DRAFT: FLEXURE-PIVOT OSCILLATOR RESTORING TORQUE NONLINEARITY AND ISOCHRONISM DEFECT

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ABSTRACT

The timebase of mechanical watches is a harmonic oscillator consisting of a balance wheel attached to a hairspring pivoting on a jeweled bearing. The jeweled bearing has frictional losses limiting watch autonomy and oscillator quality factor. Replacement by flexure pivots leads to a drastic reduction in friction and an order of magnitude increase in quality factor. However, flexure pivots have drawbacks including gravity sensitivity, restoring torque nonlinearity, limited stroke and parasitic shift. These properties have been studied by the authors in a previous article for two flexure pivots: the generalized cross-spring pivot (GCSP), which is widely used in the field of compliant mechanisms, and the novel gravity insensitive flexure pivot (GIFP), designed by the authors to solve the problem of gravity sensitivity. However no analytical solution for the restoring torque nonlinearity has been found. This property is crucial for oscillators as it directly affects isochronism, the capacity of an oscillator to have a constant frequency regardless of its amplitude. This paper addresses the issue by finding an empirical quadratic expression for restoring torque nonlinearity through numerical simulation. We use this expression to improve the analytical formula for rotational stiffness of GCSP and GIFP. We give an explicit formula for the effect of restoring torque nonlinearity on isochronism.

INTRODUCTION

Flexure pivots as mechanical watch oscillators

Classical mechanical watches use a harmonic oscillator consisting of a spiral spring attached to a balance wheel as timebase. The balance wheel pivots on jeweled bearings which have significant friction. Flexure pivots have been introduced in watchmaking to reduce this friction [5] [6], thus supressing the need for lubrication, increasing the watch autonomy and increasing the oscillator quality factor, the quantity believed to be the most significant indicator of chronometric performance [2]. In addition to their rotational bearing function, flexure pivots provide an elastic restoring torque which can be used as spring for harmonic oscillators. Hence one single part, monolithically fabricated, can replace classical balance wheel, spring and bearing.

However some issues intrinsic to flexure mechanisms limit their application to timebases.

Limitation 1. Gravity sensitivity: spring stiffness can be affected by the orientation of gravity load.

Limitation 2. Restoring torque nonlinearity: spring restoring torque can be a nonlinear function of rotation angle leading to an isochronism defect.

Limitation 3. Limited stroke: stroke of flexure bearings is limited by the yield stress of the material. Limited stroke makes it difficult to maintain and count oscillation using classical watch escapements.

Limitation 4. Parasitic shift: by construction, the kinematics of flexure pivots closely approximate rotational motion around a fixed axis but small translation can occur as angular rotation increases.

Studied flexure pivots

We study the common flexure pivot called *Generalized Cross-Spring Pivot* (GCSP, see Fig.1) and our novel flexure pivot presented by the authors in [10] called *Gravity Insensitive Flexure Pivot* (GIFP, see Fig.2). We showed in [10] that the behavior of these pivots can be described by a geometric parameter δ.

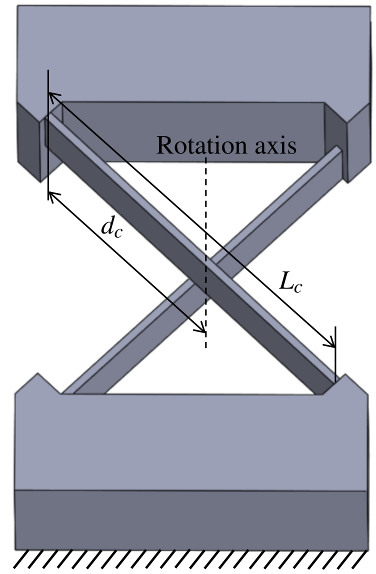
In the case of GCSP, $\delta = d_c/L_c$, where L_c is the length of the leaf springs and d_c is the distance between the rotation axis and the mobile end of the springs, see Fig.1. When $\delta \leq 0$, the axis of rotation passes through the leaf springs, see Fig.1(a), this pivot first described by Wittrick [15] is sometimes called *cross-spring pivot* in literature [9] [4] [11] [7]. When $\delta > 0$, the axis of rotation is outside of the physical spring structure, see Fig.1(b), this pivot is sometimes called *Remote Center Compliance* (RCC) in literature [4]. Note that only the configuration where the flexure beams cross at an angle of 90 is studied.

