COMPETITION AND COLLUSION IN BILATERAL MARKETS [x)]

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Research Memorandum No. 1o7
May, 1976

x) This research was partially supported by a grant given to New York University, Department of Economics, by the Office of Naval Research. A first version of the paper was presented at the Third World Congress of the Econometric Society, Toronto, Aug. 2o-26, 1975, under the title "Symmetric Solutions of Bilateral Market Games".

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Abstract

In this paper, the core of a market game which constitutes the set of equilibria in the process of competitive contracting and recontracting is criticized as a solution concept for not being immune against "theory absorption" in the sense that knowledge of the core on part of the traders may result in a collusive stabilization of some dominated imputation. It is pointed out that a stable set (or, von Neumann-Morgenstern) solution does not suffer from this deficiency. Moreover, it is argued that stable set solutions provide an adequate analytical framework for the study of collusion, and are in this respect superior to the approach (relying on the core concept) chosen by Aumann in his work on disadvantageous monopolies. For symmetric bilateral market games - generated by markets involving the exchange of only two commodities, one of which also serves as a means of side payment and utility transfer, among two types of traders - with one seller and one up to three buyers all symmetric solutions are determined. Furthermore, a symmetric solution for markets with equal, but otherwise arbitrary, numbers of sellers and buyers is given. The symmetric stable sets of imputations are interpreted as rational standards of behavior providing the consistent and defensible rules of division necessary to make a cartel agreement viable.

1. INTRODUCTION

Since VON NEUMANN and MORGENSTERN [1944] for the first time applied the then newly discovered theory of games and their intricate concept of a solution of an n-person cooperative game to the problem of market exchange and the formation of prices, mathematical economic theory has experienced a fast and impressive development. Although the theory of games and the mathematical techniques emerging in its wake have had without doubt a profound impact on the modern treatment f.i. of oligopoly and general economic equilibrium, mathematical economics has been conceptually dominated, however, by two notions which had been well known in economic theory long before the theory of games (but have since undergone, under its influence, a thorough generalization): namely, the non-cooperative equilibrium point of a game in normal (or, strategic) form - which is essentially nothing else than Cournot's solution of the oligopoly problem -, and the core of a cooperative game in characteristic-function form - which was originally introduced by EDGEWORTH [1881] under the name "contract curve" and was also used by BÖHM-BAWERK [1888] in his theory of "marginal pairs".

The notion of core in particular has gained central importance in general equilibrium analysis (for a survey see SCHOTTER [1973], also HILDENBRAND and KIRMAN [1976]) mainly because of its intimate relationship to the concept of

Walrasian competitive equilibrium, a connection (constituted by the convergence of the core to the set of Walrasian competitive equilibrium allocations under rather general conditions) which, incidentally, was also already well known to Edgeworth.

In this paper it is argued that, from the point of view of game theory, the core suffers from a serious conceptual deficiency. The theory of games seeks to define, prove the feasibility of, and compute rational behavior in multi-person decision-making situations involving conflict and cooperation. One of the main requirements of a satisfactory, or at least acceptable, concept of solution (i.e. rational behavior) is that it ought not to be invalidated by the knowledge of the theory on part of the participants in the game - it should be immune against "theory absorption". The core, defined as the set of non-dominated imputations of a cooperative game, does not, in general, satisfy this condition (because there will be, in general, imputations outside the core which are not dominated by any imputation in the core). In terms of economic theory this means the following: The core is the set of imputations (corresponding to a set of price profiles) where the competitive process of contracting and recontracting, in which each trader is assumed to be always willing to annul any contract for the prospect of higher profit in another trading arrangement, comes to an end. It can be easily shown that in this process traders may be played off against each other and manoeuvred into core imputations with which they are worse off than with certain imputations outside the core. Thus, for some traders it proves profitable to stop the process of recontracting at some dominated imputation which they are able to agree upon. Collusion is defined as the practice of stabilizing dominated imputations by means of precontracts between traders of the same type (cartels or, in Edgeworth's terminology, combinations), whereas competition may be defined as the absence of any combinations. This reasoning leads to the somewhat paradoxical statement: If the traders are rational - in the sense of always striving for higher profits - and if they knew that the core were the only stable outcome of the bargaining process, then the core would not be stable. The conclusion is that competition can only prevail if the behavior of the traders is characterized by a peculiar mixture of rationality, complete information about the opportunities the market offers, and short-sightedness. In general, the core as a predictive concept yields a self-defeating prophecy.

The solution concept originally propounded by VON NEUMANN and MORGENSTERN [1944], the so-called stable set solution, overcomes these difficulties: By definition, a stable set of imputations not only possesses "internal stability" (any two elements of a stable set do not dominate each other) but also "external stability" (any imputation outside the solution is dominated by at least one element of the stable set). In this paper it is contended that the rational standard of behavior described by a stable set of imputations provides a consistent and defensible rule of division not only for the case of competitive behavior (the core is always part of the solution) but also for all conceivable collusive arrangements. It is argued that the analysis of collusive phenomena in terms of stable sets is conceptually superior to an approach chosen by AUMANN [1973], who followed a suggestion made by EDGEWORTH [1881], according to which a cartel is treated as a single agent and the core of the correspondingly reduced market game is computed. Besides the avoidance of certain paradoxical results that may come up in the Edgeworth-Aumann approach, the stable set analysis yields a deeper insight into the distributional conditions a collusive precontract has to satisfy in order to make a cartel viable.

In this study, we investigate the class of symmetric bilateral market games generated by markets involving the exchange of only two commodities (one of which also serves as a means of side payment and utility transfer) among two types of traders (sellers and buyers of the one, "non-monetary" commodity). These bilateral market games are a special case of the class of market games studied by SHAPLEY and SHUBIK [1969, 1975] with respect to the core. In some sense, the present investigation is a direct continuation of the analysis begun by VON NEUMANN and MORGENSTERN [1944, pp. 555 ff.]

who gave a complete account of the solutions for bilateral monopolies and markets with one seller and two buyers, and, furthermore, described the market game with one seller and an arbitrary number of buyers (without giving solutions for this general problem of monopoly). In the present paper, all symmetric stable sets for markets with one seller and one up to three buyers are determined. Furthermore, a symmetric solution for markets with equal, but otherwise arbitrary, numbers of sellers and buyers is given. In confining ourselves to the study of symmetric solutions only, this investigation is more special in scope than von Neumann's and Morgenstern's original undertaking. On the other hand, its object is a more general type of bilateral markets than those whose symmetric solutions were studied by SHAPLEY [1959].

2. MARKETS AND MARKET GAMES

A two-sided (or, bilateral) market consists of a finite set T of traders dealing in two different kinds of commodities.[1] The traders are characterised by the commodity bundles in their possession at the beginning of exchange. There are two types of these initial endowments:

$$(a_i, 0) \in \mathbb{R}_+^2 \quad \text{for } i \in M \subset T,$$
$$(0, b_j) \in \mathbb{R}_+^2 \quad \text{for } j \in N \subset T,$$

where $M \cup N = T$ and $M \cap N = \emptyset$. We call $M = \{1, \ldots, m\}$ the set of sellers and $N = \{m + 1, \ldots, m + n\}$ the set of buyers.

Each trader possesses a continuous, non-decreasing, and concave utility function $U_i : \mathbb{R}_+^2 \to \mathbb{R}$ mapping the two-dimensional commodity space into the real numbers. We assume that the utility scales U_i are invariant up to linear transformations and, moreover, that one of the two commodities (say, the second) serves as an ideal standard of value such that the utility functions may be written as

$$U_i\,(x_i, w_i) = R_i(x_i) + w_i, \; i \in T.$$

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1) A market in more than two types of goods is called multilateral.

Thus, utility is measured in terms of that ideal "money" [SHAPLEY and SHUBIK, 1966], which enables a group of traders to redistribute utility among them by means of side payments without changing the specific allocation of the (non-monetary) commodity agreed upon. This assumption of utility functions that are separable in the two commodities and linear in the one which is called money is of course rather strong (it is tantamount to, in the context of the von Neumann-Morgenstern theory of utility, assuming risk neutrality of the traders with respect to money income), but enables us to restrict the domain of the functions R_i, $i \in T$, to the set $\mathbb{Z}_+$ of non-negative integers, i.e. to deal with indivisible units of the first commodity, without disturbing the theory. In general, the assumption of infinitely divisible commodities - which has become customary especially in the mathematical theory of general economic equilibrium but can only in few cases be regarded as an acceptable approximation to reality - is not necessary for the theory of market games as long as there exists one commodity that is infinitely divisible and performs one of the classical functions of money, namely, to serve as a means of compensation in the exchange of indivisible units of the other goods.[2)]

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2) Indivisibilities without the possibility of money side payments, however, may cause analytical difficulties (see, f.i., HILDENBRAND and KIRMAN [1976, pp. 78 ff.]). See also VON NEUMANN and MORGENSTERN [1944, pp. 555 ff.] where a bilateral market game has been defined for finitely divisible goods.

