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An equivariant Quillen theorem

Bernhard Hanke [a,*], Michael Wiemeler [b]

[a] *Institut für Mathematik, Universität Augsburg, D-86135 Augsburg, Germany*
[b] *Mathematisches Institut, WWU Münster, Einsteinstr. 62, D-48149 Münster, Germany*

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A classical theorem due to Quillen (1969) identifies the unitary bordism ring with the Lazard ring, which represents the universal one-dimensional commutative formal group law. We prove an equivariant generalization of this result by identifying the homotopy theoretic $\mathbb{Z}/2$-equivariant unitary bordism ring, introduced by tom Dieck (1970), with the $\mathbb{Z}/2$-equivariant Lazard ring, introduced by Cole–Greenlees–Kriz (2000). Our proof combines a computation of the homotopy theoretic $\mathbb{Z}/2$-equivariant unitary bordism ring due to Strickland (2001) with a detailed investigation of the $\mathbb{Z}/2$-equivariant Lazard ring.

* Corresponding author.
E-mail addresses: hanke@math.uni-augsburg.de (B. Hanke), wiemelerm@uni-muenster.de (M. Wiemeler).

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1. Introduction

Around 50 years ago Daniel Quillen gave the following algebraic description of the unitary bordism ring.

Theorem 1.1 *([5]). The canonical map*

$$L_* \to \mathrm{MU}_*$$

from the Lazard ring to the coefficient ring of unitary bordism theory is an isomorphism.

Recall that the Lazard ring is the representing ring of the universal one-dimensional commutative formal group law [3], and that the unitary bordism ring $\mathrm{MU}_* = \mathrm{MU}^{-*}$ is the representing ring of a one-dimensional formal group law

$$F(x,y) \in \mathrm{MU}^{-*}[[x,y]] = \mathrm{MU}^{-*}(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

given by the pull back of the universal Chern class in $\mathrm{MU}^2(\mathbb{C}P^\infty)$ under the classifying map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ of the tensor product of the universal line bundles on each factor of $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$.

Today Quillen's theorem is one of the organizational principles of stable homotopy theory. Establishing equivariant analogues of this result is therefore a reasonable goal. Tom Dieck [8] defined homotopy theoretic G-equivariant unitary bordism theories MU_*^G for compact Lie groups G, using arbitrary unitary G-representations as suspension coordinates. Further information on the foundations of equivariant stable homotopy theory is contained in [4], for instance.

For compact abelian Lie groups A the notion of A-equivariant formal group laws was introduced in [1], and subject to an extensive theoretical investigation in [7]. Similarly to the non-equivariant situation there is a universal one-dimensional commutative A-equivariant formal group law, see [1, Cor. 14.3], together with a representing ring L_A, and a classifying map

$$\lambda_A : L_A \to \mathrm{MU}_*^A .$$

An equivariant version of Quillen's theorem amounts to λ_A being an isomorphism.

Let A be a finite abelian group. By use of a localization-completion pull back square due to tom Dieck (for cyclic A this is [8, Theorem 5.1]), together with the classification of Euler-complete and Euler local equivariant formal group laws, one can show that λ_A is surjective with each element in the kernel being Euler torsion and infinitely Euler divisible, see [2, Theorem 13.1].

Strickland [6] presented an algebraic description of $\mathrm{MU}_*^{\mathbb{Z}/2}$ in terms of generators and relations, and stated without proof the existence of a section $\mathrm{MU}_*^{\mathbb{Z}/2} \to L_{\mathbb{Z}/2}$ of the classifying map $\lambda_{\mathbb{Z}/2}$, establishing $\mathrm{MU}_*^{\mathbb{Z}/2}$ as a retract of $L_{\mathbb{Z}/2}$.

In spite of these positive results, injectivity of λ_A has remained elusive for any non-trivial A. This problem has been raised at several places, see e.g. [2, Questions 16.8] and [7, Chapter 13]. We remark that there do exist non-additive $\mathbb{Z}/2$-equivariant formal group laws with representing rings containing non-zero, infinitely Euler divisible and Euler torsion elements, see Example 4.1. Hence injectivity of λ_A does not follow merely from the results mentioned before, but requires some new structural insight concerning the ring L_A itself. This is what we will achieve in the paper at hand for the simplest non-trivial case $A = \mathbb{Z}/2$, providing the first instance of an equivariant Quillen theorem:

Theorem 1.2. *The map* $\lambda_{\mathbb{Z}/2} : L_{\mathbb{Z}/2} \to \mathrm{MU}_*^{\mathbb{Z}/2}$ *is an isomorphism.*

Our argument starts with the construction of an explicit section $\mu_{\mathbb{Z}/2}$ of the classifying map $\lambda_{\mathbb{Z}/2}$. This allows us to introduce structure constants $\rho_{ij} \in L_{\mathbb{Z}/2}$ in the kernel of $\lambda_{\mathbb{Z}/2}$, which measure the deviation from $\lambda_{\mathbb{Z}/2}$ being an isomorphism.

The proof of the vanishing of all ρ_{ij} rests on two major lines of thought, developed in Sections 6 and 7 of our paper. The first one is of a conceptual nature and based on the construction of a *normalization functor*, turning $\mathbb{Z}/2$-equivariant formal group laws into ones with Euler classes equal to 1. Applied to the universal $\mathbb{Z}/2$-equivariant formal group law we can hence derive efficient upper bounds on the Euler torsion of the elements ρ_{ij}, see Theorem 5.2.

The second one is an explicit computation of a particular $\mathbb{Z}/2$-equivariant formal group law, which is obtained from $L_{\mathbb{Z}/2}$ by dividing out the ideal $\overline{J}$ generated by the images under $\mu_{\mathbb{Z}/2}$ of the positive degree generators of $\mathrm{MU}_*^{\mathbb{Z}/2}$. It turns out that the resulting $\mathbb{Z}/2$-equivariant formal group law is the additive one. In other words (Theorem 5.3): The structure constants ρ_{ij} lie in $\overline{J}$.

Combining these two results in a bootstrap like manner forces vanishing of the ρ_{ij}. See Section 5 for more details.

On the one hand we are optimistic that our approach can be generalized to more general A, for example cyclic groups of prime order. On the other hand we feel that the proof of an equivariant Quillen theorem for all compact abelian A requires some additional insight, which, among others, avoids explicit computations of MU_*^A in terms of generators and relations. We leave this topic for future research.

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2. Recollections on equivariant formal group laws

For the notion of equivariant formal group laws, and the basics of the corresponding theory we refer the reader to [1]. Here we only recall some of the most important features and fix some notation. Let A be a compact abelian Lie group. We denote by $A^* = \mathrm{Hom}(A, S^1)$ the group of irreducible unitary A-representations. The trivial one-dimensional representation is denoted by ϵ. An A-*equivariant formal group law* is given by a quintuple $(k, R, \Delta, \theta, y(\epsilon))$ with a commutative ring k, a complete commutative topological k-algebra R, a continuous comultiplication

$$\Delta : R \to R \widehat{\otimes} R,$$

an augmentation $\theta : R \to k^{A^*}$ and an orientation $y(\epsilon) \in R$, such that the following axioms are satisfied, see [1, Def. 11.1], which we here only recall for finite abelian A.

(1) The comultiplication Δ is a map of k-algebras, co-commutative, co-associative and co-unital.
(2) The augmentation θ is a map of k-algebras compatible with the coproduct, so that $\ker \theta$ defines the topology.
(3) $y(\epsilon) \in R$ is a regular element in the kernel of $\theta(\epsilon)$, and $\theta(\epsilon)$ induces an isomorphism $R/(y(\epsilon)) \cong k$.

We obtain an action l of A^* on R by the formula

$$l_\alpha(c) = (\theta(\alpha^{-1}) \otimes \mathrm{id})(\Delta(c))$$

for $\alpha \in A^*$ and $c \in R$. Moreover, corresponding to $\alpha \in A^*$ we have coordinates $y(\alpha) := l_\alpha(y(\epsilon)) \in R$. Finally, we define Euler classes

$$e(\alpha) := \theta(\epsilon)(y(\alpha)) \in k.$$

For explicit computations it is necessary to choose a basis of the topological k-module R. Recall [1, Notation 12.1] that a *complete flag* $F = (V^0 \subset V^1 \subset V^2 \subset \dots)$ is a sequence of r-dimensional complex A-representations V^r such that $V^r \subset V^{r+1}$ and each finite dimensional complex A-representation is isomorphic to a subrepresentation of some V^r. Given a complete flag $(V^r)_{r\in\mathbb{N}}$ we obtain a k-basis $(y(V^r))_{r\in\mathbb{N}}$ of the topological k-module R, see [1, Lemma 13.2]. In this basis the coproduct Δ is given by

$$\Delta(y(V^r)) = \sum_{i,j\geq 0} \beta_{i,j}^{(r)} \cdot y(V^i) \otimes y(V^j)$$

with structure constants $\beta_{i,j}^{(r)} \in k$. Let $k' \subset k$ be the subring generated by the coefficients $\beta_{i,j}^{(1)}$ and the Euler classes $e(\alpha)$, $\alpha \in A^*$, and let $R' \subset R$ be the free k'-module with basis $(y(V^r))_{r\in\mathbb{N}}$. By an argument similar as for the proof of [1, Theorem 16.1] the coproduct Δ restricts to a coproduct on R' and thus we obtain an induced $\mathbb{Z}/2$-equivariant formal group law $(k', R', \Delta, \theta, y(\epsilon))$. We can hence assume without loss of generality that the underlying ring k of a formal group law (k, R) is generated by the structure constants

$\beta^{(1)}_{i,j} \in k$ and the Euler classes $e(\alpha)$. In this case we briefly call k the *representing ring* of the A-equivariant formal group law.

