

# OPTIMAL LIPSCHITZ CRITERIA AND LOCAL ESTIMATES FOR NON-UNIFORMLY ELLIPTIC PROBLEMS

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To Vladimir Gilelevich Maz'ya,  
master of analysis across two different worlds and centuries,  
on his 80th birthday

**ABSTRACT.** We report on new techniques and results in the regularity theory of general non-uniformly elliptic variational integrals. By means of a new potential theoretic approach we reproduce, in the non-uniformly elliptic setting, the optimal criteria for Lipschitz continuity known in the uniformly elliptic one and provide a unified approach between non-uniformly and uniformly elliptic problems.

## 1. INTRODUCTION AND RESULTS

Non-uniformly elliptic equations and functionals are a classical topic in partial differential equations and in the Calculus of Variations. They emerge in many different contexts, and they are often stemming from geometric and physical problems [22, 31, 44, 49, 50]. The study of regularity of their solutions involves a wealth of methods and techniques. Here we announce a few results from [5], concerning the regularity of minimizers of non-uniformly elliptic variational integrals of the type

$$(1) \quad W^{1,1}(\Omega; \mathbb{R}^N) \ni w \mapsto \mathcal{F}(w; \Omega) := \int_{\Omega} [F(Dw) - f \cdot w] \, dx ,$$

where  $\Omega \subset \mathbb{R}^n$  is an open subset,  $n \geq 2$ ,  $N \geq 1$ , and the integrand  $F: \mathbb{R}^{N \times n} \rightarrow \mathbb{R}$  satisfies suitable convexity and growth conditions.

The aim of [5] is twofold, specifically:

- To identify sharp conditions on the datum  $f$  in order to guarantee that minimizers are locally Lipschitz continuous.
- To introduce a new potential theoretic technique allowing to reduce the treatment of non-uniformly elliptic problems to the one of uniformly elliptic ones. This technique yields optimal and new local estimates already in the case  $f \equiv 0$ .

Let us immediately clarify the meaning or the terminology concerning the non-uniform ellipticity. The Euler–Lagrange equation of the functional in (1) formally reads as

$$-\operatorname{div} \partial F(Du) = f .$$

Ellipticity is then tested by looking at the so-called second variation equation, and therefore at the ellipticity ratio

$$(2) \quad \mathcal{R}(z) := \frac{\text{highest eigenvalue of } \partial^2 F(z)}{\text{lowest eigenvalue of } \partial^2 F(z)} .$$

The term non-uniform ellipticity refers to the situation when  $\mathcal{R}(z)$  is not bounded. This is the case of interest here and therefore is not present in more classical situations as for instance the one of the

$p$ -Laplacean operator  $F(z) =: |z|^p/p$ , for  $p > 1$  (see [35, 47, 48] for basic regularity results). More in general, when considering functionals of the type

$$(3) \quad w \mapsto \int_{\Omega} [A(|Dw|) - f \cdot w] \, dx, \quad \text{where } A(t) := \int_0^t \tilde{a}(s)s \, ds,$$

then the non-uniform ellipticity is connected to the absence of the double-side bound

$$(4) \quad -1 < i_a \leq \frac{\tilde{a}'(t)t}{\tilde{a}(t)} \leq s_a < \infty$$

(considered e.g. in [1, 11, 12, 32]). This implies the validity of the so-called  $\Delta_2$ -condition

$$(5) \quad A(2t) \lesssim A(t),$$

and therefore immediately allows for polynomial controls for  $t \mapsto A(t)$ . In the classical uniformly elliptic case (3)-(4), the question of determining sharp conditions on  $f$  implying the local and global Lipschitz continuity of minimizers has found a satisfying solution only in the last few years. Results are available both for the scalar and for the vectorial case, see e.g. [12, 13, 23, 27, 28, 30, 42]. The most general answer can be obtained in terms of Riesz potentials, and we refer to [29] for an overview of nonlinear potential estimates. In terms of function spaces, a neat and sharp condition on  $f$  makes use of Lorentz spaces  $L(n, 1)(\Omega)$ , that is

$$(6) \quad f \in L(n, 1)(\Omega) : \Longleftrightarrow \|f\|_{L(n, 1)(\Omega)} := \int_0^\infty |\{x \in \Omega : |f(x)| > \lambda\}|^{1/n} \, d\lambda < \infty.$$

The remarkable feature is that condition (6) does not depend on the functional under consideration. In particular, when considering the  $p$ -Laplacean equation  $-\operatorname{div}(|Du|^{p-2}Du) = f$ , i.e., for  $F(z) = |z|^p/p$  in (1), the condition to ensure the local Lipschitz continuity of minimizers is in fact independent of  $p$ , as first observed in [23]. In particular, condition (6) is sharp for the Poisson equation  $-\Delta u = f$ , as a consequence of a well-known theorem of Stein [46] and the work of Cianchi [9]. Notice that the Lorentz space  $L(n, 1)$  is an intermediate, borderline space in the sense that  $L^{n+\varepsilon} \subset L(n, 1) \subset L^n$  holds for every  $\varepsilon > 0$  (and all such inclusions are actually strict). Concerning the non-uniformly elliptic case, essentially nothing is known.