We may interpret

$$U_j(x_j, w_j) - b_j = R_j(x_j) - p_j x_j, \; j\varepsilon N,$$

where $w_j = b_j - p_j x_j$ and p_j is the average money price paid by buyer j for one unit of the commodity, as the profit he is able to make in a different market of his own by selling the commodity there; R_j is his concave revenue function in that market.

Instead of using concave utility (or, revenue) functions R_i, $i\varepsilon M$, defined for the quantities $x_i = a_i - y_i$ that remain unsold, we may as well endow the sellers (or, producers) of the commodity with convex disutility (or, cost) functions $C_i(y_i) = R_i(a_i) - R_i(a_i - y_i)$, $i\varepsilon M$, defined for the quantities y_i produced (or, sold); the a_i's may be regarded as production capacity limits. Thus, we may write

$$U_i(x_i, w_i) - R_i(a_i) = p_i y_i - C_i(y_i), \; i\varepsilon M,$$

where $w_i = p_i y_i$ and p_i is the average money price obtained by seller i for one unit of the commodity, and interpret this expression as standing for producer i's profit made by selling y_i units of the commodity not in some external market (characterised by the revenue function R_i) but in the market under consideration.

After suitable additive transformations of the traders' utility functions, a bilateral <u>market game</u> is defined by the so-called characteristic function $v : 2^T \to \mathbb{R}$ with

$$v(S) = v[(M\cap S)\cup(N\cap S)] = \max_{\substack{\Sigma x_j = \Sigma y_i \\ j\in N\cap S\ i\in M\cap S \\ 0 \leqq y_i \leqq a_i}} \Big[\sum_{j\in N\cap S} R_j(x_j) - \sum_{i\in M\cap S} C_i(y_i)\Big], S\subset T,$$

where $v(S)$ is the maximum utility or profit a coalition $S\subset T$ of traders can achieve by trading among themselves (see VON NEUMANN and MORGENSTERN [1944], SHAPLEY and SHUBIK [1966, 1969]). This simple formulation of the characteristic function v crucially hinges upon our assumption that the revenue $R_i(x_i)$ a trader i is able to make in another market by selling x_i units of the commodity there, is independent of the quantities the other traders might sell externally (implying that the external markets which provide the basis für the traders' valuations of the commodity in the given market have to be either perfectly "competitive" or completely isolated from each other).[3)]

In the following we are dealing with the case of absolute symmetry within the group M of sellers and the group N of buyers, i.e., we assume

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3) If there is interaction among the traders also in these external markets the bilateral market game ceases to be a "game of orthogonal coalitions" [SHAPLEY and SHUBIK, 1973] where the worth $v(S)$ of a coalition S does not depend on the actions of its complement $T\setminus S$. In this more general case $v(S)$ has to be defined as the value of the two-person zero-sum game the trading coalition S plays against $T\setminus S$ whereby the function v loses some of its plausibility.

$$C_i = C \text{ and } a_i = a > 0 \text{ for all } i \in M,$$

$$R_j = R \text{ and } b_j = b \geqq R(ma) \text{ for all } j \in N;$$

furthermore, we impose the normalisation condition

$$R(0) = C(0) = 0,$$

and the condition

$$R(\xi) - C(\xi) > 0 \text{ for some } 0 < \xi \leqq a,$$

ensuring that profitable trade can take place at all (else, the market game is inessential).

Let us denote by $s = |M \cap S|$ and $t = |N \cap S|$ the numbers of sellers and buyers, respectively, in a group S of traders. The corresponding symmetric market game f is derived from a characteristic function v with

$$(2.1) \qquad v(S) = \max_{0 \leqq z \leqq sa} \left[t\, R\left(\frac{z}{t}\right) - sC\left(\frac{z}{s}\right)\right]$$

$$= t\, R\left(\frac{z_{st}}{t}\right) - sC\left(\frac{z_{st}}{s}\right)$$

$$= f(s, t) \text{ for all } S \subset T,$$

thus expressing the fact that under the assumptions made the characteristic function depends only on the numbers of buyers and sellers; z_{st} is the (not necessarily unique) optimum total volume of transactions among a group of traders composed of s sellers and t buyers (by virtue of concavity and convexity, respectively, of revenue and cost functions, z_{st}

is distributed equally among buyers and sellers).

The function f has the property

$$f(s,t) \begin{cases} = 0 \text{ for } s = 0 \text{ or } t = 0 \\ > 0 \text{ otherwise } (s = 1, \ldots, m;\ t = 1, \ldots, n), \end{cases}$$

the game being obviously non-constant-sum, demonstrated f.i. by $f(m, 0) + f(0, n) < f(m, n)$. Moreover, f is the restriction to Z_+^2 of a real function which is homogeneous of degree one, and non-decreasing and concave in each of its arguments.[4)]

An <u>imputation</u> in f is a point $\alpha \in \mathbb{R}^{m+n}$ which satisfies

$$(2.2) \qquad \alpha_i \geqq 0 \text{ for all } i \in T,$$

and

$$(2.3) \qquad \alpha(T) = f(m, n),$$

where $\alpha(S) = \sum_{i \in S} \alpha_i$, $S \subset T$. The set of all imputations in f is an $(m+n-1)$-dimensional simplex, denoted by I. An imputation

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4) f is a generalization of SHAPLEY's [1959] symmetric market game $f(s, t) = \min(s, t)$ which can be derived from a market where each seller possesses exactly one unit of an indivisible commodity and none of the buyers has any use for more than one unit.

is a distribution of the maximum total profit $nR(\frac{z^*}{n}) - mC(\frac{z^*}{m})$, where z^* is the optimum total volume of transactions, among the m+n traders. For $\alpha \epsilon I$,

$$(2.4) \qquad \alpha_i = p_i \frac{z^*}{m} - C(\frac{z^*}{m}), \; i \epsilon M,$$

$$(2.5) \qquad \alpha_j = R(\frac{z^*}{n}) - p_j \frac{z^*}{n}, \; j \epsilon N.$$

Thus, with every imputation $\alpha = (\alpha_1, \ldots, \alpha_m; \alpha_{m+1}, \ldots, \alpha_{m+n})$ there is associated an n-tuple of (average, personal) prices $p = (p_1, \ldots, p_m; p_{m+1}, \ldots, p_{m+n})$ for which

$$(2.6) \qquad p_i \geqq C(\frac{z^*}{m}) / \frac{z^*}{m} \quad \text{for all} \quad i \epsilon M,$$

$$(2.7) \qquad p_j \leqq R(\frac{z^*}{n}) / \frac{z^*}{n} \quad \text{for all} \quad j \epsilon N,$$

$$(2.8) \qquad \frac{z^*}{m} \sum_{i \epsilon M} p_i = \frac{z^*}{n} \sum_{j \epsilon N} p_j$$

hold. Is it possible to say more about the price profiles p emerging in such markets than that the prices accepted by the buyers will not exceed their average revenues, and the prices obtained by the sellers have to cover at least their average costs of production, and further that the total sum of money paid equals the sum of money received? Or, in other words, is every imputation $\alpha \epsilon I$ equally likely to arise as an outcome of the bargaining process among the traders? This question leads to the examination of certain concepts of solution studied in the theory of n-person cooperative games.[5)]

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5) One possibility of selecting a specific imputation would be to impose a so-called "competitive" equilibrium price p^* (perhaps more appropriately termed "parametric" or "administrative" equilibrium price) which balances demand and supply if both buyers and sellers act as utility (profit) maximizing price takers:

(i) $p^* y^* - C(y^*) \geqq p^* y_i - C(y_i)$, $0 \leqq y_i \leqq a$, for all $i \varepsilon M$,

(ii) $R(x^*) - p^* x^* \geqq R(x_j) - p^* x_j$ for all $j \varepsilon N$,

(iii) $\sum_{j \varepsilon N} x_j = \sum_{i \varepsilon M} y_i$.