It is sometimes important to consider *graded* A-equivariant formal group laws (k, R). This means that k and R are $\mathbb{Z}$-graded rings, where for $r \geq 0$ the basis element $y(V^r)$ sits in degree $-2r$, and the coproduct, the A^*-action and the augmentation are grading preserving. In this case the structure constants $\beta^{(r)}_{i,j}$ are homogeneous of degree $2(i+j-r)$ and the Euler classes have degree -2. The most prominent example is the universal A-equivariant formal group law (L_A, R). Here we notice that the construction of (L_A, R) in [1, Cor. 14.3] in fact produces a graded A-equivariant formal group law. With this grading the classifying map

$$L_A \to \mathrm{MU}^A_*$$

of the formal group law associated with A-equivariant unitary bordism theory is grading preserving. We remark that this grading structure of L_A, which plays an important role for our argument, is not present in [1].

3. Coordinate change

Let $A = \mathbb{Z}/2$ and let $(k, R, \Delta, \theta, y(\epsilon))$ be an A-equivariant formal group law. We will work out some explicit formulas relating expansions of elements in R with respect to different complete flags. Let $\eta \in A^*$ be the unique non-trivial one dimensional unitary A-representation and define $e := e(\eta) \in k$ as the corresponding Euler class. For $n \geq 0$ we define a complete flag $F_n = (V^r_n)_{r\geq 0}$ as follows:

- $V^r_n = \epsilon^r$ for $r \leq n$,
- $V^{n+2p}_n = \epsilon^n \oplus (\eta \oplus \epsilon)^p$ for $p \geq 0$,
- $V^{n+2p+1}_n = \epsilon^n \oplus (\eta \oplus \epsilon)^p \oplus \eta$ for $p \geq 0$.

In many cases we will work with the so-called *alternating flag* F_1, whose subquotients V^{r+1}_1/V^r_1, $r \geq 0$, alternate between ϵ and η, starting with ϵ. It has been studied before in [1, Appendix C].

We denote by $d_i \in k$ the coefficients of the expansion of $y(\epsilon)$ in the topological basis induced by F_0,

$$y(\epsilon) = \sum_{i\geq 0} d_i \cdot y(V^i_0).$$

By applying the action l_η to both sides we obtain

$$y(\eta) = \sum_{i\geq 0} d_i \cdot y(V^i_1).$$

Note that $d_0 = e$ by definition of the Euler class.

We will now study the coordinate change induced by passing from the flag F_{n+1} to the flag F_n. For $n \geq 0$ a basis element induced by F_{n+1} has one of the following forms:

- $y(\epsilon), \dots, y(\epsilon)^n$,
- $y(\epsilon)^{n+1}(y(\eta)y(\epsilon))^p$ with $p \geq 0$,
- $y(\epsilon)^{n+1}(y(\eta)y(\epsilon))^p y(\eta)$ with $p \geq 0$.

The basis elements of the first and last type are also part of the basis induced by F_n. Therefore we only have to express the basis elements of the second type in the basis induced by F_n,

$$\begin{aligned} y(\epsilon)^{n+1}(y(\eta)y(\epsilon))^p &= y(\epsilon)^n(y(\eta)y(\epsilon))^p \sum_i d_i \cdot y(V_0^i) \\ &= y(\epsilon)^n(y(\eta)y(\epsilon))^p \sum_i \big(d_{2i} \cdot (y(\epsilon)y(\eta))^i + d_{2i+1} \cdot (y(\epsilon)y(\eta))^i y(\eta)\big) \\ &= \sum_i \big(d_{2i} \cdot y(\epsilon)^n (y(\epsilon)y(\eta))^{i+p} + d_{2i+1} \cdot y(\epsilon)^n (y(\epsilon)y(\eta))^{i+p} y(\eta)\big). \end{aligned}$$

This implies the following coordinate change formula.

Lemma 3.1. *Let $n \geq 0$ and*

$$\sum_{i\geq 0} \gamma_i^{n+1} \cdot y(V_{n+1}^i) = \sum_{i \geq 0} \gamma_i^n \cdot y(V_n^i) \in R,$$

with coefficients γ_i^{n+1} and γ_i^n in k. Then we have

- $\gamma_i^n = \gamma_i^{n+1}$ *for $i < n$,*
- $\gamma_n^n = \gamma_n^{n+1} + e\gamma_{n+1}^{n+1}$.

Given two flags F_n and F_m, where $n, m \geq 0$, we have a topological basis $(y(V_n^i) \otimes y(V_m^j))_{i,j \geq 0}$ of the complete k-module $R \widehat{\otimes} R$. We denote by $\beta_{i,j}^{n,m} \in k$ the coefficients in the expansion of $\Delta(y(\epsilon))$ with respect to this basis,

$$\Delta(y(\epsilon)) = \sum_{i,j \geq 0} \beta_{i,j}^{n,m} \cdot y(V_n^i) \otimes y(V_m^j). \tag{3.1}$$

Note that we can study coordinate change formulas separately for the index pairs (n, i) and (m, j) while fixing the other index pair. Keeping Lemma 3.1 in mind we will now introduce some special elements in k:

- For $0 \leq i < n$ and $0 \leq j < m$ the coefficient $\beta_{i,j}^{n,m}$ is independent of n, m and we denote this element by $\alpha_{i,j}$.

- For $0 \le n$ and $0 \le j < m$ the coefficient $\beta^{n,m}_{n,j}$ is independent of m and we denote this element by $\sigma_{n,j}$.
- For $0 \le m$ we set $\tau_m = \beta^{0,m}_{0,m}$.

Lemma 3.2. *The elements $\alpha_{i,j}$, $\sigma_{n,j}$ and τ_m satisfy the following relations:*

a) $\tau_0 = 0$, $\sigma_{0,0} = e$, $\sigma_{1,0} = 1$ *and* $\sigma_{n,0} = 0$ *for* $n > 1$,
b) $\sigma_{n,j} - \alpha_{n,j} = e\sigma_{n+1,j}$ *for all* $j, n \ge 0$,
c) $\tau_m - \sigma_{0,m} = e\tau_{m+1}$ *for all* $m \ge 0$.

Proof. By the equivariance of the comultiplication with respect to the action l, and using $l_\eta^2 = l_{\eta^2} = \mathrm{id}$, we have

$$\Delta(y(\epsilon)) = \sum_{i,j\ge 0} \beta^{1,1}_{i,j} \cdot l_\eta y(V_1^i) \otimes l_\eta y(V_1^j) = \sum_{i,j\ge 0} \beta^{1,1}_{i,j} \cdot y(V_0^i) \otimes y(V_0^j)\,.$$

We conclude $\tau_0 = \beta^{0,0}_{0,0} = \beta^{1,1}_{0,0} = 0$, the last equation by the co-unitality of the coproduct. Also, for $n \ge 0$, we have

$$\begin{aligned} y(\epsilon) &= l_\epsilon(y(\epsilon)) = (\theta(\epsilon) \otimes \mathrm{Id}) \circ \Delta(y(\epsilon)) \\ &= (\theta(\epsilon) \otimes \mathrm{Id})\Big(\sum_{i,j} \beta^{n+1,n}_{i,j} y(V^i_{n+1}) \otimes y(V^j_n)\Big) \\ &= \sum_j \beta^{n+1,n}_{0,j} y(V^j_n). \end{aligned}$$

Moreover we have $\sigma_{n,0} = \beta^{n,n+1}_{n,0} = \beta^{n+1,n}_{0,n}$, the first equation by definition, the second equation by symmetry of the coproduct. Since for $n \ge 1$ we have $y(V^1_n) = y(\epsilon)$, and $y(\epsilon) = \sum_i d_i \cdot y(V^i_0)$ with $d_0 = e$ by the definition of the Euler class, the remaining parts of assertion a) follow.

Assertion b) and c) follow from the second coordinate change formula in Lemma 3.1, which implies, for $n \ge 0$ and $j < m$,

$$\alpha_{n,j} = \beta^{n+1,m}_{n,j} = \beta^{n,m}_{n,j} - e\beta^{n+1,m}_{n+1,j} = \sigma_{n,j} - e\sigma_{n+1,j}\,,$$

and for all $m \ge 0$

$$\sigma_{0,m} = \beta^{0,m+1}_{0,m} = \beta^{0,m}_{0,m} - e\beta^{0,m+1}_{0,m+1} = \tau_m - e\tau_{m+1}. \quad \square$$

Definition 3.3. The given equivariant formal group law is called *tame* if $d_i = 0$ for all $i > 1$ and $d_1 = 1$, i.e. if

$$y(\eta) = y(\epsilon) + e \text{ and } y(\epsilon) = y(\eta) + e.$$

Note that for tame equivariant formal group laws we have $2e = 0$. The additive equivariant formal group law (cf. [1, Appendix A]) is tame. Another tame group law will be described in Example 4.1 below.

Lemma 3.4. *For a tame equivariant formal group law the coefficients appearing in Lemma 3.1 satisfy the following relations:*

$$\gamma_i^{n+1} = \begin{cases} \gamma_i^n, \text{ if } i < n \text{ or } i \not\equiv n \mod 2\,, \\ \gamma_i^n - e \cdot \gamma_{i+1}^n, \text{ if } i \geq n \text{ and } i \equiv n \mod 2\,. \end{cases}$$

Proof. This follows by the calculation preceding Lemma 3.1 together with the fact that $d_0 = e$, $d_1 = 1$ and $d_i = 0$ for all $i > 1$ for tame equivariant group laws. □

From this we derive the following coordinate change formula.

Lemma 3.5. *Let the given equivariant formal group law be tame. If $i \geq n$, then in the formula*

$$\gamma_i^{n+1} = \sum_{\ell=0}^{n} x_{i,n,\ell} \cdot e^\ell \cdot \gamma_{i+\ell}^1,$$

which is implied by Lemma 3.4, the coefficients $x_{i,n,\ell}$ satisfy the congruence

$$x_{i,n,\ell} \equiv \binom{\ell + [(n-\ell)/2]}{\ell} \mod 2,$$

if $i + \ell$ is even. Here $[-]$ denotes the Gauß bracket.

