Structure-Preserving Reduced Basis Methods for Hamiltonian Systems with a Nonlinear Poisson Structure

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Abstract

We develop structure-preserving reduced basis methods for a large class of problems by resorting to their semi-discrete formulation as Hamiltonian dynamical systems. In this perspective, the phase space is naturally endowed with a Poisson manifold structure which encodes the physical properties, symmetries and conservation laws of the dynamics. We design reduced basis methods for the general case of nonlinear state-dependent degenerate Poisson structures based on a two-step approach. First, via a local approximation of the Poisson tensor we split the Hamiltonian dynamics into an "almost symplectic" part and the trivial evolution of the Casimir invariants. Second, canonically symplectic reduced basis techniques are applied to the nontrivial component of the dynamics, whereas the local Poisson tensor kernel is preserved exactly. The global Poisson structure and the conservation properties of the phase flow are retained by the reduced model in the constant-valued case and up to errors in the Poisson tensor approximation in the state-dependent case. The proposed reduction scheme is combined with a discrete empirical interpolation method (DEIM) to deal with nonlinear Hamiltonian functionals and ensure a computationally competitive reduced model. A priori error estimates for the solution of the reduced system are established. A set of numerical simulations is presented to corroborate the theoretical findings.

Keywords. Hamiltonian dynamics, Poisson manifolds, symplectic structure, invariants of motion, structure-preserving schemes, reduced basis methods (RBM).

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1 Introduction

During the last decade there has been substantial developments of model order reduction techniques to efficiently solve parameterized partial differential equations in computationally intensive scenarios such as real-time and many-query simulations. Reduced basis methods (RBM) aim at pruning the computational effort by replacing the original high-dimensional problems with models of significantly reduced dimensionality without compromising the overall accuracy. For time-dependent parametric problems, an approximation space of low dimension, the so-called *reduced space*, is constructed from a collection of full-order solutions at sampled values of time and parameters during a computationally intensive offline phase. The reduced space is spanned by the modes associated with the dominant components of the dynamics. In the online phase the reduced order model is then solved at a substantially reduced computational cost for any parameter query.

The development and analysis of reduced basis techniques for the efficient solution of PDEs is well-established in the context of linear elliptic and parabolic problems. For nonlinear and hyperbolic

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problems, model order reduction is less well understood. Available methods are the method of freezing [46], Lax pairs [26] and dictionary-based approximations [30], L^1-norm minimization techniques [1], model reduction tracking discontinuities [50], the Gauss–Newton with approximated tensors (GNAT) method [14], RBM for finite volume discretizations [31], with conservation properties [13]. Most of these approaches are conceived to tackle ad hoc features of the problems considered, hence not striving to be sufficiently general, let alone provide a sound strategy to preserve the structure intrinsic to the original model.

Hamiltonian Formulation of Nonlinear and Hyperbolic Problems. The field theory formalism, grounded in the mathematical description of physical quantities via integral action functionals, provides a unified perspective encompassing partial differential equations of interest in a broad range of applications. Examples of problems that can be derived from action principles include Maxwell's equations [9], Schrödinger's equation, Korteweg–de Vries [42] and wave equations, compressible [37] and incompressible [4] Euler equations, Vlasov–Poisson and Vlasov–Maxwell equations [43], etc. The action principle yields a formulation of the constitutive equations as Hamiltonian systems whose phase space is naturally endowed with a differentiable Poisson manifold structure. The algebraic structure of the phase space, which is generally *degenerate* and nonlinearly *state-dependent*, underpins the physical properties of the system. Most prominently, Poisson structures encode a family of conserved quantities, which by Noether's theorem, are related to symmetries of the Hamiltonian. In addition, the degeneracy of the Poisson brackets entails the conservation of families of invariants.

The preservation, at the discrete level, of the algebraic and topological structure of physical problems have received considerable attention during the last decades. Both in the context of spatial discretization and temporal integration (with the so-called geometric numerical integrators), it has become apparent that structure-preserving strategies can yield approximate solutions with superior stability and accuracy properties. On the other hand, the geometric structure of continuous and discrete Hamiltonian systems is in general thwarted during model order reduction, resulting in the onset of spurious and unphysical artifacts which may trigger instabilities and qualitatively wrong solution behavior.

Recently, the promising performance of structure-preserving techniques have fostered the development of reduced basis methods tantamount to geometric numerical integration and compatible spatial discretizations. In the context of nondissipative Hamiltonian dynamical systems Lall, Krysl, and Marsden [34] pioneered the use of a Galerkin projection on the Euler–Lagrange equations to devise reduced order models preserving Lagrangian structures. A similar approach was later pursued and improved in [15]. Dealing directly with the Hamiltonian formulation, reduced basis methods preserving the canonical symplectic structure of dynamical systems were developed in [48], and [3]. A similar technique has been adopted in [45] in the study of dynamical low-rank methods for the approximation of the stochastic wave equation.

To the best of our knowledge, none of the aforementioned works address the case of degenerate and/or state-dependent nonlinear Poisson structures. A naïve extension of the available structure-preserving reduction techniques to these cases is hindered by the intrinsic nonlinearity and degeneracy of the structure which, among others, fails to provide a pseudo-inner product and a path to endow the reduced model with a Poisson phase flow.

Our Contribution: Novelties and Outline. In this work we develop and analyze structure-preserving reduced basis methods for Hamiltonian dynamics with state-dependent and possibly degenerate Poisson manifold structures. Pursuing a method of lines approach, we consider the ordinary differential equations ensuing from a suitable spatial approximation of the Hamiltonian dynamics. The latter is assumed to be structure-preserving in the sense of yielding a semi-discrete system of the form

$$\frac{du}{dt} = \mathcal{J}_N(u)\,\nabla_u \mathcal{H}_N(u), \tag{1.1}$$

where the unknown u depends on time and possibly on a set of parameters, $\mathcal{H}_N$ is the discrete system Hamiltonian and $\mathcal{J}_N(u)$ is a finite-dimensional operator describing the Poisson manifold structure.

The gist of our method is to perform a local splitting of the Hamiltonian dynamics into a canonically symplectic component and a trivial evolution of the invariants associated with the kernel of the Poisson structure. The splitting of the dynamics is motivated by a result of Darboux [23] which, roughly speaking, demonstrates the existence of *local* charts in which any Poisson structure has the canonical form. The rationale is that canonical Poisson structures are more amenable to model order reduction since the nonlinearity has been removed from the structure and its kernel singled out. Since an analytic expression of the Darboux charts is usually unavailable, we rely on piecewise approximations by leveraging the local linearization introduced by the timestepping. The original dynamics is then approximated in a lower-dimensional manifold, foliated by the Poisson tensor kernel and a reduced symplectic component. The latter is derived via canonically symplectic reduced basis techniques adapted from [48, 3]. The resulting reduced dynamics retains the Poisson structure of the phase flow up to the approximation error of the Darboux charts.

The remainder of the paper is organized as follows. In Section 2, the algebraic structure underlying Hamiltonian systems on finite-dimensional Poisson manifolds is described. Section 3 pertains the case of degenerate constant-valued parametric Poisson structures for which the manifold splitting is performed *globally*. The resulting reduced problem is Hamiltonian with a Poisson manifold structure and inherits the physical properties of the high-fidelity model in terms of conservation of the Hamiltonian, preservation of the Casimir invariants, and Lyapunov stability, as shown in Section 3.2.1. In Section 3.3 the structure-preserving reduced basis method is coupled to a symplectic DEIM for the efficient treatment of the nonlinear terms. State space error bounds for the solution of the reduced system in terms of the projection error into the reduced space are presented in Section 3.3.1. Next, Section 4 is devoted to the more challenging case of state-dependent nonlinear Poisson structures. The reduced model built upon piecewise linear approximations of the Darboux charts is shown to be structure-preserving up to approximation errors of the Darboux maps, Section 4.2. A priori error estimates for the fully discrete reduced problem are established in Section 4.3 and in Section 4.3.1 when the nonlinear terms are approximated via a DEIM strategy. In Section 5 a set of numerical experiments is presented and conclusions are drawn in Section 6.

2 Dynamical Systems with Poisson Structure

In this Section we briefly describe the topological and algebraic structure underlying the phase space of Hamiltonian dynamical systems of the form (1.1).

Definition 2.1 (Poisson Structure). Let $\mathcal{V}_N$ be a finite N-dimensional smooth manifold. A *Poisson structure* on $\mathcal{V}_N$ is a bilinear operation $\{\cdot,\cdot\}_N : C^\infty(\mathcal{V}_N) \times C^\infty(\mathcal{V}_N) \to C^\infty(\mathcal{V}_N)$, called a *bracket*, with the following properties: for all $\mathcal{F},\mathcal{G},\mathcal{I} \in C^\infty(\mathcal{V}_N)$ and $u \in \mathcal{V}_N$,

(i) Skew-symmetry: $\{\mathcal{F},\mathcal{G}\}_N(u) = -\{\mathcal{G},\mathcal{F}\}_N(u)$.

(ii) Leibniz rule: $\{\mathcal{F}\mathcal{G},\mathcal{I}\}_N(u) = \{\mathcal{F},\mathcal{I}\}_N(u)\mathcal{G}(u) + \mathcal{F}(u)\{\mathcal{G},\mathcal{I}\}_N(u)$.

(iii) Jacobi identity: $\{\mathcal{F},\{\mathcal{G},\mathcal{I}\}_N\}_N(u) + \{\mathcal{G},\{\mathcal{I},\mathcal{F}\}_N\}_N(u) + \{\mathcal{I},\{\mathcal{F},\mathcal{G}\}_N\}_N(u) = 0$.

A manifold endowed with a Poisson structure is called a *Poisson manifold*.

The space $C^\infty(\mathcal{V}_N)$ of real-valued smooth functions over the Poisson manifold $(\mathcal{V}_N,\{\cdot,\cdot\}_N)$ together with the bracket $\{\cdot,\cdot\}_N$ forms a Lie algebra [2, Proposition 3.3.17], called the *Poisson algebra* of $\mathcal{V}_N$.

Owing to the bilinearity of $\{\cdot,\cdot\}_N$ and the Leibniz rule, given an analytic function $\mathcal{H} \in C^\infty(\mathcal{V}_N)$, the map $\mathcal{F} \in C^\infty(\mathcal{V}_N) \mapsto \{\mathcal{F},\mathcal{H}\}_N \in C^\infty(\mathcal{V}_N)$ defines a differentiation on the Poisson manifold $\mathcal{V}_N$. Hence, there exists a locally *unique* vector field $X_\mathcal{H}(u) \in T_u\mathcal{V}_N$ such that $\mathsf{L}_{X_\mathcal{H}}\mathcal{F} = \{\mathcal{F},\mathcal{H}\}_N$, where L_X denotes the Lie derivative with respect to the velocity field X. The vector $X_\mathcal{H}(u)$ is called the *Hamiltonian vector field* of the functional $\mathcal{H} \in C^\infty(\mathcal{V}_N)$, and characterizes the dynamics of the evolution problem (1.1), as explained in the forthcoming Section 2.1. The map $\mathcal{H} \in C^\infty(\mathcal{V}_N) \mapsto X_\mathcal{H} \in T\mathcal{V}_N$ is a (anti)homomorphism between Lie algebras [39, Proposition 10.2.2].

If $\mathsf{d}\mathcal{H}$ is the 1-form given by the exterior derivative of the functional $\mathcal{H} \in C^\infty(\mathcal{V}_N)$, its Hamiltonian vector field $X_\mathcal{H}$ can be obtained as the image of $\mathsf{d}\mathcal{H}$ under the vector bundle morphism $\mathcal{J}_N(u)$ defined, for any $u \in \mathcal{V}_N$, as

$$\begin{aligned} \mathcal{J}_N(u) : T^*\mathcal{V}_N &\longrightarrow T\mathcal{V}_N \\ \mathsf{d}\mathcal{H} &\longmapsto X_\mathcal{H}(u) := \mathcal{J}_N(u)\mathsf{d}\mathcal{H}. \end{aligned} \tag{2.1}$$

The Poisson bracket can be expressed in terms of $\mathcal{J}_N$ as

$$\{\mathcal{F},\mathcal{G}\}_N(u) = {}_{T^*\mathcal{V}_N}\langle \mathsf{d}\mathcal{F}, \mathcal{J}_N(u)\,\mathsf{d}\mathcal{G}\rangle_{T\mathcal{V}_N}, \qquad \forall\, \mathcal{F},\mathcal{G}\in C^\infty(\mathcal{V}_N),\ \forall\, u\in\mathcal{V}_N, \tag{2.2}$$

where ${}_{T^*\mathcal{V}_N}\langle\cdot,\cdot\rangle_{T\mathcal{V}_N}$ denotes the duality pairing between the cotangent and the tangent bundle. The application $\mathcal{J}_N$ is a contravariant 2-tensor on the manifold $\mathcal{V}_N$, commonly referred to as *Poisson tensor*. The tensor $\mathcal{J}_N$ is skew-symmetric with respect to the metric g on $\mathcal{V}_N$ defined as $g(\nabla\mathcal{F},\cdot) := {}_{T^*\mathcal{V}_N}\langle \mathsf{d}\mathcal{F},\cdot\rangle_{T\mathcal{V}_N}$, and ∇ is the Riemannian gradient. Hence, in local coordinates, the Poisson bracket reads

$$\{\mathcal{F},\mathcal{G}\}_N(u) = \nabla_u\mathcal{F}(u)^\top\mathcal{J}_N(u)\nabla_u\mathcal{G}(u), \qquad \forall\, \mathcal{F},\mathcal{G}\in C^\infty(\mathcal{V}_N),\ \forall\, u\in\mathcal{V}_N.$$

In view of the relationship between the bracket $\{\cdot,\cdot\}_N$ and the tensor $\mathcal{J}_N$, the Poisson manifold structure on $\mathcal{V}_N$ can be equivalently characterized as follows.

Lemma 2.2. *Let $\mathcal{V}_N$ be a finite N-dimensional smooth manifold and let $\mathcal{J}_N$ be the vector bundle map defined in (2.1). Then the bracket (2.2) is a Poisson structure as per Definition 2.1, if and only if $\mathcal{J}_N$ is skew-symmetric and satisfies the Jacobi identity*

$$\sum_{\ell=1}^{N}\left(\frac{\partial(\mathcal{J}_N(u))_{i,j}}{\partial u_\ell}(\mathcal{J}_N(u))_{\ell,k} + \frac{\partial(\mathcal{J}_N(u))_{j,k}}{\partial u_\ell}(\mathcal{J}_N(u))_{\ell,i} + \frac{\partial(\mathcal{J}_N(u))_{k,i}}{\partial u_\ell}(\mathcal{J}_N(u))_{\ell,j}\right) = 0, \tag{2.3}$$

for all $u\in\mathcal{V}_N$ and $i,j,k = 1,\dots,N$.

It immediately follows that any constant-valued, skew-adjoint operator gives a Poisson structure.

In general, the vector bundle map (2.1) is not an isomorphism: its rank at a given point $u\in\mathcal{V}_N$ defines the *rank* of the Poisson manifold $\mathcal{V}_N$ at u. By the skew-symmetry of the Poisson bracket, the rank of a Poisson manifold is always an even (non-negative) integer. Moreover, if $\mathcal{V}_N$ is not full rank then its Poisson structure is said to be *degenerate*. Degeneracy of Poisson structures generates conservation laws of the Hamiltonian dynamics on the phase space $\mathcal{V}_N$, *cf.* Definition 2.6.

The notion of rank characterizes *symplectic* manifolds as the Poisson manifolds which have maximal global rank. Symplectic manifolds are endowed with a nondegenerate, closed 2-form ω, called *symplectic structure*. The Poisson bracket on the symplectic manifold $(\mathcal{V}_N,\omega)$ is defined as

$$\{\mathcal{F},\mathcal{G}\}_N(u) = \omega(X_\mathcal{F},X_\mathcal{G}) := \omega(\mathcal{J}_N(u)\,\mathsf{d}\mathcal{F},\mathcal{J}_N(u)\,\mathsf{d}\mathcal{G}), \qquad \forall\, \mathcal{F},\mathcal{G}\in C^\infty(\mathcal{V}_N),\ \forall\, u\in\mathcal{V}_N.$$

Since ω is nondegenerate, the map $\omega^\flat_{X_\mathcal{H}} : Y\in T\mathcal{V}_N \mapsto \omega(X_\mathcal{H},Y) = (\mathsf{i}_{X_\mathcal{H}}\omega)(Y)$ is injective (i_X denotes the contraction by X), and the Hamiltonian vector field $X_\mathcal{H}$ of the functional $\mathcal{H}\in C^\infty(\mathcal{V}_N)$ satisfies $d\mathcal{H} = \mathsf{i}_{X_\mathcal{H}}\omega$.

2.1 Hamiltonian Dynamics

The Hamiltonian vector field $X_\mathcal{H}$ characterizes the evolution problem (1.1), whose dynamics preserves the Poisson structure of the phase space.

Applications between Poisson manifolds, consistent with the structure in the sense of preserving the bracket, are called Poisson maps.

Definition 2.3 (Poisson Map)**.** Let $(\mathcal{V}_N,\{\cdot,\cdot\}_N)$ and $(\mathcal{V}_n,\{\cdot,\cdot\}_n)$ be Poisson manifolds of finite dimension N and n respectively, with $n\leqslant N$. A smooth application $\Psi : (\mathcal{V}_N,\{\cdot,\cdot\}_N)\to(\mathcal{V}_n,\{\cdot,\cdot\}_n)$ is called a *Poisson map* if

$$(\Psi^*\{\mathcal{F},\mathcal{G}\}_n)(u) = \{\Psi^*\mathcal{F},\Psi^*\mathcal{G}\}_N(u), \qquad \forall\, \mathcal{F},\mathcal{G}\in C^\infty(\mathcal{V}_n), \forall\, u\in\mathcal{V}_N.$$

A vector field $X_\mathcal{H}$ on a manifold $\mathcal{V}_N$ determines a phase flow, namely a one-parameter group of diffeomorphisms $\Phi^t_{X_\mathcal{H}} : \mathcal{V}_N\to\mathcal{V}_N$ satisfying $d_t\Phi^t_{X_\mathcal{H}}(u) = X_\mathcal{H}(\Phi^t_{X_\mathcal{H}}(u))$ for all $t\in\mathcal{T}$ and $u\in\mathcal{V}_N$, with $\Phi^0_{X_\mathcal{H}}(u) = u$. The flow map $\Phi^t_{X_\mathcal{H}}$ of a vector field $X_\mathcal{H}\in T\mathcal{V}_N$ is Hamiltonian if $\Phi^t_{X_\mathcal{H}}$ is a Poisson map (on its domain). The reverse is also true.

Proposition 2.4 ([39, Proposition 10.2.3]). *Let $(\mathcal{V}_N, \{\cdot,\cdot\}_N)$ be a Poisson manifold and $\mathcal{H} \in C^\infty(\mathcal{V}_N)$. Then, the map $\Phi^t_X : \mathcal{V}_N \to \mathcal{V}_N$ satisfies*

$$\frac{d}{dt}(\mathcal{F} \circ \Phi^t_X) = \{\mathcal{F}, \mathcal{H}\}_N \circ \Phi^t_X, \qquad \forall\, \mathcal{F} \in C^\infty(\mathcal{V}_N),$$

if and only if it is the flow of $X_{\mathcal{H}}$.

With the definitions introduced hitherto, we can recast the dynamical system (1.1) as a Hamiltonian initial value problem as follows. Let $\mathcal{T} := (t_0, T]$ be a temporal interval and let $\mathcal{V}_N$ be an N-dimensional Poisson manifold with Poisson tensor $\mathcal{J}_N(u)$. For $u_0 \in \mathcal{V}_N$, find $u \in C^1(\mathcal{T}, \mathcal{V}_N)$ such that

$$\begin{cases} d_t u(t) = \mathcal{J}_N(u(t))\nabla_u \mathcal{H}_N(u(t)), & \text{for } \ t \in \mathcal{T}, \\ u(t_0) = u_0, \end{cases} \tag{2.4}$$

where $\mathcal{H}_N \in C^\infty(\mathcal{V}_N)$ is the Hamiltonian functional.

We can regard $\mathcal{V}_N$ as a submanifold of $\mathbb{R}^N$ equipped with the standard Euclidean metric whose induced norm is denoted henceforth by $\|\cdot\|$. Local well-posedness of (2.4) is guaranteed by assuming that the operator $F(t, u(t)) := X_{\mathcal{H}_N(u)}$ is Lipschitz continuous in u uniformly in $t \in \mathcal{T}$ in the $\|\cdot\|$-norm, in the spirit of Picard-Lindelöf result.

In addition to possessing a Poisson phase flow, Hamiltonian dynamics is characterized by the existence of differential invariants, and symmetry-related conservation laws.

Definition 2.5 (Invariants of Motion). A functional $\mathcal{I} \in C^\infty(\mathcal{V}_N)$ is an *invariant of motion* of the dynamical system (2.4) with flow map $\Phi^t_{X_{\mathcal{H}_N}}$, if $\{\mathcal{I}, \mathcal{H}_N\}_N(u) = 0$ for all $u \in \mathcal{V}_N$. Consequently, $\mathcal{I}$ is constant along the orbits of $X_{\mathcal{H}_N}$.

The Hamiltonian functional (if time-independent) is an invariant of motion. A particular subset of the invariants of motion of a dynamical system is given by the *Casimir invariants*, functionals on $\mathcal{V}_N$ which $\{\cdot,\cdot\}_N$-commute with every other functionals in $C^\infty(\mathcal{V}_N)$.

Definition 2.6 (Casimir Invariants). If $\mathfrak{g}$ is a Lie algebra with Lie product $\{\cdot,\cdot\}$, the *centralizer* of a subset S of $\mathfrak{g}$ is defined as $\mathsf{C}_{\mathfrak{g}}(S) := \{\mathcal{C} \in \mathfrak{g} : \{\mathcal{C}, \mathcal{F}\} = 0 \text{ for all } \mathcal{F} \in S\}$. The centralizer $\mathsf{C}_{\mathfrak{g}}(\mathfrak{g})$ of the Lie algebra itself is called the *center* of $\mathfrak{g}$ and its elements are called Casimir functions.