Obviously, x^* and y^* are maximizing the sum of utilities for the whole set of traders, i.e., $x^* = z^* / n$ and $y^* = z^* / m$, so that p^* determines the imputation α^* with $\alpha_i^* = p^* z^*/m - C(z^*/m)$, $i \varepsilon M$, $\alpha_j^* = R(z^*/n) - p^* z^*/n$, $j \varepsilon N$. If R is strictly concave and C strictly convex, then p^* and α^* are uniquely determined [SHAPLEY and SHUBIK, 1975]. In this paper, however, we are not so much interested in such exogenously given, parametric equilibrium prices, but in negotiated prices resulting from the traders' free haggling over acceptable profit shares.

3. SOLUTION CONCEPTS: COLLUSION VERSUS COMPETITION

Let α and β be some imputations in f, then α is said to dominate β via coalition $S \subset T$, denoted $\alpha\ \mathrm{dom}_S\ \beta$, if

(3.1) $\alpha_i > \beta_i$ for all $i \in S$, and

(3.2) $\alpha(S) \leqq f(s, t)$ for $s = |M \cap S|$ and $t = |N \cap S|$.

An α satisfying (3.2) is said to be S-effective, and S is said to be effective for α. (We speak of strict effectiveness, exact effectiveness, or ineffectiveness if the strict inequality, the equality or neither holds in (3.2), respectively.) If there exists some (non-empty) set $S \subset T$ for which $\alpha\ \mathrm{dom}_S\ \beta$ holds, we say that α dominates β, denoted by α dom β. The dominion $\mathrm{Dom}_S\ \alpha$ of an imputation α via the coalition S is defined as

$$\mathrm{Dom}_S\ \alpha = \{\beta \in I \mid \alpha\ \mathrm{dom}_S\ \beta\},$$

and the dominion Dom α as

$$\mathrm{Dom}\ \alpha = \bigcup_{S \subset T} \mathrm{Dom}_S\ \alpha.$$

Similarly, for any $A \subset I$ we define the dominions

$$\mathrm{Dom}_S A = \bigcup_{\alpha \in A} \mathrm{Dom}_S\ \alpha$$

and

$$\mathrm{Dom}\ A = \bigcup_{S \subset T} \mathrm{Dom}_S\ A$$

(All these dominions are open subsets of I.)

In case of full freedom to trade with anyone, i.e., freedom for every trader to join any coalition, we may expect no trader to be satisfied with his profit share α_i in an imputation α if he is able to find partners offering better terms. We call a market (or, rather, the behavior of buyers and sellers) <u>competitive</u> if every trader is always willing to desert any coalition for the prospect of higher profit in another trading arrangement. <u>Competition</u> can be regarded as the process of eliminating dominated imputations by contracting and recontracting [EDGEWORTH, 1881] which may be thought of as a fictitious tâtonnement-like procedure or may, under certain assumptions of stationarity, take place in real time.[6)] Under competitive conditions so defined, it is clear that only undominated imputations are equilibrium states in the sense that once reached the process of recontracting would stop. The (closed) set $G \subset I$ of all undominated imputations in f, defined by

$$(3.3) \quad G = I \setminus \mathrm{Dom}\, I = \{\alpha \in I \mid \alpha(S) \geqq f(s, t) \text{ for all } S \subset T \text{ with } s = |M \cap S|, \; t = |N \cap S|\}$$

6) WALKER [1973] has convincingly argued that Edgeworth himself eventually preferred to interpret the traders' contracting and recontracting as a non-hypothetical, non-tâtonnement process, an interpretation which restricts the range of the theory to markets where the preferences of the traders do not change over time and no stocks can be carried over from one period to the next - these assumptions are sufficient for what MORGENSTERN [1948] has called the reconstitution of demand and supply.

is called the core[7)8)] of the (symmetric market) game. For bilateral markets in infinitely divisible commodities the notion of the core conceptually coincides with EDGEWORTH's [1881] contract curve [SHUBIK, 1959].[9)] For a bilateral market in an indivisible commodity the core corresponds to BÖHM-BAWERK's [1888] theory of "marginal pairs" (see also VON NEUMANN and MORGENSTERN [1944, pp. 555 ff.] and SHAPLEY and SHUBIK [1972]).

One of the main attractions the concept of core has held for general equilibrium theorists is its peculiar relationship to the notion of Walrasian competitive equilibrium: The imputations associated with Walrasian parametric equilibrium prices (in case of our bilateral markets, the prices that equate demand and supply of the commodity if buyers and sellers are behaving as price takers) are always undominated, i.e., elements of the core [SHAPLEY and SHUBIK, 1975]. Moreover, under certain conditions the core "shrinks" with increasing numbers of traders and converges to the set of Walrasian equilibrium allocations (see, f.i., EDGEWORTH [1881], SHUBIK [1959], HILDENBRAND and KIRMAN [1976]. But one has to be careful in the assessment of the

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7) See SHUBIK [1959], SHAPLEY and SHUBIK [1966, 1969]. The core has also been termed the set of detached imputations by VON NEUMANN and MORGENSTERN [1944].

8) There are games (f.i., all constant-sum games) for which the core is empty. This does not hold for market games. On the contrary, it has been proved that every market game possesses a non-empty core, and, in fact, more than that: It has been shown that the class of games every subgame (i.e., restriction of v to a subset $S \subset T$) of which has a non-empty core and the class of market games are equivalent [SHAPLEY and SHUBIK, 1969].

9) Except that Edgeworth did not assume side payments.

impact of increasing numbers of competitors on the size of the core and the extent of indeterminacy in prices and quantities: Let us, for example, consider a market where only one seller offers exactly one unit of an indivisible good and one buyer demands exactly one unit. In this case the core corresponds to the set of all prices between the respective estimates of the buyer and seller (there is, of course, no indeterminacy with respect to the quantity transferred).
If a second buyer willing to pay the same amount of money as his competitor enters the market, the core will be immediately reduced to a single price (namely, the buyers' valuation of the commodity) but it will now be left open which one of the two potential buyers will actually get the commodity; if another seller with the same estimate of the value of the commodity as the first one comes in, the original indeterminacy in prices will however reappear. If, on the other hand, the second buyer who enters the bilateral monopoly values the commodity less than the first buyer does, the core will only be reduced to the interval between the two buyers' estimates (but, of course, the first buyer will be able to get the commodity); any additional buyer entering the market can only have an influence on the core if his valuation is at least above the second buyer's estimate.

Thus, in case of a Böhm-Bawerkian market a mere increase in the number of traders of the same type (keeping the balance between the numbers of buyers and sellers) does not yield any reduction of indeterminacy - the reason for this being that for a bilateral market in an indivisible commodity the price range predicted by the core coincides with the set of Walrasian equilibrium prices for any number of traders. For an Edgeworthian bilateral market, however, i.e., a market in infinitely divisible commodities and with traders possessing continuous and convex preferences, the core for $m = n = 1$ is generally larger than the set of Walrasian equilibrium allocations and shrinks with an increase in the number of traders of each type; for $m \to \infty$, $n \to \infty$ it converges to the set of Walrasian equilibrium allocations.[10)] It is mainly because of this property that the core has been held in such high esteem in current economic theory (see, f.i., HILDENBRAND and KIRMAN [1976]).

