Reductive overgroups of distinguished unipotent elements in simple algebraic groups

THÈSE N^{O} 8362 (2018)

PRÉSENTÉE LE 25 MAI 2018

À LA FACULTÉ DES SCIENCES DE BASE
GROUPE TESTERMAN
PROGRAMME DOCTORAL EN MATHÉMATIQUES

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

POUR L'OBTENTION DU GRADE DE DOCTEUR ÈS SCIENCES

PAR

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ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Suisse
2018

Abstract

Let G be a simple linear algebraic group over an algebraically closed field K of characteristic $p \geq 0$. In this thesis, we investigate closed connected reductive subgroups $X < G$ that contain a given distinguished unipotent element u of G. Our main result is the classification of all such X that are maximal among the closed connected subgroups of G.

When G is simple of exceptional type, the result is easily read from the tables computed by Lawther in [Law09]. Our focus is then on the case where G is simple of classical type, say $G = \mathrm{SL}(V)$, $G = \mathrm{Sp}(V)$, or $G = \mathrm{SO}(V)$. We begin by considering the maximal closed connected subgroups X of G which belong to one of the families of the so-called *geometric subgroups*. Here the only difficult case is the one where X is the stabilizer of a tensor decomposition of V. For $p = 2$ and $X = \mathrm{Sp}(V_1) \otimes \mathrm{Sp}(V_2)$, we solve the problem with explicit calculations; for the other tensor product subgroups we apply a result of Barry [Bar15].

After the geometric subgroups, the maximal closed connected subgroups that remain are the $X < G$ such that X is simple and V is an irreducible and tensor indecomposable X-module. The bulk of this thesis is concerned with this case. We determine all triples (X, u, φ) where X is a simple algebraic group, $u \in X$ is a unipotent element, and $\varphi : X \to G$ is a rational irreducible representation such that $\varphi(u)$ is a distinguished unipotent element of G. When $p = 0$, this was done in previous work by Liebeck, Seitz and Testerman [LST15].

In the final chapter of the thesis, we consider the more general problem of finding all connected reductive subgroups X of G that contain a distinguished unipotent element u of G. This leads us to consider connected reductive overgroups X of u which are contained in some proper parabolic subgroup of G. Testerman and Zalesski [TZ13] have shown that when u is a regular unipotent element of G, no such X exists. We give several examples which show that their result does not generalize to distinguished unipotent elements. As an extension of the Testerman-Zalesski result, we show that except for two known examples which occur in the case where $(G, p) = (C_2, 2)$, a connected reductive overgroup of a distinguished unipotent element of order p cannot be contained in a proper parabolic subgroup of G.

Keywords: group theory, representation theory, algebraic groups, unipotent elements, classical groups.

Résumé

Soit G un groupe algébrique simple sur un corps algébriquement clos K de caractéristique $p \geq 0$. Dans cette thèse, nous nous intéressons aux sous-groupes $X < G$ fermés réductifs connexes qui contiennent un élément unipotent distingué u de G. Notre principal résultat est une classification de tels X qui sont maximaux parmi les sous-groupes fermés connexes de G.

Quand G est simple de type exceptionnel, le resultat peut facilement être trouvé dans les tableaux calculés par Lawther dans [Law09]. L'accent est donc porté sur le cas où G est simple de type classique, disons $G = \mathrm{SL}(V)$, $G = \mathrm{Sp}(V)$, ou $G = \mathrm{SO}(V)$. Nous commençons par examiner les sous-groupes X maximaux parmi les sous-groupes fermés connexes de G, qui appartiennent à l'une des familles dénommées *sous-groupes géométriques*. Ici le seul cas difficile est celui où X est un stabilisateur d'une décomposition tensorielle de V. Pour $p = 2$ et $X = \mathrm{Sp}(V_1) \otimes \mathrm{Sp}(V_2)$, nous résolvons le problème par des calculs explicites; pour les autres sous-groupes tensoriels nous appliquons un résultat de Barry [Bar15].

Aprés les sous-groupes géométriques, les sous-groupes X maximaux parmi les sous-groupes fermés connexes de G restants sont les $X < G$ tels que X est simple et V est un X-module irréductible et indécomposable en produit tensoriel. La majeure partie de cette thèse se rapporte à ce cas. Nous déterminons tous les triplets (X, u, φ) où X est un groupe algébrique simple, $u \in X$ est un élément unipotent, et $\varphi : X \to G$ est une représentation rationnelle irréductible telle que $\varphi(u)$ est un élément unipotent distingué de G. Dans le cas $p = 0$, cela a été fait dans des travaux antérieurs de Liebeck, Seitz et Testerman [LST15].