The GIFP depicted in Fig.2 consists of a rigid-body (1) attached to the ground (0) by five beams: four bending beams (2), (3), (4) and (5), and a single torsional beam (6). The single degree of freedom is rotation around the torsional beam axis [10]. The geometric parameter is $\delta = d_g/L_g$, where L_g is the length of the bending beams and d_g is the distance between the rotation axis and the mobile end of the beams. Similarly to GCSP, when $\delta \leq 0$, the bending beams cross the rotation axis (see Fig.2(a)) and when $\delta > 0$ the beams do not intersect it, see Fig.2(b).

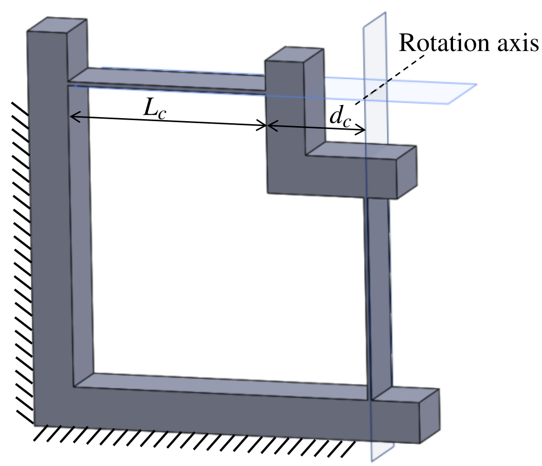
Focus of the study

We addressed limitations 1 and 3 for GCSP and GIFP in [10] by deriving analytical formulas for gravity sensitivity and stroke and validating them by finite element analysis (FEA). We gave special values of geometric parameter δ to overcome limitations 1 and 3 and showed that for any value of δ, GIFP stiffness variation due to gravity load affects the timebase precision in the order of 1 second per day, which is acceptable in watchmaking. Limitation 4, parasitic shift of flexure pivot center of rotation, is a well-studied subject, see [9] [11], and is already addressed by minimizing gravity sensitivity. Indeed, since gravity sensitivity is caused by the work of gravity load acting along the parasitic shift of the center of gravity, minimizing gravity sensitivity also minimizes parasitic shift.

Limitation 2 has not been adressed yet and is the focus of this paper. The analytical formula for restoring torque nonlinearity derived in [10] does not describe the behavior observed in FEA simulation. Since this expression is crucial for designing timebases, we derive an empirical expression for the nonlinearity curve of pivots with crossed beams obtained by FEA.



(a) $\delta \leq 0$ (Cross-spring pivot)

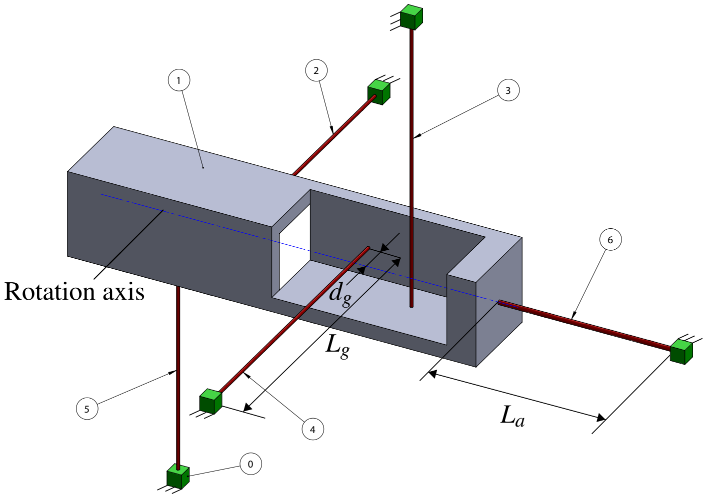


(b) $\delta > 0$ (Remote center compliance pivot)