The Casimir invariants of the Poisson manifold $\mathcal{V}_N$ form the center of the Lie algebra $C^\infty(\mathcal{V}_N)$. Hence they are independent of the dynamics and only depend on the Poisson structure of the manifold, in particular its degeneracy. The number of Casimir invariants without functional relations among them, called *independent* Casimir invariants, is equal to the rank of the Lie algebra.

Henceforth, we assume that $\mathcal{V}_N$ is a *regular* Poisson manifold, namely $\mathrm{rank}(\mathcal{J}_N(u)) = 2R$, for all $u \in \mathcal{V}_N$, with $R \in \mathbb{N}$, $2R \leqslant N$. We denote with $q := N - 2R$ the dimension of the center of the Lie algebra, i.e. the number of independent Casimir invariants of $(\mathcal{V}_N, \mathcal{J}_N(u))$. Considering only regular Poisson manifolds is not restrictive, since the Hamiltonian systems we are interested in ensue from problems characterized by globally conserved quantities, such as energy, angular momentum, vorticity, etc.

2.2 Canonical Form of Poisson Structures

In the theory of Hamiltonian systems of classical mechanics, canonical forms on cotangent bundles are of great relevance. On an even dimensional manifold, a symplectic structure is given by the *canonical* symplectic 2-form defined as the exterior derivative of the tautological 1-form, see e.g. [12, Section 2.3]. Resorting to a coordinate system, the canonical structure on a symplectic manifold can be characterized as in the following result.

Proposition 2.7 ([2, Proposition 3.3.21]). *Let $(\mathcal{V}_{2R}, \omega)$ be a symplectic manifold and (U, ψ) a cotangent coordinate chart $\psi(u) = (q^1(u), \ldots, q^R(u), p_1(u), \ldots, p_R(u))$, for all $u \in U$. Then (U, ψ) is a* symplectic canonical chart *if and only if $\{q^i, q^j\} = \{p_i, p_j\} = 0$, and $\{q^i, p_j\} = \delta_{i,j}$ on U for all $i, j = 1, \ldots, R$, where $\{\cdot,\cdot\}$ is the Poisson bracket on $(\mathcal{V}_{2R}, \omega)$.*

Every finite-dimensional symplectic manifold admits local coordinates in which the local symplectic form is canonical. This result is known as Darboux's theorem [23].

Theorem 2.8. *Let $\mathcal{V}_{2R}$ be a finite $2R$-dimensional symplectic manifold. For each $u \in \mathcal{V}_{2R}$ there exists a chart $(\mathcal{B}_u, \Psi_u)$ in which a nondegenerate closed 2-form is locally isomorphic to the canonical form. The manifold $\mathcal{V}_{2R}$ can be covered by such charts.*

Note that this result can be extended to the infinite-dimensional case only under special assumptions and in general not if the symplectic structure ω on the manifold is only *weakly* nondegenerate, i.e. the map $\omega_X^\flat$ is injective but not necessarily onto. We refer to [51, 38, 47] for further details on the topic.

In order to derive the canonical form of Poisson structures one has to first deal with the kernel of the vector bundle map. Every Poisson manifold can be foliated by injectively immersed submanifolds corresponding to the equivalence classes under the following relation: two points on a Poisson manifold belong to the same class if there exists a piecewise smooth curve joining them consisting of segments of integral curves of Hamiltonian vector fields.

Definition 2.9 (Manifold Foliation)**.** Let $\mathcal{V}_N$ be an N-dimensional manifold. A *foliation* F of class C^p and of dimension q on $\mathcal{V}_N$ is a decomposition of $\mathcal{V}_N$ into disjoint connected subsets $\mathsf{F} = \{f_\alpha\}_\alpha$, called the *leaves* of the foliation, with the following property: each point of $\mathcal{V}_N$ has a neighborhood $\mathcal{B}$ and a system of C^p coordinates $\mathcal{B} \to z := (z_s, z_c) \in \mathbb{R}^q \times \mathbb{R}^{N-q}$ such that for each leaf f_α, the components of $\mathcal{B} \cap f_\alpha$ are described by the equations $(z_c)_1 =$ constant, ..., $(z_c)_{N-q} =$ constant.

The embedding of each symplectic leaf in a Poisson manifold is an injective Poisson map, and the phase flow of a Hamiltonian vector field preserves the symplectic structures on the leaves.

The combination of Darboux's theorem with the foliation properties of Poisson manifolds (*cf.* also the symplectic stratification theory [5, Chapter 2]) provides a way to bring degenerate Poisson structures into canonical form.

Theorem 2.10 (Lie-Weinstein Splitting Theorem [35, 52])**.** *Let $(\mathcal{V}_N, \{\cdot,\cdot\}_N)$ be an N-dimensional Poisson manifold. For each $u \in \mathcal{V}_N$ there exists a neighborhood $\mathcal{B}_u \subset \mathcal{V}_N$ of u, in which the rank of $\mathcal{V}_N$ is equal to $2R$, and an isomorphism $\Psi_u : \mathcal{B}_u \to \mathcal{S} \times \mathcal{N}$[1] where $\mathcal{S} = \Psi_s(\mathcal{B}_u)$ is a symplectic manifold and $\mathcal{N} = \Psi_c(\mathcal{B}_u)$ is a Poisson manifold whose rank vanishes at $\Psi_c(u)$. The factors $\mathcal{S}$ and $\mathcal{N}$ are unique up to local isomorphisms. Moreover, there exist local coordinates $\{q^1, \dots, q^R, p_1, \dots, p_R, c^1, \dots, c^{N-2R}\}$ which are canonical, i.e. $\{q^i, q^j\}_N = \{p_i, p_j\}_N = \{q^i, c^k\}_N = \{p_i, c^k\}_N = 0$, and $\{q^i, p_j\}_N = \delta_{i,j}$ for all $i, j = 1, \dots, R$ and $k = 1, \dots, N-2R$.*

On the neighborhood $\mathcal{B}_u$, the coordinates $\{c^k\}_k$ correspond to the Casimir invariants, whereas $\{(q^i, p_i)\}_i$ are the symplectic canonical coordinates, sometimes referred to as Clebsch variables [19]. In the canonical coordinates, the vector bundle map (2.1) takes the form

$$\mathcal{J}_N^c := \begin{pmatrix} & \mathrm{Id} & \\ -\mathrm{Id} & & \\ & & 0 \end{pmatrix} : T^*\mathcal{V}_R \times T^*\mathcal{V}_R \times T^*\mathcal{V}_{N-2R} \longrightarrow T\mathcal{V}_N,$$

where Id and 0 denote the identity and zero map, respectively.

There are many advantages for using canonical coordinates, see e.g. [47], most prominently, the possibility of bringing the Poisson tensor into constant-valued form and isolate its kernel. The design of our structure-preserving reduced basis methods for (2.4) hinges upon canonical forms obtained via exact or approximate Darboux maps, *cf.* Sections 3.1 and 4.1.

[1]The Cartesian product of two Poisson manifolds is endowed with a Poisson structure given by the Poisson map property of the projection on each factor, and by requiring that the pullbacks of the Poisson algebras on each factor form commuting subalgebras of the Poisson algebra of the Cartesian product.

2.3 Construction of Global Darboux's Map

On finite-dimensional Poisson manifolds $\mathcal{V}_N$, endowed with a *constant-valued* Poisson structure $\mathcal{J}_N$, the Darboux map from Theorem 2.8 is global. An analytic expression for the Darboux map can be derived by reverting to well-known results on matrix decompositions.

Proposition 2.11. *Every skew-symmetric matrix $M \in \mathbb{R}^{N,N}$ with $\mathrm{rank}(M) = 2R < N$ admits a decomposition of the form*

$$M = U\mathcal{J}_N^c U^\top, \tag{2.5}$$

where $U \in \mathbb{R}^{N,N}$ is invertible (but not orthogonal in general), and $\mathcal{J}_N^c \in \mathbb{R}^{N,N}$ is the matrix representation of the Poisson tensor in canonical form, namely

$$\mathcal{J}_N^c := \begin{pmatrix} \mathcal{J}_{2R}^c & 0_{2R,q} \\ 0_{q,2R} & 0_q \end{pmatrix}, \qquad \mathcal{J}_{2R}^c := \begin{pmatrix} 0_R & I_R \\ -I_R & 0_R \end{pmatrix},$$

where $q := N - 2R$ is the dimension of the null space of M, $0_R \in \mathbb{R}^{R,R}$ and $I_R \in \mathbb{R}^{R,R}$ denote the zero and the identity matrix, respectively.

The factorization (2.5) is unique up to transformations in the symplectic group $\mathrm{Sp}(2R, \mathbb{R})$.

Proof. We propose a constructive proof by steps: The implementation on the numerical experiments of Section 5 will mimic this argument.

Step 1. Every skew-symmetric square matrix can be brought into canonical form by a unitary congruence transformation, namely there exists $Q \in \mathbb{R}^{N,N}$ orthogonal such that $M = QSQ^\top$. The so-called Youla form S [55] is formed by blocks along the main diagonal, each 2×2 block formed by the complex part of a conjugate pair of complex eigenvalues of M, $\{\pm i\delta_j\}_{j=1}^R$, $\delta_j > 0$, and zeros for $j > R$. The proof of this result can be found in [55, Corollary 2] or [25, Theorem 2].

The Youla decomposition in not unique: the factor S can be fixed by computing a decomposition for a given ordering of the eigenvalues of M, see e.g. [10]. However, the orthogonal matrix Q is not unique.

Step 2. The block diagonal matrix $S \in \mathbb{R}^{N,N}$ can be further decomposed as $S = \widehat{D}\widehat{S}\widehat{D}$ where the matrix $\widehat{D}$ is diagonal with diagonal equal to $(\sqrt{\delta_1}, \sqrt{\delta_1}, \ldots, \sqrt{\delta_R}, \sqrt{\delta_R}, 0, \ldots, 0)$, while each element of $\widehat{S} \in \mathbb{R}^{N,N}$ is the sign of the corresponding element of S, i.e. the upper left block $\widehat{S}_{2R} \in \mathbb{R}^{2R,2R}$ of S is formed by R blocks along the main diagonal, each 2×2 block containing ± 1 as off-diagonal elements. Combining the first two steps, one has $M = Q\widehat{D}\widehat{S}\widehat{D}Q^\top$.

Step 3. As a last step, we construct a permutation matrix such that $\widehat{S}_{2R}$ is similar to $\mathcal{J}_{2R}^c$. Let $\widehat{P}_{2R} \in \mathbb{R}^{2R,2R}$ be the perfect shuffle permutation matrix in $\mathbb{R}^{2R}$, namely

$$\widehat{P}_{2R} := [\, e_1 \,|\, e_3 \,|\, \ldots \,|\, e_{2R-1} \,|\, e_2 \,|\, e_4 \,|\, \ldots \,|\, e_{2R} \,],$$

where e_j is the j-th canonical column vector in $\mathbb{R}^{2R}$. Then $\widehat{S}_{2R} = \widehat{P}_{2R}\mathcal{J}_{2R}^c\widehat{P}_{2R}^\top$, and we have that $M = Q\widehat{D}\widehat{P}\mathcal{J}_N^c\widehat{P}^\top\widehat{D}Q^\top$, where $\widehat{P} \in \mathbb{R}^{N,N}$ is the zero extension of $\widehat{P}_{2R}$.

Since we would like the transformation that brings M into canonical form to be invertible, we introduce the modified matrices

$$D := \begin{pmatrix} \widehat{D}_{2R} & 0_{2R,q} \\ 0_{q,2R} & I_q \end{pmatrix}, \qquad P := \begin{pmatrix} \widehat{P}_{2R} & 0_{2R,q} \\ 0_{q,2R} & I_q \end{pmatrix}. \tag{2.6}$$

The matrices D and P are invertible, and the extension of $\widehat{P}_{2R}$ by the identity I_q makes P into an orthogonal matrix. With the modified matrices P and D, the decomposition still holds, namely $M = QDP\mathcal{J}_N^c P^\top DQ^\top$. The conclusion (2.5) follows by setting $U = QDP$.

It can be easily verified that the factorization is not unique: if $Y \in \mathbb{R}^{N,N}$ satisfies $Y\mathcal{J}_N^c Y^\top = \mathcal{J}_N^c$ and it is nonsingular, then (2.5) holds with UY in lieu of U. □

2.4 Geometric Temporal Discretizations

Reduced basis methods for dynamical systems, designed with the aim to preserve the algebraic and geometric structure of the phase flow, cannot leave out of considerations the importance of relying on structure-preserving time integrators.

Definition 2.12. Let $\Phi_h^t : (\mathcal{V}_N, \mathcal{J}_N(u)) \to (\mathcal{V}_N, \mathcal{J}_N(u))$ be the discrete flow map associated with a temporal approximation of problem (2.4) with initial condition $u_0 \in (\mathcal{V}_N, \mathcal{J}_N(u))$. A numerical time discretization $u_1 = \Phi_h^t(u_0)$ is a $\mathcal{J}_N(u)$-*Poisson integrator* if the discrete flow map Φ_h^t is a Poisson map and preserves the Casimir invariants. If the manifold is symplectic then the time integrator associated with Φ_h^t is symplectic if Φ_h^t is a symplectomorphism.

While the literature on canonically symplectic numerical schemes is vast, for the case of Poisson systems with non-constant nonlinear structure, general structure-preserving integrators are largely unavailable, *cf.* e.g. [32] for a comprehensive treatise on the topic. However, since the study of geometric numerical integrators is outside of the scope of the present work, we assume the availability of a Poisson solver for the dynamical system (2.4).

3 Constant-Valued Degenerate Poisson Structures

In the present Section we develop reduced basis methods for parametric Hamiltonian systems on Poisson manifolds with a constant-valued Poisson structure. To fix the notation $\mathcal{V}_N$ is assumed to be an N-dimensional Poisson manifold with bracket $\{\cdot,\cdot\}_N$ and constant-valued tensor $\mathcal{J}_N$ with $\operatorname{rank}(\mathcal{J}_N) = 2R$. Moreover, $\mathcal{V}_N$ is endowed with a vector space structure given by the ℓ^2-norm $\|\cdot\|$.

3.1 Splitting of Poisson Dynamics

Let $\Lambda \subset \mathbb{R}^d$, with $d \geqslant 1$, be a compact set of possible parameters. For each $\mu \in \Lambda$, we consider the initial value problem: For $u_0(\mu) \in \mathcal{V}_N$, find $u(\cdot,\mu) \in C^1(\mathcal{T}, \mathcal{V}_N)$ such that

$$\begin{cases} \partial_t u(t,\mu) = \mathcal{J}_N \nabla_u \mathcal{H}_N(u(t,\mu);\mu), & \text{for } t \in \mathcal{T}, \\ u(t_0,\mu) = u_0(\mu). \end{cases} \tag{3.1}$$

To ensure well-posedness of (3.1), we assume that, for any $\mu \in \Lambda$, $\nabla \mathcal{H}_N$ is Lipschitz continuous in u uniformly in $t \in \mathcal{T}$ in the ℓ^2-norm.

As a first step towards the development of reduced basis methods for (3.1), we perform a *global* splitting of the Poisson manifold $(\mathcal{V}_N, \mathcal{J}_N)$ as describe in Section 2.2, namely

$$\Psi : \mathcal{V}_N \longrightarrow \mathcal{V}_{2R} \times \mathcal{N},$$

where $\mathcal{V}_{2R} = \Psi_s(\mathcal{V}_N)$ is a symplectic manifold of dimension $2R$ and $\mathcal{N} = \Psi_c(\mathcal{V}_N)$ is a submanifold whose dimension equals q, the number of independent Casimir invariants of $\{\cdot,\cdot\}_N$. The map Ψ exists, is linear and bijective in view of Proposition 2.11, and satisfies $\Psi \mathcal{J}_N \Psi^\top = \mathcal{J}_N^c$. The splitting preserves the Poisson structure of $\mathcal{V}_N$.

Proposition 3.1. *Let $\{\cdot,\cdot\}_{cN} : C^\infty(\mathcal{V}_N) \times C^\infty(\mathcal{V}_N) \to C^\infty(\mathcal{V}_N)$ be the bracket defined by $\{\mathcal{F},\mathcal{G}\}_{cN}(u) := \nabla_u \mathcal{F}(u)^\top \mathcal{J}_N^c \nabla_u \mathcal{G}(u)$, for all $\mathcal{F},\mathcal{G} \in C^\infty(\mathcal{V}_N)$ and $u \in \mathcal{V}_N$. The manifold $(\mathcal{V}_N, \{\cdot,\cdot\}_{cN})$ is Poisson. Moreover, the map $\Psi : (\mathcal{V}_N, \{\cdot,\cdot\}_N) \longrightarrow (\mathcal{V}_N, \{\cdot,\cdot\}_{cN})$ and its inverse are Poisson.*

Proof. It can be easily verified that the operator $\mathcal{J}_N^c$ satisfies the assumptions of Lemma 2.2 and therefore it is a Poisson structure.

To prove that the map $\Psi : (\mathcal{V}_N, \{\cdot,\cdot\}_N) \longrightarrow (\mathcal{V}_N, \{\cdot,\cdot\}_{cN})$ is Poisson we need to show that $(\Psi^*\{\mathcal{F},\mathcal{G}\}_{cN})(u) = \{\Psi^*\mathcal{F}, \Psi^*\mathcal{G}\}_N(u)$, for all $u \in \mathcal{V}_N$ and $\mathcal{F},\mathcal{G} \in C^\infty(\mathcal{V}_N)$. Let $z := \Psi\, u$, then

$$\begin{aligned} \{\Psi^*\mathcal{F}, \Psi^* G\}_N(u) &= \nabla_u(\Psi^*\mathcal{F})(u)^\top \mathcal{J}_N \nabla_u(\Psi^*\mathcal{G})(u) = (\Psi^*\nabla_u\mathcal{F})(u)^\top \mathcal{J}_N (\Psi^*\nabla_u\mathcal{G})(u) \\ &= (\nabla_z\mathcal{F})(\Psi u)^\top \Psi \mathcal{J}_N \Psi^\top (\nabla_z\mathcal{G})(\Psi u) = (\nabla_z\mathcal{F})(\Psi u)^\top \mathcal{J}_N^c (\nabla_z\mathcal{G})(\Psi u) \\ &= \{\mathcal{F},\mathcal{G}\}_{cN}(\Psi u). \end{aligned}$$

An analogous reasoning shows that $((\Psi^{-1})^*\{\mathcal{F},\mathcal{G}\}_N)(u) = \{(\Psi^{-1})^*\mathcal{F},(\Psi^{-1})^*\mathcal{G}\}_{cN}(u)$ for all $u \in \mathcal{V}_N$ and $\mathcal{F},\mathcal{G} \in C^\infty(\mathcal{V}_N)$. □

The dynamics $\Phi^t_{X_{\mathcal{H}_N}}$ can then be decoupled into the dynamics on the symplectic leaf and the trivial dynamics of the Casimir invariants. More in details, the system (3.1) can be recast in canonical form as: Find $z(\cdot,\mu) \in C^1(\mathcal{T},\mathcal{V}_N)$ such that

$$\begin{cases} \partial_t z(t,\mu) = \mathcal{J}^c_N \nabla_z \mathcal{H}^c_N(z(t,\mu);\mu), & \text{for } t \in \mathcal{T}, \\ z(t_0,\mu) = \Psi u_0(\mu), \end{cases} \tag{3.2}$$

where $\mathcal{H}^c_N := (\Psi^{-1})^*\mathcal{H}_N$ for every $\mu \in \Lambda$.

Since Ψ is linear and bijective, Proposition 3.1 entails that Ψ is a Poisson isomorphism: for any $t \in \mathcal{T}$ and any fixed parameter $\mu \in \Lambda$, $z(t,\mu)$ is a solution of (3.2) if and only if $z(t,\mu) = \Psi u(t,\mu)$, where $u(t,\mu)$ is a solution of (3.1).

Moreover, since Poisson maps preserve the Poisson bracket, the invariants of $\Phi^t_{X_{\mathcal{H}_N}}$ are in one-to-one correspondence with the invariants of $\Phi^t_{X_{\mathcal{H}^c_N}}$.

Corollary 3.2. *For any fixed parameter $\mu \in \Lambda$, let $\Phi^t_{X_{\mathcal{H}_N}}$ and $\Phi^t_{X_{\mathcal{H}^c_N}}$ be the flow maps associated with (3.1) and (3.2), respectively. The functional $\mathcal{I} \in C^\infty(\mathcal{V}_N)$ is an invariant of motion of $\Phi^t_{X_{\mathcal{H}_N}}$ if and only if $(\Psi^{-1})^*\mathcal{I} \in C^\infty(\mathcal{V}_N)$ is an invariant of $\Phi^t_{X_{\mathcal{H}^c_N}}$. Conversely, $\mathcal{I} \in C^\infty(\mathcal{V}_N)$ is an invariant of motion of $\Phi^t_{X_{\mathcal{H}^c_N}}$ if and only if $\Psi^*\mathcal{I} \in C^\infty(\mathcal{V}_N)$ is an invariant of $\Phi^t_{X_{\mathcal{H}_N}}$.*

Note that all *independent* Casimir invariants of a constant-valued degenerate Poisson tensor are linear. Indeed the Casimir invariants of $\{\cdot,\cdot\}_{cN}$ are the functionals $\{\mathcal{I}_m : z \in \mathcal{V}_N \mapsto z_m := (\Psi_c u)_m\}_{m=1}^q$. In view of Proposition 3.1 and Corollary 3.2, the functionals $\{\Psi^*\mathcal{I}_m\}_m$ are Casimir invariants of $\{\cdot,\cdot\}_N$ and since Ψ is linear, they are linear in u.

3.2 Reduced Basis Methods Preserving Poisson Structures

Exploiting the splitting of the dynamics introduced in Section 3.1, we seek a structure-preserving symplectic model order reduction on the symplectic manifold $\mathcal{V}_{2R}$, while leaving unchanged the submanifold $\mathcal{N}$ associated with the center of the Lie algebra $C^\infty(\mathcal{V}_N)$.

