Characteristic Classes and Rozansky-Witten Invariants of Compact Hyperkähler Manifolds

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Kurzzusammenfassung. Huybrechts zeigt in [**18**], daß die Eulercharakteristik eines holomorphen Geradenbündels L auf einer irreduziblen kompakten Hyperkählermannigfaltigkeit durch $\chi(X,L) = \sum_{k=0}^{n} a_{2k}/(2k)!q(L)^k$ gegeben ist, wobei die a_{2k} universelle Konstanten sind, welche nur von X abhängen und q die (nicht normalisierte) Beauville-Bogomolov-Form von X ([**3**]) ist.

Nach den Ideen von Hitchin und Sawon ([**15**]) machen wir von der Theory der Rozansky-Witten-Invarianten Gebrauch, um eine geschlosse Formel zu gewinnen, welche die a_{2k} durch polynomiale Ausdrücke in gewissen Chernzahlen von X ausdrückt. Unsere Methoden können benutzt werden, um diejenigen Anteile gewisser charakteristischer Klassen von X zu bestimmen, welche in dem von $\mathrm{H}^2(X,\mathbb{C})$ erzeugten Unterring im Kohomologiering liegen.

Ein Kapitel dieser Arbeit beschäftigt sich mit der Berechnung der Chernzahlen der verallgemeinerten Kummervarietäten ([**3**]). Wir geben eine Formel an, welche sie zu den Chernzahlen der Hilbertschemata von Punkten auf einer Fläche in Beziehung setzt. Wir verwenden diese Formel, um die Chernzahlen der verallgemeinerten Kummervarietäten bis zur Dimension zwanzig zu bestimmen.

Abstract. Huybrechts showed in [**18**] that the Euler characteristic of a holomorphic line bundle L on an irreducible compact hyperkähler manifold X is given by $\chi(X,L) = \sum_{k=0}^{n} a_{2k}/(2k)!q(L)^k$ where the a_{2k} are universal constants depending only on X, and q is the (unnormalised) Beauville-Bogomolov form of X ([**3**]).

Similar to the ideas of Hitchin and Sawon ([**15**]), we use the theory of Rozansky-Witten invariants to develop a closed formula that expresses the a_{2k} by polynomial expressions in certain Chern numbers of X. Our methods can be used to determine those components of certain characteristic classes of X that lie in the subring generated by $\mathrm{H}^2(X,\mathbb{C})$ inside the cohomology ring.

One chapter of this thesis is concerned with the calculation of the Chern numbers of the generalised Kummer surfaces ([**3**]). We give a formula that links them to the Chern numbers of Hilbert schemes of points on a surface. We use this formula to compute the Chern numbers of all generalised Kummer varieties up to dimension twenty.

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Vorwort

Diese Arbeit ist über einen Zeitraum von zweieinviertel Jahren entstanden. An dieser Stelle möchte ich mich bei all denen bedanken, die zu diesem Entstehen in der einen oder der anderen Weise beigetragen haben.

An erster Stelle möchte ich Daniel Huybrechts danken. Als ich ihn Mitte 1999 fragte, ob er bereit wäre, nach Abschluß meiner Diplomarbeit mein Doktorvater zu werden, kannte ich ihn nur durch die Teilnahme an seiner Vorlesung „Turbo-Einführung in die Komplexe Geometrie“ und an einem Seminar über derivierte Kategorien. Ich bin dankbar, daß er sofort bereit war, meine Betreuung zu übernehmen und diese Aufgabe stets ernstgenommen hat. Mit Fragen konnte ich jederzeit zu ihm kommen.

Auch zahlreiche weitere Mathematiker, die im Laufe der Zeit am Institut tätig waren, nahmen sich bereitwillig Zeit für meine Fragen und trugen so zum schnellen Gelingen meiner Arbeit bei. Hervorheben möchte ich hier Manfred Lehn, bei dem ich immer ein offenes Ohr fand, so daß ich eigentlich zwei volle Betreuer für meine Doktorarbeit hatte. Er wird der zweite Gutachter für meine Dissertation sein.

Die viele Zeit seit Februar 2000, die ich in meinem Büro im Mathematischen Institut verbracht habe, um meine Arbeit zu schreiben oder meinen Pflichten als wissenschaftliche Hilfskraft nachzukommen, habe ich gerne dort zugebracht. Das habe ich zum großen Teil Michael Britze zu verdanken, mit dem ich das Büro teile. Es hat mir sehr viel Spaß gemacht, neben ihm zu arbeiten, mit ihm zu scherzen oder auch ernsthaft zu diskutieren, mal über Mathematik, mal über Politik, mal über das Leben an sich.

Ich bin froh darüber, daß ich meine Frau Bettina gefunden habe. Ihr danke ich, daß sie mich mit viel Verständnis auf dem Weg durch die Promotion begleitet hat. Sie war es, die mich auch immer wieder angetrieben hat, wenn meine natürliche Trägheit mich von irgendwelchem notwendigen „Papierkram“ abgehalten hat. Dadurch, daß es sie gibt, weiß ich, daß ich jederzeit jemandem habe, der zu mir hält, und dem ich hundertprozentig vertrauen kann.

Ferner danke ich meiner Familie und allen meinen Freunden, die unerwähnt geblieben sind.

Köln, im April 2002 Marc A. Nieper-Wißkirchen

Zusammenfassung

Kompakte Kählermannigfaltigkeiten, welche eine (eindeutige) *holomorphe* symplektische 2-Form tragen, scheinen relativ selten zu sein. In der Dimension zwei ist das einzige Beispiel dieser *irreduziblen holomorphen symplektischen Mannigfaltigkeiten* die wohlbekannte K3-Fläche. In den Dimensionen vier, sechs, acht, und so weiter sind jeweils nur zwei oder drei Typen holomorpher symplektischer Mannigfaltigkeiten bekannt.

Es ist wünschenswert, möglichst viele Eigenschaften zu kennen, die alle holomorphen symplektischen Mannigfaltigkeiten gemeinsam haben und die sie von anderen kompakten Kählermannigfaltigkeiten unterscheiden. Vielleicht wird das schließlich zu einem Klassifikationstheorem für holomorphe symplektische Mannigfaltigkeiten führen. Die Theorie begann sich erst richtig mit Beauvilles Artikel [**3**] zu entfalten. Er definierte eine quadratische Form auf der zweiten Kohomologiegruppe einer holomorphen symplektischen Mannigfaltigkeit, welche als Verallgemeinerung der Schnittpaarung auf der mittleren Kohomologie einer K3-Fläche angesehen werden kann, und bewies schon das „Lokale Torelli-Theorem“. Von den folgenden Jahren an bis heute ist eine allgemeine Theorie um diese Mannigfaltigkeiten entstanden, welche zu einem großen Teil der Theorie der K3-Flächen entspricht (zu dieser Theorie findet sich viel in [**4**]).

Holomorphe symplektische Mannigfaltigkeiten können von differential-geometrischen Objekten, den sogenannten *kompakten Hyperkählermannigfaltigkeiten* konstruiert werden. Nach dem Holonomieprinzip besitzt jede Riemannsche Mannigfaltigkeit (X, g) mit einer $\mathrm{U}(n)$-Holonomie eine komplexe Struktur, so daß X eine Kählermannigfaltigkeit mit Kählermetrik g wird. Eine (irreduzible) kompakte Hyperkählermannigfaltigkeit ist eine Riemannsche Mannigfaltigkeit (X, g) mit der symplektischen Gruppe $\mathrm{Sp}(n)$ als Holonomiegruppe. Es folgt, daß eine ganze 2-Sphäre komplexer Strukturen auf X existiert, für die jeweils X eine Kählermannigfaltigkeit mit Kählermetrik g ist. Darüber hinaus zeigt eine lokale Rechnung, daß alle diese komplexen Mannigfaltigkeiten irreduzible holomorphe symplektische Mannigfaltigkeiten sind. Auf der anderen Seite läßt sich mit dem Existenzsatz über Calabi-Yau-Metriken ([**31**]) zeigen, daß alle irreduziblen holomorphen symplektischen Mannigfaltigkeiten auf diese Weise entstehen. Aufgrund dieser Verbindung zwischen kompakten Hyperkähler- und holomorphen symplektischen Mannigfaltigkeiten, ist es wichtig zu wissen, welche der Eigenschaften irreduzibler holomorpher symplektischer bzw. kompakter Hyperkählermannigfaltigkeiten topologische, holomorphe oder Riemannsche sind.

Meine Arbeit für diese Dissertation ist durch die allgemeine Theorie der kompakten Hyperkähler- bzw. der holomorphen symplektischen Mannigfaltigkeiten angeregt worden. Die von mir erhaltenen Ergebnisse, welche in dieser Arbeit enthalten sind, sind größtenteils kohomologischer Natur. Ich möchte jetzt den Inhalt dieser Dissertation genauer beschreiben:

Sowohl, um es dem Leser einfacher zu machen, als auch um die nötigen Bezeichnungen einzuführen, habe ich Kapitel 2 eingebunden. Es enthält die exakte Definition einer kompakten Hyperkählermannigfaltigkeit, einer holomorphen symplektischen und ihre oben erwähnte Verbindung, sowie eine Aufstellung aller bekannten Beispiele. Es enthält außerdem einen Abschnitt über die Chernklassen dieser Mannigfaltigkeiten. Dort wird gezeigt, daß diese eigentlich topologische Invarianten sind. Alle Ergebnisse dieses Abschnittes sind wohlbekannt und zum Beispiel auch in [**18**] zu finden.

Das nächste Kapitel (Kapitel 3) befaßt sich mit der Beauville-Bogomolov-Form, welche auf jeder irreduziblen kompakten Hyperkählermannigfaltigkeit existiert. Ein wichtiges Anliegen in diesem Kapitel ist es zu zeigen, daß diese Form eine topologische Invariante ist. Das ist zwar schon bekannt, ich habe dieses Resultat aber nirgendwo explizit gefunden. Die quadratische Form kann dazu verwendet werden, den von allen Klassen der zweiten Kohomologiegruppe einer kompakten Hyperkählermannigfaltigkeiten erzeugten Unterring SH^2 des Kohomologieringes zu beschreiben. Es wird gezeigt, daß der Raum der Kohomologieklassen in SH^2, welche konstanten Hodge-Typs $(2j, 2j)$ auf jeder Deformation sind, eindimensional ist. Es finden sich einige explizite Berechnungen in diesem Kapitel, welche zum Beispiel in Verbindung mit den Ergebnissen aus Kapitel 6 dazu verwendet werden können, zu entscheiden, ob die zweite Chernklasse c_2 in SH^2 liegt.

Huybrechts zeigte (in [**18**]), daß für jede irreduzible holomorphe symplektische Mannigfaltigkeit der komplexen Dimension $2n$ ein Polynom P_X vom Grad n über $\mathbb{C}$ existiert, so daß die Eulercharakteristik eines holomorphen Geradenbündels L auf X durch

$$\chi(X, L) = P_X(q(c_1(L))) \tag{1}$$

gegeben ist, wobei q die (nicht normalisierte) Beauville-Bogomolov-Form von X ist. Eines der wichtigsten Ergebnisse dieser Dissertation ist die Bestimmung von P_X für gegebenes X. Es zeigt sich, daß P_X nur von den charakteristischen Zahlen von X abhängt, d.h. von seiner komplexen Kobordismusklasse. Kapitel 4–6 drehen sich um den Beweis einer geschlossenen Formel für P_X in Termen der charakteristischen Zahlen von X. Die Formel ist

$$P_X(\lambda) = \int_X \exp\left(-2\sum_{k=1}^{\infty} b_{2k} s_{2k} T_k \left(1 + \frac{\lambda}{2}\right)\right), \tag{2}$$

wobei die b_{2k} die modifizierten Bernoulli-Zahlen, die s_{2k} die Komponenten des Cherncharakters von X (also Polynome in den Chernklassen) und die T_k die Tschebyscheff-Polynome sind.

Der Beweis folgt den Ideen, die Hitchin und Sawon in [**15**] benutzten, um einen Ausdruck für die L^2-Norm des Riemannschen Krümmungstensors einer irreduziblen kompakten Hyperkählermannigfaltigkeit zu beweisen. Er macht von der Theorie der *Rozansky-Witten-Invarianten* kompakter Hyperkählermannigfaltigkeiten, welche in [**27**] eingeführt wurden, ausführlichen Gebrauch. Jeder eckenorientierte unitrivalente Graph Γ, in dieser Dissertation ein *Jacobi-Diagramm* genannt, kann zusammen mit der Atiyahklasse einer holomorphen symplektischen Mannigfaltigkeiten benutzt werden, um eine Kohomologieklasse $\mathrm{RW}(\Gamma)$ kanonisch zu definieren. Wie Hitchin und Sawon bemerkten, entstehen alle Chernklassen auf diese Art und Weise. Es gibt eine bestimmte Homologierelation auf dem Raum aller Jacobi-Diagramme. Zwei homologe Graphen liefern dieselben Rozansky-Witten-Invarianten. Das wird in dieser Dissertation benutzt, um Relationen auf der Ebene der Graphenhomologie auf die Ebene der Kohomologie einer holomorphen symplektischen Mannigfaltigkeit zu drücken. Eine wichtige Graphenhomologie-Relation ist durch das „Wheeling Theorem“ (in [**29**]) gegeben.

In Kapitel 4 führen wir den Raum der Graphenhomologie ein, welcher Objekt vielfältiger mathematischer Untersuchungen ist. Wir werden einen umfassenden Formalismus entwickeln, welcher sich aber in den folgenden Kapiteln als sehr vorteilhaft herausstellen wird. Das „Wheeling Theorem“ liefert uns viel mehr Relationen auf der Graphenhomologie als wir für die Beweise unserer Resultate in den nächsten Kapiteln benötigen. Deswegen habe ich in Kapitel 4 das Theorem 3 aufgenommen, welches in gewisser Weise ein Korollar des „Wheeling Theorem“ ist. Allerdings ist es mir gelungen, das Theorem 3 mittels elementarer Methoden zu beweisen, während das „Wheeling Theorem“ mit Mitteln der Knotentheorie bewiesen wird. Daher entschied ich mich, Theorem 3 zusammen mit seinem Beweis in diese Dissertation einzubinden. Am Ende von Kapitel 4 wird eine gewisse $\mathfrak{sl}_2$-Operation auf einem erweiterten Graphenhomologieraum untersucht. Momentan weiß ich nicht, ob das irgendwelche Implikationen für die Theorie der Graphenhomologie nach sich zieht. Auf der anderen Seite ist diese Struktur mit einer bestimmten $\mathfrak{sl}_2$-Operation auf der Kohomologie einer irreduziblen holomorphen symplektischen Mannigfaltigkeit verträglich.

Kapitel 5 befaßt sich mit der Definition der *Rozansky-Witten-Klassen* einer irreduziblen holomorphen symplektischen Mannigfaltigkeit und mit ihren Eigenschaften. Bisher ist in der Literatur nur die Bezeichnung von *Rozansky-Witten-Invarianten* aufgetaucht. Diese sind im Prinzip Integrale über Produkte von Rozansky-Witten-Klassen, wie sie hier definiert sind, mit der holomorphen symplektischen Form und ihrem komplex-Konjugierten. Für jede holomorphe Mannigfaltigkeit X zusammen mit einer fixierten holomorphen 2-Form σ erhalten wir eine lineare Abbildung $\mathrm{RW} : \hat{\mathcal{B}} \to \mathrm{H}^*(X, \Omega^*_X)$, wobei $\hat{\mathcal{B}}$ der Raum der Graphenhomologie ist. Beide Räume, Definitions- und Wertebereich von RW, tragen zusätzliche Strukturen: beide sind doppelt-graduierte $\mathbb{Q}$-Algebren, besitzen eine bestimmte Bilinearform und sind auf kanonische Weise $\mathfrak{sl}_2$-Moduln. Wir zeigen, daß die Abbildung RW mit diesen Strukturen verträglich ist.

Unter Benutzung der Theorie, die in den zwei vorhergehenden Kapiteln entwickelt worden ist, können wir in Kapitel 6 den Beweis der Formel (2) für die holomorphe Eulercharakteristik eines Geradenbündels präsentieren. Wir beweisen sogar einiges mehr. Zusammen mit den Ergebnissen aus Kapitel 3 können wir die Anteile gewisser charakteristischer Klassen in SH^2 berechnen. Außerdem, quasi als Nebenprodukt, zeigen wir die Existenz der Beauville-Bogomolov-Form in diesem Kapitel. Das ist natürlich kein neues Resultat. Es wird lediglich mit einer vollständig anderen Methode als der üblichen bewiesen.

Das letzte Kapitel dieser Dissertation (Kapitel 7) ist mehr oder weniger in sich abgeschlossen und nicht eng mit den vorhergehenden Kapiteln verwoben. Das Hauptresultat hier ist eine Formel, welche benutzt werden kann, um eine Chernzahl einer *verallgemeinerten Kummervarietät* universell in Termen von Chernzahlen der Hilbertschemata von Punkten einer gegebenen Fläche X mit $\int_X c_1(X)^2 \neq 0$ auszudrücken. Die verallgemeinerten Kummervarietäten bilden zusammen mit den Hilbertschemata von Punkten einer K3-Fläche die beiden Beispielserien irreduzibler holomorpher symplektischer Mannigfaltigkeiten. Das Wissen der Chernzahlen einer irreduziblen holomorphen symplektischen Mannigfaltigkeit scheint wichtig zu sein, da zum Beispiel das oben erwähnte Polynom P_X nur von den Chernzahlen von X abhängt. Nach einem Ergebnis von Milnor ([**23**]) und Hirzebruchs Arbeit [**14**] ist das Wissen aller Chernzahlen einer kompakten komplexen Mannigfaltigkeit äquivalent mit dem Wissen ihrer rationalen komplexen Kobordismusklasse.

In [**9**] zeigen Ellingsrud, Göttsche und Lehn daß die komplexe Kobordismusklasse des Hilbertschematas $X^{[n]}$ der nulldimensionalen Unterschemata der Länge

n einer glatten projektiven Fläche X über den komplexen Zahlen nur von der Kobordismusklasse von X, d.h. von $c_1(X)^2$ und $c_2(X)$ abhängt. (Hier und später sind Top-Schnittprodukte auf Flächen als Schnittzahlen zu verstehen.) Die Autoren zeigen, wie dieses Resultat benutzt werden kann, um die Chernzahlen eines jeden solchen Hilbertschemas $X^{[n]}$ zu berechnen, wenn die Chernzahlen der Varietäten $(\mathbb{P}^2)^{[k]}$ und $(\mathbb{P}^1 \times \mathbb{P}^1)^{[k]}$ bekannt sind, welche wiederum mittels der Bottschen Restformel (dazu [**10**] und [**11**]) berechnet werden können. Damit können die Chernzahlen der Hilbertschemata von Punkten einer K3-Fläche effizient berechnet werden. Allerdings ist keine explizite Formel bekannt. Diese Zahlen können zum Beispiel benutzt werden, um die Vermutung von [**8**] über das elliptische Geschlecht der Hilbertschemata $X^{[k]}$ zu überprüfen, wobei X eine K3-Fläche ist.

Durch die Resultate im Kapitel 7 haben wir eine Methode, um die Chernzahlen der verallgemeinerten Kummervarietäten zu berechnen. Bisher sind nur Teilergebnisse in dieser Richtung in der Literatur veröffentlicht worden. Die χ_y-Geschlechter der verallgemeinerten Kummervarietäten sind von Göttsche und Soergel ([**13**]) berechnet worden. Indem wir mit der Hirzebruch-Riemann-Roch-Formel dieses Geschlecht in Termen der Chernzahlen ausdrücken, erhalten wir genügend Informationen, um alle Chernzahlen der verallgemeinerten Kummervarietäten bis zur Dimension sechs zu bestimmen. Unter Benutzung der Theorie der Rozansky-Witten-Invarianten ist es Sawon gelungen, eine weitere Relation zu bestimmen, welche es ihm erlaubt hat, alle Chernzahlen bis zur Dimension acht zu bestimmen. Die Chernzahlen der zehndimensionalen Kummervarietät sind von M. Britze und mir in [**5**] mit Hilfe des Hauptresultates aus Kapitel 6 dieser Arbeit berechnet worden. J. Sawon hat uns informiert, daß er ebenfalls diese Zahlen berechnet hat. Insgesamt sind aber alle diese Methoden nicht ausreichend, um alle Chernzahlen in Dimensionen zwölf oder höher zu bestimmen.

Wir haben die in Kapitel 7 entwickelte Formel benutzt, um die Chernzahlen aller verallgemeinerten Kummervarietäten bis zur Dimension zwanzig aus den Chernzahlen der Hilbertschemata von Punkten der projektiven Ebene zu gewinnen, welche wiederum mit Hilfe der Bottschen Restformel gewonnen wurden. Die Ergebnisse sind in Anhang B aufgenommen worden. Es sei beachtet, daß wir diese Methode im Prinzip nutzen können, um die Chernzahlen einer beliebigen verallgemeinerten Kummervarietät zu bestimmen.

Contents

CHAPTER 1

Introduction

Compact Kähler manifolds that possess a (unique) *holomorphic* symplectic two-form seem to be relatively rare. In dimension two, the only example of these *irreducible holomorphic symplectic manifolds* is the well-known K3 surface. In every dimension four, six, eight, and so on, only two or three types of holomorphic symplectic manifolds are known.

It is desirable to know many of the properties that all holomorphic symplectic manifolds have in common, and that distinguish them from other compact Kähler manifolds. Maybe this will eventually lead to a classification theorem for holomorphic symplectic manifolds. The theory really started with Beauville's article [**3**]. He defined a quadratic form on the second cohomology group of holomorphic symplectic manifolds that can be seen as a generalization of the intersection pairing on the middle cohomology group of a K3 surface, and already proved the "Local Torelli Theorem". In the following years up to now a general theory around these manifolds has been developed, which resembles to a large extent the theory of the K3 surfaces (for this theory, see for example [**4**]).

Holomorphic symplectic manifolds can be constructed from differential-geometric objects, namely from *compact hyperkähler manifolds.* By the holonomy principle, every Riemannian manifold (X, g) with $\mathrm{U}(n)$-holonomy possesses a complex structure such that X is a Kähler manifold with Kähler metric g. An (irreducible) compact hyperkähler manifold is a Riemannian manifold (X, g) with holonomy the symplectic group $\mathrm{Sp}(n)$. It follows that there is a two-sphere of complex structures on X for which X is a Kähler manifold with Kähler metric g. Moreover, a local calculation shows that all these complex manifolds are irreducible holomorphic symplectic manifolds. On the other hand, by the existence theorem on Calabi-Yau metrics ([**31**]), one can prove that all irreducible holomorphic symplectic manifolds arise in this way. Due to this connection between compact hyperkähler and holomorphic symplectic manifolds, it is important to know which properties of irreducible holomorphic symplectic manifolds resp. compact hyperkähler manifolds are really topological properties or of holomorphic or Riemannian nature.