Proof. The assertion is clear for $n = 0$. In the induction step we assume the assertion holds for $n = n_0$. Assume that $i \geq n_0 + 1$, $0 \leq \ell \leq n_0 + 1$ and $i + \ell$ is even. We distinguish the following cases:

- n_0 and i (and hence also n_0 and ℓ) have the same parity.
- n_0 and i (and hence also n_0 and ℓ) have different parities.

In the first case we have

$$\gamma_i^{n_0+2} = \gamma_i^{n_0+1}$$

such that by the induction assumption and the fact that n_0 and ℓ have the same parity

$$x_{i,n_0+1,\ell} = x_{i,n_0,\ell} \equiv \binom{\ell + [(n_0-\ell)/2]}{\ell} = \binom{\ell + [((n_0+1)-\ell)/2]}{\ell} \mod 2,$$

completing the induction step. In the second case we first notice that the assertion of the lemma is clear for $\ell = 0$. In the case $\ell > 0$ we have

$$\gamma_i^{n_0+2} = \gamma_i^{n_0+1} - e\gamma_{i+1}^{n_0+1} \equiv \gamma_i^{n_0+1} + e\gamma_{i+1}^{n_0+1} \mod 2$$

such that, again by the induction assumption,

$$\begin{aligned} x_{i,n_0+1,\ell} &= x_{i,n_0,\ell} + x_{i+1,n_0,\ell-1} \\ &\equiv \binom{\ell + [(n_0-\ell)/2]}{\ell} + \binom{\ell - 1 + [(n_0-(\ell-1))/2]}{\ell - 1} \mod 2. \end{aligned}$$

Using the fact that n_0 and ℓ have different parities the last sum is equal to

$$\binom{\ell - 1 + [((n_0+1)-\ell)/2]}{\ell} + \binom{\ell - 1 + [((n_0+1)-\ell)/2]}{\ell - 1} = \binom{\ell + [((n_0+1)-\ell)/2]}{\ell},$$

completing the induction step in the second case. □

4. A section of the classifying map $\lambda_{\mathbb{Z}/2} : L_{\mathbb{Z}/2} \to \mathrm{MU}_*^{\mathbb{Z}/2}$

Let $A = \mathbb{Z}/2$ and let the coproduct of the universal non-equivariant formal group law be given by $\Delta(z) = \sum_{ij} a_{ij} \cdot z^i \otimes z^j$, where the elements a_{ij}, $i, j \geq 0$, generate the non-equivariant Lazard ring L [3]. By [6, Section 2] the coefficient ring MU_*^A of A-equivariant unitary bordism is given as an algebra over L by generators s_{nj}, $n, j \geq 0$, and t_m, $m \geq 0$, and relations

- $t_0 = 0$, $s_{10} = 1$ and $s_{n0} = 0$ for $n > 1$,
- $s_{nj} - a_{nj} = es_{n+1,j}$,
- $t_m - s_{0m} = et_{m+1}$.

Here e is an abbreviation for s_{00}, and this element corresponds to the Euler class in MU_{-2}^A associated to the representation η.

Example 4.1. Introducing the additional relations

- $a_{ij} = 0$ for $i + j \geq 2$,
- $s_{01} = 1$, $s_{0j} = 0$ for $j \geq 2$, $s_{nj} = 0$ for $j \neq 2$ and $n \geq 1$,
- $t_1 = 1$, $t_m = 0$ for $m \geq 1$,

we obtain a tame $\mathbb{Z}/2$-equivariant formal group law with a representing ring which is given as a $\mathbb{Z}[e]/(2e)$-algebra by generators s_{n2}, $n \geq 1$, and relations $es_{12} = 0$, $s_{n2} = es_{n+1,2}$ for $n \geq 1$. When viewed as a graded equivariant formal group law all elements of positive degree in this ring are infinitely e-divisible and e-torsion.

We now combine this description of MU_*^A with the calculus developed in Section 3.

Proposition 4.2. *The assignment*

- $a_{ij} \mapsto \alpha_{ij}$,
- $s_{nj} \mapsto \sigma_{nj}$,
- $t_m \mapsto \tau_m$

defines a graded ring map $\mu_A : \mathrm{MU}_*^A \to L_A$ *which satisfies* $\lambda_A \circ \mu_A = \mathrm{id}$.

Proof. By the relations for the generators for MU_*^A given by Strickland and by Lemma 3.2 above we indeed get a well defined ring map $\mathrm{MU}_*^A \to L_A$.

It remains to check that the canonical map $\lambda_A : L_A \to \mathrm{MU}_*^A$ sends the elements α_{ij}, σ_{nj} and τ_m to the elements a_{ij}, s_{nj} and t_m. Consider the commutative diagram

$$\begin{array}{ccc} L_A & \xrightarrow{\lambda_A} & \mathrm{MU}_*^A \\ \downarrow & & \downarrow \\ \widehat{L_A} & \xrightarrow{\widehat{\lambda_A}} & \widehat{\mathrm{MU}_*^A} \end{array}$$

relating the map λ_A to the induced map of completions at the ideal (e).

The map $\widehat{\lambda_A}$ can be identified with the identity $L[[e]]/[2](e) \to L[[e]]/[2](e)$, using the canonical isomorphism $\widehat{L_A} \cong L[[e]]/[2](e)$ from [2, Cor. 6.6], which is induced by $\alpha_{ij} \mapsto a_{ij}$, $e \mapsto e$, and the L-algebra isomorphism $\widehat{\mathrm{MU}_*^A} \cong L[[e]]/[2](e)$ from [6, Section 4].

Furthermore the completion map

$$\mathrm{MU}_*^A \to \widehat{\mathrm{MU}_*^A} \cong L[[e]]/[2](e) = \mathrm{MU}^{-*}(B\mathbb{Z}/2)$$

appearing as the right hand vertical map in the above diagram can be identified with a “bundling map” of tom Dieck, which, in the case relevant for us, was shown to be injective in [8, Prop. 6.1 and preceding explanations]. Alternatively the injectivity of the completion map follows from the results in [6].

We therefore need to show that σ_{nj} and s_{nj} on the one hand, and τ_m and t_m on the other, are mapped to the same elements under the left and right hand vertical maps in the above diagram. By the recursive formulas for σ_{nj} and τ_m from Lemma 3.2 and the corresponding formulas for s_{nj} and t_m from [6, Section 4] we arrive at the equation

$$\sigma_{nj} = \sum_{\ell \geq 0} a_{n+\ell,j} e^\ell = s_{nj} \in L[[e]]/[2](e)$$

and this implies, in a similar way,

$$\tau_m = \sum_{\ell \geq 0} \sigma_{0,m+\ell} e^\ell = \sum_{\ell \geq 0} s_{0,m+\ell} e^\ell = t_m \in L[[e]]/[2](e). \quad \square$$

5. Proof of the main theorem

In this section we explain the proof of Theorem 1.2. Let us write the coproduct of the equivariant formal group law of $\mathbb{Z}/2$-equivariant unitary bordism as

$$\Delta(y(\epsilon)) = \sum_{i,j\geq 0} \beta_{ij} \cdot y(V^i) \otimes y(V^j),$$

with $\beta_{ij} \in \mathrm{MU}_*^{\mathbb{Z}/2}$, where we use the alternating flag $(V^r) = (V_1^r)$. Setting

$$\gamma_{ij} := \mu_{\mathbb{Z}/2}(\beta_{ij}) \in L_{\mathbb{Z}/2},$$

with the map $\mu_{\mathbb{Z}/2} : \mathrm{MU}_*^{\mathbb{Z}/2} \to L_{\mathbb{Z}/2}$ from the previous section, the coproduct of the universal $\mathbb{Z}/2$-equivariant formal group law takes the form

$$\Delta(y(\epsilon)) = \sum_{i,j} (\gamma_{ij} + \rho_{ij}) \cdot y(V^i) \otimes y(V^j)\,.$$

This defines new structure constants $\rho_{ij} \in L_{\mathbb{Z}/2}$ measuring the deviation from $\mu_{\mathbb{Z}/2}$ being surjective (and hence from $\lambda_{\mathbb{Z}/2}$ being injective). Note that $\rho_{ij} = 0$ for $i + j \leq 1$ and $\rho_{ij} \in \ker \lambda_{\mathbb{Z}/2}$ for all i, j. Hence each ρ_{ij} is infinitely e-divisible and e-torsion [2]. In particular,

$$\rho_{ij} \cdot \rho_{pq} = 0$$

for all $i, j, p, q \geq 0$.

Lemma 5.1. *The kernel of the canonical map $\lambda_{\mathbb{Z}/2} : L_{\mathbb{Z}/2} \to \mathrm{MU}_*^{\mathbb{Z}/2}$ is equal to the square zero ideal generated by ρ_{pq}, $p, q \geq 0$.*

Proof. It is clear that the given ideal is contained in $\ker \lambda_{\mathbb{Z}/2}$. Conversely, note that $L_{\mathbb{Z}/2}$ is generated as an $\mathbb{Z}[e, \gamma_{ij}]$-module by the elements 1 and ρ_{pq}. If $x \in L_{\mathbb{Z}/2}$ lies in the kernel of $\lambda_{\mathbb{Z}/2}$, then the coefficient of 1 in some expansion of x as a linear combination of these generators is equal to 0, because each ρ_{pq} lies in $\ker \lambda_{\mathbb{Z}/2}$ and $\lambda_{\mathbb{Z}/2}$ in injective on $\mathbb{Z}[e, \gamma_{ij}] \cdot 1 \subset L_{\mathbb{Z}/2}$. Hence x lies in the ideal generated by the elements ρ_{pq}. □

The proof of Theorem 1.2 is based on the following two results, the first of which provides an efficient estimate of the order of the e-power torsion of ρ_{ij}.

Theorem 5.2. *We have*

$$e^{i+j+1}\rho_{ij} = 0$$

for all $i, j \geq 0$.