The reduced basis solution is the linear combination of a suitably chosen finite collection of solution trajectories computed from the high-fidelity model in canonical form, to provide an optimal decomposition in the sense of representing the dominant components of the dynamics. This is done via a weak greedy strategy, discussed in Section 3.2.2. The reduced basis functions are constructed to span an n-dimensional space $\mathcal{V}_n$, for $n \ll N$, with the following properties:

- $\mathcal{V}_n$ is a manifold endowed with the canonical Poisson structure $\{\cdot,\cdot\}_{cn}$.
- The rank of the canonical Poisson tensor $\mathcal{J}^c_n$ on $\mathcal{V}_n$, $\operatorname{rank}(\mathcal{J}^c_n) =: 2r$, satisfies $n - 2r = q$, namely the dimension of the center of the Lie algebras $C^\infty(\mathcal{V}_N)$ and $C^\infty(\mathcal{V}_n)$ coincides.
- $\mathcal{V}_n$ has a vector space structure given by the ℓ^2-norm.

To compute the evolution of the coefficients of the expansion in the reduced basis we rely on a Galerkin projection of the original dynamical system (2.4). This ensures that the expansion coefficients are uniquely determined by the basis. To preserve the Poisson structure, the projection is constructed to be symplectic on the symplectic leaf of $\mathcal{V}_N$ and to preserve the kernel of the Poisson tensor $\mathcal{J}_N$.

Let $\pi_+ : \mathcal{V}_N \to \mathcal{V}_n$ be a *surjective* map which is assumed to be *linear*. Since π_+ is surjective there exists a linear map $\pi : \mathcal{V}_n \to \mathcal{V}_N$ such that $\pi_+ \circ \pi : \mathcal{V}_n \to \operatorname{Im}(\pi) \subset \mathcal{V}_N \to \mathcal{V}_n$ is the identity on $\mathcal{V}_n$.

Lemma 3.3. *The map $\pi_+ : (\mathcal{V}_N, \{\cdot,\cdot\}_{cN}) \to (\mathcal{V}_n, \{\cdot,\cdot\}_{cn})$ is Poisson if and only if*

$$\pi_+ \mathcal{J}^c_N \pi_+^\top = \mathcal{J}^c_n.$$

Proof. We need to show that the pullback of π_+ preserves the Poisson bracket, namely that $(\pi_+^*\{\mathcal{F},\mathcal{G}\}_{cn})(u) = \{\pi_+^*\mathcal{F},\pi_+^*\mathcal{G}\}_{cN}(u)$, for all $u \in \mathcal{V}_N$ and $\mathcal{F},\mathcal{G} \in C^\infty(\mathcal{V}_n)$. Let $y := \pi_+ u$, rewriting the bracket using the canonical vector bundle map $\mathcal{J}_N^c$, results in

$$\begin{aligned}\{\pi_+^*\mathcal{F},\pi_+^*\mathcal{G}\}_{cN}(u) &= \nabla_u(\pi_+^*\mathcal{F})(u)^\top \mathcal{J}_N^c \nabla_u(\pi_+^*\mathcal{G})(u) = (\pi_+^*\nabla_u\mathcal{F})(u)^\top \mathcal{J}_N^c(\pi_+^*\nabla_u\mathcal{G})(u)\\ &= (\nabla_y\mathcal{F})(\pi_+u)^\top \pi_+\mathcal{J}_N^c\pi_+^\top(\nabla_y\mathcal{G})(\pi_+u) = \{\mathcal{F},\mathcal{G}\}_{cn}(\pi_+u),\end{aligned}$$

where the last equality holds if and only if $\pi_+\mathcal{J}_N^c\pi_+^\top = \mathcal{J}_n^c$. □

Following the splitting approach described in Section 3.1, the map π_+ can be constructed as

$$\pi_+ : \mathcal{V}_{2R}\times\mathcal{N} \longrightarrow \mathcal{V}_{2r}\times\mathcal{N}, \qquad \pi_+ = \pi_+^s \times \mathrm{Id}\,,$$

where π_+^s is taken to be a surjective ℓ^2-orthogonal symplectic application, i.e. $\pi_+^s\mathcal{J}_{2R}^c(\pi_+^s)^\top = \mathcal{J}_{2r}^c$.

Remark 3.4. The map π cannot be a Poisson map between the regular Poisson manifolds $(\mathcal{V}_n, \{\cdot,\cdot\}_{cn})$ and $(\mathcal{V}_N, \{\cdot,\cdot\}_{cN})$. Indeed, if that was the case, by a simple counting argument $\mathrm{rank}(\mathcal{J}_N^c) \leqslant \min\{\mathrm{rank}(\mathcal{J}_n^c), \mathrm{rank}(\pi)\}$, which cannot hold under the assumption $r \ll R$.

Definition 3.5. The *Poisson projection* onto $\mathrm{Im}(\pi^s)\times\mathcal{N} \subset \mathcal{V}_N$ is defined as the map $\mathcal{P} = \mathcal{P}_s \times \mathrm{Id}$: $\mathcal{V}_{2R}\times\mathcal{N} \to \mathrm{Im}(\pi)\times\mathcal{N}$ such that, for any $z_s \in (\mathcal{V}_{2R}, \mathcal{J}_{2R}^c)$,

$$\omega(\mathcal{P}_s z_s - z_s, \xi) = 0, \qquad \forall\,\xi \in \mathrm{Im}(\pi^s),$$

where ω is the canonical symplectic 2-form on the symplectic vector space $(\mathcal{V}_{2R}, \mathcal{J}_{2R}^c)$.

The reduced problem is derived via the Poisson projection $\mathcal{P} := \pi \circ \pi_+$ onto $\mathrm{Im}(\pi) \subset \mathcal{V}_N$ of the canonical Poisson dynamical system (3.2), namely for $t \in \mathcal{T}$ and $\mu \in \Lambda$,

$$\partial_t z_{\mathrm{rb}}(t,\mu) = \mathcal{P}(\mathcal{J}_N^c\nabla_z\mathcal{H}_N(\Psi^{-1}z_{\mathrm{rb}}(t,\mu);\mu)), \qquad z_{\mathrm{rb}}(t_0,\mu) = z^0(\mu).$$

On the n-dimensional Poisson manifold $\mathcal{V}_n$, the function $y(t,\mu) = \pi_+ z_{\mathrm{rb}}(t,\mu)$ satisfies

$$\begin{cases}\partial_t y(t,\mu) = \mathcal{J}_n^c\nabla_y\mathcal{H}_n(y(t,\mu);\mu), & \text{for } \ t\in\mathcal{T},\\ y(t_0,\mu) = \pi_+\Psi u_0(\mu),\end{cases} \tag{3.3}$$

where $\mathcal{H}_n := \pi^*\mathcal{H}_N^c$. Problem (3.3) is a dynamical system in canonical Poisson form on the manifold $(\mathcal{V}_n, \mathcal{J}_n^c)$. The assumption on the Lipschitz continuity of $\nabla\mathcal{H}_N$ ensures that $\nabla\mathcal{H}_n$ is also Lipschitz continuous with constant $\|\Psi^{-1}\|^2 L_{\delta\mathcal{H}}^\mu$, where $L_{\delta\mathcal{H}}^\mu$ is the Lipschitz constant of $\nabla\mathcal{H}_N$ for parameter $\mu \in \Lambda$. This guarantees the well-posedness of the reduced problem (3.3).

Remark 3.6. In principle, instead of relying on an orthogonal Poisson map π_+, one can envision π_+ to be only ℓ^2-orthogonal. In this case $\pi_+\mathcal{J}_N\pi_+^\top$ is constant-valued and skew-symmetric since $\mathcal{J}_N$ is skew-symmetric, hence it generates a Poisson bracket on $\mathcal{V}_n$. A Galerkin projection of the original Poisson dynamics yields a reduced Poisson system in noncanonical form. Using Proposition 2.11 one can bring such a system into canonical form via a bijective global change of coordinates. However, if $\mathcal{J}_N = \mathcal{J}_N(u)$ this reasoning fails since $\pi_+\mathcal{J}_N(u)\pi_+^\top$ is still skew-symmetric for every $u \in \mathcal{V}_N$, but it may not satisfy the Jacobi identity (2.3).

3.2.1 Stability and Conservation Properties of the Reduced Problem

By construction, the Hamiltonian $\mathcal{H}_n \in C^\infty(\mathcal{V}_n)$ of the reduced systems is obtained by pullback from the Hamiltonian $\mathcal{H}_N^c \in C^\infty(\mathcal{V}_N)$, namely $\mathcal{H}_n = \pi^*\mathcal{H}_N^c$, for all $\mu \in \Lambda$. This has the important consequence that, for any fixed $\mu \in \Lambda$, whenever $\mathcal{H}_N$ is a Lyapunov function with equilibria $\{u_e\}_e$ [2, Chapter 3 p. 207], then $\mathcal{H}_N^c$ is a Lyapunov function with equilibria $\{\Psi u_e\}_e$, and the reduced dynamics preserve the Lyapunov stable equilibria $\{\Psi u_e\}_e$ contained in $\mathrm{Im}(\pi)$. Indeed it can be shown that $\mathcal{H}_n$ is Lyapunov function with equilibria given by the image of the equilibria of the canonical Poisson system under π_+.

Concerning the preservation of the invariants of motion of $\Phi^t_{X_{\mathcal{H}_N^c}}$ after the reduction, we introduce the following concepts.

Definition 3.7. The model order reduction described by $\mathcal{P} = \pi \circ \pi_+$ is said to be *invariant-preserving* if the Hamiltonian of the high-fidelity canonical problem (3.2) satisfies $\mathcal{H}_N^c \in \mathrm{Im}(\pi_+^*)$, for all $\mu \in \Lambda$.

A weaker condition is that the error in the Hamiltonian vanishes only along solution trajectories: the model order reduction is said to be *Hamiltonian-preserving* if

$$\Delta\mathcal{H}_N^c(\mathcal{P},\mu) := |\mathcal{H}_N^c(z(t,\mu);\mu) - \mathcal{H}_N^c(\pi y(t,\mu);\mu)| = 0, \qquad \forall\, t \in \mathcal{T},\ \mu \in \Lambda,$$

with z solution of (3.2) and y solution of the reduced problem (3.3).

If the model order reduction is invariant-preserving then $\mathcal{H}_N^c = \pi_+^*\mathcal{H}_n$, since π_+^* is injective.

Note that since the map π_+^* acts as the identity on the center of the Lie algebra $C^\infty(\mathcal{V}_N)$, which is therefore not affected by the reduction, the Casimir invariants of the bracket $\{\cdot,\cdot\}_N$ are exactly conserved in the reduced problem. Moreover, the Poisson map π_+ provides a Hamiltonian-preserving model reduction.

Proposition 3.8. *For any $\mu \in \Lambda$ fixed, let $z \in C^1(\mathcal{T},(\mathcal{V}_N,\mathcal{J}_N^c))$ be a solution of the high-fidelity model (3.2) and $y \in C^1(\mathcal{T},(\mathcal{V}_n,\mathcal{J}_n^c))$ be a solution of the reduced model (3.3). Then the reduced basis method given by $\mathcal{P} = \pi \circ \pi_+$ is Hamiltonian-preserving in the sense of Definition 3.7.*

Proof. Since the Hamiltonian is an invariant of motion, it holds

$$\begin{aligned}\Delta\mathcal{H}_N^c(\mathcal{P},\mu) &= |\mathcal{H}_N^c(z(t,\mu);\mu) - \mathcal{H}_n(y(t,\mu);\mu)| = |\mathcal{H}_N^c(z_0(\mu);\mu) - \mathcal{H}_n(y_0(\mu);\mu)| \\ &= |\mathcal{H}_N^c(z_0(\mu);\mu) - (\pi_+^*\mathcal{H}_n)(z_0(\mu);\mu)|,\end{aligned}$$

for z and y being solutions of (3.2) and (3.3), respectively, with $z_0(\mu) := \Psi u_0(\mu)$. This implies that the reduced model is Hamiltonian-preserving if $z_0(\mu) \in \mathrm{Im}(\pi)$ for all $\mu \in \Lambda$.

Let $\mu \in \Lambda$ be fixed. Introducing the shifted variable $z^p(t;\mu) := z(t;\mu) - z_0(\mu) \in \mathcal{V}_N$ for all $t \in \mathcal{T}$, the high-fidelity canonical problem (3.2) can be cast as

$$\begin{cases} \partial_t z^p(t,\mu) = \mathcal{J}_N^c \nabla_z \mathcal{H}_N^{c,p}(z^p(t,\mu);\mu), & \text{for } \ t \in \mathcal{T}, \\ z^p(t_0;\mu) = 0, \end{cases} \tag{3.4}$$

where $\mathcal{H}_N^{c,p}(z^p(t;\mu);\mu) := \mathcal{H}_N^c(z^p(t;\mu) + z_0(\mu);\mu)$ for all $t \in \mathcal{T}$. Let us apply the splitting of the dynamics and the $\mathcal{J}_N^c$-Poisson reduced basis method, described in the previous Sections, to (3.4): The resulting reduced basis method is Hamiltonian-preserving for every parameter, i.e. $\Delta\mathcal{H}_N^{c,p}(\mathcal{P},\mu) = 0$ for all $\mu \in \Lambda$. This follows from the fact that the initial condition $z^p(t_0;\mu) = 0 \in \mathrm{Im}(\pi)$ for all $\mu \in \Lambda$ since the map π is linear. Note that the invariants of motion $\{\mathcal{I}_m^p\}_m$, associated with the Hamiltonian vector field of $\mathcal{H}_N^{c,p}$, are in one-to-one correspondence with the invariants $\{\mathcal{I}_m\}_m$ of $\mathcal{H}_N^c$ via $\mathcal{I}_m^p(z) = \mathcal{I}_m(z - z_0)$ for all $z \in \mathcal{V}_N$. □

With the exception of the Hamiltonian, even if $\mathcal{I} \in C^\infty(\mathcal{V}_N)$ is an invariant of motion of the canonical Poisson system (3.2), $\pi^*\mathcal{I} \in C^\infty(\mathcal{V}_n)$ is not necessarily an invariant of the system (3.3) in $(\mathcal{V}_n, \{\cdot,\cdot\}_{cn})$, since π is not a Poisson map. However, if the reduced model is invariant-preserving, it is possible to characterize the invariants of motion of the high-fidelity model belonging to $\mathrm{Im}(\pi_+^*)$ in terms of the invariants of the reduced dynamical system.

Lemma 3.9. *Let $\mu \in \Lambda$ be fixed. Assume that the model order reduction is invariant-preserving, namely $\mathcal{H}_N^c(\cdot,\mu) \in \mathrm{Im}(\pi_+^*)$. Then, $\mathcal{I} \in C^\infty(\mathcal{V}_n)$ is an invariant of $\Phi^t_{X_{\mathcal{H}_n}}$ if and only if $\pi_+^*\mathcal{I} \in C^\infty(\mathcal{V}_N)$ is an invariant of $\Phi^t_{X_{\mathcal{H}_N^c}}$ in $\mathrm{Im}(\pi_+^*) \subset C^\infty(\mathcal{V}_N)$.*

Proof. Let $\widehat{\mathcal{I}} \in C^\infty(\mathcal{V}_N) \cap \mathrm{Im}(\pi_+^*)$ be an invariant of (3.2), and $\widehat{\mathcal{I}} = \pi_+^*\mathcal{I}$. Since π_+^* is injective such $\mathcal{I} \in C^\infty(\mathcal{V}_n)$ is unique. We seek to show that the functional $\mathcal{I}$ is an invariant of (3.3), i.e. $\{\mathcal{I},\mathcal{H}_n\}_{cn}(y) = 0$ for all $y \in \mathcal{V}_n$. By the surjectivity of π_+, there exists at least one $z \in \mathcal{V}_N$ such that $y = \pi_+ z$. Since π_+ is a Poisson map, it holds

$$\{\mathcal{I},\mathcal{H}_n\}_{cn}(y) = \{\mathcal{I},\mathcal{H}_n\}_{cn}(\pi_+ z) = (\pi_+^*\{\mathcal{I},\mathcal{H}_n\}_{cn})(z) = \{\pi_+^*\mathcal{I},\pi_+^*\mathcal{H}_n\}_N(z) = \{\widehat{\mathcal{I}},\pi_+^*\mathcal{H}_n\}_N(z).$$

The result follows from the fact that $\mathcal{H}_N^c = \pi_+^* \mathcal{H}_n$ by assumption.

For the reverse implication, assume that $\mathcal{I} \in C^\infty(\mathcal{V}_n)$ is such that $\{\mathcal{I}, \mathcal{H}_n\}_{cn}(y) = 0$ for all $y \in \mathcal{V}_n$. Let $\mathcal{D}(\pi_+)$ be the preimage of π_+ in $\mathcal{V}_N$. An analogous reasoning yields,

$$0 = \{\mathcal{I}, \mathcal{H}_n\}_{cn}(\pi_+ z) = (\pi_+^* \{\mathcal{I}, \mathcal{H}_n\}_{cn})(z) = \{\pi_+^* \mathcal{I}, \pi_+^* \mathcal{H}_n\}_N(z), \qquad \forall z \in D(\pi_+).$$

Hence $\pi_+^* \mathcal{I}$ is an invariant of (3.1) in $\mathrm{Im}(\pi_+^*) \subset C^\infty(\mathcal{V}_N)$. □

3.2.2 Reduced Basis Generation via Symplectic Greedy Algorithm

As in a standard reduced basis approach, we build the set of reduced basis functions from a set of sampled high-fidelity solutions, called *snapshots*. Let us define the set of solutions of the dynamical system (3.1) as $\mathcal{U} := \{u(t,\mu) = \Phi^t_{X_{\mathcal{H}_N(\cdot,\mu)}}(u_0(\mu)) \in (\mathcal{V}_N, \mathcal{J}_N) : t \in \mathcal{T},\, \mu \in \Lambda\}$.

Let us consider a time discretization $\Phi^t_{h,\mu}$ of (3.1) on the uniform partition of $\mathcal{T}$ into $M \in \mathbb{N}$ elements given by $\mathcal{T}_h := \bigcup_{j \in \Upsilon_h} \mathcal{T}_j$, with $\mathcal{T}_j := (t^j, t^{j+1}]$ and $\Upsilon_h := [0, M) \cap \mathbb{N}$. Let Λ_h be a finite subset of the parameter set Λ and let $\overline{\Upsilon}_h := [0, M] \cap \mathbb{N}$. Consider the following sets of solution trajectories, obtained at sample time instants and parameters:

$$\begin{aligned}
\mathcal{U}_N &:= \{u^j(\mu) := \Phi^{t^j}_{h,\mu}(u_0(\mu)),\, j \in \overline{\Upsilon}_h,\, \mu \in \Lambda_h\}, && \text{sampled solution set of (3.1);}\\
\mathcal{Z}_N &:= \Psi(\mathcal{U}_N) = \{z^j(\mu) := \Psi u^j(\mu),\, j \in \overline{\Upsilon}_h,\, \mu \in \Lambda_h\}, && \text{sampled solution set of (3.2);}\\
\mathcal{Z}_N^s &:= \Psi_s(\mathcal{U}_N) = \{\Psi_s u^j(\mu),\, j \in \overline{\Upsilon}_h,\, \mu \in \Lambda_h\}, && \text{symplectic component of } \mathcal{Z}_N.
\end{aligned} \tag{3.5}$$

As explained previously, the model order reduction is applied only to the canonical symplectic leaf $(\mathcal{V}_{2R}, \mathcal{J}_{2R}^c)$ of $\mathcal{V}_N$. Hence, the reduced basis functions are generated from the snapshots in $\mathcal{Z}_N^s$ to form an ℓ^2-orthogonal and canonically $\mathcal{J}_{2R}^c$-symplectic set.

Definition 3.10 (Orthosymplectic Basis)**.** Let $(\mathcal{V}_{2R}, \omega)$ be a $2R$-dimensional symplectic vector space and let ω be the canonical symplectic form. Then the set of vectors $\{e_i\}_{i=1}^{2R}$ is said to be *orthosymplectic* in $\mathcal{V}_{2R}$ if

$$\omega(e_i, e_j) = (\mathcal{J}_{2R}^c)_{i,j}\,, \qquad \text{and} \qquad (e_i, e_j) = \delta_{i,j}\,, \qquad \forall i, j = 1 \ldots, 2R, \tag{3.6}$$

where $(\cdot,\cdot)$ is the Euclidean inner product and $\mathcal{J}_{2R}^c$ is the canonical symplectic tensor on $\mathcal{V}_{2R}$.

A subspace of a symplectic vector space $(\mathcal{V}_{2R}, \omega)$ is called *Lagrangian* if it coincides with its symplectic complement in $\mathcal{V}_{2R}$. As a consequence of the fact that any basis of a Lagrangian subspace of a symplectic vector space $(\mathcal{V}_{2R}, \omega)$ can be extended to a symplectic basis in $(\mathcal{V}_{2R}, \omega)$, every symplectic vector space admits an orthosymplectic basis. Numerical algorithms to build a canonically symplectic reduced basis include the POD-like strategies developed in [48] (cotangent lift, complex SVD, and nonlinear programming) and the symplectic greedy of [3] which couples a weak greedy strategy to select the snapshots to a symplectic Gram–Schmidt [49] procedure to enforce the constraints in (3.6). Here we opt for a greedy strategy since it gives us larger leeway in the choice of the orthosymplectic reduced basis when compared to a symplectic POD strategy [48].

The greedy approach consists of building a sequence of nested symplectic manifolds $V_{2k} \subset V_{2r}$ and an orthogonal $\mathcal{J}_{2k}^c$-symplectic basis by minimizing, at each iteration k, the projection error $\|\mathcal{Z}_N^s - \mathcal{P}_{2k}\mathcal{Z}_N^s\|$ and enforcing the constraints $\pi_+^s \mathcal{J}_{2R}^c (\pi_+^s)^\top = \mathcal{J}_{2r}^c$, and $\pi_+^s (\pi_+^s)^\top = \mathrm{Id}$. In this way the reduced space provides a good approximation of the sampled solution manifold $\mathcal{Z}_N^s$, whereas the constraints ensure that the dynamics in the lower dimensional space has the canonical orthosymplectic Hamiltonian structure (3.3). For the sake of completeness we report in Algorithm 1 the pseudoalgorithm for the weak greedy approach, adapted from [3, Algorithm 2].