My work for this thesis has been inspired by the general theory of compact hyperkähler manifolds resp. holomorphic symplectic manifolds. The results I have obtained and which are included in this thesis are mostly of cohomological nature. Let me now describe the contents of this thesis in detail.

For convenience of the reader and to introduce the necessary notions, I included chapter 2. It contains the precise definition of a compact hyperkähler manifold, and a holomorphic symplectic one, their connection mentioned above, and the list of all known examples. It also includes a section on Chern classes of these manifolds. There is shown that these are actually topological invariants. All the results of this section are well-known and can be found for example also in [**18**].

The next chapter (chapter 3) is dedicated to the Beauville-Bogomolov quadratic form which exists on every irreducible compact hyperkähler manifold. One main purpose of this chapter is to show that it is actually a topological invariant. This has been known before, although I haven't found this result stated explicitely. One can use the quadratic form to determine the subring SH^2 generated by all classes in the second cohomology of a compact hyperkähler's cohomology ring. It is shown that the space of cohomology classes in SH^2 that are of constant Hodge type $(2j, 2j)$ on every deformation is one-dimensional. There are some explicit calculations in this chapter, which together with the results in chapter 6 can be used to determine e.g. if the second Chern class c_2 is in SH^2.

Huybrechts has shown (see [**18**]) that for every irreducible holomorphic symplectic manifold X of complex dimension $2n$, there exists a polynomial P_X of degree n over $\mathbb{C}$ such that the Euler characteristic of a holomorphic line bundle L on X is given by

$$\chi(X, L) = P_X(q(c_1(L))) \tag{3}$$

where q is the (unnormalised) Beauville-Bogomolov quadratic form on X. One of the main issues of this thesis is the determination of P_X for given X. It turns out that P_X depends only on the charactistic numbers of X, i.e. on its complex cobordism class. Chapters 4–6 are concerned with the proof of a closed formula for P_X in terms of the characteristic numbers of X. The formula is

$$P_X(\lambda) = \int_X \exp\left(-2\sum_{k=1}^{\infty} b_{2k} s_{2k} T_k\left(1 + \frac{\lambda}{2}\right)\right) \tag{4}$$

where the b_{2k} are the modified Bernoulli numbers, the s_{2k} are the components of the Chern character of X (i.e. polynomials in the Chern classes), and the T_k are the Chebyshev polynomials.

The proof goes along the ideas Hitchin and Sawon used in [**15**] to prove an expression for the L^2-norm for the Riemannian curvature tensor of an irreducible compact hyperkähler manifold. It makes heavy use of the theory of *Rozansky-Witten invariants* of compact hyperkähler manifolds, introduced in [**27**]. Every vertex-oriented unitrivalent graph Γ, called a *Jacobi diagram* in this thesis, can be used to form from the Atiyah class of a holomorphic symplectic manifold a cohomology class $\mathrm{RW}(\Gamma)$. As Hitchin and Sawon remarked, all the Chern classes arise in this way. There is a certain homology relation on the space of Jacobi diagrams. Two homologous graphs yield the same Rozansky-Witten invariants. This will be used in this thesis to push down relations on the level of graph homology to the level of the cohomology of a holomorphic symplectic manifold. An important graph homology relation is given by the "Wheeling Theorem" (see [**29**]).

In chapter 4 we introduce the graph homology space, which has been subject to many research activities. We develop quite an amount of formalism here which will prove to be very handy in the following chapters. The Wheeling Theorem gives us much more relations on the graph homology space than we need for the proofs of our results in the following chapters. Therefore, in chapter 4 I included Theorem 3, which is somewhat a corollary of the Wheeling Theorem. However, I managed to proof Theorem 3 by elementary methods while the Wheeling Theorem is proven by means of knot theory. This is the reason why I decided to include Theorem 3 with its proof in this thesis. At the end of chapter 4 a certain $\mathfrak{sl}_2$-action on an enlarged space of graph homology is exhibited. At the moment, I don't know if this will have any implications for the theory of graph homology. On the other hand, this structure nicely compares with a certain $\mathfrak{sl}_2$-structure on the cohomology space of an irreducible holomorphic symplectic manifold.

Chapter 5 is dedicated to the definition of the *Rozansky-Witten classes* of irreducible holomorphic symplectic manifolds and their properties. In the literature so far, only the notion of *Rozansky-Witten invariants* occures. These are basically integrals over products of the Rozansky-Witten classes as defined here with the holomorphic symplectic form and its complex-conjugate. For a holomorphic symplectic manifold X together with a fixed holomorphic symplectic form σ, we obtain a linear map $\mathrm{RW} : \hat{\mathcal{B}} \to \mathrm{H}^*(X, \Omega_X^*)$, where $\hat{\mathcal{B}}$ is the space of graph homology. Both spaces, the domain and the codomain of RW, have additional structures: they are double-graded $\mathbb{Q}$-algebras, possess a certain bilinear form, and are canonically $\mathfrak{sl}_2$-modules. We show that the map RW respects these properties.

Using the theory developed in the two previous chapters, we can present in chapter 6 the proof of the expression (4) for the holomorphic Euler characteristic of a line bundle. In fact, we prove even more. Together with the results of chapter 3 we can compute the components of certain characteristic classes that lie in SH^2. Furthermore, as a byproduct, we show the existence of the Beauville-Bogomolov quadratic form in this chapter. Of course, this is no new result. It is just proven by a completely different method than usual.

The last chapter of this thesis (chapter 7) is more or less self-contained and not deeply linked with the previous chapters. The main result here is a formula which can be used to express a Chern number of a *generalised Kummer variety* universally in terms of Chern numbers of the Hilbert schemes of points on a given surface X with $\int_X c_1(X)^2 \neq 0$. The generalised Kummer varieties together with the Hilbert schemes of points on a K3 surface form the two main series of examples of irreducible holomorphic symplectic manifolds. Knowledge of the Chern numbers of an irreducibe holomorphic symplectic manifold seems to be important as, for example, the polynomial P_X mentioned above depends only on the Chern numbers of X. By a result of Milnor ([**23**]) and Hirzebruch's work [**14**], the knowledge of all Chern numbers of a compact complex manifold is equivalent with the knowledge of its rational complex cobordism class.

In [**21**] Ellingsrud, Göttsche and Lehn proved that the complex cobordism class of the Hilbert scheme $X^{[n]}$ of zero-dimensional subschemes of length n on a smooth projective surface X over the complex numbers depends only on the cobordism class of the surface X, i.e. on $c_1(X)^2$ and $c_2(X)$. (Here and later on, top intersections on surfaces are to be understood as intersection numbers.) The authors showed how this result can be used to calculate the Chern numbers of any such Hilbert scheme $X^{[n]}$ if one knows the Chern numbers of the varieties $(\mathbb{P}^2)^{[k]}$ and $(\mathbb{P}^1 \times \mathbb{P}^1)^{[k]}$, which in turn can be calculated by means of Bott's residue formula (cf. [**10**] and [**11**]). Therefore, the Chern numbers of the Hilbert schemes of points on a K3 surface can be efficiently calculated though no explicit formula is known. These numbers can, for example, be used to check the conjecture of [**8**] about the elliptic genus of the Hilbert schemes $X^{[k]}$ where X is a K3 surface (see [**9**]).

Due to the results of chapter 7, we have a method for computing the Chern numbers of the generalised Kummer varieties. Only partial results in this direction have appeared in the literature so far: The χ_y-genera of the generalised Kummer varieties have been calculated by Göttsche and Soergel ([**13**]). Expressing this genus in terms of Chern numbers by using the Hirzebruch-Riemann-Roch formula gives us enough information to deduce the Chern numbers for all generalised Kummer varieties of dimension less or equal six. Using the theory of Rozansky-Witten invariants, Sawon [**28**] produced a further relation that allowed him to compute all the Chern numbers for dimensions up to eight. The Chern numbers of the ten-dimensional generalised Kummer variety were calculated by M. Britze and me in [**5**] using the main result of chapter 6 of this work. J. Sawon informed us that he also had computed these numbers. However, all these methods are not sufficient to compute all Chern numbers for dimensions twelve or higher.

We used the formula developed in chapter 7 to compute all Chern numbers of the generalised Kummer varieties up to dimension twenty from the Chern numbers of the Hilbert schemes of points on the projective plane which have been calculated by means of Bott's residue formula. The results are included in Appendix B. Note that, in principle, we can use these methods to compute the Chern numbers of an arbitrary generalised Kummer variety.

CHAPTER 2

Compact hyperkähler manifolds

In this chapter, we will define our main topic of interest, *compact hyperkähler manifolds*, and review some of their properties. There are two ways to look at these manifolds, from a differential-geometric and from a complex-geometric point of view.

Our presentation is based on the first chapters of [**17**], resp. [**18**].

1. Compact hyperkähler manifolds as Riemannian manifolds

DEFINITION 1. A $4n$-dimensional Riemannian manifold (X, g) is called an *(irreducible) compact hyperkähler manifold* if it is compact and its holonomy is contained in (equals) $\mathrm{Sp}(n)$.

REMARK 1. Mainly for simplicity, we will only work with irreducible compact hyperkähler manifolds. By abuse of notion, we will omit the adjective "irreducible", i.e. from now on, by a compact hyperkähler manifold, we will always mean a manifold with holonomy equal to $\mathrm{Sp}(n)$. This is no big limitation since there is the decomposition theorem for compact Ricci-flat Kähler manifolds (see [**18**]).

By the holonomy principle, on any compact hyperkähler manifold (X, g), there are three complex structures I, J and K with

$$IJ = K = -JI, \qquad JK = I = -KJ, \qquad \text{and} \qquad KI = J = -IK \tag{5}$$

such that g is a Kähler metric with respect to all of these structures. These structures correspond to the three complex structures on the quaternions. With respect to these structures, the holonomy is contained in $\mathrm{SU}(n)$; therefore, X has a trivial canonical bundle with respect to all the structures I, J and K. Note that this implies $c_1(X) = 0$.

2. Irreducible holomorphic symplectic manifolds

DEFINITION 2. An *irreducible holomorphic symplectic manifold* X is a simply connected compact Kähler manifold such that $\mathrm{H}^0(X, \Omega_X^2)$ is generated by an everywhere non-degenerate holomorphic two-form σ. Here, we call σ *everywhere non-degenerate* if σ induces an isomorphism $\mathcal{T}_X \to \Omega_X$.

It follows that every irreducible holomorphic symplectic manifold X has trivial canonical bundle whose sections are multiples of σ^n, and, therefore, vanishing first Chern class.

The following theorem due to Beauville (see [**3**], or [**17**], [**18**]) shows that compact hyperkähler manifolds and irreducible holomorphic symplectic manifolds are basically the same objects:

THEOREM 1. *If X is an irreducible holomorphic symplectic manifold and $\omega \in \mathrm{H}^2(X, \mathbb{R})$ is a Kähler class, then there exists a unique Kähler metric g with Kähler class ω such that (X, g) is a compact hyperkähler manifold.*

If (X, g) is a compact hyperkähler manifold, then X is an irreducible holomorphic symplectic manifold with respect to any of the complex structures I, J and K given on X.

Remark 2. We do not want to reproduce the proof of this theorem, which makes use of Yau's theorem on the existence of Ricci flat metrics for compact complex manifolds with vanishing first Chern class ([**31**]).

However, given a compact hyperkähler manifold (X, g), we want to describe how the symplectic form σ on X with respect to I can be constructed:

Since (X, g) is Kähler with respect to I, J and K, there are three corresponding Kähler forms ω_I, ω_J and ω_K. By a local computation, one shows that $\sigma := \omega_J + i\omega_K$ is an everywhere non-degenerate holomorphic two-form for X with respect to I.

3. Examples

In this section we collect all examples of deformation types of compact hyperkähler manifolds known up to now (April 2002):

Example 1 (K3 surfaces). The only irreducible holomorphic symplectic manifolds of dimension two are the K3 surfaces.

Proof. By definition, a K3 surface is a surface with trivial canonical bundle, and vanishing first Betti number ([**4**]). Therefore, every irreducible holomorphic symplectic manifold of dimension two has to be a K3 surface. On the other hand, every K3 surface is simply-connected and Kähler ([**4**]). Note further that $\mathrm{H}^{2,0}$ of a K3 surface is spanned by a holomorphic two-form σ. That the induced morphism $\mathcal{T} \to \Omega$ is an isomorphism, follows directly from the triviality of the canonical bundle. □

The following two series of examples are due to Beauville ([**3**]).

Example 2 (Hilbert schemes of points on K3 surfaces). For every K3 surface X the Hilbert scheme $X^{[n]}$ of zero-dimensional subspaces ξ of X of length n is an irreducible holomorphic symplectic manifold of dimension $2n$ ([**3**]). Its second Betti number is always 23 except for the case $n = 1$, when the Hilbert scheme is just the K3 surface itself, and therefore $b_2 = 22$.

Remark 3. The Hilbert scheme $X^{[n]}$ is a scheme only if X is algebraic. Otherwise, it is just a complex space, also called the Douady-space of X.

For the next series of examples, let us briefly recall the construction of the generalised Kummer varieties.

For X a smooth complex surface, the Hilbert scheme $X^{[n]}$ can be viewed as a resolution $\rho : X^{[n]} \to X^{(n)}$ of the n-fold symmetric product $X^{(n)} := X^n/\mathfrak{S}_n$ of X. The morphism ρ, sending closed points, i.e. subspaces of X, to their support counting multiplicities, is called the Hilbert-Chow morphism.

Now, let n be a positive integer. If A is a complex torus of dimension two, there is an obvious summation morphism $A^{(n)} \to A$. We denote its composition with the Hilbert-Chow morphism $\rho : A^{[n]} \to A^{(n)}$ by $\sigma : A^{[n]} \to A$.

Definition 3. The n^{th} *generalised Kummer variety* $A^{[[n]]}$ is the fibre of σ over $0 \in A$.

For $n = 2$, the generalised Kummer variety coincides with the Kummer model of a K3 surface (therefore the name). For $n > 2$, we have $b_2(A^{[[n]]}) = 7$.

Example 3 (Generalised Kummer varieties). For every complex torus A of dimension two, the generalised Kummer variety $A^{[[n]]}$ is an irreducible holomorphic symplectic manifold of dimension $2n - 2$ ([**3**]).

Besides these two series, only two further examples of non-deformation equivalent compact hyperkähler manifolds are known:

(1) O'Grady's 6-dimensional example [**26**] with $b_2 = 8$, and
(2) O'Grady's 10-dimensional example [**25**] with $b_2 \geq 24$.

In all examples given, we included the value of the second Betti number. This shows that the example manifolds are really in different deformation classes.

In chapter 7 of this work, we will look at the cobordism class of the generalised Kummer varieties.

4. Characteristic classes of hyperkähler manifolds

Let us take a brief look at the (rational) Chern classes of irreducible holomorphic symplectic manifolds. We have already seen that c_1 vanishes since the canonical bundle is trivial. In fact, all odd Chern classes vanish up to two-torsion.

PROPOSITION 1. *Let X be an irreducible holomorphic symplectic manifold. For all $i \in \mathbb{N}_0$, the Chern classes $c_{2i+1}(X)$ vanish modulo two-torsion.*

PROOF. Remember that $\mathcal{T}_X$ is isomorphic to its dual bundle $\mathcal{T}_X^* = \Omega_X$ by means of the holomorphic two-form $\sigma \in \mathrm{H}^0(X, \Omega_X^2)$. Then we use that $c_k(E) = (-1)^k c_k(E^*)$ for any holomorphic vector bundle E on any complex manifold.

Putting this together yields:

$$c_{2i+1}(X) = c_{2i+1}(\mathcal{T}_X) = -c_{2i+1}(\mathcal{T}_X) = -c_{2i+1}(X),$$

and therefore, $2c_{2i+1}(X) = 0$ as to be proven. □

The rational Chern classes of an irreducible holomorphic symplectic manifold X are actually topological invariants. This can be seen by relating them to the Pontrjagin classes of X.

DEFINITION 4. Let X be any differentiable manifold with tangent bundle $\mathrm{T}X$. Following [**14**], we define *the i^{th} Pontrjagin class $p_i(X)$ of X* to be

$$p_i(X) := (-1)^i c_{2i}(\mathrm{T}X \otimes \mathbb{C}) \in \mathrm{H}^{4i}(X, \mathbb{C}), \tag{6}$$

where $\mathrm{T}X \otimes \mathbb{C}$ is the complexified tangent bundle of X.

REMARK 4. Apparently, the Pontrjagin classes are invariants of the differentiable structure. In fact, they are topological invariants, i.e. they cannot distinguish between different differentiable structures on the topological manifold X since they only fix the oriented cobordism class of X which is a topological invariant.

On a complex manifold, the Pontrjagin classes are linked to the Chern classes as follows:

PROPOSITION 2. *Let X be a complex manifold. For $i \in \mathbb{N}_0$ we have*

$$p_i(X) = (-1)^i \sum_{k=0}^{2i} (-1)^k c_k(X) c_{2i-k}(X). \tag{7}$$

PROOF. Let $\mathcal{T}_X$ be the holomorphic tangent bundle. We have:

$$\begin{aligned} p_i(X) =& (-1)^i c_{2i}(\mathrm{T}X \otimes \mathbb{C}) = (-1)^i c_{2i}(\mathcal{T}_X \oplus \overline{\mathcal{T}}_X) \\ =& (-1)^i \sum_{k=0}^{2i} c_k(\overline{\mathcal{T}}_X) c_{2i-k}(\mathcal{T}_X) = (-1)^i \sum_{k=0}^{2i} (-1)^k c_k(\mathcal{T}_X) c_{2i-k}(\mathcal{T}_X) \\ =& (-1)^i \sum_{k=0}^{2i} (-1)^k c_k(X) c_{2i-k}(X), \end{aligned}$$

by the Whitney sum formula for the Chern classes. □

COROLLARY 1. *For a complex manifold X, the collection of its Chern classes determine the Pontrjagin classes of the underlying differentiable manifold.*

EXAMPLE 4. For a complex manifold X, we have

$$p_0(X) = 1, \quad p_1(X) = -2c_2(X) + c_1^2(X), \\ p_2(X) = 2c_4(X) - 2c_3(X)c_1(X) + c_2^2(X), \quad \dots . \tag{8}$$

On an irreducible holomorphic symplectic manifold, we can invert these relations:

PROPOSITION 3. *Let X be an irreducible holomorphic symplectic manifold. The rational Pontrjagin classes of the underlying differentiable manifold determine the rational Chern classes of X.*

PROOF. Since the odd rational Chern classes of X vanish (see Proposition 1), we just have to express the even rational Chern classes in terms of the rational Pontrjagin classes. Looking at (7), we see that

$$c_{2i}(X) = \frac{(-1)^i}{2} p_i(X) + P(c_2(X), \dots, c_{2i-2}(X)), \tag{9}$$

where P is a polynomial in $i-1$ variables over $\mathbb{Q}$. Inductively, we can therefore express the $c_{2i}(X)$ in terms of the $p_k(X)$. □

REMARK 5. It follows that the rational Chern classes of an irreducible holomorphic symplectic manifold are in fact (differential-)topological invariants. So every other structure of an irreducible holomorphic symplectic manifold on the differentiable manifold X leads to the same rational Chern classes. In particular, the rational Chern classes of a compact hyperkähler manifold are well-defined: Just take the rational Chern classes of any associated irreducible holomorphic symplectic manifold structure. Of course, it also follows that the Chern classes modulo two-torsion cannot distinguish between different compact hyperkähler structures on X.

At the end of this section, we want to introduce one more notation, which will be of use in chapter 5 and chapter 6.

Let u be any formal power series in the "universal Chern classes" c_i over $\mathbb{C}$. For any $\alpha \in \mathbb{C}$, we define

$$u(\alpha) := \sum_{i=0}^{\infty} u_i \alpha^i, \tag{10}$$

where u_i is the component of u that is of total weight i. Later, we will apply this definition to $\mathrm{td}^{1/2}$, the square root of the universal Todd genus.

5. A well-known result on the second cohomology group

This section contains a well-known result on the second cohomology group of a compact hyperkähler manifold, which we will need later on. Due to its importance for our results, a proof is included.

PROPOSITION 4. *Let X be a $4n$-dimensional differentiable manifold which can be equipped with the structure of a compact hyperkähler manifold. Let further $f : \mathrm{H}^2(X, \mathbb{C}) \to \mathbb{C}$ be any homogeneous polynomial over $\mathbb{C}$ such that*

$$f(\sigma + \bar{\sigma}) = 0 \tag{11}$$

for all $\sigma \in \mathrm{H}^2(X,\mathbb{C})$ such that there exists a structure of an irreducible holomorphic symplectic manifold on X for which σ is the cohomology class of the symplectic form. Then

$$f(\alpha) = 0 \tag{12}$$

for all $\alpha \in \mathrm{H}^2(X,\mathbb{C})$, i.e. f is zero.

Proof. Let $\Sigma \subset \mathrm{H}^2(X,\mathbb{C})$ be the set of all cohomology classes σ such that a structure of an irreducible holomorphic symplectic manifold exists on X for which σ is the class of the symplectic form. We have to show that the Zariski closure of the set $U := \{[\sigma+\bar\sigma] | \sigma \in \Sigma\}$ in $\mathbb{P}(\mathrm{H}^2(X,\mathbb{C}))$ is the whole projective space.

Equip X with a structure of a compact hyperkähler manifold. Let $\mathcal{X} \to \mathrm{Def}(X)$ be the universal deformation (which exists since $\mathrm{H}^0(X,\mathcal{T}_X) = 0$, and is unobstructed since $c_1(X) = 0$; see e.g. [**18**]). One has *the period map* $P_X : \mathrm{Def}(X) \to \mathbb{P}(\mathrm{H}^2(X,\mathbb{C}))$, which maps a $t \in \mathrm{Def}(X)$ to $[\sigma_t]$, where σ_t is up to a multiple constant the unique holomorphic symplectic two-form of the fibre $\mathcal{X}_t$ of t. Let $g : \mathrm{H}^2(X,\mathbb{C}) \to \mathbb{C}, \alpha \mapsto \int_X \alpha^{2n}$. By considering the holomorphic degree of σ_t in any fibre, it follows that the image of P_X is contained in an irreducible component Q_X of the hypersurface defined by g in $\mathbb{P}(\mathrm{H}^2(X,\mathbb{C}))$. Now the "Local Torelli Theorem" (Beauville, [**3**]) says that $P_X : \mathrm{Def}(X) \to Q_X$ is a local isomorphism. To show this, one calculates the differential of P_X which happens to be injective. By comparing dimensions ($\dim \mathrm{Def}(X) = \dim \mathrm{H}^1(X,\mathcal{T}_X) = b_2(X) - 2 = \dim Q_X$), one concludes the Local Torelli Theorem.