This is proven in Section 6, where we introduce and investigate a *normalization functor*, which turns any $\mathbb{Z}/2$-equivariant formal group law into a $\mathbb{Z}/2$-equivariant formal group law with Euler class equal to 1.

Let $a_{ij} \in L$ denote the structure constants of the universal non-equivariant formal group law, considered as elements in $\mathrm{MU}_*^{\mathbb{Z}/2}$ as in Section 4. Recall $a_{ij} = a_{ji}$, $a_{0j} = \delta_{1j}$ and a_{ij} carries a grading equal to $2(i+j-1)$. Next, let $J \subset MU_*^{\mathbb{Z}/2}$ be the ideal generated by a_{ij}, s_{nj}, and t_m, with $i+j \geq 2$, $n+j \geq 2$, and $m \geq 2$. In particular this ideal is generated by elements in strictly positive degrees. By the calculations in [6] the remaining generators of $\mathrm{MU}_*^{\mathbb{Z}/2}$, as an algebra over $L[e]$, satisfy the relations $t_1 = 1 + e(s_{11} + t_2)$ and $s_{01} = t_1 - et_2 = 1 + es_{11}$, and hence we get

$$\mathrm{MU}_*^{\mathbb{Z}/2}/J \cong \mathbb{Z}[e]/(2e).$$

Let $\overline{J} \subset L_{\mathbb{Z}/2}$ be the ideal generated by $\mu_{\mathbb{Z}/2}(J) \subset L_{\mathbb{Z}/2}$, or, in other words, generated by α_{ij}, σ_{nj}, and τ_m, with $i+j \geq 2$, $n+j \geq 2$, and $m \geq 2$. We obtain an induced $\mathbb{Z}/2$-equivariant formal group law with representing ring $L_{\mathbb{Z}/2}/\overline{J}$. In Section 7 we will show by an explicit computation that this is in fact the additive $\mathbb{Z}/2$-equivariant formal group law. This implies the following fact.

Theorem 5.3. *$L_{\mathbb{Z}/2}/\overline{J} \cong \mathbb{Z}[e]/(2e)$.*

After these preparations we are in a position to prove Theorem 1.2. Let some $i, j \geq 0$ be given. We claim $\rho_{ij} = 0$. This holds for $i + j \leq 1$ by the co-unitality of the coproduct on $L_{\mathbb{Z}/2}$. We therefore can assume $i + j \geq 2$, which implies that the degree of ρ_{ij} is positive.

We abbreviate $\lambda_{\mathbb{Z}/2}$ by λ and $\mu_{\mathbb{Z}/2}$ by μ. Since the degree of ρ_{ij} is positive, Theorem 5.3 implies

$$\rho_{ij} = \sum_\ell \gamma_\ell \cdot x_\ell$$

with a finite sum on the right hand side, where each $\gamma_\ell \in \mu(J)$ and $x_\ell \in L_{\mathbb{Z}/2}$. Let us define

$$\delta_\ell := \mu \circ \lambda(x_\ell) \text{ and } x'_\ell := x_\ell - \delta_\ell\,.$$

Because each $x'_\ell \in \ker \lambda$ we have $\lambda(\sum \gamma_\ell \cdot x'_\ell) = 0$ and hence

$$\sum \gamma_\ell \cdot \delta_\ell = (\mu \circ \lambda)(\sum \gamma_\ell \cdot \delta_\ell) = (\mu \circ \lambda)(\sum \gamma_\ell \cdot x_\ell) = \mu \circ \lambda(\rho_{ij}) = 0.$$

The first equation holds because $\sum \gamma_\ell \cdot \delta_\ell \in \operatorname{im} \mu$. We conclude

$$\rho_{ij} = \sum_\ell \gamma_\ell \cdot x'_\ell$$

where each x'_ℓ is in the ideal $\ker\lambda$, which is equal to the ideal generated by the elements ρ_{pq} by Lemma 5.1. Repeating this process several times we conclude that for each $N > 0$ there is a relation of the form

$$\rho_{ij} = \sum_{p,q} c_{pq} \cdot \gamma_{pq} \cdot \rho_{pq}$$

where $c_{pq} \in L_{\mathbb{Z}/2}$ and $\gamma_{pq} \in \mu(J)^N$ the N-th power of $\mu(J)$. Because each generator of J has degree at least 2 we conclude, by comparing degrees of the right and left hand sides of the last equation, that $c_{pq} \cdot \gamma_{pq}$ must be divisible by $e^{p+q-1+N-(i+j-1)}$. For $N = i+j+1$ the exponent satisfies

$$p + q - 1 + N - (i + j - 1) = p + q + 1\,.$$

Hence for $N = i + j + 1$ we get $c_{pq} \cdot \gamma_{pq} \cdot \rho_{pq} = 0$ for all p, q by Theorem 5.2. We must therefore have $\rho_{ij} = 0$, as required.

6. Normalization functor

Let $(k, R, \Delta, \theta, y(\epsilon))$ be a (graded or ungraded) $\mathbb{Z}/2$-equivariant formal group law. As before the Euler class is denoted by e, which sits in degree -2, if (k, R) happens to be graded. Passing to the quotient ring $k/(e-1)$ we obtain a new, ungraded $\mathbb{Z}/2$-equivariant formal group law with Euler class equal to one. In this section we will present a different way to associate to $(k, R, \Delta, \theta, y(\epsilon))$ a $\mathbb{Z}/2$-equivariant formal group law $(k', R', \Delta', \theta', y'(\epsilon))$ with the following properties:

- k' is a subring of $k/\operatorname{Ann}(e^2)$, where $\operatorname{Ann}(e^2)$ is the annihilator ideal of the multiplication with e^2. If (k, R) is graded, then k' is concentrated in degree 0.
- The Euler class of (k', R') is equal to 1.
- The construction is functorial in $(k, R, \Delta, \theta, y(\epsilon))$.
- For the formal group law $(k = \mathrm{MU}_*^{\mathbb{Z}/2}, R)$, associated to $\mathbb{Z}/2$-equivariant unitary bordism, the formal group law (k', R') is the universal $\mathbb{Z}/2$-equivariant formal group law with Euler class equal to 1.

Definition 6.1. We call $(k', R', \Delta', \theta', y'(\epsilon))$ the *normalization* of $(k, R, \Delta, \theta, y(\epsilon))$.

Our construction is based on the description of equivariant formal group laws relative to flags, see [1, Section 12]. We work with the alternating flag $(V^r) = (V_1^r)$ throughout. Let the coproduct of (k, R) be given by

$$\Delta(y(V^r)) = \sum_{i,j\geq 0} f_{i,j}^{(r)} \cdot y(V^i) \otimes y(V^j).$$

Furthermore let

$$y(\eta) = \sum_{i\geq 0} d_i y(V^i) = e + \sum_{i\geq 1} d_i y(V^i).$$

Now consider the free topological k-module T with topological basis $(z(V^i))_{i\geq 0}$ where $z(V^0) := 1$ and define the elements

$$z(\epsilon) := z(V^1) \text{ and } z(\eta) := 1 + \sum_{i\geq 1} e^{i-1} d_i z(V^i)$$

of T. We define a k-bilinear multiplication on T as follows. If at least one of r and s is even, then we set

$$z(V^r) \cdot z(V^s) := z(V^{r+s}).$$

Furthermore, we set

$$z(V^1) \cdot z(V^1) := z(V^1) + \sum_{i\geq 1} e^{i-1} d_i z(V^{1+i})$$

and define

$$z(V^{2p+1}) \cdot z(V^{2q+1}) := z(V^{2p+2q}) z(V^1) z(V^1).$$

This determines a continuous, k-bilinear product on all of T. It is easy to check that it is commutative and associative, where the last point follows from the equation

$$\big(z(V^1) \cdot z(V^1)\big) \cdot z(V^1) = z(V^1) \cdot \big(z(V^1) \cdot z(V^1)\big)$$

which is implied by commutativity of the product.

We first have to check that the topological ring T satisfies the conditions (Flag) and (Ideal) from [1, Section 12]. By definition we have

$$z(V^{2p}) z(\epsilon) = z(V^{2p}) z(V^1) = z(V^{2p+1})$$

which is consistent with (Flag), whereas by definition of $z(\eta)$

$$z(V^1) z(\eta) = z(V^1)(1 + \sum_{i\geq 1} e^{i-1} d_i z(V^i)).$$

If this element is equal to $z(V^2)$, then the condition (Flag) is satisfied on T by the definition of the product on T.

For this and later calculations we set

$$\overline{z}(V^r) := e^r z(V^r) \in T$$

for all $r \geq 0$ and consider the map $\Psi : R \to T$ of topological k-modules given on basis elements by

$$y(V^r) \mapsto \overline{z}(V^r).$$

If the original formal group law (k, R) is graded and we consider all $z(V^r) \in T$ as sitting in degree 0, then the degree of $\overline{z}(V^r)$ is equal to $-2r$ and the map Ψ is grading preserving. Moreover the definition of the ring structure on T together with the calculation

$$\begin{aligned}\Psi(y(V^1) \cdot y(V^1)) &= \Psi\big(ey(V^1) + \sum_{i \geq 1} d_i y(V^{1+i})\big) = e\overline{z}(V^1) + \sum_{i \geq 1} d_i \overline{z}(V^{1+i}) \\ &= \overline{z}(V^1) \cdot \overline{z}(V^1)\end{aligned}$$

shows that the map Ψ is a ring map. Note that the elements $\overline{z}(\epsilon) := ez(\epsilon)$ and

$$\overline{z}(\eta) := ez(\eta) = \sum_{i \geq 0} d_i \overline{z}(V^i)$$

are the images of $y(\epsilon)$ and $y(\eta)$ under the map Ψ. After these preparations we can calculate

$$\begin{aligned}e^2 \cdot z(V^1)z(\eta) &= \overline{z}(V^1)\overline{z}(\eta) = \Psi(y(V^1)) \cdot \Psi(y(\eta)) = \Psi(y(V^1) \cdot y(\eta)) = \Psi(y(V^2)) \\ &= e^2 \cdot z(V^2).\end{aligned}$$

The fourth equation follows from the relation (Flag) in R. In other words: The coefficients of $z(V^1)z(\eta) - z(V^2) \in T$ are in the annihilator ideal $\operatorname{Ann}(e^2) \subset k$. Hence the normalization condition (Flag) is satisfied in T, if we pass from the ring k to the quotient ring $k/\operatorname{Ann}(e^2)$. In this case also the condition (Ideal) follows immediately.