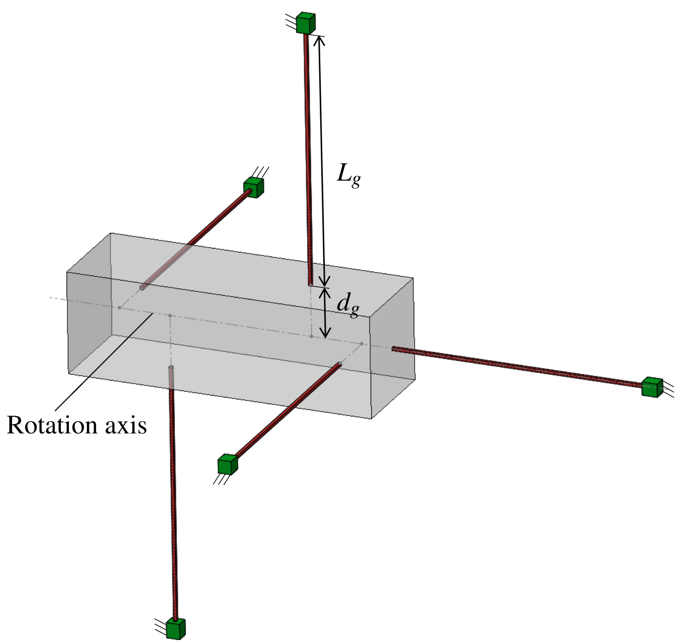
FIGURE 1: TWO CONFIGURATIONS OF THE GENERALIZED CROSS SPRING PIVOT

Effect of restoring torque nonlinearity on isochronism

In order for an oscillator to have the chronometric performance necessary for a timebase, its frequency must be as constant as possible. In the case of a mechanical harmonic oscillators, it must obey Hooke's Law which means that spring restoring torque should be a linear function of displacement, in other words spring stiffness should be constant. If this property is respected, a harmonic oscillator with constant inertia and no ex-



(a) GIFP with $\delta \leq 0$



(b) GIFP with $\delta > 0$

FIGURE 2: TWO CONFIGURATIONS OF GIFP

ternal influences will have a constant frequency regardless of its oscillation amplitude. This property called *isochronism* in horology [3] is essential for chronometric performance. Pendulums, such as the ones used in clocks, are isochronous only for small angles. The spiral spring associated to a balance wheel by Christiaan Huygens in 1675 significantly improved the precision of portable timekeepers by having a low sensitivity to gravity orientation and a quasi-linear restoring torque for a wide range of amplitude. The first flexure pivot used as a mechanical watch timebase, a cross-spring pivot oscillator introduced in 2014 [1], uses a a special geometry [8] which minimizes the effect of gravity on stiffness and a separate mechanism called *isochronism corrector* to compensate for its nonlinearity. The GIFP oscillator on the other hand can be isochronous by choosing the right value of design parameter δ for which it is linear and gravity insensitive, see [10].

For a pivot whose rotational stiffness varies with respect to angular amplitude, we defined in [10] *relative nonlinearity* to be the relative deviation of the rotational stiffness from the nominal value. For small amplitudes of the pivot, its stiffness can be expressed by a power series with first two terms $k = k_0 + k_2\theta^2 + \mathscr{O}\left(\theta^4\right)$ and the relative nonlinearity is defined to be

$$\mu = \frac{k_2}{k_0}. \tag{1}$$

The effect of restoring torque nonlinearity on oscillator rate can be derived. Given an oscillator with nominal amplitude α_n and nominal angular frequency ω_n, the *daily rate* of the same oscillator at amplitude α with corresponding frequency ω is defined to be

$$\rho = 86400\,\frac{\omega - \omega_n}{\omega_n}, \tag{2}$$

see [13], where ρ is described in second per day (s/day) and 86400 is the number of seconds in one day. Daily rate is essentially the gain or loss of the timekeeper with respect to time at nominal frequency after one day of running. The frequency ω of a non-linear oscillator at amplitude α with relative nonlinearity μ can be derived using [12, Eqn. 2.3.34]:

$$\omega(\alpha) = \omega_0(1 + \frac{3\mu}{8}\alpha^2) \tag{3}$$

where $\omega_0 = \sqrt{k_0/J}$ is the frequency for infinitesimal amplitude and J is the moment of inertia of the oscillator.