Remark 3.11 (A posteriori error estimates)**.** A posteriori error estimates are crucial in reduced basis methods to certify the accuracy of the reduced basis approximation online, and for rigorous and efficient error control in the greedy sampling procedure offline, to allow exploration of much larger subsets of the parameter domain. In the context of dynamical systems, a posteriori error estimators obtained via adjoint problems or via time integration of residual relations are known to exhibit poor long time behavior, in

Algorithm 1 Symplectic Greedy. Input: $\{\mathcal{Z}_N^s, z_0, \mu_1, \mathrm{tol}_\gamma, \mathrm{tol}_\delta\}$. Output: π^{2j}.

Set $j = 1$.
Given the initial condition z_0, and μ_1 take $e_1 = z_0(\mu_1)/\|z_0(\mu_1)\|$ and $\pi^{2j} = [e_1, (\mathcal{J}_{2R}^c)^\top e_1]$.
Compute the pseudoinverse $\pi_+^{2j} = (\mathcal{J}_{2j}^c)^\top \pi^{2j} \mathcal{J}_{2R}^c$.
Compute the error in the symplecticity $\delta_{2j} = \|(\pi^{2j})^\top \mathcal{J}_{2R}^c \pi^{2j} - \mathcal{J}_{2j}^c\|_\infty$.
Initialize the maximum projection error $\gamma_{2j}^{\max} = 1$.
while $j < R$, and $\gamma_{2j}^{\max} > \mathrm{tol}_\gamma$, and $\delta_{2j} < \mathrm{tol}_\delta$ **do**
    Compute the projection error of all snapshots $\gamma_{2j}(z) = \|z - \pi^{2j}\pi_+^{2j} z\|$, for all $z \in \mathcal{Z}_N^s$.
    Select the new basis element $z^*(\mu_*) = \mathrm{argmax}_{z \in \mathcal{Z}_N^s} \gamma_{2j}(z)$.
    Update the maximum projection error $\gamma_{2j}^{\max} = \gamma_{2j}(z^*(\mu_*))$.
    Apply symplectic Gram–Schmidt to $z^*(\mu_*)$ and normalize $e_{j+1} = z^*(\mu_*)/\|z^*(\mu_*)\|$.
    $j = j + 1$.
    Update the matrix $\pi^{2j} = [e_1, \dots, e_j, (\mathcal{J}_{2R}^c)^\top e_1, \dots, (\mathcal{J}_{2R}^c)^\top e_j]$.
    Compute the pseudoinverse $\pi_+^{2j} = (\mathcal{J}_{2j}^c)^\top \pi^{2j} \mathcal{J}_{2R}^c$.
    Update the error in the symplecticity $\delta_{2j} = \|(\pi^{2j})^\top \mathcal{J}_{2R}^c \pi^{2j} - \mathcal{J}_{2j}^c\|_\infty$.
end while

particular for hyperbolic or singularly perturbed problems [53]. Although we acknowledge the paramount importance of efficient and reliable a posteriori error indicators, especially in a greedy approach, in this work we are mainly concerned with the structure-preserving properties of the reduced basis method and less with the efficiency or optimality of the algorithm. We therefore postpone the investigation of the topic to a later time.

The approximability properties of the solution sets (3.5) by linear subspaces of lower dimension n can be expressed by the Kolmogorov width [33]. The Kolmogorov n-width of a compact subset $\mathcal{U}_N$ of $(\mathcal{V}_N, \|\cdot\|)$ is defined as

$$d_n(\mathcal{U}_N) := \inf_{\substack{W_n \subset \mathcal{V}_N \\ \dim W_n = n}} \sup_{u \in \mathcal{U}_N} \inf_{w \in W_n} \|u - w\|. \tag{3.7}$$

We can bound the Kolmogorov width of the solution set of the canonical problem (3.2) in terms of the Kolmogorov width of $\mathcal{U}_N$, independently of the sampling of the temporal and parameter spaces. This is expressed in the following Lemma.

Lemma 3.12. *Let $\mathcal{U}_N$ and $\mathcal{Z}_N$ be the sampled solution sets introduced in (3.5). The Kolmogorov n-width of the solution set $\mathcal{Z}_N$ of the dynamical system (3.2) satisfies*

$$d_n(\mathcal{Z}_N) \leqslant \frac{1}{\min_{1 \leqslant j \leqslant N} \sqrt{|\lambda_j(\mathcal{J}_N)|}}\, d_n(\mathcal{U}_N),$$

where $\{\lambda_j \in \mathbb{C}\}_{j=1}^N$ are the eigenvalues of the constant-valued Poisson tensor $\mathcal{J}_N$.

Proof. Let $\Psi : (\mathcal{V}_N, \|\cdot\|, \mathcal{J}_N) \longrightarrow (\mathcal{V}_N, \|\cdot\|, \mathcal{J}_N^c)$ be the Darboux map associated with the Poisson tensor $\mathcal{J}_N$ and derived as in Proposition 2.11. Since Ψ is a linear bijection between finite-dimensional vector spaces, is bounded. Therefore the Kolmogorov n-width of $\Psi(\mathcal{U}_N)$ can be bounded as

$$d_n(\Psi(\mathcal{U}_N)) \leqslant \|\Psi\|\, d_n(\mathcal{U}_N),$$

where $\|\cdot\|$ denotes the operator 2-norm. Let $U \in \mathbb{R}^{N,N}$ be the matrix representation of the linear map Ψ^{-1}. From Proposition 2.11, Ψ is the composition of linear maps: $U^{-1} = (QDP)^{-1}$ where $Q \in \mathbb{R}^{N,N}$ is orthogonal, $D \in \mathbb{R}^{N,N}$ is diagonal and $P \in \mathbb{R}^{N,N}$ is the extension of a permutation matrix by the identity. Hence, it can be inferred that $\|U^{-1}\| \leqslant \|D^{-1}\| = \max\{1, \max_{1 \leqslant j \leqslant N} 1/\sqrt{|\lambda_j(\mathcal{J}_N)|}\}$, where $\{\lambda_j \in \mathbb{C}\}_{j=1}^N$ are the eigenvalues of the Poisson structure $\mathcal{J}_N$. Note that each eigenvalue λ_j is of the form $\lambda_j = \pm i\delta_j$ with $\delta_j \geqslant 0$. Since the modified matrix D in (2.6) is an arbitrary nonsingular extension of the matrix $\widehat{D}_{2R}$, one could in principle extend $\widehat{D}_{2R}$ by $(\min_j \sqrt{\delta_j})\, I_q$. The resulting D is nonsingular and $\|D^{-1}\| = 1/\min_{1 \leqslant j \leqslant N} \sqrt{|\lambda_j(\mathcal{J}_N)|}$. $\square$

Proposition 3.13 (Convergence of the Weak Symplectic Greedy Algorithm). *Let $\mathcal{U}_N$ and $\mathcal{Z}_N$ be the sampled solution sets introduced in (3.5). Assume that $\mathcal{U}_N$ has Kolmogorov n-width $d_n(\mathcal{U}_N)$. Then, the reduced space $\mathcal{V}_n = \mathcal{V}_{2r} \times \mathcal{N}$, with $\mathcal{V}_{2r}$ constructed via Algorithm 1, satisfies*

$$\|z - \mathcal{P}z\| \leqslant \frac{C\, 3^{r+1}(r+1)}{\min_{1\leqslant j\leqslant N} \sqrt{|\lambda_j(\mathcal{J}_N)|}}\, d_n(\mathcal{U}_N), \qquad \forall\, z \in \mathcal{Z}_N,$$

where $C > 0$ is a constant independent of n, r and N.

Proof. The convergence estimates for the weak greedy algorithm, derived in [11] and adapted to the symplectic case in [3, Section 4.1.3], result in

$$\|z_s - \mathcal{P}_s z_s\| \leqslant C 3^{r+1}(r+1)\, d_{2r}(\mathcal{Z}_N^s), \qquad \forall\, z_s \in \mathcal{Z}_N^s.$$

Let us define the Kolmogorov n-width of $\mathcal{Z}_N$ restricted to subspaces of $\mathcal{V}_n$ of the form $\mathcal{V}_{2r} \times \mathcal{N}$ as,

$$\widehat{d_n}(\mathcal{Z}_N) := \inf_{\substack{\widehat{W}_n \subset \mathcal{V}_N \\ \dim W_n = n}} \sup_{z\in\mathcal{Z}_N} \inf_{w\in W_n} \|z - w\|,$$

where $\widehat{W}_n := \{w \in \mathcal{V}_n : w = (w_s, \Psi_c u),\, w_s \in \mathcal{V}_{2r},\, u \in \mathcal{V}_N\}$. If $d_{2r}(\mathcal{Z}_N^s)$ denotes the Kolmogorov $2r$-width (3.7) of the symplectic component of the solution set $\mathcal{Z}_N$, it holds $d_{2r}(\mathcal{Z}_N^s) = \widehat{d_n}(\mathcal{Z}_N) \leqslant d_n(\mathcal{Z}_N)$. The definition of the Poisson projection from Definition 3.5 together with Lemma 3.12 yields the conclusion. ☐

Note that, depending on the decay of the Kolmogorov width, sharper convergence estimates can be derived following [8].

3.3 Discrete Empirical Interpolation for Poisson Dynamics

In the context of projection-based reduced order models, the discrete empirical interpolation methods introduced in [17] provide a well-established technique to evaluate nonlinear terms at a computational cost proportional to the dimension n of the reduced problem.

Let the parameter $\mu \in \Lambda$ be fixed. Let us assume that the dynamical system (3.1) can be written by separating a linear and a nonlinear part, namely let $\nabla_u \mathcal{H}_N(u;\mu) = \mathcal{L}_N u + \mathcal{M}_N(u)$, where $\mathcal{L}_N$ denotes a linear operator and $\mathcal{M}_N$ a nonlinear operator on $\mathcal{V}_N$, (the dependence of $\mathcal{L}_N$ and $\mathcal{M}_N$ on μ is omitted for the sake of readability). Then (3.1) can be recast as

$$\partial_t u(t,\mu) = \mathcal{J}_N \mathcal{L}_N u + \mathcal{J}_N \mathcal{M}_N(u), \quad u(t_0,\mu) = u_0(\mu). \tag{3.8}$$

Analogously, we can rewrite the canonical problem (3.2) as $z(t_0,\mu) = \Psi u_0(\mu)$ and

$$\partial_t z(t,\mu) = \mathcal{J}_N^c \nabla_z \mathcal{H}_N(\Psi^{-1} z(t,\mu);\mu) = \mathcal{J}_N^c \mathcal{L}_N^c z(t,\mu) + \mathcal{J}_N^c \mathcal{M}_N^c(z(t,\mu)),$$

where $\mathcal{L}_N^c := \Psi^{-\top}\mathcal{L}_N \Psi^{-1}$ and $\mathcal{M}_N^c(z) := \Psi^{-\top}\mathcal{M}_N(\Psi^{-1}z)$. The reduced problem (3.3) becomes $y(t_0,\mu) = \pi_+ z_0(\mu)$,

$$\begin{aligned}\partial_t y(t,\mu) &= \pi_+ \mathcal{J}_N^c \Psi^{-\top}\mathcal{L}_N \Psi^{-1}\pi y(t,\mu) + \pi_+ \mathcal{J}_N^c \Psi^{-\top}\mathcal{M}_N(\Psi^{-1}\pi y(t,\mu)),\\ &= \mathcal{J}_n^c \mathcal{L}_n y(t,\mu) + \mathcal{J}_n^c \pi^\top \Psi^{-\top}\mathcal{M}_N(\Psi^{-1}\pi y(t,\mu)),\end{aligned}$$

where $\mathcal{L}_n := \pi^\top \Psi^{-\top}\mathcal{L}_N \Psi^{-1}\pi$. Adopting a DEIM strategy we approximate the nonlinear term as $\mathcal{M}_N(u) \approx U(P^\top U)^{-1}P^\top \mathcal{M}_N(u)$ for all $u \in \mathcal{V}_N$ as in [17, Eq. 3.5]. Analogously to the a symplectic DEIM approach [3, Section 4.2] we take $U^\top = \pi_+$ so that

$$\Psi^{-\top}\mathcal{M}_N(\Psi^{-1}\pi y) \approx \pi_+^\top (P^\top \pi_+^\top)^{-1} P^\top \Psi^{-\top}\mathcal{M}_N(\Psi^{-1}\pi y).$$

The reduced problem in the SDEIM formulation thus reads

$$\partial_t y(t,\mu) = \mathcal{J}_n^c \mathcal{L}_n y(t,\mu) + \mathcal{J}_n^c (P^\top \pi_+^\top)^{-1}\mathcal{M}_n(y(t,\mu)), \quad y(t_0,\mu) = \pi_+ z_0(\mu), \tag{3.9}$$

where $\mathcal{M}_n(y) := P^\top \Psi^{-\top}\mathcal{M}_N(\Psi^{-1}\pi y)$.

3.3.1 A Priori Error Estimates for Symplectic RBM with DEIM

Taking the cue from the convergence analysis in [18], an a priori error estimate can be derived for the state approximation error between the high-fidelity solution and the reduced solution obtained by applying the symplectic DEIM to the Poisson systems (3.1) and (3.3), respectively.

We derive L^2-error estimates in both time and parameter space.

Proposition 3.14. *For any given $\mu \in \Lambda$, let $u(\cdot,\mu) \in C^1(\mathcal{T},(\mathcal{V}_N,\mathcal{J}_N))$ be the solution of (3.8) and let $u_{\mathrm{rb}} := \Psi^{-1}\pi y$ where $y(\cdot,\mu) \in C^1(\mathcal{T},(\mathcal{V}_n,\mathcal{J}_n^c))$ is the solution of the reduced system (3.9). Assume that for every $\mu \in \Lambda$ the nonlinear operator $\mathcal{M}_N$ is Lipschitz continuous in the norm $\|\cdot\|$ with constant $L_{\mathcal{M}}(\mu)$. Then,*

$$\begin{aligned}\|u-u_{\mathrm{rb}}\|^2_{L^2(\mathcal{T}\times\Lambda;\mathcal{V}_N)} \leqslant{}& \|\Psi^{-1}\|\, C_1(T,\alpha(\mu))\|\Psi u - \mathcal{P}\Psi u\|^2_{L^2(\mathcal{T}\times\Lambda;\mathcal{V}_N)}\\ &+ \|\Psi^{-1}\|^2\, C_2(T,\alpha(\mu))\|\mathcal{M}_N(u) - \pi_+^\top\pi_+\mathcal{M}_N(u)\|^2_{L^2(\mathcal{T}\times\Lambda;\mathcal{V}_N)},\end{aligned} \tag{3.10}$$

where $\alpha(\mu) := \|\Psi^{-\top}\mathcal{L}_N(\mu)\Psi^{-1}\| + \beta\|\Psi^{-1}\|^2 L_{\mathcal{M}}(\mu)$, and $\beta := \|(P^\top\pi_+^\top)^{-1}\|$, $\Delta\mathcal{T} := |T-t_0|$ and

$$C_1(T,\alpha(\mu)) := 2\Delta\mathcal{T}\max_{\mu\in\Lambda}(\alpha(\mu)(e^{2\alpha(\mu)\Delta\mathcal{T}}-1)+1), \quad C_2(T,\alpha(\mu)) := 2\Delta\mathcal{T}\beta^2\max_{\mu\in\Lambda}(\alpha(\mu)^{-1}(e^{2\alpha(\mu)\Delta\mathcal{T}}-1)).$$

Proof. The error between the high-fidelity and reduced solution can be bounded by the reduction error associated with the dynamical systems in canonical form. Indeed,

$$\|u-u_{\mathrm{rb}}\|^2_{L^2(\mathcal{T}\times\Lambda;\mathcal{V}_N)} = \int_\Lambda\int_{\mathcal{T}} \|\Psi^{-1}(\Psi u(t,\mu) - \pi y(t,\mu))\|\,dt\,d\mu \leqslant \|\Psi^{-1}\|\|z-\pi y\|^2_{L^2(\mathcal{T}\times\Lambda;\mathcal{V}_N)}, \tag{3.11}$$

where z is the solution of the high-fidelity model in canonical form (3.2) and y is the solution of the reduced problem (3.3).

At each time t and $\mu\in\Lambda$, let $z-\pi y = (z-\mathcal{P}z)+(\mathcal{P}z-\pi y) =: e_p + e_h$. Then, if $W := \pi_+^\top(P^\top\pi_+^\top)^{-1}P^\top$,

$$\begin{aligned}\partial_t e_h(t,\mu) &= \mathcal{P}\partial_t z(t,\mu) - \pi\partial_t y(t,\mu) = \mathcal{P}\mathcal{J}_N^c(\mathcal{L}_N^c z + \mathcal{M}_N^c(z) - \mathcal{L}_N^c\pi y - W\mathcal{M}_N^c(\pi y))\\ &= \mathcal{P}\mathcal{J}_N^c\mathcal{L}_N^c e_h + \mathcal{P}\mathcal{J}_N^c\mathcal{L}_N^c e_p + \mathcal{P}\mathcal{J}_N^c(\mathcal{M}_N^c(z) - W\mathcal{M}_N^c(\pi y))\\ &=: \mathcal{O}(\mu)e_h + \mathcal{Q}(t,\mu).\end{aligned}$$

Using the fact that $(I-W)\pi_+^\top\pi_+\mathcal{M}_N^c(z) = 0$, we can bound $\mathcal{Q}$ as,

$$\begin{aligned}\|\mathcal{Q}(t,\mu)\| &\leqslant \|\mathcal{L}_N^c e_p\| + \|(I-W)\mathcal{M}_N^c(z)\| + \|W(\mathcal{M}_N^c(z) - \mathcal{M}_N^c(\pi y))\|\\ &\leqslant \|\mathcal{L}_N^c\|\|e_p\| + \|(I-W)(\mathcal{M}_N^c(z) - \pi_+^\top\pi_+\mathcal{M}_N^c(z))\| + \|W(\mathcal{M}_N^c(z) - \mathcal{M}_N^c(\pi y))\|\\ &\leqslant \|\mathcal{L}_N^c\|\|e_p\| + \|I-W\|\|w\| + \|W\|L_{\mathcal{M}}(\mu)\|\Psi^{-1}\|^2(\|e_p\|+\|e_h\|).\end{aligned}$$

where $w(t,\mu) := \mathcal{M}_N^c(z(t,\mu)) - \pi_+^\top\pi_+\mathcal{M}_N^c(z(t,\mu))$. The error satisfies the evolution equation

$$\begin{aligned}\partial_t\|e_h\| &= \frac{1}{\|e_h\|}(\partial_t e_h, e_h)_{\mathcal{V}} = \frac{1}{\|e_h\|}(\mathcal{O}e_h, e_h)_{\mathcal{V}} + \frac{1}{\|e_h\|}(\mathcal{Q}(t,\mu), e_h)_{\mathcal{V}}\\ &\leqslant \|\mathcal{O}\|\|e_h\| + \|\mathcal{Q}\| \leqslant \alpha(\mu)\|e_h\| + b(t,\mu),\end{aligned} \tag{3.12}$$

where $\alpha(\mu) := \|\mathcal{L}_N^c\| + \|W\|L_{\mathcal{M}}(\mu)\|\Psi^{-1}\|^2$ and $b(t,\mu) := \alpha(\mu)\|e_p(t,\mu)\| + \beta\|w(t,\mu)\|$, $\beta := \|I-W\|$. Since W is a projector $\beta = \|I-W\| = \|W\|$: the norm of W is bounded [17, Lemma 3.2], and depends on the DEIM selection of indices in P [17, Section 3.2]. From (3.12), Gronwall's inequality [29] gives

$$\begin{aligned}\|e_h(t,\mu)\| &\leqslant \|e_h(t_0,\mu)\|\,e^{\alpha(\mu)t} + \int_{t_0}^t e^{\alpha(\mu)(t-s)}b(s,\mu)\,ds\\ &\leqslant \left(\int_{t_0}^t e^{2\alpha(\mu)(t-s)}\,ds\right)^{1/2}\left(2\int_{t_0}^t \alpha(\mu)^2\|e_p(s,\mu)\|^2 + \beta^2\|w(s,\mu)\|^2\,ds\right)^{1/2}.\end{aligned}$$

Hence, for all $t \in \mathcal{T}$,

$$\|e_h(t,\mu)\|^2 \leqslant c(t,\alpha(\mu)) \int_{t_0}^{t} (\alpha(\mu)^2 \|e_p(s,\mu)\|^2 + \beta^2 \|w(s,\mu)\|^2)\, ds,$$

where $c(t,\alpha(\mu)) := 2\alpha(\mu)^{-1}(e^{2\alpha(\mu)(t-t_0)} - 1)$ assuming $\alpha(\mu) \neq 0$, for all $\mu \in \Lambda$. This implies that,

$$\begin{aligned}
\|z - \pi y\|^2_{L^2(\mathcal{T}\times\Lambda;\mathcal{V}_N)} &\leqslant \int_\Lambda \int_{t_0}^{T} \|e_p(t,\mu)\|^2 \, dt\, d\mu \\
&\quad + \Delta\mathcal{T} \int_\Lambda c(T,\alpha(\mu)) \int_{t_0}^{T} (\alpha^2(\mu)\|e_p(t,\mu)\|^2 + \beta^2 \|w(t,\mu)\|^2)\, dt\, d\mu \\
&\leqslant 2\Delta\mathcal{T}\beta^2 \int_\Lambda \alpha(\mu)^{-1}(e^{2\alpha(\mu)\Delta\mathcal{T}} - 1)\|\mathcal{M}_N^c(z(\cdot,\mu)) - \pi_+^\top \pi_+ \mathcal{M}_N^c(z(\cdot,\mu))\|^2_{L^2(\mathcal{T};\mathcal{V}_N)}\, d\mu \\
&\quad + \int_\Lambda (\Delta\mathcal{T}\alpha(\mu)(e^{2\alpha(\mu)\Delta\mathcal{T}} - 1) + 1)\|z(\cdot,\mu) - \mathcal{P}z(\cdot,\mu)\|^2_{L^2(\mathcal{T};\mathcal{V}_N)}\, d\mu \\
&\leqslant C_1(T,\alpha(\mu))\|z - \mathcal{P}z\|^2_{L^2(\mathcal{T}\times\Lambda;\mathcal{V}_N)} \\
&\quad + \|\Psi^{-1}\| C_2(T,\alpha(\mu))\|\mathcal{M}_N(\Psi^{-1}z) - \pi_+^\top \pi_+ \mathcal{M}_N(\Psi^{-1}z)\|^2_{L^2(\mathcal{T}\times\Lambda;\mathcal{V}_N)}.
\end{aligned}$$

The conclusion follows by combining the above estimate with (3.11). □

A few observations are in order. The projection error appearing in the estimate of Proposition 3.14 can be written as

$$\begin{aligned}
\|z - \mathcal{P}z\|^2_{L^2(\mathcal{T}\times\Lambda;\mathcal{V}_N)} &\leqslant \left| \|z - \mathcal{P}z\|^2_{L^2(\mathcal{T}\times\Lambda;\mathcal{V}_N)} - \sum_{j\in\overline{\Upsilon}_h, i\leqslant\sharp\Lambda} w_{j,i}\|z(t^j,\mu_i) - \mathcal{P}z(t^j,\mu_i)\|^2 \right| \\
&\quad + \sum_{j\in\overline{\Upsilon}_h, i\leqslant\sharp\Lambda} w_{j,i}\|z(t^j,\mu_i) - \mathcal{P}z(t^j,\mu_i)\|^2.
\end{aligned}$$

The term in absolute value is a quadrature error ($\{w_{j,i}\}_{j\in\overline{\Upsilon}_h, i\leqslant\sharp\Lambda}$ are quadrature weights), and depends on the number and choice of the snapshots $\mathcal{Z}_N$, the smoothness of the integrand in the temporal variable and in the parameter, etc. The second term is controlled by the greedy algorithm according to Proposition 3.13.

