Twisted Cohomology of the Hilbert Schemes of Points on Surfaces

Marc A. Nieper-Wisskirchen[1]

Received: February 1, 2008
Revised: April 23, 2009

Communicated by Thomas Peternell

Abstract. We calculate the cohomology spaces of the Hilbert schemes of points on surfaces with values in local systems. For that purpose, we generalise I. Grojnoswki's and H. Nakajima's description of the ordinary cohomology in terms of a Fock space representation to the twisted case. We make further non-trivial generalisations of M. Lehn's work on the action of the Virasoro algebra to the twisted and the non-projective case.

Building on work by M. Lehn and Ch. Sorger, we then give an explicit description of the cup-product in the twisted case whenever the surface has a numerically trivial canonical divisor.

We formulate our results in a way that they apply to the projective and non-projective case in equal measure.

As an application of our methods, we give explicit models for the cohomology rings of the generalised Kummer varieties and of a series of certain even dimensional Calabi–Yau manifolds.

2000 Mathematics Subject Classification: 14C05; 14C15
Keywords and Phrases: Hilbert schemes of points on surfaces, rational cohomology ring, locally constant systems, generalised Kummer varieties

[1]The author would like to thank the Max Planck Institute for Mathematics in Bonn for its hospitality and support during the preparation of this paper. He would also like to thank the referee for some very helpful suggestions.

Contents

1.	Introduction and results	750
2.	The Fock space description	754
3.	The Virasoro algebra in the twisted case	758
4.	The boundary operator	760
5.	The ring structure	762
6.	The generalised Kummer varieties	767
References		769

1. Introduction and results

Let X be a quasi-projective smooth surface over the complex numbers. We denote by $X^{[n]}$ the Hilbert scheme of n points on X, parametrising zero-dimensional subschemes of X of length n. It is a quasi-projective variety ([Gro61]) and smooth of dimension $2n$ ([Fog68]). Recall that the Hilbert scheme $X^{[n]}$ can be viewed as a resolution of the n-th symmetric power $X^{(n)} := X^n/\mathfrak{S}_n$ of the surface X by virtue of the Hilbert–Chow morphism $\rho\colon X^{[n]} \to X^{(n)}$, which maps each zero-dimensional subscheme ξ of X to its support $\operatorname{supp}\xi$ counted with multiplicities.

Let L be a local system (always over the complex numbers and of rank 1) over X. We can view it as a functor from the fundamental groupoid Π of X to the category of one-dimensional complex vector spaces.

The fundamental groupoid $\Pi^{(n)}$ of $X^{(n)}$ is the quotient groupoid of Π^n by the natural $\mathfrak{S}_n$-action by [Bro88]. (Recall from [Bro88] that the quotient groupoid of a groupoid P on which a group G is acting (by functors) is a groupoid P/G together with a functor $p\colon P \to P/G$ that is invariant under the G-action and so that $p\colon P \to P/G$ is universal with respect to this property.)

Readers who prefer to think in terms of the fundamental group (as opposed to the fundamental groupoid) can find a description of the fundamental group of $L^{(n)}$ in [Bea83].

By the universal property of $\Pi^{(n)}$, we can thus construct from L a local system $L^{(n)}$ on $X^{(n)}$ by setting

$$L^{(n)}(x_1, \ldots, x_n) := \bigotimes_i L(x_i),$$

for each $(x_1, \ldots, x_n) \in X^{(n)}$ (for the notion of the tensor product over an unordered index set see, e.g., [LS03]). This induces the locally free system $L^{[n]} := \rho^* L^{(n)}$ on $X^{[n]}$.

We are interested in the calculation of the direct sum of cohomology spaces $\bigoplus_{n\geq 0} H^*(X^{[n]}, L^{[n]}[2n])$. Besided the natural grading given by the cohomological degree it carries weighting (see remark 1.1 below) given by the number of points n. Likewise, the symmetric algebra $S^*(\bigoplus_{\nu\geq 1} H^*(X, L^\nu[2]))$ carries a grading by cohomological degree and a weighting, which is defined so that $H^*(X, L[2]^\nu)$ is of pure weight ν.

Remark 1.1. Here, a *weighting* is just another name for a second grading. A *weight space* is a homogeneous subspace to a given degree with respect to this second grading. Being of pure weight means being homogeneous with respect to the second grading.
In the context of super vector spaces, however, me make a difference between a grading and a weighting: Write $V = V^0 \oplus V^1$ for the decomposition of a super vector space into its even and odd part. Recall that for a grading $V = \bigoplus_{n\in\mathbf{Z}} V_n$ on V we have $V^i = \bigoplus_n V_n^{i+n \pmod 2}$.
For a weighting, on the contrary, we want to adopt the following convention: If $V = \bigoplus_{n\in\mathbf{Z}} V(n)$ is the decomposition of a weighted super vector space into its weight spaces, one has $V^i = \bigoplus_n V(n)^i$, i.e. the weighting does not interfere with the $\mathbf{Z}/(2)$-grading.
This difference is important, for example, for the notion of (super-)commutativity.

The first result of this paper is the following:

Theorem 1.2. *There is a natural vector space isomorphism*

$$\bigoplus_{n\geq 0} H^*(X^{[n]}, L^{[n]}[2n]) \to S^*\left(\bigoplus_{\nu\geq 1} H^*(X, L^\nu[2])\right)$$

that respects the grading and weighting.

For $L = \mathbf{C}$, the trivial system, this result has already appeared in [Gro96] and [Nak97]).
Theorem 1.2 is proven by defining a Heisenberg Lie algebra $\mathfrak{h}_{X,L}$, whose underlying vector space is given by

$$\bigoplus_{n\geq 0} H^*(X, L^n[2]) \oplus \bigoplus_{n\geq 0} H_c^*(X, L^{-n}[2]) \oplus \mathbf{C}\boldsymbol{c} \oplus \mathbf{C}\boldsymbol{d}$$

and by showing that $\bigoplus_{n\geq 0} H^*(X^{[n]}, L^{[n]}[2n])$ is an irreducible lowest weight representation of this Lie algebra, as is done in [Nak97] for the untwisted case.
Let $p\colon \hat{X} \to X$ be a finite abelian Galois covering over the surface X with Galois group G. The direct image $M := p_*\mathbf{C}$ of the trivial local system on $\hat{X}$ is a local system on X of rank $|G|$, the order of G. Note that G acts naturally on M. As G is abelian, there is a decomposition $M \cong \bigoplus_{\chi\in G^\vee} L_\chi$, where $G^\vee = \mathrm{Hom}(G, \mathbf{C}^\times)$ is the character group of G and L_χ is the subsystem of M on which G acts via χ. In fact, each L_χ is a local system of rank one.
Consider $M^{[n]} := \bigoplus_{\chi\in G^\vee} L_\chi^{[n]}$. This is a local system of rank $|G|$ on $X^{[n]}$. Let $q\colon \widehat{X^{[n]}} \to X^{[n]}$ be a finite abelian Galois covering of $X^{[n]}$ such that $q_*\mathbf{C} = M^{[n]}$. Using the Leray spectral sequence for q, which already degenerates at the E_2-term, the cohomology of $\widehat{X^{[n]}}$ can be computed by Theorem 1.2:

COROLLARY 1.3. *There is a natural vector space isomorphism*

$$\bigoplus_{n\geq 0} H^*(\widehat{X^{[n]}}, \mathbf{C}[2n]) \to \bigoplus_{\chi\in G^\vee} S^*\left(\bigoplus_{\nu\geq 1} H^*(X, L_\chi^\nu[2])\right)$$

that respects the grading and weighting.

We then proceed in the paper by defining a twisted version $\mathfrak{v}_{X,L}$ of the Virasoro Lie algebra, whose underlying vector space will be given by

$$\bigoplus_{n\geq 0} H^*(X, L^n) \oplus \bigoplus_{n\geq 0} H_c^*(X, L^{-n}) \oplus \mathbf{C}\boldsymbol{c} \oplus \mathbf{C}\boldsymbol{d}.$$

(Note the different grading compared to $\mathfrak{h}_{X,L}$.) We define an action of $\mathfrak{v}_{X,L}$ on $\bigoplus_{n\geq 0} H^*(X^{[n]}, L^{[n]}[2n])$ by generalising results of [Leh99] to the twisted, not necessarily projective case. As in [Leh99], we calculate the commutators of the operators in $\mathfrak{h}_{X,L}$ with the boundary operator ∂ that is given by multiplying with $-\frac{1}{2}$ of the exceptional divisor class of the Hilbert–Chow morphism. It turns out that the same relations as in the untwisted, projective case hold.
The next main result of the paper is a decription of the ring structure whenever X has a numerically trivial canonical divisor. Following ideas in [LS03], we introduce a family of explicitely described graded unital algebras $H^{[n]}$ associated to a G-weighted (non-counital) graded Frobenius algebra H of degree d. For example, $H = \bigoplus_{L\in G^\vee} H^*(X, L_\chi[2])$ is such a Frobenius algebra of degree 2 where G is as above. The following holds for each $n \geq 0$:

THEOREM 1.4. *Assume that X has a numerically trivial canonical divisor. Then there is a natural isomorphism*

$$\bigoplus_{\chi\in G^\vee} H^*(X^{[n]}, L_\chi^{[n]}[2n]) \to \left(\bigoplus_{\chi\in G^\vee} H^*(X, L_\chi[2])\right)^{[n]}$$

of (G-weighted) graded algebras of degree $2n$.