The classical horological definition of isochronism considers oscillator rate with respect to oscillator amplitude [3]. We favour the approach described in [14] which analyzes how oscillator rate varies as oscillator energy E varies. This method is valid for any kind of oscillator and is equivalent to to the horological definition since in that case energy is proportional to the square of oscillator amplitude. Using this proportionality, the relative energy variation with respect to oscillator energy at nominal amplitude E_n expressed in percentage is

$$E_{\%} = \frac{100\,(E - E_n)}{E_n} = \frac{100\left(\alpha^2 - \alpha_n^2\right)}{\alpha_n^2}. \tag{4}$$

Finally, isochronism defect σ is evaluated by dividing the rate ρ by the relative energy variation $E_{\%}$. Inserting Eqn. (3) into Eqn. (2) we obtain

$$\sigma = \frac{\rho}{E_{\%}} = 864 \frac{\frac{3}{8}\mu\alpha_n^2}{1+\frac{3}{8}\mu\alpha_n^2}. \qquad (5)$$

We arrive to a simpler expression using Taylor series expansion around $\alpha_n^2 = 0$

$$\sigma = 324\mu\alpha_n^2 + \mathcal{O}\left(\alpha_n^4\right). \qquad (6)$$

In practice, isochronism can be obtained from the slope of the plot of daily rate σ vs relative energy variation expressed in percentage $E_{\%}$. Such curves can be seen in the "Results" section.

METHODOLOGY

We use nonlinear FEA to find the nonlinear torque-angle relationship of flexure pivots with different values of geometric parameter δ. For each geometry, 100 incremental displacement values are applied on the mobile part of the pivot and the reaction torque on the fixed frame is measured. The simulations are performed using ANSYS®Workbench, Release 18.2, with shell elements for the flexible blades. The simulations are done for values of δ ranging from -0.5 to 1. Due to interchangeability of the rotating rigid body and the fixed frame, to analyze a pivot with $\delta < -0.5$, one can use the results presented here with $\delta' = -\delta - 1$. The behavior of the pivots for values of $\delta > 1$ is not investigated due to high stiffness, high nonlinearity and short stroke of these pivots limiting their application. The results obtained should however also be valid in this range.

We use the analytical formula for nominal stiffness of the pivots to validate the finite element model. The nominal stiffness of GIFP for an infinitesimal rotation in absence of gravity, is

$$k_0 = k_a + \frac{16EI_g}{L_g}\left(3\delta^2 + 3\delta + 1\right), \qquad (7)$$

where E and I_g are the Young's modulus and the area moment of inertia of the bending beams and k_a is the torsional stiffness of the torsional beam. It has been found in [10] that the stiffness of the torsional beam does not play a major role in the restoring torque nonlinearity of the pivot. The rest of the analysis is thus done neglecting this beam. The resulting normalized nominal stiffness is

$$\bar{k}_0 = 3\delta^2 + 3\delta + 1, \qquad (8)$$

which is the same formula as the normalized nominal stiffness of GCSP. It corresponds to the normalized nominal stiffness of the cross-spring pivot when $\delta \leq 0$ in [10, Eqn. 9], and the normalized stiffness of the RCC pivot when $\delta > 0$ in [4, Eqn. 5.6]. The numerical simulations are thus done for the GCSP and the results are applicable to GIFP. Figure 3 shows a good match between FEA results and analytical nominal stiffness validating the finite element model.