As a solution concept, however, the core has one serious flaw: There are, in general, imputations outside the core which are not dominated by any imputation in the core. SHAPLEY's [1959] symmetric market game $f(s, t) = \min(s, t)$ which is the model of a Böhm-Bawerkian market where the estimates of all sellers and all buyers, respectively, are equal, may serve as an extreme

10) In case of our market games where monetary side payments are admissible, the core is more indeterminate than the Walrasian equilibrium only with respect to prices but not with respect to the quantities exchanged.

example: For $m = 1$ and $n > 1$, the case of monopoly, the core consists of the single imputation (1; 0, ..., 0) which does not dominate any other imputation in the game. If the bargaining starts with an imputation $\alpha \neq (1; 0, \ldots, 0)$, in which some buyers are necessarily better off than in the core, and the buyers are aware that the procedure of free contracting and recontracting (i.e., competition among them) works by offering only temporary gains to some of them while making all worse off in the long run, they can be expected to conclude that it is profitable to stop the process of contracting and recontracting at some imputation outside the core which they are able to agree upon. Such a "precontract between two or more contractors" (traders of the same type) "that none of them will recontract without the consent of all" has been termed a combination [EDGEWORTH, 1881, p. 19]. We call a market (or, again, the behaviour of buyers and sellers) collusive, if combinations play a role in - and have an impact on the outcome of - the bargaining process; collusion is the practice of stabilizing dominated imputations by means of combinations (cartels, trade unions, etc.) while competition may be defined as the absence of any combinations. The above considerations may be summed up in the following, somewhat paradoxical, statement: If the traders are rational (in the sense of always striving for higher profits) and if they knew that the core were the only stable outcome of the bargaining process (and what the core looks like), then the core would not be stable! Thus, competition can only be expected to prevail if the behaviour of the traders is characterised by a peculiar mixture of rationality, complete information about the opportunities the market offers, and short-sightedness.[11)]

A solution concept which overcomes these difficulties has been proposed by VON NEUMANN and MORGENSTERN [1944]: A <u>stable set solution</u> (or, <u>von Neumann-Morgenstern solution</u>) of a game is defined as a subset $V \subset I$ satisfying

(3.4) $$V \cap \text{Dom } V = \emptyset ,$$

and

(3.5) $$V \cup \text{Dom } V = I,$$

or, in one expression,

(3.6) $$V = I \setminus \text{Dom } V.$$

In other words, a stable set solution is a (closed) set of imputations which do not dominate each other - property (3.4), so-called "internal stability" - and, for any imputation not in this set, contains some imputation dominating it - property (3.5), so-called "external stability".

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11) This had been pointed out already in MORGENSTERN [1935], where several aspects of the incompatibility of competitive equilibrium and perfect foresight were discussed in some detail. One might also say that the core is a game-theoretic solution concept - i.e., a definition of rational behavior - which is not immune against "theory absorption": It is an example of a theory of rational action the knowledge of which (on the part of the actors) destroys its predictive validity.

The relationship between the core of a game and a stable set of imputations is somewhat involved and so far not yet completely understood: The core obviously possesses the property of internal stability (by definition, imputations in the core cannot dominate each other), it is, however, not necessarily externally stable(which is exactly the deficiency discussed above). Since the core is the set of undominated imputations it must be a subset of any stable set solution. If the core itself is a stable set, which is for example the case for the market game $f(s, t) = \min(s, t)$ if $m = n$ [SHAPLEY, 1959], then it is the unique stable set solution of the game - because no stable set can by definition be a proper subset of another stable set. There are, however, games with unique stable set solutions different from the core, there are games with a multitude of stable sets - apparently the most frequent case - which may or may not have a non-empty core, and games that do not possess any stable set solution. Moreover, the ten-person game for which LUCAS [1968] has found that no stable set exists, has been shown to be a market game [SHAPLEY and SHUBIK, 1969, 1973] which, therefore, has a non-empty core.

A stable set has been interpreted by VON NEUMANN and MORGENSTERN [1944] as a rational "standard of behavior". What does this mean in the case of market games? Combinations may be conceived of as exogenously given - as part of the rules of the game, so to speak. EDGEWORTH [1881] seemed to have had in mind this sort of combinations when inquiring into the possible imperfections of competition that prevent the contract curve from shrinking when the number of traders is increased - a combination is viewed as existing irrespective of the actual distribution of profits among its members and may, therefore, be treated as a separate player. Hence, in this case, the introduction of combinations into a market game just amounts to a reduction in the number of traders. Recently, several authors [AUMANN, 1973; POSTLEWAITE and ROSENTHAL, 1974] have treated the phenomenon of collusion by studying the core of a market game in which a certain subgroup of traders is regarded as a single agent (termed "syndicate") and no coalition is considered that contains some but not all members of the syndicate. This approach yields the somewhat paradoxical result that syndication may be disadvantageous in the sense that when a subgroup S of traders is syndicated, there are imputations in the core that are worse for S than the core imputation that is the most disadvantageous for the traders in S when they are unsyndicated.

The following example which is due to POSTLEWAITE and ROSENTHAL [1974] elucidates this technique of analysing the consequences of collusion. We consider a market with two sellers and three buyers where each seller owns exactly two units of an indivisible commodity, each buyer has no use for more than one unit, and the respective valuations of the sellers and buyers are equal. The market can then be represented by a characteristic function

$$f(s, t) = \min(2s, t) \text{ for } s = 0, 1, 2; t = 0, 1, 2, 3.$$

The core of this market game consists of the single imputation (0, 0; 1, 1, 1) meaning that fierce competition among the sellers will drive the price of the commodity down to the lower limit they have set for themselves (their opportunity costs) : Whenever one of the sellers wants to raise his price, his competitor will accept a contract offered by two of the buyers interested in taking his two units at a lower price. Thus, the process of contracting and recontracting will stop only at a price equal to the sellers' estimates. What effect would the formation of a cartel consisting of all the buyers have? If, in following the authors mentioned above, we were to consider the market in which the buyers had joined in a cartel as a game with now only three players, namely two sellers and one buyer (willing to take at most three units of the good), represented by the characteristic function

$$f^{*}(s, t) = \min(2s, 3t) \text{ for } s = 0, 1, 2; t = 0, 1;$$

we would find that the core of f^* is the convex hull of the four points (0, 0; 3), (1, 0; 2), (0, 1; 2), and (1, 1, 1). This means that competition among the sellers would not drive down the price necessarily to their opportunity cost - because in order to acquire three units of the commodity, the cartel needs the cooperation of both sellers and cannot, acting as a single entity, split itself to play off seller 1 and seller 2 against each other. As such, this is a highly interesting result showing that being a monopolist under conditions of otherwise free contracting and recontracting may not always turn out to be an advantage. But, in our opinion, this so-called paradox of "disadvantageous syndicates" only underlines the profound distinction between a monopoly proper which is rightly treated as a single player, and a so-called collective monopoly or cartel consisting of several distinct decision-making units (players) which do have the option of leaving the cartel whenever they see fit.

In reality we observe cartels, unions, etc. disintegrating when certain conditions of equitable distribution of utility are not met. Thus, another, more sophisticated and realistic approach to the analysis of collusion and combinations is to look upon the latter as coalitions which may be viable as long as the distribution of the proceeds is satisfactory to each one of their members but may break up if the profit shares do not appeal to some of them. A stable set solution V of a market game is a standard of behavior specifying not only those profit distributions (or price profiles) which are equilibrium states in case of competitive behavior - the core G -, but also those which both provide an incentive for collusion (by not being dominated by any imputation in G) and enable some combinations to uphold them - the set of collusive imputations $V \setminus G$.

In contrast to the core which can be given a predictive interpretation (as we have seen it might even happen that the core of a market game yields a unique imputation) - though it will generally be a self-destroying prophecy - a stable set solution is not meant to predict a specific imputation (or price profile) or subset of imputations. A set containing a single imputation can never be a stable set solution of an essential game (for inessential games the imputation space I is a one-element set, therefore any solution will have to consist of this imputation) because a one-element set, while being trivially internally stable, will not possess external stability. Also, a single element of a stable set does not have any meaning as such. It acquires meaning only in connection with the other imputations in the solution with which it forms a "rational" standard of behavior. Thus, the collusive part $V \setminus G$ of a stable set solution of a market game - which does not show any tendency to shrink when the number of traders is increased - describes a rule of division of the proceeds from forming a cartel which is, in the sense of internal and external stability, both consistent and defensible. Because of its internal stability it can never lead to the paradoxical conclusion that cartelisation is disadvantageous.

4. SYMMETRIC STABLE SETS

A. Preliminaries

A set $A \subset I$ of imputations is called symmetric (in the sets M and N of sellers and buyers, respectively) if

$$(\alpha_1, \ldots, \alpha_m; \ \alpha_{m+1}, \ldots, \alpha_{m+n}) \in A \iff$$

$$(\alpha_{\pi(1)}, \ldots, \alpha_{\pi(m)}; \ \alpha_{\rho(m+1)}, \ldots, \alpha_{\rho(m+n)}) \in A$$

for any permutations $\pi : M \to M$ and $\rho : N \to N$.