A main outcome of [5] is that the effectiveness of condition (6) is extended to the case of non-uniformly elliptic problems, thereby filling a huge gap in the literature. We also notice that, when looking at non-uniformly elliptic functionals, the results in [5] are new already in the case when  $f$  is more regular than in (6). The usual cases treated in the literature are indeed  $f \equiv 0$  or  $f \in L^\infty$ , where the latter one was only considered under polynomial growth conditions (see [36]). As we shall see in Section 4 below, the techniques of [5] yield new results in the classical uniformly elliptic case with (3)-(4) as well. The class of functionals treated in [5] is large, and essentially covers all the main models appearing in the literature as far as autonomous functionals are considered. It includes functionals with polynomial, yet unbalanced growth conditions, i.e., when  $F$  satisfies different growth and coercivity conditions such as

$$(7) \quad |z|^p \lesssim F(z) \lesssim |z|^q + 1, \quad 1 < p \leq q.$$

These are usually called  $(p, q)$ -growth conditions, according to Marcellini's terminology from [36]. We also cover the case of functionals with very fast growth as for instance

$$(8) \quad w \mapsto \int_{\Omega} \left[ \exp(\exp(\dots \exp(|Dw|^p) \dots)) - f \cdot w \right] \, dx, \quad p \geq 1,$$

and for which the  $\Delta_2$ -condition (5) fails.

In the following, we report a few results from [5], essentially referring to the two main model situations (7) and (8). These are in turn consequences of a more general result still from [5] whose formulation turns out to be rather technical as it is devised to cover large and different classes of model cases. For a reasonably updated review of the existing regularity literature on the subject we refer to [41].

For the rest of the paper, in view of condition (6), we assume that  $f \in L^n(\Omega; \mathbb{R}^N)$ . Moreover, we recall that a function  $u \in W_{\text{loc}}^{1,1}(\Omega; \mathbb{R}^N)$  is a *local minimizer* of the functional  $\mathcal{F}$  in (1) with  $f \in L_{\text{loc}}^n(\Omega; \mathbb{R}^N)$  if, for every open subset  $\tilde{\Omega} \Subset \Omega$ , we have that  $\mathcal{F}(u; \tilde{\Omega}) < \infty$  and moreover  $\mathcal{F}(u; \tilde{\Omega}) \leq \mathcal{F}(w; \tilde{\Omega})$  holds whenever  $w \in u + W_0^{1,1}(\tilde{\Omega}; \mathbb{R}^N)$ . As  $f \in L_{\text{loc}}^n(\Omega; \mathbb{R}^N)$ , Sobolev embedding theorem implies  $f \cdot u \in L_{\text{loc}}^1(\Omega)$  so that  $F(Du) \in L_{\text{loc}}^1(\Omega)$ . In the following we denote by  $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$  the open ball with center  $x_0$  and radius  $r > 0$ ; the center will be omitted ( $B_r \equiv B_r(x_0)$ ) when clear from the context. If not otherwise stated, different balls in the same context will share the same center. With  $\mathcal{B} \subset \mathbb{R}^n$  being a measurable subset with bounded positive measure  $0 < |\mathcal{B}| < \infty$ , and with  $g : \mathcal{B} \rightarrow \mathbb{R}^k$ ,  $k \geq 1$ , being a measurable map, we shall denote its integral average by

$$(g)_{\mathcal{B}} \equiv \oint_{\mathcal{B}} g(x) dx := \frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} g(x) dx .$$

## 2. A MODEL RESULT FOR POLYNOMIAL GROWTH CONDITIONS

In this section we give a model result covering the case of functionals under  $(p, q)$ -growth conditions as in (7). They have been studied in detail in the literature and, in the setting of the Calculus of Variations, an extensive treatment has been provided by Marcellini [36]; see also [45] for the vectorial case. In this setting, the integrand  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is convex function and locally  $C^2$ -regular in  $\mathbb{R}^n \setminus \{0\}$ . The crucial growth and ellipticity properties of  $F$  are described as follows:

$$(9) \quad \begin{cases} \nu(|z|^2 + \mu^2)^{p/2} \leq F(z) \leq L(|z|^2 + \mu^2)^{q/2} + L(|z|^2 + \mu^2)^{p/2} \\ (|z|^2 + \mu^2)|\partial^2 F(z)| \leq L(|z|^2 + \mu^2)^{q/2} + L(|z|^2 + \mu^2)^{p/2} \\ \nu(|z|^2 + \mu^2)^{(p-2)/2}|\xi|^2 \leq \langle \partial^2 F(z)\xi, \xi \rangle , \end{cases}$$

for every choice of  $z, \xi \in \mathbb{R}^n$  with  $z \neq 0$  and for exponents  $1 \leq p \leq q$ . Here  $0 < \nu \leq 1 \leq L$  are fixed ellipticity constants and  $\mu \in [0, 1]$  serves to distinguish the so-called degenerate case ( $\mu = 0$ ) and non-degenerate case ( $\mu > 0$ ). In this case the ellipticity ratio in (2) is controlled as follows:

$$(10) \quad \mathcal{R}(z) \lesssim (|z|^2 + \mu^2)^{(q-p)/2} + 1 .$$

It therefore becomes unbounded as  $|z| \rightarrow \infty$  for the gradient variable, with a speed which is proportional to the so-called gap  $q - p$ . The main result for  $(p, q)$ -growth functionals is the following