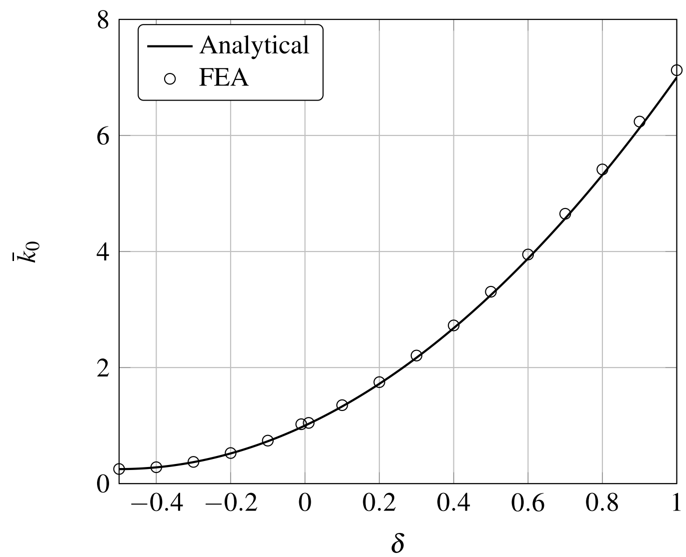


FIGURE 3: NORMALIZED NOMINAL STIFFNESS $\bar{k}_0$ OF THE GCSP VERSUS GEOMETRIC PARAMETER δ.

RESULTS

The restoring torque nonlinearity is obtained by fitting an odd cubic polynomial to the torque-angle relationship obtained by numerical simulations for chosen values of δ. The relative nonlinearity $\mu = k_2/k_0$ is extracted from the torque-angle relationship $M(\theta) = k_0\theta + k_2\theta^3$ according to Eqn. (1). The results are shown in 4. A quadratic curve fits the data well with a coefficient of determination $R^2 = 0.9999$. The resulting empirical expression for stiffness nonlinearity is

$$\mu = 0.0803 + 1.00\delta + 1.02\delta^2. \qquad (9)$$

Remark: Geometric parameter $\delta = 0.088$ solves the equation $\mu = 0$ and cancels the nonlinearity.

The analytical solution derived by Haringx for the nonlinear torque-angle relationship of GCSP with $\delta = -0.5$ (see [7, Eqn. 37]) is plotted on Fig.4 and matches the FEA results. Note

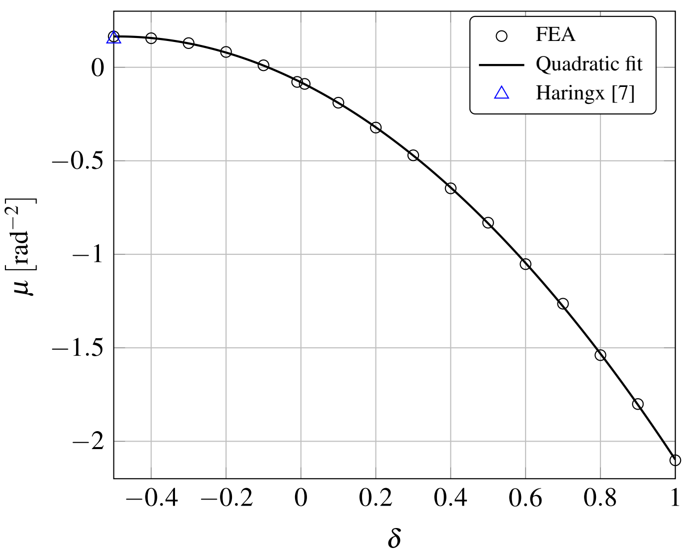


FIGURE 4: RELATIVE RESTORING TORQUE NONLINEARITY μ OF GCSP VERSUS GEOMETRIC PARAMETER δ.

that Haringx's analytical model is limited to the generalized cross-spring pivot with $\delta = -0.5$ since he solved the nonlinear equations using the inherent symmetry which holds only for this configuration.