Dans le dernier chapitre de cette thèse, nous considérons un probléme plus général de trouver tous les sous-groupes $X < G$ fermés réductifs connexes qui contiennent un élément unipotent distingué u de G. Cela nous amène à étudier les sous-groupes X qui contiennent u et qui sont contenus dans un sous-groupe parabolique propre de G. Testerman et Zalesski [TZ13] ont montré que si u est régulier, il n'existe aucun tel X. Nous donnons plusieurs exemples qui démontrent que leur resultat ne se généralise pas aux éléments unipotents distingués. Comme une extension du résultat de Testerman-Zalesski, nous montrons, sauf dans deux cas où $(G, p) = (C_2, 2)$, qu'un sous-groupe fermé réductif connexe qui contient un élément unipotent distingué d'ordre p ne peut pas être contenu dans un sous-groupe parabolique propre de G.

Mots-clefs: théorie des groupes, théorie des représentations, groupes algébriques, éléments unipotents, groupes classiques.

Acknowledgements

The biggest thanks of all are due to my thesis director, Donna Testerman, who introduced me to the world of algebraic groups and supported me throughout this work. She has been very generous with her time, and I thank her for many fruitful discussions. She read patiently and carefully a countless number of pages which eventually became this thesis; I am grateful for her many useful comments, suggestions, and guidance.

I would like to thank Alexander Premet, Alexander Zalesski, and Jacques Thévenaz for agreeing to be on the thesis jury, and Kathryn Hess for acting as the president of the jury.

Special thanks are due to Frank Lübeck, who on request extended the tables of weight multiplicities on his website, which allowed me to shorten the proof of Lemma 5.1.9 and Lemma 5.1.10. I am thankful to Alastair Litterick and Adam Thomas for informing me of their ongoing work. I would also like to thank several other mathematicians for small comments, discussions, clarifications, etc. related to my thesis work: Michael Barry, Tim Burness, George McNinch, Ross Lawther, Gary Seitz, and Irina Suprunenko.

Many thanks also to past and present colleagues who have made working at EPFL pleasant, especially those within GR-TES: Mikaël Cavallin, Claude Marion, Jay Taylor, Nathan Scheinmann, and Neil Saunders. Finally, to my friends, family, and Fengda: warm thanks for all of your encouragement and support.

Contents

Chapter 1

Introduction

This thesis concerns the subgroup structure and representation theory of simple linear algebraic groups G over an algebraically closed field K. Two major open problems for researchers working in this subject are

(1) Understanding the rational irreducible representations of G;

(2) Classifying all positive-dimensional reductive subgroups of G up to conjugacy in G.

When K is a field of characteristic zero, both problems have been well understood for a fairly long time. For (1), we have classical results such as the Weyl character formula and Freudenthal's formula for weight multiplicities. For (2), the work of Dynkin [Dyn52a] [Dyn52b] gives a full list of all maximal closed connected subgroups of G. Furthermore, it follows from a theorem of Mostow [Mos56] that any maximal *connected reductive* subgroup of G is either a maximal closed connected subgroup, or a Levi subgroup of G. Consequently in characteristic zero, we have a recursive way of classifying the connected reductive subgroups of G.

In the case where K is a field of positive characteristic, considerable progress has been made towards both problems in the past decades. One key result is in the work of Seitz and Testerman, who considered maximal subgroups for G of classical type [Sei87] [Tes88] [Tes89]. Except for the so-called *geometric subgroups* (e.g. parabolic subgroups, stabilizers of tensor decompositions), any maximal closed connected subgroup of G is a simple algebraic group X which acts irreducibly and tensor indecomposably on the natural module of G [Sei87, Theorem 3]. In their work, Seitz and Testerman proved that apart from a known list of exceptions[1], any such X is a maximal closed connected subgroup of G. This is analogous to Dynkin's result [Dyn52a] in characteristic zero, but in positive characteristic the "known list of exceptions" is much larger and different techniques are needed for establishing the list. Later for G simple of exceptional type, Liebeck and Seitz [LS04] found all maximal positive-dimensional closed subgroups of G.

