# Modelling competition in demand-based optimization models

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# Outline

1. Motivation
2. Modelling the problem
3. Future work

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2. Modelling the problem
3. Future work

# Competition

- In many markets competition is present in the form of oligopolies (regulations, barriers to entry, mergers, acquisitions, alliances).
- In transportation, deregulation often led to oligopolistic markets.
    - Airlines
    - Railways
    - Buses
    - Multi-modal networks

# How to study competitive transport markets?

- Modelling demand
- Modelling supply
- Modelling competition

# Demand

- Each customer chooses the alternative that maximizes his/her utility.
- Customers have different tastes and socioeconomic characteristics that influence their choice.

# Supply

- Operators take decisions that optimize their objective function (e.g. revenue maximization).
- Decisions can be related to pricing, capacity, frequency, availability ...
- Decisions are influenced by:
  - The preferences of the customers
  - The decisions of the competitors

# Competition

- We consider non-cooperative games.
- We aim at understanding the Nash equilibrium solutions of such games, i.e. stationary states of the system in which no competitor has an incentive to change its decisions.

# The framework

Three elements to be modelled: customers, operators and market.

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Three elements to be modelled: customers, operators and market.

1. **Customers**: discrete choice models take into account preference heterogeneity and model individual decisions.
2. **Operators**: a mixed integer program can maximize any relevant objective function.
3. **Market**: Nash equilibrium solutions are found by enforcing best response constraints.

# The framework: customer level

**Non-linear formulation**:

- The probability of customer $n \in N$ choosing alternative $i \in I$ depends on the discrete choice model specification.
- For the logit model, there exists a closed-form expression:

$$P_{in} = \frac{\exp(V_{in})}{\sum_{j \in I} \exp(V_{jn})}$$

- For other discrete choice models, there is no closed-form expression.

# The framework: customer level

**Linear formulation**:

- A linear formulation can be obtained by relying on simulation to draw from the distribution of the error term of the utility function.
- For all customers and all alternatives, $R$ draws of are extracted from the error term distribution. Each $\xi_{inr}$ corresponds to a different behavioral scenario.

$$U_{inr} = \beta_{in} p_{in} + q_{in} + \xi_{inr}$$

- In each scenario, customers choose the alternative with the highest utility:

$$w_{inr} = 1 \text{ if } U_{inr} = \max_{j \in I} U_{jnr}, \text{ and } w_{inr} = 0 \text{ otherwise}$$

- Over multiple scenarios, the probability of customer $n$ choosing alternative $i$ is given by

$$P_{in} = \frac{\sum_{r \in R} w_{inr}}{R}.$$

# The framework: operators level

- We assume that an operator $k \in K$ can decide on the price $p_{in}$ of each alternative $i \in I$ for all customers $n \in N$.
- Stackelberg game: the operator (the leader) knows the best response of the customers ("collective" follower) to all strategies.
- Objective function to be maximized by operator $k$:

$$V_k = \sum_{i \in I} \sum_{n \in N} p_{in} P_{in}$$

# Non-linear optimization model for a single operator

$$\begin{aligned}
\max \quad & V = \sum_{i \in I} \sum_{n \in N} p_{in} P_{in} & \\
s.t. \quad & P_{in} = \frac{\exp(U_{in})}{\sum_{j \in I} \exp(U_{jn})} & \forall i \in I, \forall n \in N \\
& U_{in} = \beta_{in} p_{in} + q_{in} & \forall i \in I, \forall n \in N
\end{aligned}$$

# Linear optimization model for a single operator

$$
\begin{aligned}
\max \quad & V = \sum_{i \in I} \sum_{n \in N} p_{in} P_{in} & \\
s.t. \quad & P_{in} = \frac{\sum_{r \in R} w_{inr}}{R} & \forall i \in I, \forall n \in N \\
& U_{inr} = \beta_{in} p_{in} + q_{in} + \xi_{inr} & \forall i \in I, \forall n \in N, \forall r \in R \\
& U_{inr} \leq U_{nr} & \forall i \in I, \forall n \in N, \forall r \in R \\
& U_{nr} \leq U_{inr} + M_{U_{nr}} (1 - w_{inr}) & \forall i \in I, \forall n \in N, \forall r \in R \\
& \sum_{i \in I} w_{inr} = 1 & \forall n \in N, \forall r \in R \\
& w_{inr} \in \{0, 1\} & \forall i \in I, \forall n \in N, \forall r \in R
\end{aligned}
$$

# The framework: market level

- The payoff of an operator also depends on the strategies of the competitors.
- Let's define as $X_k$ the set of strategies that can be played by operator $k \in K$.
- Condition for Nash equilibrium (best response constraints):

$$V_k = V_k^* = \max_{x_k \in X_k} V_k(x_k, x_{K \setminus \{k\}}) \qquad \forall k \in K$$

- Nash (1951): finite games have at least one mixed strategy equilibrium solution.
- Finite/infinite strategy sets; pure/mixed strategies; continuous/discrete payoff function.

# A fixed-point iteration method

- Sequential algorithm to find Nash equilibrium solutions of a k-player game:
  - Initialization: players select an initial feasible strategy.
  - Iterative phase: players take turns and each plays its best response pure strategy to the current solution.
  - Termination criterion: either a Nash equilibrium or a cyclic equilibrium is reached.