**Theorem 2.1** (Scalar  $(p, q)$ -estimates). *Let  $u \in W_{\text{loc}}^{1,1}(\Omega)$  be a local minimizer of the functional  $\mathcal{F}$  in (1) under assumptions (9) with  $1 < p \leq q$  and  $n > 2$ . Assume*

$$(11) \quad \frac{q}{p} < 1 + \min \left\{ \frac{2}{n}, \frac{4(p-1)}{p(n-2)} \right\} \quad \text{and} \quad f \in L(n, 1)(\Omega) .$$

*Then  $Du$  is locally bounded in  $\Omega$ . Moreover, the local a priori estimate*

$$(12) \quad \|Du\|_{L^\infty(B/2)} \leq c \left( \oint_B F(Du) dx + \|f\|_{L(n,1)(B)}^{\frac{p}{p-1}} \right)^{\frac{1}{p}} \\ + c \left( \oint_B F(Du) dx + \|f\|_{L(n,1)(B)}^{\frac{p}{p-1}} \right)^{\frac{2}{(n+2)p-nq}} + c [\|f\|_{L(n,1)(B)}]^{\frac{4}{4(p-1)-(n-2)(q-p)}}$$

*holds for a constant  $c \equiv c(n, p, q, \nu, L)$ , whenever  $B \Subset \Omega$  is a ball. When  $p \geq 2 - 4/(n+2)$  or when  $f \equiv 0$ , condition (11) can be replaced by*

$$(13) \quad \frac{q}{p} < 1 + \frac{2}{n} .$$

The above result is valid for scalar minimizers. When passing to the vectorial case, singularities may in general emerge, cf. [41]. Minimizers are regular outside so-called singular sets whose dimension can be estimated see e.g. [26, 45]; so called partial regularity theory comes into the play.

However, assuming a suitable, quasidiagonal structure, one is able to recover everywhere regularity together with the scalar results. This is known since the fundamental work of Uhlenbeck [47]; as for the non-homogeneous case, we refer to the recent paper [30] that also contains potential estimates. In our situation we indeed have

**Theorem 2.2** (Vectorial  $(p, q)$ -estimates). *Let  $u \in W_{\text{loc}}^{1,1}(\Omega; \mathbb{R}^N)$  be a local minimizer of the functional  $\mathcal{F}$  in (1) under assumptions (9) with  $1 < p \leq q$  and  $n > 2$ . Assume*

$$(14) \quad \frac{q}{p} < 1 + \min \left\{ \frac{1}{n}, \frac{2(p-1)}{p(n-2)} \right\} \quad \text{and} \quad f \in L(n, 1)(\Omega),$$

*and that there exists a  $C_{\text{loc}}^1[0, \infty) \cap C_{\text{loc}}^2(0, \infty)$ -regular function  $\tilde{F}: [0, \infty) \rightarrow [0, \infty)$  such that  $F(z) = \tilde{F}(|z|)$  for every  $z \in \mathbb{R}^{N \times n}$ . Finally, assume that the function  $t \mapsto (\mu^2 + t^2)^{(2-\gamma)/2} \tilde{F}'(t)/t$  is non-decreasing for some  $\gamma > 1$ ; in particular,  $\gamma = p$  is admissible. Then  $Du$  is locally bounded in  $\Omega$  and an estimate similar to (12) holds. In the case that  $t \mapsto \tilde{F}'(t)/t$  is non-decreasing, assumption (14) can be relaxed to (11).*

Conditions (11), (13) and (14) are standard in the present setting and serve to bound the rate of possible blow-up of  $\mathcal{R}(z)$  as described in (10). They occur in various forms, cp. [2, 3, 6, 8, 14, 15, 17, 34, 36], and counterexamples show that they are in fact necessary [36]. In particular, (13) is assumed in several papers starting from [36].

Estimate (11) is general enough to reproduce several classical results when applied to particular settings. Indeed, when dealing with the  $p$ -Laplacean equation  $-\text{div}(|Du|^{p-2} Du) = f$ , we then take  $p = q$  and  $\mu = 0$ , and (12) gives the local estimate (see for instance [23, 28, 30])

$$(15) \quad \|Du\|_{L^\infty(B/2)} \lesssim \left( \int_B |Du|^p dx \right)^{\frac{1}{p}} + \|f\|_{L(n,1)(B)}^{\frac{1}{p-1}}.$$

This is nothing but the classical  $L^\infty - L^p$  estimate for  $p$ -harmonic functions [35] when  $f \equiv 0$ . Instead, keeping  $p \neq q$  as in (11), but considering the homogeneous case  $f \equiv 0$ , it gives back the following classical bound of Marcellini [36, Theorem 3.1] (there  $\mu = 1$ ,  $q \geq p \geq 2$  and (13) are assumed):

$$\|Du\|_{L^\infty(B/2)} \lesssim \left( \int_B F(Du) dx \right)^{\frac{2}{(n+2)p-nq}} + 1.$$