The symmetry of the bilateral market games defined in (2.1) suggests the study of their symmetric stable set solutions which may be interpreted as standards of behavior offering equal opportunities to all traders also in the case of collusion (the core of a symmetric game is always a symmetric set of imputations). It should be noted, however, that the symmetry of a game does not enforce a corresponding symmetry in each of its stable sets.[12] There will be, in general, standards of behavior that do not reflect the full symmetry of the game or show no symmetry at all. A special case of

12) Nor does the symmetry of a stable set solution necessarily mean that all players who are given equal opportunities are actually treated equally. As we will see in the case of the market with one seller and tree buyers, equal opportunities (symmetry of the standard of behavior) may be in keeping with inequalities in the results.

non-symmetric standards of behavior are the so-called discriminatory stable set solutions [VON NEUMANN and MORGENSTERN, 1944, pp. 289 f.] where one or more players are completely excluded from any participation in the negotiations and are assigned shares in the joint profit that remain constant throughout the whole range of imputations prescribed by the stable set solution.

An imputation α is called <u>ordered</u> if

$$\alpha_1 \geqq \alpha_2 \geqq \ldots \geqq \alpha_m \quad \text{and} \quad \alpha_{m+1} \geqq \alpha_{m+2} \geqq \ldots \geqq \alpha_{m+n}.$$

We say that an ordered imputation α dominates an ordered imputation β via (s,t)-coalitions $S, S' \subset T$ with $|M \cap S| = |M \cap S'| = s$, $|N \cap S| = |N \cap S'| = t$, denoted $\alpha \text{ dom}_{s,t}\, \beta$, if

(i) $\alpha_i > \beta_{\pi(i)}$ for all $i \in M \cap S$, $\pi(i) \in M \cap S'$ and some permutation π of M;

(ii) $\alpha_j > \beta_{\rho(j)}$ for all $j \in N \cap S$, $\rho(j) \in N \cap S'$ and some permutation ρ of N;

(iii) $\alpha(S) \leqq f(s,t)$.

An ordered imputation α dominates an ordered imputation β, denoted $\alpha \text{ dom } \beta$, if there exist such (s,t)-coalitions for some $s = 1, \ldots, m$, $t = 1, \ldots, n$.

Since, in the following, we are looking for symmetric stable sets, our task will always be to decide whether at least one imputation α in a symmetric set A dominates some imputation β. For this purpose, it will suffice to find out whether some ordered imputation $\alpha \varepsilon A$ dominates the ordered imputation β' we get from β. Hence, we have only to deal with domination between ordered imputations [VON NEUMANN and MORGENSTERN, 1961; HART, 1973]. In the remainder of the paper, if not explicitly stated otherwise, α, β, etc. always denote ordered imputations, I denotes the set of all ordered imputations, and "dom" refers to domination between ordered imputations.

B. The Case of Monopoly

First, we are going to analyse the case of monopoly or monopsony, $\min(m, n) = 1$. Without loss of generality, we may assume $m = 1$ and $n \geq 1$.

Let us replace, for the sake of convenience, the characteristic function f, as defined in (2.1), by g where

(4.1) $$g(t) = f(s, t) \quad \text{for} \quad s = 1,\ t = 0, \ldots, n;$$

g is an increasing, concave[13] function with $g(0) = 0$.

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13) In the sense that g is the restriction of a concave function to the domain $\mathbb{Z}_+$.

We denote by ν_t, with

$$(4.2) \qquad \nu_t = \frac{1}{t}\,[g(n) - g(n - t)],\ t = 1, \ldots, n,$$

$$\nu_0 = 0,$$

the average marginal utility of the last t buyers (or, if we interpret g as a kind of production function, the average marginal product of the last t units of the only variable factor of production), and observe that, by virtue of the concavity of g,

$$(4.3) \qquad \nu_0 \leqq \ldots \leqq \nu_n.$$

An ordered imputation $\alpha = (\alpha_1; \alpha_2, \ldots, \alpha_{n+1}) \epsilon\, \mathbb{R}^{n+1}$ satisfies

$$(4.4) \qquad \alpha_1', \alpha_{n+1} \geqq 0,$$

$$(4.5) \qquad \alpha_1 + \alpha(N) = g(n) = n\nu_n.$$

<u>P r o p o s i t i o n 1:</u>

The set of imputations

$$(4.6) \qquad G_{1,n} = \{\alpha \epsilon I \mid \nu_1 \geqq \alpha_2\}$$

is the core of the market game defined in (2.1) with $|M| = 1$ and $|N| = n$.

Proof:

A necessary condition for α belonging to the core is $\alpha_1 + \alpha(N \setminus \{i\}) \geqq g(n-1)$, $i \in N$. This implies $\nu_1 = g(n) - g(n-1) \geqq \alpha_2$.

Conversely, $\nu_1 \geqq \alpha_2$ implies $\nu_t \geqq \alpha_i$ for all $i \in N$ and $t = 1, \ldots, n$. From this we get

$$g(n) - g(n-t) \geqq \alpha(L) \text{ with } |L| = t,$$

$$\alpha_1 + \alpha(N \setminus L) \geqq g(n-t)$$

for all $L \subset N$ with $|L| = t = 1, \ldots, n$, which are precisely the core conditions.

Q.E.D.

Thus, if the behavior of the traders on the demand side of the market is competitive, the buyers - played off against each other by the monopolist - will not be able to get more than what the last buyer contributes to the maximum joint profit obtainable in the market.[14)]
What will be the structure of a symmetric standard of behavior which does not only provide for a competitive

14) For the already mentioned special case of a market game, $f(s, t) = \min(s, t)$ and $n \geqq 2$, $\nu_1 = 0$ and $G_{1,n} = \{\alpha \varepsilon I \mid \alpha_2 = 0\}$; the price is driven up to the average revenue the buyers are able to obtain in their external markets.

conduct but also for collusive arrangements on part of the buyers? In order to study this question we have to investigate the set $I \setminus G_{1,n} = \text{Dom } I$ of imputations.
We note that α dom β implies $\alpha_1 > \beta_1$ (because every effective coalition has to contain the monopolist).
Furthermore, we observe that for every $\alpha \in \text{Dom } I$ there exists a positive integer $t < n$ such that

$$\alpha_2 + \ldots + \alpha_{t+1} > t\nu_t ,$$

which is equivalent to

$$g(n-t) > \alpha_{t+2} + \ldots + \alpha_{n+1} + \alpha_1 ,$$

and

$$\alpha_2 + \ldots + \alpha_{t+2} \leqq (t+1)\, \nu_{t+1} ,$$

which is equivalent to

$$g(n-t-1) \leqq \alpha_{t+3} + \ldots + \alpha_{n+1} + \alpha_1 .$$

I.e., there are coalitions with n-t or more buyers (excluding t or less buyers) that are strictly effective for α, whereas all coalitions with n-t-1 or less buyers (excluding the first t+1 or more buyers) are exactly effective or ineffective for α. Thus, for every $\alpha \in \text{Dom } I$ there is an integer $0 < t < n$ such that α is dominated via $(n-t, 1)$ - or larger coalitions, and cannot be dominated via any $(n-t-1,1)$ - or smaller coalitions.

For n = 1, the case of bilateral monopoly, no imputation can be dominated, thus, $G_{1,1} = V_{1,1} = I$ [VON NEUMANN and MORGENSTERN, 1944, pp. 560 ff.].

For n = 2, we obtain the following result[15]:

P r o p o s i t i o n 2:

The set of imputations

(4.7) $V_{1,2} = \{\alpha\varepsilon I \mid \nu_1 \geqq \alpha_2\} \cup \{\alpha\varepsilon I \mid \nu_2 \geqq \alpha_2 = \alpha_3 \geqq \nu_1\}$

is the unique symmetric stable set solution of the market game defined in (2.1) with $|M| = 1$ and $|N| = 2$.

Proof:

The internal stability of the set $V_{1,2}$ is obvious. In order to show its external stability it is easy to verify that every imputation β with $\beta_2 > \nu_1 > \beta_3$ is dominated by some α with $\nu_1 = \alpha_2$, and every β with $\beta_2 > \beta_3 \geqq \nu_1$ is dominated by some α with $\beta_2 > \alpha_2 = \alpha_3 > \beta_3$. No symmetric set of imputations β with $\nu_2 \geqq \beta_2 > \beta_3 \geqq \nu_1$ can be internally stable, for there is always an α with $\nu_2 \geqq \alpha_2 > \alpha_3 \geqq \nu_1$ dominating β via some (1,1)-coalition. This proves the uniqueness of $V_{1,2}$.

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15) This finding is not essentially different from SHAPLEY's [1959] result for the special case f(s,t) = min (s,t).