It follows that the Zariski closure of $\{[\sigma] | \sigma \in \Sigma\}$ in $\mathbb{P}(\mathrm{H}^2(X,\mathbb{C}))$ is the whole hypersurface Q_X. Let $[\sigma]$ be in Q_X. Then $[\sigma+\bar\sigma]$ lies in the Zariski closure $\overline{U}$ of U. Substituting σ with $e^{-i\phi/2}\sigma$, it follows that for all $\phi \in \mathbb{R}$, the element $[\sigma + e^{i\phi}\bar\sigma]$ lies in $\overline{U}$ and therefore, $[\sigma + z\bar\sigma] \in \overline{U}$ for all $z \in \mathbb{C}$. In particular, $Q_X \subset \overline{U}$. Furthermore, $[\sigma + z\bar\sigma] \notin Q_X$ for all $z \neq 0$, so the codimension of $\overline{U}$ is strictly less than the codimension of Q_X, which is one. This proves $\overline{U} = \mathbb{P}(\mathrm{H}^2(X,\mathbb{C}))$. □

6. An $\mathfrak{sl}_2$-action on the cohomology of an irreducible holomorphic symplectic manifold

We want to introduce an $\mathfrak{sl}_2$-action on the cohomology of any irreducible symplectic manifold besides the usual Lefschetz action defined by the Kähler class ω. Let X be such a manifold and σ its symplectic form which induces a symplectic form σ_x on each vector space $W := \bigwedge^* \mathcal{T}_{X,x} \otimes A$ for $x \in X$ with $A := \bigwedge^* \overline{\mathcal{T}}_{X,x}$. As shown in Appendix A there are operators Π_x and δ_x on W ((V,σ) of Appendix A corresponding to $(\mathcal{T}_{X,x},\sigma_x)$ here) such that $(\sigma_x, -\delta_x, \Pi_x)$ acts as an $\mathfrak{sl}_2$-triple on W, i.e. these three operators obey the commutator relations of the standard $\mathfrak{sl}_2$-generators X, Y and H. On the level of cohomology, these operators induce an operator L_σ, which is multiplication by σ, an operator $\Lambda_{\sigma/4} := -\delta$, which is up to a sign the contraction by σ, and Π, which maps $\alpha \in \mathrm{H}^{p,q}(X,\mathbb{C})$ for $p,q \in \mathbb{N}_0$ to $(p-n)\alpha$. It follows that $(\mathrm{L}_\sigma, \Lambda_{\sigma/4}, \Pi)$ acts as an $\mathfrak{sl}_2$-triple on $\mathrm{H}^{*,*}(X,\mathbb{C})$.

Remark 6. The operator $\Lambda_{\sigma/4}$ is one-forth of the operator Λ_σ introduced by Huybrechts in [**17**], [**18**].

Remark 7. The $\mathfrak{sl}_2$-action we have exhibited so far, is part of the Lie algebra-action of the Lie algebra $\mathfrak{so}(q_X \oplus H)$, where H is the hyperbolic plane, on $\mathrm{H}^*(X,\mathbb{C})$, described by Looijenga and Lunts ([**22**]). (q_X will be defined in the next chapter.)

CHAPTER 3

The Beauville-Bogomolov quadratic form

In this chapter, we want to introduce the Beauville-Bogomolov quadratic form of compact hyperkähler manifolds. It consists of two sections. The first one is dedicated to general facts about quadratic forms, the second section contains applications of these results to the case of compact hyperkähler manifolds.

1. Linear and multilinear algebra of quadratic forms

Definition 5. Let V be a finite-dimensional $\mathbb{R}$-vector space and $q : V \to \mathbb{R}$ a non-degenerate quadratic form on V. For every $n \in \mathbb{N}$, we define a linear form

$$\begin{aligned} &q^{(n)} : \mathrm{S}^{2n}V \to \mathbb{R}, \\ &v_1 \cdot \dots \cdot v_{2n} \mapsto \frac{1}{(2n)!} \sum_{\pi \in \mathfrak{S}_{2n}} q(v_{\pi(1)}, v_{\pi(2)}) \cdot \dots \cdot q(v_{\pi(2n-1)}, v_{\pi(2n)}) \end{aligned} \tag{13}$$

on the $(2n)^{\text{th}}$ symmetric power of V.

Remark 8. The linear forms $q^{(n)}$ are normalised in such a way that $q^{(n)}(v^{2n}) = q(v)^n$ for every $v \in V$.

Let us equip the vector space S^nV with the following canonical non-degenerate scalar product

$$\langle v_1 \cdot \dots \cdot v_n, w_1 \cdot \dots \cdot w_n \rangle := \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} q(v_1, w_{\pi(1)}) \cdot \dots \cdot q(v_n, w_{\pi(n)}). \tag{14}$$

I.e. for $v \in V$ and $n \in \mathbb{N}$, we have $\langle v^n, v^n \rangle = q^{(n)}(v^{2n})$.

Definition 6. Let V be a finite-dimensional vector space equipped with a non-degenerated quadratic form q. We define the *Casimir (element) $\hat{q}$ of the quadratic form q* to be the unique element $\hat{q} \in \mathrm{S}^2V$ such that

$$\langle v, \hat{q} \rangle = q^{(1)}(v)$$

for all $v \in \mathrm{S}^2V$.

Proposition 5. *Let $e_1, \dots, e_l$ be an orthonormal basis for the non-degenerate quadratic form (V, q), i.e. $e_1, \dots, e_l$ is a basis with $q(e_i, e_j) = 0$ for $i \neq j$ and $q(e_i) = \varepsilon_i \in \{-1, 1\}$.*

Then the Casimir element is given by

$$\hat{q} = \sum_{i=1}^{l} \varepsilon_i e_i^2. \tag{15}$$

Proof. For $v \in V$ we have

$$\langle v^2, \hat{q} \rangle = \sum_{i=1}^{l} \varepsilon_i \langle v^2, e_i^2 \rangle = \sum_{i=1}^{l} \varepsilon_i q(v, e_i)^2 = q(v) = q^{(1)}(v^2).$$

$\square$

REMARK 9. From this representation of $\hat{q}$, one easily concludes that $q^{(1)}(\hat{q}) = l = \dim V$.

PROPOSITION 6. *Let be $k, n \in \mathbb{N}_0$ with $k \leq n$. We have*

$$q^{(n)}(v^{2n-2k}\hat{q}^k) = \left(\prod_{i=1}^{k} \frac{l+2n-2i}{2n-2i+1}\right) q(v)^{n-k} \tag{16}$$

for every $v \in V$, where $l := \dim V$.

PROOF. First, we show that (16) holds for $k = 1$. Using the representation of $\hat{q}$ in terms of an orthonormal basis given as in Proposition 5, one calculates

$$\begin{aligned} q^{(n)}(v^{2n-2}\hat{q}) &= \sum_{i=1}^{l} \varepsilon_i q^{(n)}(v^{2n-2}e_i^2) \\ &= \sum_{i=1}^{l} \varepsilon_i \left(\frac{1}{2n-1} q(v)^{n-1} q(e_i) + \frac{2n-2}{2n-1} q(v)^{n-2} q(e_i, v)^2\right) \\ &= \frac{l+2n-2}{2n-1} q(v)^{n-1}. \end{aligned}$$

By linear extension,

$$q^{(n)}(v^{2n-2k}\hat{q}^k) = \frac{l+2n-2}{2n-1} q^{(n-1)}(v^{2n-2k}\hat{q}^{k-1}).$$

From this, the general formula follows easily by induction over n. □

Later, we want to apply our results on quadratic forms to quadratic forms defined on cohomology groups. Therefore, it is convenient to introduce the following notion:

DEFINITION 7. A *graded $\mathbb{R}$-Frobenius algebra of degree d* is a finite dimensional (super-)commutative graded $\mathbb{R}$-algebra $A = \bigoplus_{k=0}^{d} A^d$ together with a degree-preserving linear form $\int : A \to \mathbb{R}[d]$ (i.e. $\int$ vanishes on A^k for $k \neq d$) such that the induced *Poincaré pairing*

$$A \otimes A \to \mathbb{R}[d], a \otimes a' \mapsto \int (aa')$$

is non-degenerate.

EXAMPLE 5. Let X be a differentiable manifold of dimension d. The cohomology ring $\mathrm{H}^*(X, \mathbb{R})$ is a graded $\mathbb{R}$-Frobenius algebra of degree d.

PROPOSITION 7. *Let A be a graded $\mathbb{R}$-Frobenius algebra of degree $4n$, $n \in \mathbb{N}_0$. Let A^2 be equipped with a non-degenerate quadratic form q such that there exists a positive real constant $C > 0$ with*

$$\int a^{2n} = C \cdot q(a)^n \tag{17}$$

for all $a \in A^2$. Let $\mathrm{S}A^2$ be the graded subalgebra of A generated by A^2 and $\mu : \mathrm{S}^{2k}A^2 \to (\mathrm{S}A)^{4k}$ the canonical multiplication.

We have

(1) $C = \prod_{i=1}^{n} \frac{2n-2i+1}{l+2n-2i} \int \mu(\hat{q})^n$.
(2) $A = \mathrm{S}A^2 \oplus (\mathrm{S}A^2)^\perp$, *where $(\mathrm{S}A^2)^\perp$ is the orthogonal complement of $\mathrm{S}A^2$ with respect to the Poincaré pairing. Equivalently, $\int$ restricted to $\mathrm{S}A^2$ is non-degenerate.*

(3) *For all $k \in \{0, \dots, n\}$, there exists exactly one element $c \in (\mathrm{S}A^2)^{4k}$ such that the linear forms*

$$v_1 \cdot \dots \cdot v_{2n-2k} \mapsto \int v_1 \cdot \dots \cdot v_{2n-2k} \cdot c \tag{18}$$

and

$$q^{(n-k)} \tag{19}$$

on $\mathrm{S}^{2n-2k}V$ coincide.

We can express c by

$$c = \left(\prod_{i=k+1}^{n} \frac{l + 2n - 2i}{2n - 2i + 1} \right) \mu(\hat{q})^k / \int \mu(\hat{q})^n. \tag{20}$$

Proof. The formula for C follows from (16) with $k = n$.

The non-degeneracy of $\int$ restricted to $\mathrm{S}A^2$ is a direct consequence of (17) and the fact that the quadratic form q is non-degenerate.

That c is unique in case of its existence, follows from the fact that the Poincaré pairing restricted to $\mathrm{S}A^2$ remains non-degenerate. Therefore, the linear forms on $(\mathrm{S}A^2)^{4n-4k}$ correspond exactly to elements of $(\mathrm{S}A)^{4k}$.

Eventually, for $v \in A^2$, using (16), we calculate

$$\begin{aligned} \int v^{2n-2k} \mu(\hat{q})^k = \int \mu(v^{2n-2k} \hat{q}^k) = C q^{(n)} (v^{2n-2k} \hat{q}^k) \\ = C \left(\prod_{i=1}^{k} \frac{l + 2n - 2i}{2n - 2i + 1} \right) q(v)^{n-k}, \end{aligned}$$

which proves the expression given for c. □

2. The quadratic form on a compact hyperkähler manifold

Let us apply the results of the last section to the Beauville-Bogomolov quadratic form of compact hyperkähler manifolds which we will define by the following proposition:

Proposition 8. *Let X be a $4n$-dimensional compact differentiable manifold. Then there exists at most one quadratic form*

$$q_X : \mathrm{H}^2(X, \mathbb{R}) \to \mathbb{R} \tag{21}$$

with the following properties:

(1) *q_X is induced by a primitive quadratic form on $\mathrm{H}^2(X, \mathbb{Z})$.*
(2) *There exists an $\alpha \in \mathrm{H}^2(X, \mathbb{R})$ with $q_X(\alpha) \neq 0$, and for all $\alpha \in \mathrm{H}^2(X, \mathbb{R})$ with $q_X(\alpha) \neq 0$, we have that*

$$\left(\int_X p_1(X) \alpha^{2n-2} \right) / q_X(\alpha)^{n-1} < 0. \tag{22}$$

(3) *There exists a positive real constant $c > 0$ such that $q_X(\alpha)^n = c \int_X \alpha^{2n}$ for all $\alpha \in \mathrm{H}^2(X, \mathbb{R})$.*

In case of the existence of such a q_X, we call q_X the Beauville-Bogomolov quadratic form *of X.*

Proof. Note first that there exists up to a sign at most one q_X which fulfills (3) with $c = 1$. This can be seen by the following consideration: In the ring $\mathrm{S}^* \mathrm{H}^2(X, \mathbb{R})^*$, there are at most n solutions x of the equation

$$x^n = y$$

for a fixed $y \in \mathrm{S}^{2n}\mathrm{H}^2(X,\mathbb{R})^*$. Since all these solutions differ only be a n^{th}-root of unity, there is actually up to sign at most one x with $x^n = y$. We apply this to $y = (\alpha^{2n} \mapsto \int_X \alpha^{2n}) \in \mathrm{S}^{2n}\mathrm{H}^2(X,\mathbb{R})^*$.

Thus, we know that q_X — if it exists — is unique up to a rational constant. Property (1) fixes this constant up to a sign, property (2) fixes this sign if n is even, property (3) fixes this sign if n is odd. □

EXAMPLE 6. Let X be a four-manifold. Then X possesses a Beauville-Bogomolov form exactly when $\operatorname{sign}(X) < 0$, where $\operatorname{sign}(X) = b_2^+(X) - b_2^-(X)$ is its signature. Its Beauville-Bogomolov form is given by the intersection pairing

$$H^2(X,\mathbb{R}) \times \mathrm{H}^2(X,\mathbb{R}) \to \mathbb{R}, (\alpha,\beta) \mapsto \int_X \alpha \cup \beta. \tag{23}$$

PROOF. By property (3) we know that q_X has to be a positive multiple of the intersection pairing. Since the intersection pairing on the real cohomology comes from a primitive pairing on the cohomology with $\mathbb{Z}$-coefficients, by property (1), q_X has to be the intersection pairing, at least up to a sign. The only condition, property (2) imposes, is that $\int_X p_1(X) < 0$ which is, by the Hirzebruch signature theorem ($\int_X p_1(X) = \operatorname{sign}(X)/3$, see [**14**]) equivalent to $\operatorname{sign}(X) < 0$ (from which the existence of $\alpha \in \mathrm{H}^2(X,\mathbb{R})$ with $\int_X \alpha^2 \neq 0$ follows). □

EXAMPLE 7. Let $n \in \mathbb{N}$ be an integer. The projective space $\mathbb{P}^{4n}$ has a Beauville-Bogomolov quadratic form given by

$$q_{\mathbb{P}^{4n}}(H) = -1, \tag{24}$$

where H is the class of a hyperplane.

In the literature, the notion “Beauville-Bogomolov quadratic form” appears only for compact hyperkähler manifolds. In the hyperkähler case, such a quadratic form always exists, and to be honest, I don’t know if the notion of a Beauville-Bogomolov form on an arbitrary manifold is of any further use.

THEOREM 2. *If X can be equipped with the structure of a compact hyperkähler manifold, it possesses a Beauville-Bogomolov quadratic form.*

This theorem is due to Beauville, Bogomolov and Fujiki. Later (Remark 22) we will give a non-standard proof of this theorem, which makes use of Rozansky-Witten classes.

REMARK 10. For every compact hyperkähler manifold X, the second cohomology group $\mathrm{H}^2(X,\mathbb{Z})$ is a free $\mathbb{Z}$-module.

To prove the existence of q_X, it suffices to show that there exists a rational quadratic form $q : \mathrm{H}^2(X,\mathbb{Q}) \to \mathbb{Q}$ with properties (2) and (3) since then there exist exactly one $c \in \mathbb{Q}_{>0}$ such that cq is induced by a primitive integral form on $\mathrm{H}^2(X,\mathbb{Z})$.

We will now cite some of the properties of this quadratic form that we will need in the following:

PROPOSITION 9. *The Beauville-Bogomolov quadratic form of a compact hyperkähler manifold is definite and of index $(3, b_2 - 3)$, where b_2 is the second Betti number of X.*

PROOF. (See, e.g. [**18**].) The result can be proven by the Hodge-Riemann bilinear relations for the Kähler form ω_I and using the defining properties of the quadratic form. □

Because the quadratic form of a compact hyperkähler manifold is definite, we can form the Casimir element $\hat{q}_X \in \mathrm{S}^2\mathrm{H}^2(X, \mathbb{Q})$. By abuse of notion, we also denote its image under the canonical multiplication map $\mathrm{S}^2\mathrm{H}^2(X, \mathbb{Q}) \to \mathrm{H}^4(X, \mathbb{Q})$ by $\hat{q}_X$.

Proposition 10. *Let $\alpha \in \mathrm{H}^{4j}(X, \mathbb{C})$ be of type $(2j, 2j)$ on all small deformations on X. Then there is a constant $c_\alpha \in \mathbb{C}$, depending on α, such that*

$$\int_X \alpha \beta^{2(n-j)} = c_\alpha q_X(\beta)^{n-j}. \tag{25}$$

Proof. See, e.g. [**18**]. □

Remark 11. Of course, the proposition remains true if we replace q_X by a non-vanishing multiple. This also holds true for the following Propositions 12 and 13.

Example 8. The Chern classes $c_{2j}(X)$ of X are examples of such classes which are of type $(2j, 2j)$ on all small deformations of X. This is due to the fact that $c_{2j}(X) = c_{2j}(X')$ for X' being an deformation of X (see Remark 5).

Proposition 11. *Let X be an irreducible holomorphic symplectic manifold, and $q : \mathrm{H}^2(X, \mathbb{Q}) \to \mathbb{Q}$ a quadratic form that fulfills property (3). Then, (2) is fulfilled by q whenever $q(\sigma + \bar{\sigma}) > 0$, with σ being the holomorphic symplectic form of X.*

Proof. Let q be such a form, i.e. q is up to a non-vanishing rational factor the Beauville-Bogomolov quadratic form.

Let $\alpha \in \mathrm{H}^2(X, \mathbb{R})$ be any class with $q(\alpha) \neq 0$. We have to show that

$$\int_X c_2(X) \alpha^{2n-2} / q(\alpha)^{n-1} > 0$$

(remember that $c_2(X) = -\frac{1}{2} p_1(X)$).

By Proposition 10 we know that there exists a constant $c \in \mathbb{C}$ with

$$\int_X c_2(X) \alpha^{2n-2} = c q(\alpha)^{n-1}$$

and

$$\int_X c_2(X)(\sigma + \bar{\sigma})^{2n-2} = c q(\sigma + \bar{\sigma})^{n-1}.$$

Therefore, we have to show that $c > 0$, which is equivalent to $\int_X c_2(X)(\sigma+\bar{\sigma})^{2n-2} > 0$. This inequality follows from a remark of Hitchin and Sawon [**15**] (see Remark 21 of this thesis). □

Next, we want to ask which cohomology classes in $\mathrm{H}^*(X, \mathbb{C})$ fulfill the assumption of Proposition 10. As the full cohomology ring of a compact hyperkähler manifold is not well understood yet, we will restrict ourselves to classes in $\mathrm{SH}^2(X, \mathbb{C})$. For these classes, we can give a definite answer:

Proposition 12. *Let X be a $4n$-dimensional manifold which can be equipped with the structure of a compact hyperkähler manifold. The subring of all classes in $\mathrm{SH}^2(X, \mathbb{C})$ which are of type $(2j, 2j)$ with respect to all structures of irreducible holomorphic symplectic manifolds we can impose on X is generated by $\hat{q}_X$ and canonically isomorphic to $\mathbb{C}[\hat{q}_X]/(\hat{q}_X^{n+1})$.*

Proof. We apply Proposition 7 to the Frobenius algebra $\mathrm{H}^*(X, \mathbb{C})$: Let α be any class in $(\mathrm{SH}^2(X, \mathbb{C}))^{4j}$ which is of Hodge type $(2j, 2j)$ with respect to all irreducible holomorphic symplectic structures on X. By Proposition 10 the form $\mathrm{H}^2(X, \mathbb{C}) \to \mathbb{C}, \beta \mapsto \int_X \alpha \beta^{2n-2j}$ is a multiple of q_X^{n-j}. By Proposition 7, it follows that α is a multiple of $\hat{q}_X^j$.

It remains to show that $\hat{q}_X$ is of Hodge type $(2,2)$ with respect to all irreducible holomorphic symplectic structures on X. This can be seen as follows: Let $\pi : \mathrm{H}^*(X,\mathbb{C}) \to \mathrm{SH}^2(X,\mathbb{C})$ be the orthogonal projection. It is compatible with the Hodge decomposition. By the considerations made so far, we know that $\pi(c_2(X))$ is a non-vanishing multiple of $\hat{q}_X$. This proves the claim about the Hodge type of $\hat{q}_X$. □

Let $\pi : \mathrm{H}^*(X,\mathbb{C}) \to \mathrm{SH}^2(X,\mathbb{C})$ be the orthogonal projection again. Proposition 12 implies that for any class $\alpha \in \mathrm{H}^{4j}(X,\mathbb{C})$ which is of constant Hodge type $(2j,2j)$ with respect to all irreducible holomorphic symplectic structures there exists a real number u with $\pi(\alpha) = u\hat{q}_X^j$. Calculating with Proposition 7 yields:

Proposition 13. *Let X be a $4n$-dimensional differentiable manifold which can be equipped with the structure of a compact hyperkähler manifold and $j \in \{0,\dots,n\}$. Let $\alpha \in \mathrm{H}^{4j}(X,\mathbb{C})$ be a cohomology class such there exist a constant $c \in \mathbb{C}$ with*

$$\int_X \alpha\beta^{2n-2j} = cq_X(\beta)^{n-j} \tag{26}$$

for all $\beta \in \mathrm{H}^2(X,\mathbb{C})$. Then

$$\pi(\alpha) = \frac{c}{C}\left(\prod_{i=1}^{j} \frac{2n-2i+1}{l+2n-2i}\right)\hat{q}_X^j = \frac{c}{\int_X \hat{q}_X^n}\left(\prod_{i=j+1}^{n} \frac{l+2n-2i}{2n-2i+1}\right)\hat{q}_X^j \tag{27}$$

where C is the positive constant with $\int_X \alpha^{2n} = Cq_X(\alpha)^n$ for all $\alpha \in \mathrm{H}^2(X,\mathbb{C})$.

CHAPTER 4

Graph homology

This chapter is concerned with the space of graph homology classes of unitrivalent graphs. A very detailed discussion of this space and other graph homology spaces can be found in [**2**]. Further aspects of graph homology can be found in [**29**], and, with respect to Rozansky-Witten invariant, in [**15**].

1. The graph homology space

In this thesis, *graph* means a collection of vertices connected by edges, i.e. every edge connects two vertices. We want to call a half-edge (i.e. an edge together with an adjacent vertex) of a graph a *flag*. So, every edge consists of exactly two flags. Every flag belongs to exactly one vertex of the graph. On the other hand, a vertex is given by the set of its flags. It is called *univalent* if there is only one flag belonging to it, and it is called *trivalent* if there are exactly three flags belonging to it. We shall identify edges and vertices with the set of their flags. A graph is

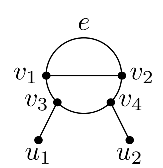


FIGURE 1. This Jacobi diagram has four trivalent vertices $v_1, \ldots, v_4$, and two univalent vertices u_1 and u_2, and e is one of its 7 edges.

called *vertex-oriented* if, for every vertex, a cyclic ordering of its flags is fixed.