Finally, let us comment on the two dimensional case  $n = 2$ . For this we use a different, almost sharp characterization making use of suitable Orlicz spaces. We recall that, given a Young function  $A: [0, \infty) \rightarrow [0, \infty)$ , with  $\Omega \subset \mathbb{R}^n$ , the Orlicz space  $L^A(\Omega)$  is defined as the vector space of measurable maps  $g$  such that  $\|A(|g|)\|_{L^1(\Omega)} < \infty$  the following Luxemburg norm is finite:

$$\|w\|_{L^A(\Omega)} := \inf \left\{ \lambda > 0 : \int_\Omega A \left( \frac{|w|}{\lambda} \right) dx \leq 1 \right\}.$$

This is a Banach space. The case of interest for us is given by the choice  $A(t) = t^2 \log^\alpha(e + t)$  for  $\alpha \geq 1$ , for which the standard notation is  $L^A \equiv L^2(\text{Log } L)^\alpha$ . For the Poisson equation  $-\Delta u \in L^2(\text{Log } L)^\alpha$  it is known that  $\alpha > 1$  is sufficient for local Lipschitz continuity of  $u$ , and this result is sharp in this scale of spaces. In [5] we are able to reproduce almost the same criterion; indeed it holds the following:

**Theorem 2.3** (The two dimensional case). *Let  $u \in W_{\text{loc}}^{1,1}(\Omega)$  be a local minimizer of the functional in (1) under assumptions (9) with  $1 < p \leq q$  and  $n = 2$ . Assume*

$$q < 2p \quad \text{and} \quad f \in L^2(\text{Log } L)^\alpha(\Omega) \text{ for some } \alpha > 2,$$

*Then  $Du$  is locally bounded in  $\Omega$ .*

The previous result comes along with local estimates and extends to the vectorial case (under suitable structure conditions mentioned above) as well; see [5].

## 3. NON-POLYNOMIAL GROWTH CONDITIONS AND NATURAL ESTIMATES

Here we deal with non-polynomial growth conditions, and we present the results directly in the vectorial case. An important example of a functional with fast growth conditions appears in (8). It belongs to a more general family which is described as follows. Let  $\{p_k\}$  be a sequence of real numbers such that  $p_0 > 1$ ,  $p_k > 0$  for every  $k \in \mathbb{N}$ . By inductions, we define  $\mathbf{e}_{p_k} : [0, \infty) \rightarrow \mathbb{R}$  as

$$\begin{cases} \mathbf{e}_{k+1}(t) &:= \exp[(\mathbf{e}_k(t))^{p_{k+1}}] \\ \mathbf{e}_0(t) &:= \exp(t^{p_0}) . \end{cases}$$

and consider the variational integrals

$$(16) \quad W^{1,1}(\Omega; \mathbb{R}^N) \ni w \mapsto \mathcal{E}_k(w) := \int_{\Omega} [\mathbf{e}_k(|Dw|) - fw] \, dx .$$

We then have

**Theorem 3.1** (Exponential estimates). *Let  $k \geq 0$  be an integer and let  $u \in W_{\text{loc}}^{1,1}(\Omega; \mathbb{R}^N)$  be a local minimizer of the functional  $\mathcal{E}_k$  in (16).*

- *If  $f \in L(n, 1)(\Omega; \mathbb{R}^N)$  and  $n > 2$ , then  $Du$  is locally bounded in  $\Omega$ .*
- *When  $n = 2$  the same conclusion holds provided  $f \in L^2(\text{Log } L)^\alpha(\Omega)$  for some  $\alpha > 2$ .*
- *Finally, when  $f \equiv 0$ , the local estimate*

$$(17) \quad \|Du\|_{L^\infty(B/2)} \leq c \mathbf{e}_k^{-1} \left( \int_B \mathbf{e}_k(|Du|) \, dx \right) + c$$

*holds for a constant  $c \equiv c(n, N, k, p_0, \dots, p_k)$  and for every ball  $B \Subset \Omega$ .*

The above theorem features new and relevant information already in the case  $f \equiv 0$  (originally treated by Marcellini [37, 38]; see also [22, 33]). To understand the progress, let us consider the simplest example given by

$$w \mapsto \int_{\Omega} \exp(|Dw|^p) \, dx, \quad p > 1 .$$

For this case the best a priori estimate available up to now, obtained in [37, 38], reads as

$$(18) \quad \|Du\|_{L^\infty(B/2)} \leq c_\varepsilon \left( \int_B \exp(|Du|^p) \, dx \right)^{1+\varepsilon} + c_\varepsilon, \quad \text{for every } \varepsilon > 0 ,$$

whereas our estimate (17), when applied in this situation, gives

$$\|Du\|_{L^\infty(B/2)}^p \leq c \log \left( \int_B \exp(|Du|^p) \, dx \right) + c, \quad c \equiv c(n, N, p) .$$