For G the trivial group, and X projective, this theorem is the main result in [LS03].
The idea of the proof of Theorem 1.4 is not to reinvent the wheel but to study how everything can already be deduced from the more special case considered in [LS03].
Again by the Leray spectral sequence, Theorem 1.4 also has a natural application to the cohomology ring of coverings of $X^{[n]}$:

COROLLARY 1.5. *There is a natural isomorphism*

$$H^*(\widehat{X^{[n]}}, \mathbf{C}[2n]) \to \left(\bigoplus_{\chi\in G^\vee} H^*(X, L_\chi[2])\right)^{[n]}$$

of graded unital algebras of degree $2n$.

We want to point out at least two applications of our results. The first one is the computation of the cohomology ring of certain families of Calabi–Yau manifolds of even dimension: Let X be an Enriques surface. Let $\hat{X}$ be its universal covering, which is a K3 surface. Then $G \simeq \mathbf{Z}/(2)$. We denote the local system corresponding to the non-trivial element in G by L. The Hodge diamonds of $H^*(X, \mathbf{C}[2])$ and $H^*(X, L[2])$ are given by

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 10 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 10 & & 1, \\ & 0 & & 0 & \\ & & 0 & & \end{array}$$

respectively.

Denote by $X^{\{n\}}$ the (two-fold) universal cover of $X^{[n]}$. By Remark 2.7, the isomorphism of Corollary 1.3 is in fact an isomorphism of Hodge structures. It follows that

$$H^{k,0}(X^{\{n\}}, \mathbf{C}) = \begin{cases} \mathbf{C} & \text{for } k = 0 \text{ or } k = 2n, \text{ and} \\ 0 & \text{for } 0 < k < 2n. \end{cases}$$

In conjunction with Corollary 1.5, we have thus proven:

Proposition 1.6. *For $n > 1$, the manifold $X^{\{n\}}$ is a Calabi–Yau manifold in the strict sense. Its cohomology ring $H^*(X^{\{n\}}, \mathbf{C}[2n])$ is naturally isomorphic to $(H^*(X, \mathbf{C}[2]) \oplus H^*(X, L[2]))^{[n]}$.*

Our second application is the calculation of the cohomology ring of the generalised Kummer varieties $X^{[[n]]}$ for an abelian surface X. (A slightly less explicit description of this ring has been obtained by more special methods in [Bri02].) Recall from [Bea83] that the generalised Kummer variety $X^{[[n]]}$ is defined as the fibre over 0 of the morphism $\sigma\colon X^{[n]} \to X$, which is the Hilbert–Chow morphism followed by the summation morphism $X^{(n)} \to X$ of the abelian surface. The generalised Kummer surface is smooth and of dimension $2n-2$ ([Bea83]).

As above, let H be a G-weighted graded Frobenius algebra of degree d. Assume further that H is equipped with a compatible structure of a Hopf algebra of degree d. For each $n > 0$, we associate to such an algebra an explicitely described graded unital algebra $H^{[[n]]}$ of degree n.

In the following Theorem, we view $H^*(X, \mathbf{C}[2])$ as such an algebra (the Hopf algebra structure is given by the group structure of X), where we give $H^*(X, \mathbf{C}[2])$ the trivial G-weighting for the group $G := X[n]$, the character group of the group of n-torsion points on X. We prove the following:

Theorem 1.7. *There is a natural isomorphism*

$$H^*(X^{[[n]]}, \mathbf{C}[2n]) \to (H^*(X, \mathbf{C}[2]))^{[[n]]}$$

of graded unital algebras of degree $2n$.

We should remark that most of the "hard work" that is hidden behind the scenes is the work of [Gro96], [Nak97], [Leh99], [LQW02], [LS03], etc. Our own contribution is to generalise and apply the ideas and results in the cited papers to the twisted and to the non-projective case.

Remark 1.8. Let us finally mention that the restriction to algebraic, i.e. quasi-projected surfaces, is just a matter of convenience. Our methods work equally well when we replace X by any complex surface. In this case, the Hilbert schemes become the Douady spaces ([Dou66]).

2. The Fock space description

In this section, we prove Theorem 1.2 for a local system L on X by the method that is used in [Nak97] for the untwisted case, i.e. by realising the cohomology space of the Hilbert schemes (with coefficients in a local system) as an irreducible representation of a Heisenberg Lie algebra.
Let $l \geq 0$ and $n \geq 1$ be two natural numbers. Set

$$X^{(l,n)} := \left\{(\underline{x}', x, \underline{x}) \in X^{(n+l)} \times X \times X^{(l)} \mid \underline{x}' = \underline{x} + nx\right\}$$

(we write the union of unordered tuples additively). We further define the reduced subvariety

$$X^{[n,l]} := \left\{(\xi', x, \xi) \in X^{[n+l]} \times X \times X^{[l]} \mid \xi \subset \xi', (\rho(\xi'), x, \rho(\xi)) \in X^{(l,n)}\right\}$$

in $X^{[n+l]} \times X \times X^{[l]}$. This incidence variety has already been considered in [Nak97]. In contrast to the Hilbert schemes, these incidence varieties are almost never smooth. Its image under the Hilbert–Chow morphism is again $X^{(l,n)}$.
We denote the projections of $X^{(l+n)} \times X \times X^{(l)}$ onto its three factors by $\tilde{p}$, $\tilde{q}$ and $\tilde{r}$, respectively. Likewise, we denote the three projections of $X^{[l+n]} \times X \times X^{[l]}$ by p, q and r.

Lemma 2.1. *We have a natural isomorphism* $q^* L^n \otimes r^* L^{[l]}|_{X^{[n,l]}} \cong p^* L^{[l+n]}|_{X^{[n,l]}}$.

Proof. Firstly, we have a natural isomorphism $\tilde{q}^* L^n \otimes \tilde{r}^* L^{(l)}|_{X^{(n,l)}} \cong \tilde{p}^* L^{(l+n)}|_{X^{(n,l)}}$. This follows from

$$\begin{aligned}(\tilde{q}^* L^n &\otimes \tilde{r}^* L^{(l)})(\underline{x} + nx, x, \underline{x}) \\ &= L(x)^{\otimes n} \otimes \bigotimes_{x' \in \underline{x}} L(x') = \bigotimes_{x' \in \underline{x} + nx} L(x') = \tilde{p}^* L^{(l+n)}(\underline{x} + nx, x, \underline{x})\end{aligned}$$

for every $(\underline{x} + nx, x, \underline{x}) \in X^{(l,n)}$. By pulling back everything to the Hilbert schemes, the Lemma follows. □

Due to Lemma 2.1 and the fact that $p|_{X^{[l,n]}}$ is proper ([Nak97]), the operator (a correspondence, see [Nak97])

$$N\colon H^*(X, L^n[2]) \times H^*(X^{[l]}, L^{[l]}[2l]) \to H^*(X^{[l+n]}, L^{[l+n]}[2(l+n)]),$$
$$(\alpha, \beta) \mapsto \mathrm{PD}^{-1} p_*((q^*\alpha \cup r^*\beta) \cap [X^{[l,n]}])$$

is well-defined. Here,

$$\mathrm{PD}\colon H^*(X^{[l+n]}, L^{[n+l]}[2(l+n)]) \to H_*^{\mathrm{BM}}(X^{[l+n]}, L^{[n+l]}[-2(l+n)])$$

is the Poincar -duality isomorphism between the cohomology and the Borel–Moore homology. (The degree shifts are chosen in a way that N is an operator of degree 0, see [LS03].)

Remark 2.2. Note that although the variety $X^{[l,n]}$ is not smooth in general, it nevertheless possesses a fundamental class $[X^{[l,n]}] \in H_*^{\mathrm{BM}}(X^{[l,n]}, \mathbf{C})$. (This is actually true for every analytic variety, see e.g. the appendices of [PS08].)

Furthermore, $q \times r|_{X \times X^{[l]}}$ is proper ([Nak97]). Thus we can also define an operator the other way round:

$$N^\dagger\colon H_c^*(X, L^{-n}[2]) \times H^*(X^{[n+l]}, L^{[l+n]}[2]) \to H^*(X^{[l]}, L^{[l]}[2l]),$$
$$(\alpha, \beta) \mapsto (-1)^n \mathrm{PD}^{-1} r_*(q^*\alpha \cup p^*\beta \cap [X^{[l,n]}])$$

As in [Nak97], we will use these operators to define an action of a Heisenberg Lie algebra on

$$V_{X,L} := \bigoplus_{n \geq 0} H^*(X^{[n]}, L^{[n]}[2n]).$$

For this, let A be a weighted, graded Frobenius algebra of degree d (over the complex numbers), that is a weighted and graded vector space over $\mathbf{C}$ with a (graded) commutative and associative multiplication of degree d and weight 0 and a unit element 1 (necessarily of degree $-d$ and weight 0) together with a linear form $\int\colon A \to \mathbf{C}$ of degree $-d$ and weight 0 such that for each weight $\nu \in \mathbf{Z}$ the induced bilinear form $\langle \cdot, \cdot \rangle\colon A(\nu) \times A(-\nu) \to \mathbf{C}, (a, a') \mapsto \int_A aa'$ is non-degenerate (of degree 0). Here $A(\nu)$ denotes the weight space of weight ν. In particular, all weight spaces are finite-dimensional. In the case of a trivial weighting, this notion of a graded Frobenius algebra has already appeared in [LS03].

