8-1935

The Solution of Ordinary & Partial Differential Equations in Series

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1935

THE SOLUTION OF
ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS
IN SERIES

BY

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A THESIS
SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF ARTS

WESTERN KENTUCKY STATE TEACHERS COLLEGE

AUGUST, 1935

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INTRODUCTION

A very small percentage of all the classes of ordinary and partial differential equations can be solved by simple or elementary methods, compared with the large number of classes. Many, in fact, most of the differential equations of major importance to scientists and mathematicians in the study of applied science can not be solved completely by the general methods or by the many special methods of solution of ordinary and partial differential equations.

It is possible, however, that the solutions of these equations may be found and expressed in the form of infinite series. The power series is used in many cases in finding numerical approximations to solutions. Some common examples of series which take special forms are Legendre's Coefficients, or Zonal Harmonics; Laplace's Coefficients, or Spherical Harmonics; Bessel's Functions, or Cylindrical Harmonics; Lamé's Functions, or Ellipsoidal Harmonics, etc.[1] These functions are named after the men who have studied them exhaustively. Fourier, Riccati, Gauss, Cauchy, and others have also done pioneer work in the study of solutions in the form of series.

The purpose of this thesis is to compile and discuss some of the methods of solution of both ordinary and partial differential equations, whose solutions are expressible in the form of a series. An exhaustive study is not attempted. A few of the methods of most common occurrence for finding solutions in series

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[1] W. E. Byerly, An Elementary Treatise on Fourier's Series and Spherical, Cylindrical, and Ellipsoidal Harmonics (New York, Ginn and Co., 1893), p. 4.

are discussed and examples illustrating these methods are presented.

PART I

ORDINARY DIFFERENTIAL EQUATIONS

1. The development of a series.- A differential equation expresses a relation between the dependent variable, y, and all successive derivatives, included in the equation. If we consider the equation solved for the derivative of highest order, we may consider that one of the highest order as being expressed in terms of those of lower orders. That is, an equation of the second order would give $\frac{d^2y}{dx^2}$ in terms of $\frac{dy}{dx}$ and y. If we differentiate once we get $\frac{d^3y}{dx^3}$, in terms of $\frac{d^2y}{dx^2}$, $\frac{dy}{dx}$, and y; but since $\frac{d^2y}{dx^2}$ is given in terms of $\frac{dy}{dx}$ and y we can find $\frac{d^3y}{dx^3}$ also in terms of these two. In like manner each of the differential coefficients of higher order can be expressed in terms of $\frac{dy}{dx}$ and y; but no relation between $\frac{dy}{dx}$ and y is given by the differential equation. Suppose that when x takes the value x_0, $y = A$ and $\frac{dy}{dx} = B$, where A and B are arbitrary constants; then the successive derivatives when $x = x_0$ will be in terms of A and B. Let these be represented by C, D, E, . . . If $y = f(x)$, and we assume this function expansible by Taylor's theorem in a converging series of ascending powers of $(x - x_0)$, then when expanded in the neighborhood of x_0, we have

$$y = f(x) = f[x_0 + (x - x_0)]$$

$$= f(x_0) + (x - x_0)\left(\frac{df}{dx}\right)_0 + \frac{(x-x_0)^2}{\underline{|2}}\left(\frac{d^2f}{dx^2}\right)_0 + \frac{(x-x_0)^3}{\underline{|3}}\left(\frac{d^3f}{dx^3}\right)_0 + \cdots\cdots,$$

where $\left(\frac{d^nf}{dx^n}\right)_0$ represents the value of $\frac{d^nf(x)}{dx^n}$ after the differentiation and the substitution of $x = x_0$. Substituting for the differential coefficients their values as determined above, we get

$$y = f(x) = A + B(x-x_0) + C\,\frac{(x-x_0)^2}{\lfloor 2} + D\,\frac{(x-x_0)^3}{\lfloor 3} + \cdots\cdots.$$

This series is a solution of the given differential equation.

Since all of the coefficients are determined in terms of A and B in the above solution of a second order equation, we have only two arbitrary constants. If our equation had been of the first order, the differential coefficients would have been determined in terms of the one arbitrary constant substituted for y. In a differential equation of the third order three arbitrary constants enter the solution, and in an equation of the order n we find n arbitrary constants in the complete solution.

As an illustration of this method let us solve the second order equation

$$\frac{d^2y}{dx^2} = x\frac{dy}{dx} + y.$$

The successive derivatives are:

$$\frac{d^3y}{dx^3} = x\frac{d^2y}{dx^2} + 2\frac{dy}{dx},$$

$$\frac{d^4y}{dx^4} = x\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2},$$

$$\frac{d^5y}{dx^5} = x\frac{d^4y}{dx^4} + 4\frac{d^3y}{dx^3},$$

$$\cdots\cdots\cdots\cdots\cdots.$$

If, when x = o, y takes the value c_0, the expansion of y = f(x) by Taylor's theorem becomes

$$y = C_0 + C_1 x + \frac{C_2}{\lfloor 2}x^2 + \frac{C_3}{\lfloor 3}x^3 + \cdots\cdots .$$

From our differential equation each differential coefficient of order two or higher can be determined in terms of y and $\frac{dy}{dx}$, which in this case are c_0 and c_1. Substituting c_0 and c_1 respectively when $x = o$, we get

$$\frac{d^2y}{dx^2} = C_2 = C_0 ,$$

$$\frac{d^3y}{dx^3} = C_3 = 2C_1 ,$$

$$\frac{d^4y}{dx^4} = C_4 = 3C_0 ,$$

$$\frac{d^5y}{dx^5} = C_5 = 8C_1 ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots .$$

Hence

$$y = C_0 + C_1 x + \frac{C_0}{2}x^2 + \frac{2C_1}{\lfloor 3}x^3 + \frac{3C_0}{\lfloor 4}x^4 + \frac{8C_1}{\lfloor 5}x^5 + \cdots ,$$

or

$$y = C_0\left(1 + \frac{x^2}{2} + \frac{3x^4}{\lfloor 4} + \cdots\right) + C_1\left(x + \frac{2x^3}{\lfloor 3} + \frac{8x^5}{\lfloor 5} + \cdots\right)$$

is the complete solution.

2. Equations of the first order.- The theorem of the existence of an integral for a differential equation of the first order $f(x,y,\frac{dy}{dx}) = o$ or $\frac{dy}{dx} = F(x,y)$ is: [1]

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[1] A. Cohen, An Elementary Treatise on Differential Equations (New York, D. C. Heath & Co., 1906), p. 165.

If $F(x,y)$ is finite, continuous, and single valued, and has a finite partial derivative with respect to y, as long as x and y are restricted to certain regions, then if x_0 and y_0 are a pair of values lying in these regions, there is one integral y, and only one, which will take the value y_0 when x takes the value x_0.[2]

The solution of the equation is expressed, in the proof of the existence theorem, in the form of an infinite series

$$y = y_0 + c_1x + c_2x^2 + c_3x^3 + \cdots\cdots + c_nx^n + \cdots\cdots.$$

One arbitrary constant enters in the solution of a first order equation since y_0 is chosen arbitrarily in certain regions.

If the equation

(1) $$\frac{dy}{dx} = f(x,y)$$

satisfies the conditions of the existence theorem, i.e., $f(x,y)$ is finite, continuous, and single valued, and has a finite partial derivative with respect to y, its solution may be expressed in the form of the above power series. From the series, an approximation to a solution can often be obtained when it is impossible to obtain a solution by more elementary methods.

The general method of solution is to substitute

(2) $$y = \sum_{n=1}^{\infty} c_nx^n = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots$$

in the differential equation (1), equate coefficients of like powers of x, and calculate the value of as many c's in (2) as necessary. Three cases may arise:

1. A general law of the coefficients in (2) appears, and as

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[2] For a proof of the existence theorem see F. S. Woods, Advanced Calculus (New York, Ginn & Co., 1926), p. 91.
D. A. Murray, Introductory Course in Differential Equations (New York, Longmans, Green & Co., 1930), p. 190.

many terms as desired may be written, since the general term is known.

2. All the coefficients after a certain one become zero, and we have a finite series.

3. No general law of coefficients is apparent, and the solution can only be approximated. This is the case of most common occurrence.

As an illustration of the first case, we shall solve the first order equation

$$\frac{dy}{dx} = x + 2xy. \tag{3}$$

Here f(x,y) is finite, continuous, and single valued for all values of x and y, and the partial derivative with respect to y exists; therefore we can write the solution in the form of a power series. Let the series

$$y = c_0 + c_1 x + c_2 x^2 + \cdots\cdots + c_n x^n + \cdots\cdots \tag{4}$$

represent the form of the solution. Since the equation (3) is of the first order, one arbitrary constant will appear. Our solution is complete if we are able to determine all the coefficients in terms of some one of them.