This estimate is new and parallels the one for  $p$ -harmonic functions in (15) in the sense that it exhibits a gain of an exponential scale with respect to (18). The gain with respect to the estimates in [37, 38] increases when considering functionals of even faster growth as in (8). Estimate (17) is a special occurrence of a more general result obtained in [5] which concerns the special situation of functionals defined in Orlicz spaces of the type

$$(19) \quad w \mapsto \int_{\Omega} A(|Dw|) \, dx ,$$

where  $A : [0, \infty) \rightarrow [0, \infty)$  is a Young function. The case  $A(t) = t^p/p$  for  $p \geq 1$  is the most basic example. A natural question is of course whether or not natural a priori estimates of the type

$$\|Du\|_X \lesssim A^{-1} \left( \int_B A(|Du|) \, dx \right) + 1$$

hold for function spaces  $X$  that embed into  $L^A(B/2)$ . The answer is known to be positive [1, 10, 24] provided the  $\Delta_2$ -condition (5) holds. The techniques in [5] allow to give a general positive answer to

this question also in the more difficult situation when the  $\Delta_2$ -condition is dropped. The results in [5] cover the most important case of  $X \equiv L^\infty(B/2)$  and a large class of functionals of the type (19).

#### 4. REVISITING UNIFORMLY ELLIPTIC OPERATORS

The results in [5] allow to get new conclusions in the classical uniformly elliptic case too. We indeed have the following:

**Theorem 4.1** (Natural growth estimates under  $\Delta_2$ -condition). *Let  $u \in W_{\text{loc}}^{1,1}(\Omega; \mathbb{R}^N)$  be a local minimizer of the functional in (3), with  $A: [0, \infty) \rightarrow [0, \infty)$  being of class  $C_{\text{loc}}^1[0, \infty) \cap C_{\text{loc}}^2(0, \infty)$ . Assume that the uniform ellipticity assumptions (4) hold and that  $f \in L(n, 1)(\Omega; \mathbb{R}^N)$  for  $n > 2$ . Then  $Du \in L_{\text{loc}}^\infty(\Omega; \mathbb{R}^{N \times n})$  and the estimate*

$$(20) \quad \|Du\|_{L^\infty(B/2)} \leq cA^{-1} \left( \int_B A(|Du|) dx \right) + cA^{-1} \left( \|f\|_{L(n,1)(B)}^{\frac{i_a+2}{i_a+1}} \right) + c_2$$

holds for every ball  $B \Subset \Omega$ , for constants  $c, c_2$  depending only on  $n, N, i_a, s_a$  and  $\tilde{a}(1)$ . When  $n = 2$  a similar results holds assuming that  $f \in L^2(\text{Log } L)^\alpha(\Omega; \mathbb{R}^N)$  for some  $\alpha > 2$ . In the case it is

$$i_2 := \liminf_{t \rightarrow 0} \frac{\tilde{a}(t)}{t^{i_a}} > 0,$$

then we can take  $c_2 = 0$  in (20) and the constant  $c$  depends also on  $i_2$ .

The above result extends those of Baroni [1] and Lieberman [32] when the problem is vectorial. It also provides a local analog to the global bounds of Cianchi & Maz'ya [11, 12, 13]. It is worth remarking that, when applied to the usual  $p$ -Laplacean case  $A(|z|) \equiv |z|^p/p$ , estimate (20) reduces to the classical one in (15).

#### 5. EXISTENCE AND REGULARITY FOR EQUATIONS UNDER $(p, q)$ -GROWTH CONDITIONS

When considering general equations under  $(p, q)$ -growth conditions, not necessarily arising from variational integrals, it is still possible to prove existence of locally Lipschitz continuous solutions. Specifically, let us consider Dirichlet problems of the type

$$(21) \quad \begin{cases} -\text{div } a(Du) = f & \text{in } \Omega \\ u \equiv u_0 & \text{on } \partial\Omega, \end{cases} \quad u_0 \in W^{1, \frac{p(q-1)}{p-1}}(\Omega).$$

Here we additionally assume that  $\Omega$  is a bounded and Lipschitz regular domain. The vector field  $a: \mathbb{R}^n \mapsto \mathbb{R}^n$  is assumed to be  $C^1$ -regular outside the origin and such that

$$(22) \quad \begin{cases} |a(z)| + (|z|^2 + \mu^2)^{\frac{1}{2}} |\partial a(z)| \leq L(|z|^2 + \mu^2)^{\frac{q-1}{2}} + L(|z|^2 + \mu^2)^{\frac{p-1}{2}} \\ \nu(|z|^2 + \mu^2)^{\frac{p-2}{2}} |\xi|^2 \leq \langle \partial a(z) \xi, \xi \rangle \end{cases}$$

holds for every choice of  $z, \xi \in \mathbb{R}^n$  with  $z \neq 0$ , and for exponents  $1 \leq p \leq q$ , ellipticity constants  $0 < \nu \leq 1 \leq L$  and  $0 \leq \mu \leq 1$ . Marcellini's by now classical result [36] states the existence of locally Lipschitz continuous solutions to (22) assuming that  $f \in L^\infty(\Omega)$ . Marcellini's result is upgraded in [5] to the sharp level  $f \in L(n, 1)$ . The precise statement of the result is as follows:

**Theorem 5.1** (Existence of locally Lipschitz solutions). *With  $1 < p \leq q$ ,  $n > 2$ , and under assumptions (14) and (22), there exists a solution  $u \in W_{\text{loc}}^{1,\infty}(\Omega) \cap W^{1,p}(\Omega)$  to the Dirichlet problem in (21). Moreover, the a priori estimate*

$$(23) \quad \|Du\|_{L^\infty(B/2)} \leq c \frac{\mathcal{D}}{|B|} + c \left( \frac{\mathcal{D}}{|B|} \right)^{\frac{p}{p-(q-p)n}} + c \|f\|_{L(n,1)(B)}^{\frac{1}{p-1}} + c \|f\|_{L(n,1)(B)}^{\frac{2}{2(p-1)-(q-p)(n-2)}}$$

holds for every ball  $B \Subset \Omega$ , where  $c \equiv c(n, p, q, \nu, \Lambda)$ , and

$$\mathcal{D}^p := \int_\Omega (|Du_0|^2 + 1)^{\frac{p(q-1)}{2(p-1)}} dx + c|\Omega| \|f\|_{L^n(\Omega)}^{p/(p-1)}.$$

When  $n = 2$  and  $f \in L^2(\text{Log } L)^\alpha(\Omega)$  holds for some  $\alpha > 2$ , again  $Du$  is locally bounded in  $\Omega$  and an estimate similar to (23) holds upon replacing  $\|f\|_{L(n,1)(B)}$  by  $\|f\|_{L^2(\text{Log } L)^\alpha(B)}$ .

We notice that, in the case when  $p = q$  and  $\Omega$  coincides with a ball  $B$ , estimate (23) gives

$$\|Du\|_{L^\infty(B/2)}^p \lesssim \int_B (|Du_0|^p + 1) dx + \|f\|_{L(n,1)(B)}^{p/(p-1)},$$

which is the usual a priori local estimate for the Dirichlet problems; see [30] and compare with (15). Apart from the optimal regularity  $f \in L(n, 1)(\Omega)$  of the right-hand side, further differences compared with [36] rely in the fact that we are also able to treat the degenerate case  $\mu = 0$ , while we are not assuming any control on the antisymmetric part of  $\partial a(\cdot)$ . Specifically, we are assuming no inequality of the type  $|\partial a(z) - [\partial a(z)]^t| \lesssim |z|^{(q+p-2)/2}$ , as indeed done in [36]. This is essentially reflected in the more restrictive bound (14) on  $q/p$  with respect to the one considered in [36], which is in fact (13).

## 6. POTENTIAL THEORETIC TECHNIQUES

At the core of the new approach developed in [5] there lies a set of nonlinear potential theoretic techniques that find their origins in the seminal papers of Maz'ya and Havin [39, 40]. Potential estimates for solutions to nonlinear partial differential equations have first been developed by Kilpeläinen & Malý [25], in turn relying on the fundamental methods of De Giorgi [21]. Gradient potential estimates have been derived in [1, 28, 29] in the scalar case and in [30] for the vectorial one. We refer to [27] for an overview on potential estimates. Such techniques have been successfully applied to the uniformly elliptic case, whilst a general extension to the non-uniformly elliptic one has always appeared problematic. The main novelty of [5] is indeed to find a way to use a potential theoretic approach in the non-uniformly elliptic setting. This involves the use of a modified Riesz potential of the right-hand side data  $f$ , originally introduced in [23], defined by

$$\mathbf{P}_1^f(x_0, R) := \int_0^R \left( \varrho^2 \int_{B_\varrho(x_0)} |f|^2 dx \right)^{1/2} \frac{d\varrho}{\varrho}.$$

for  $x_0 \in \Omega$  and  $R > 0$ . The potential  $\mathbf{P}_1^f(\cdot, R)$  plays in the present context the role of the (truncated) Riesz potential which is instead defined by

$$\mathbf{I}_1^f(x_0, R) := \int_0^R \frac{|f|(B_\varrho(x_0))}{\varrho^{n-1}} \frac{d\varrho}{\varrho} = \int_0^R \frac{1}{\varrho^n} \int_{B_\varrho(x_0)} |f| dx d\varrho,$$

provided the argument function  $f$  is at least locally  $L^1$ -regular. This can be easily seen by using Hölder inequality to estimate  $\mathbf{I}_1^f(x_0, R) \leq |B_1| \mathbf{P}_1^f(x_0, R)$  and observing that the two quantities share the same homogeneity and scaling properties. In [5] we prove an estimate that allows to bound, locally, the  $L^\infty$ -norm of  $Du$  in terms of the  $L^\infty$ -norm of  $\mathbf{P}_1^f(\cdot, R)$  (modulo controllable terms). The catch with Lorentz spaces and local estimates then comes from the relation

$$\|\mathbf{P}_1^f(\cdot, R)\|_{L^\infty(B)} \lesssim \|f\|_{L(n,1)(B_{2R})}$$

that holds for every ball  $B \subset \mathbb{R}^n$ . Similar relations hold in the two dimensional case for the space  $L^2(\text{Log } L)^\alpha$  with  $\alpha > 2$ . The a priori estimates are then combined with an approximation argument aimed at by-passing the fact that in [5] we never assume the validity of the  $\Delta_2$ -condition (5). Actually different approximations of the functional are needed in the scalar and in the vectorial case. The fact that we are not assuming the  $\Delta_2$ -condition (5) poses additional technical difficulties also when building the approximation argument.