DEFINITION 8. A *Jacobi diagram* is a vertex-oriented graph with only uni- and trivalent vertices. A *connected Jacobi diagram* is a Jacobi diagram which is connected as a graph. A *trivalent Jacobi diagram* is a Jacobi diagram with no univalent vertices.

We define the *degree of a Jacobi diagram* to be the number of its vertices. It is always an even number.

We identify two graphs if they are isomorphic as vertex-oriented graphs in the obvious sense.

EXAMPLE 9. The empty graph is a Jacobi diagram, denoted by 1. The unique Jacobi diagram consisting of two univalent vertices (which are connected by an edge) is denoted by ℓ.

REMARK 12. There are different names in the literature for what we call a "Jacobi diagram", e.g. unitrivalent graphs, chord diagrams, Chinese characters, Feynman diagrams. The name chosen here is also used by Thurston in [**29**]. The name comes from the fact that the IHX relation in graph homology defined later is essentially the well-known Jacobi identity for Lie algebras.

FIGURE 2. The Jacobi diagram ℓ with its two univalent vertices u_1 and u_2.

With our definition of the degree of a Jacobi diagram, the algebra of graph homology defined later will be commutative in the graded sense. Further, the map RW that will associate to each Jacobi diagram a Rozansky-Witten class will respect this grading. But note that often the degree is defined to be *half* of the number of vertices which still is an integer.

We can always draw a Jacobi diagram in a planar drawing so that it looks like a planar graph with vertices of valence 1, 3 or 4. Each 4-valent vertex has to be interpreted as a crossing of two non-connected edges of the drawn graph and not as one of its vertices. Further, we want the counter-clockwise ordering of the flags at each trivalent vertex in the drawing to be the same as the given cyclic ordering.

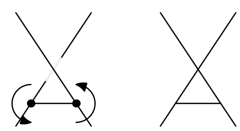


FIGURE 3. These two graphs depict the same one.

In drawn Jacobi diagrams, we also use a notation like $\cdots \overset{n}{-} \cdots$ for a part of a graph which looks like a long line with n univalent vertices ("legs") attached to it, for example $\ldots \perp\perp\perp \ldots$ for $n = 3$. The position of n indicates the placement of the legs relative to the "long line".

DEFINITION 9. Let S be a totally ordered set with n elements. We set

$$\varepsilon(S) := s_1 \wedge \cdots \wedge s_n \in \bigwedge\nolimits_{\mathbb{Q}} S. \tag{28}$$

Here, $s_1, \ldots, s_n$ are the elements of S in increasing order. (For the definition of $\bigwedge_{\mathbb{Q}} S$, see appendix A.)

DEFINITION 10. Let S be a cyclicly ordered set with an odd number n of elements. We set

$$\varepsilon(S) := s_1 \wedge \cdots \wedge s_n \in \bigwedge\nolimits_{\mathbb{Q}} S. \tag{29}$$

Here, $s_1, \ldots, s_n$ are the elements of S in an order compatible with the given cyclic one. The Definition of $\varepsilon(S)$ does not depend on this order as long as the compatibility condition is fulfilled.

EXAMPLE 10. If t is a trivalent vertex in a Jacobi diagram, it makes sense to write $\varepsilon(t)$ because we have said that we identify a vertex with the set of the flags belonging to it, for which a cyclic ordering has been fixed.

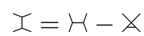
DEFINITION 11. Let Γ be a Jacobi diagram with k trivalent and l univalent vertices, so $m := \frac{3k+l}{2}$ is the number of its edges. Let F be the set of its flags, E the set of its edges, T the set of its trivalent vertices, and U the set of its univalent vertices.

A choice of total orderings of the sets T, U and every set $e \in E$ (recall that an edge is identified with the set of the two flags belonging to it) is said to be *compatible with the orientation of* Γ if the equality

$$\varepsilon(t_1) \wedge \cdots \wedge \varepsilon(t_k) \wedge u_1 \wedge \cdots \wedge u_l = \varepsilon(e_1) \wedge \cdots \wedge \varepsilon(e_m) \tag{30}$$

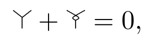
holds in the Graßmann algebra $\bigwedge_{\mathbb{Q}} F$ generated by the elements of F. Here, $t_1, \ldots, t_k$ are the elements of T in increasing order, $u_1, \ldots, u_l$ are the elements of U in increasing order, and $e_1, \ldots, e_m$ are all the edges in arbitrary order. (Changing the order of the edges E does not change the right hand side of (30) since the $\varepsilon(e_k)$ are of even degree.)

Definition 12. We define $\mathcal{B}$ to be the $\mathbb{Q}$-vector space spanned by all Jacobi diagrams modulo the IHX relation



(31)

and the anti-symmetry (AS) relation



(32)

which can be applied anywhere within a diagram. (For this definition see also [**2**] and [**29**].) Two Jacobi diagrams are said to be *homologous* if they are in the same modulo the IHX and AS relation.

Furthermore, let $\mathcal{B}'$ be the subspace of $\mathcal{B}$ spanned by all Jacobi diagrams not containing ℓ as a component, and let $\mathcal{B}^{\mathrm{t}}$ be the subspace of $\mathcal{B}'$ spanned by all trivalent Jacobi diagrams. All these are graded and double-graded. The grading is induced by the degree of Jacobi diagrams, the double-grading by the number of univalent and trivalent vertices.

The completion of $\mathcal{B}$ (resp. $\mathcal{B}'$, resp. $\mathcal{B}^{\mathrm{t}}$) with respect to the grading will be denoted by $\hat{\mathcal{B}}$ (resp. $\hat{\mathcal{B}}'$, resp. $\hat{\mathcal{B}}^{\mathrm{t}}$).

We define $\mathcal{B}_{k,l}$ to be the subspace of $\hat{\mathcal{B}}$ generated by graphs with k trivalent and l univalent vertices. $\mathcal{B}'_{k,l}$ and $\mathcal{B}^{\mathrm{t}}_k := \mathcal{B}^{\mathrm{t}}_{k,0}$ are defined similarly.

All these spaces are called *graph homology spaces* and their elements are called *graph homology classes* or *graphs* for short.

Remark 13. The subspaces $\mathcal{B}_k$ of $\hat{\mathcal{B}}$ spanned by the Jacobi diagrams of degree k are always of finite dimension. The subspace $\mathcal{B}_0$ is one-dimensional and spanned by the graph homology class 1 of the empty diagram 1.

Remark 14. We have $\hat{\mathcal{B}} = \prod_{k,l \geq 0} B_{k,l}$. Further, $\hat{\mathcal{B}} = \hat{\mathcal{B}}'[[\ell]]$. Due to the AS relation, the spaces $\mathcal{B}'_{k,l}$ are zero for $l > k$. Therefore, $\hat{\mathcal{B}}' = \prod_{k=0}^{\infty} \bigoplus_{l=0}^{k} \mathcal{B}'_{k,l}$.

Example 11. If γ is a graph which has a part looking like $\cdots \overset{n}{-} \cdots$, it will become $(-1)^n \gamma$ if we substitute the part $\cdots \overset{n}{-} \cdots$ by $\cdots \underset{n}{-} \cdots$ due to the anti-symmetry relation.

2. Operations with graphs and special graphs

Definition 13. Disjoint union of Jacobi diagrams induces a bilinear map

$$\hat{\mathcal{B}} \times \hat{\mathcal{B}} \to \hat{\mathcal{B}}, (\gamma, \gamma') \mapsto \gamma \cup \gamma'. \tag{33}$$

By mapping $1 \in \mathbb{Q}$ to $1 \in \hat{\mathcal{B}}$, the space $\hat{\mathcal{B}}$ becomes a graded $\mathbb{Q}$-algebra, which has no components in odd degrees. Often, we omit the product sign "$\cup$". $\mathcal{B}$, $\mathcal{B}'$, $\mathcal{B}^{\mathrm{t}}$, and so on are subalgebras.

Definition 14. Let $k \in \mathbb{N}$. We call the graph homology class of the Jacobi diagram $\overset{2k}{\bigcirc}$ the $2k$-*wheel* w_{2k}, i.e. $w_2 = -\!\circ\!-$, $w_4 = ¤$, and so on. It has $2k$ univalent and $2k$ trivalent vertices. The expression w_0 will be given a meaning later, see section 4.

For $k_1, k_2 \in \mathbb{N}_0$, we call the graph homology class of the Jacobi diagram $\overset{k_1}{\underset{k_2}{\ominus}}$ a *double-wheel*, denoted by w_{k_1,k_2}. In particular, $w_{0,0} = \ominus$.

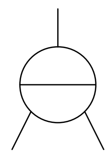


Figure 4. These double-wheel $w_{1,2}$.

Remark 15. The wheels w_k with k odd and the double-wheels w_{k_1,k_2} with $k_1 + k_2$ odd vanish in $\hat{\mathcal{B}}$ due to the IHX and AS relations.

Next, we define a very special element Ω of the graph homology space with very remarkable properties, which have been proven by Bar-Natan, Le and Thurston (see [**29**]).

Definition 15. Let the series $(b_{2k})_k \in \mathbb{C}^{\mathbb{N}_0}$ of the *modified Bernoulli numbers* (c.f. [**29**]) be defined by

$$\sum_{k=0}^{\infty} b_{2k} x^{2k} = \frac{1}{2} \ln \frac{\sinh \frac{x}{2}}{\frac{x}{2}}. \tag{34}$$

(Actually, all the b_{2k} are rational numbers.) Let $\Omega \in \hat{\mathcal{B}}'$ be the image of the element

$$\exp\left(\sum_{k=1}^{\infty} b_{2k} x_{2k}\right) \in \mathbb{Q}[[(x_{2k})_{k \in \mathbb{N}}]] \tag{35}$$

under the morphism $\mathbb{Q}[[(x_{2k})]] \to \hat{\mathcal{B}}'$ of $\mathbb{Q}$-algebras that maps x_{2k} to w_{2k}.

For any $\mu \in \mathbb{C}$ we set $\Omega(\mu) := \sum_{k=0}^{\infty} \Omega_k \mu^k$, where Ω_k is the homogeneous component of degree $2k$ of Ω. (Note that $\Omega_k = 0$ for odd k.) The element Ω lies in the ring $\hat{\mathcal{B}}' \otimes_{\mathbb{Q}} \mathbb{C}$. In what follows, by abuse of notion, we will use symbol $\hat{\mathcal{B}}$ for the ring as defined so far as well for the base-changed ring $\hat{\mathcal{B}} \otimes_{\mathbb{Q}} \mathbb{C}$. This should not lead to any confusion.

The following remark has already been stated in [**29**].

Remark 16. The modified Bernoulli numbers are connected to the usual Bernoulli numbers $B_1, B_2, B_3, \ldots$ via

$$b_{2k} = \frac{B_{2k}}{4k(2k)!} \tag{36}$$

for all $k \in \mathbb{N}$. In addition to this, $b_0 = 0$.

The generating function of the (usual) Bernoulli numbers is given by

$$\sum_{k=0}^{\infty} \frac{B_k}{k!} t^k = \frac{t}{e^t - 1}. \tag{37}$$

Note that $B_k = 0$ for $k > 1$ and k odd. Furthermore, $B_0 = 1$ and $B_1 = -\frac{1}{2}$.

The connection between the Bernoulli numbers and the modified ones can be proven easily by considering the two defining power series.

Let Γ be a Jacobi diagram and u, u' be two different univalent vertices of Γ. These two should not be the two vertices of a component ℓ of Γ. Let v (resp. v') be the vertex u (resp. u') is attached to. The process of *gluing the vertices u and u'* means to remove u and u' together with the edges connecting them to v resp. v' and to add a new edge between v and v'. Thus, we arrive at a new graph $\Gamma/(u, u')$, whose number of trivalent vertices is the number of trivalent vertices of Γ and whose number of univalent vertices is the number of univalent vertices of Γ minus two. To make it a Jacobi diagram we define the cyclic orientation of the flags at v (resp. v') to be the cyclic orientation of the flags at v (resp. v') in Γ with the flag belonging to the edge connecting v (resp. v') with u (resp. u') replaced by the flag belonging to the added edge.

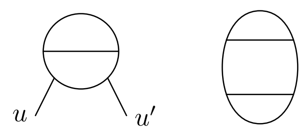


FIGURE 5. Gluing the two univalent vertices u and u' of the left graph produces the right one.

For example, gluing the two univalent vertices of w_2 leads to the graph $\ominus$.

Of course, the process of gluing two univalent vertices given above does not work if u and u' are the two univalent vertices of ℓ, thus our assumption on Γ.

DEFINITION 16. Let Γ, Γ' be two Jacobi diagrams, at least one of them without ℓ as a component and $U = \{u_1, \ldots, u_n\}$ resp. U' the sets of their univalent vertices. We define

$$\hat{\Gamma}(\Gamma') := \sum_{\substack{f:U\hookrightarrow U' \\ \text{injective}}} (\Gamma \cup \Gamma')/(u_1, f(u_1))/\ldots/(u_n, f(u_n)), \tag{38}$$

viewed as an element in $\hat{\mathcal{B}}$.

This induces for every $\gamma \in \hat{\mathcal{B}}$ a $\hat{\mathcal{B}}^{\mathrm{t}}$-linear map

$$\hat{\gamma} : \hat{\mathcal{B}}' \to \hat{\mathcal{B}}', \gamma' \mapsto \hat{\gamma}(\gamma'). \tag{39}$$

EXAMPLE 12. Set $\partial := \frac{1}{2}\hat{\ell}$. It is is an endomorphism of $\hat{\mathcal{B}}'$ of degree -2. For Example, $\partial{-\!\circ\!-} = \ominus$. By setting

$$\partial(\gamma, \gamma') := \partial(\gamma \cup \gamma') - \partial(\gamma) \cup \gamma' - \gamma \cup \partial(\gamma') \tag{40}$$

for $\gamma, \gamma' \in \hat{\mathcal{B}}'$, we have the following formula for all $\gamma \in \hat{\mathcal{B}}'$:

$$\partial(\gamma^n) = \binom{n}{1}\partial(\gamma)\gamma^{n-1} + \binom{n}{2}\partial(\gamma, \gamma)\gamma^{n-2}. \tag{41}$$

This shows that ∂ is a differential operator of order two acting on $\hat{\mathcal{B}}'$.

Acting by ∂ on a Jacobi diagram means to glue two of its univalent vertices in all possible ways, acting by $\partial(\cdot, \cdot)$ on two Jacobi diagrams means to connect them by gluing a univalent vertex of the first with a univalent vertex of the second in all possible ways.

For example, we have

$$\partial(w_{2k}) = k \sum_{n=0}^{2k-2} w_{n,2k-2-n} \tag{42}$$

for $k \in \mathbb{N}$, and by the IHX relation,

$$\partial(w_{2k_1}, w_{2k_2}) = 8k_1k_2w_{2k_1-1,2k_2-1} \tag{43}$$

for $k_1, k_2 \in \mathbb{N}$.

DEFINITION 17. Let Γ, Γ' be two Jacobi diagrams, at least one of them without ℓ as a component, and $U = \{u_1, \dots, u_n\}$ resp. U' the sets of their univalent vertices. We define

$$\langle \Gamma, \Gamma' \rangle := \sum_{\substack{f:U\to U' \\ \text{bijective}}} (\Gamma \cup \Gamma')/(u_1, f(u_1))/\dots/(u_n, f(u_n)), \tag{44}$$

viewed as an element in $\hat{\mathcal{B}}^{\mathrm{t}}$.

This induces a $\hat{\mathcal{B}}^{\mathrm{t}}$-bilinear map

$$\langle \cdot, \cdot \rangle : \hat{\mathcal{B}}' \times \hat{\mathcal{B}} \to \hat{\mathcal{B}}^{\mathrm{t}}, \tag{45}$$

which is symmetric on $\hat{\mathcal{B}}' \times \hat{\mathcal{B}}'$.

Note that $\langle \Gamma, \Gamma' \rangle$ is zero unless Γ and Γ' have equal numbers of univalent vertices. In this case, the expression is the sum over all possibilities to glue the univalent vertices of Γ with univalent vertices of Γ'.

PROPOSITION 14. *The map $\langle 1, \cdot \rangle : \hat{\mathcal{B}} \to \hat{\mathcal{B}}^{\mathrm{t}}$ is the canonical projection map, i.e. it removes all non-trivalent components from a graph. Furthermore, for $\gamma \in \hat{\mathcal{B}}'$ and $\gamma' \in \hat{\mathcal{B}}$, we have*

$$\left\langle \gamma, \frac{\ell}{2}\gamma' \right\rangle = \langle \partial\gamma, \gamma' \rangle \,. \tag{46}$$

For $\gamma, \gamma' \in \hat{\mathcal{B}}'$, we have the following (combinatorial) formula:

$$\langle \exp(\partial)(\gamma\gamma'), 1 \rangle = \langle \exp(\partial)\gamma, \exp(\partial)\gamma' \rangle \,. \tag{47}$$

PROOF. The formula (46) should be clear from the definitions.

Let us investigate (47) a bit more. We can assume that γ and γ' are Jacobi diagrams with l resp. l' univalent vertices and $l + l' = 2n$ with $n \in \mathbb{N}_0$. So we have to prove

$$\frac{\partial^n}{n!}(\gamma\gamma') = \sum_{\substack{m,m'=0 \\ l-2m=l'-2m'}}^{\infty} \left\langle \frac{\partial^m}{m!}\gamma, \frac{\partial^{m'}}{m'!}\gamma' \right\rangle,$$

since $\langle \cdot, 1 \rangle : \hat{\mathcal{B}} \to \hat{\mathcal{B}}^{\mathrm{t}}$ means to remove the components with at least one univalent vertex. Recalling the meaning of $\langle \cdot, \cdot \rangle$, it should be clear that (47) follows from the fact that applying $\frac{\partial^k}{k!}$ on a Jacobi diagram means to glue all subsets of $2k$ of its univalent vertices to k pairs in all possible ways. $\square$

3. Ω as an eigenvector of the operator ∂

As said in the previous section, the element Ω plays a central role in the "Wheeling Theorem" (see [**29**], where Bar-Natan, Le and Thurston gave a knot theoretical proof of this theorem). Hitchin and Sawon ([**15**]) discovered that this Theorem together with the ideas of Rozansky and Witten ([**27**]) can be used to deduce some interesting facts about characteristic classes on irreducible holomorphic symplectic manifolds.

There is actually a second product "$\times$" on the space $\hat{\mathcal{B}}$ of graph homology, which coincides with the previously defined product "$\cup$" on the space $\hat{\mathcal{B}}^{\mathrm{t}}$ of trivalent graphs. The Wheeling Theorem says that the map

$$\hat{\Omega} : (\hat{\mathcal{B}}, \cup) \to (\hat{\mathcal{B}}, \times) \tag{48}$$

is an isomorphism of algebras. Therefore, the map

$$\hat{\Omega} : (\hat{\mathcal{B}}, \cup) \to (\hat{\mathcal{B}}^{\mathrm{t}}, \cup), \gamma \mapsto \langle \Omega, \alpha \rangle = \left\langle \hat{\Omega}(\alpha), 1 \right\rangle \tag{49}$$

is a homomorphism of algebras. This corollary has been used by Hitchin and Sawon to express $\int_X c_2 \alpha^{2n-2}$ in terms of $\int_X \alpha^{2n}$ and $\int_X \mathrm{td}^{\frac{1}{2}}(X)$ for any $\alpha \in \mathrm{H}^2(X, \mathbb{C})$ on any irreducible holomorphic symplectic manifold X.

We want to extend the methods of Hitchin and Sawon in the next chapter. This will eventually lead to a general Hirzebruch-Riemann-Roch formula for line bundles on irreducible holomorphic symplectic manifolds. Like Hitchin and Sawon we shall not need the full Wheeling Theorem. In fact, we only need the corollary of the "Wheeling Theorem" given as Theorem 3 below to prove our results. Similarly, everything stated in [**15**] that is based on the Wheeling Theorem can also be based on the following Theorem 3. It is a direct corollary of Lemma 6.2 in [**29**]. Nevertheless, we give another proof here, which does not use any knot theory.

Theorem 3. *For each $\mu \in \mathbb{C}$, the graph $\Omega(\mu)$ is an eigenvector of the endomorphism $\partial : \hat{\mathcal{B}}' \to \hat{\mathcal{B}}'$ to the eigenvalue $\frac{\mu^2}{48}\ominus$, i.e.*

$$\partial \Omega(\mu) = \frac{\mu^2}{48} \ominus \Omega(\mu). \tag{50}$$

The proof goes along the following idea: We apply a certain linear map P from the space of graph homology to a polynomial algebra to both sides of (50). In general, such a linear form on the space of graph homology is called a *weight system*. We show that the images of the left and the right hand side under the map P are equal. If we have chosen P to be injective, the theorem is proven.

Let us say a few words about the construction of the map P. Given a Lie algebra $\mathfrak{g}$ together with an $\mathrm{ad}_{\mathfrak{g}}$-invariant symmetric non-degenerate bilinear form σ and a representation, a weight system can be constructed from this data. This construction is analogous to the construction of Rozansky-Witten classes (see Chapter 5) with the Atiyah class substituted by the Lie bracket $[\cdot, \cdot] \in \mathfrak{g}^{\otimes 3}$. Here we have identified $\mathfrak{g}^*$ with $\mathfrak{g}$ using σ.

For our purposes, we take for $\mathfrak{g}$ the Lie algebra $\mathfrak{gl}(N)$ for large N, and set $\sigma(x, y) := \mathrm{tr}(xy)$. Instead of working with a specific representation, we use universal Casimir elements x_i accounting for all representations at once.

This gives us a map P as described above. Chmutov and Duzhin calculated this map on a certain subspace of graph homology in [**6**]. We will only need their result on how P acts on double-wheels, so we do not have to go into more detail on the construction of P as a Lie algebra weight system here.

Lemma 1. *Let $\mathcal{W}$ be the subspace of $\hat{\mathcal{B}}$ that is spanned by all graphs $w_{i,j}$ with $i, j \in \mathbb{N}_0$. Let $P : \mathcal{W} \to \mathrm{S}^3_{\mathbb{Q}}((x_n)_{n \in \mathbb{N}_0})$ be defined by*

$$P(w_{i,j}) = \begin{cases} 2 \sum_{l,m=0}^{\infty} (-1)^{l+m} \binom{i}{l} \binom{j}{m} x_l x_m x_{i+j-l-m} & \text{for } i+j \text{ even,} \\ 0 & \text{for } i+j \text{ odd.} \end{cases} \tag{51}$$

Then P is a well-defined, injective map.