Replacing y in (3) by the series (4), we must have

$$c_1 + 2c_2 x + 3c_3 x^2 + \cdots\cdots + nc_n x^{n-1} + \cdots\cdots$$

$$\equiv x + 2c_0 x + 2c_1 x^2 + 2c_2 x^3 + \cdots\cdots + 2c_{n-2} x^{n-1} + \cdots$$

Since the two series are to be identically equal, the coefficients of the corresponding terms must be equal.[3]

Equating coefficients, we have

$$c_1 = 0, \qquad \therefore\ c_1 = 0.$$

$$2c_2 = 1 + 2c_0, \qquad \therefore\ c_2 = \frac{1+2c_0}{2}.$$

$$3c_3 = 2c_1, \qquad c_3 = \frac{2}{3}c_1, \qquad \therefore\ c_3 = 0.$$

$$4c_4 = 2c_2, \qquad c_4 = \frac{1}{2}c_2, \qquad \therefore\ c_4 = \frac{1+2c_0}{2}\cdot\frac{1}{\underline{|2}}.$$

$$5c_5 = 2c_3, \qquad c_5 = \frac{2^2}{3\cdot 5}c_1, \qquad \therefore\ c_5 = 0.$$

$$6c_6 = 2c_4, \qquad c_6 = \frac{1}{3}c_4, \qquad \therefore\ c_6 = \frac{1+2c_0}{2}\cdot\frac{1}{\underline{|3}}.$$

$$7c_7 = 2c_5, \qquad c_7 = \frac{2^3}{3\cdot 5\cdot 7}c_1, \qquad \therefore\ c_7 = 0.$$

$$8c_8 = 2c_6, \qquad c_8 = \frac{1}{4}c_6, \qquad \therefore\ c_8 = \frac{1+2c_0}{2}\cdot\frac{1}{\underline{|4}}.$$

. . . .

$$nc_n = 2c_{n-2}$$

When n is even $\quad c_n = \frac{1+2c_0}{2}\cdot\frac{1}{\underline{|n/2}},$

and when n is odd $\quad c_n = \frac{2^{\frac{n-1}{2}}}{1\cdot 3\cdot 5\cdots n}c_1.$

Here the coefficients of the terms involving even powers of x can be determined in terms of c_0, and the coefficients of the terms involving odd powers of x can be determined in terms of c_1. We notice that, since c_1 is a factor of the coefficients of the odd powers of x and is also equal to zero, the terms involving odd powers of x vanish. We can calculate each successive term and write the general term, therefore the whole series is known.

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[3] M. Bocher, Introduction to Higher Algebra (New York, The Macmillan Co., 1933), Theorem 5, p. 3.

Substituting for the c's in (4) their equivalents in terms of c_0 and c_1.

$$y = C_0 + C_1 x + C_2 x^2 + \cdots\cdots + C_n x^n + \cdots$$

becomes

$$y = C_0 + \frac{1+2C_0}{2} x^2 + \frac{1+2C_0}{2} \cdot \frac{x^4}{\lfloor 2} + \frac{1+2C_0}{2} \cdot \frac{x^6}{\lfloor 3} + \cdots + \frac{1+2C_0}{2} \cdot \frac{x^{2n}}{\lfloor n} + \cdots.$$

A simpler form of solution may be obtained by replacing c_0 by its equal

$$C_0 = -\frac{1}{2} + \frac{1+2C_0}{2}$$

and factoring out $\frac{1+2C_0}{2}$ We get

$$y = -\frac{1}{2} + \frac{1+2C_0}{2}\left(1 + x^2 + \frac{x^4}{\lfloor 2} + \frac{x^6}{\lfloor 3} + \cdots + \frac{x^{2n}}{\lfloor n} + \cdots\right).$$

The series in parenthesis is e^{x^2} developed as a power series, and our solution may be expressed as

$$y = -\frac{1}{2} + Ae^{x^2},$$

where

$$A = \frac{1+2C_0}{2}.$$

Since c_0 is arbitrary, A is arbitrary.

Equation (3) is of the first order, therefore only one arbitrary constant appears in the general solution.

The solution of the equation

(5) $$(1+2x^2)\frac{dy}{dx} = 4xy$$

illustrates the case in which the series is finite. Substituting the series (4) for y in this equation, we get

$$c_1 + 2c_2x + (2c_1 + 3c_3)x^2 + (4c_2 + 4c_4)x^3 + (6c_3 + 5c_5)x^4 + \cdots$$

$$\equiv 4c_0x + 4c_1x^2 + 4c_2x^3 + 4c_3x^4 + \cdots.$$

Equating coefficients,

$$c_1 = 0, \qquad \therefore\ c_1 = 0.$$

$$2c_2 = 4c_0, \qquad \therefore\ c_2 = 2c_0.$$

$$2c_1 + 3c_3 = 4c_1, \quad c_3 = \tfrac{2}{3}c_1, \qquad \therefore\ c_3 = 0.$$

$$4c_2 + 4c_4 = 4c_2, \qquad \therefore\ c_4 = 0.$$

$$6c_3 + 5c_5 = 4c_3, \quad c_5 = -\tfrac{2}{5}c_3, \qquad \therefore\ c_5 = 0.$$

Each succeeding coefficient is zero, since it has a factor equal to zero. Replacing the c's in (4) by their equivalents in terms of c_0 and o, the solution of (5) is found to be

$$y = c_0 + 2c_0x^2, \quad \text{or} \quad y = c_0(1 + 2x^2)$$

when c_0 is the arbitrary constant.

As an illustration of the case in which no general law of coefficients appears, we shall solve the equation

(6) $$\frac{dy}{dx} = x + y^2.$$

This is a special case of Riccati's equation

$$\frac{dy}{dx} + by^2 = cx^m,$$

where b = -1, c = 1, and m = 1.

The right hand member of (6), $x + y^2$, satisfies the restrictions on f(x,y) in the existence theorem; therefore the solution of (6) can be written in the form of a series. As in the preceding illustrations we replace y in (6) by the series (4). We must have

$$c_1 + 2c_2 x + 3c_3 x^2 + \cdots \equiv x + (c_0 + c_1 x + c_2 x^2 + \cdots)^2.$$

Equating coefficients, we have

$$c_1 = c_0^2, \qquad \therefore \quad c_1 = c_0^2.$$

$$2c_2 = 2c_0 c_1 + 1, \qquad \therefore \quad c_2 = \tfrac{1}{2} + c_0^3.$$

$$3c_3 = 2c_0 c_2 + c_1^2, \qquad \therefore \quad c_3 = \tfrac{1}{3} c_0 + c_0^4.$$

$$4c_4 = 2c_0 c_3 + 2c_1 c_2, \qquad \therefore \quad c_4 = \tfrac{5}{12} c_0^2 + c_0^5.$$

$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots.$$

Each coefficient can be determined in terms of the one next preceding it, and therefore all can be found in terms of the first. No general law for finding the coefficients is evident and we can only write the result to include as many terms as may be desired. The solution is

$$y = c_0 + c_0^2 x + (\tfrac{1}{2} + c_0^3)x^2 + (\tfrac{1}{3} c_0 + c_0^4)x^3 + (\tfrac{5}{12} c_0^2 + c_0^5)x^4 + \cdots.$$

This solution contains one arbitrary constant c_0.

3. Equations of higher order than the first.- We can solve n differential equations of the first order in n dependent variables for the derivatives of these variables. Our result may be written in the form

$$(1) \qquad \begin{aligned} \frac{dy}{dx} &= f_1(x, y, z, \cdots, w), \\ \frac{dz}{dx} &= f_2(x, y, z, \cdots, w), \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

. . . .

$$\frac{dw}{dx} = f_n(x, y, z, \cdots w).$$

The general existence theorem for n equations of the above form is:

If $f_1, f_2, \ldots f_n$ can each be expanded by Taylor's theorem in a power series which converges in certain regions, then if $x_0, y_0, z_0, \ldots w_0$ are in these regions, one and only one set of functions $y, z, \ldots w$ can be found to satisfy the system of equations and to take the values $y_0, z_0, \ldots w_0$ respectively when x takes the value x_0.

In the proof of this existence theorem the solutions of the equations are expressed in the form

$$\begin{aligned}
(2) \qquad y &= y_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots, \\
z &= z_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots + b_n x^n + \cdots, \\
&\text{- -} \\
W &= W_0 + k_1 x + k_2 x^2 + k_3 x^3 + \cdots + k_n x^n + \cdots.
\end{aligned}$$

A differential equation of the nth order may be transformed into n equations of the first order. If we solve for the derivative of highest order, we have

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots, \frac{d^{n-1}y}{dx^{n-1}}\right).$$

We may put

$$\frac{dy}{dx} = y_1, \quad \frac{d^2y}{dx^2} = \frac{dy_1}{dx} = y_2, \ldots \frac{d^ny}{dx^n} = \frac{dy_{n-1}}{dx} = y_n,$$

when y_1, y_2, y_3, . . . y_n are to be regarded as new variables. Then

(3) $$\frac{dy}{dx} = y_1 ,$$

$$\frac{dy_1}{dx} = y_2 ,$$

$$\frac{dy_2}{dx} = y_3 ,$$

$$\cdots\cdots\cdots$$

$$\frac{dy_{n-2}}{dx} = y_{n-1} ,$$

$$\frac{dy_{n-1}}{dx} = y_n = f(x, y, y_1, y_2, y_3, \cdots y_{n-1})$$

are n equations of the first order involving n dependent variables. Since the existence theorem is applicable to the n equations of the first order, it will also be applicable to the equivalent equation of the nth order.

Since the solution of the system of n equations (1) in n dependent variables, which is a system of the same form as equations (3), involve n arbitrary constants, and since we have just seen that an equation of the nth order can be replaced by n equations of the first order, it follows that the general solution of a differential equation of the nth order involves n arbitrary constants.