## 7. MORE ON NON-UNIFORMLY ELLIPTIC PROBLEMS

The extension of results as those in Theorem 2.1 to the case of general non-autonomous functionals of the type

$$w \mapsto \mathcal{F}(w; \Omega) := \int_{\Omega} [H(x, Dw) - f \cdot w] \, dx$$

is not an easy task and it is the object of ongoing investigation [19]. Examples of energies of this type are obviously given by functionals with coefficients of the type

$$w \mapsto \mathcal{F}(w; \Omega) := \int_{\Omega} [c(x)F(Dw) - f \cdot w] \, dx, \quad 0 < \nu \leq c(x) \leq L,$$

where  $F(\cdot)$  is of the type considered for instance in Theorem 2.1, or it is one of the exponential growth. Intermediate classes of non-autonomous functionals can be considered as well. These are somehow more interesting as they present a special form of non-uniform ellipticity that can be detected only in a non-local fashion. Indeed, while considering the ratio in (2) gives a bounded quantity, the non-local ratio

$$\mathcal{R}(z, B) := \frac{\sup_{x \in B} \text{highest eigenvalue of } \partial^2 H(x, z)}{\inf_{x \in B} \text{lowest eigenvalue of } \partial^2 H(x, z)},$$

does not. This is for instance the case of energies of the type

$$(24) \quad \begin{cases} w \mapsto \mathcal{F}(w; \Omega) := \int_{\Omega} [|Dw|^p + a(x)|Dw|^q - f \cdot w] \, dx \\ 1 < p \leq q, \quad 0 \leq a(x) \leq L, \end{cases}$$

as the coefficient  $a(x)$  is allowed to be zero. For such energies, ad hoc methods can be developed; see for instance [2, 4, 15, 14, 16, 20]. Another class of relevant non-autonomous functionals is the one involving a variable growth exponent. In this case the relevant energy is given by

$$w \mapsto \int_{\Omega} [|Dw|^{p(x)} - f \cdot w] \, dx$$

and requires less stringent assumptions on the exponent  $p(x)$  than those needed on the coefficient  $a(x)$  in (24). The regularity problems have been in this case the object of intensive investigation. See the survey [41] for a rapid review and also [43] for more general results. Recently, manifold constrained problems have been investigated in the case of variable growth exponent functionals [18] too. In some cases, under more restrictive assumptions, also constrained  $(p, q)$ -growth conditions can be considered [17].

## REFERENCES

1. P. Baroni, *Riesz potential estimates for a general class of quasilinear equations*, Calc. Var. Partial Differential Equations **53** (2015), 803–846.
2. P. Baroni, M. Colombo and G. Mingione, *Harnack inequalities for double phase functionals*, Nonlinear Anal. **121** (2015), 206–222.
3. P. Baroni, M. Colombo and G. Mingione, *Regularity for general functionals with double phase*, Calc. Var. Partial Differential Equations **57** (2018) Art. 62, 48 pp.
4. P. Baroni, M. Colombo and Y. Youn, *Non-autonomous functionals with mild phase transition and gradient regularity*, Preprint 2018.
5. L. Beck and G. Mingione, *Lipschitz bounds and non-uniformly elliptic problems*, Preprint 2018.
6. M. Bildhauer and M. Fuchs, *Partial regularity for variational integrals with  $(s, \mu, q)$ -growth*, Calc. Var. Partial Differential Equations **13** (2001), 537–560.
7. S.S. Byun and J. Oh, *Global gradient estimates for non-uniformly elliptic equations*, Calc. Var. Partial Differential Equations **56**, No. 2, 36 Pages, 2017.
8. M. Carozza, J. Kristensen, A. Passarelli Di Napoli, *Regularity of minimizers of autonomous convex variational integrals*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (V) **13** (2014), 1065–1089.
9. A. Cianchi, *Maximizing the  $L^\infty$  norm of the gradient of solutions to the Poisson equation*, J. Geom. Anal. **2** (1992), 499–515.