The term $\|\mathcal{M}_N(u) - \pi_+^\top \pi_+ \mathcal{M}_N(u)\|^2_{L^2(\mathcal{T}\times\Lambda;\mathcal{V}_N)}$ in (3.10) can be controlled during the assembling of the reduced basis from the nonlinear snapshots $\{\mathcal{M}_N(u^j(\mu))\}_{j\in\overline{\Upsilon}_h, i\leqslant\sharp\Lambda}$, see [17, 18, Section 2.1] and [3, Section 4.2].

Finally, observe that the bound in (3.10) depends exponentially on the final time T. A linear dependence on T can be obtained in special cases, for example when $\nabla\mathcal{H}_N$ is uniformly negative monotone or when the linear part of (3.8) has a logarithmic norm $\mu(\mathcal{L}_N) := \lim_{h\searrow 0}(\|\operatorname{Id} + h\mathcal{L}_N\| - 1)/h$ [22] bounded by $\|(P^\top \pi_+^\top)^{-1}\| L_{\mathcal{M}}$, as in [18, Sections 3 and 4].

4 State-Dependent Poisson Structures

The Hamiltonian formulation of most problems in fluid and plasma physics possesses a Poisson structure which is not only degenerate but depends on the state variable. The difficulty in dealing with such problems stems from the time dependence and nonlinearity intrinsic to the manifold structure.

As for the constant-valued case, Darboux's Theorem 2.8 suggests a change of coordinates to bring the structure into a canonical form more amenable to discretization and model order reduction. However, in the state-dependent case, the Darboux charts have a local nature and the corresponding global change of coordinates is nonlinear. For these reasons, aside from very particular cases [36, 40], it is generally non-trivial to derive such nonlinear maps.

On the other hand, resorting to approximation techniques requires particular care. Indeed the use of too crude an approximation of the Poisson tensor, e.g. by expanding the state u in a power series of a small parameter and then truncate the expansion of $\mathcal{J}_N(u)$, destroys the underlying Poisson structure since the Jacobi identity (2.3) generally fails to hold for the approximate tensor. In the context of Hamiltonian perturbation theory, the authors of [44] advocate a near identity change of variables in the neighborhood of a stable equilibrium to bring the Poisson tensor in constant form pointwise. This approach is, however, limited to weakly nonlinear Hamiltonian systems describing the dynamics near equilibria, and introduces a local approximation of the Poisson structure by truncating the expansion of the Poisson tensor.

We propose to perform a piecewise linear approximation of the Darboux map in each discrete time interval and subsequently derive a reduced basis method for the resulting, locally canonical, structure.

To keep the presentation focused, we next consider dynamical systems which do not depend on a parameter. We further comment on the extension of the results obtained in the forthcoming Sections to the parameter-dependent case in Section 4.4.

4.1 Linear Approximation of Darboux's Charts

We exploit the linearization introduced by the timestepping to derive piecewise linear approximations of the Darboux map Ψ, and construct a locally finite cover of $\mathcal{V}_N$ along the solution trajectory, using time intervals and linear approximations of the homomorphisms $\{\Psi_u\}$, from Theorem 2.10, on each interval.

Let us define the submanifolds $\mathcal{V}_{N,j} := \{u(t) \in \mathcal{V}_N : t \in \mathcal{T}_j = (t^j, t^{j+1}]\} \subset \mathcal{V}_N$, associated with the temporal mesh $\mathcal{T}_h = \bigcup_{j\in\Upsilon_h} \mathcal{T}_j$. Time discretization of (2.4) yields: For $u_0 \in \mathcal{V}_N$, find $\{u^{j+1}\}_{j\in\Upsilon_h} \subset \mathcal{V}_N$ such that

$$\begin{cases} u^{j+1} = u^j + \Delta t\, \mathcal{J}_N(\widehat{u}^j)\nabla_u \mathcal{H}_N(\widehat{u}^j), \qquad \text{for } \; j \in \Upsilon_h, \\ u^0 = u_0, \end{cases} \tag{4.1}$$

where $\widehat{u}^j \in \mathcal{V}_{N,j}$ is determined by the temporal discretization of choice, and can be a state or a combination of them. Alternative discretizations of the Poisson tensor and of the Hamiltonian can be considered. This choice will affect the convergence estimates and the restriction of the time step in Theorem 4.8 and Theorem 4.9, but not the approximation of the Darboux map, nor the derivation of the reduced basis method.

Definition 4.1. On each submanifold $\mathcal{V}_{N,j}$, with $j \in \Upsilon_h$, the local approximation of the Darboux map Ψ is defined to be the linear function $\psi_{j+1/2} : \mathcal{V}_{N,j} \to \mathcal{V}_{N,j}$, which satisfies $\psi_{j+1/2}\mathcal{J}_N(\widehat{u}^j)\psi_{j+1/2}^\top = \mathcal{J}_N^c$ at the state(s) $\widehat{u}^j \in \mathcal{T}_j$ dictated by the temporal discretization (4.1). Each map $\psi_{j+1/2}$ provides the local splitting $\psi_{j+1/2} : \mathcal{V}_{N,j} \to \mathcal{V}_{2R} \times \mathcal{N}_j$, with $\psi^s_{j+1/2}(\mathcal{V}_{N,j}) = \mathcal{V}_{2R}$ being a $2R$-dimensional subspace of $\mathcal{V}_N$ and $\psi^c_{j+1/2}(\mathcal{V}_{N,j}) = \mathcal{N}_j$ the approximation of the subspace associated with the kernel of the Poisson tensor at $\widehat{u}^j$.

Let $\overline{\mathcal{V}_{N,j}} := \{u(t) \in \mathcal{V}_N : t \in \overline{\mathcal{T}_j}\}$ for all $j \in \Upsilon_h$. *Transition maps* between neighboring intervals are $T_j : \psi_{j-1/2}(\mathcal{V}_{N,j-1} \cap \overline{\mathcal{V}_{N,j}}) \longrightarrow \psi_{j+1/2}(\mathcal{V}_{N,j-1} \cap \overline{\mathcal{V}_{N,j}})$, defined as $T_j := \psi_{j+1/2} \circ \psi^{-1}_{j-1/2}$, for $j \in \Upsilon_h \backslash \{0\}$ with $T_0 := \mathrm{Id}$. A sketch of the approximated Darboux's charts is presented in Figure 1.

We denote $\psi : \mathcal{V}_N \to \mathcal{V}_N$ as the global map collecting the linear functions $\psi_{j+1/2}$ on each $\mathcal{V}_{N,j}$, $j \in \Upsilon_h$. For any $j \in \Upsilon_h$ fixed, the map $\psi_{j+1/2}$ is in general *not* Poisson on $\mathcal{V}_{N,j}$. However, provided the time discretization (4.1) preserves the Casimir invariants (see Definition 2.12), the map ψ preserves the rank of the Poisson structure since $\dim \mathcal{N}_j = q$ for all $j \in \Upsilon_h$.

With the local change of coordinates introduced by $\psi : \mathcal{V}_N \to \mathcal{V}_N$, we can recast the fully discrete problem (4.1) as: For $u_0 \in \mathcal{V}_N$, find $\{z^{j+1}\}_{j\in\Upsilon_h} \subset \mathcal{V}_N$ such that

$$\begin{cases} z^{j+1} = T_j z^j + \Delta t\, \mathcal{J}_N^c \nabla_z \mathcal{H}_N^j(\widehat{z}^j), \qquad \text{for } \; j \in \Upsilon_h, \\ z^0 = \psi_{1/2} u_0, \end{cases} \tag{4.2}$$

where $\widehat{z}^j := \psi_{j+1/2}\widehat{u}^j$ and $\mathcal{H}_N^j(z) := \mathcal{H}_N(\psi^{-1}_{j+1/2} z)$ for all $z \in \mathcal{V}_{N,j}$.

The fact that the approximation of the Darboux map is based on the linearization, introduced by the timestepping, ensures that the Poisson structure is not jeopardized by recasting the discrete problem (4.1) as (4.2).

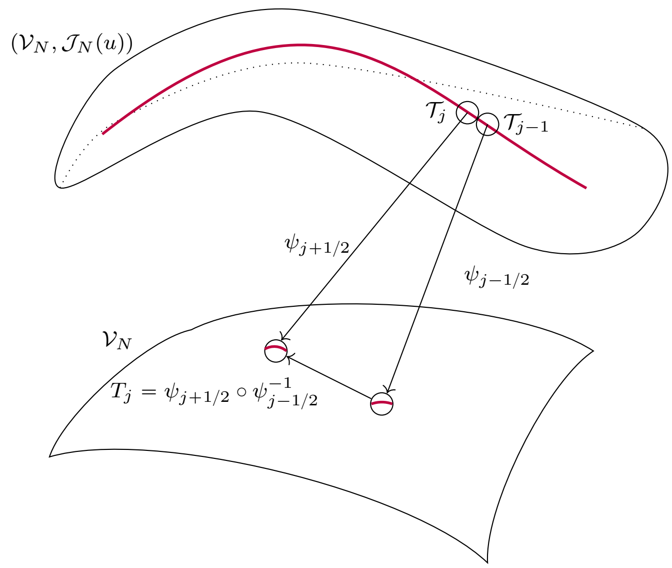

Figure 1: Sketch of Darboux's charts approximation on the Poisson manifold $(\mathcal{V}_N, \mathcal{J}_N(u))$.

Proposition 4.2. *The discrete problem (4.2) is well-posed. Moreover, let Φ^t_h be the discrete flow map associated with the time discretization (4.1) of (2.4) on $\mathcal{T}_h$. Let $\Phi^t_{h,cN}$ be the discrete flow map associated with the discrete system (4.2). Then $\psi^{-1}_{M-1/2} \circ \Phi^T_{h,cN} \circ \psi_{1/2} = \Phi^T_h$.*

Proof. It is enough to show that, for every $j \in \Upsilon_h$, the following diagram

$$\begin{array}{ccc} u^j \in \mathcal{V}_{N,j} & \xrightarrow{\Phi_h} & u^{j+1} \in \mathcal{V}_{N,j} \\ {\scriptstyle \psi_{j+1/2}} \downarrow & & \uparrow {\scriptstyle \psi^{-1}_{j+1/2}} \\ z^j \in \mathcal{V}_{N,j} & \xrightarrow{\Phi_{h,cN}} & z^{j+1} \in \mathcal{V}_{N,j} \end{array}$$

commutes. Here Φ_h and $\Phi_{cN,h}$ denote one step of the temporal integrators Φ^t_h and $\Phi^t_{cN,h}$, respectively. Without loss of generality we can show the commutativity property on the first time interval $\mathcal{T}_1$. By construction, if u^1 is a numerical solution of (4.1) in $\mathcal{T}_1$, i.e. $u^1 = \Phi_h(u_0)$, then $\psi_{1/2}u^1$ is a numerical solution of (4.2), guaranteeing the existence of solutions to (4.2). Conversely, let z^1 be a numerical solution of (4.2) in $\mathcal{T}_1$, i.e. $z^1 = \Phi_{h,cN}(z^0) = \Phi_{h,cN}(\psi_{1/2}u^0)$. From (4.2) we have

$$\begin{aligned} \psi^{-1}_{1/2} z^1 &= \psi^{-1}_{1/2} T_0 z^0 + \Delta t\, \psi^{-1}_{1/2} \mathcal{J}^c_N \nabla_z \mathcal{H}_N(\psi^{-1}_{1/2} \widehat{z}^0) \\ &= \psi^{-1}_{1/2} z^0 + \Delta t\, \psi^{-1}_{1/2} \mathcal{J}^c_N \psi^{-\top}_{1/2} \nabla_{\psi^{-1}_{1/2} z} \mathcal{H}_N(\psi^{-1}_{1/2} \widehat{z}^0) \\ &= u^0 + \Delta t\, \mathcal{J}_N(\widehat{u}) \nabla_u \mathcal{H}_N(\widehat{u}^0), \end{aligned}$$

provided $\widehat{z}^0 = \psi_{1/2}\widehat{u}^0$. This implies that $\psi^{-1}_{1/2} z^1 = u^1 := \Phi_h(u_0)$, ensuring that the solution of (4.2) in $\mathcal{T}_1$ is unique. Furthermore, applying a similar reasoning to any given interval $\mathcal{T}_j$, $j \in \Upsilon_h$, we have $\Phi_h = \psi^{-1}_{j+1/2} \circ \Phi_{h,cN} \circ \psi_{j+1/2}$. Hence,

$$\Phi^T_h = \psi^{-1}_{M-1/2} \circ \Phi_{h,cN} \circ T_{M-1} \circ \Phi_{h,cN} \circ T_{M-2} \circ \ldots \circ T_1 \circ \Phi_{h,cN} \circ \psi_{1/2}.$$

□

This result implies that: $\mathcal{I} \in C^\infty(\mathcal{V}_N)$ is an invariant of the motion of $\Phi^t_{X_{\mathcal{H}_N}}$ if and only if $\mathcal{I}^j := \psi^*_{j+1/2}\mathcal{I} \in C^\infty(\mathcal{V}_N)$ is a (local) invariant of $\Phi^t_{X_{\mathcal{H}^j_N}}$ for all $j \in \Upsilon_h$.

Corollary 4.3. *The Hamiltonian functional* $\mathcal{H}_N$ *of* (4.1) *is preserved if and only if* (4.2) *is locally Hamiltonian-preserving i.e.* $\mathcal{H}_N^j(z^{j+1}) = \mathcal{H}_N^j(T_j z^j)$ *for every* $j \in \Upsilon_h$*. This holds true for any invariant of motion of* $\Phi^t_{X_{\mathcal{H}_N}}$.

Proof. Let $\{z^j\}_{j\in\overline{\Upsilon}_h}$ be numerical solutions of (4.2) in each interval $\mathcal{T}_j$. In view of Proposition 4.2, it holds $\mathcal{H}_N^j(z^{j+1}) = \mathcal{H}_N(\psi^{-1}_{j+1/2} z^{j+1}) = \mathcal{H}_N(u^{j+1})$. If (4.2) is locally Hamiltonian-preserving, using the definition of transition maps and the local conservation properties, we recover

$$\begin{aligned}\mathcal{H}_N^j(z^{j+1}) = \mathcal{H}_N^j(T_j z^j) = \mathcal{H}_N(\psi^{-1}_{j+1/2} T_j z^j) = \mathcal{H}_N(\psi^{-1}_{j+1/2}\psi_{j+1/2}\psi^{-1}_{j-1/2} z^j) = \mathcal{H}_N^{j-1}(z^j)\\ = \mathcal{H}_N^{j-1}(T_{j-1} z^{j-1}) = \ldots = \mathcal{H}_N^0(z^1) = \mathcal{H}_N^0(T_0 z^0) = \mathcal{H}_N(u_0).\end{aligned}$$

Conversely, if (4.1) is (globally) Hamiltonian preserving, then $\mathcal{H}_N(u_0) = \ldots = \mathcal{H}_N(u^j) = \mathcal{H}_N(u^{j+1})$ for all $j \in \Upsilon_h$. The conclusion follows from $\mathcal{H}_N(u^j) = \mathcal{H}_N(\psi^{-1}_{j-1/2} z^j) = \mathcal{H}_N(\psi^{-1}_{j+1/2} T_j z^j)$. □

The global evolution equation for z is not $\mathcal{J}_N^c$-Poisson, due to the transition between neighboring intervals, notwithstanding that (4.2) is canonically $\mathcal{J}_N^c$-Poisson on each time interval $\mathcal{T}_j$. Furthermore, the initial condition $T_j z^j$ on each $\mathcal{T}_j$ does not in general belong to the canonical Poisson manifold $(\mathcal{V}_{N,j}, \mathcal{J}_N^c)$. Likewise the solution z^{j+1} of (4.2) on $\mathcal{T}_j$ in general does not belong to $(\mathcal{V}_{N,j}, \mathcal{J}_N^c)$, i.e. the splitting of the dynamics is clearly not exact. One might consider a "correction" of the initial condition $T_j z^j$ to reduce the distance between $(\mathcal{V}_{N,j}, \mathcal{J}_N^c)$ and the space where the local dynamics is taking place. However, this might introduce an error in the approximation of the solution of the original problem (4.1) and, more importantly, a loss in the preservation of the original Poisson structure $\mathcal{J}_N(u)$, in view of Proposition 4.2 and Corollary 4.3. We therefore discard this option. This consideration is supported by the observation that the global evolution of z cannot "drift away" from the canonical Poisson manifold $(\mathcal{V}_{N,j}, \mathcal{J}_N^c)$ provided each $\psi_{j+1/2}$ is a sufficiently accurate approximation of the Darboux map Ψ on the whole interval $\mathcal{T}_j$. Indeed, in view of Proposition 4.2, the distance (induced by the $\|\cdot\|$-norm) of the solution of (4.2) in $\mathcal{T}_j$ from the canonical Poisson manifold $(\mathcal{V}_N, \mathcal{J}_N^c)$ is bounded by $\|\psi_{j+1/2} u^{j+1} - \Psi(u^{j+1})\|$. This error is local, independent of the dynamics and of the space-time discretization, and only depends on the approximation properties of each $\psi_{j+1/2}$, which are clearly problem-dependent, but controllable.

4.2 Reduced Basis Methods for State-Dependent Structures

To develop reduced basis methods for the discrete dynamical system (4.1) we can now apply a *local* reduction approach, similar to that in Section 3.2. We have built an approximate cover of the high-dimensional Poisson manifold $(\mathcal{V}_N, \mathcal{J}_N(u)) \approx \bigcup_{j\in\Upsilon_h}(\mathcal{V}_{N,j}, \mathcal{J}_N(\widehat{u}^j))$ and generated the local splittings $\psi_{j+1/2} : \mathcal{V}_{N,j} \to \mathcal{V}_{2R} \times \mathcal{N}_j$ via the piecewise linear approximations of the Darboux map.

As a lower dimensional space we construct an n-dimensional Poisson manifold, $n \ll N$, endowed with the canonical $\mathcal{J}_n^c$-Poisson bracket such that $n - \operatorname{rank}(\mathcal{J}_n^c) = q$ and the dimension of the null space of $\mathcal{J}_N(u)$ is conserved in the model order reduction. Analogously to the splitting approach described in Section 3.1, this is achieved through a *global* linear surjective map π_+ such that, for every $j \in \Upsilon_h$,

$$\pi_+ : \mathcal{V}_{2R} \times \mathcal{N}_j \longrightarrow \mathcal{V}_{2r} \times \mathcal{N}_j, \qquad \pi_+ = \pi_+^s \times \mathrm{Id},$$

where π_+^s is taken to be an ℓ^2-orthogonal symplectic application, i.e. $\pi_+^s \mathcal{J}_{2R}^c (\pi_+^s)^\top = \mathcal{J}_{2r}^c$.

The map $\pi_+ : (\mathcal{V}_N, \{\cdot,\cdot\}_{cN}) \to (\mathcal{V}_n, \{\cdot,\cdot\}_{cn})$ is Poisson since $\pi_+ \mathcal{J}_N^c \pi_+^\top = \mathcal{J}_n^c$. However, since the set of solution snapshots does not possess a global Poisson structure, the low-dimensional space $\mathcal{V}_n$ is recovered as a linear subspace of $\mathcal{V}_N$, with the latter considered as a normed vector space, being the structure $\{\cdot,\cdot\}_{cN}$ constant-valued.

Lemma 4.4. *The map* $\mathcal{P} = \pi \circ \pi_+ : \mathcal{V}_N \to \mathrm{Im}(\pi) \subset \mathcal{V}_N$ *is* ℓ^2*-orthogonal and it is a projection.*

Proof. A straightforward application of the properties of π and of its pseudoinverse π_+, yields the result. Using $\pi_+ \circ \pi = \mathrm{Id}$ and the surjectivity of π_+ results in $\mathcal{P} \circ \mathcal{P} = \mathcal{P}$.

The ℓ^2-orthogonality of $\mathcal{P}$ simply follows from the fact that, by construction, the pseudoinverse π_+^s and the adjoint of π^s coincide. □

The orthogonality of $\mathcal{P}$ guarantees the inclusion of $\mathcal{V}_n = \operatorname{Im}(\pi)$ in $\mathcal{V}_N$, and hence the approximation properties of the reduced solution, while the symplecticity of π_+^s ensures that the nontrivial phase flow is a symplectomorphism and that the local kernel $\{\mathcal{N}_j\}_j$ is preserved.

