$

Since $\mathcal{M}_{[0, R]}$ is compact, the operation Θ_R does not depend on R , and since \mathcal{M}_0 consists of the constant maps to Δ_n^{tot} , Θ_0 is a Morse homological realization of the cup product. By assumption, Θ_0 does not vanish, and

hence all the maps Θ_R are non-zero. Using this along a sequence $R \rightarrow \infty$ one shows as in [3, §4] that \mathbb{A}_K has at least $k + 1$ critical points.

7. IMPROVEMENTS

In this paper we have only used basic results on Lagrangian Floer homology in rather simple settings. More sophisticated versions imply stronger forms of Theorem 1.2. We here describe three such improvements.

For this, the following lemma will be useful. Recall that a pair of submanifolds S_0, S_1 of a manifold X is said to be relative spin if there exists a cohomology class $\alpha \in H^2(X; \mathbb{Z}_2)$ such that $\iota_i^* \alpha = w_2(S_i)$ for $i = 0, 1$. Here, $w_2(S_i)$ is the second Stiefel–Whitney class of the tangent bundle TS_i , and $\iota_i: S_i \rightarrow X$ are the inclusions.

Lemma 7.1. *The pair $\Delta_n^0, \Delta_n^1 \subset M_n$ is relatively spin, for every $n \geq 1$.*

Proof. Abbreviate the 2^n -fold product $1 \otimes \cdots \otimes 1 \in H^0(M_n; \mathbb{Z}_2)$ by 1_n , and let $w = w_2(M) \in H^2(M; \mathbb{Z}_2)$ be the second Stiefel–Whitney class of TM .

For $n = 1$, $\Delta_1^0 = \Delta_1^1 = \Delta_1$ is spin in $M \times M = M_1$. We can take

$$\alpha_1 = w \otimes 1 \quad \text{or} \quad \bar{\alpha}_1 = 1 \otimes w.$$

Indeed, let $\iota: \Delta_1 \rightarrow M \times M$ be the inclusion, let $\text{pr}_1: M \times M \rightarrow M$ be the projection to the first factor, and let $f_1: \Delta_1 \rightarrow M$ be the diffeomorphism $(x, x) \mapsto x$ to the first factor. Then $\text{pr}_1 \circ \iota = f_1$ and $f_1^* w = w_2(\Delta_1)$. Hence

$$\iota^* \alpha_1 = \iota^*(w \otimes 1) = \iota^*(\text{pr}_1^* w) = f_1^* w = w_2(\Delta_1).$$

Similarly, working with the second factor of $M \times M$, we find $\iota^* \bar{\alpha}_1 = w_2(\Delta_1)$.

For $n = 2$, $\Delta_2^0 = \Delta_1 \times \Delta_1$ and $\Delta_2^1 = \Delta_2$ are relatively spin in M_2 . We can take

$$\alpha_2 = \alpha_1 \otimes 1_1 + 1_1 \otimes \bar{\alpha}_1 \quad \text{or} \quad \bar{\alpha}_2 = \bar{\alpha}_1 \otimes 1_1 + 1_1 \otimes \alpha_1.$$

Proceeding inductively we define $\alpha_{n+1}, \bar{\alpha}_{n+1} \in H^2(M_{n+1}; \mathbb{Z}_2)$ by

$$\alpha_{n+1} = \alpha_n \otimes 1_n + 1_n \otimes \bar{\alpha}_n \quad \text{and} \quad \bar{\alpha}_{n+1} = \bar{\alpha}_n \otimes 1_n + 1_n \otimes \alpha_n.$$

Using (6) and elementary properties of Stiefel–Whitney classes one sees that both α_n and $\bar{\alpha}_n$ define relative spin structures for $\Delta_n^0, \Delta_n^1 \subset M_n$. \square

Lemma 7.1 implies that after a perturbation (making them manifolds), all the moduli spaces of Floer strips used in the proofs in Section 6 are orientable, see [11, §8] or [22]. This leads to the following improvements of Theorem 1.2.

1. *The lower bounds in Theorem 1.2 can be replaced by $\text{cuplength}(M; \mathbb{F}) + 1$ and by $\dim H_*(M; \mathbb{F})$, for every field \mathbb{F} .*

Proof. The Morse- and Floer moduli spaces used in the proofs of (i) and (ii) are orientable, and Pozniak’s theorem holds over \mathbb{F} , see [22]. \square

2. *In the situation of Theorem 1.2 assume that $n = 1$. Then the lower bound $\text{cuplength}(M) + 1$ in assertion (i) can be replaced by $\dim M + 1$.*

Proof. The (perturbed) spaces \mathcal{M}_r of solutions of (23) with boundary on Δ_1 are orientable manifolds, and the same is true for the space \mathcal{M}_∞ of uncapped solutions of equation (23) with $\beta \equiv 1$. As in [14] one finds that the evaluation map $\text{ev}: \mathcal{M}_\infty \rightarrow M \times M$, $u \mapsto u(0, 0)$ induces an injection $\text{ev}^*: H^*(M \times M; G) \rightarrow H^*(\mathcal{M}_\infty; G)$ for every coefficient group G . Since $[\omega]|_{\pi_2(M)} = 0$, the Lusternik–Schnirelmann category of M is $\dim M + 1$, see [21]. Hence, by the proof of Theorem A in [20], the number of points in $\phi_K(\Delta_1) \cap \Delta_1$ is at least $\dim M + 1$. \square

3. Giving up the standard assumption of the paper, we now consider a closed symplectic manifold M with $[\omega]|_{\pi_2(M)} \neq 0$. Then the action functional \mathbb{A}_K is well-defined only on the cover of \mathcal{P}_n^c consisting of pairs $(w, [\bar{w}])$ of paths $w \in \mathcal{P}_n^c$ together with a homotopy class of filling half disks \bar{w} . We expect that the Floer homology of \mathbb{A}_K can still be constructed as a Morse–Novikov theory of this cover, with coefficients in a Novikov ring Λ over \mathbb{Q} , by combining [22] with [12]. This uses Lemma 7.1, that both Δ_n^0 and Δ_n^1 are fixed point sets of (the obvious) anti-symplectic involutions, and heavy technical machinery dealing with the occurrence of bubbling of spheres and disks. A further complication is that the critical manifold $\Delta_n^{\text{tot}} \subset M_n$ now lifts to a disjoint union of copies of Δ_n^{tot} , whence the Floer strips can now interact between different connected components of the critical manifold.

For assertion (ii) of Theorem 1.2, Floer homology should still compute the homology of M , but with coefficients in the rational Novikov ring Λ , and so assertion (ii) with lower bound $\dim H_*(M; \mathbb{Q})$ should hold for all closed symplectic manifolds. For $n = 1$ this is a theorem, see [12, Theorem 1.9], and for $n \geq 2$ this should follow by combining [22] with [12].

For assertion (i), however, the cuplength estimate has to be replaced by a rational quantum-type cuplength estimate, similar to the lower bounds given in [16, 23], see [12, Theorem 1.9] for the case $n = 1$.

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