The interpretation of this result is straightforward. Besides the core, the symmetric standard of behavior contains as its collusive part only the set of imputations $\{\alpha\epsilon I \mid \nu_2 \geqq \alpha_2 = \alpha_3 \geqq \nu_1\}$ meaning that the only possibility for the buyers to obtain more than in the core is to form a cartel which has to distribute its profit equally between its two members. It is instructive to compare this conclusion with the result from the Edgeworth-Aumann approach: If we treat the coalition consisting of the two buyers as a single player and compute the core of the resulting reduced game we find that the payoff to the buyers' cartel will lie in the interval $2\nu_2 \geqq \alpha_2 + \alpha_3 \geqq 0$ which is compatible with the symmetric standard of behavior but of lower informational content (because the Edgeworth-Aumann analysis does not give any rule of division to be observed by the cartel).

The contrast between the Edgeworth-Aumann method and the stable set approach to the analysis of collusion is even more conspicuous for the case n = 3 :

P r o p o s i t i o n 3:

The sets of imputations

(4.8) $$V^{c,\varphi}_{1,3} = G_{1,3} \cup V' \cup V''_{c,\varphi} \cup V'''_{c}$$

with

$$G_{1,3} = \{\alpha\epsilon I \mid \nu_1 \geqq \alpha_2\},$$

$$V' = \{\alpha\epsilon I \mid \nu_2 \geqq \alpha_2 = \alpha_3 \geqq \nu_1 \geqq \alpha_4\},$$

$$V''_{c,\varphi} = \{\alpha\varepsilon I \mid c \geqq \alpha_2 = \alpha_3 \geqq \nu_2,\ 2\nu_2 - c \geqq \alpha_4 \geqq \nu_1,\ \alpha_4 \ \varepsilon\varphi(\alpha_2)\},$$

$$V'''_c = \{\alpha\varepsilon I \mid \nu_3 \geqq \alpha_2 = \alpha_3 = \alpha_4 \geqq c\},$$

where φ is any continuous, convex-valued, and monotonically non-decreasing[16)] mapping from $[\nu_2, c]$ onto $[\nu_1, 2\nu_2 - c]$ and c is any real number in $[\nu_2, \nu_3]$ with $c \leqq 2\nu_2 - \nu_1$, constitute the family of all symmetric stable set solutions of the market game defined in (2.1) with $|M| = 1$ and $|N| = 3$.

<u>Proof:</u>

a) Let us first show the internal stability of a set $V^{c,\varphi}_{1,3}$: Any $\alpha\varepsilon V'$ can only be dominated by some imputation β with $\beta_2 > \beta_3 \geqq \nu_1$; thus, $\beta \notin V^{c,\varphi}_{1,3}$.

Any $\alpha \ \varepsilon \ V''_{c,\varphi}$ can only be dominated by some β with either $\beta_2, \beta_3 > \alpha_2$ and $\beta_4 < \alpha_4$, or $\beta_2, \beta_3 < \alpha_2$, $\beta_2 + \beta_3 \geqq 2\nu_2$, and $\beta_4 > \alpha_4$; thus, $\beta \notin V^{c,\varphi}_{1,3}$ because $\beta_4 \notin \varphi(\beta_2)$. (Observe that an imputation α with $c \geqq \alpha_2 = \alpha_3 \geqq \nu_2$, but $\alpha_4 > 2\nu_2 - c$ may be dominated by some imputation β with $\beta_2 = \beta_3 < \alpha_2$ and $\alpha_4 > \beta_4 > 2\nu_2 - c$, so that in this case the condition of monotonic variation of α_4 with α_2 would not suffice to preserve internal stability.)

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16) $\alpha_4 \ \varepsilon \ \varphi(\alpha_2)$, $\beta_4 \ \varepsilon \ \varphi(\beta_2)$, $\alpha_2 > \beta_2 \Rightarrow \alpha_4 \geqq \beta_4$.

Any $\alpha \in V_c'''$ can only be dominated by some β with $\beta_2 > \alpha_2 > \beta_4$, implying that $\beta \notin V_{1,3}^{c,\varphi}$. Moreover, any symmetric set of imputations violating the rules of distribution given by the sets V', $V''_{c,\varphi}$, V'''_c in any one of the respective ranges is not internally stable.

b) In order to prove the external stability of a set $V_{1,3}^{c,\varphi}$ we have to show that any $\beta \in \mathrm{Dom}\ I \setminus V_{1,3}^{c,\varphi}$ is dominated by some $\alpha \in V_{1,3}^{c,\varphi}$.

We consider first the set $A = \{\alpha \varepsilon I \mid \alpha_2 > \nu_1,\ \alpha_2 + \alpha_3 \leqq 2\nu_2\}$. For $\beta \in A$ with $\nu_1 > \beta_3$ there always exists an $\alpha \in G_{1,3}$ with α dom β.

For $\beta \in A \setminus V'$ with $\beta_3 \geqq \nu_1$ and $\beta_4 < \nu_1$ there is an $\alpha \in V'$ with $\alpha_4 < \nu_1$ dominating β.

For $\beta \in A \setminus V'$ with $\beta_4 \geqq \nu_1$ there exists an $\alpha \in V'$ with $\alpha_4 = \nu_1$ and α dom β.

Next, we consider the set $B = \{\alpha \varepsilon I \mid \alpha_2 + \alpha_3 > 2\nu_2, \alpha_2 + \alpha_3 + \alpha_4 \leqq 3\nu_3\}$. For $\beta \in B$ with $\beta_2 > \nu_2 > \beta_3$ there is an $\alpha \in G_{1,3} \cup V'$ dominating it.

For $\beta \in B$ with $\beta_3 \geqq \nu_2$ and $\nu_1 > \beta_4$ there is an $\alpha \in V'$ with $\alpha_2 = \alpha_3 = \nu_2$, $\nu_1 \geqq \alpha_4$, dominating β.

Let us now consider an imputation $\beta \in B$ with $c \geqq \beta_2 = \beta_3 \geqq \nu_2$, $2\nu_2 - c \geqq \beta_4 \geqq \nu_1$, but $\beta_4 \notin \varphi(\beta_2)$. There exists an α with $\alpha_4 \in \varphi(\alpha_2)$, for some given φ, and either $\beta_2 > \alpha_2 = \alpha_3 \geqq \nu_2$ and $2\nu_2 - c \geqq \alpha_4 > \beta_4$, or $c \geqq \alpha_2 = \alpha_3 > \beta_2$ and $\beta_4 > \alpha_4 \geqq \nu_1$, such that α dom β. It is obvious that without continuity and convex-valuedness of the mapping there may be some β which is not dominated by any $\alpha \in V''_{c,\varphi}$ (see figures 1 and 2 below, for a geometrical illustration of the argument).