The same general procedure followed in solving differential equations of the first order may be applied in finding the solutions of equations of higher order. However, the series

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots + C_n x^n + \cdots$$

is a general form of solution only when all exponents in the series are positive. For example, the solution of the equation

$$(x - x^2)\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 2y = 0$$

is

$$y = A\left(1 - \frac{x}{2} + \frac{x^2}{10}\right) + B(x^{-3} - 5x^{-2} + 10x^{-1} - 10 + 5x - x^2),$$

but if the above series is substituted for y in the differential equation, only the first integral appears. A more general series for substitution is of the form

(4)
$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots.$$

If this series is used, both integrals can be found.

As an illustration of this substitution of the general series, we shall solve the equation

$$\frac{d^2y}{dx^2} - xy = 0.$$

Solving this for $\frac{d^2y}{dx^2}$ we get immediately

(5)
$$\frac{d^2y}{dx^2} = xy.$$

Upon replacing y by the series (4) equation (5) becomes

(6)
$$(m-1)ma_0 x^{m-2} + m(m+1)a_1 x^{m-1} + (m+1)(m+2)a_2 x^m$$
$$+ (m+2)(m+3)a_3 x^{m+1} + (m+3)(m+4)a_4 x^{m+2} + \cdots$$
$$\equiv a_0 x^{m+1} + a_1 x^{m+2} + a_2 x^{m+3} + \cdots.$$

These two series are identically equal, therefore the coefficients of like powers of x are equal and we get

(7) $$(m-1)ma_0 = 0,$$

(8) $$m(m+1)a_1 = 0,$$

(9) $$(m+1)(m+2)a_2 = 0,$$

(10) $$(m+2)(m+3)a_3 = a_0,$$

(11) $$(m+3)(m+4)a_4 = a_1,$$

. . . .

If the term x^m appears in (4), $a_0 \neq 0$; but from (7), $(m-1)ma_0 = 0$, therefore

$$(m-1)m = 0 \quad \text{and} \quad m = 1 \text{ or } 0.$$

We obtain a solution for each value of m. When m = 1, we get from (8), (9), . . .

$$a_1 = 0,$$

$$a_2 = 0,$$

$$a_3 = \frac{1}{(m+2)(m+3)} a_0 = \frac{1}{3\cdot 4} a_0 = \frac{2}{\lfloor 4} a_0,$$

$$a_4 = \frac{1}{(m+3)(m+4)} a_1 = 0,$$

$$a_5 = \frac{1}{(m+4)(m+5)} a_2 = 0,$$

$$a_6 = \frac{1}{(m+5)(m+6)} a_3 = \frac{1}{6\cdot 7} a_3 = \frac{2\cdot 5}{\lfloor 7} a_0,$$

. . . .

(12)

$$a_n = \frac{1}{(m+n-1)(m+n)} a_{n-3} = \quad = \frac{2\cdot5\cdot8\cdots(n-1)\,a_0}{\lfloor(n+1)},$$

Therefore . . . .

(13) $$y = a_0 x + \frac{2}{\lfloor 4} a_0 x^4 + \frac{2\cdot5}{\lfloor 7} a_0 x^7 + \cdots + \frac{2\cdot5\cdot8\cdots(n-1)}{\lfloor(n+1)} a_0 x^{n+1} + \cdots$$

is a solution. Here n is a multiple of three, since all the other terms, whose power in x is different from one plus a multiple of three, have either an a_1 or a_2 factor, both of which are zero. Let us call (13)

$$y = Ay_1,$$

where A is an arbitrary constant.

When m = o, from (8), (9), . . . we get

$$a_1 = 0,$$ [4]

$$a_2 = 0,$$

$$a_3 = \frac{1}{(m+2)(m+3)} a_0 = \frac{1}{\lfloor 3} a_0,$$

$$a_4 = \frac{1}{(m+3)(m+4)} a_1 = 0,$$

$$a_5 = \frac{1}{(m+4)(m+5)} a_2 = 0,$$

$$a_6 = \frac{1}{(m+5)(m+6)} a_3 = \frac{1}{5\cdot6} a_3 = \frac{1\cdot4}{\lfloor 6} a_0,$$

. . . .

. . . .

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[4] a_1 might not be equal to zero and another integral could be obtained but this integral is included in the general solution obtained by considering a_1 equal to zero.

$$a_n = \frac{1}{(m+n-1)(m+n)}\, a_{n-3} = \cdots = \frac{1\cdot 4\cdot 7\cdots(n-2)}{\lfloor n}\, a_0,$$

Therefore . . . .

(14) $$y = a_0 + \frac{1}{\lfloor 3}\, a_0 x^3 + \frac{1\cdot 4}{\lfloor 6}\, a_0 x^6 + \cdots + \frac{1\cdot 4\cdot 7\cdots(n-2)}{\lfloor n}\, a_0 x^n$$

is a solution. Let us call it

$$By_2,$$

where B is an arbitrary constant. The complete solution

$$y = Ay_1 + By_2$$

is obtained by adding the two solutions.

The labor of computing coefficients which will be zero may be eliminated by using a special form of series in the assumed solution when the substitution of x^m for y reduces the differential equation to an equation having only two powers of x.

If we again take the equation

(15) $$\frac{d^2y}{dx^2} - xy = 0$$

and substitute $y = x^m$ in the left-hand member, we get

(16) $$m(m-1)x^{m-2} - x^{m+1} = 0.$$

Here we have two distinct powers of x which differ by 3, and $(m-2)$ is the smaller exponent. We shall assume

(17) $$y = c_0 x^m + c_1 x^{m+3} + c_2 x^{m+6} + \cdots$$

to be a solution and find the conditions under which it will be a solution. If we replace y in the differential equation by this series term by term, we get

(18)

$$[m(m-1)c_0 x^{m-2} - c_0 x^{m+1}]$$

$$+\left[(m+3)(m+2)c_1 x^{m+1} - c_1 x^{m+4}\right]$$
$$+\left[(m+6)(m+5)c_2 x^{m+4} - c_2 x^{m+7}\right]$$
$$+ \cdots\cdots\cdots\cdots\cdots$$
$$+\left[(m+3r)(m+3r-1)c_r x^{m+3r-2} - c_r x^{m+3r+1}\right]$$
$$+\left[(m+3r+3)(m+3r+2)c_{r+1} x^{m+3r+1} - c_{r+1} x^{m+3r+4}\right]$$
$$+ \cdots\cdots\cdots\cdots\cdots \equiv 0.$$

It will be noticed that since the exponents of (16) differ by 3 and the exponent of each successive term of (17) is 3 greater than the last, the last term in each of the expressions in brackets in (18) will be of the same degree in x as the first term in the following expression. Hence if

(19) $$m(m-1)c_0 = 0$$

this being the coefficient of the only term in x^{m-2}, and if

(20)
$$(m+3)(m+2)c_1 - c_0 = 0,$$
$$(m+6)(m+5)c_2 - c_1 = 0,$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$(m+3r)(m+3r-1)c_r - c_{r-1} = 0,$$

i.e., the coefficients of like powers of x cancel each other in pairs, the left-hand member of (18) will be identically zero and (17) will be a solution of (15). If (17) has a term in x^m, $c_0 \neq 0$ and in order for (19) to be zero

$$m(m-1) = 0 \qquad \therefore m = 0 \text{ or } 1.$$

From the general equation (20)

(21) $$c_r = \frac{1}{(m+3r)(m+3r-1)} c_{r-1}.$$

Where $n = 3r$ equation (21) is identical with (12); for $m = o$ solution (17) becomes (14) above and for $m = 1$ solution (17 becomes (13). Hence, our solution

$$y = Ay_1 + By_2.$$

If the right-hand member of the differential equation is not zero, the particular integral can be found by a method similar to the above. Suppose the equation had been

$$\frac{d^2y}{dx^2} - xy = 12x^2 \tag{22}$$

Since the result of putting $y = c_0 x^m$ in the left-hand member is

$$c_0 m(m-1)x^{m-2} - c_0 x^{m+1}, \tag{23}$$

we shall assume as a particular integral

$$y = c_0 x^m + c_1 x^{m+3} + c_2 x^{m+6} + \cdots. \tag{24}$$

As in the finding of the complementary function above, if the series (24) be substituted for y in (22) the resulting equation of the form (18) must be an identity and

$$c_0 m(m-1)x^{m-2} \equiv 12x^2 \tag{25}$$

while the coefficients of like powers of x, as in (20), cancel each other in pairs.

If (25) is to be satisfied, both the exponents and the coefficients of x must be equal respectively. Hence

$$m - 2 = 2, \qquad \therefore m = 4.$$

$$c_0 m(m-1) = 12, \qquad \therefore c_0 = 1.$$

The other terms will cancel each other in pairs of like powers of x if, as in (20),

$$(m+3r)(m+3r-1)c_r - c_{r-1} = 0$$

When $m = 4$ and $c_0 = 1$

$$c_1 = \frac{1}{7\cdot 6} c_0 \qquad \therefore \; c_1 = \frac{\lfloor 5}{\lfloor 7}$$

$$c_2 = \frac{1}{10\cdot 9} c_1 \qquad \therefore \; c_2 = \frac{\lfloor 5 \cdot 8}{\lfloor 10}$$

$$c_3 = \frac{1}{13\cdot 12} c_2 \qquad \therefore \; c_3 = \frac{\lfloor 5 \cdot 8 \cdot 11}{\lfloor 13}$$

. . . .

$$c_r = \frac{1}{(3r+4)(3r+3)} c_{r-1} \quad \therefore \; c_r = \frac{\lfloor 5 \cdot 8 \cdot 11 \cdots (3r+2)}{\lfloor (3r+4)}$$

Hence, the particular integral of equation (22) is

$$x^4 + \frac{\lfloor 5}{\lfloor 7} x^7 + \frac{\lfloor 5 \cdot 8}{\lfloor 10} x^{10} + \frac{\lfloor 5 \cdot 8 \cdot 11}{\lfloor 13} x^{13} + \cdots + \frac{\lfloor 5 \cdot 8 \cdot 11 \cdots (3r+2)}{\lfloor (3r+4)} x^{3r+4} + \cdots$$

The above methods suggest a more general method for finding both the complementary function and the particular integral.