Example 2.3. Consider the vector space

$$A_{X,L} := \bigoplus_{\nu \geq 0} H^*(X, L^\nu[2]) \oplus \bigoplus_{\nu \geq 0} H_c^*(X, L^{-\nu}[2]).$$

It inherits a grading from the cohomological grading of its pieces $H^*(X, L^\nu[2])$. We endow $A_{X,L}$ also with a weighting by defining $H^*(X, L^\nu[2])$ to be of pure weight ν for $\nu \geq 0$ and $H_c^*(X, L^{-\nu}[2])$ to be of pure weight $-\nu$.
Recall that there is a natural linear map $\phi\colon H_c^*(X, M) \to H^*(X, M)$ for every local system M on X. (In the de Rham-model of cohomology, it is induced by the inclusion of the (co-)complex of forms with compact support into the

complex of forms with arbitrary support.) This linear map is compatible with the — non-unitary in the case of compact support — algebra structures on $H_c^*(X, \cdot)$ and $H^*(X, \cdot)$ and the module structure of $H_c^*(X, \cdot)$ over $H^*(X, \cdot)$. With this, we mean that

$$(1) \qquad \phi(am) = a\phi(m), \qquad \phi(mn) = \phi(m)\phi(n), \qquad \phi(m)n = mn$$

for all $a \in H^*(X, M)$ and $m, n \in H_c^*(X, M)$.
This allows us to define a commutative multiplication map of degree 2 ($= \dim X$) and weight 0 on $A_{X,L}$ as follows: For elements $\alpha, \beta \in A_{X,L}$ of pure weight, we set

$$\alpha \cdot \beta := \begin{cases} \alpha \cup \beta & \text{for } \alpha \in H^*(X, L^\nu[2]), \beta \in H^*(X, L^\mu[2]) \\ \alpha \cup \beta & \text{for } \alpha \in H_c^*(X, L^{-\nu}[2]), \beta \in H_c^*(X, L^{-\mu}[2]) \\ \alpha \cup \beta & \text{for } \alpha \in H^*(X, L^\nu[2]), \beta \in H_c^*(X, L^{-\mu}[2]) \text{ and } \nu \leq \mu \\ \phi(\alpha \cup \beta) & \alpha \in H^*(X, L^\nu[2]), \beta \in H_c^*(X, L^{-\mu})[2] \text{ and } \nu > \mu \end{cases}$$

for $\nu, \mu \geq 0$. By (1), it follows immediately that this multiplication map is associative, i.e. defines on $A_{X,L}$ the structure of a weighted, graded, unital, commutative and associative algebra of degree 2.
We proceed by extending the linear form $\int_X \colon H_c^*(X, \mathbf{C}[2]) \to \mathbf{C}$ of degree -2 given by evaluating a class of compact support on the fundamental class of X trivially (that is by extending by zero) on $A_{X,L}$ and call the resulting linear form $\int \colon A_{X,L} \to \mathbf{C}$.
We claim that this endows $A_{X,L}$ with the structure of a weighted, graded Frobenius algebra of degree 2: In fact, given a class $\alpha \in H^*(X, L^\nu)$, we can always find a class $\beta \in H_c^*(X, L^{-\nu})$ and vice versa so that $\int \alpha \cdot \beta = \int_X \alpha \cup \beta \neq 0$.

For any weighted, graded Frobenius algebra A we set

$$\mathfrak{h}_A := A \oplus \mathbf{C}\boldsymbol{c} \oplus \mathbf{C}\boldsymbol{d}.$$

We define the structure of a weighted, graded Lie algebra on $\mathfrak{h}_A$ by defining $\boldsymbol{c}$ to be a central element of weight 0 and degree 0, $\boldsymbol{d}$ an element of weight 0 and degree 0 and by setting the following commutator relations: $[\boldsymbol{d}, a] := n \cdot a$ for each element $a \in A$ of weight n, and $[a, a'] = \langle [\boldsymbol{d}, a], a' \rangle \boldsymbol{c}$ for elements $a, a' \in A$.

Definition 2.4. The Lie algebra $\mathfrak{h}_A$ the *Heisenberg algebra associated to A*.

For $A = A_{X,L}$, we set $\mathfrak{h}_{X,L} := \mathfrak{h}_A$. We define a linear map

$$q \colon \mathfrak{h}_{X,L} \to \operatorname{End}(V_{X,L})$$

as follows: Let $l \geq 0$ and $\beta \in V_{X,L}(l) = H^*(X^{[l]}, L^{[l]}[2l])$. We set $q(\boldsymbol{c})(\beta) := \beta$, and $q(\boldsymbol{d})(\beta) := l\beta$. For $n \geq 0$, and $\alpha \in A_{X,L}(\nu) = H^*(X, L^\nu[2])$, we set $q(\alpha)(\beta) := N(\alpha, \beta)$. For $\alpha \in A_{X,L}(-\nu) = H_c^*(X, L^{-\nu}[2])$, we set $q(\alpha)(\beta) := N^\dagger(\alpha, \beta)$. Finally, we set $q(\alpha)(\beta) = 0$ for $\alpha \in A_{X,L}(0) = H^*(X, \mathbf{C}) \oplus H_c^*(X, \mathbf{C})$.

Proposition 2.5. *The map q is a weighted, graded action of $\mathfrak{h}_{X,L}$ on $V_{X,L}$.*

Proof. This Proposition is proven in [Nak97] for the untwisted case, i.e. for $L = \mathbf{C}$. The proof there is based on calculating commutators on the level of cycles of the correspondences defined by the incidence schemes $X^{[l,n]}$. These commutators are independent of the local system used. Thus the proof in [Nak97] also applies to this more general case. □

Example 2.6. Let $\alpha = \sum \alpha_{(1)} \otimes \cdots \otimes \alpha_{(n)} \in H^*(X^{(n)}, L^{(n)}[2n]) = S^n H^*(X, L[2]) = (H^*(X, L[2])^{\otimes n})^{\mathfrak{S}_n}$ (we use the Sweedler notation to denote elements in tensor products). The pull-back of α by the Hilbert–Chow morphism $\rho\colon X^{[n]} \to X^{(n)}$ is then given by

$$\rho^*\alpha = \frac{1}{n!} \sum q(\alpha_{(1)}) \cdots q(\alpha_{(n)})|0\rangle,$$

where $|0\rangle$ is the unit $1 \in H^*(X^{[0]}, \mathbf{C}) = \mathbf{C}$.

We will use Proposition 2.5 to prove our first Theorem.

Proof of Theorem 1.2. The vector space $\tilde{V}_{X,L} := S^*(\bigoplus_{\nu \geq 1} H^*(X, L^\nu[2]))$ carries a unique structure of an $\mathfrak{h}_{X,L}$-module such that $\boldsymbol{c}$ acts as the identity, $\boldsymbol{d}$ acts by multiplying with the weight, $\alpha \in H^*(X, L^n)$ for $n \geq 1$ acts by multiplying with α, and $\alpha \in H^*(X, \mathbf{C}) \oplus H^*_c(X, \mathbf{C})$ acts by zero. By the representation theory of the Lie algebras of Heisenberg type, this is an irreducible lowest weight representation of $\mathfrak{h}_{V,L}$, which is generated by the lowest weight vector 1 of weight 0.
The $\mathfrak{h}_{V,L}$-module $V_{X,L}$ also has a vector of weight 0, namely $|0\rangle$. Thus, there is a unique morphism $\Phi\colon \tilde{V}_L \to V_L$ of $\mathbf{h}_L$-modules that maps 1 to $|0\rangle$. This will be the inverse of the isomorphism mentioned in Theorem 1.2. It remains to show that Φ is bijective. The injectivity follows from the fact that $\tilde{V}_{X,L}$ is irreducible as an $\mathfrak{h}_{X,L}$-module.
In order to prove the surjectivity, we will derive upper bounds on the dimensions of the weight spaces of the right hand side $V_{X,L}$ (see also [Leh04] about this proof method). By the Leray spectral sequence associated to the Hilbert–Chow morphism $\rho\colon X^{[n]} \to X^{(n)}$, such an upper bound is provided by the dimension of the spectral sequence's E_2-term $H^*(X^{(n)}, \mathbf{R}^*\rho_* L[2n])$. As shown in [GS93], it follows from the Beilinson–Bernstein–Deligne–Gabber decomposition theorem that

$$\mathbf{R}^*\rho_*\mathbf{Q}[2n] = \bigoplus_{\lambda \in \mathbf{P}(n)} (i_\lambda)_*\mathbf{Q}[2\ell(\lambda)].$$