# A mixed integer model for the fixed-point problem

- We can write a model that minimizes the distance between two consecutive fixed-point iterations.
- A solution for a two-operator problem: $(x_1^b, x_2^b)$
- Optimization problems for the operators:

$$x_1^* = \arg\max_{x_1 \in X_1} V_1(x_1, x_2^b)$$
$$x_2^* = \arg\max_{x_2 \in X_2} V_2(x_1^b, x_2)$$

- Fixed-point problem:

$$\min_{x_1, x_2, x_1^*, x_2^*} \|x_1^* - x_1^b\| + \|x_2^* - x_2^b\|$$

# Initial configuration

- No optimization at operator level: any feasible strategy could be selected.
- Constraints:
  - Customer choice
    - Continuous (MINLP), or
    - Binary (MILP)
  - Customer utility maximization

$$
\begin{aligned}
&\sum_{i \in I} w_{inr}^b = 1 && \forall n \in N, \forall r \in R \\
&U_{inr}^b = \beta_{in} p_{in}^b + q_{in} + \xi_{inr} && \forall i \in I, \forall n \in N, \forall r \in R \\
&U_{inr}^b \leq U_{nr}^b && \forall i \in I, \forall n \in N, \forall r \in R \\
&U_{nr}^b \leq z_{inr}^b + M(1 - w_{inr}^b) && \forall i \in I, \forall n \in N, \forall r \in R
\end{aligned}
$$

# Best response configurations

- Each operator solves an optimization problem having the following constraints:
  - Customer choice
    - Continuous (MINLP), or
    - Binary (MILP)
  - Customer utility maximization
  - Best response

# Best response configurations

Best response constraints:

$$V_{ks} = \frac{1}{R} \sum_{i \in I_k} \sum_{n \in N} \sum_{r \in R} p_{ins} w^a_{inrs} \qquad \forall k \in K, \forall s \in S_k$$

$$V_{ks} \leq V^{max}_{ks} \qquad \forall s \in S_k$$

$$V^{max}_{ks} \leq V_{ks} + M(1 - x_{ks}) \qquad \forall s \in S_k$$

$$\sum_{s \in S} x_{ks} = 1$$

Customer constraints:

$$U^a_{inrs} = \beta_{in} p_{ins} + q_{in} + \xi_{inr} \qquad \forall i \in I_k, \forall n \in N, \forall r \in R, \forall s \in S_k$$

$$U^a_{inrs} = U^b_{inr} \qquad \forall i \in I \setminus I_k, \forall n \in N, \forall r \in R, \forall s \in S_k$$

$$\sum_{i \in I} w^a_{inrs} = 1 \qquad \forall n \in N, \forall r \in R, \forall s \in S_k$$

$$U^a_{inrs} \leq U^a_{nrs} \qquad \forall i \in I, \forall n \in N, \forall r \in R, \forall s \in S_k$$

$$U^a_{nrs} \leq z^a_{inrs} + M(1 - w^a_{inrs}) \qquad \forall i \in I, \forall n \in N, \forall r \in R, \forall s \in S_k$$

# Objective function

- Minimization problem:

$$z^* = \min_{x_1, x_2, x_1^*, x_2^*} \|x_1^* - x_1^b\| + \|x_2^* - x_2^b\|$$

- If $z^* = 0$, we have an equilibrium. What can we say about this equilibrium?
  If $z^* > 0$, can we conclude something? Are we in an equilibrium region?

# Numerical experiments

- Case study: 3 parking choices. 2 owned by 2 different operators and 1 opt-out option. Parameter estimation available in the literature.
- Tests: non-linear and linear formulations with logit and mixed logit specifications.

| | Non-linear | Linear |
|---|---|---|
| Logit | - | $\xi$ |
| Mixed logit | $\beta$ | $\beta$, $\xi$ |

Figure: Random draws needed in the different sets of experiments

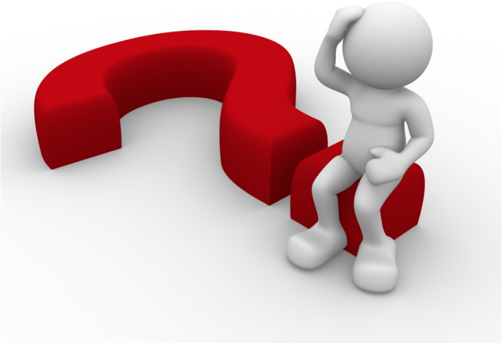
# Numerical experiments

- Model specification: drawing from the error term distribution ($R = 50, 100, 200$) gives good approximation of the choice probabilities found with the logit formula.
- Preliminary results: equilibrium solutions are found for all tested instances with the MILP formulation.
- Computational times:
  - Logit:
    - The non-linear model is faster (no need for simulation).
    - The time required by the linear model increases with the number of draws.
  - Mixed logit:
    - The non-linear model (highly non convex) does not converge for larger instances.
    - The linear model outperforms the non-linear model on larger instances.

# Open questions and future work

- The current model uses finite strategy sets (i.e. price discretization). Is it possible to reformulate the problem with the help of complementarity?
- How can the structure of the problem be exploited to efficiently search for equilibria in the solution space?
- Is it possible to compare different equilibrium solution or to prove uniqueness within a region of the solution space?

# Questions?



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