If the substitution of $y = x^m$ reduces the differential equation to an equation of the form

$$f(m) x^h + \phi(m) x^{h+l} = 0$$

where l is a positive integer, the complementary function may be found in the following manner:

If we let

(26) $$y = c_0 x^m + c_1 x^{m+l} + c_2 x^{m+2l} + \cdots + c_r x^{m+rl} + \cdots,$$

and substitute this series in the differential equation, the equation becomes

(27) $$c_0 f(m) x^h + c_0 \phi(m) x^{h+l} + c_1 f(m+l) x^{h+l} + c_1 \phi(m+l) x^{h+2l}$$

$$+ C_2 f(m+2l)x^{h+2l} + C_2 \phi(m+2l)x^{h+3l}$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ C_{r-1} f(m+[r-1]l)x^{h+[r-1]l} + C_{r-1}\phi(m+[r-1]l)x^{h+rl}$$

$$+ C_r f(m+rl)x^{h+rl} + C_r \phi(m+rl)x^{h+[r+1]l}$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \equiv 0.$$

If the coefficients are so determined as to make the coefficients of like powers of x cancel each other, and if the coefficient of the single term $c_0 f(m)x^h$ is equal to zero, (27) is an identity and (26) is a solution of the original equation. If $f(m)$ is of degree n in m, $f(m) = o$ has n roots, $m_1, m_2, \ldots m_n$, each of which will, in general, give a solution.

$a_0 \neq o$, $\therefore f(m) = o$ and $m = m_1, m_2, \ldots m_n$.

If the coefficients are to cancel each other in pairs, we must have

$$C_r f(m+rl)x^{h+rl} + C_{r-1}\phi(m+[r-1]l)x^{h+rl} \equiv 0 \text{ for } \begin{cases} r = 1, 2, \cdots \infty \\ m = m_1, \cdots m_n \end{cases}$$

Then

$$C_r = -\frac{\phi(m+[r-1]l)}{f(m+rl)} C_{r-1}$$

$$= (-1)^r \frac{\phi(m+[r-1]l)\phi(m+[r-2]l)\cdots\phi(m+l)\phi(m)}{f(m+rl)f(m+[r-1]l)\cdots f(m+2l)f(m+l)} C_0.$$

If any c vanishes, all the following ones do, and our series is finite. For each value of m, in general, we get a particular solution, and the sum of all these particular solutions gives the general solution. If two of the m's are equal, only one particular integral is obtained for both of them. If two of the roots differ by an integral multiple of l, one of the coefficients in the particular integral corresponding to one of these roots will become infinite, unless the numerator also has a zero factor, i.e., if $m_2 = m_1 + gl$, where g is an integer, the coefficient of c_g will be infinite since the denominator has the factor $f(m_2) = f(m_1 + gl) = 0$. Then, in general, we get only as many particular solutions as we have distinct values of m, no two of which differ by a multiple of l.

If f(m) is of degree less than n in m and $\varphi(m)$ is of degree n, the general method must be altered slightly since that method will give less than n particular solutions in this case, while the general solution must have n particular solutions. $\varphi(m) = 0$ will be satisfied by n values of m; we shall call them $m = m_1'$, m_2', . . . m_n'. If we substitute the series

(28) $$y = c_0 x^m + c_{-1} x^{m-l} + c_{-2} x^{m-2l} + \cdots + c_{-n} x^{m-nl} + \cdots$$

in the differential equation, we get, corresponding to (27),

(29)
$$\begin{aligned}
&c_0 \varphi(m) x^{k+l} + c_0 f(m) x^k \\
&+ c_{-1} \varphi(m-l) x^k + c_{-1} f(m-l) x^{k-l} \\
&+ c_{-2} \varphi(m-2l) x^{k-l} + c_{-2} f(m-2l) x^{k-2l} \\
&+ \cdots\cdots\cdots\cdots\cdots \\
&+ c_{-n+1} \varphi(m-[n-1]l) x^{k-[n-2]l} + c_{-n+1} f(m-[n-1]l) x^{k-[n-1]l} \\
&+ c_{-n} \varphi(m-nl) x^{k-[n-1]l} + c_{-n} f(m-nl) x^{k-nl}
\end{aligned}$$

$$+ \cdots \cdots \cdots \cdots \cdots \equiv 0$$

Exactly as in (27) this will be zero if the left-hand member vanishes, i.e., if, since we assume c_0 to be different from zero,

$$\phi(m) = 0 \qquad \therefore m = m_1', m_2', \cdots m_n',$$

(30)

$$C_{-r} = -\frac{f(m-[r-1]l)}{\phi(m-rl)} C_{-r+1} \text{ for } \begin{cases} m = m_1', m_2', \cdots m_n' \\ r = 1, 2, 3, \cdots \infty \end{cases}$$

$$= (-1)^r \frac{f(m-[r-1]l) f(m-[r-2]l) \cdots f(m-l) f(m)}{\phi(m-rl)\phi(m-[r-1]l) \cdots \phi(m-2l)\phi(m-l)} C_0.$$

Our particular solution for each value of m will take the form of the series (28).

The general method for finding the particular integral when the right-hand member is a power of x, consider it Ax^s, follows.

If f(m) is of degree n in m, we must have, from (27)

$$C_0 f(m) x^h + C_0 \phi(m) x^{h+l} + C_1 f(m+l) x^{h+l}$$
$$+ C_1 \phi(m+l) x^{h+2l} + C_2 f(m+2l) x^{h+2l}$$
$$+ \cdots \cdots \cdots \cdots \equiv Ax^s.$$

Equating the first term of the left-hand member to the right-hand member gives

$$C_0 f(m) x^h \equiv Ax^s.$$

The equation,

$$h = s$$

determines one value of m, say m_s, since, from (26), the exponent of x was a linear function of m and differentiation would merely make it a new linear function.

We must also have $\quad C_0 f(m_s) = A.$

The remaining coefficients are determined exactly as in the first general method above. This method will not give a particular integral if $f(m_s) = o$, since then every term would be zero.

If $\varphi(m)$ is of degree n we follow the second general method above. Then

$$C_0 \varphi(m) x^{h+l} \equiv A x^s$$

$$h + l = s \quad \text{determines } m = m'_s$$

$$C_0 \varphi(m'_s) = A$$

The rest of the coefficients are determined as in (30) above using m'_s for m.

PART II

PARTIAL DIFFERENTIAL EQUATIONS. NUMERICAL APPROXIMATIONS

4. <u>The expansion of $E^{mxD'}$ in series.</u>- Partial differential equations that have been solved by integration in series have required individual methods applicable only to particular equations. Individual methods of solution have been applied to many equations in mathematical physics.

In the Analytic Theory of Heat we have, for the change of temperature of a slab of infinite length with parallel plane faces, where the temperature can be regarded as a function of one coordinate,

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

Before attempting a solution let us examine a simpler equation for a solution. The equation

$$\frac{\partial u}{\partial t} = a \frac{\partial u}{\partial x} \tag{1}$$

is also an equation in the field of heat. It is seen by actual substitution that

$$u = f(x + at)$$

is a solution of (1). If $f(x+at)$ is expanded into a Taylor Series, the solution takes the form

$$u = f(x) + at f'(x) + \frac{a^2 t^2}{\lfloor 2} f''(x) + \cdots + \frac{a^n t^n}{\lfloor n} f^n(x). \tag{2}$$

Expressing the right-hand member as a symbolic operator operating on $f(x)$ equation (2) becomes

$$u = \left(1 + atD + \frac{a^2t^2D^2}{\lfloor 2} + \frac{a^3t^3D^3}{\lfloor 3} + \cdots + \frac{a^nt^nD^n}{\lfloor n}\right) f(x)$$

and by analogy of form with the expansion into a Maclauren's series of e^x we may represent the operator by

Then e^{atD}.

(3) $$u = e^{atD} f(x)$$

is a solution of (1).

In order to verify this result we shall replace u in equation (1) by the series (2). Differentiating (2) with respect to t, we get

(4) $$\frac{\partial u}{\partial t} = af'(x) + a^2tf''(x) + \frac{a^3t^2}{\lfloor 2} f'''(x) + \cdots .$$

Differentiating (2) with respect to x and multiplying by a, we get

(5) $$a\frac{\partial u}{\partial x} = af'(x) + a^2tf''(x) + \frac{a^3t^2}{\lfloor 2} f'''(x) + \cdots .$$

Upon substituting the values of $\frac{\partial u}{\partial t}$ and $a\frac{\partial u}{\partial x}$ from (4) and (5) in (1) we get two series that are identically equal, since the coefficients of like terms are equal, and our solution (2) is verified, provided the series converge.