Here, $P(n)$ is the set of all partitions of n, $\ell(\lambda) = r$ is the length of a partition $\lambda = (\lambda^1, \lambda^2, \ldots, \lambda^r)$, $X^{(\lambda)} := \{\sum_{i=1}^r \lambda_i x_i \mid x_i \in X\} \subset X^{(n)}$, and $i_\lambda\colon X^{(\lambda)} \to X^{(n)}$ is the inclusion map.
Set $L^{(\lambda)} := i_\lambda^* L^{(n)}$. By the projection formula, it follows that $\mathbf{R}^*\rho_* L[2n] = \bigoplus_{\lambda \in \mathbf{P}(n)} (i_\lambda)_* L^{(\lambda)}[2\ell(\lambda)]$.
Thus, an upper bound on the dimension of $H^*(X^{[n]}, L^{[n]}[2n])$ is provided by the dimension of $\bigoplus_{\lambda \in \mathbf{P}(n)} H^*(X^{(\lambda)}, L^{(\lambda)}[2\ell(\lambda)])$. By [GS93], this can be seen

to be isomorphic to

$$\bigoplus_{\sum_{i\geq 1} i\nu_i = n} \bigotimes_{i\geq 1} S^{\nu_i} H^*(X, L^i[2]),$$

where each $\nu_i \geq 0$. It follows that the upper bound given by the E_2-term is exactly the dimension of the n-th weight space of $\tilde{V}_{X,L}$. Thus the dimension of the weight spaces of $V_{X,L}$ cannot be greater than the dimensions of the weight spaces of $\tilde{V}_{X,L}$. Thus the Theorem is proven. □

Remark 2.7. Assume that X is projective. In this case, the (twisted) cohomology spaces of X and its Hilbert schemes $X^{[n]}$ carry pure Hodge structures. As the isomorphism of Theorem 1.2 is defined by algebraic correspondences (i.e. by correspondences of Hodge type (p,p)), it follows that the isomorphism in Theorem 1.2 is compatible with the natural Hodge structures on both sides. In terms of Hodge numbers, the following equation encodes our result:

$$\sum_{n\geq 0} \prod_{i,j} h^{i,j}(X^{[n]}, L^{[n]}[2n]) p^i q^j z^n = \prod_{m\geq 1} \prod_{i,j} (1 - (-1)^{i+j} p^i q^j z^m)^{-(-1)^{i+j} h^{i,j}(X, L^m[2])}$$

3. The Virasoro algebra in the twisted case

To each weighted, graded Frobenius algebra A of degree d, we associate a skew-symmetric form $e\colon A \times A \to \mathbf{C}$ of degree d as follows:
Let $n \in \mathbf{Z}$. We note that $A(n)$ and $A(-n)$ are dual to each other via the linear form $\int$. Thus we can consider the linear map $\Delta(n)\colon \mathbf{C} \to A(n) \otimes A(-n)$ dual to the bilinear form $\langle\cdot,\cdot\rangle\colon A(n) \otimes A(-n) \to \mathbf{C}$. Write $\Delta(n)1 = \sum e_{(1)}(n) \otimes e_{(2)}(n)$ in Sweedler notation. Then we define e by setting

$$e(\alpha, \beta) := \sum_{\nu=0}^{n} \frac{\nu(n-\nu)}{2} \int \sum e_{(1)}(\nu) e_{(2)}(\nu) \alpha\beta$$

for all $\alpha \in A(n)$ whenever $n \geq 0$. We shall call this form the *Euler form of A*.

Example 3.1. Assume that $A(n) \equiv A(0)$ for all $n \in \mathbf{Z}$. In this case, we have

$$e(\alpha, \beta) = \frac{n^3 - n}{12} \int e\alpha\beta$$

for $\alpha \in A(n)$ with $e := \int \sum e_{(1)}(0) e_{(2)}(0)$ ([Leh99]).

We use the Euler form to define another Lie algebra associated to A. We set

$$\mathfrak{v}_A := A[-2] \oplus \mathbf{C}\boldsymbol{c} \oplus \mathbf{C}\boldsymbol{d}.$$

We define the structure of a weighted, graded Lie algebra on $\mathfrak{v}_A$ be defining $\boldsymbol{c}$ to be a central element or weight 0 and degree 0, $\boldsymbol{d}$ an alement of weight 0 and degree 0 and by introducing the following commutator relations: $[\boldsymbol{d}, a] := n \cdot a$ for each element $a \in A[-2]$ of weight n, and $[a, a'] := (\boldsymbol{d}a)a' - a(\boldsymbol{d}a') - e(a, a')$ for elements $a, a' \in A$.

Definition 3.2. The Lie algebra $\mathfrak{v}_A$ is the *Virasoro algebra associated to A*.

For $A = A_{X,L}$, we set $\mathfrak{v}_{X,L} := \mathfrak{v}_A$. The whole construction is a generalisation to the twisted case of the Virasoro algebra found in [Leh99].
We now define a linear map $L\colon \mathfrak{v}_{X,L} \to \mathrm{End}(V_{X,L})$ as follows: We define $L(\boldsymbol{c})$ to be the identity, $L(\boldsymbol{d})$ to be multiplication with the weight, and for $\alpha \in A[-2]$ we set

$$L(\alpha) := \frac{1}{2} \sum \sum_{\nu \in \mathbf{Z}} {:}q(e_{(1)}(\nu))q(e_{(2)}(\nu)\alpha){:},$$

where the *normal ordered product* $:aa':$ of two operators is defined to be aa' if the weight of a is greater or equal to the weight of a' and is defined to be $a'a$ if the weight of a' is greater than the weight of a.
The following Lemma is proven for the untwisted case in [Leh99].

Lemma 3.3. *For $\alpha \in A_{V,L}[-2]$ and $\beta \in A_{V,L}$, we have*

$$[L(\alpha), q(\beta)] = -q(\alpha[\boldsymbol{d}, \beta]).$$

Proof. Let $\alpha \in A_{V,L}[2](n)$ and $\beta \in A_{V,L}(m)$ with $n, m \in \mathbf{Z}$. In the following calculations we omit all Koszul signs arising from commuting the graded elements α and β. By definition, we have $[L(\alpha), q(\beta)] = \frac{1}{2}\sum\sum_\nu [{:}q(e_{(1)}(\nu))q(e_{(2)}(\nu)\alpha){:}, q(\beta)]$, where ν runs through all integers. As the commutator of two operators in $\mathfrak{h}_{V,L}$ is central, we do not have to pay attention to the order of the factors of the normally ordered product when calculating the commutator:

$$\begin{aligned}&[{:}q(e_{(1)}(\nu))q(e_{(2)}(\nu)\alpha){:}, q(\beta)]\\ &\qquad = \nu\langle e_{(1)}(\nu), \beta\rangle q(e_{(2)}(\nu)\alpha) + (n-\nu)\langle e_{(2)}(\nu)\alpha, \beta\rangle q(e_{(1)}(\nu)).\end{aligned}$$

As $\langle\cdot,\cdot\rangle$ is of weight zero, the first summand is only non-zero for $\nu = -m$, while the second summand is only non-zero for $\nu = n + m$. Thus we have

$$\begin{aligned}[L(\alpha), q(\beta)] = -\frac{m}{2}\sum \big(&\langle e_{(1)}(-m), \beta\rangle q(e_{(2)}(-m)\alpha)\\ &+\langle e_{(2)}(n+m)\alpha, \beta\rangle q(e_{(1)}(n+m))\big).\end{aligned}$$

As $e_{(1)}(\cdot)$ is the dual basis to $e_{(2)}(\cdot)$, the right hand side simplifies to $-mq(\alpha\beta)$, which proves the Lemma. □

We use Lemma 3.3 to prove the following Proposition, which has already appeared in [Leh99] for the untwisted, projective case:

Proposition 3.4. *The map L is a weighted, graded action of the Virasoro algebra $\mathfrak{v}_{X,L}$ on $V_{X,L}$.*

Proof. Let $\alpha \in A[-2](m)$ and $\beta \in A[-2](n)$ with $m, n \in \mathbf{Z}$. We have to prove that $[L(\alpha), L(\beta)] = (m-n)L(\alpha\beta) - e(\alpha, \beta)$. We follow ideas in [FLM88]. In all summations below, ν runs through all integers if not specified otherwise.

We begin with the case $n \neq 0$ and $m+n \neq 0$. In this case, by Lemma 3.3, it is

$$[L(\alpha), L(\beta)] = \frac{1}{2}\left[L(m), \sum\sum_{\nu} q(e_{(1)}(\nu))q(e_{(2)}(\nu)\beta)\right] =$$

$$= \frac{1}{2}\left(\sum\sum_{\nu}(-\nu)q(e_{(1)}(\nu)\alpha)q(e_{(2)}(\nu)\beta) + \sum\sum_{\nu}(\nu-n)q(e_{(1)}(\nu))q(e_{(2)}(\nu)\alpha\beta)\right).$$

As $\sum q(e_{(1)}(\nu)(\alpha)q(e_{(2)}(\nu)\beta) = q(e_{(1)}(\nu+m))q(e_{(2)}(\nu-m)\alpha\beta)$, the right hand side is equal to

$$\frac{1}{2}\sum\sum_{\nu}\big((-\nu)q(e_{(1)}(\nu+m))q(e_{(2)}(\nu+m)\alpha\beta)+ \\ +(\nu-n)q(e_{(1)}(\nu))q(e_{(2)}(\nu)\alpha\beta)\big),$$

which is nothing else than $(m-n)L(\alpha\beta)$. Note that $e(\alpha,\beta) = 0$ in this case.