Next we define an A^*-action on the topological ring T. We set $l_\epsilon := \mathrm{id}$,

$$l_\eta z(V^{2p}) := z(V^{2p}),$$

and

$$l_\eta z(V^{2p+1}) := z(V^{2p}) \cdot z(\eta).$$

This map is extended k-linearly onto T. We need to examine the compatibility of l_η with the multiplication on T defined before as well as the property $l_\eta^2 = \mathrm{id}$.

By definition we have

$$l_\eta(z(V^r) \cdot z(V^s)) = l_\eta z(V^r) \cdot l_\eta z(V^s),$$

if either r or s is even. If both $r = 2p + 1$ and $s = 2q + 1$ are odd, then, by definition,

$$l_\eta(z(V^r)\cdot z(V^s)) = l_\eta\big(z(V^{2p+2q})\cdot z(V^1)\cdot z(V^1)\big) = z(V^{2p+2q})\cdot l_\eta\big(z(V^1)\cdot z(V^1)\big)$$

and

$$l_\eta(z(V^r))\cdot l_\eta(z(V^s)) = z(V^{2p+2q})\cdot l_\eta z(V^1)\cdot l_\eta z(V^1).$$

Thus we need to show

$$l_\eta\big(z(V^1)\cdot z(V^1)\big) = l_\eta(z(V^1))\cdot l_\eta(z(V^1)).$$

For this we calculate

$$\begin{aligned} e^2\cdot l_\eta\big(z(V^1)\cdot z(V^1)\big) &= l_\eta\big(\overline{z}(V^1)\cdot \overline{z}(V^1)\big) = l_\eta(\overline{z}(V^1))\cdot l_\eta(\overline{z}(V^1)) = \overline{z}(\eta)\cdot\overline{z}(\eta) \\ &= e^2\cdot z(\eta)z(\eta). \end{aligned}$$

The second equation uses the fact that the above ring map Ψ is compatible with the map l_η. For this assertion we notice that indeed $l_\eta(\overline{z}(\epsilon)) = el_\eta(z(\epsilon)) = ez(\eta) = \overline{z}(\eta)$ and

$$l_\eta(\overline{z}(\eta)) = l_\eta(\sum_{i\geq 0} d_i\overline{z}(V^i)) = e + \overline{z}(\eta)(\sum_{i\geq 1} d_i\overline{z}(V^{i-1})) = \overline{z}(\epsilon),$$

where the last equation follows from the corresponding relation in R and application of the ring map Ψ. In summary we see that l_η is a ring map on T after passing to the coefficient ring $k/\operatorname{Ann}(e^2)$.

The equation $l_\eta\circ l_\eta = \mathrm{id}$ on T holds after passing to the coefficient ring $k/\operatorname{Ann}(e^2)$, because

$$e\cdot l_\eta z(\eta) = l_\eta\overline{z}(\eta) = \overline{z}(\epsilon) = e\cdot z(\epsilon)$$

and $l_\eta z(\epsilon) = z(\eta)$, by definition.

Let us turn to the definition of the coproduct on T.

Lemma 6.2. *For the structure constants of the coproduct in R we have*

$$f^{(2)}_{0,j} = \delta_{2,j} \text{ and } ef^{(2)}_{1,j} = 0$$

for all $j\geq 0$.

Proof. Recall

$$\Delta(y(V^2)) = \sum_{i,j\geq 0} f^{(2)}_{i,j}\cdot y(V^i)\otimes y(V^j).$$

The first equality of the lemma holds by the co-unitality of Δ. For the second equality we use the fact that the coproduct Δ is compatible with l_η:

$$\Delta(y(V^2)) = \Delta(l_\eta y(V^2)) = \sum_{i,j\geq 0} f^{(2)}_{i,j} \cdot (l_\eta y(V^i)) \otimes y(V^j)\,.$$

Hence the second equation in the lemma follows from $l_\eta(V^1) = e + \sum_{i\geq 1} d_i y(V^i)$ and $l_\eta y(V^r) \in (y(V^1))$ (the ideal spanned by $y(V^1)$) for all $r \geq 2$, by comparing coefficients and using the first part of the lemma. □

Now we set

$$\Delta(z(\epsilon)) := \sum_{i,j\geq 0} e^{i+j-1} f^{(1)}_{i,j} \cdot z(V^i) \otimes z(V^j),$$

$$\Delta(z(V^2)) := \sum_{i,j\geq 0} e^{i+j-2} f^{(2)}_{i,j} \cdot z(V^i) \otimes z(V^j).$$

Note that on the right hand sides only non-negative exponents occur at e, by the co-unitality of the coproduct Δ on R, compare the first part of Lemma 6.2. We now define, for $p \geq 1$,

$$\Delta(z(V^{2p})) := \Delta(z(V^2))^p,$$

and

$$\Delta(z(V^{2p+1})) := \Delta(z(V^{2p})) \cdot \Delta(z(\epsilon)).$$

It follows from the definition of the multiplication on T and from the second part of Lemma 6.2 that this extends to a continuous map

$$\Delta : T \to T \widehat{\otimes} T$$

after passing to the coefficient ring $k/\operatorname{Ann}(e^2)$. We also notice that this coproduct Δ is compatible with the coproduct on R and the ring map $\Psi : R \to T$. By definition

$$\Delta(z(V^r) \cdot z(V^s)) = \Delta(z(V^r)) \cdot \Delta(z(V^s))$$

if at least one of r and s is even. If r and s are odd, then we only need to check

$$\Delta\big(z(V^1) \cdot z(V^1)\big) = \Delta(z(V^1)) \cdot \Delta(z(V^1)).$$

Applying the same reasoning as before we know that this equation holds after multiplication with e^2, by using the corresponding relation for the coproduct on R. Hence this equation holds after passing to the coefficient ring $k/\operatorname{Ann}(e^2)$.

Next we check compatibility of Δ with the left A^*-action on T. Because this action is by ring maps, it is enough to check

- $\Delta(z(\eta)) = (l_\eta \otimes \mathrm{id})(\Delta(z(\epsilon))) = (\mathrm{id} \otimes l_\eta)(\Delta(z(\epsilon)))$,
- $\Delta(z(\epsilon)) = (l_\eta \otimes \mathrm{id})(\Delta(z(\eta))) = (\mathrm{id} \otimes l_\eta)(\Delta(z(\eta)))$,
- $\Delta(z(\epsilon)) = (l_\eta \otimes l_\eta)(\Delta(z(\epsilon))$,
- $\Delta(z(\eta)) = (l_\eta \otimes l_\eta)(\Delta(z(\eta))$.

All of these equations are true after multiplication with e, hence we are fine if we work over the coefficient ring $k/\operatorname{Ann}(e^2)$. Finally we check co-associativity. Because Δ is multiplicative on T, if we work over the coefficient ring $k/\operatorname{Ann}(e^2)$, it is enough to consider the equations

$$(\Delta \otimes \mathrm{id}) \circ \Delta(z(\epsilon)) = (\mathrm{id} \otimes \Delta) \circ \Delta(z(\epsilon)),$$

and

$$(\Delta \otimes \mathrm{id}) \circ \Delta(z(V^2)) = (\mathrm{id} \otimes \Delta) \circ \Delta(z(V^2)).$$

The first equation holds after multiplication with e, and the second equation holds after multiplication with e^2. Hence both equations hold after passing to the coefficient ring $k/\operatorname{Ann}(e^2)$.

In summary, using the discussion of [1, Section 12], we have defined a $\mathbb{Z}/2$-equivariant formal group law $(k/\operatorname{Ann}(e^2), T, \Delta, \theta, z(\epsilon))$, where we write T instead of $T/\operatorname{Ann}(e^2)$ by a slight abuse of notation. The augmentation $\theta : T \to (k/\operatorname{Ann}(e^2))^{A^*}$ is given by the constant term in the expansion relative to the flag (V_1^r) (resp. relative to the flag (V_0^r)), at the representation ϵ (resp. at the representation η).

By definition this formal group law has Euler class equal to 1. Now we let $k' \subset k/\operatorname{Ann}(e^2)$ be the subring generated by the coefficients $e^{i+j-1}f_{i,j}^{(1)}$, $i, j \geq 0$, of the coproduct on T (regarded as elements in $k/\operatorname{Ann}(e^2)$) and define R' as the free topological k'-module with basis $(z(V^r))_{r\geq 0}$. If (k, R) is graded, then k' is indeed concentrated in degree 0. Regarding R' as a subset of T we note that the product, coproduct and A^*-action on T restrict to corresponding structures on R', compare [1, Theorem 16.1]. Also the augmentation θ restricts to an augmentation $\theta' : R' \to (k')^{A^*}$. Setting $y'(\epsilon) := z(\epsilon)$ this concludes the construction of $(k', R', \Delta', \theta', y'(\epsilon))$. The functoriality of this construction is clear.

The next result highlights an important example.

Proposition 6.3. *The normalized formal group law R' associated to $\mathbb{Z}/2$-equivariant unitary bordism $k = \mathrm{MU}_*^{\mathbb{Z}/2}$ is the universal $\mathbb{Z}/2$-equivariant formal group law with Euler class* 1.