Solution (3) may be written in a manner analogous to the following:

The solution of

(6) $$(D'' - \alpha)u = 0 \qquad \left(D'' = \frac{\partial}{\partial x}\right)$$

where α is independent of t, is

$$u = e^{at}A,$$ [5]

A also being independent of t. This solution is readily verified by substituting the value of u in the differential equation (6).

Transposing the right-hand member in equation (1), the equation becomes

$$\frac{\partial u}{\partial t} - a\frac{\partial u}{\partial x} = 0.$$

Replacing $\frac{\partial}{\partial t}$ by D" and $\frac{\partial}{\partial x}$ by D, we have

$$(D'' - aD)u = 0.$$

In the same way that we wrote the solution of (6), we may write, for a solution of this,

$$u = e^{atD}f(x).$$

This is the same as (3) and has been verified as a solution.

Now to solve our original heat equation

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} \tag{7}$$

transpose the right hand member, replace $\frac{\partial}{\partial t}$ by D" and $\frac{\partial}{\partial x}$ by D, and express as an operator on u,

$$(D'' - a^2D^2)u = 0.$$

Using the method of the preceding discussion, we shall assume our solution to be

$$u = e^{a^2tD^2}f(x). \tag{8}$$

This solution may be verified in the following manner:

Replacing the operator by its expansion corresponding to the expansion of e^x, we get the series

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5 A. R. Forsyth, A Treatise on Differential Equations (London, The Macmillan Company, 1921), p. 512

$$u = \left[1 + a^2tD^2 + \frac{a^4t^2D^4}{\lfloor 2} + \frac{a^6t^3D^6}{\lfloor 3} + \cdots\right]f(x).$$

Indicating the operations on f(x), term by term, we get

(9)
$$u = f(x) + a^2tD^2f(x) + \frac{a^4t^2D^4}{\lfloor 2}f(x) + \frac{a^6t^3D^6}{\lfloor 3}f(x) + \cdots.$$

This is the solution in the form of a series in powers of t.

Differentiating (9) with respect to t, we get

(10)
$$\frac{\partial u}{\partial t} = a^2D^2f(x) + a^4tD^4f(x) + \frac{a^6t^2D^6}{\lfloor 2}f(x) + \cdots.$$

Differentiating (9) with respect to x, we get

$$\frac{\partial u}{\partial x} = Df(x) + a^2tD^3f(x) + \frac{a^4t^2D^5}{\lfloor 2}f(x) + \cdots.$$

Differentiating again with respect to x, we have

(11)
$$\frac{\partial^2 u}{\partial x^2} = D^2f(x) + a^2tD^4f(x) + \frac{a^4t^2D^6}{\lfloor 2}f(x) + \cdots.$$

Substituting (10) and (11) in the original equation (7) we have

$$a^2D^2f(x) + a^4tD^4f(x) + a^6t^2D^6f(x) + \cdots$$
$$\equiv a^2D^2f(x) + a^4tD^4f(x) + a^6t^2D^6f(x) + \cdots.$$

The two series are identically equal, since the coefficients of the terms are equal, and our solution (9) and therefore (8) is verified, provided the series are convergent. This solution contains only one arbitrary function and is not the most general solution.

5. Trigonometric series.- Many partial differential equations have solutions that may be expressed in the form of trig-

onometric series. We may assume

$$u = e^{ay+\beta x}$$

where a and β are constants. This assumption is only tentative and must be verified by substituting in the equation. It can be accepted only if it leads to a solution.

As an illustration of the development of a series let us take a problem of the permanent state of temperatures in a thin rectangular slab of infinite length and breadth π whose long edges are at a constant temperature of zero, and one of the short edges, taken as a base, is held at a temperature of 100 degrees. We assume that the temperature decreases as it recedes from the base.

If we place the base along the x axis with the left corner at the origin, then the left side will lie along the positive y axis. Our solution must be in a form that will enable us to find the temperature at any point in the plate. The equation of the temperature in a rectangular plate is

(1) $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The following conditions must be satisfied:

(1) $u = 0$ when $x = 0$

(2) $u = 0$ " $x = \pi$

(3) $u = 0$ " $y = \infty$

(4) $u = 100$ " $y = 0$

Assume

$$u = e^{ay+\beta x}.$$

Substituting in equation (1), we get

$$\beta^2 e^{ay+\beta x} + a^2 e^{ay+\beta x} = 0.$$

Dividing by $e^{ay+\beta x}$, we get

$$\beta^2 + a^2 = 0, \text{ and } \beta = \pm ai. \quad (i = \sqrt{-1})$$

When $\beta = \pm ai$, $U = e^{ay+\beta x}$ is a solution.

Hence

(2) $$u = e^{ay \pm axi}$$

is a solution for all values of **a**.

If we add the two solutions

$$u = e^{ay} e^{axi} \quad \text{and} \quad u = e^{ay} e^{-axi}$$

and divide by 2, we get

$$u = e^{ay} \frac{e^{axi} + e^{-axi}}{2},$$

or

(3) $$u = e^{ay} \cos ax.$$

We have thus eliminated the imaginary unit from our solution of (1). If we subtract the solutions (2) and divide by 2i, the result is

$$u = e^{ay} \frac{e^{axi} - e^{-axi}}{2i},$$

or

(4) $$u = e^{ay} \sin ax.$$

It is now necessary to build from one of these a solution that will satisfy conditions (1), (2), (3), and (4).

The value of u in the equation

$$u = Ae^{ay} \sin ax$$

where A is any constant, is zero for $x = o$ for all values of a, since $\sin o = o$. Hence the first condition is satisfied. It is zero for $x = \pi$, since $\sin \pi = o$. The second condition is satisfied. If a is negative, $u = o$ when $y = \infty$. Therefore

$$u = \sum_{a=1}^{\infty} A_a e^{-ay} \sin ax$$

is a solution satisfying the first three conditions. This may be written

(5) $$u = A_1 e^{-y} \sin x + A_2 e^{-2y} \sin 2x + A_3 e^{-3y} \sin 3x + \cdots$$

where A_1, A_2, A_3, . . . are undetermined constants.

If $y = o$ (5) becomes

(6) $$u = A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + \cdots .$$

If we can write a series similar to (6) and equal to 100 when $o < x < \pi$ our four conditions are satisfied and the solution is complete when the coefficients of the series equal to 100 are substituted in (5) for the constants. It is known that

$$1 = \frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

for all values of x between o and π. Therefore, the solution is

(7) $$u = \frac{400}{\pi}\left(e^{-y}\sin x + \frac{1}{3}e^{-3y}\sin 3x + \frac{1}{5}e^{-5y}\sin 5x + \cdots\right)$$

As an application of the above problem we shall compute the temperature at the point $\left(\frac{\pi}{6}, 1\right)$, correct to the nearest degree.

Substitute $\frac{\pi}{6}$ for x and 1 for y in equation (7) and calculate three terms. (This will be sufficient to make our result accurate to the nearest degree, since each succeeding term is rapidly approaching zero as a limit.) Equation (7) becomes

$$u = \frac{400}{\pi}\left(e^{-1}\sin 30 + \frac{1}{3}e^{-3}\sin 90 + \frac{1}{5}e^{-5}\sin 150 + \cdots\right)$$

$$u = \frac{400}{3.1416}\left(\frac{1}{2.718}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{(2.718)^3}1 + \frac{1}{5}\cdot\frac{1}{(2.718)^5}\frac{1}{2} + \cdots\right).$$

$u = 25.63$ or 26 degrees to the nearest degree, the temperature at the point $(\frac{\pi}{6}, 1)$.

The series in equation (6) is known as Fourier's half-range series. Fourier made an extensive study of the theory of heat. Equation (1) is very important to that study. Fourier's complete series takes the form

$$u = a_0 + a_1\sin x + a_2\sin 2x + a_3\sin 3x + \cdots$$
$$+ b_1\cos x + b_2\cos 2x + b_3\cos 3x + \cdots.$$

The solution in the form of a trigonometric series lends itself readily to some problems restricted to certain definite regions.

In acoustics the equation

(1) $$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$$

is of value in studying the transmission of plane sound waves through the air, or the transverse vibrations of a stretched elastic string fastened at both ends.

As an illustration of this we shall consider the transverse vibrations of an elastic string of length 1. We shall take the position of equilibrium of the string as the x axis with one end of the string at the origin and the other at the point (1,0), and we shall also assume the string to be initially distorted into a curve whose equation $y = f(x)$ is given.

We must find an expression for y which will be a solution of (1) and also satisfy the conditions

(a) $\quad y = 0 \quad$ when $\quad x = 0,$

(b) $\quad y = 0 \quad$ " $\quad x = l,$

(c) $\quad y = f(x) \quad$ " $\quad t = 0,$

(d) $\quad \frac{dy}{dt} = 0 \quad$ " $\quad t = 0,$

the last condition meaning that the string starts from rest, since $\frac{dy}{dt}$, when $t = o$, is the initial velocity.

As in the problem on heat, we shall assume

(2) $$y = e^{\alpha x + \beta t}$$

and substitute this in equation (1). This gives us

$$\beta^2 e^{\alpha x + \beta t} = a^2 \alpha^2 e^{\alpha x + \beta t}.$$

Divide by $e^{\alpha x + \beta t}$ and we get

$$\beta^2 = a^2 \alpha^2.$$

Then when $\beta = \pm a\alpha$ (2) is a solution and becomes

(3) $$y = e^{\alpha x \pm a \alpha t}.$$

The trigonometric series is preferred to either the exponential or the hyperbolic series, and we can derive one by taking an imaginary value for α in (3). Equation (3) becomes

(4) $$y = e^{(x \pm at)\alpha i}.$$

If we replace α in (3) by a negative, imaginary value $-\alpha i$ we get

(5) $$y = e^{-(x \pm at)\alpha i}.$$

If we add (4) and (5) and divide by 2, we get $\cos \alpha(x \pm at)$.