The next case we study is $m > 0$ and $n = -m$. In order to ensure convergence in the following calculations we have to split up $L(\beta)$ as follows:

$$L(\beta) = \sum\sum_{\nu\geq m} q(e_{(1)}(\nu)\beta)q(e_{(2)}(\nu)) + \sum\sum_{\nu<m} q(e_{(2)}(\nu))q(e_{(1)}(\nu)\beta)$$

Calculating the commutator $[L(\alpha), L(\beta)]$ thus yields the four terms:

$$\frac{1}{2}\sum\sum_{\nu\geq m}(m-\nu)q(e_{(1)}(\nu)\alpha\beta)q(e_{(2)}(\nu)) + \frac{1}{2}\sum\sum_{\nu\geq m}\nu q(e_{(1)}(\nu)\beta)q(e_{(1)}(\nu)\alpha)+$$

$$+\frac{1}{2}\sum\sum_{\nu<m}\nu q(e_{(2)}(\nu)\alpha)q(e_{(1)}(\nu)\beta) + \frac{1}{2}\sum\sum_{\nu<m}(m-\nu)q(e_{(2)}(\nu))q(e_{(1)}(\nu)\alpha\beta).$$

As in the first case, we now move α and β rightwards. Then we can split off an infinite part given by a multiple of $L(\alpha\beta)$ and are left over with the finite sum

$$[L(\alpha), L(\beta)] - 2mL(\alpha\beta) \\ = \frac{1}{2}\sum\sum_{\nu=0}^{m}(m-\nu)\left(q(e_{(2)}(\nu))q(e_{(1)}(\nu)\alpha\beta) - q(e_{(1)}(\nu))q(e_{(2)}(\nu)\alpha\beta)\right).$$

The right side is exactly $e(\alpha,\beta)$.

The remaining cases either follow from the above by exchanging n and m or are trivial ($n = m = 0$). □

4. The boundary operator

We proceed as in [Leh99] by introducing a boundary operator on $V_{X,L}$. Recall the definition of the tautological classes of the Hilbert scheme [LQW02]: Let Ξ^n be the universal family over $X^{[n]}$, which is a subscheme of $X^{[n]} \times X$. We denote the projections of $X^{[n]} \times X$ onto its factors by p and q. To each $\alpha \in H^*(X, \mathbf{C})$ we associate the *tautological classes*

$$\alpha^{[n]} := p_*(\mathrm{ch}(\mathcal{O}_{\Xi^n}) \cup q^*(\mathrm{td}(X) \cup \alpha))$$

in $H^*(X^{[n]}, \mathbf{C})$.

Remark 4.1. Note that the tautological classes live in the cohomology with untwisted coefficients, and we have not generalised this concept to the twisted case.

Each $\alpha \in H^*(X, \mathbf{C})$ defines an operator $m(\alpha) \in \operatorname{End}(V_{X,L})$, which is given by $m(\alpha)(\beta) := \alpha^{[n]} \cup \beta$ for all $\beta \in H^*(X^{[n]}, L^{[n]})$. It is an operator of weight zero. As it does not respect the grading, we split it up into its homogeneous components $m(\alpha) = \sum m^*(\alpha)$ with respect to the grading. Following [Leh99], we set $\partial := m^2(1)$ and call it the *boundary operator*. It is an operator of weight 0 and degree 2. For each operator $p \in \operatorname{End}(V_{X,L})$, we set $p' := [\partial, p]$ and call it the *derivative of* p.

The main theorem in [Leh99] is the calculation of the derivatives of the Heisenberg operators in the untwisted, projective case. In the sequel, we will do this in our more general case:

Let K be the canonical divisor class of X. We make it into an operator $K\colon A_{X,L} \to A_{X,L}[-2]$ of weight zero by setting

$$K(\alpha) := \frac{|n| - 1}{2} K\alpha$$

for $\alpha \in H^*(X, L^n[2])$.

Proposition 4.2. *For all $\alpha, \beta \in A_{X,L}$ the following holds:*

$$[q'(\alpha), q(\beta)] = -q([\boldsymbol{d}, \alpha][\boldsymbol{d}, \beta]) - \int K([\boldsymbol{d}, \alpha])[\boldsymbol{d}, \beta].$$

Proof. Let us first consider the case of $\alpha \in A(m)$ and $\beta \in A(n)$ with $n+m \neq 0$. We have to show that $[q'(\alpha), q(\beta)] = -nmq(\alpha\beta)$. This is proven in [Leh99] for the projective, untwisted case. The proof in [Leh99] is based on calculating the commutator on the level of cycles. As these calculations are local in X, the result remains true for non-projective X. Furthermore, the proof literally works in the twisted case.

The case $n + m = 0$ remains. Here we have to show that $[q'(\alpha), q(\beta)] = m^2 \frac{|m|-1}{2} \int K\alpha\beta$. In [Leh99] the following intermediate result is formulated for the projective, untwisted case: For all $m \in \mathbf{Z}$, there exists a class $K_m \in H^*(X, \mathbf{C})$ such that $[q'(\alpha), q(\beta)] = m^2 \mathrm{id} \int K_m \alpha\beta$. As above the proof for this intermediate result that is given in [Leh99] also works in the twisted and non-projective case. The classes K_m do not depend on the choice of L, i.e. are universal for the surface. In [Leh99], the classes K_m are computed for the projective case, namely $K_m = \frac{|m|-1}{2} K$, where K is the class of the canonical divisor. All that remains is to calculate the classes K_m for the non-projective (untwisted) case. As $[q'(\alpha), q(\beta)] = [q'(\beta), q(\alpha)]$ (up to Koszul signs), it is enough to calculate K_m for $m > 0$:

Let $\beta \in A_{X,\mathbf{C}}(-m) = H^*_c(X, \mathbf{C}[2])$. Consider an open embedding $j\colon X \to \hat{X}$ of X into a smooth, projective surface $\hat{X}$. We denote the corresponding embeddings $X^{[n]} \to \hat{X}^{[n]}$ also by the letter j. Denote the 1 in $A_{\hat{X},\mathbf{C}}(m) =$

$H^*(X, \mathbf{C}[2])$ by $1(m)$. As all constructions considered so far are functorial (in the appropriate senses) with respect to open embeddings, we have

$$j^*[q'(1(m)), q(j_*\beta)]|0\rangle = [q'(j^*1(m)), q(\beta)]|0\rangle.$$

The right hand side is given by $m^2 \int K_m\beta$, where K_m is the class corresponding to X. By the calculations in [Leh99], the left hand side is given by $m^2\frac{|m|-1}{2}\int K_{\hat{X}} j_*\beta$, where $K_{\hat{X}}$ is the canonical divisor class of $\hat{X}$. As $j^*K_{\hat{X}} = K_X$, we see that $K_m = \frac{|m|-1}{2}K$ also holds in the non-projective case, which proves the Proposition. $\square$

COROLLARY 4.3. *For all $\alpha \in A_{X,L}$, the following holds:*

$$q'(\alpha) = L([\boldsymbol{d}, \alpha]) + q(K([\boldsymbol{d}, \alpha])).$$

Proof. This can be deduced from 4.2 as the respective statement for the untwisted, projective case is proven in [Leh99]. $\square$

5. The ring structure

From now on, we assume that the canonical divisor of X is numerically trivial.

Example 5.1. Let H be a graded Frobenius algebra of degree d. Recall the symmetric non-degenerate bilinear form $\langle\cdot,\cdot\rangle\colon H\otimes H \to \mathbf{C}, h\otimes h' \to \int hh'$. It defines an isomorphism between H and its dual $H^\vee$. We can use this to dualise the multiplication map $H\otimes H \to H$ to a map $\Delta\colon H \to H\otimes H, h \mapsto \sum h_{(1)}\otimes h_{(2)}$ (in Sweedler notation) of degree d. It is coassociative and cocommutative (this follows from the associativity and commutativity of the multiplication map of H). Further, this map is characterised by

$$\sum \langle h_{(1)}, e\rangle \langle h_{(2)}, f\rangle = \langle h, ef\rangle$$

for all $e, f \in H$. It follows that $\Delta(gh) = \sum (gh_{(1)}) \otimes h_{(2)}$ for all $g \in H$. Thus Δ is a homomorphisms of H-modules when we view $H\otimes H$ as a left H-algebra by scalar multiplication on the first factor. (By the cocommutativity of Δ we could have equally chosen the analogously defined right H-algebra structure on $H\otimes H$.)