Proof. Set $k = \mathrm{MU}_*^{\mathbb{Z}/2}$ and let R be the topological k-algebra of the associated $\mathbb{Z}/2$-equivariant formal group law. We work with the notation from [6], repeated in Section 4 above. By [6, Cor. 10] the annihilator ideal $\operatorname{Ann}(e^2) \subset k$ is generated by $(t_1 + 1)$, and in fact equal to the annihilator ideal of the multiplication with e. By Section 4 we can identify the distinguished generators a_{ij}, t_m and s_{nj} of k with certain coefficients of the coproduct $\Delta(y(\epsilon))$ in R developed with respect to suitable flags. By our definition of the coproduct on R' the ring k' is therefore the subring of $k/(t_1+1)$ generated by the elements

$$\overline{a}_{ij} := e^{i+j-1}a_{ij}\ ,\ \overline{s}_{nj} := e^{n+j-1}s_{nj}\ ,\ \overline{t}_m := e^{m-1}t_m,$$

with $i+j \geq 1$, $n+j \geq 1$, and $m \geq 1$. These elements are only subject to the relations

$$\overline{t}_m - \overline{s}_{0m} = \overline{t}_{m+1}\ ,\ \overline{s}_{nj} - \overline{a}_{nj} = \overline{s}_{n+1,j}$$

for all m, j, and n. This implies that k' is generated as a $\mathbb{Z}$-algebra by the elements $\overline{a}_{ij}$, $i+j \geq 2$, and $\overline{s}_{0m}$, $m \geq 1$, where the generators $\overline{a}_{ij}$ satisfy the same relations as in the non-equivariant Lazard ring and $\overline{s}_{0m}$ are free polynomial generators. Using Strickland's calculation of k we hence conclude, on the one hand, that the quotient map $k \mapsto k/(e-1)$ induces an isomorphism

$$k' \cong k/(e-1).$$

On the other hand we observe that $L_{\mathbb{Z}/2}/(e-1)$ is the underlying ring of the universal $\mathbb{Z}/2$-equivariant formal group law with Euler class equal to 1. But the classifying map

$$\lambda_{\mathbb{Z}/2} : L_{\mathbb{Z}/2} \to k$$

is surjective and elements in the kernel are Euler torsion, see [2]. It hence induces an isomorphism

$$L_{\mathbb{Z}/2}/(e-1) \cong k/(e-1).$$

This finishes the proof of Proposition 6.3. $\square$

We are now in a position to prove Theorem 5.2. Consider the classifying map

$$\lambda : L_{\mathbb{Z}/2} \to k := \mathrm{MU}_*^{\mathbb{Z}/2}$$

of the equivariant formal group law of $\mathbb{Z}/2$-equivariant unitary bordism and the section μ of this map constructed in Section 4. Applying the normalization functor we obtain induced maps $\lambda' : L'_{\mathbb{Z}/2} \to k'$ and $\mu' : k' \to L'_{\mathbb{Z}/2}$ satisfying $\lambda' \circ \mu' = \mathrm{id}$, by functoriality. Furthermore we have

$$\mu'(e^{i+j-1}\beta_{ij}) = e^{i+j-1}\gamma_{ij}$$

by the definition of β_{ij} and γ_{ij} in Section 5. Proposition 6.3 implies that the equivariant formal group law for $L'_{\mathbb{Z}/2}$ is classified by a ring map

$$\phi : k' \to L'_{\mathbb{Z}/2}.$$

Because both λ' and ϕ are classifying maps, they are inverse to each other, and because $\lambda' \circ \mu' = \mathrm{id}$ we in fact have $\phi = \mu'$. Since

$$\phi(e^{i+j-1}\beta_{ij}) = e^{i+j-1}(\gamma_{ij} + \rho_{ij})$$

this implies

$$e^{i+j-1}\rho_{ij} = 0 \in L'_{\mathbb{Z}/2}.$$

Finally, because $L'_{\mathbb{Z}/2} \subset L_{\mathbb{Z}/2}/\operatorname{Ann}(e^2)$, this implies the equation

$$e^{i+j+1}\rho_{ij} = 0$$

in $L_{\mathbb{Z}/2}$, and hence the assertion of Theorem 5.2.

7. Computation of a particular $\mathbb{Z}/2$-equivariant formal group law

Let $(L_{\mathbb{Z}/2}, R, \Delta, \theta, y(\epsilon))$ denote the universal $\mathbb{Z}/2$-equivariant formal group law. We consider the graded ideal $\overline{J} \subset L_{\mathbb{Z}/2}$ defined in Section 5, spanned by the homogeneous elements a_{ij}, σ_{nj} and τ_m for $i+j \geq 2$, $n+j \geq 2$, and $m \geq 2$. The resulting $\mathbb{Z}/2$-equivariant formal group law $(L_{\mathbb{Z}/2}/\overline{J}, R/(\overline{J}), \Delta)$ has the form

$$\Delta(y(V_1^1)) = y(V_1^1) \otimes 1 + 1 \otimes y(V_1^1) + \sum_{i,j \geq 0} \rho_{ij} \cdot y(V_1^i) \otimes y(V_1^j)$$

with respect to the alternating flag. By abuse of notation we here denote by $\rho_{ij} \in L_{\mathbb{Z}/2}/\overline{J}$ the images of the structure constants ρ_{ij} introduced in Section 5. In particular we have $\rho_{ij} = 0$ for $i + j \leq 1$ and the ρ_{ij} are all infinitely e-divisible and e-torsion, such that all products $\rho_{ij} \cdot \rho_{pq}$ are equal to 0. Also recall that $\rho_{ij} = \rho_{ji}$ for all $i, j \geq 0$.

The section $\mu_{\mathbb{Z}/2} : \mathrm{MU}_*^{\mathbb{Z}/2} \to L_{\mathbb{Z}/2}$ induces a section $\mathrm{MU}_*^{\mathbb{Z}/2}/J \to L_{\mathbb{Z}/2}/\overline{J}$ of the canonical map $\lambda : L_{\mathbb{Z}/2}/\overline{J} \to \mathrm{MU}_*^{\mathbb{Z}/2}/J$. We can hence consider $\mathrm{MU}_*^{\mathbb{Z}/2}/J = \mathbb{Z}[e]/(2e)$ as a subring of $L_{\mathbb{Z}/2}/\overline{J}$. In particular $2e = 0$ in $L_{\mathbb{Z}/2}/\overline{J}$ and therefore

$$2 \cdot \rho_{ij} = 0$$

for all $i, j \geq 1$, by the e-divisibility of ρ_{ij}. This will simplify the following computations considerably.

Notice that the kernel of $\lambda : L_{\mathbb{Z}/2}/\overline{J} \to \mathrm{MU}_*^{\mathbb{Z}/2}/J$ is generated by the structure constants ρ_{ij}. In the remainder of this section we will show that all $\rho_{ij} = 0 \in L_{\mathbb{Z}/2}/\overline{J}$. This assertion implies Theorem 5.3.

The structure constants $\rho_{ij} \in L_{\mathbb{Z}/2}/\overline{J}$ underly the following restrictions:

Restriction (1): The comultiplication Δ is co-associative.
Restriction (2): The elements $\sigma_{n,j} \in L_{\mathbb{Z}/2}$, $n+j \geq 2$, and $\tau_m \in L_{\mathbb{Z}/2}$, $m \geq 2$, map to 0 in the quotient ring $L_{\mathbb{Z}/2}/\overline{J}$.

At first we will explore Restriction (1), which results in Proposition 7.1. Then, in Propositions 7.3, 7.4 and 7.5 we will derive implications from Restriction (2), making use of the coordinate change formulas from Section 3. After these preparations the proof of $\rho_{ij} = 0$ will be completed at the end of this section.

As a shorthand we use the notation $z^r := y(V_1^r) \in R/(\overline{J})$, $r \geq 0$, for the basis elements corresponding to the alternating flag. We warn the reader that in the ring $R/(\overline{J})$ we cannot assume the relation $z^{r_1} \cdot z^{r_2} = z^{r_1+r_2}$ for $r_1, r_2 \geq 1$.

The $\mathbb{Z}/2$-equivariant additive formal group law $(k_a, R_a, \Delta_a, \theta_a, y_a(\epsilon))$ with representing ring $k_a = \mathbb{Z}[e]/(2e)$ defines structure constants $f_{ij}^{(r)} \in \mathbb{Z}[2]/(2e)$ for $r \geq 0$ by the equation

$$\Delta_a(y_a(V_1^r)) = \sum_{i,j \geq 0} f_{ij}^{(r)} y_a(V_1^i) \otimes y_a(V_1^j).$$

We now define $\rho_{ij}^{(r)} \in L_{\mathbb{Z}/2}/\overline{J}$ by the equation

$$\Delta(z^r) = \sum_{i,j \geq 0} (f_{ij}^{(r)} + \rho_{ij}^{(r)}) \cdot z^i \otimes z^j \,.$$

In particular $\rho_{i,j}^{(0)} = 0$, $\rho_{i,j}^{(1)} = \rho_{ij}$ for all i,j, and all $\rho_{ij}^{(r)}$ are in the kernel of $\lambda : L_{\mathbb{Z}/2}/\overline{J} \to \mathrm{MU}_*^{\mathbb{Z}/2}/J$ and hence lie in the ideal spanned by ρ_{pq}, $p,q \geq 0$. In particular $2 \cdot \rho_{ij}^{(r)} = 0$ and $\rho_{ij}^{(r)} \cdot \rho_{pq}^{(s)} = 0$ for all i,j,r and p,q,s.