If we subtract (4) and (5) and divide by $2i$, we get $\sin \alpha(x \pm at)$.

Then

$$y = \cos \alpha(x + at),$$
$$y = \cos \alpha(x - at),$$
$$y = \sin \alpha(x + at),$$
$$y = \sin \alpha(x - at)$$

are solutions of equation (1). If we write y equal to half the sum of the first two, half their difference, half the sum of the last two, and half their difference, respectively, we get four new solutions:

$$y = \cos \alpha x \cos \alpha a t,$$
$$y = \sin \alpha x \sin \alpha a t,$$
$$y = \sin \alpha x \cos \alpha a t,$$
$$y = \cos \alpha x \sin \alpha a t.$$

If we take the third form

$$y = \sin \alpha x \cos \alpha a t,$$

it will satisfy conditions (a) and (d) for all values of α , and it will satisfy (b) if we assign to α a value which will make

αx a multiple of π when $x = 1$. This value of α is $\frac{m\pi}{l}$, where m is an integer. If we assign successive positive integral values to m and introduce the undetermined constants A_1, A_2, . . . we build up a solution in the form of the following series:

$$(6) \qquad y = A_1 \sin\frac{\pi x}{l}\cos\frac{\pi a t}{l} + A_2 \sin\frac{2\pi x}{l}\cos\frac{2\pi a t}{l} + \cdots.$$

This series satisfies the first, second, and fourth conditions. When $t = 0$ (6) becomes, since $\cos 0 = 1$,

$$(7) \qquad y = A_1 \sin\frac{\pi x}{l} + A_2 \sin\frac{2\pi x}{l} + A_3 \sin\frac{3\pi x}{l} + \cdots.$$

If we can now expand f(x) into a series of the form (7), (6) will be a solution satisfying all four conditions when the A_1, A_2, . . . are replaced by the coefficients of the new series. Since f(x) is a known function, this expansion is the Fourier sine series.

6. Numerical approximations to solutions.- In most of the practical applications of differential equations, the solutions are required to abide by certain, previously fixed conditions such as passing through a fixed point, being confined to a definitely bounded region or having a particular slope at a given point. As an example, let us find a particular solution of

$$\frac{dy}{dx} = f(x, y)$$

passing through the point (x_0, y_0). If the solution is found in the form of an infinite series only an approximation of the result can be obtained. We shall illustrate two of these methods of approximation here.

If the successive derivatives with respect to the independent variable can be readily determined for a fixed point, use may be made of the Taylor expansion as discussed in Section I to find a particular solution meeting the required conditions. To illustrate this case, let us find the solution of

$$\frac{dy}{dx} = 2x - y^2$$

passing through the point (o,1). We shall find the successive derivatives at the point (o,1) and substitute the value of x at the point.

$$\frac{dy}{dx} = 2x - y^2 = 1,$$

$$\frac{d^2y}{dx^2} = 2 - 2y\frac{dy}{dx} = 4,$$

$$\frac{d^3y}{dx^3} = -2\left(\frac{dy}{dx}\right)^2 - 2y\frac{d^2y}{dx^2} = -10,$$

$$\frac{d^4y}{dx^4} = -6\frac{dy}{dx}\frac{d^2y}{dx^2} - 2y\frac{d^3y}{dx^3} = 44,$$

$$\frac{d^5y}{dx^5} = -6\left(\frac{d^2y}{dx^2}\right)^2 - 8\frac{dy}{dx}\frac{d^3y}{dx^3} - 2y\frac{d^4y}{dx^4} = -264,$$

. . . .

By Taylor's Theorem, i.e.,

$$y = y_0 + (x - x_0)\left(\frac{dy}{dx}\right)_0 + \frac{(x-x_0)^2}{\lfloor 2}\left(\frac{d^2y}{dx^2}\right)_0 + \frac{(x-x_0)^3}{\lfloor 3}\left(\frac{d^3y}{dx^3}\right)_0 + \cdots$$

where $x_0 = o$, $y_0 = 1$, $\left(\frac{dy}{dx}\right)_0 = -1$, $\left(\frac{d^2y}{dx^2}\right)_0 = 4$, etc., we get

$$y = 1 - x + \frac{4x^2}{\lfloor 2} - \frac{10x^3}{\lfloor 3} + \frac{44x^4}{\lfloor 4} - \frac{264x^5}{\lfloor 5} + \cdots$$

$$= 1 - x + 2x^2 - \frac{5}{3}x^3 + \frac{11}{6}x^4 - \frac{11}{5}x^5 + \cdots$$

as a solution satisfying the given condition.

The second method of approximating the result that we shall consider here is the one derived by Picard and bearing his name.

Let us consider the equation

(1) $$\frac{dy}{dx} = f(x, y).$$

We shall assume

$$y = \phi(x)$$

to be the solution which passes through the point (x_0, y_0). If we replace y in the differential equation by this function of x, we have

$$\phi'(x) = f[x, \phi(x)]\,dx.$$

Integrating between the limits x_0 and x, we get

$$\int_{x_0}^{x} \phi'(x) = \int_{x_0}^{x} f[x, \phi(x)]\,dx,$$

$$\phi(x) - \phi(x_0) = \int_{x_0}^{x} f[x, \phi(x)]\,dx,$$

i.e.,

$$y - y_0 = \int_{x_0}^{x} f[x, \phi(x)]\,dx,$$

or

(2) $$y = y_0 + \int_{x_0}^{x} f[x, \phi(x)]\,dx.$$

The last equation gives the exact value of y for any point on the curve when the exact values of x and $\varphi(x)$ are used, but if we take only an approximate value of $\varphi(x)$, the corresponding value of y will be an approximation, the accuracy of which depends upon the accuracy of the approximate value taken for $\varphi(x)$. Since the solution is to pass through the point (x_0, y_0), we shall start by assigning to $\varphi(x)$ the value $\varphi(x_0)$ or y_0, then (2) becomes

(3) $$y_1 \equiv y_0 + \int_{x_0}^{x} f(x, y_0)\,dx.$$

Since y_0 is only an approximate value of y for any other point in the neighborhood of (x_0, y_0), y_1 is not equal to y, but it is a closer approximation than y_0 is. Now, replacing y_0 in (3) by the new approximation y_1, we get

$$y_2 \equiv y_0 + \int_{x_0}^{x} f(x, y_1)\,dx.$$

Proceeding as before, we find

$$y_3 \equiv y_0 + \int_{x_0}^{x} f(x, y_2)\,dx,$$

$$y_4 \equiv y_0 + \int_{x_0}^{x} f(x, y_3)\,dx,$$

. . . .

. . . .

(4) $$y_n \equiv y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx.$$

Hence, we get the n functions of x: y_0, y_1, y_2, . . . y_n, all of which take the value y_0 when $x = x_0$, since a definite integral vanishes when its limits are equal. These y's are not exact solutions of equation (1) but merely approximations, and the farther along in the sequence the y is taken, the nearer the approximation approaches the exact value of y.

Furthermore, since

$$\frac{\partial}{\partial b}\int_a^b f(x)\,dx = f(b),$$

we have

$$\frac{dy_1}{dx} = f(x, y_0) \qquad \therefore \left(\frac{dy_1}{dx}\right)_0 = f(x_0, y_0).$$

(5)

$$\frac{dy_2}{dx} = f(x, y_1) \qquad \therefore \left(\frac{dy_2}{dx}\right)_0 = f(x_0, y_0).$$

$$\frac{dy_3}{dx} = f(x, y_2) \qquad \therefore \left(\frac{dy_3}{dx}\right)_0 = f(x_0, y_0).$$

. . . .

$$\frac{dy_n}{dx} = f(x, y_{n-1}). \qquad \therefore \left(\frac{dy_n}{dx}\right)_0 = f(x_0, y_0).$$

We see from (5) that, just as was the case of the ordinates in (4), the slopes of the tangents to the curve are found approximately at all the points except at (x_0, y_0), where the slope takes on its exact value.

We shall illustrate Picard's method with the solution of the equation

(6) $$\frac{dy}{dx} = 2x + y^2$$

passing through the origin.

We see immediately that

(7) $$f(x, y_0) = 2x.$$

Then from (7) and (3), we get

$$y_1 = 0 + \int_0^x 2x\,dx = x^2,$$

$$y_2 = 0 + \int_0^x (2x + x^4)\,dx = x^2 + \frac{x^5}{5},$$

$$y_3 = 0 + \int_0^x \left(2x + \left[x^2 + \frac{x^5}{5}\right]^2\right)dx = x^2 + \frac{x^5}{5} + \frac{x^8}{20} + \frac{x^{11}}{275},$$

$$y_4 = 0 + \int_0^x \left(2x + \left[x^2 + \frac{x^5}{5} + \frac{x^8}{20} + \frac{x^{11}}{275}\right]^2\right)dx$$

$$= x^2 + \frac{x^5}{5} + \frac{x^8}{20} + \frac{7x^{11}}{550} + \frac{3x^{14}}{1540} + \frac{87x^{17}}{374,000} + \frac{x^{20}}{55000} + \frac{2x^{23}}{1,739,375}.$$

When $x = 1$, we find from y_4 that $y = 1.2649258$ approximately.