The reduced problem is derived from the canonical Poisson dynamical systems (4.2) by a local Poisson projection onto $\operatorname{Im}(\pi)\cap\mathcal{V}_{N,j}\subset\mathcal{V}_N$. On the n-dimensional Poisson manifold $\mathcal{V}_n$, the fully discrete problem reads: For $u_0\in\mathcal{V}_N$, find $\{y^{j+1}\}_{j\in\Upsilon_h}\subset\mathcal{V}_n$ such that

$$\begin{cases} y^{j+1} = \tau_j y^j + \Delta t\,\mathcal{J}_n^c\nabla_y\mathcal{H}_n^j(\widehat{y}^j), & \text{for } j\in\Upsilon_h,\\ y^0 = \pi_+\psi_{1/2}u_0, \end{cases} \tag{4.3}$$

where $\mathcal{H}_n^j(y) := \mathcal{H}_N(\psi_{j+1/2}^{-1}\pi y)$ for all $y\in\mathcal{V}_n$, and the reduced transition maps τ_j are defined as $\tau_j := \pi_+\circ T_j\circ\pi$ for all $j\in\Upsilon_h\backslash\{0\}$, and $\tau_0 := \operatorname{Id}$. A sufficient condition for the well-posedness of (4.3) is that $\nabla\mathcal{H}_N$ is Lipschitz continuous, where $\mathcal{H}_N$ is the Hamiltonian of the high-fidelity problem (4.1).

Problem (4.3) can be seen as the temporal discretization of an evolution equation which is canonically $\mathcal{J}_n^c$-Poisson on each time interval $\mathcal{T}_j$. Indeed $y^{j+1}\in\mathcal{V}_n$ is the numerical approximation of the solution of

$$\begin{cases} d_t y = \mathcal{J}_n^c\nabla_y\mathcal{H}_n^j(y), & \text{for } t\in\mathcal{T}_j,\\ y^0 = \tau_j y(t^j), \end{cases} \tag{4.4}$$

where $y(t^j)\in\mathcal{V}_n$ is the numerical solution at time t^j. The reduced phase flow is no longer globally Poisson: the Hamiltonian $\mathcal{H}_N$ of the high-fidelity problem (2.4) is conserved up to approximation error of the local Darboux map.

Proposition 4.5. *Let u_0 be the initial condition of the dynamical system (2.4). For $j\in\Upsilon_h$ fixed, let $u_{\mathrm{rb}}(t^{j+1})$ be the time-continuous solution of the reduced problem given by $u_{\mathrm{rb}}(t^{j+1}) = \psi_{j+1/2}^{-1}\pi y(t^{j+1})$ where $y(t)$ is the solution of (4.4) at time $t\in\mathcal{T}_h$. If the Hamiltonian $\mathcal{H}_N$ of (2.4) is Lipschitz continuous with constant $L_{\mathcal{H}}$ then*

$$\left|\mathcal{H}_N(u_{\mathrm{rb}}(t^{j+1})) - \mathcal{H}_N(u_0)\right| \leqslant L_{\mathcal{H}}\sum_{k=1}^{j}\|\psi_{k+1/2}^{-1}\|\|\psi_{k-1/2}\|\|T_k - Id\,\|\|u_{\mathrm{rb}}(t^k)\|.$$

Proof. Let $y(t^k)\in\mathcal{V}_n$ be the solution of the reduced system (4.4) at time $t^k\in\mathcal{T}_h$. Since the system is locally $\mathcal{J}_N^c$-Poisson, the Hamiltonian is an invariant of the local motion, namely $\mathcal{H}_n^k(y(t^{k+1})) = \mathcal{H}_N(\psi_{k+1/2}^{-1}\pi y(t^{k+1})) = \mathcal{H}_n^k(\tau_k y(t^k))$. However, the global Hamiltonian $\mathcal{H}_N$ is not preserved at the interface between intervals. Indeed,

$$\begin{aligned}\left|\mathcal{H}_n^k(\tau_k y(t^k)) - \mathcal{H}_n^{k-1}(y(t^k))\right| &= \left|\mathcal{H}_N(\psi_{k+1/2}^{-1}\pi\tau_k y(t^k)) - \mathcal{H}_N(\psi_{k-1/2}^{-1}\pi y(t^k))\right|\\ &= \left|\mathcal{H}_N(\psi_{k+1/2}^{-1}\mathcal{P}T_k\pi y(t^k)) - \mathcal{H}_N(\psi_{k-1/2}^{-1}\pi y(t^k))\right|.\end{aligned}$$

If $T_k\pi y(t^k)\in\operatorname{Im}(\pi)$ for all $k\in\Upsilon_h$, then $\psi_{k+1/2}^{-1}\mathcal{P}T_k\pi y(t^k) = \psi_{k-1/2}^{-1}T_k^{-1}\mathcal{P}T_k\pi y(t^k) = \psi_{k-1/2}^{-1}\pi y(t^k)$, and, hence, the Hamiltonian would be conserved.

Under the assumption that the Hamiltonian $\mathcal{H}_N$ is Lipschitz continuous with constant $L_{\mathcal{H}}$ it holds,

$$\begin{aligned}\left|\mathcal{H}_n^k(\tau_k y(t^k)) - \mathcal{H}_n^{k-1}(y(t^k))\right| &\leqslant L_{\mathcal{H}}\|\psi_{k+1/2}^{-1}\mathcal{P}T_k\pi y(t^k) - \psi_{k-1/2}^{-1}\pi y(t^k)\|\\ &= L_{\mathcal{H}}\|\psi_{k+1/2}^{-1}\mathcal{P}(T_k - \operatorname{Id})\pi y(t^k) - \psi_{k+1/2}^{-1}(T_k - \operatorname{Id})\mathcal{P}\pi y(t^k)\|\\ &\leqslant L_{\mathcal{H}}\|\psi_{k+1/2}^{-1}\|\|\psi_{k-1/2}\|\|T_k - \operatorname{Id}\|\|u_{\mathrm{rb}}(t^k)\|.\end{aligned}$$

Hence, the error in the conservation of the Hamiltonian at time t^j can be bounded as

$$\begin{aligned}\left|\mathcal{H}_N(u_{\mathrm{rb}}(t^j)) - \mathcal{H}_N(u_0)\right| = \left|\mathcal{H}_n^{j-1}(y(t^j)) - \mathcal{H}_N(u_0)\right| &\leqslant \sum_{k=1}^{j-1} \left|\mathcal{H}_n^k(y(t^{k+1})) - \mathcal{H}_n^{k-1}(y(t^k))\right| \\ &\leqslant \sum_{k=1}^{j-1} \left|\mathcal{H}_n^k(\tau_k y(t^k)) - \mathcal{H}_n^{k-1}(y(t^k))\right| \\ &\leqslant L_{\mathcal{H}} \sum_{k=1}^{j-1} \|\psi_{k+1/2}^{-1}\| \|\psi_{k-1/2}\| \|T_k - \mathrm{Id}\,\| \|u_{\mathrm{rb}}(t^k)\|.\end{aligned}$$

□

Since, by construction, the map π_+ acts as the identity on $\mathcal{N}_j$ for all $j \in \Upsilon_h$, the approximation of the center of the Lie algebra $C^\infty(\mathcal{V}_N)$, given by $\mathrm{span}\{(\psi u)_m,\, m = 1, \dots, q\}$, is not affected by the reduction. This means that the error made in the conservation of the Casimir invariants of the bracket $\{\cdot,\cdot\}_N$ is only attributable to the approximation of the Darboux charts.

Concerning the stability properties of the problem, since the Poisson system (2.4) and its canonical form obtained through Darboux's map are in one-to-one correspondence, u_e is Lyapunov stable equilibrium of (2.4) if and only if $\Psi(u_e)$ is Lyapunov stable equilibrium of the corresponding canonical system. When resorting to the piecewise linear approximation of Ψ, as introduced in Definition 4.1, Lyapunov stable equilibria are preserved by the discrete problem since, by Proposition 4.2, $\psi_{j-1/2}^{-1} z^j = u^j$ for all $j \in \Upsilon_h$ with u^j numerical solution of (4.1) and z^j numerical solution of (4.2) in $\mathcal{T}_{j-1}$. Note that the property of preserving the Lyapunov equilibria at the discrete level depends on the temporal solver, see e.g. [28] and references therein.

Furthermore, if $\Psi^*\mathcal{H}_N$ is Lyapunov function, a reasoning analogous to the one of Section 3.2.1 allows to show that the global reduced system associated with the exact Darboux map preserves the Lyapunov stable equilibria belonging to $\mathrm{Im}(\pi)$. However, $y_e := \pi_+ \psi u_e \approx \pi_+ \Psi(u_e)$ is generally not an equilibrium of (4.3). Ideally one would want to have that $\|\psi_{j-1/2}^{-1} \pi y^j - u_e\|$ is uniformly bounded for all $j \in \Upsilon_h$, where y^j is the numerical solution of (4.3) in $\mathcal{T}_{j-1}$. It holds

$$\begin{aligned}\|\psi_{j-1/2}^{-1}\pi y^j - u_e\| = \|\psi_{j-1/2}^{-1}\pi y^j - \Psi^{-1}(\pi y_e)\| &\leqslant \|\psi_{j-1/2}^{-1}(\pi y^j - \pi y_e)\| + \|\psi_{j-1/2}^{-1}\pi y_e - \Psi^{-1}(\pi y_e)\| \\ &\leqslant \|\psi_{j-1/2}^{-1}\| \|y^j - y_e\| + \|\psi_{j-1/2}^{-1}\pi y_e - \Psi^{-1}(\pi y_e)\|.\end{aligned}$$

The second term is the approximation error of the Darboux map, while the term $\|y^j - y_e\|$ can be bounded by the approximation error associated with solving as reduced problem (4.3) instead of the reduced system obtained from the exact Darboux map Ψ. Although the reduced solution is not guaranteed to belong to an arbitrary small neighborhood of u_e, the term $\|y^j - y_e\|$ does not depend on the reduction but only on the approximation properties of the Darboux map.

4.2.1 Convergence of the Weak Symplectic Greedy Algorithm

For the derivation of the reduced basis, we rely on the weak greedy algorithm, described in Section 3.2.2, with the following modifications. A set of snapshots $\mathcal{U}_N = \{u^j = \Phi_{h,N}^{t^j}(u^0),\, j \in \overline{\Upsilon}_h\}$ is computed from the high-fidelity problem (4.1) (with discrete flow map $\Phi_{h,N}^t$) together with the linear approximation maps $\{\psi_{j+1/2}\}_{j\in\Upsilon_h}$. The image of each snapshot under the corresponding $\psi_{j+1/2}$ supplies the solution of the local system (4.2) in every time interval. By extracting the symplectic part and excluding the contribution of the Casimir invariants, we define

$$\mathcal{Z}_N^s := \{\psi_{j+1/2}^s u^{j+1},\, j \in \Upsilon_h\} \cup \{\psi_{1/2}^s u_0\}. \tag{4.5}$$

We finally build an orthogonal $\mathcal{J}_{2R}^c$-symplectic reduced basis from $\mathcal{Z}_N^s$ via Algorithm 1 and the Poisson projection $\mathcal{P} := \pi \circ \pi_+$ from π^s and π_+^s.

Theorem 4.6 (Convergence of the Weak Symplectic Greedy Algorithm)**.** *Let $\Phi^t_{h,cN}$ be the discrete flow map associated with (4.2). Assume that the solution set $\mathcal{Z}_N := \{z^j = \Phi^{t^j}_{h,cN}(z^0),\, j \in \overline{\Upsilon}_h\}$ has Kolmogorov n-width $d_n(\mathcal{Z}_N)$. Then, the reduced space $\mathcal{V}_n = \mathcal{V}_{2r} \times \mathcal{N}$, with $\mathcal{V}_{2r}$ derived via the symplectic weak greedy Algorithm 1, satisfies*

$$\|z - \mathcal{P}z\| \leqslant C3^{r+1}(r+1)d_n(\mathcal{Z}_N), \qquad \forall\, z \in \mathcal{Z}_N,$$

where the finite constant $C > 0$ is independent of n, r and N.

Proof. Let $\mathcal{Z}^s_N$ be the set (4.5) containing the symplectic part of the solution trajectory at time instants $\{t^j\}_{j\in\overline{\Upsilon}_h}$. The greedy Algorithm 1 iteratively generates a hierarchy of subspaces of $\mathcal{V}_{2R}$ such that the projection $\mathcal{P}_s$ is ℓ^2-orthogonal, see Lemma 4.4. In the context of an orthogonal reduced basis generation via a greedy strategy we can revert to the a priori convergence estimates derived in [11] and [8]. The argument proposed here is a straightforward modification of the proof presented in [11, Section 2] by taking into account the form of the orthosymplectic reduced basis (Definition 3.10), and it is therefore relegated to Appendix A. ☐

Remark 4.7. We are making the tacit assumption that the Kolmogorov n-width of the solution set $\mathcal{Z}_N$ has a sufficiently fast decay. Unlike the constant case, see Section 3.2.2 and Lemma 3.12, the Kolmogorov width of the solution set $\mathcal{Z}_N$ associated with the system (4.2) *cannot* easily be bounded by the Kolmogorov width of the solution set of the original system (4.1). That would require stronger conditions on the global nonlinear Darboux map Ψ, see e.g. [20], which are generally not guaranteed by Darboux's Theorem 2.8.

4.3 A Priori Convergence Estimates for the Reduced Solution

For state-dependent Poisson structures we perform model order reduction in a local perspective. We therefore derive a priori estimates for the error between the high-fidelity solution and the reduced solution for the fully discrete system in each temporal interval. The total error at a given time is controlled by the projection error at all the previous time steps.

Note that the error of the reduced solution is computed with respect to the solution of the fully discrete high-fidelity system and not with respect to the exact solution of (2.4). Hence the estimate (4.6) does not include the approximation error ensuing from the temporal discretization.

Theorem 4.8. *Let $j \in \Upsilon_h$ be fixed. Let u^{j+1} be the numerical solution of (4.1) at time t^{j+1} and let u^{j+1}_{rb} be the numerical solution of the reduced problem, obtained as $u^{j+1}_{\text{rb}} = \psi^{-1}_{j+1/2}\pi y^{j+1}$, where y^{j+1} is the solution of (4.3) at time t^{j+1}. Assume that $\nabla\mathcal{H}_N$ is Lipschitz continuous in the $\|\cdot\|$-norm with constant $L_{\delta\mathcal{H}}$. If the numerical discretization of the Hamiltonian in (4.1) is (semi-)implicit, and the time step Δt satisfies*

$$\Delta t\, L_{\delta\mathcal{H}} C_1 \|\psi^{-1}_{j+1/2}\|^2 < 1, \qquad \textit{for all } j \in \Upsilon_h,$$

where the finite constant $C_1 > 0$ depends only on the discretization of the Hamiltonian, then,

$$\|u^{j+1} - u^{j+1}_{\text{rb}}\| \leqslant \frac{\|\psi^{-1}_{j+1/2}\|}{1 - \Delta t\, L_{\delta\mathcal{H}} C_1 \|\psi^{-1}_{j+1/2}\|^2} \left(\|z^{j+1} - \mathcal{P}z^{j+1}\| + \sum_{k=1}^{j} \gamma_k \beta_k \|z^k - \mathcal{P}z^k\| \right). \tag{4.6}$$

Here $z^k = \psi_{k-1/2}u^k$, $\beta_k := \|T_k - Id\| + \Delta t\, L_{\delta\mathcal{H}} C_2 \|\psi^{-1}_{k+1/2}\|^2$, the constant $C_2 > 0$ depends only on the discretization of the Hamiltonian, and

$$\gamma_k := \begin{cases} 1 + \sum\limits_{m=k+1}^{j} \beta_m, & \textit{if } k \leqslant j-1, \\ 1, & \textit{if } k = j. \end{cases}$$

Proof. Let us split the error at time t^j, for $j \in \overline{\Upsilon}_h$, as

$$e^j := z^j - \pi y^j = (z^j - \mathcal{P}z^j) + (\mathcal{P}z^j - \pi y^j) =: e^j_p + e^j_h.$$

Subtracting problem (4.2) and the reduced problem (4.3), the approximation error e_h^{j+1} at time t^{j+1} can be written as,

$$\begin{aligned} e_h^{j+1} &= \mathcal{P} z^{j+1} - \pi y^{j+1} \\ &= e_h^j + \mathcal{P}(T_j - \mathrm{Id})z^j + \pi y^j - \mathcal{P} T_j \pi y^j + \Delta t\, \mathcal{P} \mathcal{J}_N^c (\nabla_z \mathcal{H}_N^j(\widehat{z}^j) - \nabla_{\pi y} \mathcal{H}_N^j(\pi \widehat{y}^j)) \\ &= (\mathrm{Id} + \mathcal{P}(T_j - \mathrm{Id}))e_h^j + \mathcal{P}(T_j - \mathrm{Id})e_p^j + \Delta t\, \mathcal{P} \mathcal{J}_N^c (\nabla_z \mathcal{H}_N^j(\widehat{z}^j) - \nabla_{\pi y} \mathcal{H}_N^j(\pi \widehat{y}^j)). \end{aligned}$$

The total error at time t^{j+1} is bounded as,

$$\begin{aligned} \|e^{j+1}\| &\leqslant \|e_p^{j+1}\| + \|e_h^{j+1}\| \\ &\leqslant \|\mathrm{Id} + \mathcal{P}(T_j - \mathrm{Id})\| \|e_h^j\| + \|\mathcal{P}(T_j - \mathrm{Id})\| \|e_p^j\| + \Delta t\, \|R_j\| + \|e_p^{j+1}\| \\ &\leqslant (1 + \|T_j - \mathrm{Id}\|)\|e_h^j\| + \|T_j - \mathrm{Id}\| \|e_p^j\| + \Delta t\, \|R_j\| + \|e_p^{j+1}\|, \end{aligned} \tag{4.7}$$

where $R_j := \mathcal{P} \mathcal{J}_N^c (\nabla_z \mathcal{H}_N^j(\widehat{z}^j) - \nabla_{\pi y} \mathcal{H}_N^j(\pi \widehat{y}^j))$. Since $\nabla \mathcal{H}_N$ is Lipschitz continuous by assumption, using the definition of the local Hamiltonian $\mathcal{H}_N^j := (\psi_{j+1/2}^{-1})^* \mathcal{H}_N$, the term R_j satisfies

$$\begin{aligned} \|R_j\| \leqslant \|\nabla_z \mathcal{H}_N^j(\widehat{z}^j) - \nabla_{\pi y} \mathcal{H}_N^j(\pi \widehat{y}^j)\| &\leqslant L_{\delta \mathcal{H}} \|\psi_{j+1/2}^{-1}\|^2 \|\widehat{z}^j - \pi \widehat{y}^j\| \\ &\leqslant L_{\delta \mathcal{H}} C_1 \|\psi_{j+1/2}^{-1}\|^2 \|e^{j+1}\| + L_{\delta \mathcal{H}} C_2 \|\psi_{j+1/2}^{-1}\|^2 \|e^j\|, \end{aligned} \tag{4.8}$$

where the finite non-negative constants C_1 and C_2 depend only on the temporal discretization of (4.1) (e.g. for the implicit Euler scheme $C_1 = 1$ and $C_2 = 0$, for the implicit midpoint rule $C_1 = C_2 = 1/2$, etc.). Hence, the total error at time t^{j+1} satisfies

$$\begin{aligned} \|e^{j+1}\| \leqslant &\Delta t\, L_{\delta \mathcal{H}} C_1 \|\psi_{j+1/2}^{-1}\|^2 \|e^{j+1}\| + (1 + \alpha_j + \Delta t\, L_{\delta \mathcal{H}} C_2 \|\psi_{j+1/2}^{-1}\|^2)\|e_h^j\| \\ &+ (\alpha_j + \Delta t\, L_{\delta \mathcal{H}} C_2 \|\psi_{j+1/2}^{-1}\|^2)\|e_p^j\| + \|e_p^{j+1}\|. \end{aligned}$$

where $\alpha_j := \|T_j - \mathrm{Id}\|$. Under the condition that the time step Δt satisfies $\Delta t\, L_{\delta \mathcal{H}} C_1 \|\psi_{j+1/2}^{-1}\|^2 < 1$ for all $j \in \Upsilon_h$, the total error at time t^{j+1} is controlled by the projection error e_p^k at all previous time steps $k \leqslant j + 1$; thereby,

$$\|e^{j+1}\| \leqslant \frac{1}{1 - \Delta t\, L_{\delta \mathcal{H}} C_1 \|\psi_{j+1/2}^{-1}\|^2} \left(\sum_{k=1}^{j} \beta_k \|e_p^k\| + \|e_p^{j+1}\| + \sum_{k=1}^{j-1} \left(\sum_{m=k+1}^{j} \beta_m \right) \beta_k \|e_p^k\| \right),$$

where $\beta_m := \alpha_m + \Delta t\, L_{\delta \mathcal{H}} C_2 \|\psi_{m+1/2}^{-1}\|^2$.

The conclusion follows from the fact that $u^j - u_{\mathrm{rb}}^j = \psi_{j-1/2}^{-1} e^j$, owing to Proposition 4.2. □

4.3.1 Discrete Empirical Interpolation for State-Dependent Structures

For state-dependent Poisson structures we can apply a discrete empirical interpolation strategy to the nonlinear terms of the fully discrete dynamical system (4.3). The reasoning proceeds similarly to that of Section 3.3 but is carried out locally, on each time interval. Observe that, except for the evaluation of the nonlinear terms, the local reduced basis technique, described in the previous sections, does not incur a computational cost proportional to N during the online phase since the evaluation of the reduced transition maps $\{\tau_j\}_j$ can be performed offline.