Proof. First, we show that P is well-defined. For every $n \in \mathbb{N}_0$, let $\mathcal{W}_n$ be the subspace of $\mathcal{W}$ spanned by all $w_{i,j}$ with $i + j = n$. It is enough to show that for all $n \in \mathbb{N}_0$ there exists an injective map $P_n : \mathcal{W}_n \to \mathrm{S}^3_{\mathbb{Q}}((x_n)_{n \in \mathbb{N}_0})$ that fulfills (51). Further, we can restrict ourselves to the case of even n due to Lemma 6.2 of [**7**], which says that $w_{i,j}$ is homologous to zero for odd n. (This follows at once from the anti-symmetry relation.)

For any $N > n$, the map $P_{\mathfrak{gl}(N)}$ defined in section 3.1 of [**6**] fulfills (51) when restricted to $\mathcal{W}_n$. This is because of Proposition 4.5 of [**6**], where it is shown that $P_{\mathfrak{gl}(N)}$ evaluated at $w_{i,j}$ equals the right hand side of (51).

It remains to show that P_n is injective. This can be proven by a dimension argument: By Lemma 6.2 and Lemma 6.8 of [**7**] the image of $\mathcal{W}_n$ under P_n has at least the dimension of $\mathcal{W}_n$, so P_n is injective. $\square$

The following lemma is also of combinatorial nature.

LEMMA 2. *Let $B_0, B_1, B_2, \ldots$ denote the Bernoulli numbers. The following formula holds in $\mathrm{S}^3_{\mathbb{Q}}((x_n)_{n\in\mathbb{N}_0})$:*

$$\sum_{k=2}^{\infty} \frac{B_k}{k!} \sum_{n=0}^{k-2} \sum_{l,m=0}^{\infty} (-1)^{l+m} \binom{n}{l} \binom{k-2-n}{m} x_l x_m x_{k-2-l-m}$$
$$+ \sum_{i,j=2}^{\infty} \frac{B_i}{i!} \frac{B_j}{j!} \sum_{l,m=0}^{\infty} (-1)^{l+m} \binom{i-1}{l} \binom{j-1}{m} x_l x_m x_{i+j-2-l-m} = \frac{1}{12} x_0^3. \quad (52)$$

PROOF. In $\mathbb{Q}[X_1, X_2, X_3]$, we calculate

$$\sum_{\pi\in\mathfrak{S}_3} \left(\sum_{k=2}^{\infty} \frac{B_k}{k!} \sum_{n=0}^{k-2} \sum_{l,m=0}^{\infty} (-1)^{l+m} \binom{n}{l} \binom{k-2-n}{m} X_{\pi(1)}^l X_{\pi(2)}^m X_{\pi(3)}^{k-2-l-m} \right.$$
$$\left. + \sum_{i,j=2}^{\infty} \frac{B_i}{i!} \frac{B_j}{j!} \sum_{l,m=0}^{\infty} (-1)^{l+m} \binom{i-1}{l} \binom{j-1}{m} X_{\pi(1)}^l X_{\pi(2)}^m X_{\pi(3)}^{i+j-2-l-m} \right)$$
$$= \sum_{\pi\in\mathfrak{S}_3} \left(\sum_{k=2}^{\infty} \frac{B_k}{k!} \sum_{n=0}^{k-2} \left(X_{\pi(3)} - X_{\pi(1)}\right)^n \left(X_{\pi(3)} - X_{\pi(2)}\right)^{k-2-n} \right.$$
$$\left. + \sum_{i,j=2}^{\infty} \frac{B_i}{i!} \frac{B_j}{j!} (X_{\pi(3)} - X_{\pi(1)})^{i-1} (X_{\pi(3)} - X_{\pi(2)})^{j-1} \right)$$
$$= \sum_{\pi\in\mathfrak{S}_3} \left(\frac{1}{X_{\pi(1)} - X_{\pi(2)}} \left(\begin{aligned} &\frac{1}{X_{\pi(3)} - X_{\pi(2)}} \sum_{k=2}^{\infty} \frac{B_k}{k!} \left(X_{\pi(3)} - X_{\pi(2)}\right)^k \\ &\qquad - \frac{1}{X_{\pi(3)} - X_{\pi(1)}} \sum_{k=2}^{\infty} \frac{B_k}{k!} \left(X_{\pi(3)} - X_{\pi(1)}\right)^k \end{aligned} \right) \right.$$
$$+ \frac{1}{(X_{\pi(3)} - X_{\pi(1)})(X_{\pi(3)} - X_{\pi(2)})}$$
$$\left. \cdot \left(\sum_{k=2}^{\infty} \frac{B_k}{k!} \left(X_{\pi(3)} - X_{\pi(1)}\right)^k \right) \left(\sum_{k=2}^{\infty} \frac{B_k}{k!} \left(X_{\pi(3)} - X_{\pi(2)}\right)^k \right) \right)$$
$$= \sum_{\pi\in\mathfrak{S}_3} \left(\frac{1}{X_{\pi(1)} - X_{\pi(2)}} \left(\begin{aligned} &\frac{1}{\exp\left(X_{\pi(3)} - X_{\pi(2)}\right) - 1} - \frac{1}{X_{\pi(3)} - X_{\pi(2)}} \\ &\qquad - \frac{1}{\exp\left(X_{\pi(3)} - X_{\pi(1)}\right) - 1} + \frac{1}{X_{\pi(3)} - X_{\pi(1)}} \end{aligned} \right) \right.$$
$$+ \left(\frac{1}{\exp\left(X_{\pi(3)} - X_{\pi(1)}\right) - 1} - \frac{1}{X_{\pi(3)} - X_{\pi(1)}} + \frac{1}{2} \right)$$
$$\left. \cdot \left(\frac{1}{\exp\left(X_{\pi(3)} - X_{\pi(2)}\right) - 1} - \frac{1}{X_{\pi(3)} - X_{\pi(2)}} + \frac{1}{2} \right) \right) = \frac{1}{2}.$$

This proves the lemma because there is a well-defined $\mathbb{Q}$-linear map

$$\mathrm{S}^3_{\mathbb{Q}}((x_n)_{n\in\mathbb{N}_0}) \to \mathbb{Q}[X_1,X_2,X_3], x_i x_j x_k \mapsto \sum_{\pi\in\mathfrak{S}_3} X^i_{\pi(1)} X^j_{\pi(2)} X^k_{\pi(3)},$$

which is injective. $\square$

Proof of Theorem 3. Since ∂ is a linear operator of degree -2 on $\hat{\mathcal{B}}'$ and $\Omega(\mu) = \sum_{k=0}^\infty \Omega_{2k}\mu^{2k}$, we can assume that $\mu = 1$. Set $\Gamma := \sum_{k=1}^\infty b_{2k} w_{2k}$. We have

$$\begin{aligned}\partial\Omega = \partial\exp(\Gamma) &= \sum_{n=0}^\infty \frac{\partial(\Gamma^n)}{n!}\\ &= \sum_{n=1}^\infty \partial(\Gamma)\frac{\Gamma^{n-1}}{(n-1)!} + \sum_{n=2}^\infty \frac{\partial(\Gamma,\Gamma)}{2}\frac{\Gamma^{n-2}}{(n-2)!}\\ &= \left(\partial\Gamma + \frac{1}{2}\partial(\Gamma,\Gamma)\right)\Omega,\end{aligned}$$

so we see that Ω is an eigenvector. (The above calculation is actually based only on the fact that ∂ is a differential operator of order two.)

Now, we have to calculate the eigenvalue:

$$\begin{aligned}\partial\Gamma + \frac{1}{2}\partial(\Gamma,\Gamma) &= \sum_{k=1}^\infty b_{2k}\partial w_{2k} + \sum_{i,j=1}^\infty b_{2i}b_{2j}\frac{\partial(w_{2i},w_{2j})}{2}\\ &= \sum_{k=1}^\infty k b_{2k}\sum_{n=0}^{2k-2} w_{n,2k-2-n} + \sum_{i,j=1}^\infty 4ij b_{2i}b_{2j} w_{2i-1,2j-1}\\ &= \frac{1}{4}\sum_{k=1}^\infty \frac{B_{2k}}{(2k)!}\sum_{n=0}^{2k-2} w_{n,2k-2-n} + \frac{1}{4}\sum_{i,j=1}^\infty \frac{B_{2i}}{(2i)!}\frac{B_{2j}}{(2j)!} w_{2i-1,2j-1}\\ &= \frac{1}{4}\sum_{k=2}^\infty \frac{B_k}{k!}\sum_{n=0}^{k-2} w_{n,k-2-n} + \frac{1}{4}\sum_{i,j=2}^\infty \frac{B_i}{i!}\frac{B_j}{j!} w_{i-1,j-1}.\end{aligned}$$

Applying the (injective) map P of Lemma 1, and using Lemma 2 yields

$$\begin{aligned}P\left(\partial\Gamma + \frac{1}{2}\partial(\Gamma,\Gamma)\right) &= \frac{1}{4}P\left(\sum_{k=2}^\infty \frac{B_k}{k!}\sum_{n=0}^{k-2} w_{n,k-2-n} + \sum_{i,j=2}^\infty \frac{B_i}{i!}\frac{B_j}{j!} w_{i-1,j-1}\right)\\ &= \frac{1}{24}x_0^3 = P\left(\frac{1}{48}w_{0,0}\right) = P\left(\frac{1}{48}\ominus\right).\end{aligned}$$

Because of the injectivity of P, this proves the theorem. $\square$

4. An $\mathfrak{sl}_2$-action on the space of graph homology

In this short section we want to extend the space of graph homology slightly. This is mainly due to two reasons: When we defined the expression $\hat{\Gamma}(\Gamma)$ for two Jacobi diagrams Γ and Γ', we restricted ourselves to the case that Γ or Γ' does not contain a component with an ℓ. Secondly, we have not given the *zero-wheel* w_0 a meaning yet.

We do this by adding an element $\bigcirc$ to the various spaces of graph homology.

Definition 18. *The extended space of graph homology* is the space $\hat{\mathcal{B}}[[\bigcirc]]$. Further, we set $w_0 := \bigcirc$, which, at least picturally, is in accordance with the definition of w_k for $k > 0$.

Note that this element is not depicting a Jacobi diagram as we have defined it. Nevertheless, we want to use the notion that $\bigcirc$ has no univalent and no trivalent vertices, i.e. the homogeneous component of degree zero of $\hat{\mathcal{B}}[[\bigcirc]]$ is $\mathbb{C}[[\bigcirc]]$.

When defining $\Gamma/(u,u')$ for a Jacobi diagramm Γ with two univalent vertices u and u', i.e. gluing u to u', we assumed that u and u' are not the vertices of one component ℓ of Γ. Now we extend this definition by defining $\Gamma/(u,u')$ to be the extended graph homology class we get by replacing ℓ with $\bigcirc$, whenever u and u' are the two univalent vertices of a component ℓ of Γ.

Doing so, we can give the expression $\hat{\gamma}(\gamma') \in \hat{\mathcal{B}}[[\bigcirc]]$ a meaning with no restrictions on the two graph homology classes $\gamma, \gamma' \in \hat{\mathcal{B}}$, i.e. every $\gamma \in \hat{\mathcal{B}}[[\bigcirc]]$ defines a $\hat{\mathcal{B}}^{\mathrm{t}}[[\bigcirc]]$-linear map

$$\hat{\gamma} : \hat{\mathcal{B}}[[\bigcirc]] \to \hat{\mathcal{B}}[[\bigcirc]]. \tag{53}$$

Example 13. We have

$$\partial \ell = \bigcirc. \tag{54}$$

Remark 17. We can similarly extend $\langle \cdot, \cdot \rangle : \hat{\mathcal{B}}' \times \hat{\mathcal{B}} \to \hat{\mathcal{B}}^{\mathrm{t}}$ to a $\hat{\mathcal{B}}^{\mathrm{t}}[[\bigcirc]]$-bilinear form

$$\langle \cdot, \cdot \rangle : \hat{\mathcal{B}}[[\bigcirc]] \times \hat{\mathcal{B}}[[\bigcirc]] \to \hat{\mathcal{B}}^{\mathrm{t}}[[\bigcirc]]. \tag{55}$$

Both $\ell/2$ and ∂ are two operators acting on the extended space of graph homology, the first one just multiplication with $\ell/2$. By calculating their commutator, we show that they induce a natural structure of an $\mathfrak{sl}_2$-module on $\hat{\mathcal{B}}[[\bigcirc]]$.

Proposition 15. *Let $H : \hat{\mathcal{B}}[[\bigcirc]] \to \hat{\mathcal{B}}[[\bigcirc]]$ be the linear operator which acts on $\gamma \in \hat{\mathcal{B}}_{k,l}[[\bigcirc]]$ by*

$$H\gamma = \left(\frac{1}{2} \bigcirc + l \right) \gamma. \tag{56}$$

We have the following commutator relations in $\operatorname{End} \hat{\mathcal{B}}[[\bigcirc]]$*:*

$$[\ell/2, \partial] = -H, \tag{57}$$

$$[H, \ell/2] = 2 \cdot \ell/2, \tag{58}$$

and

$$[H, \partial] = -2\partial, \tag{59}$$

i.e. the triple $(\ell/2, -\partial, H)$ defines a $\mathfrak{sl}_2$-operation on $\hat{\mathcal{B}}[[\bigcirc]]$.

Proof. Equations (58) and (59) follow from the fact that multiplying by $\bigcirc$ commutes with $\ell/2$ and ∂, and from the fact that $\ell/2$ is an operator of degree 2 with respect to the grading given by the number of univalent vertices, whereas ∂ is an operator of degree -2 with respect to the same grading.

It remains to look at (57). For $\gamma \in \hat{\mathcal{B}}_{k,l}[[\bigcirc]]$, we calculate

$$[\ell, \partial]\gamma = \ell\partial(\gamma) - \partial(\ell\gamma) = \ell\partial(\gamma) - \partial(\ell)\gamma - \ell\partial(\gamma) - \partial(\ell, \gamma) = - \bigcirc \gamma - 2l\gamma = -2H\gamma. \tag{60}$$

$\square$

Remark 18. Since $\hat{\mathcal{B}}[[\bigcirc]]$ is infinite-dimensional, we have unfortunately difficulties to apply the standard theory of $\mathfrak{sl}_2$-representations to this $\mathfrak{sl}_2$-module. For example, there are no eigenvectors for the operator H.

CHAPTER 5

Rozansky-Witten classes

The idea to associate to every graph Γ and every hyperkähler manifold X a cohomology class $\mathrm{RW}_X(\Gamma)$ is due to Rozansky and Witten (c.f. [**27**]). Kapranov showed in [**20**] that the metric structure of a hyperkähler manifold is not nessessary to define these classes. It was his idea to build the whole theory upon the Atiyah class and the symplectic structure of an irreducible holomorphic symplectic manifold. We will make use of his definition of Rozansky-Witten classes in this section. A very detailed text on defining Rozansky-Witten invariants is the thesis by Sawon [**28**].

1. Rozansky-Witten classes in general

Let k be a field of characteristic zero, V a finite-dimensional k-vector space, $A = \bigoplus_{i=0}^{\infty} A_i$ a $\mathbb{Z}$-graded (super-)commutative k-algebra and σ a symplectic form on V.

We shall use this general setting later in the case when $V = \mathcal{T}_{X,x}$ will be the holomorphic tangent space of a complex manifold X at a point x and $A = \overline{\Omega}^*_{X,x}$ the Graßmann algebra of anti-holomorphic forms at x.

For every Jacobi diagram Γ with k trivalent and l univalent vertices, and every $\alpha \in \mathrm{S}^3 V \otimes_k A_1$, we define an element

$$\mathrm{RW}_{\sigma,\alpha}(\Gamma) \in \bigwedge^l V^* \otimes A_k \tag{61}$$

by the following procedure:

Let T denote the set of trivalent vertices, U the set of univalent vertices, E the set of edges, and F the set of flags of Γ. So $l = |U|$, and $k = |T|$. The Jacobi diagram Γ defines a map

$$\begin{aligned}
\Phi^\Gamma : (\mathrm{S}^3 V \otimes A_1)^{\otimes T} \otimes (\operatorname{End} V)^{\otimes U} &\xrightarrow{(1)} \left(\bigotimes_{t \in T} V^{\otimes t}\right) \otimes A_1^{\otimes T} \otimes V^{\otimes U} \otimes (V^*)^{\otimes U} \\
&\xrightarrow{(2)} V^{\otimes F} \otimes (V^*)^{\otimes U} \otimes A_1^{\otimes T} \\
&\xrightarrow{(3)} \bigotimes_{e \in E} V^{\otimes e} \otimes (V^*)^{\otimes U} \otimes A_1^{\otimes T} \\
&\xrightarrow{(4)} (V^{\otimes 2})^{\otimes \frac{3k+l}{2}} \otimes (V^*)^{\otimes l} \otimes (A_1)^{\otimes k} \\
&\xrightarrow{(5)} \bigwedge^l V^* \otimes A_k,
\end{aligned} \tag{62}$$

where

(1) is induced by the inclusions of the symmetric tensors $\mathrm{S}^3 V$ in the spaces $V^{\otimes t}$ with $t \in T$ (note again that t is a set of three elements), and the canonical identification $\operatorname{End} V = V \otimes V^*$,

(2) is induced by the canonical bijection $U \amalg \coprod_{t \in T} t \to F$ which is on each $t \in T$ the inclusion of the subset t in F and which maps each $u \in U$ to the flag belonging to u,
(3) is induced by the associativity of the tensor product (note that $\coprod_{e \in E} e = F$),
(4) is induced by choosing an arbitrary ordering of the set E, and total orderings of the sets $e \in E$, the set U and the set T that are compatible with the orientation of the graph Γ, and
(5) is given by $((v_1 \otimes v_2) \otimes \cdots \otimes (v_{3k+l-1} \otimes v_{3k+l})) \otimes (\alpha_1 \otimes \cdots \otimes \alpha_l) \otimes (a_1 \otimes \cdots \otimes a_k) \mapsto \sigma(v_1, v_2) \cdot \cdots \cdot \sigma(v_{3k+l-1}, v_{3k+l}) \cdot (\alpha_1 \wedge \cdots \wedge \alpha_l) \otimes (a_1 \cdot \cdots \cdot a_k)$.

The map Φ^Γ is independent of the specific choices made in (4). Especially changing the ordering of E does not change the map though the ordering of E is not mentioned in the compatibility condition.

Further, we define

$$\Phi^{\bigcirc} : k \to k, 1 \mapsto -\dim V, \tag{63}$$

the "Jacobi diagram" $\bigcirc$ being the element used in the construction of the extended graph homology space.

One defines

$$\mathrm{RW}_{\sigma,\alpha}(\Gamma) := \Phi^\Gamma(\alpha^{\otimes T} \otimes (\mathrm{id}_V)^{\otimes U}); \tag{64}$$

in particular,

$$\mathrm{RW}_{\sigma,\alpha}(\bigcirc) = -\dim V. \tag{65}$$

The following proposition summarises some of the properties of the maps Φ^Γ we need later on. All of them follow directly from the definitions.

PROPOSITION 16. *Identifying the set of the univalent vertices of ℓ with the set $\{1, 2\}$, we have*

$$\Phi^\ell : (\mathrm{End}\, V)^{\otimes 2} \to \bigwedge^2 (V^*), (v_1 \otimes \lambda_1) \otimes (v_2 \otimes \lambda_2) \mapsto \sigma(v_1, v_2) \cdot (\lambda_1 \wedge \lambda_2) \tag{66}$$

for all $v_1, v_2 \in V$, $\lambda_1, \lambda_2 \in V^*$.

Let $k \in \mathbb{N}$. Identifying both the set of the univalent vertices and the set of the trivalent vertices of the wheel w_{2k} (viewed as a Jacobi diagram) with the set $\{1, \ldots, 2k\}$, we have

$$\begin{gathered}\Phi^{w_{2k}} : (\mathrm{S}^3 V \otimes A_1)^{\otimes 2k} \otimes (\mathrm{End}\, V)^{\otimes 2k} \to \bigwedge^{2k} (V^*) \otimes A_{2k}, \\ \bigotimes_{i=1}^{2k} (v_i^3 \otimes a_i) \otimes \bigotimes_{i=1}^{2k} (w_i \otimes \lambda_i) \mapsto - \prod_{i \in \mathbb{Z}/(2k)} (\sigma(v_i, v_{i+1}) \cdot \sigma(v_i, w_i)) \cdot \bigwedge_{i=1}^{2k} \lambda_i \otimes \prod_{i=1}^{2k} a_i\end{gathered} \tag{67}$$

for all $v_i, w_i \in V$, $\lambda_i \in V^*$ and $a_i \in A_1$.

Let Γ and Γ' be two Jacobi diagrams with univalent vertices U and U' and trivalent vertices T and T'. Let the cardinalities of T, T', U, U' be k, k', l, l'.

Then the diagram

$$
\begin{array}{ccc}
\big((\mathrm{S}^3V\otimes A_1)^{\otimes T}\otimes(\operatorname{End}V)^{\otimes U}\big) & & \Big(\bigwedge^{2l}(V^*)\otimes A_{2k}\Big) \\
\qquad\otimes\Big((\mathrm{S}^3V\otimes A_1)^{\otimes T'}\otimes(\operatorname{End}V)^{\otimes U'}\Big) & \xrightarrow{\Phi^\Gamma\otimes\Phi^{\Gamma'}} & \qquad\otimes\Big(\bigwedge^{2l'}(V^*)\otimes A_{2k'}\Big) \\
\big\downarrow & & \big\downarrow \\
(\mathrm{S}^3V\otimes A_1)^{\otimes(T\amalg T')}\otimes(\operatorname{End}V)^{\otimes(U\amalg U')} & \xrightarrow[\Phi^{\Gamma\cup\Gamma'}]{} & \bigwedge^{2(l+l')}(V^*)\otimes A_{2(k+k')},
\end{array}
\tag{68}
$$

where the vertical maps are the canonical ones, and the diagram

$$
\begin{array}{ccc}
\big((\mathrm{S}^3V\otimes A_1)^{\otimes T}\otimes(\operatorname{End}V)^{\otimes U}\big) & & \Big(\bigwedge^{2l}(V^*)\otimes A_{2k}\Big) \\
\qquad\otimes\Big((\mathrm{S}^3V\otimes A_1)^{\otimes T'}\otimes(\operatorname{End}V)^{\otimes U'}\Big) & \xrightarrow{\Phi^\Gamma\otimes\Phi^{\Gamma'}} & \qquad\otimes\Big(\bigwedge^{2l'}(V^*)\otimes A_{2k'}\Big) \\
\big\uparrow & & \big\downarrow \\
(\mathrm{S}^3V\otimes A_1)^{\otimes T}\otimes(\mathrm{S}^3V\otimes A_1)^{\otimes T'} & \xrightarrow[\Phi^{\langle\Gamma,\Gamma'\rangle}]{} & A_{2(k+k')},
\end{array}
\tag{69}
$$

where the left vertical map is induced by tensoring with the identity on V and the right one is just (137) *(see Appendix A), commute. If both Γ and Γ' contain a component ℓ, we have, of course, to work in the extended space of graph homology.*

2. Rozansky-Witten classes of irreducible holomorphic symplectic manifolds

Let X be an irreducible holomorphic symplectic manifold, and σ a fixed holomorphic symplectic form on X. We denote by $A^k(X,E)$ the space of differentiable $(0,k)$-forms with values in a holomorphic vector bundle E. We set $A^{l,k}(X) := A^k(X,\Omega^l_X)$.