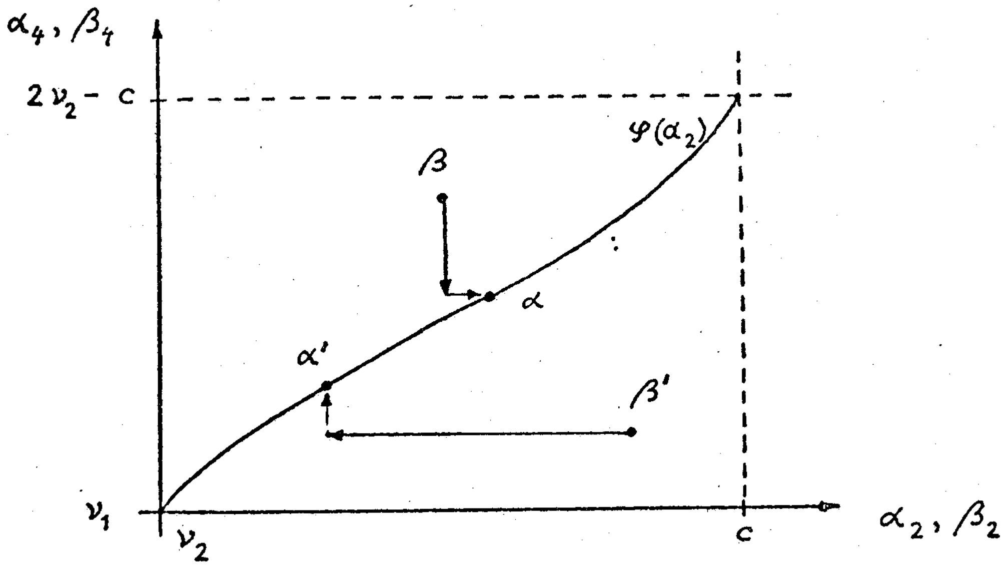

Figure 1

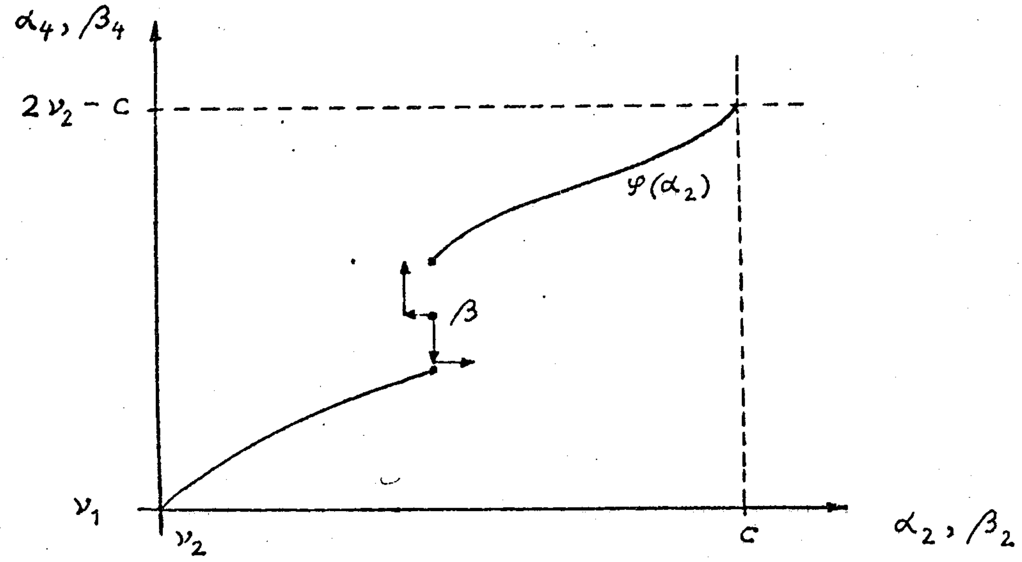

Figure 2

An analogous argument holds for imputations $\beta \varepsilon B \setminus V''_{c,\mathcal{S}}$ with $c \geqq \beta_2 > \beta_3 \geqq \nu_2$, $\beta_4 \geqq \nu_1$; with $c > \beta_2 = \beta_3 \geqq \nu_2$, $\beta_4 > 2\nu_2 - c$; and with $\beta_2, \beta_3 \geqq c$, $2\nu_2 - c > \beta_4 \geqq \nu_1$.

For $\beta \varepsilon B$ with $\beta_2, \beta_3 \geqq c > \beta_4 \geqq 2\nu_2 - c$ and $\beta \notin V''_{c,\mathcal{S}}$ it is easy to see that there exists an imputation $\alpha \varepsilon V''_{c,\mathcal{S}}$ with $\alpha_2 = \alpha_3 = c$ and $\alpha_4 = 2\nu_2 - c$ such that α dom β.

For any $\beta \varepsilon B \setminus V'''_c$ and $\beta_4 \geqq c$ it is clear that there is always some $\alpha \varepsilon V'''_c$ with $\alpha_4 > c$ with dominates β.

Q.E.D.

If we interpret, again following the Edgeworth-Aumann proposal, collusion simply as a competitive process involving contractors some of which are (exogenously given) unions of traders, and, therefore, apply the core concept to the appropriately reduced market game (or, what is essentially equivalent, disregard those core conditions for the original game which correspond to coalitions not containing the "syndicates") we obtain the following results for the market with one seller and three buyers: If we assume that two of the three buyers, say traders 2 and 3, form a cartel the core analysis tells us that their share will lie in the interval

$2\nu_2 \geqq \alpha_2 + \alpha_3 \geqq 0$ while their competitor will be confined to $\nu_1 \geqq \alpha_4 \geqq 0$. If a cartel consisting of all three buyers is formed the outcome predicted by the core of the corresponding two-person game, $3\nu_3 \geqq \alpha_2 + \alpha_3 + \alpha_4 \geqq 0$, is even more indeterminate. The analysis of collusion in terms of symmetric stable sets is again not incompatible with but rather more informative than the above stated results: The part V' of a symmetric standard of behavior specifies the one-half-each rule of division to be observed by a cartel consisting of two buyers. The part $V''_{c,\varphi} \cup V'''_c$ corresponds to the formation of a cartel comprising all three buyers; it shows that the one-third-each rule (V'''_c) that one might expect only holds in case the cartel's joint proceeds are equal to or exceed the amount of 3 c (where c is an arbitrary real constant chosen from the interval $[\nu_2, \nu_3]$ and satisfying $c \leqq 2\nu_2 - \nu_1$). If the cartel settles for a joint profit between $2\nu_2 + \nu_1$ and $2\nu_2 + c$ it has to split its proceeds unevenly among its members such that two of them receive equal shares lying between ν_2 and c, and the third is worse off with a profit of at most $2\nu_2 - c$ (where for any c the exact rule of division is given by the form of the mapping φ). However, in case the cartel as a whole succeeds in securing a bigger profit, $V''_{c,\varphi}$ stipulates that none of its members must lose (this is the meaning of the monotonicity of φ). The analysis also permits the conclusion that, given our assumption of equal opportunities for all traders, the buyers will never settle for joint proceeds lying in the interval $(2\nu_2 + c, 3c)$ because there is no symmetric standard of behavior providing an acceptable rule of division for this range of profits.

C. Same Number of Traders on Both Sides

Next, we analyse the case where the number of sellers equals the number of buyers, $m = n$.

We denote by $\mu_s(t)$, with

$$(4.9) \quad \mu_s(t) = \frac{1}{s}\left[f(m,t) - f(m-s,t)\right], \quad s = 1, \ldots, m; \; t = 0, \ldots, m,$$

$$\mu_0(t) = 0, \qquad t = 0, \ldots, m,$$

the average marginal utility of the last s sellers joining an (m,t)-coalition.

Analogously,

$$(4.10) \quad \nu_t(s) = \frac{1}{t}\left[f(s,m) - f(s,m-t)\right], \quad t = 1, \ldots, m; \; s = 0, \ldots, m,$$

$$\nu_0(s) = 0, \qquad s = 0, \ldots, m,$$

is the average marginal utility of the last t buyers in an (s,m)-coalition.

We observe that, by virtue of the properties of $f(s,t)$,

$$(4.11)\quad \begin{array}{ll} \mu_0(t) \leqq \ldots \leqq \mu_m(t) & \text{for } t = 0, \ldots, m, \\ \nu_0(s) \leqq \ldots \leqq \nu_m(s) & \text{for } s = 0, \ldots, m, \end{array}$$

and

$$(4.12)\quad \begin{array}{ll} \mu_s(0) \leqq \ldots \leqq \mu_s(m) & \text{for } s = 0, \ldots, m, \\ \nu_t(0) \leqq \ldots \leqq \nu_t(m) & \text{for } t = 0, \ldots, m. \end{array}$$

An ordered imputation $\alpha = (\alpha_1, \ldots, \alpha_m; \alpha_{m+1}, \ldots, \alpha_{2m}) \in \mathbb{R}^{2m}$ satisfies

$$(4.13)\qquad \alpha_m, \alpha_{2m} \geqq 0,$$

$$(4.14)\qquad \alpha(M) + \alpha(N) = m\mu_m(m) = m\nu_m(m).$$

<u>Proposition 4:</u>

The set of imputations

$$(4.15)\quad G_{m,m} = \{\alpha\varepsilon I \mid \mu_1(m) \geqq \alpha_i = \alpha' \geqq \mu_1(m-1),$$
$$\nu_1(m) \geqq \alpha_j = \alpha'' \geqq \nu_1(m-1),\ i\varepsilon\, M,\ j\varepsilon N\}$$

is the core of the market game defined in (2.1) with $|M| = |N| = m$.

Proof:

A necessary condition for $\alpha\varepsilon I$ being undominated is $\alpha_i + \alpha_j \geqq f(1,1)$ for all $i\varepsilon M$, $j\varepsilon N$. From this and $\alpha(M) + \alpha(N) = f(m,m) = mf(1,1)$ follows that $\alpha_i + \alpha_j = f(1,1)$, which implies $\alpha_i = \alpha'$, $\alpha_j = \alpha''$, $\alpha' + \alpha'' = f(1,1)$ for all $i\varepsilon M$, $j\varepsilon N$.