Despite these and plenty of other advances made during the years, many problems still remain open and a complete solution to the two problems (1) and (2) above seems to be unattainable for the foreseeable future. For example, the consensus among researchers seems to be that even finding the dimensions of the

[1]However, we should mention that there is a recently discovered mistake in [Sei87], which caused a family of examples to be missing from the "known list of exceptions" given in [Sei87, Table 1]. This mistake is corrected by Cavallin and Testerman in their preprint [CT17].

rational irreducible G-modules will remain an open problem for some time. Hence it is unrealistic to try to classify all possible reductive subgroups of all simple linear algebraic groups.

In order to get a glimpse of the complete picture, one has to then focus on more specific problems, such as the classification of certain types of reductive subgroups. The main contribution of this thesis is one such classification result: we give a complete classification of the maximal closed connected reductive subgroups of G that contain *distinguished* unipotent elements (see Definition 1.1.1 and Problem 1.1.2 below). This generalizes previous results found in [Sup95, Theorem 1.9], [SS97], and [LST15].

Roughly, one can think of the main topic of this thesis as a part of a general approach or philosophy for studying subgroup structure, where one classifies the subgroups containing elements that are "special" in some sense, see [Sax98] for a survey of some results. Besides being helpful for understanding the subgroup structure of linear groups, solutions to such problems have been motivated by applications in recognition algorithms for linear groups [NP92] and the Inverse Galois Problem in number theory [GM14].

More specifically, the problem of classifying overgroups of specific classes of unipotent elements is a topic which many researchers have considered, one example is the work of Liebeck and Seitz [LS94] on subgroups containing root elements. It is perhaps slightly surprising that in many cases a full classification is feasible, and just the presence of a single unipotent element of specific type in a reductive subgroup can tell us a lot about its structure. For example, all connected reductive subgroups of $\mathrm{SL}(V)$ containing a full Jordan block can be classified [SS97] [TZ13].

1.1 Main problem and statements of results

We now introduce the main problem of this thesis and make some preliminary obversations. Later in this section we state our main results. To begin, we need the following definition.

Definition 1.1.1. Let $u \in G$ be a unipotent element. We say that $u \in G$ is *distinguished* if $C_G(u)$ does not contain a nontrivial torus[2].

Basic properties of distinguished unipotent elements that are needed in this work will be given in Chapter 2. The main problem we are concerned with (and which we will solve in this thesis) is the following.

Problem 1.1.2 (Main problem)**.** *Let $u \in G$ a distinguished unipotent element. Classify all maximal closed connected subgroups X of G that contain u, up to G-conjugacy.*

Recall that as a corollary of the Borel-Tits theorem, any maximal closed connected subgroup X of G is either parabolic or reductive. Furthermore, it is well known that all parabolic subgroups of G contain a representative of every unipotent conjugacy class of G. Therefore Problem 1.1.2 is solved in the case where X is parabolic.

[2]Equivalently, we can say that a unipotent element $u \in G$ is distinguished if the identity component of $C_G(u)$ is unipotent. Another equivalent formulation is that u is not contained in any proper Levi factor of G.

Suppose then that X is a *reductive* maximal closed connected subgroup of G. If G is simple of exceptional type, a complete list of such X is given by Liebeck and Seitz in [LS04]. Furthermore, for each such X, Lawther gives in [Law09] for each unipotent element $u \in X$ the conjugacy class of u in G. Thus when G is simple of exceptional type, the solution of Problem 1.1.2 is easily read from the tables in [Law09].

What our work will focus on then is the case where G is simple of classical type. It is easy to see (Lemma 2.1.2) that the isogeny type of G will not make any difference in the solution Problem 1.1.2. Therefore we can assume that $G = \mathrm{SL}(V)$, $G = \mathrm{Sp}(V)$, or $G = \mathrm{SO}(V)$, where V is a finite-dimensional vector space over K. For what follows, set $\operatorname{char} K = p \geq 0$. The maximal closed connected subgroups of G can be described with [Sei87, Theorem 3], as follows[3]:

Theorem 1.1.3. *Let $G = \mathrm{SL}(V)$ ($\dim V \geq 2$), $G = \mathrm{Sp}(V)$ ($\dim V \geq 2$), or $G = \mathrm{SO}(V)$ ($\dim V \geq 5$), where V is a finite-dimensional vector space over K. Let X be a proper closed connected subgroup of G. Then X is maximal among the closed connected subgroups of G if and only if one of the following holds:*