Let $\tilde\alpha \in A^1(X,\Omega_X\otimes\operatorname{End}\mathcal{T}_X)$ be a Dolbeault representative of the Atiyah class of X, i.e. $\tilde\alpha$ represents the extension class of the sequence

$$0 \longrightarrow \Omega_X\otimes\mathcal{T}_X \longrightarrow \mathrm{J}^1\mathcal{T}_X \longrightarrow \mathcal{T}_X \longrightarrow 0 \tag{70}$$

in $\operatorname{Ext}^1_X(\mathcal{T}_X,\Omega_X\otimes\mathcal{T}_X) = \mathrm{H}^1(X,\Omega_X\otimes\operatorname{End}\mathcal{T}_X)$. Here, $\mathrm{J}^1\mathcal{T}_X$ is the bundle of one-jets of sections of $\mathcal{T}_X$ (for more on this, see [**20**]). The Atiyah class can also be viewed as the obstruction for a global holomorphic connection to exist on $\mathcal{T}_X$.

We can use σ to identify the tangent bundle $\mathcal{T}_X$ of X with the cotangent bundle Ω_X. Doing this, $\tilde\alpha$ can be viewed as an element of $A^1(X,\mathcal{T}_X^{\otimes 3})$. Now the point is that it $\tilde\alpha$ is not any such element: The following proposition was proven by Kapranov in [**20**].

Proposition 17. *The representative $\tilde\alpha$ can be chosen to lie in $A^1(X,\mathrm{S}^3\mathcal{T}_X)$, i.e. the values of $\tilde\alpha$ are symmetric tensors.*

From now on, let $\tilde\alpha$ be such an element.

Definition 19. For every Jacobi diagram Γ with k trivalent and l univalent vertices, one defines

$$\mathrm{RW}_\sigma(\Gamma) \in \mathrm{H}^k(X,\Omega^l) \tag{71}$$

to be the Dolbeault cohomology class of the ($\bar{\partial}$-)closed (l, k)-form

$$(x \mapsto \mathrm{RW}_{\sigma_x, \alpha_x}(\Gamma)) \in A^{l,k}(X), \tag{72}$$

where

$$\alpha := \frac{i}{2\pi}\tilde{\alpha}. \tag{73}$$

For a $\mathbb{C}$-linear combination γ of Jacobi diagrams, $\mathrm{RW}_\sigma(\gamma)$ is defined by linear extension.

Remark 19. That the form defined in (72) is $\bar{\partial}$-closed follows from the fact that σ and α are $\bar{\partial}$-closed.

Similarly, changing α by an $\bar{\partial}$-exact form leaves $\mathrm{RW}_\sigma(\Gamma)$ invariant up to an $\bar{\partial}$-exact form. Therefore, $\mathrm{RW}_\sigma(\Gamma)$ depends only on the cohomology class of α.

In [**20**], Kapranov also showed the following Proposition, which is crucial for the next definition. It follows from a Bianchi-identity for the Atiyah class.

Proposition 18. *If γ is a $\mathbb{Q}$-linear combination of Jacobi diagrams that is zero modulo the anti-symmetry and IHX relations, then* $\mathrm{RW}_\sigma(\gamma) = 0$.

Definition 20. We define a double-graded linear map

$$\mathrm{RW}_\sigma : \hat{\mathcal{B}} \to \mathrm{H}^*(X, \Omega^*), \tag{74}$$

which maps $\mathcal{B}_{k,l}$ into $\mathrm{H}^k(X, \Omega^l)$ by mapping a homology class of a Jacobi diagram Γ to $\mathrm{RW}_\sigma(\Gamma)$.

The values of the just defined map RW_σ are called *Rozansky-Witten classes of the irreducible holomorphic symplectic manifold X*.

We can extend the Definition of Rozansky-Witten classes associated to graph homology classes to the extended space of graph homology by setting $\mathrm{RW}_\sigma(w_0) = \mathrm{RW}_\sigma(\bigcirc) = -\operatorname{rk}(\mathcal{T}_X) = -\dim X$ in accordance with the definition of $\Phi^\bigcirc$. Doing so, we extend RW_σ to a double-graded map

$$\mathrm{RW}_\sigma : \hat{\mathcal{B}}[\bigcirc] \to \mathrm{H}^*(X, \Omega^*_X). \tag{75}$$

3. Examples of Rozansky-Witten classes

Let X and σ be as before.

Example 14. The Dolbeault cohomology class $[\sigma] \in \mathrm{H}^{2,0}(X)$ is a Rozansky-Witten class; more precisely, we have

$$\mathrm{RW}_\sigma(\ell) = 2\sigma, \tag{76}$$

which follows from (66).

The following example is due to Hitchin and Sawon [**15**]. It is of great importance for their and our results.

Example 15. Let $\mathrm{ch}(X) = \sum_{k=0}^\infty s_{2k}/(2k)!$, $s_{2k} \in \mathrm{H}^{2k,2k}(X)$, be the Chern character of X. Then

$$\mathrm{RW}_\sigma(w_{2k}) = -s_{2k} \tag{77}$$

for all $k \in \mathbb{N}_0$. (Note that for a holomorphic symplectic manifold, $\mathrm{ch}(X)$ has no term in degree (k, k) for k odd as said in chapter 2.)

Since the algebra of characteristic classes of X is spanned by the classes s_{2k}, every characteristic class is a Rozansky-Witten class due to Proposition 19 below.

A proof of (77) is given by Hitchin and Sawon in [**15**], where the Rozansky-Witten invariants are defined by using the Riemann curvature tensor of a hyperkähler metric of X instead of the Atiyah class. An idea of the proof in our context is given below.

IDEA OF THE PROOF OF (77). We can assume $k > 0$ (the case that is considered by Hitchin and Sawon) since the statement for $k = 0$ follows directly from the definitions.

As remarked in [**20**] by Kapranov, a Dolbeault representative of the Atiyah class $\tilde{\alpha} \in A^1(X, \Omega_X \otimes \operatorname{End} \mathcal{T}_X)$ on every Kähler manifold X is related to the characteristic classes $s_{2k} \in \mathrm{H}^{2k,2k}(X)$ of X in the following way: A Dolbeault representative of s_{2k} is given by $\operatorname{Alt}(\operatorname{tr}(\alpha^k))$. Here, $\alpha^k \in A^k(X, \Omega_X^{\otimes k} \otimes \operatorname{End} \mathcal{T}_X)$ means taking the k-th product of α viewed as $(0,1)$-form, giving an element of $A^k(X, \Omega_X^{\otimes k} \otimes_X (\operatorname{End} \mathcal{T}_X)^{\otimes k})$, and then using the associative algebra structure of $\operatorname{End} \mathcal{T}_X$. Further, tr means taking the trace on $\operatorname{End} \mathcal{T}_X$, and Alt is induced by the canonical projection $\Omega_X^{\otimes k} \to \Omega_X^k = \bigwedge^k \Omega_X$.

That this procedure on holomorphic symplectic manifolds is essentially (i.e. up to a sign) the same as taking the Rozansky-Witten class of a wheel follows from (67). $\square$

EXAMPLE 16. The Todd genus of a holomorphic symplectic manifold X is given by

$$\operatorname{td}(X) = \exp\left(-2\sum_{k=0}^{\infty} b_{2k} s_{2k}\right), \tag{78}$$

with b_{2k} being a modified Bernoulli number (see [**15**] for this statement). Thus,

$$\operatorname{td}(X) = \mathrm{RW}_\sigma(\Omega^2) \tag{79}$$

and

$$\operatorname{td}^{\frac{1}{2}}(\mu)(X) = \mathrm{RW}_\sigma(\Omega(\mu)) \tag{80}$$

for all $\mu \in \mathbb{C}$ (see (10)). Here, we have used the following Proposition 19.

4. Properties of Rozansky-Witten classes

PROPOSITION 19. *The map* $\mathrm{RW}_\sigma : \hat{\mathcal{B}}[\bigcirc] \to \mathrm{H}^{*,*}(X)$ *is a morphism of graded* $\mathbb{Q}$*-algebras.*

PROOF. The statement follows from (68). $\square$

PROPOSITION 20. *Let* $\gamma, \gamma' \in \hat{\mathcal{B}}[\bigcirc]$ *such that* $\langle \gamma, \gamma' \rangle \in \hat{\mathcal{B}}[\bigcirc] \subset \hat{\mathcal{B}}[[\bigcirc]]$*. Then*

$$\mathrm{RW}_\sigma(\langle \gamma, \gamma' \rangle) = \langle \mathrm{RW}_\sigma(\gamma), \mathrm{RW}_\sigma(\gamma') \rangle\,, \tag{81}$$

where $\langle \gamma, \gamma' \rangle$ *is given by* (28) *lifted on the level of cohomology.*

PROOF. The statement follows from (69). $\square$

The following proposition is also stated in [**15**] in a slightly different notation. With the formalism we have introduced so far, we can give a compact proof.

PROPOSITION 21. *For the Rozansky-Witten class of* $\ominus$*, we have*

$$\mathrm{RW}_\sigma(\ominus) = \frac{2\int_X c_2(X)\exp(\sigma + \bar{\sigma})}{n\int_X \exp(\sigma + \bar{\sigma})} \cdot [\bar{\sigma}]. \tag{82}$$

PROOF. Due to the irreducibility of X, i.e. $\mathrm{H}^{2k}(X,\mathcal{O}_X)=\mathbb{C}\cdot[\bar\sigma]^k$ for all $k\in\mathbb{N}_0$, we can write

$$\alpha=\frac{\int_X \alpha\exp(\sigma+\bar\sigma)}{n\int_X\exp(\sigma+\bar\sigma)}\cdot[\bar\sigma] \tag{83}$$

for all $\alpha\in\mathrm{H}^{0,2}(X)$. Using this, we have

$$\begin{aligned}\mathrm{RW}_\sigma(\ominus)&=\mathrm{RW}_\sigma\left(\frac{1}{2}\langle w_2,\ell\rangle\right)=\frac{1}{2}\langle \mathrm{RW}_\sigma(w_2),\mathrm{RW}_\sigma(\ell)\rangle\\ &=\langle -s_2,\sigma\rangle=2\langle c_2(X),\exp\sigma\rangle=\frac{2\int_X\langle c_2(X),\exp\sigma\rangle\exp(\sigma+\bar\sigma)}{n\int_X\exp(\sigma+\bar\sigma)}\cdot[\bar\sigma],\end{aligned} \tag{84}$$

which proves the proposition because of Proposition 29 (in Appendix A). □

Inspecting the action of the graph homology operator ∂ on the level of cohomology, we see that it is just contracting by the symplectic form (see also appendix A). Therefore, the following diagram commutes:

$$\begin{CD}\hat{\mathcal{B}}[\bigcirc] @>{-\partial}>> \hat{\mathcal{B}}[\bigcirc]\\ @V{\mathrm{RW}_\sigma}VV @VV{\mathrm{RW}_\sigma}V\\ \mathrm{H}^*(X,\Omega^*_X) @>>{\Lambda_{\sigma/4}}> \mathrm{H}^*(X,\Omega^*_X).\end{CD} \tag{85}$$

Since also the diagrams

$$\begin{CD}\hat{\mathcal{B}}[\bigcirc] @>{\ell/2}>> \hat{\mathcal{B}}[\bigcirc]\\ @V{\mathrm{RW}_\sigma}VV @VV{\mathrm{RW}_\sigma}V\\ \mathrm{H}^*(X,\Omega^*_X) @>>{\mathrm{L}_\sigma}> \mathrm{H}^*(X,\Omega^*_X)\end{CD} \tag{86}$$

and

$$\begin{CD}\hat{\mathcal{B}}[\bigcirc] @>{H}>> \hat{\mathcal{B}}[\bigcirc]\\ @V{\mathrm{RW}_\sigma}VV @VV{\mathrm{RW}_\sigma}V\\ \mathrm{H}^*(X,\Omega^*_X) @>>{\Pi}> \mathrm{H}^*(X,\Omega^*_X)\end{CD} \tag{87}$$

commute, the following proposition is proven:

PROPOSITION 22. *The map* $\mathrm{RW}_\sigma:\hat{\mathcal{B}}[\bigcirc]\to\mathrm{H}^*(X,\Omega_X)$ *is a morphism of* $\mathfrak{sl}_2$*-modules where the* $\mathfrak{sl}_2$*-module structure on* $\hat{\mathcal{B}}[\bigcirc]$ *is given by* $(\ell/2,-\partial,H)$*, and the* $\mathfrak{sl}_2$*-module structure on* $\mathrm{H}^*(X,\Omega_X)$ *is given by* $(\mathrm{L}_\sigma,\Lambda_{\sigma/4},H)$.

CHAPTER 6

The Euler characteristic of a line bundle in terms of the quadratic form

1. Some calculations in the graph homology space

For convenience of the reader, we recall the definition of the Chebyshev polynomials:

DEFINITION 21. For every $n \in \mathbb{N}_0$ the n^{th} Chebyshev polynomial is that polynomial with $\mathbb{Q}$-coefficients that fulfills

$$T_n(x) = \cos(n \arccos x) \tag{88}$$

for all $x \in \mathbb{R}$ for which $\arccos x$ is defined.

We shall need the Chebyshev polynomials due to the following fact:

LEMMA 3. *Let be $q, z \in \mathbb{C}$ such that*

$$q + q^{-1} = z. \tag{89}$$

Then

$$q^k + q^{-k} = 2T_k\left(\frac{z}{2}\right) \tag{90}$$

for all $k \in \mathbb{N}_0$.

PROOF. We can assume that $z \in [-2, 2]$. Then $q = e^{\pm ix}$ with $x = \arccos \frac{z}{2}$. Therefore,

$$q^k + q^{-k} = e^{ikx} + e^{-ikx} = 2\cos(kx) = 2\cos(k \arccos \frac{z}{2}) = 2T_k\left(\frac{z}{2}\right).$$

□

COROLLARY 2. *Let $\alpha, \beta, \lambda, \mu \in \mathbb{C}$ such that*

$$(\alpha\mu)^2 + (\beta/\mu)^2 = \alpha^2 + \beta^2 + \lambda. \tag{91}$$

It follows that

$$(\alpha\mu)^{2k} + (\beta/\mu)^{2k} = 2(\alpha\beta)^k T_k\left(\frac{\alpha^2 + \beta^2 + \lambda}{2\alpha\beta}\right) \tag{92}$$

REMARK 20. Note that the right hand side of (92) is a polynomial in $\alpha^2, \beta^2, \lambda$ of degree $2k$ since T_k is an odd (if k is odd) or an even (if k is even) polynomial of degree k.

PROOF. We can assume that $\alpha, \beta \neq 0$. Set $q := (\alpha/\beta)\mu^2$. It follows that $q + q^{-1} = \alpha/\beta + \beta/\alpha + \lambda/(\alpha\beta)$. Now use Lemma 3 and multiply both sides of (90) with $(\alpha\beta)^k$. □

PROPOSITION 23. *Let $\alpha, \beta, \lambda \in \mathbb{C}$. In $\hat{\mathcal{B}}$, we have*

$$\langle \Omega(\alpha)\Omega(\beta), \exp(\ell/2)\rangle \exp\left(\frac{\lambda}{48}\ominus\right) = \left\langle \exp\left(2\sum_{k=1}^{\infty} b_{2k}w_{2k}(\alpha\beta)^k T_k\left(\frac{\alpha^2+\beta^2+\lambda}{2\alpha\beta}\right)\right), \exp(\ell/2)\right\rangle. \quad (93)$$

PROOF. Without loss of generality, we can assume that $\alpha, \beta \neq 0$. Let us choose a $\mu \in \mathbb{C}$ with

$$(\alpha\mu)^2 + (\beta/\mu)^{-2} = \alpha^2 + \beta^2 + \lambda.$$

Note that $\langle \Omega(\alpha), \Omega(\beta)\rangle = \langle \Omega(\alpha\mu), \Omega(\beta/\mu)\rangle$. By Proposition 14, Theorem 3, and the preceeding Corollary 2:

$$\begin{aligned}
&\langle \Omega(\alpha)\Omega(\beta), \exp(\ell/2)\rangle \exp\left(\frac{\lambda}{48}\ominus\right)\\
&= \langle \exp(\partial)\Omega(\alpha), \exp(\partial)\Omega(\beta)\rangle \exp\left(\frac{\lambda}{48}\ominus\right) = \langle \Omega(\alpha), \Omega(\beta)\rangle \exp\left(\frac{\alpha^2+\beta^2+\lambda}{48}\ominus\right)\\
&= \langle \Omega(\alpha\mu), \Omega(\beta/\mu)\rangle \exp\left(\frac{\alpha^2+\beta^2+\lambda}{48}\ominus\right) = \langle \exp(\partial)\Omega(\alpha\mu), \exp(\partial)\Omega(\beta/\mu)\rangle\\
&= \langle \Omega(\alpha\mu)\Omega(\beta/\mu), \exp(\ell/2)\rangle = \left\langle \exp\left(\sum_{k=1}^{\infty} b_{2k}w_{2k}\left((\alpha\mu)^{2k} + (\beta/\mu)^{2k}\right)\right), \exp(\ell/2)\right\rangle\\
&= \left\langle \exp\left(2\sum_{k=1}^{\infty} b_{2k}w_{2k}(\alpha\beta)^k T_k\left(\frac{\alpha^2+\beta^2+\lambda}{2\alpha\beta}\right)\right), \exp(\ell/2)\right\rangle.
\end{aligned} \quad (94)$$

□

2. Applications to irreducible holomorphic symplectic manifolds

Let X be an irreducible holomorphic symplectic manifold with symplectic form σ, $\dim X = 2n$.

DEFINITION 22. For every $\alpha \in \mathrm{H}^2(X, \mathbb{C})$ let us define

$$\lambda(\alpha) := \begin{cases} \frac{24n\int_X \exp(\alpha)}{\int_X c_2(X)\exp(\alpha)} & \text{if well-defined,}\\ 0 & \text{otherwise.}\end{cases} \quad (95)$$

For L a line bundle on X, we set $\lambda(L) := \lambda(c_1(L))$.

REMARK 21. Expressing (82) in terms of λ gives

$$\frac{\lambda(\sigma+\bar\sigma)}{48}\,\mathrm{RW}_\sigma(\ominus) = [\bar\sigma]. \quad (96)$$

Here, we have used that $\int_X c_2(X)\exp(\sigma+\bar\sigma) > 0$ since this expression equals the L^2-norm of the Riemann curvature tensor of X (having been equipped with a hyperkähler metric compatible with the given symplectic structure) up to a positive constant (see [**15**]). But if the Riemann curvature tensor vanished, X would be a torus, which contradicts the assumption on irreducibility. Therefore $\lambda(\sigma+\bar\sigma) > 0$.

Applying Proposition 23 to Rozansky-Witten classes of X leads to the following:

THEOREM 4. *For all* $\alpha, \beta \in \mathbb{C}$ *and all* $\omega \in \mathrm{H}^2(X, \mathbb{C})$, *we have*

$$\int_X \mathrm{td}^{\frac{1}{2}}(\alpha)(X)\mathrm{td}^{\frac{1}{2}}(\beta)(X)\exp(\omega) = \int_X \exp\left(-2\sum_{k=1}^{\infty} b_{2k}s_{2k}(\alpha\beta)^k T_k\left(\frac{\alpha^2+\beta^2+\lambda(\omega)}{2\alpha\beta}\right)\right). \quad (97)$$

PROOF. We calculate step by step:

$$\int_X \mathrm{td}^{\frac{1}{2}}(\alpha)(X)\mathrm{td}^{\frac{1}{2}}(\beta)(X)\exp(\sigma+\bar\sigma)$$

by (80) and Proposition 29 (in Appendix A):

$$= \int_X \langle \mathrm{RW}_\sigma(\Omega(\alpha)\Omega(\beta)), \exp(\sigma)\rangle \exp(\sigma+\bar\sigma)$$

due to (76), (81), Remark 21, and Proposition 19:

$$= \int_X \mathrm{RW}_\sigma\left(\langle \Omega(\alpha)\Omega(\beta), \exp(\ell/2)\rangle \exp\left(\frac{\lambda(\sigma+\bar\sigma)}{48}\ominus\right)\right)\exp(\sigma)$$

by applying Proposition 23:

$$= \int_X \mathrm{RW}_\sigma\left(\left\langle \exp\left(2\sum_{k=1}^{\infty} b_{2k}w_{2k}(\alpha\beta)^k T_k\left(\frac{\alpha^2+\beta^2+\lambda(\sigma+\bar\sigma)}{2\alpha\beta}\right)\right), \exp(\ell/2)\right\rangle\right)\cdot$$
$$\cdot \exp(\sigma)$$

by (77), (81), and Proposition (19):

$$= \int_X \exp\left(-2\sum_{k=1}^{\infty} b_{2k}s_{2k}(\alpha\beta)^k T_k\left(\frac{\alpha^2+\beta^2+\lambda(\sigma+\bar\sigma)}{2\alpha\beta}\right)\right)\exp(\sigma)$$

by degree reasons:

$$= \int_X \exp\left(-2\sum_{k=1}^{\infty} b_{2k}s_{2k}(\alpha\beta)^k T_k\left(\frac{\alpha^2+\beta^2+\lambda(\sigma+\bar\sigma)}{2\alpha\beta}\right)\right).$$

Thus, we have proven the proposition for the case $\omega = \sigma+\bar\sigma$. Due to Proposition 4 and the deformation invariance of the Chern classes, this suffices to prove that the proposition holds for all $\omega \in \mathrm{H}^2(X,\mathbb{C})$ (by scaling $\sigma+\bar\sigma$ we can split the equation in its homogeneous parts). □

COROLLARY 3. *For all* $\omega \in \mathrm{H}^2(X,\mathbb{C})$ *we have*

$$\int_X \mathrm{td}^{\frac{1}{2}}(X)\exp(\omega) = \int_X \exp\left(-\sum_{k=1}^{\infty} b_{2k}s_{2k}(1+\lambda(\omega))^k\right) = (1+\lambda(\omega))^n \int_X \mathrm{td}^{\frac{1}{2}}(X) \quad (98)$$

PROOF. Set $\alpha = 1$, $\beta = 0$ in (97). Then use $T_k(x) = 2^{k-1}x^k + \ldots$ (up to terms of lower degree) for $k > 0$. This proves the first equality. The second follows from the fact that integrals over products of the s_{2k} vanish if the total degree doesn't sum up to the dimension of X. □

COROLLARY 4. *For all* $\omega \in \mathrm{H}^2(X,\mathbb{C})$ *we have*

$$\int_X \mathrm{td}^{\frac{1}{2}}(X)\omega^2 = 2n\lambda(\omega)\int_X \mathrm{td}^{\frac{1}{2}}(X), \quad (99)$$

and

$$\int_X \omega^{2n} = (2n)!\lambda(\omega)^n \int_X \mathrm{td}^{\frac{1}{2}}(X). \tag{100}$$

In particular, $\int_X \mathrm{td}^{\frac{1}{2}}(X) \neq 0$.