Since the quadratic curve for the relative nonlinearity fits the data well, we can use it to give a new formula for the stiffness of our pivots. Substituting Eqn. (9) into [10, Eqn. 11] which has been validated for describing the effect of normalized external load $\bar{N}$ on the stiffness of GIFP, we obtain the new formula for GIFP stiffness

$$\begin{aligned} k_g = & \frac{16EI_g}{L_g}\left(3\delta^2+3\delta+1\right)\left[1-\left(0.0803+1.00\delta+1.02\delta^2\right)\theta^2\right] \\ & -\frac{EI_g}{12600L_g}\left(9\delta^2+9\delta+11\right)\bar{N}^2+k_a+\mathscr{O}\left(\theta^4\right) \\ & +\mathscr{O}\left(\theta^2\bar{N}^2\right)+\mathscr{O}\left(\bar{N}^4\right). \end{aligned} \tag{10}$$

Similarly we obtain the new formula for stiffness of GCSP

$$\begin{aligned} k_c = & \frac{8EI_c}{L_c}\left(3\delta^2+3\delta+1\right)\left[1-\left(0.0803+1.00\delta+1.02\delta^2\right)\theta^2\right] \\ & +\frac{2EI_c}{15L_c}\left(9\delta^2+9\delta+1\right)\bar{N}\left(\sin\varphi+\cos\varphi\right) \\ & -\frac{EI_c}{6300L_c}\left(9\delta^2+9\delta+11\right)\bar{N}^2+\mathscr{O}\left(\theta^2\bar{N}\right)+\mathscr{O}\left(\theta^4\right)+\mathscr{O}\left(\bar{N}^3\right) \end{aligned} \tag{11}$$

where E, I_c are the Young's modulus and area moment of inertia and of the leaf springs and φ is the angle between a normalized external load $\bar{N}$ and the mid-plane of one of the leaf springs in undeflected position.

Figure 5 shows isochromism curves for 5 chosen values of δ. Isochronism defect is obtained from the slope of the linear curves of daily rate σ vs relative energy variation. We can see that the special value for geometric parameter $\delta = 0.088$ which cancels the restoring torque nonlinearity in Eqn. (9) shows no isochronism defect. Note that the sign of nonlinearity defines the sign of isochronism defect.

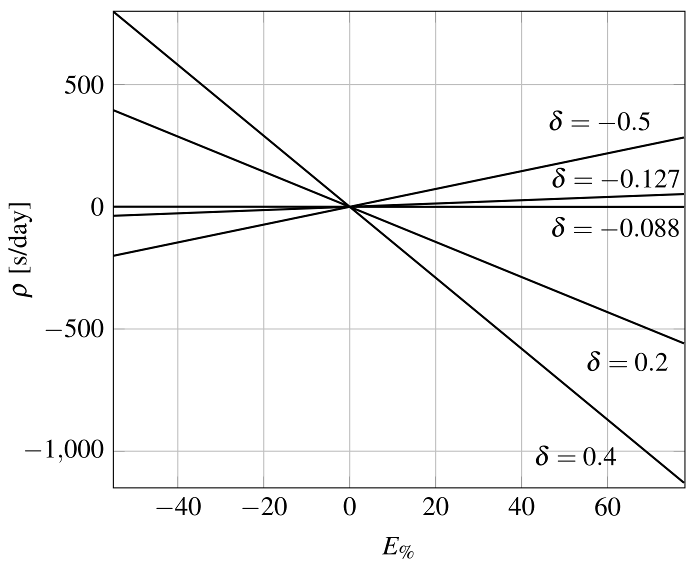


FIGURE 5: DAILY RATE ρ OF GCSP WITH DIFFERENT VALUES OF GEOMETRIC PARAMETER δ FOR AMPLITUDES α BETWEEN 10° AND 20° AND REFERENCE AMPLITUDE $\alpha_n = 15°$.

CONCLUSION

An empirical quadratic formula for the restoring torque nonlinearity of GIFP and GCSP has been found which closely matches the results obtained by finite element simulation. This property is crucial when using flexure pivots as oscillators for mechanical watch timebase as it is directly linked to isochronism defect, the variation of oscillator frequency with amplitude which deteriorates chronometric performance. An analytical formula is given to predict isochronism defect from restoring torque nonlinearity and resulting isochronism curves are shown.

The restoring torque nonlinearity results have been combined with the existing analytical formulas describing the effect of external load on rotational stiffness to provide an improved description of rotational stiffness of GIFP and GCSP.

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