The example leads us to the following definition when we forget about the linear form $\int$:

A *non-counital graded Frobenius algebra H of degree d (over the complex numbers)* is a graded vector space over $\mathbf{C}$ with a (graded) commutative and associative multiplication of degree d and a unit element 1 (of degree $-d$) together with a coassociative and cocommutative H-module homomorphism $\Delta\colon H \to H\otimes H$ of degree d where we regard $H\otimes H$ as a left H-algebra by multiplying on the left factor. The map Δ is called the *diagonal*.

By example 5.1, every graded Frobenius algebra is in particular a non-counital graded Frobenius algebra.

Let G be a finite abelian group. A *G-weighting on H* is an action of G on H compatible with the Frobenius structure on H. In other words, H comes together with a weight decomposition of the form $\bigoplus_{\chi \in G^\vee} H(\chi)$, where each $g \in G$ acts on $H(\chi)$ by multiplication with $\chi(g)$.

Example 5.2. Let $p\colon \hat{X} \to X$ be a finite abelian Galois covering of X with Galois group G. Then G acts on $M := p_* \mathbf{C}$. Write $M = \bigoplus_{\chi \in G^\vee} L_\chi$, where G acts on each L_χ by multiplication via the character χ. The multiplication on the local system on $\mathbf{C}$ induces a multiplication map $M \otimes M \to M$ and thus isomorphisms $L_\chi \otimes L_{\chi'} \cong L_{\chi\chi'}$ of local systems on X that are commutative and associative in a certain sense. Thus we may assume without loss of generality that these isomorphisms are in fact equalities.
The G-weighted vector space

$$H_{X,G} := \bigoplus_{\chi \in G^\vee} H^*(X, L_\chi[2])$$

is naturally a non-counital G-weighted, graded Frobenius algebra of degree 2 as follows: the grading is given by the cohomological grading. The multiplication is given by the cup product. The diagonal is given by the proper push-forward $\delta_*\colon H_{X,G} \to H_{X,G} \otimes H_{X,G}$ that is induced by the diagonal map $\delta\colon X \to X \times X$. (The map δ_* is indeed a module homomorphism with respect to the left (or, equivalently, right) module structure on $H_{X,G} \otimes H_{X,G}$ as one can see as follows: Let $\pi\colon X \times X \to X$ be the projection onto the left factor. Then one has by the projection formula that

$$\delta_*(\alpha \cup \beta) = \delta_*((\delta^* \pi^* \alpha) \cup \beta) = (\pi^* \alpha) \cup (\delta_* \beta)$$

for all $\alpha, \beta \in H_{X,G}$.)

By iterated application, Δ induce maps $\Delta\colon H \to H^{\otimes n}$ with $n \geq 1$. We denote the restriction of $\Delta\colon H \to H^{\otimes n}$ to $H(L_\chi^n)$, $\chi \in G^\vee$, followed by the projection onto $H(L_\chi)^{\otimes n}$ by $\Delta(\chi)\colon H(L_\chi^n) \to H(L_\chi)^{\otimes n}$. The element $e := (\nabla \circ \Delta(1))(1) \in H$ is called the *Euler class of* H, where $\nabla\colon H \otimes H \to H$ is the multiplication map.
There is a construction given in [LS03] that associates to each graded Frobenius algebra H of degree of d a sequence of graded Frobenius algebras $H^{[n]}$ (whose degrees are given by nd). We extend this construction to G-weighted not necessarily counital Frobenius algebras as follows: For each $\chi \in G^\vee$, set

$$H_n(\chi) := \bigoplus_{\sigma \in \mathfrak{S}_n} \left(\bigotimes_{B \in \sigma \backslash [n]} H(L_\chi^{|B|}) \right) \boldsymbol{\sigma} \quad \text{and} \quad H_n := \bigoplus_{\chi \in G^\vee} H_n(\chi),$$

where $[n] := \{1, \ldots, n\}$ and $\sigma \backslash [n]$ is the set of orbits of the action of the cyclic group generated by σ on the set $[n]$. (Note that $H_n(1) = H(1)\{\mathfrak{S}_n\}$ in the terminology of [LS03].) The symmetric group $\mathfrak{S}_n$ acts on H_n. The graded vector space of invariants, $H_n^{\mathfrak{S}_n}$, is denoted by $H^{[n]}$.

Let $f\colon I \to J$ a surjection of finite sets and $(n_i)_{i\in I}$ a tuple of integers. Fibrewise multiplication yields ring homomorphisms

$$\nabla^{I,J} := \nabla^f \colon \bigotimes_{i\in I} H(L_\chi^{n_i}) \to \bigotimes_{j\in J} H\left(L_\chi^{\sum_{f(i)=j} n_i}\right)$$

of degree $d(|I| - |J|)$. (These correspond to the ring homomorphism $f^{I,J}$ in [LS03].) Dually, by using the diagonal morphisms $\Delta(\chi)$ and relying on their coassociativity and cocommutativity, we can define ∇^f-module homomorphisms

$$\Delta_{J,I} := \Delta_f \colon \bigotimes_{j\in J} H\left(L_\chi^{\sum_{f(i)=j} n_i}\right) \to \bigotimes_{i\in I} H(L_\chi^{n_i}),$$

which are also of degree $d(|I| - |J|)$. (These correspond to the module homomorphisms $f_{J,I}$ in [LS03]).

Let $\sigma, \tau \in \mathfrak{S}_n$ be two permutations. By $\langle\sigma,\tau\rangle$ we denote the subgroup of $\mathfrak{S}_n$ generated by the two permutations. Note that there are natural surjections $\sigma\backslash[n] \to \langle\sigma,\tau\rangle\backslash[n]$, $\tau\backslash[n] \to \langle\sigma,\tau\rangle\backslash[n]$, and $(\sigma\tau)\backslash[n] \to \langle\sigma,\tau\rangle\backslash[n]$. The corresponding ring and module homomorphism are denoted by $\nabla^{\sigma,\langle\sigma,\tau\rangle}$, etc., and $\Delta_{\langle\sigma,\tau\rangle,\sigma}$, etc.

Let $\chi, \chi' \in G^\vee$. We define a linear map

$$m_{\sigma,\tau}\colon \bigotimes_{B\in\sigma\backslash[n]} H(L_\chi^{|B|}) \otimes \bigotimes_{B\in\tau\backslash[n]} H(L_{\chi'}^{|B|}) \to \bigotimes_{B\in(\sigma\tau)\backslash[n]} H(L_{\chi\chi'}^{|B|})$$

by

$$m_{\sigma,\tau}(\alpha\otimes\beta) = \Delta_{\langle\sigma,\tau\rangle,(\sigma\tau)}(\nabla^{\sigma,\langle\sigma,\tau\rangle}(\alpha)\nabla^{\tau,\langle\sigma,\tau\rangle}(\beta)e^{\gamma(\sigma,\tau)}),$$

where the expression $e^{\gamma(\sigma,\tau)}$ is defined as in [LS03] (we have to use our Euler class e, which is defined above). This defines a product $H_n \otimes H_n \to H_n$ which is given by

$$(\alpha\boldsymbol{\sigma})\cdot(\beta\boldsymbol{\tau}) := m_{\sigma,\tau}(\alpha,\beta)\boldsymbol{\sigma\tau}$$

for $\alpha\boldsymbol{\sigma} \in H_n(L_\chi)$ and $\beta\boldsymbol{\tau} \in H_n(L_{\chi'})$. This product is associative, $\mathfrak{S}_n$-equivariant, and of degree nd, which can be proven exactly as the corresponding statements about the product of the rings $H\{\mathfrak{S}_n\}$, which are defined in [LS03]. The product becomes (graded) commutative when restricted to $H^{[n]}$. Thus we have made $H^{[n]}$ a graded commutative, unital algebra of degree nd.

Definition 5.3. The algebra $H^{[n]}$ is the *n-th Hilbert algebra of H.*

In case G is trivial, the n-th Hilbert algebra of H defined here is exactly the algebra $H^{[n]}$ of [LS03]. For non-trivial G, this is no longer true.

The underlying graded vector space of $\bigoplus_{n\geq 0} H^{[n]}(L_\chi)$ is naturally isomorphic to $S(L_\chi) := S^*(\bigoplus_{n\geq 1} H(L_\chi^n))$, namely as follows: Firstly, we introduce linear maps $H_n(L_\chi) \to S(L_\chi)$, which are defined by mapping an element of the form $\sum_{\sigma\in\mathfrak{S}_n}\bigotimes_{B\in\sigma\backslash[n]}\alpha_{\sigma,B}\boldsymbol{\sigma}$ to $\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\prod_{B\in\sigma\backslash[n]}\alpha_{\sigma,B}$. The restrictions of these morphisms to the $\mathfrak{S}_*$-invariant parts define a linear map $\bigoplus_{n\geq 0} H^{[n]}(L_\chi) \to$

$S(L_\chi)$. This map is an isomorphism, which can be proven exactly as it is in [LS03] for trivial G.
Recall that $H^*(X, \mathbf{C}[2])$ is a (trivially weighted) graded non-counital Frobenius algebra of degree d.