A further necessary condition for α belonging to the core is $(m-1)\ \alpha' + m\alpha'' \geqq f(m-1,\ m)$ implying that $\alpha' \leqq f(m,m) - f(m-1,m) = \mu_1(m)$ and $\alpha'' \geqq f(m-1,\ m) - f\ (m-1,\ m-1) = \nu_1(m-1)$.

By the same argument $m\alpha' + (m-1)\ \alpha'' \geqq f(m,\ m-1)$ implies $\alpha'' \leqq f(m,\ m) - f(m,\ m-1) = \nu_1(m)$ and $\alpha' \geqq f(m,\ m-1) - f\ (m-1,m-1) = \nu_1(m-1)$.

On the other hand, because of

$$f(m-1,\ m) \geqq f(m-t,\ m-t) + f\ (t-1,t),$$

we have for $t > s$,

$$\alpha' \leqq \mu_1(m) \leqq f(t,t) - f(t-1,\ t) \leqq \frac{1}{t-s}\left[f(t,t) - f(s,t)\right],$$

hence,

$$(t-s)\ \alpha' \leqq f(t,t) - f(s,t),$$

(i) $f(s,t) \leqq s\ \alpha' + t\alpha''$ for all $s,t,t > s$.

Analogously,

$$f(m, m-1) \geqq f(m-s, m-s) + f(s, s-1)$$

implies

$$\alpha'' \leqq \nu_1(m) \leqq f(s,s) - f(s,s-1) \leqq \frac{1}{s-t}\left[f(s,s) - f(s,t)\right]$$

for s > t, hence,

$$(s-t)\ \alpha'' \leqq f(s,s) - f(s,t),$$

(ii) $f(s,t) \leqq s\ \alpha' + t\alpha''$ for all s,t, s > t.

From $\alpha' + \alpha'' = f(1,1)$ follows

(iii) $f(s,t) = s\alpha' + t\alpha''$ for all s,t,s = t.

Conditions (i), (ii), (iii) are exactly the core conditions.

Q. E. D.

We note that in the core of a two-sided, symmetric market game with equal numbers of sellers and buyers each trader is allotted a profit the maximum and minimum of which is given by his respective marginal contributions (as the last trader of his type) to trading coalitions comprising m and m-1 traders of the other type. Moreover, in this case, a single price

$$p = \frac{\alpha' + C(y^*)}{y^*} = \frac{R(y^*) - \alpha''}{y^*}$$

for the commodity corresponds to any undominated imputation.

In order to construct a stable set solution we have to look, as a first step, for those dominated imputations which are not dominated by any element of the core. It is easy to show that every (ordered) imputation $\alpha \in \text{Dom } I$ with $\alpha_m < \mu_1(m)$ and $\alpha_{2m} < \nu_1(m)$ is dominated by some imputation in the core, and is therefore not an element of any conceivable stable set. This follows from $\alpha_m + \alpha_{2m} < f(1,1)$ which is in turn a consequence of $\alpha_1 > \alpha_m$ or $\alpha_{m+1} > \alpha_{2m}$. $\text{Dom } I \setminus \text{Dom } G_{m,m}$ is therefore the set of imputations α with $\alpha_1 = \alpha_m > \mu_1(m)$ or $\alpha_1 > \alpha_m \geq \mu_1(m)$ (implying $\alpha_{2m} < \nu_1(m-1)$), or, $\alpha_{m+1} = \alpha_{2m} > \nu_1(m)$ or $\alpha_{m+1} > \alpha_{2m} \geq \nu_1(m)$ (implying $\alpha_m < \mu_1\ (m-1)$).

Moreover, we observe that any imputation α for which $\alpha_1 > \alpha_m$ or $\alpha_{m+1} > \alpha_{2m}$ hold, is dominated via (1,1)-coalitions by some imputation β with $\beta_i = \beta'$, $i \in M$, and $\beta_j = \beta''$, $j \in N$. Furthermore, it is obvious that any two imputations $\alpha, \beta \in I$ with $\alpha_i = \alpha'$, $\beta_i = \beta'$, $i \in M$, and $\alpha_j = \alpha''$, $\beta_j = \beta''$, $j \in N$, do not dominate each other. These observations prove

P r o p o s i t i o n 5:

The set of imputations

$$(4.16)\quad V_{m,m} = \{\alpha\varepsilon I \mid \mu_m(m) \geqq \alpha_i = \alpha' \geqq 0,\ \nu_m(m) \geqq \alpha_j = \alpha'' \geqq 0,\ i\varepsilon M,\ j\varepsilon N\}$$

is a symmetric stable set of the market game defined in (2.1) with $|M| = |N| = m$. [17)]

$V_{m,m}$ is a standard of behavior which allows only combinations comprising all sellers or all buyers. The part of $V_{m,m}$ which satisfies the condition

$$\mu_m(m) \geqq \alpha' > \mu_1(m) \text{ and } \nu_1(m-1) > \alpha'' \geqq 0$$

corresponds to the sellers' cartel; analogously, the part of $V_{m,m}$ fulfilling

$$\nu_m(m) \geqq \alpha'' > \nu_1(m) \text{ and } \mu_1(m-1) > \alpha' \geqq 0$$

describes the combination involving all buyers. Since this standard of behavior prescribes an egalitarian rule of division for the respective cartel which is, moreover, not

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17) A similar result was reached by SHUBIK [1959] for bilateral market games with identical preferences for all traders. For the special case considered by SHAPLEY [1959], $f(s,t) = \min(s,t)$, we get $V_{m,m} = G_{m,m}$ as the unique stable set solution of the market game.

able to discriminate among the traders on the other side of the market but has to offer the same terms to each contractor, there is only one (of course, indeterminate) price in the market also in the case of collusion.

5. CONCLUDING REMARKS

Since the main purpose of the present paper is not so much the mathematically exhaustive investigation of the solutions of market games than the economic interpretation of the notion of a stable set of imputations in terms of competition and collusion, and the clarification of the latter concepts, it certainly leaves more problems open than it answers. The few concrete results obtained in section 4 ought to be regarded as examples that may stimulate further interest in the structure of rational standards of behavior for other market games. Thus, as a next step, one could try to determine the symmetric stable sets of monopolies with $m = 1$ and $n \geq 4$ (or, better, for general n), the complete class of symmetric stable sets for the case $m = n$, and eventually, of symmetric bilateral market games with general m and n. A further step would be to relax and eventually to drop the assumption of symmetry. The symmetry assumptions, with respect to the structure of the underlying market and the configuration of the solution itself, serve a twofold purpose: on the one hand, they are useful in simplifying the mathematical analysis, and on the other hand, conceptually even more important, they enable us to demonstrate that rational standards of behavior ensuring equal opportunities for equals need not result in actual equal treatment of the market participants (which means that observed inequalities do not necessarily reveal asymmetries or discrimination). It will, however, be of interest to inquire into the effects of various asymmetries both in market structure (f.i., production capacities, cost functions, revenue functions of the traders) and standards of behavior. Finally, the analysis might be extended to the most general case of multilateral market games involving the exchange of more than one non-monetary commodity.

In the course of exploring, in the search for stable set solutions, broader and broader classes of market games we will sooner or later be confronted with a market game which does not possess any stable set of imputations (the ten-person game discovered by LUCAS [1968] is of a highly asymmetrical structure) - a phenomenon that is still lacking a satisfactory and definitive economic interpretation. What can we expect to happen in a market for which no rational standard of behavior exists? One possibility is that in this case competitive behavior prevails because, after all, a non-empty core exists and there is no consistent and defensible rule of division indispensable for stabilizing any collusive combination. This may be an acceptable interpretation for the case where the traders have no choice but to participate in the market game with no solution. If, however, the market under consideration is just a potential trading arrangement into which actual traders may or may not enter (the "players" in the game being the roles offered to the traders by the rules of the game, i.e. the market structure), the consequence of the lack of a rational standard of behavior probably is that the traders who would not be willing to embark on a recontracting procedure resulting in a core imputation will refuse to participate at all in the market. They will try to substitute another game, for which a rational standard of behavior exists, for the game without a solution, i.e., they will try to alter the market structure. For this purpose, it would be of practical importance to know those structural characteristics of a market which are resposible for the non-existence of a solution - a knowledge we do not yet have [SHAPLEY and SHUBIK, 1969, 1973] and which we will only be able to acquire by a careful and systematic study of more and more comprehensive classes of market games.

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