Picard's method may prove to be unsatisfactory in actual practice, mainly because of the difficulty encountered in performing the successive integrations.

PART III

SPECIAL APPLICATIONS

7. Legendre's equation.- The solution of the problem of potential due to a wire ring is based on Laplace's equation expressed in spherical coordinates. Two transformations[6] of that equation changed it into

$$(1) \qquad (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0,$$

where n is a constant.
This equation is commonly known as Legendre's equation, and it is its solution in which we are now interested.

The substitution of the series

$$(2) \qquad y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots\cdots$$

for y in the equation will give a solution of equation (1) if we can determine m and evaluate the coefficients in terms of any two of them. (The complete solution will have two arbitrary constants, since equation (1) is of the second order.)

Computing $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from (2) and forming the terms of (1), we get

$$\frac{d^2y}{dx^2} = m(m-1)a_0 x^{m-2} + m(m+1)a_1 x^{m-1}$$

$$(3) \qquad + (m+2)(m+1)a_2 x^m + \cdots\cdots,$$

$$-x^2\frac{d^2y}{dx^2} = \quad -m(m-1)a_0 x^m - \cdots\cdots,$$

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[6] W. E. Byerly, An Elementary Treatise on Fourier's Series and Spherical, Cylindrical, and Ellipsoidal Harmonics (New York, Ginn and Co., 1893), p. 8.

$$-2x\frac{dy}{dx} = -2ma_0x^m - \cdots,$$

$$n(n+1)y = n(n+1)a_0x^m + \cdots.$$

The sum of the right hand members must be identically zero, since the substitution of (2) in (1) is to give an identity. Therefore, the coefficients of like powers of x in the sum of the right hand members must vanish.

(4) $$m(m-1)a_0 = 0,$$

(5) $$m(m+1)a_1 = 0,$$

(6) $$(m+2)(m+1)a_2 - (m-n)(m+n+1)a_0 = 0.$$

Equation (4) gives us $m = 1$ or 0, assuming from (2) that $a_0 \neq 0$. If we take $m = 0$, from equation (5) we get a_1 arbitrary and from equation (6) we get

$$a_2 = -\frac{n(n+1)}{2}a_0 .$$

The successive coefficients may be obtained by taking more terms of (3). Let us find the general law of coefficients. If the general term of (2) is $a_r x^{m+r}$, we have

$$\frac{d^2y}{dx^2} = \cdots + (m+r)(m+r-1)a_r x^{m+r-2} + \cdots,$$

$$-x^2\frac{d^2y}{dx^2} = \cdots - (m+r-3)(m+r-2)a_{r-2}x^{m+r-2} - \cdots,$$

$$-2x\frac{dy}{dx} = \cdots - 2(m+r-2)a_{r-2}x^{m+r-2} - \cdots,$$

$$n(n+1)y = \cdots + n(n+1)a_{r-2}x^{m+r-2} + \cdots,$$

and since the coefficient of x^{m+r-2} in the sum of these terms also must vanish, we have, after factoring

$$(m+r)(m+r-1)a_r + (n-m-r+2)(n+m+r-1)a_{r-2} = 0.$$

Since $m = 0$,

$$a_r = -\frac{(n-r+2)(n+r-1)}{r(r-1)}a_{r-2}.$$

From this general term we are able to determine any coefficient from the second term preceding it. Hence, a_0 and a_1 being the two arbitrary constants, the general solution of (1) is

$$(7) \qquad y = a_0\left(1 - \frac{n(n+1)}{\lfloor 2}x^2 + \frac{n(n-2)(n+1)(n+3)}{\lfloor 4}x^4 - \cdots\right)$$
$$+ a_1\left(x - \frac{(n-1)(n+2)}{\lfloor 3}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{\lfloor 5}x^5 - \cdots\right).$$

If we take the value $m = 1$, we get from (5) that a_1 is zero and from (6) that

$$a_2 = -\frac{(n-1)(n+2)}{\lfloor 3}a_0.$$

If we proceed to find more terms, we see that they form the second series in our solution (7).

When either a_0 or a_1 is zero, the solution (7) reduces to a single series. If n is a positive even integer, the first series reduces to a polynomial, the degree of whose last term is equal to this particular value of n. This is true since the following term has a factor in the numerator which becomes zero. In like manner, if n is a positive odd integer, the second series reduces to a polynomial whose number of terms is determined in the same way. If we assign to a_0 or a_1, depending on which series we are using, a value which makes the polynomial take the value unity when x is unity, we obtain a system of polynomials known as the Legendre polynomials. Denoting the value of n by the subscript of the polynomial, a few of these polynomials are:

$$P_0(x) = 1,$$

$$P_1(x) = x,$$

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2},$$

$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x,$$

$$P_4(x) = \frac{7\cdot 5}{4\cdot 2}x^4 - 2\frac{5\cdot 3}{4\cdot 2}x^2 + \frac{3\cdot 1}{4\cdot 2},$$

$$P_5(x) = \frac{9\cdot 7}{4\cdot 2}x^5 - 2\frac{7\cdot 5}{4\cdot 2}x^3 + \frac{5\cdot 3}{4\cdot 2}x.$$

The symbol $P_m(x)$ for the particular solution when $n = m$ represents what is also known as Legendre's Coefficient, or as a Surface Zonal Harmonic and is of great value in the solution of many important applications.

8. Bessel functions.- The Bessel equation

(1) $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

where n is a constant, can be solved by the same procedure used in finding the solution of Legendre's equation. We again assume the series

(2)
$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots.$$

If this series (2) is substituted in (1), the terms of (1) become

$$x^2\frac{d^2y}{dx^2} = (m-1)ma_0x^m + (m+1)ma_1x^{m+1} + (m+2)(m+1)a_2x^{m+2} + \cdots,$$

$$x\frac{dy}{dx} = \qquad ma_0x^m + (m+1)a_1x^{m+1} + (m+2)a_2x^{m+2} + \cdots,$$

$$-n^2y = \quad -n^2a_0x^m - n^2a_1x^{m+1} - n^2a_2x^{m+2} - \cdots,$$

$$x^2y = \qquad\qquad\qquad\qquad\qquad\qquad a_0x^{m+2} + \cdots.$$

Since the sum of the left-hand members is zero, the sum of the right-hand members must be identically zero. Equating to zero the coefficients of like powers of x, we get

(3)
$$(m^2 - n^2)a_0 = 0,$$

(4)
$$[(m+1)^2 - n^2]a_1 = 0,$$

(5)
$$[(m+2)^2 - n^2]a_2 + a_0 = 0,$$

. . . .

In order to find the general expression of relation between coefficients, we take

$$x^2\frac{d^2y}{dx^2} = \cdots + (m+r)(m+r-1)a_rx^{m+r} + \cdots,$$

$$x\frac{dy}{dx} = \cdots\cdots (m+r)a_rx^{m+r} + \cdots,$$

$$-n^2y = \cdots\cdots - n^2a_rx^{m+r} - \cdots,$$

$$x^2y = \cdots\cdots + a_{r-2}x^{m+r} + \cdots.$$

Equating to zero the coefficient of x^{m+r}, we get

$$[(m+r)^2-n^2]\,a_r + a_{r-2} = 0.$$

Therefore

(6) $$a_r = -\frac{1}{(m+r)^2-n^2}\,a_{r-2}.$$

Assuming that (2) has a term in x^m, $a_0 \neq 0$; then from (3)

$m^2 - n^2 = 0$, and $m = \pm n$. When $m = n$, (4), (5), and (6) give us

$$a_1 = 0,$$

$$a_2 = -\frac{a_0}{2(2n+2)},$$

. . . .

$$a_r = -\frac{a_{r-2}}{r(2n+r)}.$$

Thus determining our coefficients, the solution (2) becomes, after factoring out a_0x^m;

(7) $$y_1 = a_0x^n\left(1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2\cdot4(2n+2)(2n+4)} - \frac{x^6}{2\cdot4\cdot6(2n+2)(2n+4)(2n+6)} + \cdots\right).$$

In like manner, replacing m by -n, we get the solution

(8) $$y_2 = a_0x^{-n}\left(1 + \frac{x^2}{2(2n-2)} + \frac{x^4}{2\cdot4(2n-2)(2n-4)}\right.$$

$$+ \frac{x^6}{2\cdot4\cdot6(2n-2)(2n-4)(2n-6)} + \cdots).$$

In case $n = 0$ solutions (7) and (8) are identical. If n is a positive integer, the second solution is meaningless, since a factor in the denominator of each term after a certain one is zero, making the series infinite. If n is a negative integer, solution (7) is meaningless for the same reason. Hence, if n is zero or an integer, we get only one solution of our equation. If n is neither zero nor an integer, we get two particular solutions each containing one arbitrary constant. Therefore

$$y = Ay_1 + By_2$$

is a complete solution, and it contains two arbitrary constants.