Lemma 5.4. *There is a natural isomorphism $H^*(X, \mathbf{C}[2])^{[n]} \to H^*(X^{[n]}, \mathbf{C}[2n])$ of graded unital algebras of degree nd.*

Proof. Recall the just defined isomorphism between the spaces $\bigoplus_{n\geq 0} H^*(X, \mathbf{C}[2])^{[n]}$ and $S^*(\bigoplus_{\nu>0} H^*(X, \mathbf{C}[2]))$ (for the trivial character $\chi = 1$). The composition of this isomorphism with isomorphism between the spaces $S^*(\bigoplus_{\nu>0} H^*(X, \mathbf{C}[2]))$ and $\bigoplus_{n\geq 0} H^*(X^{[n]}, \mathbf{C}[2n])$ of Theorem 1.2 induces by restriction the claimed isomorphism of the Lemma on the level of graded vector spaces.
That this isomorphism is in fact an isomorphism of unital algebras, is proven in [LS03] for X being projective. The proof there does not use the fact that $H^*(X, \mathbf{C}[2])$ has a counit, in fact it only uses its diagonal map. It relies on the earlier work in [Leh99], which has been extended to the non-projective case above, and [LQW02], which can similarly be extended. Thus the proof in [LS03] also works in the non-projective case, when we replace the notion of a Frobenius algebra by the notion of a non-counital Frobenius algebra. $\square$

We will now deduce Theorem 1.4 from Lemma 5.4:

Proof of Theorem 1.4. Let $\chi, \chi' \in G^\vee$. Set $L := L_\chi$ and $M := L_{\chi'}$.
Let $\lambda = (\lambda_1, \dots, \lambda_l)$ be a partition of n. Let ν_i the multiplicity of i in λ, i.e. $\lambda = \sum_i \nu_i \cdot i$. Set $X^{(\lambda)} := \prod_i X^{(\nu_i)}$, and $L^{(\lambda)} := \prod_i \mathrm{pr}_i^* L^{(\nu_i)}$, where the pr_i denote the projections onto the factors $X^{(\nu_i)}$. Let $\alpha = \sum \alpha_{(1)} \cdots \alpha_{(r)} \in H^*(X^{(\lambda)}, L^{(\lambda)}[2l]) = \bigotimes_i S^{\nu_i} H^*(X, L^i[2])$.
We set

$$|\alpha\rangle := \sum q(\alpha_{(1)}) \cdots q(\alpha_{(r)}) |0\rangle.$$

By Theorem 1.2, the cohomology space $H^*(X^{[n]}, L^{[n]}[2n])$ is linearly spanned by classes of the form $|\alpha\rangle$.
Let $\mu = (\mu_1, \dots, \mu_m)$ be another partition of n and $\beta \in H^*(X^{(\mu)}, M^{(\mu)}[2m])$. In order to describe the ring structure of $H^*(X^{[n]}, L^{[n]}[2n])$, we have to calculate the classes $|\alpha \cup \beta\rangle := |\alpha\rangle \cup |\beta\rangle$ in terms of the vector space description given by Theorem 1.2.
This means that we have to calculate the numbers

$$\langle \gamma \mid \alpha \cup \beta \rangle := q(\gamma) |\alpha \cup \beta\rangle \in H^*(X^{[0]}, \mathbf{C}) = \mathbf{C}$$

for all $\gamma \in H^*_c(X^{(\kappa)}, ((LM)^{-1})^{(\kappa)}[2k])$ for all partitions $\kappa = (\kappa_1, \dots, \kappa_k)$ of n, and we have to show that they are equal to the numbers that would come out if we calculated the product of α and β by the right hand side of the claimed isomorphism of the Theorem.
The class $|\alpha\rangle$ is given by applying a sequence of correspondences to the vacuum vector: Recall from [Nak97] how to compose correspondences. It follows that

$|\alpha\rangle$ is given by

$$\mathrm{PD}^{-1}(\mathrm{pr}_1)_*(\mathrm{pr}_2^*\alpha \cap \zeta_\lambda),$$

where the symbols have the following meaning: The maps pr_1 and pr_2 are the projections of $X^{[n]} \times X^{(\lambda)}$ onto its factors $X^{[n]}$ and $X^{(\lambda)}$. Further, ζ_λ is a certain class in $H_*^{\mathrm{BM}}(Z_\lambda)$, where Z_λ is the incidence variety

$$Z_\lambda := \left\{ (\xi, (\underline{x}_1, \underline{x}_2, \dots)) \in X^{[n]} \times X^{(\lambda)} \mid \operatorname{supp} \xi = \sum_i i\underline{x}_i \right\}$$

in $X^{[n]} \times X^{(\lambda)}$. (Note that $\mathrm{pr}_1^* L^{[n]}|_{Z_\lambda} = \mathrm{pr}_2^* L^{(\lambda)}|_{Z_\lambda}$, and that $p|_{Z_\lambda}$ is proper.) For $|\beta\rangle$ and $|\gamma\rangle$ we get similar expressions. By definition of the cup-product (pull-back along the diagonal), it follows that $\langle \gamma \mid \alpha \cup \beta\rangle = \langle r^*\gamma \cup p^*\alpha \cup q^*\beta, \zeta_{\lambda,\mu,\kappa}\rangle$, where p, q, and r are the projections from $X^{(\lambda)} \times X^{(\mu)} \times X^{(\kappa)}$ onto its three factors, and $\zeta_{\lambda,\mu,\kappa}$ is a certain class in $H_*^{\mathrm{BM}}(Z_{\lambda,\mu,\kappa})$ with

$$Z_{\lambda,\mu,\kappa} := \\ = \left\{ ((\underline{x}_1, \underline{x}_2, \dots), (\underline{y}_1, \underline{y}_2, \dots), (\underline{z}_1, \underline{z}_2, \dots)) \mid \sum_i i\underline{x}_i = \sum_j j\underline{y}_j = \sum_k k\underline{z}_k \right\}.$$

(The incidence variety is proper over any of the three factors, so everything is well-defined.) The main point is now that the incidence variety $Z_{\lambda,\mu,\kappa}$ and the homology class $\zeta_{\lambda,\mu,\kappa}$ are independent of the local systems L and M. In particular, we can calculate $\zeta_{\lambda,\mu,\kappa}$ once we know the cup-product in the case $L = M = \mathbf{C}$. But this is the case that is described in Lemma 5.4, which we will analyse now.

First of all, the incidence variety is given by

$$Z_{\lambda,\mu,\kappa} = \sum_{\sigma,\tau} Z_{\sigma,\tau}$$

where σ and τ run through all permutations with cycle type λ and μ, respectively, such that $\rho := \sigma\tau$ has cycle type κ. The varieties $Z_{\sigma,\tau}$ are defined as follows:

As the orbits of the group action of $\langle\sigma\rangle$ on $[n]$ correspond to the entries of the partition λ, there exists a natural map $X^{\sigma\backslash[n]} \to X^{(\lambda)}$, which is given by symmetrising. Furthermore the natural surjection $\sigma\backslash[n] \to \langle\sigma,\tau\rangle\backslash[n]$ induces a diagonal embedding $X^{\langle\sigma,\tau\rangle\backslash[n]} \to X^{\sigma\backslash[n]}$. Composing both maps, we get a natural map $X^{\langle\sigma,\tau\rangle\backslash[n]} \to X^{(\lambda)}$. Analoguously, we get maps from $X^{\langle\sigma,\tau\rangle\backslash[n]}$ to $X^{(\mu)}$ and $X^{(\kappa)}$. Together, these maps define a diagonal embedding

$$i_{\tau,\sigma} \colon X^{\langle\sigma,\tau\rangle\backslash[n]} \to X^{(\kappa)} \times X^{(\lambda)} \times X^{(\mu)}.$$

We define $Z_{\sigma,\tau}$ to be the image of this map.

By Lemma 5.4, the class $\zeta_{\lambda,\mu,\kappa}$ is given by $\sum_{\sigma,\tau}(i_{\sigma,\tau})_*\zeta_{\sigma,\tau}$, where each class $\zeta_{\sigma,\tau} \in H_*^{\mathrm{BM}}(X^{\langle\sigma,\tau\rangle\backslash[n]})$ is Poincaré dual to $c_{\sigma,\tau}\, e^{\gamma(\sigma,\tau)}$. Here, $c_{\sigma,\tau}$ is a certain combinatorial factor (possibly depending on σ and τ), whose precise value is of no concern for us.

Having derived the value of $\zeta_{\lambda,\mu,\kappa}$ from Lemma 5.4, we have thus calculated the value $\langle\gamma \mid \alpha\cup\beta\rangle$.
Now we have to compare this value with the one that is predicted by the description of the cup-product given by the right hand side of the claimed isomorphism of the Theorem. With the same analysis as above, we find this value is also given by a correspondence on $Z_{\lambda,\mu,\kappa}$ with the class $\sum_{\tau,\sigma}(i_{\sigma,\tau})_* c_{\sigma,\tau}\mathrm{PD}(e^{\gamma(\sigma,\tau)})$ with the same combinatorial factors $c_{\sigma,\tau}$ as above. We thus find that the claimed ring structure yields the correct value of $\langle\gamma \mid \alpha\cup\beta\rangle$. $\square$

Remark 5.5. One can also define a natural diagonal map for the Hilbert algebras $H^{[n]}$ making them into graded, non-counital Frobenius algebras of degree nd. The isomorphism of Theorem 1.4 then becomes an isomorphism of graded non-counital Frobenius algebras.