Let us assume that the evolution problem (2.4) can be written by separating its linear and nonlinear parts, i.e. we write $\nabla_u \mathcal{H}_N(\widehat{u}^j) = \mathcal{L}_N \widehat{u}^j + \mathcal{M}_N(\widehat{u}^j)$, where $\mathcal{L}_N$ denotes a linear operator and $\mathcal{M}_N$ a nonlinear one. Analogously, the gradient of the Hamiltonian in the local discrete canonical formulation (4.2) can be expressed as

$$\nabla_z \mathcal{H}_N^j(\widehat{z}^j) = \psi_{j+1/2}^{-\top} \mathcal{L}_N \psi_{j+1/2}^{-1} \widehat{z}^j + \psi_{j+1/2}^{-\top} \mathcal{M}_N(\psi_{j+1/2}^{-1} \widehat{z}^j) =: \mathcal{L}_N^j \widehat{z}^j + \mathcal{M}_N^j(\widehat{z}^j).$$

The reduced problem (4.3) then becomes

$$y^{j+1} = \tau_j y^j + \Delta t\, \mathcal{J}_n^c \pi^\top (\mathcal{L}_N^j \pi \widehat{y}^j + \mathcal{M}_N^j(\pi \widehat{y}^j)) = \tau_j y^j + \Delta t\, \mathcal{J}_n^c (\mathcal{L}_n^j \widehat{y}^j + \pi^\top \mathcal{M}_N^j(\pi \widehat{y}^j)),$$

where $\mathcal{L}_n^j \widehat{y}^j := \pi^\top \psi_{j+1/2}^{-\top} \mathcal{L}_N \psi_{j+1/2}^{-1} \pi \widehat{y}^j$. Adopting a symplectic DEIM strategy we approximate the nonlinear term as $\mathcal{M}_N^j(\pi\widehat{y}^j) \approx \pi_+^\top (P^\top \pi_+^\top)^{-1} P^\top \mathcal{M}^j(\pi\widehat{y}^j)$ so that, for $\mathcal{M}_n^j(\widehat{y}^j) := P^\top \mathcal{M}_N^j(\pi\widehat{y}^j)$, the discrete reduced problem reads

$$\begin{cases} y^{j+1} = \tau_j y^j + \Delta t\, \mathcal{J}_n^c(\mathcal{L}_n^j \widehat{y}^j + (P^\top \pi_+^\top)^{-1} \mathcal{M}_n^j(\widehat{y}^j)), & \text{for } j \in \Upsilon_h, \\ y^0 = \pi_+ \psi_{1/2} u_0. \end{cases} \tag{4.9}$$

Similarly to Theorem 4.8, the approximation error can be bounded by the projection error of the previous time intervals and terms involving the nonlinear part, can be controlled during the construction of the reduced basis from the nonlinear snapshots $\{\mathcal{M}_N(\widehat{u}^j)\}_{j\in\Upsilon_h}$. Observations similar to the remarks made at the end of Section 3.3.1 apply.

Theorem 4.9. *Let $j \in \Upsilon_h$ be fixed. Let u^{j+1} be the numerical solution of (4.1) at time t^{j+1} and let u_{rb}^{j+1} be the numerical solution of the reduced problem, obtained as $u_{\text{rb}}^{j+1} = \psi_{j+1/2}^{-1} \pi y^{j+1}$ where y^{j+1} is the solution of (4.9) at time t^{j+1}. Assume that the nonlinear operator $\mathcal{M}_N$ is Lipschitz continuous with constant $L_{\mathcal{M}}$. If the numerical temporal discretization of (4.1) is (semi-)implicit, and the time step Δt satisfies the condition*

$$\Delta t\, C_1(\|\mathcal{L}_N^j\| + \|W\| \|\psi_{j+1/2}^{-1}\|^2 L_{\mathcal{M}}) < 1, \qquad \text{for all } j \in \Upsilon_h,$$

where $W := \pi_+^\top (P^\top \pi_+^\top)^{-1} P^\top$, and $C_1 > 0$ depends only on the discretization of the Hamiltonian, then, the approximation error satisfies

$$\|u^{j+1} - u_{\text{rb}}^{j+1}\| \leqslant C_j \left(\|z^{j+1} - \mathcal{P} z^{j+1}\| + \sum_{k=1}^{j} \gamma_k \beta_k \|z^k - \mathcal{P} z^k\| + \|W\| \|\mathcal{M}_n^j(\widehat{z}^j) - \pi_+^\top \pi_+ \mathcal{M}_n^j(\widehat{z}^j)\| \right).$$

Here $z^k = \psi_{k-1/2} u^k$, $\widehat{z}^j$ is determined by the temporal discretization (4.2), and

$$C_j := \frac{\|\psi_{j+1/2}^{-1}\|}{1 - \Delta t\, C_1(\|\mathcal{L}_N^j\| + \|W\| \|\psi_{j+1/2}^{-1}\|^2 L_{\mathcal{M}})}.$$

The coefficients are defined as $\beta_k := \|T_k - Id\| + \Delta t\, C_2(\|\mathcal{L}_N^k\| + \|W\| \|\psi_{k+1/2}^{-1}\|^2 L_{\mathcal{M}})$, where the constant $C_2 > 0$ depends only on the discretization of the Hamiltonian, and

$$\gamma_k := \begin{cases} 1 + \sum\limits_{m=k+1}^{j} \beta_m, & \text{if } k \leqslant j-1, \\ 1, & \text{if } k = j. \end{cases}$$

Proof. We proceed analogously to the proof of Theorem 4.8. The only difference is in the bound of the term $R_j := \mathcal{P}\mathcal{J}_N^c(\nabla_z \mathcal{H}_N^j(\widehat{z}^j) - \nabla_{\pi y} \mathcal{H}_N^j(\pi\widehat{y}^j))$. Here, taking into account the DEIM approximation of the nonlinear term $\mathcal{M}_N$, it holds

$$\begin{aligned} R_j &= \mathcal{P}\mathcal{J}_N^c(\mathcal{L}_N^j \widehat{z}^j + \mathcal{M}_N^j(\widehat{z}^j)) - \pi \mathcal{J}_n^c(\mathcal{L}_n^j \widehat{y}^j + (P^\top \pi_+^\top)^{-1} \mathcal{M}_n^j(\widehat{y}^j)) \\ &= \mathcal{P}\mathcal{J}_N^c(\mathcal{L}_N^j \widehat{z}^j + \mathcal{M}_N^j(\widehat{z}^j)) - \pi \mathcal{J}_n^c(\pi^\top \mathcal{L}_N^j \pi \widehat{y}^j + \pi^\top W \mathcal{M}_N^j(\pi\widehat{y}^j)) \\ &= \mathcal{P}\mathcal{J}_N^c \mathcal{L}_N^j(\widehat{z}^j - \pi\widehat{y}^j) + \mathcal{P}\mathcal{J}_N^c(\mathcal{M}_N^j(\widehat{z}^j) - W \mathcal{M}_N^j(\pi\widehat{y}^j)), \end{aligned}$$

since $\pi \mathcal{J}_n^c \pi^\top = \mathcal{P}\mathcal{J}_N^c$, and where $W := \pi_+^\top (P^\top \pi_+^\top)^{-1} P^\top$. In this case, using the fact that $(\text{Id} - W)\pi_+^\top \pi_+ \mathcal{M}_N^j(\widehat{z}^j) = 0$, the bound for R_j, analogous to (4.8) reads,

$$\begin{aligned} \|R_j\| &\leqslant \|\mathcal{P}\mathcal{J}_N^c \mathcal{L}_N^j\| \|\widehat{z}^j - \pi\widehat{y}^j\| + \|\mathcal{P}\mathcal{J}_N^c(\text{Id} - W)\mathcal{M}_N^j(\widehat{z}^j)\| + \|\mathcal{P}\mathcal{J}_N^c W(\mathcal{M}_N^j(\widehat{z}^j) - \mathcal{M}_N^j(\pi\widehat{y}^j))\| \\ &\leqslant \|\mathcal{P}\mathcal{J}_N^c \mathcal{L}_N^j\| \|\widehat{z}^j - \pi\widehat{y}^j\| + \|\mathcal{P}\mathcal{J}_N^c(\text{Id} - W)(\mathcal{M}_N^j(\widehat{z}^j) - \pi_+^\top \pi_+ \mathcal{M}_N^j(\widehat{z}^j))\| \\ &\quad + \|\mathcal{P}\mathcal{J}_N^c W(\mathcal{M}_N^j(\widehat{z}^j) - \mathcal{M}_N^j(\pi\widehat{y}^j))\| \\ &\leqslant (\|\mathcal{L}_N^j\| + \|W\| \|\psi_{j+1/2}^{-1}\|^2 L_{\mathcal{M}}) \|\widehat{z}^j - \pi\widehat{y}^j\| + \|\text{Id} - W\| \|\mathcal{M}_N^j(\widehat{z}^j) - \pi_+^\top \pi_+ \mathcal{M}_N^j(\widehat{z}^j)\|, \end{aligned}$$

where $L_{\mathcal{M}}$ is the Lipschitz constant of $\mathcal{M}_N$. Similar to (4.8) we recover

$$\|R_j\| \leqslant C_1 K_j \|e^{j+1}\| + C_2 K_j \|e^j\| + \|W\| \|\mathcal{M}_N^j(\widehat{z}^j) - \pi_+^\top \pi_+ \mathcal{M}_N^j(\widehat{z}^j)\|,$$

where $K_j := \|\mathcal{L}_N^j\| + \|W\| \|\psi_{j+1/2}^{-1}\|^2 L_{\mathcal{M}}$, and the finite non-negative constants C_1 and C_2 depend only on the numerical temporal discretization of (4.1). Hence, proceeding as in (4.7), the total error at time t^{j+1} can be bounded as,

$$\begin{aligned}\|e^{j+1}\| \leqslant \frac{1}{1-\Delta t\, C_1 K_j} \Bigg(\sum_{k=1}^{j} \beta_k \|e_p^k\| + \|e_p^{j+1}\| + \sum_{k=1}^{j-1} \Bigg(\sum_{m=k+1}^{j} \beta_m \Bigg) \beta_k \|e_p^k\| \\ + \|W\| \|\mathcal{M}_N^j(\widehat{z}^j) - \pi_+^\top \pi_+ \mathcal{M}_N^j(\widehat{z}^j)\| \Bigg),\end{aligned}$$

where $\beta_m := \alpha_m + \Delta t\, C_2 K_m = \alpha_m + \Delta t\, C_2 (\|\mathcal{L}_N^m\| + \|W\| \|\psi_{m+1/2}^{-1}\|^2 L_{\mathcal{M}})$.

The conclusion follows from the fact that $u^j - u_{\text{rb}}^j = \psi_{j-1/2}^{-1} e^j$, by Proposition 4.2. □

4.4 Parametric State-Dependent Poisson Structures

If we consider a parametric dynamical system similar to (3.1) but with state-dependent Poisson structure, we can extend the derivation and analysis of the reduced basis method described in the previous Sections. The main obstacle relates to the fact that the resolution of the dynamical system in the low-dimensional space (4.3) requires the knowledge of the Darboux map approximations $\{\psi_{j+1/2}\}_j$. These will inevitably depend on the parameter where the Poisson tensor is evaluated. Only the linear maps $\{\psi_{j+1/2}(\mu)\}_j$ associated with the parameters $\mu \in \Lambda_h \subset \Lambda$ will be computed in the offline phase. Therefore a way to approximate each $\psi_{j+1/2}$ at any given parameter $\mu \in \Lambda$ is indispensable. A way to perform such approximation at a computational cost proportional to the size of the low dimensional problem is to apply e.g. an empirical interpolation technique on $\psi_{j+1/2}(\mu)$ for all $\mu \in \Lambda \backslash \Lambda_h$. However, such approximation will in general affect the preservation of the Poisson structure by introducing a, pointwise in μ, error in the structure splitting of Proposition 4.2.

5 Numerical Experiments

To validate the theoretical results of the previous Sections we perform a set of one-dimensional numerical simulations. For want of Poisson integrators for general structures, we consider ad hoc test cases for which such integrators are available. The rationale is that we seek to assess the performances of the structure-preserving reduced basis method in the absence of pollution coming from the temporal discretization.

In the forthcoming numerical simulations, if not otherwise specified, we will use Newton iteration as a nonlinear solver for implicit temporal discretizations. We fix the Newton tolerance to 10^{-10} and the maximum number of nonlinear iterations to 50. In the symplectic greedy Algorithm 1, we consider a stabilized version of the symplectic Gram–Schmidt and a symplectic Gram–Schmidt with full reorthogonalization [27] to deal with cases where the snapshots matrix might be ill conditioned.

5.1 Numerical Experiments for Constant-Valued Degenerate Structures

As example of constant-valued degenerate Poisson structure we consider the Korteweg–de Vries (KdV) equation. KdV type problems are nonlinear hyperbolic equations describing the propagation of waves in nonlinear dispersive media. The KdV equation in the one-dimensional spatial domain Ω and time interval $\mathcal{T}$ reads: Find $u(t,x) : \mathcal{T} \times \Omega \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} + \mu \frac{\partial^3 u}{\partial x^3} = 0, \qquad \alpha, \mu \in \mathbb{R}. \tag{5.1}$$

The dispersive third order term provides a regularization yielding smooth solutions for smooth initial conditions. The numerical treatment of (5.1) for small values of μ, the so-called dispersion limit, is particularly challenging, and for $\mu = 0$, Burgers' equation is recovered.

The KdV equation is a completely integrable system, i.e. it has as an infinite set of invariants, and possesses a bi-Hamiltonian structure: The formulation with degenerate constant-valued Poisson tensor reads

$$\frac{\partial u}{\partial t} = \mathcal{J}\delta\mathcal{H}(u), \qquad \text{with} \qquad \mathcal{H}(u) = \int_\Omega \left(\frac{\alpha}{6}u^3 - \frac{\mu}{2}\left(\partial_x u\right)^2\right) dx, \qquad \mathcal{J} = \partial_x,$$

and where δ denotes a functional derivative. Let us consider a uniform partition of the interval $\Omega = [a, b]$, $a, b \in \mathbb{R}$ with periodic boundary conditions, into $N-1$ elements, and let $\Delta x = |b-a|/(N-1)$. The Poisson tensor $\mathcal{J}$ is discretized using centered finite differences, whereas the Hamiltonian $\mathcal{H}$ is approximated using the trapezoidal rule and forward finite differences for the first order spatial derivative, as in [6, Equation (2)]. With a small abuse of notation, u denotes henceforth the semi-discrete solution (after spatial discretization). If u_k is the nodal value of u at the k-th mesh node, then

$$\mathcal{H}_N(u) = \Delta x \sum_{k=1}^{N} \left(\frac{\alpha}{6}u_k^3 - \frac{\mu}{2}\left(\frac{u_{k+1} - u_k}{\Delta x}\right)^2\right), \tag{5.2}$$

and $(\mathcal{J}_N u)_k = u_{k+1} - u_{k-1}$, for $k = 1, \dots, N$ with $u_{N+k} = u_{1+k}$, $u_0 = u_{N-1}$ by periodicity. The Poisson tensor $\mathcal{J}_N$ has $\text{rank}(\mathcal{J}_N) = 2R$, with $2R = N - 1$ if N odd, $2R = N - 2$ if N even. The corresponding Casimir invariants are

$$\mathcal{C}_1(u) = \sum_{k=1}^{N} u_k, \qquad \mathcal{C}_2(u) = \sum_{k=1}^{N} (u_{2k} - u_{2k+1}). \tag{5.3}$$

Note that, if N is odd, then $\mathcal{C}_2(u) \equiv 0$ and $\mathcal{C}_1$ is the only Casimir invariant of the Poisson system.

Time discretization using the fully implicit midpoint rule on $\mathcal{T}_h = \bigcup_{j\in\Upsilon_h}(t^j, t^{j+1}]$ yields

$$u^{j+1} - u^j = \frac{\Delta t}{2\Delta x}\mathcal{J}_N \nabla\mathcal{H}_N(u^{j+1/2}), \qquad u^0 = \Pi_h u_0, \quad j \in \Upsilon_h, \tag{5.4}$$

where $u^{j+1/2} := (u^{j+1} + u^j)/2$ and Π_h is the nodal projection. The implicit midpoint rule provides a Poisson integrator for any constant-valued Poisson tensor. However, it does not preserve the discrete Hamiltonian (5.2) exactly.

As an alternative scheme, we consider the Average Vector Field (AVF) integration [41], which is second order accurate, preserves the Hamiltonian exactly [21, Theorem 3.1], but it is not a Poisson integrator [16]. For $j \in \Upsilon_h$, the fully discrete scheme reads

$$u^{j+1} - u^j = \frac{\alpha}{6}\frac{\Delta t}{2\Delta x}\mathcal{J}_N\left((u^{j+1})^2 + u^j u^{j+1} + (u^{j+1})^2\right) + \frac{\mu}{2\Delta x^3}\mathcal{J}_N \mathcal{F}_h(u^{j+1/2}), \tag{5.5}$$

with $u^0 = \Pi_h u_0$ and where $(\mathcal{F}_h(u))_k := (u_{k+1} - 2u_k + u_{k-1})/2$, for $k = 1, \dots, N$.

5.1.1 KdV: Long Time Stability of Double Soliton Interaction

In order to assess the stability of the reduced basis algorithm, we run a numerical test simulating solitons interaction over long time. Let us consider the KdV problem (5.1), with fixed parameters $\alpha = 6$ and $\mu = 1$, in the domain $\Omega = [-20, 20]$ and temporal interval $\mathcal{T} = (0, 500]$. Let the initial condition be the periodic function $u_0(x) = 6\,\text{sech}^2(x)$, $x \in \Omega$. The spatial discretization of the high-fidelity problem relies on the finite difference scheme (5.2) with $N = 1000$ mesh nodes. We compare the results obtained with the midpoint rule (5.4) as timestepping and the AVF scheme (5.5), both with uniform time step $\Delta t = 10^{-3}$. We select $M_s = 10000$ snapshots from the high-fidelity solution and run the symplectic greedy Algorithm 1 with tolerances $\text{tol}_\sigma = 10^{-5}$ and $\text{tol}_\delta = 10^{-12}$. The algorithm reaches convergence with $2r = 328$ for the AVF timestepping and $2r = 330$ for the midpoint rule. The need of a sufficiently large reduced space is typical of problems exhibiting propagation phenomena.

The high-fidelity solution and the reduced solution at final time are shown in Figure 2 (left), where the subscripts a and m refer to the AVF scheme and midpoint rule, respectively. The reduced solutions do not present spurious oscillations, not even over long time, and exhibit a qualitatively correct behavior in terms of phase and amplitude of the solitons, as it can be checked by comparing with [7, Example 5.2]. The solution obtained with the midpoint rule is slightly shifted with respect to the solution of the AVF scheme. This is a typical effect of numerical dispersion: the shape of the solitons is preserved but the solution is subject to a phase shift so that the solitons are wrongly located.

The error of the reduced numerical solution with respect to the high-fidelity one over time is reported in Figure 2 (right), where both the original problem (3.1) and its canonical formulation (3.2) are considered. Figure 3 reports the error of the Hamiltonian and of the Casimir invariants (5.3) over time. The AVF scheme (left) ensures almost exact preservation of the Hamiltonian and of the Casimir invariants when the canonical system is solved. For the original high-fidelity model, the Hamiltonian is conserved up to the Newton solver tolerance. The midpoint rule (Figure 3 right) preserves the linear Casimir invariants but not the (cubic) Hamiltonian, as expected.

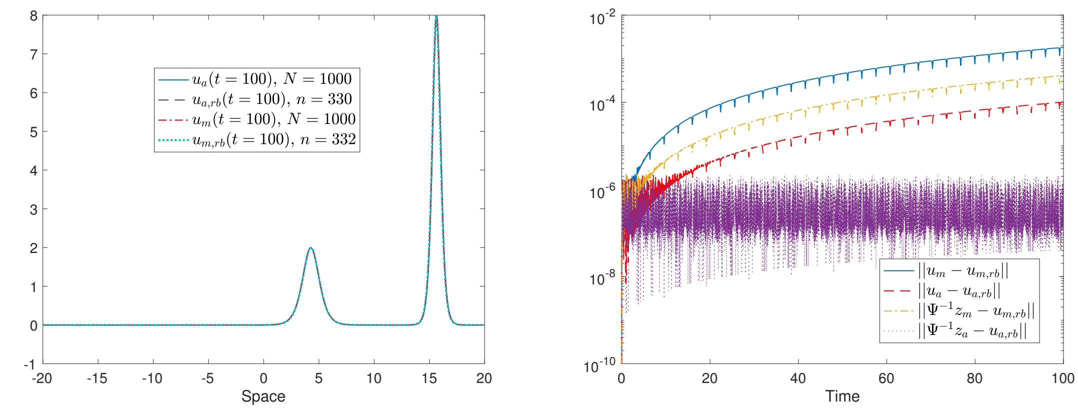

Figure 2: KdV double soliton interaction. Numerical solutions of the high-fidelity and reduced models at final time (left) for AVF timestepping and midpoint rule. Error between the numerical solution of the reduced problem and the high-fidelity solutions of the Poisson system in the original and canonical forms (right).

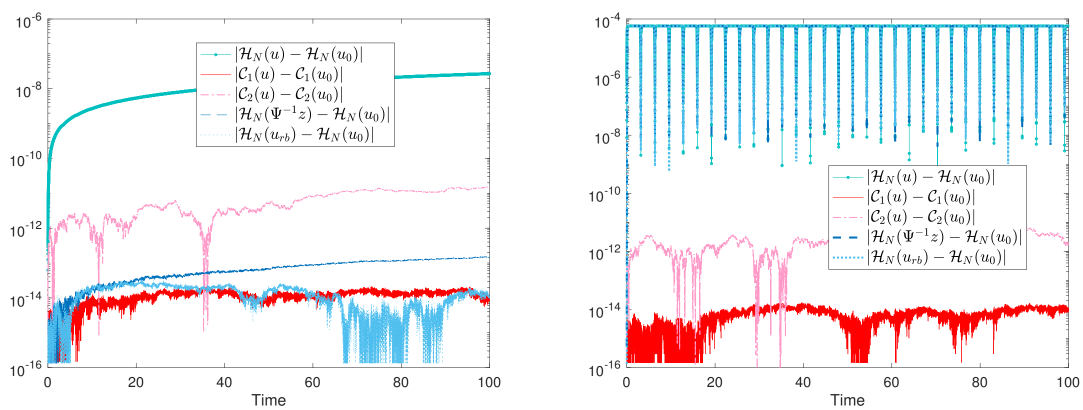

Figure 3: KdV double soliton interaction. Error of the Hamiltonian and Casimir invariants over time. Temporal discretization with AVF scheme (left) and midpoint rule (right).

5.1.2 KdV: Dispersion Limit

As a second test case, let us consider the KdV equation with varying parameter μ and solve the problem in the limit of small dispersion. Specifically, let $\alpha = 1$ and $\mu \in \Lambda := [10^{-6}, 2]$. The problem is set in the domain $\Omega = [0,1]$ and in the time interval $\mathcal{T} = (0,1]$ with initial condition $u_0(x) = 2 + 1/2\sin(2\pi(x-\mu))$, $x \in \Omega$, (a shifted variation of the test in [54, Section 4.6]). The spatial discretization relies on $N = 1600$ mesh nodes, and for the temporal approximation we use the AVF scheme (5.5) with uniform time step $\Delta t = 10^{-3}$. The kernel of the Poisson tensor has dimension $q = 2$. We select $M_s = 500$ snapshots from the high-fidelity model, and Λ_h is obtained by taking 10 equidistant points in Λ. The reduced basis algorithm uses the symplectic greedy Algorithm 1 with tolerances $\text{tol}_\sigma = 10^{-5}$ and $\text{tol}_\delta = 10^{-12}$. Convergence is reached at $2r = 556$.