(a) *$G = \mathrm{SL}(V)$, and $X = \mathrm{Sp}(V)$ or $X = \mathrm{SO}(V)$ (Exception: $X = \mathrm{SO}(V)$ and $p = 2$).*

(b) *$G = \mathrm{Sp}(V)$, $p = 2$, $\dim V > 2$, and $X = \mathrm{SO}(V)$.*

(c) *$G = \mathrm{SO}(V)$, $p = 2$, $\dim V$ is even, and X is the stabilizer of a nonsingular 1-space.*

(d) *X is a maximal parabolic subgroup of G.*

(e) *$G = \mathrm{Sp}(V)$ or $G = \mathrm{SO}(V)$, $V = W \oplus W^{\perp}$, where W is a non-degenerate subspace of V, and $X = \mathrm{stab}_G(W)^{\circ}$.*

(f) *$V = V_1 \otimes V_2$ and one of the following holds:*

   (i) *$G = \mathrm{SL}(V)$, and $X = \mathrm{SL}(V_1) \otimes \mathrm{SL}(V_2)$, where $\dim V_i \geq 2$ (Exception: $\dim V_1 = 2 = \dim V_2$).*

   (ii) *$G = \mathrm{SO}(V)$ and $X = \mathrm{Sp}(V_1) \otimes \mathrm{Sp}(V_2)$, where $\dim V_i \geq 2$.*

   (iii) *$G = \mathrm{SO}(V)$, $p \neq 2$, and $X = \mathrm{SO}(V_1) \otimes \mathrm{SO}(V_2)$, where $\dim V_i \geq 3$.*

   (iv) *$G = \mathrm{Sp}(V)$, $p \neq 2$, and $X = \mathrm{Sp}(V_1) \otimes \mathrm{SO}(V_2)$, where $\dim V_1 \geq 2$ and $\dim V_2 \geq 3$.*

(g) *X is simple, $V \downarrow X$ is simple and tensor indecomposable, and there does not exist $Y < G$ with (X, Y, V) in [Sei87, Table 1] or [CT17, Theorem 1.2].*

Our approach will be to solve Problem 1.1.2 for each of the subgroups (a) - (g) in Theorem 1.1.3. The proof will be given in Section 5.14. It will be seen

[3] Here we have modified the statement of [Sei87, Theorem 3] slightly. The dimension restrictions such as $\dim V \geq 5$ for $G = \mathrm{SO}(V)$ are set because we only consider G simple. Furthermore, as mentioned in footnote 1, it was recently discovered that there is family of examples missing from [Sei87, Table 1]; thus we have included a reference to the correction given by Cavallin and Testerman. Also, we have corrected a small inaccuracy: the statement of [Sei87, Theorem 3 (f)] excludes some subgroups which are maximal, namely the tensor product subgroups $\mathrm{Sp}_2(K) \otimes \mathrm{Sp}_{2m}(K)$ in $\mathrm{SO}_{4m}(K)$.

that in cases (a) - (e) of Theorem 1.1.3, the solution of Problem 1.1.2 will follow from basic results in the literature. In case (f) of Theorem 1.1.3, the solution will follow with the main result of Section 3.6 (Proposition 3.6.6) and a result of Barry [Bar15][4].

For the rest of this text, we will make the following assumption.

Assume that $p > 0$.

The solution of Problem 1.1.2 for the subgroups (a) - (f) in Theorem 1.1.3 is given by the following result. Below for a unipotent element $u \in G$, the notation V_d denotes the indecomposable $K[u]$-module of dimension d, on which u acts with a single Jordan block. This and other pieces of notation used in this thesis are given in 1.4 below.