We obtain

$$\begin{aligned}(\Delta \otimes \mathrm{Id}) \circ \Delta(z) &= (\Delta \otimes \mathrm{Id})(\sum_{j,k} (f_{j,k}^{(1)} + \rho_{j,k}^{(1)}) \cdot z^j \otimes z^k) \\ &= \sum_{l,m,j,k} (f_{l,m}^{(j)} + \rho_{l,m}^{(j)})(f_{j,k}^{(1)} + \rho_{j,k}^{(1)}) \cdot z^l \otimes z^m \otimes z^k \\ &= \sum_{l,m,j,k} (f_{l,m}^{(j)} f_{j,k}^{(1)} + \rho_{l,m}^{(j)} f_{j,k}^{(1)} + f_{l,m}^{(j)} \rho_{j,k}^{(1)}) \cdot z^l \otimes z^m \otimes z^k.\end{aligned}$$

Here we use the vanishing of products of ρ's. Similarly we have

$$(\mathrm{Id} \otimes \Delta) \circ \Delta(z) = \sum_{l,m,j,k} (f_{k,m}^{(j)} f_{j,l}^{(1)} + \rho_{k,m}^{(j)} f_{j,l}^{(1)} + f_{k,m}^{(j)} \rho_{j,l}^{(1)}) \cdot z^l \otimes z^m \otimes z^k.$$

Taking into account that the comultiplication Δ_a is co-associative, these calculations together with the co-associativity of Δ imply that for all $k, l, m \geq 0$ we have

$$\sum_{\nu \geq 0} (\rho^{(\nu)}_{l,m} \cdot f^{(1)}_{\nu,k} + f^{(\nu)}_{l,m} \cdot \rho^{(1)}_{\nu,k}) = \sum_{\nu \geq 0} (\rho^{(\nu)}_{k,m} \cdot f^{(1)}_{\nu,l} + f^{(\nu)}_{k,m} \cdot \rho^{(1)}_{\nu,l}) \,. \tag{7.1}$$

Observe that this is a finite sum on each side, because the comultiplication Δ is continuous.

Since the equation $2 \cdot \rho^{(r)}_{ij} = 0$ holds for all i, j, r, we need to compute the images of the elements $f^{(r)}_{pq}$ in $\mathbb{Z}/2[e]$ when evaluating Equation (7.1). In the following we use the shorthand notation $x^r := y_a(V_1^r)$, $x := x^1$. In particular $x^2 = x \cdot (x + e) = x^2 + ex$, $x^{2n} = (x^2 + ex)^n$ and $x^{2n+1} = x \cdot (x^2 + ex)^n$ for all $n \geq 0$. We have $\Delta_a(x) = x \otimes 1 + 1 \otimes x$ and

$$\begin{aligned} \Delta_a(x^2) &= \Delta_a(x) \cdot \Delta_a(x + e) = (x \otimes 1 + 1 \otimes x) \cdot (x \otimes 1 + 1 \otimes x + e \cdot (1 \otimes 1)) \\ &= x^2 \otimes 1 + 1 \otimes x^2 \,, \end{aligned}$$

after passing to the representing ring $\mathbb{Z}/2[e]$. Hence (for even and odd r) we obtain

$$\Delta_a(x^r) = (x \otimes 1 + 1 \otimes x)^r = \sum_{s=0}^{r} \binom{r}{s} x^s \otimes x^{r-s} \,.$$

This happens to be the same formula as for the additive non-equivariant formal group law with coordinate x. In $\mathbb{Z}/2[e]$ we hence obtain the equality

$$f^{(r)}_{p,q} = \begin{cases} \binom{p+q}{p} & \text{if } r = p + q, \\ 0 & \text{else.} \end{cases}$$

Recall that $\binom{p+q}{p}$ is equal to 0 modulo 2, if and only if in the binary expansions of p and q the digit 1 occurs at the same position, or, in other words, if the binary addition of p and q involves carryovers.

We arrive at the following conclusion resulting from Restriction (1).

Proposition 7.1. *$L_{\mathbb{Z}/2}/\overline{J}$ is generated over $\mathbb{Z}[e]/(2e)$ by the elements ρ_{ij}, $i + j \geq 2$, and these elements satisfy the following relations.*

a) *$\rho_{ij} \cdot \rho_{p,q} = 0$ and $2\rho_{ij} = 0$.*
b) *If $i, j \geq 1$, if either i or j is not a power of 2, and if the binary addition of i and j involves carryovers, then*

$$\rho_{ij} = 0 \in L_{\mathbb{Z}/2}/\overline{J}.$$

c) *If neither $i, j \geq 1$ nor $p, q \geq 1$ fall in the case b), and if $i + j = p + q$, then*

$$\rho_{ij} = \rho_{pq}.$$

Proof. It remains to deal with parts b) and c). Both of them follow from Equation (7.1), where we observe that the elements $\rho^{(\nu)}_{l,m}$ and $\rho^{(\nu)}_{k,m}$ can only occur with a factor 1, if $\nu = 1$, by our previous computation of $f^{(1)}_{p,q}$ as elements in $\mathbb{Z}/2[e]$.

For part b) we write

$$i = \sum_{\ell \geq 0} w_{i\ell} \cdot 2^{\ell} \text{ and } j = \sum_{\ell \geq 0} w_{j\ell} \cdot 2^{\ell}$$

with $w_{i\ell}, w_{j\ell} \in \{0, 1\}$ where i, say, is not a power of 2. We choose ℓ_1 with $w_{i\ell_1} = w_{j\ell_1} = 1$. Then the assertion follows from Equation (7.1) with $k = j$, $l = i - 2^{\ell_1}$ and $m = 2^{\ell_1}$.

Now we turn to part c). If i and j are both powers of two, then under the given assumptions the same must hold for p and q. We obtain $p = i$, $p = j$ or $p = j, q = i$, and claim c) follows from the commutativity of the formal group law Δ. It therefore remains to deal with the case that i, say, is not a power of two. Let us write

$$i = \sum_{\ell \geq 0} w_{i\ell} \cdot 2^{\ell} \text{ and } j = \sum_{\ell \geq 0} w_{j\ell} \cdot 2^{\ell}$$

with $w_{i\ell}, w_{j\ell} \in \{0, 1\}$ and $w_{i\ell} \cdot w_{j\ell} = 0$ for all ℓ. Choose ℓ_1 with $w_{i\ell_1} = 1$. Then it follows from Equation (7.1) with $k = i - 2^{\ell_1}$, $m = 2^{\ell_1}$, $l = j$, that

$$\rho_{ij} = \rho_{i-2^{\ell_1}, j+2^{\ell_1}} \,.$$

In other words, we can shift the binary digit 1 at position ℓ_1 from the left to the right hand subscript of ρ. From this claim c) in the proposition follows. □

For exploring Restriction (2) we need to work with different flags. Let us write, for $n, m \geq 1$,

$$\Delta(y(V^1_1)) = y(V^1_n) \otimes 1 + 1 \otimes y(V^1_m) + \sum_{i,j \geq 0} \rho^{n,m}_{i,j} \cdot y(V^i_n) \otimes y(V^j_m)\,.$$

Note that $y(V^1_1) = y(V^1_n) = y(V^1_m) = z^1$ by our assumption $n, m \geq 1$. By the co-unitality of Δ we therefore have $\rho^{n,m}_{i,0} = \rho^{n,m}_{0,j} = 0$ for all $i, j \geq 0$, and, using the notation introduced in Equation (3.1), we have $\rho^{n,m}_{i,j} = \beta^{n,m}_{i,j}$ for $i + j \geq 2$ (notice that $\rho^{n,m}_{1,0} = \rho^{n,m}_{0,1} = 0$, whereas $\beta^{n,m}_{1,0} = \beta^{n,m}_{0,1} = 1$ for $n, m \geq 1$). Also note that $\rho^{1,1}_{ij} = \rho_{ij}$ for $i, j \geq 0$, and all $\rho^{n,m}_{i,j}$ are in the kernel of the map $L_{\mathbb{Z}/2}/\overline{J} \to \mathrm{MU}^{\mathbb{Z}/2}/J$, and hence lie in the ideal generated by the elements ρ_{pq}. In particular all $\rho^{n,m}_{i,j}$ are 2-torsion and arbitrary products of such elements vanish.

We wish to apply the coordinate change formula in Lemma 3.4. This was originally stated for tame group laws. However, for the group law considered in this section the coefficients d_i appearing in the base change formula preceding Lemma 3.1 are equal to those of a tame equivariant law, modulo elements in the kernel of $\lambda : L_{\mathbb{Z}/2}/\overline{J} \to \mathrm{MU}_*^{\mathbb{Z}/2}/J$, which is equal to the square zero ideal generated by the elements ρ_{pq}. Hence Lemmas 3.4 and 3.5 remain valid in our case of the (potentially) non-tame group law Δ, if we apply it to coordinates of the form $\gamma_i^{n+1} := \rho_{i,j}^{n+1,m}$ or $\gamma_j^{m+1} := \rho_{i,j}^{n,m+1}$, where $n, m \geq 1$ and $i, j \geq 1$. Hence, for all $n, m, j \geq 1$ we have equations

$$\rho_{n,j}^{n,m} = \sum_{\ell=0}^{n-1} y_{n,\ell} \cdot e^{\ell} \cdot \rho_{n+\ell,j}^{1,m}\,, \tag{7.2}$$

where $y_{n,\ell} \in \mathbb{Z}/2$, and for all $n, m, i \geq 1$ we have equations

$$\rho_{i,m}^{n,m+1} = \sum_{\nu=0}^{m} x_{m,\nu} \cdot e^{\nu} \cdot \rho_{i,m+\nu}^{n,1}, \tag{7.3}$$

where each $x_{m,\nu} \in \mathbb{Z}/2$ is equal to the coefficient $x_{m,m,\nu}$ from Lemma 3.5. Notice that

$$y_{n,0} = x_{m,0} = 1$$

for all $n, m \geq 1$, by the recursive formula in Lemma 3.4. Let us compute the coefficients $x_{m,\nu}$ in some more cases.

Lemma 7.2. *For all $q \geq 0$ we have*

$$x_{2^q,2^q} = 1.$$

Furthermore, if $q \geq 2$, then for all $0 < \omega < 2^{q-1}$ we have

$$x_{2^q-\omega,\omega} = 0.$$

Note that $x_{2^q-\omega,\omega}$ is the coefficient appearing in front of $e^{\omega} \cdot \rho_{i,2^q}^{n,1}$, if we develop $\rho_{i,2^q-\omega}^{n,2^q-\omega+1}$ according to Equation (7.3). The relation $x_{2^q-\omega,\omega} = 0$ will be a crucial ingredient for proving Proposition 7.3 below.