6. The generalised Kummer varieties

Finally, we want to use Theorem 1.4 to study the cohomology ring of the generalised Kummer varieties.
Let H be a non-counital graded Frobenius algebra of degree d that is moreover endowed with a compatible structure of a cocommutative Hopf algebra of degree d. The comultiplication δ of the Hopf algebra structure is of degree $-d$. The counit of the Hopf algebra structure is denoted by ϵ and is of degree d. We further assume that H is also equipped with a G-weighting for a finite abelian group G.

Example 6.1. Let X be an abelian surface. The group structure on X induces naturally a graded Hopf algebra structure of degree 2 on the graded Frobenius algebra $H^*(X,\mathbf{C}[2])$. This algebra is also trivially $X[n]$-weighted, where $G := X[n] \simeq (\mathbf{Z}/(n))^4$ is the group of n-torsion points on X. (Trivially weighted means that the only non-trivial $X[n]$-weight space of $H^*(X,\mathbf{C}[2])$ is the one corresponding to the identity element 0.)

Let n be a positive integer. Recall the definition of the (G-weighted) Hilbert algebra $H^{[n]}$. Repeated application of the comultiplication δ induces a map $\delta\colon H \to H^{\otimes n} = H^{\mathrm{id}\backslash[n]}$, which is of degree $-(n-1)d$. Its image lies in the subspace of symmetric tensors. Thus we can define a map $\phi\colon H \to H^{[n]}$ with $\phi(\alpha) := \delta(\alpha)\mathbf{id}$. One can easily check that this map is an algebra homomorphism of degree $-(n-1)d$, making $H^{[n]}$ into an H-algebra.
Define

$$H^{[[n]]} := H^{[n]} \otimes_H \mathbf{C},$$

where we view $\mathbf{C}$ as an H-algebra of degree d via the Hopf counit ϵ. It is $H^{[[n]]}$ a (G-weighted) graded Frobenius algebra of degree nd.

Definition 6.2. The algebra $H^{[[n]]}$ is the *n-th Kummer algebra of H*.

The reason of this naming is of course Theorem 1.7.

Proof of Theorem 1.7. Let $n\colon X \to X$ denote the morphism that maps x to $n \cdot x$. There is a natural cartesian square

$$
\begin{array}{ccc}
X \times X^{[[n]]} & \xrightarrow{\nu} & X^{[n]} \\
{\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \sigma} \\
X & \xrightarrow[n]{} & X,
\end{array}
\tag{2}
$$

where p is the projection on the first factor and ν maps a pair (x, ξ) to $x + \xi$, the subscheme that is given by translating ξ by x ([Bea83]). Then G is the Galois group of n. Each element χ of $G^\vee$ corresponds to a local system L_χ on X, and we have $n_* \mathbf{C} = \bigoplus_{\chi \in G^\vee} L_\chi$. It follows that ν is an abelian Galois covering of $X^{[n]}$ with $\nu_* \mathbf{C} = \bigoplus_{\chi \in G^\vee} L_\chi^{[n]}$.
Together with Theorem 1.4, this leads to the claimed description of the cohomology ring of $X^{[[n]]}$: Firstly, there is a natural isomorphism

$$H^*(X^{[[n]]}, \mathbf{C}[2n]) \to H^*(X \times X^{[[n]]}, \mathbf{C}[2n]) \otimes_{H^*(X, \mathbf{C}[2])} \mathbf{C}$$

of unital algebras (the tensor product is taken with respect to the map p^* and the Hopf counit $H^*(X, \mathbf{C}[2]) \to \mathbf{C}$). By the Leray spectral sequence for ν and by (2), the right hand side is naturally isomorphic to

$$H^*(X^{[n]}, \nu_* \mathbf{C}[2n]) \otimes_{H^*(X, \mathbf{C}[2])} \mathbf{C} = \bigoplus_{\chi \in G^\vee} H^*(X^{[n]}, L_\chi^{[n]}[2n]) \otimes_{H^*(X, \mathbf{C}[2])} \mathbf{C}$$

(where the tensor product is taken with respect to the map σ^* and the Hopf counit).
By Theorem 1.4, the algebra $\bigoplus_{\chi \in G^\vee} H^*(X^{[n]}, L_\chi^{[n]}[2n])$ is naturally isomorphic to $\bigoplus_{\chi \in G^\vee} H^*(X, L_\chi[2])^{[n]}$. Now $H^*(X, L_\chi[2]) = 0$ unless χ is the trivial character, which follows from the fact that all classes in $H^*(X, \mathbf{C})$ are invariant under the action of the Galois group of n, i.e. correspond to the trivial character. Thus there is a natural isomorphism

$$\bigoplus_{\chi \in G^\vee} H^*(X^{[n]}, L_\chi^{[n]}[2n]) \to H^*(X, \mathbf{C}[2])^{[n]},$$

of G-weighted algebras, where we endow $H^*(X, \mathbf{C}[2])$ with the trivial G-weighting. Under this isomorphism, the map σ^* corresponds to the homomorphism ϕ defined below Example 6.1. Thus we have proven the existence of a natural isomorphism

$$H^*(X^{[[n]]}, \mathbf{C}[2n]) \to H^*(X, \mathbf{C}[2])^{[n]} \otimes_{H^*(X, \mathbf{C}[2])} \mathbf{C}$$

of unital, graded algebras. But the right hand side is nothing but $H^{[[n]]}$, thus the Theorem is proven. $\square$

References

[Bea83] Arnaud Beauville, *Variétés kähleriennes dont la première classe de Chern est nulle.*, J. Differ. Geom. 18 (1983), 755–782 (French).

[Bri02] Michael Britze, *On the Cohomology of Generalized Kummer Varieties*, Ph.D. thesis, Mathematisch-Naturwissenschaftliche Fakultät der Universität zu Köln, 2002.

[Bro88] Ronald Brown, *Topology: A geometric account of general topology, homotopy types and the fundamental groupoid. Rev., updated and expanded ed.*, Ellis Horwood Series in Mathematics and its Applications. Chichester (UK): Ellis Horwood Ltd.; New York etc.: Halsted Press., 1988.

[Dou66] A. Douady, *Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donne*, Sovremen. Probl. Teor. Analit. Funktsij, Mezhdunarod. Konf. Teor. Analit. Funktsij, Erevan 1965, 141–143, 1966.

[FLM88] Igor Frenkel, James Lepowsky, and Arne Meurman, *Vertex operator algebras and the monster.*, Pure and Applied Mathematics, 134. Boston etc.: Academic Press, Inc., 1988 (English).

[Fog68] J. Fogarty, *Algebraic families on an algebraic surface*, Am. J. Math. (1968), 511–521.

[Gro61] Alexandre Grothendieck, *Techniques de construction et théoremes d'existence en géométrie algébrique. IV: Les schemas de Hilbert*, Sem. Bourbaki 13(1960/61), No.221, 1961.

[Gro96] I. Grojnowski, *Instantons and affine algebras. I: The Hilbert scheme and vertex operators*, Math. Res. Lett. 3 (1996), no. 2, 275–291.

[GS93] Lothar Göttsche and Wolfgang Soergel, *Perverse sheaves and the cohomology of Hilbert schemes of smooth algebraic surfaces.*, Math. Ann. 296 (1993), no. 2, 235–245 (English).

[Leh99] Manfred Lehn, *Chern classes of tautological sheaves on Hilbert schemes of points on surfaces*, Invent. Math. 136 (1999), no. 1, 157–207.

[Leh04] ______, *Lectures on Hilbert schemes.*, Hurtubise, Jacques (ed.) et al., Algebraic structures and moduli spaces. Proceedings of the CRM workshop, Montréal, Canada, July 14-20, 2003. Providence, RI: American Mathematical Society (AMS). CRM Proceedings &; Lecture Notes 38, 1-30 (2004)., 2004.

[LQW02] Wei-ping Li, Zhenbo Qin, and Weiqiang Wang, *Vertex algebras and the cohomology ring structure of Hilbert schemes of points on surfaces.*, Math. Ann. 324 (2002), no. 1, 105–133 (English).

[LS03] Manfred Lehn and Christoph Sorger, *The cup product of Hilbert schemes for K3 surfaces.*, Invent. Math. 152 (2003), no. 2, 305–329 (English).

[Nak97] Hiraku Nakajima, *Heisenberg algebra and Hilbert schemes of points on projective surfaces*, Ann. of Math. (2) 145 (1997), no. 2, 379–388.

[PS08] Chris A. M. Peters and Joseph H. M. Steenbring, *Mixed Hodge Structures*, Springer-Verlag, 2008 (English).

Marc A. Nieper-Wißkirchen
Institut für Mathematik
Universität Augsburg
86157 Augsburg
Germany
marc.nieper-wisskirchen@math.uni-augsburg.de