Theorem 1.1.4. *Let* $G = \mathrm{SL}(V)$ *(*$\dim V \geq 2$*),* $G = \mathrm{Sp}(V)$ *(*$\dim V \geq 2$*), or* $G = \mathrm{SO}(V)$ *(*$\dim V \geq 5$*), where* V *is a finite-dimensional vector space over* K*. Fix a distinguished unipotent element* $u \in G$*. Let* X *be one of the maximal closed connected subgroups of* G *given in (a) - (f) of Theorem 1.1.3. Then the cases where* X *contains a* G*-conjugate of* u *are precisely the following:*

(a) $G = \mathrm{SL}(V)$*,* $X = \mathrm{Sp}(V)$ *(*$\dim V$ *even) or* $X = \mathrm{SO}(V)$ *(*$p \neq 2$ *and* $\dim V$ *odd), and* u *is a regular unipotent element.*

(b) $G = \mathrm{Sp}(V)$*,* $p = 2$*,* $\dim V > 2$*,* $X = \mathrm{SO}(V)$*, and the number of Jordan blocks of* u *is even.*

(c) $G = \mathrm{SO}(V)$*,* $p = 2$*,* $\dim V$ *is even,* X *is the stabilizer of a nonsingular* 1*-space, and* u *has a Jordan block of size* 2*.*

(d) X *is a maximal parabolic subgroup.*

(e) $G = \mathrm{Sp}(V)$ *or* $G = \mathrm{SO}(V)$*,* $V = W \oplus W^{\perp}$ *where* W *is a non-degenerate subspace of* V*,* $X = \mathrm{stab}_G(W)^{\circ}$*, and* $V \downarrow K[u] \cong \bigoplus_{i=1}^{t} V_{d_i} \oplus \bigoplus_{j=1}^{s} V_{d'_j}$ *for integers* $d_i, d'_j \geq 1$ *such that the following conditions hold:*

- $\dim W = \sum_{i=1}^{t} d_i$.
- *If* $p = 2$ *and* $G = \mathrm{SO}(V)$*, then* $t \equiv 0 \mod 2$.

(f) $V = V_1 \otimes V_2$ *with* $\dim V_1 \leq \dim V_2$ *and one of the following holds:*

  (i) $G = \mathrm{SO}(V)$*,* $p = 2$*,* $\dim V_1 = 2$*,* $X = \mathrm{Sp}(V_1) \otimes \mathrm{Sp}(V_2)$*, and the orthogonal decomposition* $V \downarrow K[u]$ *(Proposition 2.4.4) is equal to* $V(2d_1)^2 + \cdots + V(2d_t)^2$ *for some* $1 \leq d_1 < \cdots < d_t$ *such that* d_i *is odd for all* i*.*

  (ii) $G = \mathrm{Sp}(V)$ *or* $G = \mathrm{SO}(V)$*,* $p \neq 2$*,* X *is as in Theorem 1.1.3 (ii), (iii) or (iv); and for* $m = \dim V_1$ *and* $n = \dim V_2$ *one of the following conditions hold:*

  - *The pair* (m, n) *is contained in the set* $\mathscr{S}$ *of Definition 3.3.12, and* $V \downarrow K[u] \cong \bigoplus_{i=1}^{m} V_{m+n-2i+1}$*. (In this case* $V \downarrow K[u] \cong V_m \otimes V_n$*).*

[4] More precisely, the description of the set $\mathscr{S}$ that we use in Theorem 1.1.4 (f) is taken from [Bar15, Theorem 2, Theorem 3].

- *There exist integers $1 \leq n_1 < n_2 < \cdots < n_t$ such that $\sum_{i=1}^{t} n_i = n$, $n_i \equiv n_{i'} \mod 2$ for all $1 \leq i, i' \leq t$, $n_i - n_{i-1} \geq 2m$ for all $2 \leq i \leq t$, the pair (m, n_i) is contained in the set $\mathscr{S}$ of Definition 3.3.12 for all $1 \leq i \leq t$, and*

$$V \downarrow K[u] \cong \bigoplus_{j=1}^{\min(m,n_1)} V_{n_1+m-2j+1} \oplus \bigoplus_{i=2}^{t} \bigoplus_{j=1}^{m} V_{n_i+m-2j+1}.$$

(In this case $V \downarrow K[u] \cong V_m \otimes (V_{n_1} \oplus V_{n_2} \oplus \cdots \oplus V_{n_t})$).

Remark 1.1.5. We have assumed that $p > 0$, but it is easy to see that essentially the same result as Theorem 1.1.4 applies when $p = 0$. Specifically, when $p = 0$, the cases (b), (c) and (f)(i) of Theorem 1.1.4 no longer apply, and in case (f)(ii) the conditions $(m, n) \in \mathscr{S}$ and $(m, n_i) \in \mathscr{S}$ should be removed.