If we place

$$a_0 = \frac{1}{2^n \underline{|n}}$$

where n is an integer, we get the Bessel function of the first kind and of order n. This is denoted by the symbol $J_n(x)$, and we have, when n is positive,

$$J_n(x) = \frac{x^n}{2^n \underline{|n}}\left(1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^4\underline{|2}\,(n+1)(n+2)}\right.$$

$$\left. - \frac{x^6}{2^6\underline{|3}\,(n+1)(n+2)(n+3)} + \cdots\cdots\right)$$

$$= \frac{x^n}{2^n\underline{|n}} - \frac{x^{n+2}}{2^{n+2}\underline{|n+1}} + \frac{x^{n+4}}{2^{n+4}\underline{|2}\;\underline{|n+2}}$$

$$- \frac{x^{n+6}}{2^{n+6} \lfloor 3 \lfloor n+3} + \cdots .$$

The general term is

$$(-1)^k \frac{x^{n+2k}}{2^{n+2k} \lfloor k \lfloor n+k} .$$

If we take k = 0 we should get the first term of the expansion of $J_n(x)$. We get

$$\frac{x^n}{2^n \lfloor 0 \lfloor n} .$$

This will equal the first term if

$$\lfloor 0 = 1 .$$

This is justified by the general relation

$$\lfloor n-1 = \frac{\lfloor n}{n} .$$

When n = 1 this equation reduces to

$$\lfloor 0 = 1 .$$

The entire series $J_n(x)$ is

$$J_n(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{n+2k}}{2^{n+2k} \lfloor k \lfloor n+k} .$$

This holds for all positive, integral values of n and for n = 0.

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4 (\lfloor 2)^2} - \cdots + (-1)^k \frac{x^{2k}}{2^{2k} (\lfloor k)^2} + \cdots .$$

9. Gauss's equation and the hypergeometric series.- Gauss's equation is of the form

(1) $$(x^2-x)\frac{d^2y}{dx^2}+[(\alpha+\beta+1)x-\gamma]\frac{dy}{dx}+\alpha\beta y=0.$$

We shall solve this by the short method suggested in Section 3. Upon substituting $y = x^m$ in the left-hand member, we get

(2) $$-m(m-1+\gamma)x^{m-1}+(m+\alpha)(m+\beta)x^m=0.$$

This is of the form

$$f(m)x^{m-1}+\varphi(m)x^m=0$$

where

$$f(m)=-m(m-1+\gamma)$$

and

$$\varphi(m)=(m+\alpha)(m+\beta).$$

We shall assume

$$y=C_0x^m+C_1x^{m+1}+C_2x^{m+2}+\cdots\cdots.$$

If we replace y in the differential equation (1) by this series, we get

(3) $$\begin{aligned}&-C_0m(m-1+\gamma)x^{m-1}+C_0(m+\alpha)(m+\beta)x^m\\&-C_1(m+1)(m+\gamma)x^m+C_1(m+\alpha+1)(m+\beta+1)x^{m+1}\\&-\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{aligned}$$

$$-c_{r-1}(m+r-1)(m+r+\gamma-2)x^{m+r-2}+c_{r-1}(m+r+\alpha-1)(m+r+\beta-1)x^{m+r-1}$$

$$-c_r(m+r)(m+r+\gamma-1)x^{m+r-1}+c_r(m+r+\alpha)(m+r+\beta)x^{m+r}$$

$$-\ldots\ldots\ldots\ldots\ldots\ldots\equiv 0.$$

If

(4) $$-c_0 m(m-1+\gamma)=0,$$

it being the only term in x^{m-1}, and if

$$c_0(m+\alpha)(m+\beta)-c_1(m+1)(m+\gamma)=0$$

$$\vdots$$

(5) $$c_{r-1}(m+r+\alpha-1)(m+r+\beta-1)-c_r(m+r)(m+r+\gamma-1)=0,$$

i.e., if the coefficients of like powers of x cancel each other in pairs, the left-hand member of (3) will be identically zero. Assuming $c_0\neq 0$, we get, from (4)

$$-m(m-1+\gamma)=0.$$

Then

$$m=0 \text{ or } 1-\gamma.$$

From (5) we find

$$c_r=\frac{(m+r+\alpha-1)(m+r+\beta-1)}{(m+r)(m+r+\gamma-1)}c_{r-1}.$$

For $m=0$, we get

$$c_1=\frac{\alpha\cdot\beta}{1\cdot\gamma}c_0,$$

$$c_2 = \frac{(\alpha+1)(\beta+1)}{2(\gamma+1)} c_1 = \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot \gamma(\gamma+1)} c_0,$$

. . . .

$$c_n = \qquad = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{1\cdot 2\cdot 3\cdots n\cdot\gamma(\gamma+1)\cdots(\gamma+n-1)} c_0.$$

If these values of the c's are substituted for the c's in

$$y = c_0 x^m + c_1 x^{m+1} + c_2 x^{m+2} + \cdots,$$

we get, as a solution

$$y = c_0 + \frac{\alpha\cdot\beta}{1\cdot\gamma} c_0 x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)} c_0 x^2 + \cdots.$$

In the special case where $c_0 = 1$, we get the particular integral

$$y = 1 + \frac{\alpha\cdot\beta}{1\cdot\gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)} x^2 + \cdots\cdots$$

$$\cdots + \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{1\cdot 2\cdot 3\cdots n\cdot\gamma(\gamma+1)\cdots(\gamma+n-1)} x^n + \cdots.$$

This integral is known as the hypergeometric series, and is usually represented by

$$F(\alpha, \beta, \gamma, x).$$

For $m = 1-\gamma$, we get

$$c_1 = \frac{(1+\alpha-\gamma)(1+\beta-\gamma)}{1\cdot(2-\gamma)} c_0,$$

$$C_2 = \frac{(2+\alpha-\gamma)(2+\beta-\gamma)}{2(3-\gamma)} C_1 = \frac{(1+\alpha-\gamma)(2+\alpha-\gamma)(1+\beta-\gamma)(2+\beta-\gamma)}{1\cdot 2\cdot(2-\gamma)(3-\gamma)} C_0,$$

. . . .

$$C_n = \quad = \frac{(1+\alpha-\gamma)\cdots(n+\alpha-\gamma)(1+\beta-\gamma)\cdots(n+\beta-\gamma)}{1\cdot 2\cdot 3\cdots n\cdot(2-\gamma)(3-\gamma)\cdots(n+1-\gamma)} C_0.$$

Then our solution is

$$y = C_0 x^{1-\gamma}\Big[1 + \frac{(1+\alpha-\gamma)(1+\beta-\gamma)}{1\cdot(2-\gamma)} x + \frac{(1+\alpha-\gamma)(2+\alpha-\gamma)(1+\beta-\gamma)(2+\beta-\gamma)}{1\cdot 2\cdot(2-\gamma)(3-\gamma)} x^2$$

$$+ \cdots + \frac{(1+\alpha-\gamma)\cdots(n+\alpha-\gamma)(1+\beta-\gamma)\cdots(n+\beta-\gamma)}{1\cdot 2\cdot 3\cdots n\cdot(2-\gamma)(3-\gamma)\cdots(n+1-\gamma)} C_0.$$

or

$$y = C_0 x^{1-\gamma}\cdot F(1+\alpha-\gamma, 1+\beta-\gamma, 2-\gamma, x).$$

The complete solution of the original equation is

$$y = A\cdot F(\alpha,\beta,\gamma,x) + B x^{1-\gamma}\cdot F(1+\alpha-\gamma, 1+\beta-\gamma, 2-\gamma, x),$$

where A and B are the arbitrary constants.

Some well-known and important functions may be represented by the hypergeometric series. Let us examine

$$y = F\left(1, \beta, 1, \frac{x}{\beta}\right).$$

Written in the form of the series, we have

$$y = 1 + \frac{1\cdot\beta}{1}\cdot\frac{x}{\beta} + \frac{\beta(\beta+1)}{1\cdot 2}\cdot\frac{x^2}{\beta^2} + \frac{\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3}\cdot\frac{x^3}{\beta^3} + \cdots.$$

This may be written

$$y = 1 + x + \frac{1(1+\frac{1}{\beta})x^2}{1 \cdot 2} + \frac{1(1+\frac{1}{\beta})(1+\frac{2}{\beta})x^3}{1 \cdot 2 \cdot 3} + \cdots$$

$$\text{Limit}_{\beta \to \infty} y = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \cdots.$$

The right hand member is the expansion of e^x by Maclauren's theorem; therefore

$$\text{Limit}_{\beta \to \infty} y = \text{limit}_{\beta \to \infty} F(1, \beta, 1, \frac{x}{\beta}) = e^x.$$

Now let

$$y = x F(\alpha, \beta, \frac{3}{2}, -\frac{x^2}{4\alpha\beta}),$$

then from the definition of the hypergeometric series, we have

$$y = x + \frac{\alpha \cdot \beta}{1 \cdot \frac{3}{2}} \cdot x \cdot \left(-\frac{x^2}{4\alpha\beta}\right) + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \frac{3}{2} \cdot \frac{5}{2}} \cdot x \cdot \left(-\frac{x^2}{4\alpha\beta}\right)^2 + \cdots.$$

This may be written

$$y = x - \frac{x^3}{\underline{|3}} + \frac{1(1+\frac{1}{\alpha}) \cdot 1 \cdot (1+\frac{1}{\beta})}{1 \cdot 2 \cdot \frac{3}{2} \cdot \frac{5}{2}} \frac{x^5}{16} - \cdots.$$

$$\text{Limit}_{\substack{\alpha \to \infty \\ \beta \to \infty}} y = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \cdots.$$

This series is the expansion of sin x; therefore

$$\text{Limit}_{\substack{\alpha \to \infty \\ \beta \to \infty}} y = \text{limit}\; x F(\alpha, \beta, \frac{3}{2}, -\frac{x^2}{4\alpha\beta}) = \sin x.$$

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