Proof of Lemma 7.2. By Lemma 3.5 and the discussion preceding Lemma 7.2 we have

$$x_{m,\omega} = x_{m,m,\omega} = \binom{\omega + [(m-\omega)/2]}{\omega} \mod 2$$

if $m + \omega$ is even. Evaluating this formula for $m = \omega = 2^q$ shows the first assertion. We now assume $q \geq 2$, $0 < \omega < 2^{q-1}$ and set $m = 2^q - \omega$. Then $m + \omega$ is even because $q \geq 2$ and we obtain

$$x_{m,\omega} = \binom{\omega + [(2^q - 2\omega)/2]}{\omega} = \binom{2^{q-1}}{\omega} \mod 2.$$

This vanishes because $0 < \omega < 2^{q-1}$ by assumption. Hence the lemma is proven. □

We can now explore Restriction (2), saying $[\sigma_{n,j}] = [\tau_m] = 0 \in L_{\mathbb{Z}/2}/\overline{J}$ for $n + j \geq 2$ and $m \geq 2$. Let us start with the relation $[\sigma_{n,j}] = 0$.

Proposition 7.3. *Let $p \geq 1$. If $1 < j < 2^p$ is not a power of 2, then we have*

$$\rho^{1,1}_{2^p,j} = 0.$$

Proof. Let $1 < j < 2^{p-1}$ be not a power of 2 and assume inductively that we have proven $\rho^{1,1}_{2^p,j'} = 0$ for all $j < j' < 2^p$ where j' is not a power of 2 (this condition is empty for $j = 2^p - 1$). Choose $1 \leq q \leq p$ minimal with $2^q > j$ and write $j = 2^q - \omega$ where $0 < \omega < 2^{q-1}$. Using Equations (7.2) and (7.3) we obtain

$$0 = [\sigma_{2^p,2^q-\omega}] = \rho^{2^p,2^q-\omega+1}_{2^p,2^q-\omega} = \sum_{\ell=0}^{2^p-1} \sum_{\nu=0}^{2^q-\omega} y_{2^p,\ell} \cdot x_{2^q-\omega,\nu} \cdot e^{\ell+\nu} \rho^{1,1}_{2^p+\ell,2^q-\omega+\nu} \,. \tag{7.4}$$

By part b) of Proposition 7.1 $\rho^{1,1}_{2^p+\ell,2^q-\omega+\nu} \neq 0$ can only occur in one of the following cases:

i) $2^p + \ell$ and $2^q - \omega + \nu$ are both powers of 2.
ii) The binary addition of $2^p + \ell$ and $2^q - \omega + \nu$ does not involve carryovers.

Case i) is equivalent to $\ell = 0$ and $\nu - \omega = 0$, and the corresponding summand on the right hand side of Equation (7.4) is equal to $x_{2^q-\omega,\omega} \cdot e^\omega \cdot \rho^{1,1}_{2^p,2^q}$ (recall that $y_{2^p,0} = 1$). By Lemma 7.2 we have $x_{2^q-\omega,\omega} = 0$ and hence this summand vanishes.

Let us now assume that we are in case ii), but not in case i). We claim that $2^q - \omega + \nu + \ell < 2^p$. In a first step we prove $2^q - \omega + \nu < 2^p$. Here we notice $2^q - \omega + \nu < 2^{p+1}$, because $q \leq p$ and $\nu - \omega < 2^q$. Hence the assumption $2^q - \omega + \nu \geq 2^p$ together with $0 \leq \ell < 2^p$ implies that in the binary expansions of both $2^p + \ell$ and $2^q - \omega + \nu$ the digit 1 occurs at position p (corresponding to 2^p), contradicting the assumption of case ii). Because $\ell \leq 2^p - 1$ and the binary addition of $2^p + \ell$ and $2^q - \omega + \nu$ does not involve carryovers, the inequality $2^q - \omega + \nu < 2^p$ in turn implies $2^q - \omega + \nu + \ell < 2^p$, as claimed before.

Part c) of Proposition 7.1 now implies

$$\rho^{1,1}_{2^p+\ell,2^q-\omega+\nu} = \rho^{1,1}_{2^p,2^q-\omega+\nu+\ell} \,.$$

Since $2^q - \omega + \nu + \ell$ is not a power of 2 (by the assumption of case ii) and since we are not in case i)) and smaller than 2^p (as shown before), the last expression vanishes by our induction assumption, if either $\ell > 0$ or $\nu > 0$.

In summary Equation (7.4) simplifies to $0 = \rho^{1,1}_{2^p,2^q-\omega}$, finishing the induction step. □

Proposition 7.4. *If $i, j \geq 1$ and either i or j is not a power of 2, then*

$$\rho^{1,1}_{i,j} = 0.$$

If i and j are both powers of 2, then we have the relations

$$\rho^{1,1}_{i,j} + e^i \rho^{1,1}_{2i,j} = 0 \text{ and } \rho^{1,1}_{i,j} + e^j \rho^{1,1}_{i,2j} = 0.$$

Proof. The first assertion follows from Proposition 7.3 and the parts (b) and (c) of Proposition 7.1. Using the first assertion and Equations (7.2) and (7.3) we have

$$0 = [\sigma_{2^p,2^q}] = \sum_{\ell=0}^{2^p-1} \sum_{\nu=0}^{2^q} y_{2^p,\ell} \cdot x_{2^q,\nu} \cdot e^{\ell+\nu} \rho^{1,1}_{2^p+\ell,2^q+\nu} = \rho^{1,1}_{2^p,2^q} + e^{2^q} \rho^{1,1}_{2^p,2^{q+1}}$$

for all $p, q \geq 0$, where we use $y_{2^p,0} = x_{2^q,0} = x_{2^q,2^q} = 1$, the last equation by Lemma 7.2. Hence we have

$$\rho^{1,1}_{i,j} + e^j \rho^{1,1}_{i,2j} = 0$$

if i and j are powers of 2. The remaining claim follows by interchanging i and j. □

Finally we get the following uniform Euler torsion estimate. Here we use the relation $[\tau_m] = 0$ for $m \geq 2$.

Proposition 7.5. *We have*

$$e \cdot \rho^{1,1}_{1,j} = 0$$

for all $j \geq 2$.

Proof. The assertion follows from Proposition 7.4, if j is not a power of 2. It hence remains to handle the case when $j \geq 2$ is a power of 2.

First we need some preparation. Write the coproduct $\Delta(z) = \Delta(y(\epsilon))$ as in Equation (3.1) in the form

$$\Delta(y(\epsilon)) = \sum_{i,j \geq 0} \beta^{0,m}_{i,j} \cdot y(V^i_0) \otimes y(V^j_m),$$

where we henceforth assume $m \geq 1$. For all $j \geq 2$ we then have (recalling $\beta^{1,m}_{i,j} = \rho^{1,m}_{i,j}$ for $i + j \geq 2$)

$$\sum_{i\geq 0} \beta^{0,m}_{i,j} \cdot y(V_0^i) \otimes y(V_m^j) = \sum_{i\geq 0} \beta^{1,m}_{i,j} \cdot y(V_1^i) \otimes y(V_m^j) = \sum_{i\geq 0} \rho^{1,m}_{i,j} \cdot y(V_1^i) \otimes y(V_m^j) . \quad (7.5)$$

According to the base change formula preceding Lemma 3.1 we have

$$\rho^{1,m}_{i,j} \cdot y(V_1^i) \otimes y(V_m^j) = \begin{cases} \rho^{1,m}_{i,j} \cdot y(V_0^i) \otimes y(V_m^j) \text{ for even } i \\ \rho^{1,m}_{i,j} \cdot \big(e \cdot y(V_0^{i-1}) \otimes y(V_m^j) + y(V_0^i) \otimes y(V_m^j)\big) \text{ for odd } i \,, \end{cases}$$

again using the fact that modulo the ideal generated by the elements ρ_{pq} we have $d_0 = e$, $d_1 = 1$ and $d_i = 0$ for $i > 1$. Comparing coefficients of the left and right hand side in Equation (7.5) we obtain

$$\beta^{0,m}_{0,j} = \rho^{1,m}_{0,j} + e\rho^{1,m}_{1,j}$$

for all $j \geq 2$. We have $\rho^{1,m}_{0,j} = 0$ for $j \geq 0$ by the co-unitality of the coproduct Δ, hence the last equation implies

$$\beta^{0,m}_{0,j} = e\rho^{1,m}_{1,j}$$

for all $j \geq 2$.

After these preparations let $j \geq 2$ be a power of 2. Since $e\rho^{1,1}_{1,j'} = 0$, if $j' \geq 3$ is not a power of 2 (by Proposition 7.4), Equation (7.2) shows

$$\beta^{0,j}_{0,j} = e \cdot \rho^{1,j}_{1,j} = e \cdot \sum_{\ell=0}^{j-1} y_{j,\ell} \cdot e^{\ell} \cdot \rho^{1,1}_{1,j+\ell} = e \cdot \rho^{1,1}_{1,j} \,,$$

where the first equation follows from the preceding remarks. We therefore get

$$0 = [\tau_j] = \beta^{0,j}_{0,j} = e \cdot \rho^{1,1}_{1,j}$$

as required. This finishes the proof of Proposition 7.5. □

Now let $i = 2^p$, $j = 2^q$, $p, q \geq 0$, where we assume $p \leq q$ without loss of generality. Applying Proposition 7.4 several times and using Proposition 7.5 we get

$$\rho^{1,1}_{i,j} = e^j \rho^{1,1}_{2^p,2j} = e^{j-2^{p-1}} \rho^{1,1}_{2^{p-1},2j} = \cdots = e^{j-(2^p-1)} \rho^{1,1}_{1,2j} = 0.$$

This finishes the proof of Theorem 5.3.

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