The reduced solution for $\mu = 10^{-5} \notin \Lambda_h$ captures the train of soliton waves without unphysical oscillations, as shown in Figure 4. The ℓ^2-error over time of the reduced numerical solution with respect to the high-fidelity solution, obtained from the Poisson system in the non-canonical and canonical forms, is reported in Figure 5 (left). Concerning the invariants of motion, the Hamiltonian of both systems (2.4) and (3.3) is conserved up to the solver tolerance, Figure 5 (right).

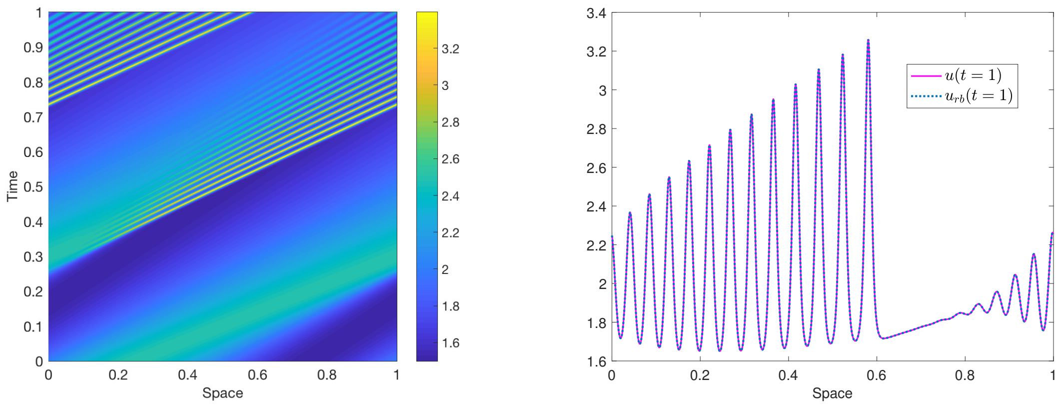

Figure 4: KdV in the dispersion limit, $\mu = 10^{-5}$. Evolution of the solution (left) and solution at final time (right) obtained with the AVF timestepping.

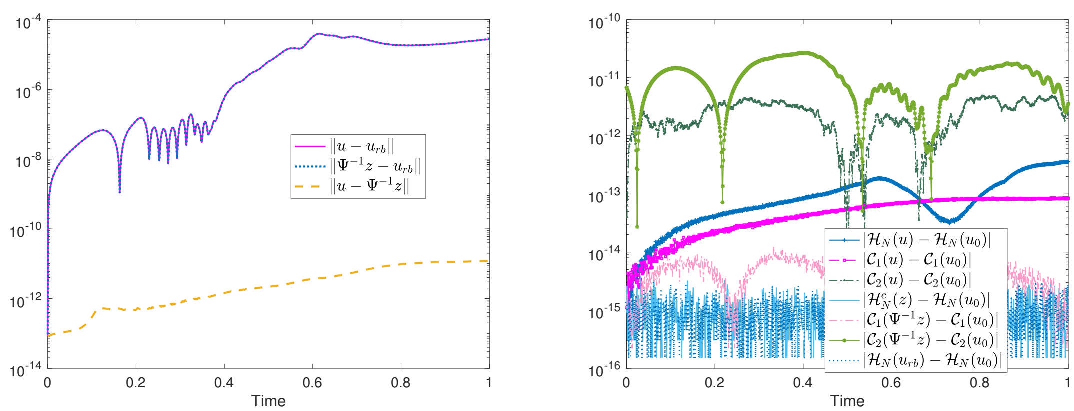

Figure 5: KdV in the dispersion limit, $\mu = 10^{-5}$. Error between the HiFi solution and the reduced solution (left). Relative error of the Hamiltonian and of the Casimir invariants (right).

5.2 Numerical Experiments for State-Dependent Structures

The multi-species generalized Lotka–Volterra problem provides an example of Hamiltonian system with state-dependent degenerate Poisson structure. The Volterra lattice equation was introduced to describe the interaction and evolution of populations of competing species. Additionally, it provides a discretization of the KdV equation or of the Logistic equation and can be used to model nonlinear control systems, lattice problems, etc. The generalized Lotka–Volterra model for N species reads

$$d_t u_k(t) = u_k(t)\Big(b_k + \sum_{\ell=1}^{N} a_{k,\ell} u_\ell(t)\Big), \qquad k = 1, \ldots, N, \quad b_k,\, a_{k,\ell} \in \mathbb{R},$$

where $u_{k+N} = u_k$ for all k, if the boundary conditions are periodic. Here we take the values of $\{b_k\}_k$ and $\{a_{k,\ell}\}_{k,\ell}$ yielding

$$d_t u_k = u_k(u_{k+1} - u_{k-1}), \qquad k = 1, \ldots, N. \tag{5.6}$$

The Lotka–Volterra system (5.6) possesses the invariants

$$\mathcal{I}_q(u) = \sum_{k=1}^{N} \left(\frac{1}{2}u_k^2 + u_k u_{k+1}\right), \qquad \mathcal{I}_c(u) = \sum_{k=1}^{N} \frac{1}{3}u_k^3 + \sum_{k=1}^{N} u_k u_{k+1}(u_k + u_{k+1} + u_{k+2}),$$

and the Casimir

$$\mathcal{C}_1(u) = \sum_{k=1}^{N} \log(u_k).$$

Furthermore, if the number N of species is even, the problem can be recast in a splitted form as follows. Let $q_k(t) = u_{2k-1}(t)$ and $p_k(t) = u_{2k}(t)$ for $k = 1, \ldots, N/2$ and $t \in \mathcal{T}$, then (5.6) is equivalent to

$$\begin{cases} d_t q_k = q_k(p_k - p_{k-1}), \\ d_t p_k = p_k(q_{k+1} - q_k). \end{cases}$$

This is a Poisson system with Hamiltonian $\mathcal{H}_N(q,p) = \sum_{k=1}^{N/2}(q_k + p_k)$, and quadratic bracket corresponding to the Poisson tensor

$$\mathcal{J}_N((q,p)) := \begin{pmatrix} 0 & q_1p_1 & & & & & -q_1p_{N/2} \\ -q_1p_1 & 0 & q_2p_1 & & & & 0 \\ & \ddots & & \ddots & & & \\ & & -q_kp_k & 0 & q_{k+1}p_k & & \\ & & & -q_kp_{k-1} & 0 & q_kp_k & \\ & & & \ddots & & \ddots & \\ q_1p_{N/2} & & & & & -q_{N/2}p_{N/2} & 0 \end{pmatrix}.$$

The dimension of the null space is $q = 2$ for all $u \in \mathcal{V}_N$.

Concerning the temporal discretization of the Lotka–Volterra problem in Hamiltonian form, the symplectic Euler method preserves the quadratic Poisson structure [24], and reads, for all $k = 1, \ldots, N/2$ and $j \in \Upsilon_h$,

$$\begin{cases} q_k^{j+1} = q_k^j + \Delta t\, q_k^j(p_k^{j+1} - p_{k-1}^{j+1}), \\ p_k^{j+1} = p_k^j + \Delta t\, p_k^{j+1}(q_{k+1}^j - q_k^j). \end{cases} \tag{5.7}$$

Let us consider a numerical simulation of problem (5.6) in the domain $\Omega = [-1, 1]$ and temporal interval $\mathcal{T} = (0, 500]$, with initial condition $u_0(x) = 1 + \mathrm{sech}^2(x)/(2N^2)$. The high-fidelity model is obtained setting $N = 1000$ and using the symplectic Euler discretization (5.7) with $\Delta t = 10^{-2}$. In the generation of the orthosymplectic reduced basis, the symplectic greedy Algorithm 1 is run with tolerances $\mathrm{tol}_\sigma = 10^{-5}$ and $\mathrm{tol}_\delta = 10^{-12}$. The algorithm reaches convergence with $2r = 210$.

In Figure 6 are reported the ℓ^2-error of the high-fidelity and reduced basis solutions at every time step (left), and the error of the Hamiltonian, the Casimir $\mathcal{C}_1$ and the invariants $\mathcal{I}_q$, $\mathcal{I}_c$ over time (right). It can be observed that the invariants of motion of the high-fidelity problem are preserved with a high degree of accuracy, similarly to [24, Figure 1]. The reduced solution produces larger, though still satisfactory, errors in the conservation of the invariants which however do not grow in time.

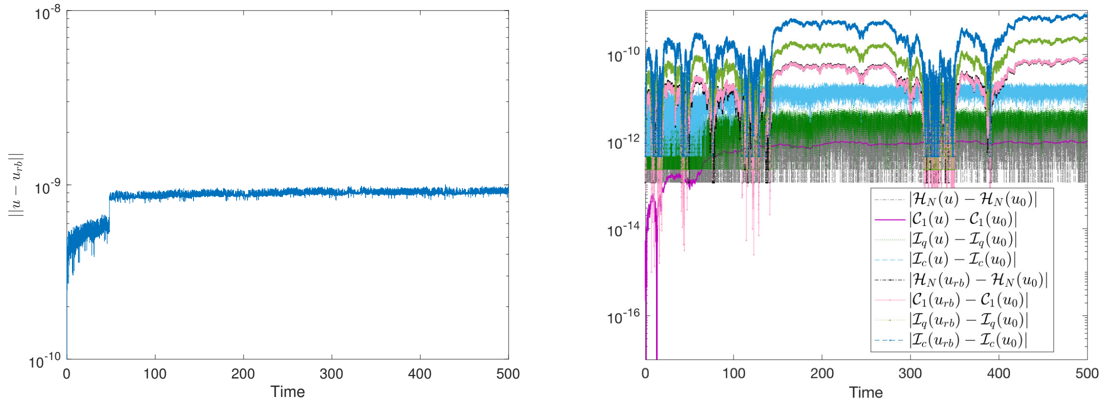

Figure 6: Lotka–Volterra lattice. Evolution of the ℓ^2-error of the high-fidelity and reduced basis solutions (left). Error in the conservation of the Hamiltonian, the Casimir $\mathcal{C}_1$ and the invariants $\mathcal{I}_q$, $\mathcal{I}_c$ over time (right).

6 Concluding Remarks

We have developed and analyzed reduced basis methods for dynamical Hamiltonian systems possessing a nonlinear state-dependent and degenerate Poisson structure. Relying on structure-preserving discretizations in space and time, the proposed reduced basis techniques are based on linear approximations of the Poisson tensor in each temporal interval followed by a model order reduction of the symplectic component of the dynamics. We have shown that the resulting reduced model retains the global Poisson structure and the conservation properties of the phase flow up to errors in the approximation of the Darboux map, it is efficient when coupled with DEIM techniques and enjoys good approximation properties. Further work may target the study of optimal and efficient approximation of the Darboux map, and corresponding approximation properties in the presence of a set of parameters in addition to time.

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A Proof of Theorem 4.6

Following Algorithm 1, the reduced basis matrix is initialized as $\pi^2 = [e_1, (\mathcal{J}_{2R}^c)^\top e_1]$ where $e_1 = z_s^0 := \psi_{1/2}^s u_0$. The projection onto $\operatorname{span}\{\pi^2\}$ is defined as $\mathcal{P}_2 = \pi^2 \circ \pi_+^2$ with $\pi_+^2 = (\mathcal{J}_{2\cdot 1}^c)^\top \pi^2 \mathcal{J}_{2R}^c$. At the r-th iteration, for $r \geqslant 1$, the greedy algorithm selects the new basis element e_{r+1} to satisfy

$$e_{r+1} = \underset{z \in \mathcal{Z}_N^s}{\operatorname{argmax}} \|z - \mathcal{P}_{2r} z\|,$$

so that $\mathcal{V}_{2(r+1)} = \operatorname{span}\{e_1, \ldots, e_{r+1}, (\mathcal{J}_{2R}^c)^\top e_1, \ldots, (\mathcal{J}_{2R}^c)^\top e_{r+1}\}$. The basis vectors are orthogonalized with respect to the ℓ^2-norm as

$$\begin{aligned}
&\xi_1 = e_1, \\
&\xi_i = e_i - \mathcal{P}_{2(i-1)} e_i, \qquad \xi_{r+i} = (\mathcal{J}_{2R}^c)^\top \xi_i, \qquad i = 2, \ldots, r+1.
\end{aligned}$$

The projection $\mathcal{P}_{2r}$ onto the symplectic manifold $\mathcal{V}_{2r}$ can be written as

$$\mathcal{P}_{2r} z = \sum_{i=1}^{r} (\alpha_i(z)\xi_i + \beta_i(z)(\mathcal{J}_{2R}^c)^\top \xi_i), \qquad \forall\, z \in \mathcal{Z}_N^s.$$

With $\mathcal{Z}_N^s$ being a subspace of the normed space $(\mathcal{V}_{2R}, \|\cdot\|)$, $\mathcal{P}_{2r}$ is an orthogonal projection onto $\mathcal{V}_{2r}$, in view of Lemma 4.4. Hence, for each $z \in \mathcal{Z}_N^s$, $\xi_\ell \in \mathcal{V}_{2r}$ and $\ell \leqslant r$,

$$\begin{aligned}
&(\mathcal{P}_{2r} z, \xi_\ell) = (z, \mathcal{P}_{2r}\xi_\ell) = (z, \xi_\ell) = \alpha_\ell(z)\|\xi_\ell\|^2, \\
&(\mathcal{P}_{2r} z, (\mathcal{J}_{2R}^c)^\top \xi_\ell) = \beta_\ell(z)\|(\mathcal{J}_{2R}^c)^\top \xi_\ell\|^2 = \beta_\ell(z)\|\xi_\ell\|^2.
\end{aligned}$$

Using the orthogonality properties of $\mathcal{P}_{2(\ell-1)}$, the fact that $\xi_\ell, (\mathcal{J}_{2R}^c)^\top \xi_\ell \in \mathcal{V}_{2\ell}$ are ℓ^2-orthogonal to $\mathcal{V}_{2(\ell-1)}$ by construction, combined with the error criterion of the greedy Algorithm 1, results in

$$\begin{aligned} |\alpha_\ell(z)| &= \frac{|(z,\xi_\ell)|}{\|\xi_\ell\|^2} = \frac{|(z-\mathcal{P}_{2(\ell-1)}z,\xi_\ell)|}{\|\xi_\ell\|^2} \leqslant \frac{\|z-\mathcal{P}_{2(\ell-1)}z\|}{\|e_\ell-\mathcal{P}_{2(\ell-1)}e_\ell\|} \leqslant 1, \\ |\beta_\ell(z)| &= \frac{|(z,(\mathcal{J}_{2R}^c)^\top\xi_\ell)|}{\|(\mathcal{J}_{2R}^c)^\top\xi_\ell\|^2} = \frac{|(z-\mathcal{P}_{2(\ell-1)}z,(\mathcal{J}_{2R}^c)^\top\xi_\ell)|}{\|(\mathcal{J}_{2R}^c)^\top\xi_\ell\|^2} \leqslant \frac{\|z-\mathcal{P}_{2(\ell-1)}z\|}{\|e_\ell-\mathcal{P}_{2(\ell-1)}e_\ell\|} \leqslant 1. \end{aligned} \tag{A.1}$$

The elements of the orthogonal basis spanning $\mathcal{V}_{2(r+1)}$, selected by the greedy algorithm, can be expanded as,

$$\xi_i = \sum_{j=1}^{i}(\gamma_j^i e_j + \delta_j^i(\mathcal{J}_{2R}^c)^\top e_j), \qquad \xi_{r+i} = (\mathcal{J}_{2R}^c)^\top \xi_i.$$

for all $i = 2, \dots, r+1$, where $\gamma_i^i = 1$, $\delta_i^i = 0$ and for $j < i$,

$$\gamma_j^i := \sum_{\ell=j}^{i-1}(-\alpha_\ell(e_i)\gamma_j^\ell + \beta_l(e_j)\delta_j^\ell), \qquad \delta_j^i := \sum_{\ell=j}^{i-1}(\alpha_\ell(e_i)\delta_j^\ell - \beta_l(e_j)\gamma_j^\ell).$$

Using (A.1), each coefficient can be bounded as $|\gamma_j^i| < 3^{i-j-1}$, $|\delta_j^i| < 3^{i-j-1}$ if $j < i$, so that

$$|\gamma_j^i| \leqslant 3^{i-j}, \quad |\delta_j^i| \leqslant 3^{i-j}, \qquad \forall\, j \leqslant i.$$

By definition of the Kolmogorov $2r$-width, given $\lambda > 1$, there exists a $2r$-dimensional space $\mathcal{W}_{2r}$ such that the angle between $\mathcal{Z}_N^s$ and $\mathcal{W}_{2r}$ satisfies $\sup_{z\in\mathcal{Z}_N^s}\inf_{w\in\mathcal{W}_{2r}}\|z-w\| \leqslant \lambda d_{2r}(\mathcal{Z}_N^s)$. Hence, for the elements of any subspace $\mathcal{V}_\ell \subset \mathcal{Z}_N^s$ with $\ell \leqslant r$, there exist $w_\ell, v_\ell \in \mathcal{W}_{2r}$ such that $\|e_\ell - w_\ell\| \leqslant \lambda d_{2r}(\mathcal{Z}_N^s)$, and $\|(\mathcal{J}_{2R}^c)^\top e_\ell - v_\ell\| \leqslant \lambda d_{2r}(\mathcal{Z}_N^s)$. For $i = 1, \dots, r$, we define the vectors

$$\mathcal{W}_{2r} \ni \zeta_i = \sum_{j=1}^{i}(\gamma_j^i w_j + \delta_j^i v_j), \qquad \zeta_{r+i} = \sum_{j=1}^{i}(-\delta_j^i w_j + \gamma_j^i v_j). \tag{A.2}$$

For $i = 1, \dots, 2r$, they satisfy

$$\|\xi_i - \zeta_i\| \leqslant \sum_{j=1}^{i}(|\gamma_j^i|\|e_j - w_j\| + |\delta_j^i|\|(\mathcal{J}_{2R}^c)^\top e_j - v_j\|), \leqslant \lambda d_{2r}(\mathcal{Z}_N^s)\sum_{j=1}^{i} 2\cdot 3^{i-j} < 3^i \lambda d_{2r}(\mathcal{Z}_N^s).$$

Let us consider the elements defined in (A.2) where we add a further pair $(\zeta_{r+1}, \zeta_{2(r+1)}) \in W_{2r}$, defined such that $w_{r+1}, v_{r+1} \in W_{2r}$ are the vectors for which $\|e_{r+1} - w_{r+1}\| \leqslant \lambda d_{2r}(\mathcal{Z}_N^s)$, and $\|(\mathcal{J}_{2R}^c)^\top e_{r+1} - v_{r+1}\| \leqslant \lambda d_{2r}(\mathcal{Z}_N^s)$. Since such a family belongs to the $2r$-dimensional space W_{2r} by construction, the vectors $\{\zeta_i\}_{i=1}^{2(r+1)}$ cannot be linearly independent: there exist $\{\sigma_i\}_{i=1}^{2(r+1)} \subset \mathbb{R}$ such that $\|\sigma\| = 1$ and $\sum_{i=1}^{2(r+1)}\sigma_i\zeta_i = 0$. Hence,

$$\begin{aligned} \left\|\sum_{i=1}^{r+1}(\sigma_i\xi_i + \sigma_{(r+1)+i}(\mathcal{J}_{2R}^c)^\top\xi_i)\right\| = \left\|\sum_{i=1}^{2(r+1)}\sigma_i(\xi_i - \zeta_i)\right\| &\leqslant \lambda d_{2r}(\mathcal{Z}_N)\sum_{i=1}^{2(r+1)}|\sigma_i|3^i \\ &\leqslant 3^{r+1}\sqrt{2(r+1)}\lambda d_{2r}(\mathcal{Z}_N^s). \end{aligned}$$

Let $1 \leqslant j \leqslant 2(r+1)$ be fixed. Define $w_j := \sigma_j^{-1}\sum_{i=1,i\neq j}^{2(r+1)}\sigma_i\xi_i$. Note that $(\xi_j, w_j) = 0$ since $\{\xi_j\}_{j=1}^{2(r+1)}$ is orthogonal, which implies $\|\xi_j\|^2 \leqslant \|\xi_j\|^2 + \|w_j\|^2 = \|\xi_j + w_j\|^2$. Furthermore,

$$\|\xi_j + w_j\| \leqslant \left\|\sigma_j^{-1}\sum_{i=1}^{2(r+1)}\sigma_i(\xi_i - \zeta_i)\right\| \leqslant |\sigma_j^{-1}|\sum_{i=1}^{2(r+1)}|\sigma_i|\|\xi_i - \zeta_i\| \leqslant 3^{r+1}\lambda d_{2r}(\mathcal{Z}_N^s)\sqrt{2(r+1)}|\sigma_j^{-1}|.$$

Since the choice of the index j is arbitrary, we select j such that $|\sigma_j| \geqslant (2(r+1))^{-1/2}$, which is possible by definition of $\{\sigma_i\}_i$. Hence, $\|\xi_j + w_j\| \leqslant 2 \cdot 3^{r+1}(r+1)\lambda d_{2r}(\mathcal{Z}_N)$. Therefore, the projection error of any $z \in \mathcal{Z}_N^s$ can be bounded as

$$\|z - \mathcal{P}_{2r}z\| \leqslant \|z - \mathcal{P}_{2(j-1)}z\| \leqslant \|e_j - \mathcal{P}_{2(j-1)}e_j\| = \|\xi_j\| \leqslant 2 \cdot 3^{r+1}(r+1)\lambda d_{2r}(\mathcal{Z}_N^s).$$

With an argument analogous to the proof of Proposition 3.13, the conclusion follows from the fact that $d_{2r}(\mathcal{Z}_N^s) \leqslant d_n(\mathcal{Z}_N)$ since, in each time interval, the subsets $\{\mathcal{N}_j\}_j$ are not affected by the reduction.

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