10. A. Cianchi and N. Fusco, *Gradient regularity for minimizers under general growth conditions*, J. reine angew. Math. (Crelles J.) **509** (1999), 15–36.
11. A. Cianchi and V.G. Maz'ya, *Global Lipschitz regularity for a class of quasilinear equations*, Comm. PDE **36** (2011), 100–133.
12. A. Cianchi and V.G. Maz'ya, *Global boundedness of the gradient for a class of nonlinear elliptic systems*, Arch. Ration. Mech. Anal. **212** (2014), 129–177.
13. A. Cianchi and V.G. Maz'ya, *Global gradient estimates in elliptic problems under minimal data and domain regularity*, Comm. Pure Appl. Anal. **14** (2015), 285–311.
14. M. Colombo and G. Mingione, *Regularity for double phase variational problems*, Arch. Ration. Mech. Anal. **215** (2015), 443–496.
15. M. Colombo and G. Mingione, *Bounded minimisers of double phase variational integrals*, Arch. Ration. Mech. Anal. **218** (2015), 219–273.
16. M. Colombo and G. Mingione, *Calderón-Zygmund estimates and non-uniformly elliptic operators*, J. Funct. Anal. **270** (2016), 1416–1478.
17. C. De Filippis, *Higher integrability for constrained minimizers of integral functionals with  $(p,q)$ -growth in low dimension*, Nonlinear Anal. **170** (2018), 1–20.
18. C. De Filippis, *Partial regularity for manifold constrained  $p(x)$ -harmonic mappings*, Preprint 2018.
19. C. De Filippis and G. Mingione, *Lipschitz bounds for non-autonomous variational problems*, to appear.
20. C. De Filippis and J. Oh, *Regularity for Multi-Phase variational problems*, Preprint 2018.
21. E. De Giorgi, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*, Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat. (III), **125** (1957), 25–43.
22. D.M. Duc and J. Eells, *Regularity of exponentially harmonic functions*, Internat. J. Math. **2** (1991), 395–408.
23. F. Duzaar and G. Mingione, *Local Lipschitz regularity for degenerate elliptic systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **27** (2010), 1361–1396.
24. N. Fusco and C. Sbordone, *Higher integrability of the gradient of minimizers of functionals with nonstandard growth conditions*, Comm. Pure Appl. Math. **43** (1990), 673–683.
25. T. Kilpeläinen and J. Malý, *The Wiener test and potential estimates for quasilinear elliptic equations*, Acta Math. **172** (1994), 137–161.
26. J. Kristensen, G. Mingione, *The singular set of Lipschitzian minima of multiple integrals*, Arch. Ration. Mech. Anal. **184** (2007), 341–369.
27. T. Kuusi and G. Mingione, *Guide to nonlinear potential estimates*, Bull. Math. Sci. **4** (2014), 1–82.
28. T. Kuusi and G. Mingione, *Linear potentials in nonlinear potential theory*, Arch. Ration. Mech. Anal. **207** (2013), 215–246.
29. T. Kuusi and G. Mingione, *The Wolff gradient bound for degenerate parabolic equations*, J. Europ. Math. Soc. (JEMS), **16** (2014), 835–892.
30. T. Kuusi and G. Mingione, *Vectorial nonlinear potential theory*, J. Europ. Math. Soc. (JEMS) **20** (2018), 929–1004.
31. O.A. Ladyzhenskaya and N.N. Ural'tseva, *Local estimates for gradients of solutions of non-uniformly elliptic and parabolic equations*, Comm. Pure Appl. Math. **23** (1970), 677–703.
32. G.M. Lieberman, *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations*, Comm. Partial Differential Equations **16** (1991), 311–361.
33. G.M. Lieberman, *On the regularity of the minimizer of a functional with exponential growth*, Comment. Math. Univ. Carolin. **33** (1992), 45–49.
34. G.M. Lieberman, *Gradient estimates for a class of elliptic systems*, Ann. Mat. Pura Appl. (IV) **164** (1993), 103–120.
35. J.J. Manfredi, *Regularity of the gradient for a class of nonlinear possibly degenerate elliptic equations*, Ph.D. Thesis. University of Washington, St. Louis, 1986.
36. P. Marcellini, *Regularity and existence of solutions of elliptic equations with  $p,q$ -growth conditions*, J. Diff. Equ. **90** (1991), 1–30.
37. P. Marcellini, *Everywhere regularity for a class of elliptic systems without growth conditions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (IV) **23** (1996), 1–25.
38. P. Marcellini, *Regularity for some scalar variational problems under general growth conditions*, J. Optim. Theory Appl. **90** (1996), 161–181.
39. V. Maz'ya, *The continuity at a boundary point of the solutions of quasi-linear elliptic equations*, Vestnik Leningrad. Univ. **25** (1970), 42–55.
40. V. Maz'ya and M. Havin, *A nonlinear potential theory*, Russ. Math. Surveys **27** (1972), 71–148.
41. G. Mingione, *Regularity of minima: an invitation to the Dark side of the Calculus of Variations*, Appl. Math. **51** (2006), 355–425.
42. G. Mingione, *Gradient potential estimates*, J. Europ. Math. Soc. **13** (2011), 459–486.
43. V. Radulescu and D. Repovš, *Partial Differential Equations with Variable Exponents: Variational Methods and Qualitative Analysis*. Chapman & Hall/CRC Monographs and Research Notes in Mathematics. 2015

44. L. Simon, *Interior gradient bounds for non-uniformly elliptic equations*, Indiana Univ. Math. J. **25** (1976), 821–855.
45. T. Schmidt, *Regularity of relaxed minimizers of quasiconvex variational integrals with  $(p, q)$ -growth*, Arch. Ration. Mech. Anal. **193** (2009), 311–337.
46. E.M. Stein, *Editor's note: the differentiability of functions in  $\mathbb{R}^n$* . Ann. of Math. (II) **113** (1981), 383–385.
47. K. Uhlenbeck, *Regularity for a class of nonlinear elliptic systems*, Acta Math. **138** (1977), 219–240.
48. N.N. Ural'tseva, *Degenerate quasilinear elliptic systems*, Zap. Na. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **7** (1968), 184–222.
49. N.N. Ural'tseva and A.B. Urdaletova, *The boundedness of the gradients of generalized solutions of degenerate quasilinear nonuniformly elliptic equations*, Vestnik Leningrad Univ. Math. **19** (1983) (russian) english. tran.: **16** (1984), 263–270.
50. V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), 675–710.

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