PROOF. Just substitute ω in (98) by $t\omega$, expand the left and the right side in t, and compare coefficients. □

REMARK 22. Firstly, (99) shows us that λ defines a quadratic form on $\mathrm{H}^2(X, \mathbb{C})$ which is induced by a rational quadratic form

$$\lambda : \mathrm{H}^2(X, \mathbb{Q}) \to \mathbb{Q}$$

since $\mathrm{td}^{\frac{1}{2}}(X)$ is a rational cohomology class.

Secondly, by (100) the n^{th}-power of λ is up to a positive multiple ($\int_X \mathrm{td}^{\frac{1}{2}}(X) > 0$, see [**15**]) the top intersection product of classes in $\mathrm{H}^2(X, \mathbb{C})$.

Since $\lambda(\sigma+\bar{\sigma}) > 0$ (Remark 21), we have proven Theorem 2 in 3 (see Remark 10 and Proposition 11).

We will call λ the *unnormalised Beauville-Bogomolov quadratic form of* X.

REMARK 23. Let us plug (99) back into (98). This gives us

$$\left(\frac{1}{(2k)!}\int_X \mathrm{td}^{\frac{1}{2}}(X)\alpha^{2k}\right)^n = \binom{n}{k}^n \left(\frac{1}{(2n)!}\int_X \alpha^{2n}\right)^k \cdot \left(\int_X \mathrm{td}^{\frac{1}{2}}(X)\right)^{n-k} \tag{101}$$

for all $k \in \{0, \dots, n\}$, a formula which has been used by Huybrechts in [**16**].

REMARK 24. By decomposing (97) in homogenous parts with respect to ω, we see that the constant $c_{\mathrm{td}^{\frac{1}{2}}(X)_{2k}\mathrm{td}^{\frac{1}{2}}(X)_{2l}}$ in (25) (here, we work with the unnormalised Beauville-Bogomolov form) with $\alpha = \mathrm{td}^{\frac{1}{2}}(X)_{2k}\mathrm{td}^{\frac{1}{2}}(X)_{2l}$ for $k, l \in \mathbb{N}_0$ is given by a universal polynomial in certain Chern numbers of X. Together with Proposition 13 this can be used to determine $\pi(\mathrm{td}^{\frac{1}{2}}(X)_{2k}\mathrm{td}^{\frac{1}{2}}(X)_{2l})$. For example, $\pi(c_2(X))$ can be calculated. If the Chern number $\int_X c_2(X)^n$ does not equal $\int_X \pi(c_2)^n = u^n \int \hat{\lambda}^n$, where $\pi(c_2(X)) = u\hat{\lambda}$, one knows that $c_2(X)$ is not entirely contained in SH^2.

COROLLARY 5. *For all* $\omega \in \mathrm{H}^2(X, \mathbb{C})$ *we have*

$$\int_X \mathrm{td}(X)\exp(\omega) = \int_X \exp\left(-2\sum_{k=1}^{\infty} b_{2k}s_{2k}T_k\left(1 + \frac{\lambda(\omega)}{2}\right)\right). \tag{102}$$

PROOF. Set $\alpha = \beta = 1$ in (97). □

3. A Hirzebruch-Riemann-Roch formula

The following theorem is actually an application of Corollary 5 for ω being the first Chern class of a line bundle.

THEOREM 5. *Let* X *be an irreducible holomorphic symplectic manifold. For every line bundle* L *on* X*, the Euler characteristic of* L *can be expressed as*

$$\chi(L) = \int_X \exp\left(-2\sum_{k=1}^{\infty} b_{2k}s_{2k}T_k\left(1 + \frac{\lambda(L)}{2}\right)\right), \tag{103}$$

which is a universal (i.e. depending only on X*) polynomial in the (unnormalised) Beauville-Bogomolov form of the first Chern class of* L*.*

PROOF. Using the usual Hirzebruch-Riemann-Roch formula for the Euler characteristic of line bundles on compact complex manifolds and the fact that $\mathrm{ch}(L) = \exp(c_1(L))$, we see that the theorem is an immediate consequence of Corollary 5. □

EXAMPLE 17. Let us write λ for $\lambda(L)$ in this example. Then the integrand of the right hand side of (103) is given by

$$
\begin{aligned}
1 &+ \frac{1}{12}\left(c_2 + \frac{1}{2}c_2 \cdot \lambda\right) \\
&+ \frac{1}{720}\left(3c_2^2 - c_4 + \left(\frac{7}{2}c_2^2 - 2c_4\right)\lambda + \left(\frac{7}{8}c_2^2 - \frac{1}{2}c_4\right)\lambda^2\right) \\
&+ \frac{1}{30240}\left(\left(5c_2^3 - \frac{9}{2}c_2c_4 + c_6\right) + \left(\frac{41}{4}c_2^3 - \frac{53}{4}c_2c_4 + \frac{9}{2}c_6\right)\lambda \right. \\
&\qquad \left. + \left(\frac{93}{16}c_2^3 - \frac{33}{4}c_2c_4 + 3c_6\right)\lambda^2 + \left(\frac{31}{32}c_2^3 - \frac{11}{8}c_2c_4 + \frac{1}{2}c_6\right)\lambda^3\right) \\
&+ \cdots,
\end{aligned}
\tag{104}
$$

where the c_i denote the Chern classes of X.

There are (at least) two possibilities to write (103) in another form. One is to define "deformed Todd genera" td_ε, $\varepsilon \in \mathbb{C}$:

DEFINITION 23. Let X be an irreducible holomorphic symplectic manifold. We set

$$
\mathrm{td}_\varepsilon(X) = \exp\left(-2\sum_{k=0}^{\infty} b_{2k}s_{2k}T_k(1+\varepsilon)\right), \tag{105}
$$

i.e. $\mathrm{td}_0(X) = \mathrm{td}(X)$.

Applying this definition to equation (103), it becomes

$$
\chi(L) = \int_X \mathrm{td}_\varepsilon(X) \tag{106}
$$

with $\varepsilon = \lambda(L)/2$.

The other possibility is to speak of a deformed tangent bundle: Let us recall some facts about the Grothendieck ring $\mathrm{K}^0(X)$ of complex vector bundles over X (see for Example [**1**]). After tensoring with $\mathbb{C}$, the Chern character gives us a ring homomorphism $\mathrm{ch} : \mathrm{K}^0(X,\mathbb{C}) \to \mathrm{H}^*(X,\mathbb{C})$ and taking the Todd class gives us a group homomorphism $\mathrm{td} : \mathrm{K}^0(X,\mathbb{C}) \to 1 + \mathrm{H}^{*>0}(X,\mathbb{C})$. Furthermore, there are the Adams operations $\psi^p : \mathrm{K}^0(X,\mathbb{C}) \to \mathrm{K}^0(X,\mathbb{C})$, $p \in \mathbb{N}$, which are ring homomorphisms with $\psi^p(L) = L^p$ for every line bundle L. The ψ^p commute, and we can write $\mathrm{K}^0(X,\mathbb{C}) = \bigoplus_{k=0}^{\infty} \mathrm{Gr}^{2k}\mathrm{K}^0(X,\mathbb{C})$ where $\mathrm{Gr}^{2k}\mathrm{K}^0(X,\mathbb{C})$ is the eigenspace of ψ^p (for $p \geq 2$) to the eigenvalue p^k. Using this grading on $\mathrm{K}^0(X,\mathbb{C})$, the Chern character becomes a homomorphism of graded rings.

If $M = \bigoplus_{i\in\mathbb{N}_0} M_i$ is a graded R-module, R a commutative ring, and $(\lambda_i)_{i\in\mathbb{N}_0}$ is a sequence in R, we define a morphism of graded modules $(\lambda_i)_{i\in\mathbb{N}_0}\cdot : M \to M, m \in M_i \mapsto \lambda_i \cdot m$. This definition will be applied to $\mathrm{K}^0(X,\mathbb{C})$:

DEFINITION 24. For every complex number $\varepsilon \in \mathbb{C}$ and complex manifold X, one defines a homomorphism

$$
\phi_\varepsilon : \mathrm{K}^0(X,\mathbb{C}) \to \mathrm{K}^0(X,\mathbb{C}), E \mapsto \left(T_{i/2}(1+\varepsilon)\right)_{i\in\mathbb{N}_0} \cdot E \tag{107}
$$

with $T_{(2j+1)/2}$ for $j \in \mathbb{N}_0$ defined arbitrarily.

Applying this definition to our irreducible holomorphic symplectic Kähler manifold X, the equation (103) becomes

$$
\chi(L) = \int_X \mathrm{td}(\phi_\varepsilon(\mathcal{T}_X)) \tag{108}
$$

with $\varepsilon = \lambda(L)/2$.

CHAPTER 7

The Chern numbers of generalised Kummer varieties

In this chapter we will develop a formula which links the genera of the Hilbert schemes of points on surfaces with the genera of the generalised Kummer varieties, leading to a method for calculation the Chern numbers of the generalised Kummer varieties.

1. More on the generalised Kummer varieties

In chapter 2, we introduced the generalised Kummer varieties. These will be the objects of interest in this chapter.

Let us recall the notation: For X a smooth projective surface over the field of complex numbers, we denote by $X^{[n]}$ the Hilbert scheme of zero-dimensional subschemes of X of length n. By a result of Fogarty ([**12**]), this scheme is smooth and projective of dimension $2n$. We continue to write $\rho : X^{[n]} \to X^{(n)}$ for the Hilbert-Chow-morphism, where $X^{(n)} := X^n/\mathfrak{S}_n$ is the n^{th}-symmetric product of X.

For A being an abelian surface, the generalised Kummer variety $A^{[[n]]}$ was defined to be the fibre of the summation morphism $\sigma : A^{[n]} \to A$ over 0, where σ factorises over $\rho : A^{[n]} \to A^{(n)}$.

Now we want to study the Hilbert schemes of points on surfaces and the generalised Kummer varieties a little bit more deeply. Let us start with the following observation:

Let A be an abelian surface again. Since A acts on itself by translation, there is also an induced operation of A on the Hilbert schemes $A^{[n]}$. Let us denote the restriction of this operation to the generalised Kummer variety $A^{[[n]]}$ by $\nu : A \times A^{[[n]]} \to A^{[n]}$. The following diagram is cartesian:

$$\begin{array}{ccc} A \times A^{[[n]]} & \xrightarrow{\nu} & A^{[n]} \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle \sigma} \\ A & \xrightarrow[n]{} & A. \end{array} \tag{109}$$

Here, $n : A \to A, a \mapsto na$ is the (multiplication by n)-morphism. It is a Galois covering of degree n^4. Therefore, also ν is a Galois covering of degree n^4.

Next, we want to introduce certain line bundles on the Hilbert schemes and generalised Kummer varieties that are constructed from line bundles on the underlying surface:

Each line bundle L on a smooth projective surface X gives us a line bundle L_n on $X^{[n]}$ in the following way: $L^{\boxtimes n}$ is a $\mathfrak{S}_n$-invariant line bundle on the n^{th}-product X^n of X. Therefore, we can define the sheaf $L^{(n)} := (\pi_*(L^{\boxtimes n}))^{\mathfrak{S}_n}$ of $\mathfrak{S}_{\mathfrak{n}}$-invariant sections of $\pi_*(L^{\boxtimes n})$ on $X^{(n)}$ where $\pi : X^n \to X^{(n)}$ is the canonical projection. The pull-back $L_n := \rho^* L^{(n)}$ by the Hilbert-Chow morphism is a line bundle on $X^{[n]}$. Note that $\mathrm{Pic}(X) \to \mathrm{Pic}(X^{[n]}), L \mapsto L_n$ is a homomorphism of groups.

This construction has already appeared for example in [**9**] and [**5**]. If X is an abelian surface, we denote by $L^{[[n]]}$ the restriction of L_n to the generalised Kummer variety $X^{[[n]]} \subseteq X^{[n]}$. By using the seesaw principle (cf. [**24**]) together with $\mathrm{H}^1(X^{[n]})$, it can be shown that

$$\nu^* L_n = L^n \boxtimes L^{[[n]]} \tag{110}$$

(cf. [**5**]).

Besides L_n one can define certain further sheaves on $X^{[n]}$ that are constructed from sheaves on X. They are called tautological sheaves of $X^{[n]}$ (cf. [**21**]). The construction is as follows: Since $X^{[n]}$ represents a functor, there is a universal family $\Xi_n \subseteq X^{[n]} \times X$ of zero-dimensional subschemes over $X^{[n]}$. Let us denote by $\mathcal{O}_n$ its structure sheaf. For any locally free sheaf F of rank r on X, we define the sheaf $F^{[n]} := p_*(\mathcal{O}_n \otimes q^* F)$ on $X^{[n]}$, where $p : X^{[n]} \times X \to X^{[n]}$ and $q : X^{[n]} \times X \to X$ are the canonical projections. $F^{[n]}$ is again locally free of rank nr.

The connection between L_n and $L^{[n]}$ is as follows: As noted in section 5 of [**9**], we have

$$\det(F^{[n]}) = \det(F)_n \otimes (\det \mathcal{O}_X^{[n]})^{\operatorname{rk} F} \tag{111}$$

for any locally free sheaf F of constant rank on X. Applying this to $F = L$ leads to

$$L_n = \det(L^{[n]}) \otimes \det(\mathcal{O}_X^{[n]})^{-1}. \tag{112}$$

2. Complex genera in general

Let $\Omega := \Omega^{\mathrm{U}} \otimes \mathbb{Q}$ denote the (rational) complex cobordism ring. By a result of Milnor ([**23**]), it is generated by the cobordism classes $[X]$ of all complex manifolds, and two complex manifolds X and Y lie in the same cobordism class if and only if they have the same Chern numbers, i.e. the Chern numbers determine the cobordism class and vice versa. Recall that the sum in the ring is induced by the disjoint union of manifolds, and the product by the cartesian product of manifolds.

A *complex genus* ϕ is a ring homomorphism $\phi : \Omega \to R$ into any $\mathbb{Q}$-algebra R. By Hirzebruch's theory of genera and multiplicative sequences ([**14**]), the R-valued complex genera are in one-to-one correspondence with the formal power series $f_\phi \in R[[x]]$ over R with constant coefficient 1. The correspondence is given as follows:

$$\phi(X) = \int_X \prod_{i=1}^n f_\phi(\gamma_i) \tag{113}$$

for all complex manifolds X, where n is the dimension of X, and $\gamma_1, \dots, \gamma_n$ are the Chern roots of its tangent bundle.

Since the cobordism class is known if one knows the value of all genera (it suffices to know the value of *the universal genus* $\mathrm{id}_\Omega : \Omega \to \Omega$), the knowledge of the values of all genera implies the knowledge of all Chern numbers.

Now, let us slightly generalise the notion of a genus.

DEFINITION 25. Let ϕ be a complex genus. For a complex manifold X together with a line bundle L on X we define

$$\phi(X, L) := \int_X e^{c_1(L)} \prod_{i=1}^n f_\phi(\gamma_i) \tag{114}$$

as *the genus* ϕ *of the pair* (X, L).

REMARK 25. Obviously, $\phi(X, \mathcal{O}_X) = \phi(X)$.

Example 18. If $\operatorname{td}(X)$ denotes the Todd genus of X, and $\chi(X, L)$ the holomorphic Euler characteristic of the line bundle L on X, we have by the Hirzebruch-Riemann-Roch Theorem that

$$\operatorname{td}(X, L) = \chi(X, L). \tag{115}$$

The genera of pairs (X, L) have the following properties, which follow directly from the appropriate properties of Chern classes/roots.

Proposition 24. *Let $\phi : \Omega \to R$ be any complex genus with values in R. We have*

(1) $\phi(X \times Y, L \boxtimes M) = \phi(X, L) \cdot \phi(Y, M)$ *for two complex manifolds X and Y together with a line bundle L resp. M.*
(2) $\phi(X, \nu^* L) = \deg(\nu) \cdot \phi(Y, L)$ *for any Galois covering $\nu : X \to Y$ and any line bundle L on Y.*

Given a genus, we can deform it in the following sense to get a new genus:

Definition 26. Let ϕ be a complex genus with values in the ring R. By ϕ_t we denote the genus with values in $R[t]$ given by

$$\phi_t(X) := \int_X \prod_{i=1}^n \left(f_\phi(\gamma_i) e^{t\gamma_i}\right) \tag{116}$$

for any complex manifold X.

Remark 26. For any integer n, we have $\phi_n(X) = \phi(X, K_X^{-n})$, where K_X is the canonical line bundle on X.

Example 19. Let χ_y be Hirzebruch's χ_y-genus, and χ_{yz} the twisted χ_y-genus (see [**30**]). In our notation, $\chi_{yz} = (\chi_y)_z$.

3. Complex genera of Hilbert schemes of points on surfaces

In this section we want to cite some of the results of [**9**] and give some corollaries which will be used later on.

Let X be a smooth projective surface. Following [**9**], we define

$$H_X := \sum_{n=0}^\infty [X^{[n]}] z^n \tag{117}$$

as an (invertible) element in the formal power series ring $\Omega[[z]]$. Analogously we define

$$K := \sum_{n=1}^\infty [A^{[[n]]}] z^n \tag{118}$$

in $\Omega[[z]]$ where A is any abelian surface. The cobordism class does not depend on the choice of A since the generalised Kummer varieties deform with A. We can reformulate our task to determine the Chern numbers of the generalised Kummer varieties by asking: What is the value $\phi(K) \in R[[z]]$ for any complex genus $\phi : \Omega \to R$?

The following lemma is a generalization of Theorem 4.2 in [**9**] for line bundles.

Lemma 4. *Let k be a nonnegative integer, $m_1, \dots, m_k \in \mathbb{Z}$, and $\phi : \Omega \to R$ be a genus. Then there exist uniquely determined universal power series $A_{i,j} \in R[[z]], 1 \leq i \leq j \leq k$, and $B_1, \dots, B_k \in R[[z]]$, and $C, D \in R[[z]]$ depending only on*

ϕ and $m_1, \dots, m_k$ such that for every smooth projective surface X, and line bundles $L_1, \dots, L_k$ on X, we have

$$\begin{aligned}&\sum_{n=0}^{\infty} \phi\left(X^{[n]}, \det(L_1^{[n]})^{m_1} \otimes \cdots \otimes \det(L_k^{[n]})^{m_k}\right) z^n \\ &= \exp\left(\sum_{1 \leq i \leq j \leq k} c_1(L_i) c_1(L_j) A_{ij} + \sum_{i=1}^{k} c_1(L_i) c_1(X) B_i + c_1(X)^2 C + c_2(X) D\right).\end{aligned} \tag{119}$$

(Remember that top intersection products on surfaces are to be understood as intersection numbers.)

Proof. First note that for $k = 1$ the statement of the lemma is just Theorem 4.2 of [**9**] for the case of line bundles with Ψ (in the notation of [**9**]) being the Chern character of the m_1^{th} power of the determinant.

Theorem 4.2 of Ellingsrud, Göttsche and Lehn, and the proof presented by them can be easily generalised for more than one bundle, i.e. for $k > 1$. Therefore, our lemma as a specialization of this generalization is proven. □

From the lemma we conclude the following:

Proposition 25. *Let $\phi : \Omega \to R$ be a genus. Then there exist uniquely determined universal power series $A_\phi, B_\phi, C_\phi, D_\phi \in R[[z]]$ depending only on ϕ such that for every smooth projective surface X together with a line bundle L on it, we have*

$$\phi(H_{X,L}) := \sum_{n=0}^{\infty} \phi\left(X^{[n]}, L_n\right) z^n \tag{120}$$

$$= \exp\left(c_1(L)^2 A_\phi + c_1(L) c_1(X) B_\phi + c_1(X)^2 C_\phi + c_2(X) D_\phi\right). \tag{121}$$

Proof. We use (112). By the previous lemma,

$$\begin{aligned}\phi(H_{X,L}) &= \sum_{n=0}^{\infty} \phi\left(X^{[n]}, \det(L^{[n]}) \otimes \det(\mathcal{O}_X^{[n]})^{-1}\right) z^n \\ &= \exp\begin{pmatrix} c_1(L)^2 A_{11} + c_1(L) c_1(\mathcal{O}_X) A_{12} + c_1(\mathcal{O}_X)^2 A_{22} \\ + c_1(L) c_1(X) B_1 + c_1(\mathcal{O}_X) c_1(X) B_2 + c_1(X)^2 C + c_2(X) D \end{pmatrix}\end{aligned}$$

for certain power series $A_{i,j}, B_i, C, D$ independent of X and L. Since $c_1(\mathcal{O}_X) = 0$, this proves the proposition with $A_\phi = A_{11}, B_\phi = B_1, C_\phi = C$ and $D_\phi = D$. □

It is possible to express the power series A_ϕ in terms of genera of Hilbert schemes of points on surfaces:

Proposition 26. *Let $\phi : \Omega \to R$ be any genus. For every smooth projective surface X,*

$$c_1(X)^2 A_\phi = \frac{1}{2} \ln \frac{\phi_1(H_X) \phi_{-1}(H_X)}{\phi(H_X)^2}. \tag{122}$$

Proof. In [**19**], it is proven that the canonical bundle of $X^{[n]}$ is K_n where K denotes the canonical bundle on X. It follows $K_n^{-m} = (K^{-m})_n$.

Using this we have by Proposition 25 that

$$\begin{aligned}\ln\phi_m(H_X) = \ln\sum_{n=0}^{\infty}\phi_m(X^{[n]})z^n = \ln\sum_{n=0}^{\infty}\phi(X^{[n]},K_n^{-m})z^n \\ = m^2A_\phi c_1(K)^2 - mB_\phi c_1(K)c_1(X) + C_\phi c_1(X)^2 + D_\phi c_2(X) \\ = (m^2A_\phi + mB_\phi + C_\phi)c_1(X)^2 + D_\phi c_2(X) \quad (123)\end{aligned}$$

for all integers m, which proves the proposition. ☐

4. Complex genera of the generalised Kummer varieties

In this section, we will relate the (generalised) complex genera of Beauville's generalised Kummer varieties to the complex genera of Hilbert schemes of points on a surface, which we studied in the previous section.

The first step in this direction is the following:

PROPOSITION 27. *Let $\phi : \Omega \to R$ be a complex genus with values in the $\mathbb{Q}$-algebra R. For every abelian surface A together with a line bundle L on it, we have*

$$c_1(L)^2\phi(A^{[[n]]},L^{[[n]]}) = 2n^2\phi(A^{[n]},L_n) \quad (124)$$

for all positive integers n.

PROOF. We will make use of (110). Recall that ν is a Galois covering of degree n^4. By Proposition 24 we have

$$\begin{aligned}\phi(A,L^n)\phi(A^{[[n]]},L^{[[n]]}) = \phi(A\times A^{[[n]]},L^n\boxtimes L^{[[n]]}) \\ = \phi(A\times A^{[[n]]},\nu^*L_n) = n^4\phi(A^{[n]},L_n), \quad (125)\end{aligned}$$

which proves the proposition, once we have shown that $\phi(A,L^n) = \frac{n^2}{2}c_1(L)^2$. This follows from the fact that the Chern classes of an abelian surface are trivial:

$$\phi(A,L^n) = \int_A f_\phi(\gamma_1)f_\phi(\gamma_2)e^{c_1(L^n)} = \int_A \frac{c_1(L^n)^2}{2} = \frac{n^2}{2}c_1(L)^2, \quad (126)$$

where we have used that f_ϕ is a power series with constant coefficient 1. ☐

In [**5**], M. Britze and the author expressed the (holomorphic) Euler characteristic of the line bundle $L^{[[n]]}$ in terms of the Euler characteristic of L in order to deduce a formula for the Euler characteristic of an arbitrary line bundle M on $A^{[[n]]}$ as a polynomial in the Beauville-Bogomolov quadratic form of $c_1(M)$. By using the analogous expression of the Euler characteristic of the line bundle L_n on $A^{[n]}$ (see [**9**]), we get the mentioned result of [**5**] as a corollary of the previous proposition:

COROLLARY 6 ([**5**]). *The holomorphic Euler characteristic of the line bundle $L^{[[n]]}$ on $A^{[[n]]}$ is given by*

$$\chi(A^{[[n]]},L^{[[n]]}) = n\binom{\chi(A,L)+n-1}{n-1}. \quad (127)$$

PROOF. By Lemma 5.1 of [**9**] we have

$$\chi(A^{[n]},L_n) = \binom{\chi(A,L)+n-1}{n}. \quad (128)$$

Using this, the corollary follows from the proposition applied to the case for ϕ being the Todd genus (remember Example 18). Also note that $\chi(A,L) = \frac{1}{2}c_1(L)^2$ by the Hirzebruch-Riemann-Roch formula. ☐

From this and our results of the previous chapter, we can conclude an explicit formula for the Euler characteristic of a line bundle L on $A^{[[n]]}$ as a polynomial in $\lambda(L)$:

Proposition 28. *For $n \in \mathbb{N}$ and L a line bundle on $A^{[[n]]}$, we have*

$$\chi(A^{[[n]]}, L) = n\binom{\frac{n}{4}\lambda(L) + n - 1}{n-1}, \tag{129}$$

where λ is the unnormalised Beauville-Bogomolov form of $A^{[[n]]}$ (see remark 22).

Proof. Since we know that the Euler characteristic of L can be expressed by a polynomial formula, all we have to show is that $\frac{n}{4}\lambda(M^{[[n]]}) = \chi(A, M) = c_1(M)^2/2$ for all line bundles M on A.

By [**3**], one knows that $\lambda(M^{[[n]]})$ coincides up to a positive factor with $c_1(M)^2$. We just have to calculate the factor. Note that for $m \in \mathbb{N}_0$, we have

$$c_1((M^m)^{[[n]]}) = c_1((M^{[[n]]})^m) = mc_1(M^{[[n]]}).$$

By the Hirzebruch-Riemann-Roch Theorem in its classical form:

$$\begin{aligned}\chi(A^{[[n]]}, (M^{[[n]]})^m) = m^{2n-2} &\int_{A^{[[n]]}} \mathrm{ch}(M^{[[n]]}) \\ &+ m^{2n-4}\int_{A^{[[n]]}} \frac{c_2(A^{[[n]]})}{12}\mathrm{ch}(M^{[[n]]}) + \dots.\end{aligned}$$

Compare this with (128) for $L = M^m$. This formula gives:

$$\begin{aligned}\chi(A^{[[n]]}, (M^{[[n]]})^m) = m^{2n-2} &\frac{n}{2^{n-1}(n-1)!}(c_1(M)^2)^{n-1} \\ &+ m^{2n-4}\frac{n^2(n-1)}{2^{n-1}(n-1)!}(c_1(M)^2)^{n-2} + \dots.\end{aligned}$$

Comparing coefficients of m^{2n-2} and m^{2n-4}, we find

$$\lambda(M^{[[n]]}) = \frac{24(n-1)\int_{A^{[[n]]}} \mathrm{ch}(M^{[[n]]})}{\int_{A^{[[n]]}} c_2(A^{[[n]]})\mathrm{ch}(M^{[[n]]})} = \frac{2}{n}c_1(M)^2.$$

This proof has already appeared in [**5**], and is due to M. Britze and the author. $\square$

If we are interested in the usual genera of the generalised Kummer varieties, i.e. the genera of the pairs $(A^{[[n]]}, \mathcal{O}_{A^{[[n]]}})$, we cannot use Proposition 27 directly since for $L = \mathcal{O}_A$ it just states $0 = 0$.

However, it is still possible to make use of the proposition. We have to look at all generalised Kummer varieties at the same time. Doing so, we get the following main result of this chapter:

Theorem 6. *Let $\phi : \Omega \to R$ be a complex genus with values in the $\mathbb{Q}$-algebra R. For every smooth projective surface X with $\int_X c_1(X)^2 \neq 0$,*

$$\phi(K) = \frac{1}{c_1(X)^2}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^2 \ln\frac{\phi_1(H_X)\phi_{-1}(H_X)}{\phi(H_X)^2}. \tag{130}$$

PROOF. Let L be any line bundle on A. We have

$$\begin{aligned}c_1(L)^2\sum_{n=1}^{\infty}\phi(A^{[[n]]},L^{[[n]]})z^n &= 2\sum_{n=1}^{\infty}n^2\phi(A^{[n]},L_n)z^n = 2\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^2\phi(H_{A,L})\\ &= 2\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^2\exp\left(c_1(L)^2A_\phi + c_1(L)c_1(A)B_\phi + c_1(A)^2C_\phi + c_2(A)D_\phi\right)\\ &= 2\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^2\exp\left(c_1(L)^2A_\phi\right) = 2\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^2 c_1(L)^2A_\phi + \mathrm{O}\left((c_1(L)^2)^2\right), \end{aligned}\tag{131}$$

which together with Proposition 26 proves the theorem, since there are line bundles on A with $c_1(L) \neq 0$. □

Let $\mathrm{id}_\Omega : \Omega \to \Omega$ be the universal genus. Like any other genus, we can deform it to a genus $(\mathrm{id}_\Omega)_t : \Omega \to \Omega[t]$. If $X \in \Omega$ is a rational complex cobordism class, we write

$$X_t := (\mathrm{id}_\Omega)_t(X) \in \Omega[t]. \tag{132}$$

With this notation, we can express the cobordism class K by

$$K = \frac{1}{c_1(X)^2}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^2 \ln\frac{(H_X)_1(H_X)_{-1}}{H_X^2} \tag{133}$$

for all surface X with $c_1(X)^2 \neq 0$.

REMARK 27. Of course, everything in this chapter still holds true if we replace the abelian surfaces A from which we constructed the generalised Kummer varieties, by an arbitrary complex torus of dimension two.

We used Theorem 6 to produce a table of all Chern numbers of the generalised Kummer varieties up to dimension twenty. It can be found in Appendix B.

APPENDIX A

Some linear algebra

In this appendix, let k denote a field of characteristic zero, V a k-vector space and A a k-algebra.

We identify the exterior algebra $\bigwedge V^* \otimes_k A$ with the space $\bigoplus_{r=0}^n \mathrm{Alt}^r(V, A)$ of alternating multilinear forms on V with values in A as vector spaces by setting

$$((\alpha_1 \wedge \cdots \wedge \alpha_r) \otimes a)(v_1, \dots, v_r) = \det((\alpha_i(v_j))_{ij}) \cdot a \tag{134}$$

for $\alpha_1, \dots, \alpha_r \in V^*$, $v_1, \dots, v_r \in V$ and $a \in A$.

Definition 27. We call an element $\sigma \in \bigwedge^2 V^*$ a *symplectic form on* V and V together with σ a *symplectic k-vector space* if the map

$$I_\sigma : V \to V^*, v \mapsto \sigma(\cdot, v) \tag{135}$$

is an isomorphism of vector spaces.

Remark 28. A symplectic vector space is always of even dimension.

If σ is a fixed symplectic form on V, we will identify V and V^* using the isomorphism I_σ. In particular, we have an induced dual symplectic form $\sigma^* \in \bigwedge^2 V$ on V^*.

For the rest of this section, let σ be a fixed symplectic form on a $2n$-dimensional k-vector space V.

Definition 28. We define a pairing

$$\langle \cdot, \cdot \rangle : \left(\bigwedge V^* \otimes_k A\right) \otimes_k \left(\bigwedge V^* \otimes_k A\right) \to A \tag{136}$$

by setting

$$\langle (\alpha_1 \wedge \cdots \wedge \alpha_r) \otimes a, (\beta_1 \wedge \cdots \wedge \beta_s) \otimes b \rangle = \delta_{rs} \det((\sigma^*(\alpha_i, \beta_j))_{ij}) \cdot ab \tag{137}$$

for $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s \in V^*$ and $a, b \in A$, where δ_{rs} is Kronecker's δ.

This defintion and the following proposition is used in the case that V is the holomorphic tangent space of a complex manifold at a point p, and A is the algebra of anti-holomorphic forms at p.

Proposition 29. *Let σ be a symplectic form on V and $\int : \bigwedge(V^*) \otimes_k A \to \bigwedge^{2n}(V^*) \otimes_k A$ the canonical projection onto the forms of top degree.*

For every $\alpha \in \bigwedge(V^*)$ we have

$$\int (\alpha \wedge \exp \sigma) = \int (\langle \alpha, \exp \sigma \rangle \exp \sigma). \tag{138}$$

Proof. We can assume that $A = k$ and $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_{2p} \in \bigwedge^{2p}(V^*)$ with $p \in \mathbb{N}_0$, $\alpha_i \in V^*$. So we have to prove

$$\alpha \wedge \sigma^{(n-p)} = \frac{(n-p)!}{p! \cdot n!} \langle \alpha, \sigma^p \rangle \sigma^n.$$

Let $e_1, \dots, e_{2n}$ be a symplectic basis of V and $\vartheta^1, \dots, \vartheta^{2n}$ the corresponding dual basis of V^*, i.e. $\sigma = \sum_{i=1}^n \vartheta^{2i-1} \wedge \vartheta^{2i}$ and $\sigma^* = \sum_{i=1}^n e_{2i-1} \wedge e_{2i}$. It follows that $\sigma^n = n! \cdot \vartheta^1 \wedge \dots \wedge \vartheta^{2n}$. It is

$$\begin{aligned}
&\left\langle \alpha \wedge \sigma^{(n-p)}, \sigma^n \right\rangle \\
&= n!(n-p)! \sum_{1 \le i_1 < \dots < i_{n-p} \le n} \begin{aligned}\big\langle \alpha_1 \wedge \dots \wedge \alpha_{2p} \wedge \vartheta^{2i_1-1} \wedge \vartheta^{2i_i} \wedge \dots \\ \dots \wedge \vartheta^{2i_{n-p}-1} \wedge \vartheta^{2i_{n-p}}, \vartheta^1 \wedge \dots \wedge \vartheta^{2n} \big\rangle\end{aligned} \\
&= n!(n-p)! \sum_{1 \le j_1 < \dots < j_p \le n} \left\langle \alpha_1 \wedge \dots \wedge \alpha_{2p}, \vartheta^{2j_1-1} \wedge \vartheta^{2j_1} \wedge \dots \wedge \vartheta^{2j_p-1} \wedge \vartheta^{2j_p} \right\rangle \\
&= \frac{n!(n-p)!}{p!} \left\langle \alpha, \sigma^p \right\rangle.
\end{aligned}$$

Since σ^n spans $\bigwedge^{2n}(V^*)$ and $\langle \sigma^n, \sigma^n \rangle = n!^2$, this proves the proposition. □

Remark 29. Hitchin and Sawon have stated this result for α being a two-form in [**15**]. Note that they identify the exterior algebra over V^* with the alternating forms on V in a different way than we do.

Extending their formula to allow an arbitrary degree of α is crucial for this work.

For every $v \in V$, we also denote by $v : \bigwedge V^* \otimes_k A \to \bigwedge V^* \otimes_k A$ *the contraction by* v, i.e.

$$v((\alpha_1 \wedge \dots \wedge \alpha_r) \otimes a) = \left(\sum_{i=1}^r (-1)^{i-1} \alpha_i(v) \cdot \alpha_1 \wedge \dots \wedge \widehat{\alpha}_i \wedge \dots \wedge \alpha_r \right) \otimes a \tag{139}$$

for $\alpha_1, \dots, \alpha_r \in V^*$, and $a \in A$. Note that v is an operator of degree -1. On the other hand, every $\alpha \in V^*$ defines an operator of degree 1 on $\bigwedge V^* \otimes_k A$ just by exterior multiplication. Let us denote this operator also by α.

There are the following (super-)commutator relations between these operators:

Proposition 30. *Let $\alpha, \beta \in V^*$ and $v, w \in V$. We have*

$$[\alpha, \beta] = 0, \qquad [v, w] = 0, \qquad \textit{and} \qquad [\alpha, v] = \alpha(v). \tag{140}$$

Note that the commutators are to be understood in the graded sense, i.e. we are actually talking about anti-commutators here.

Proof. To prove a commutator relation, just apply both sides to an element of $\bigwedge V^* \otimes_k A$, and show that you get both times the same. This is straightforward. □

Definition 29. Let $\delta : \bigwedge V^* \otimes_k A \to \bigwedge V^* \otimes_k A$ be *the contraction by* σ, i.e.

$$\delta((\alpha_1 \wedge \dots \wedge \alpha_r) \otimes a) \tag{141}$$

$$= \left(\sum_{i=1}^n \sum_{1 \le l < m \le r} (-1)^{l+m-1} \begin{aligned}(\alpha_l(e_{2i-1})\alpha_m(e_{2i}) - \alpha_l(e_{2i})\alpha_m(e_{2i-1})) \\ \cdot \alpha_1 \wedge \dots \wedge \widehat{\alpha}_l \wedge \dots \wedge \widehat{\alpha}_m \wedge \dots \wedge \alpha_r\end{aligned} \right) \otimes a \tag{142}$$

for $\alpha_1, \dots, \alpha_r \in V^*$, and $a \in A$. Here, $e_1, \dots, e_{2n}$ is again a symplectic basis for V.

Remark 30. We have $\delta = \sum_{i=1}^n e_{2i-1} e_{2i}$.

Left multiplication with σ defines an operator of degree two on $\bigwedge V^* \otimes_k A$, which we also want to denote by σ.

Proposition 31. *The commutator of σ and δ is given by*

$$[\sigma, \delta] = -\Pi, \tag{143}$$

with the operator Π acting on $\bigwedge^p V \otimes_k A$ by multiplying with $p - n$.

Proof. We have

$$\begin{aligned}[\sigma, \delta] &= \sum_{i,j=1}^{n} [\vartheta^{2i-1}\vartheta^{2i}, e_{2j-1}e_{2j}] \\ &= \sum_{i,j=1}^{n} \begin{pmatrix} -\vartheta^{2i-1}e_{2j-1}[\vartheta^{2i}, e_{2j}] + \vartheta^{2i-1}[\vartheta^{2i}, e_{2j-1}]e_{2j} \\ -e_{2j-1}[\vartheta^{2i-1}, e_{2j}]\vartheta^{2i} + [\vartheta^{2i-1}, e_{2j-1}]e_{2j}\vartheta^{2i} \end{pmatrix} = \sum_{i=1}^{2n} (1 - \vartheta^i e_i) = -\Pi,\end{aligned}$$

where we have used Proposition 30, and that $\sum_{i=1}^{2n} \vartheta^i e_i$ acts on $\bigwedge^p V^* \otimes_k A$ by multiplying with p. □

Remark 31. Since $[\Pi, \sigma] = 2\sigma$ and $[\Pi, \delta] = -2\delta$, we see that $(\sigma, -\delta, \Pi)$ is an $\mathfrak{sl}_2$-triple on $\bigwedge V^* \otimes_k A$.

We have applied this result to the situation where V is the holomorphic tangent space at a point of a complex manifold. This leads to an $\mathfrak{sl}_2$-action on the level of cohomology of an irreducible holomorphic symplectic manifold.

The following definition has nothing to do with the preceeding.

Definition 30. For every set S, we denote by $\bigwedge_k S$ the Graßmann algebra generated by the elements of S over k. If S' is a subset of S, we view $\bigwedge_k (S')$ canonically as a subalgebra of $\bigwedge_k S$. We denote by $\mathrm{S}^n_k S$ the n-th symmetric product of the k-vector space spanned by the elements of S.

APPENDIX B

The Chern numbers of the generalised Kummer varieties of dimension up to twenty

We have used Theorem 6 to compute all Chern numbers of the generalised Kummer varieties of dimension up to twenty from the Chern numbers of the Hilbert schemes of points on the projective plane. Our results are as follows:

Chern number	Evaluated on $A^{[[*]]}$
c_2	24
c_2^2	756
c_4	108
c_2^3	30208
c_2c_4	6784
c_6	448
c_2^4	1470000
$c_2^2c_4$	405000
c_4^2	111750
c_2c_6	37500
c_8	750
c_2^5	84478464
$c_2^3c_4$	26220672
$c_2c_4^2$	8141472
$c_2^2c_6$	3141504
c_4c_6	979776
c_2c_8	142560
c_{10}	2592
c_2^6	5603050432
$c_2^4c_4$	1881462016
$c_2^2c_4^2$	631808744
c_4^3	212190776
$c_2^3c_6$	268796752
$c_2c_4c_6$	90412056
c_6^2	12976376
$c_2^2c_8$	17075912
c_4c_8	5762400
c_2c_{10}	441784
c_{12}	2744
c_2^7	421414305792
$c_2^5c_4$	149664301056
$c_2^3c_4^2$	53149827072
$c_2c_4^3$	18874417152
$c_2^4c_6$	24230756352
$c_2^2c_4c_6$	8610545664
$c_4^2c_6$	3059945472

Chern number	Evaluated on $A^{[[*]]}$
$c_2c_6^2$	1397121024
$c_2^3c_8$	1914077184
$c_2c_4c_8$	681332736
c_6c_8	110853120
$c_2^2c_{10}$	71909376
c_4c_{10}	25700352
c_2c_{12}	1198080
c_{14}	7680
c_2^8	35447947999488
$c_2^6c_4$	13129602781824
$c_2^4c_4^2$	4862661530400
$c_2^2c_4^3$	1800797040144
c_4^4	666853820172
$c_2^5c_6$	2332758616128
$c_2^3c_4c_6$	864167470848
$c_2c_4^2c_6$	320117226120
$c_2^2c_6^2$	153694101888
$c_4c_6^2$	56953381608
$c_2^4c_8$	215605377504
$c_2^2c_4c_8$	79938804096
$c_4^2c_8$	29638792620
$c_2c_6c_8$	14239224576
c_8^2	1322820801
$c_2^3c_{10}$	10441752768
$c_2c_4c_{10}$	3878495784
c_6c_{10}	692780364
$c_2^2c_{12}$	254566800
c_4c_{12}	94850190
c_2c_{14}	2685636
c_{16}	9477
c_2^9	3297871360000000
$c_2^7c_4$	1262135680000000
$c_2^5c_4^2$	482990816000000
$c_2^3c_4^3$	184814229440000
$c_2c_4^4$	70712975120000
$c_2^6c_6$	240910720000000
$c_2^4c_4c_6$	92197363200000
$c_2^2c_4^2c_6$	35281909440000
$c_4^3c_6$	13500841600000
$c_2^3c_6^2$	17605804800000
$c_2c_4c_6^2$	6738177040000
c_6^3	1287476640000
$c_2^5c_8$	25082624000000
$c_2^3c_4c_8$	9603236160000
$c_2c_4^2c_8$	3676588120000
$c_2^2c_6c_8$	1835380960000
$c_4c_6c_8$	702799360000
$c_2c_8^2$	191623650000
$c_2^4c_{10}$	1459909120000
$c_2^2c_4c_{10}$	559476160000
$c_4^2c_{10}$	214406248000

Chern number	Evaluated on $A^{[[*]]}$
$c_2c_6c_{10}$	107096280000
c_8c_{10}	11208918000
$c_2^3c_{12}$	46722720000
$c_2c_4c_{12}$	17937420000
c_6c_{12}	3443000000
$c_2^2c_{14}$	774480000
c_4c_{14}	298344000
c_2c_{16}	6090000
c_{18}	18000
c_2^{10}	336252992654447616
$c_2^8c_4$	132107428736160768
$c_2^6c_4^2$	51898082311033728
$c_2^4c_4^3$	20386379301294336
$c_2^2c_4^4$	8007472661159664
c_4^5	3144990890482320
$c_2^7c_6$	26693534659013376
$c_2^5c_4c_6$	10486371945354624
$c_2^3c_4^2c_6$	4119203015724192
$c_2c_4^3c_6$	1617975749261520
$c_2^4c_6^2$	2119158341714304
$c_2^2c_4c_6^2$	832451953404192
$c_4^2c_6^2$	326987093337168
$c_2c_6^3$	168265889899008
$c_2^6c_8$	3051655882366080
$c_2^4c_4c_8$	1199055419079936
$c_2^2c_4^2c_8$	471105410929296
$c_4^3c_8$	185086417093248
$c_2^3c_6c_8$	242424490790592
$c_2c_4c_6c_8$	95252580881040
$c_6^2c_8$	19264369884144
$c_2^2c_8^2$	27756335356332
$c_4c_8^2$	10909113168228
$c_2^5c_{10}$	204371090647680
$c_2^3c_4c_{10}$	80342429404512
$c_2c_4^2c_{10}$	31583103012912
$c_2^2c_6c_{10}$	16258455456144
$c_4c_6c_{10}$	6391906873440
$c_2c_8c_{10}$	1864193494284
c_{10}^2	125480168748
$c_2^4c_{12}$	8013253087488
$c_2^2c_4c_{12}$	3153305609256
$c_4^2c_{12}$	1240853563488
$c_2c_6c_{12}$	639144656040
c_8c_{12}	73457352276
$c_2^3c_{14}$	178626056400
$c_2c_4c_{14}$	70412082840
c_6c_{14}	14310113400
$c_2^2c_{16}$	2116210140
c_4c_{16}	836469612
c_2c_{18}	11419980
c_{20}	15972

It is a remarkable fact that all Chern numbers of the varieties $A^{[[n]]}$ with $n \leq 11$ are positive and divisible by n^3. As the known Chern numbers of Hilbert schemes of points on K3 surfaces are also positive, one can wonder if, given an irreducible compact hyperkähler manifold X, all Chern numbers of X are positive.

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Teilpublikationen

(1) (mit Michael Britze.) Hirzebruch-Riemann-Roch formulae on irreducible symplectic Kähler manifolds. *arXiv:math.AG/0101062.*

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(3) On the Chern numbers of the generalised Kummer varieties. *arXiv:math.AG/0204197.*