What remains then is the case of the irreducible subgroups in Theorem 1.1.3 (g). The bulk of this thesis is devoted to solving Problem 1.1.2 in this case. To do this, it will be enough to solve the following problem, which has been solved in characteristic zero by Liebeck, Seitz and Testerman [LST15].

Problem 1.1.6. *Let $\mathscr{I}(V)$ be a connected simple classical group over K (that is, $\mathscr{I}(V) = \mathrm{SL}(V)$, $\mathscr{I}(V) = \mathrm{Sp}(V)$ or $\mathscr{I}(V) = \mathrm{SO}(V)$). Determine all closed connected simple subgroups $G < \mathscr{I}(V)$ and unipotent elements $u \in G$ such that the following hold:*

(i) *The group G acts on V irreducibly and tensor indecomposably,*

(ii) *The element $u \in G$ is a distinguished unipotent element of $\mathscr{I}(V)$.*

This is essentially a problem about the representation theory of simple algebraic groups. In what follows, we will say that a G-module V (or representation $\varphi : G \to \mathrm{GL}(V)$) is *symplectic* if V admits a non-degenerate G-invariant alternating bilinear form, and *orthogonal* if V admits a non-degenerate G-invariant quadratic form. Note that if V is irreducible, then any such form is unique up to a scalar multiple (Lemma 4.4.3 and Lemma 4.4.5). Thus if a rational irreducible representation $\varphi : G \to \mathrm{GL}(V)$ is symplectic or orthogonal, then we can write respectively $\varphi(G) < \mathrm{Sp}(V)$ or $\varphi(G) < \mathrm{SO}(V)$ without ambiguity.

We will now make the following definition to present Problem 1.1.6 in a different way.

Definition 1.1.7. Let $u \in G$ be a unipotent element and let $\varphi : G \to \mathrm{GL}(V)$ be a rational representation. For the G-module V, we say that *u acts on V as a distinguished unipotent element*, if one of the following holds.

(i) $\varphi(u)$ is a distinguished unipotent element in $\mathrm{SL}(V)$.

(ii) V is symplectic, and $\varphi(u)$ is a distinguished unipotent element in $\mathrm{Sp}(V)$, where $\mathrm{Sp}(V)$ is the stabilizer of a non-degenerate G-invariant alternating bilinear form.

(iii) V is orthogonal, and $\varphi(u)$ is a distinguished unipotent element in $\mathrm{SO}(V)$, where $\mathrm{SO}(V)$ is the identity component of the stabilizer of a non-degenerate G-invariant quadratic form.

The following lemma is immediate from Lemma 2.1.2 (ii).

Lemma 1.1.8. *Let $u \in G$ be a unipotent element. If u acts on a rational G-module as a distinguished unipotent element, then u is distinguished in G.*

With Lemma 1.1.8, Problem 1.1.6 becomes equivalent to the following problem.

Problem 1.1.9. *Let G be connected and simple and let $u \in G$ be a distinguished unipotent element. Find all irreducible, tensor-indecomposable G-modules V such that u acts on V as a distinguished unipotent element. Furthermore, for each such V, determine whether V is symplectic, orthogonal, both, or neither.*

Note that by Steinberg's tensor product theorem, it is enough to solve Problem 1.1.9 in the case where V is a p-restricted irreducible G-module. The first part of the solution to Problem 1.1.9 will follow from the following two theorems. See Definition 2.7.2 for the definition of $m_u(\lambda)$.

Theorem 1.1.10. *Assume that $p \neq 2$. Let G be simple, and let λ be a non-zero p-restricted dominant weight. A unipotent element $u \in G$ acts on the irreducible G-module $L_G(\lambda)$ of highest weight λ as a distinguished unipotent element if and only if one of the following holds:*

(i) *(u of order $> p$) G, λ, p and u occur in Table 1.1 or Table 1.2,*

(ii) *(u of order p) G, λ and u occur in Table 1.1 or Table 1.2 and $p > m_u(\lambda)$,*

(iii) *$G = G_2$, $\lambda = \omega_1 + 2\omega_2$, u is a regular unipotent element, and $p = 5$.*

Theorem 1.1.11. *Assume that $p = 2$. Let G be simple, and let λ be a non-zero 2-restricted dominant weight. A unipotent element $u \in G$ acts on the irreducible G-module $L_G(\lambda)$ of highest weight λ as a distinguished unipotent element if and only if G, λ and u occur in Table 1.3.*

For the second part of Problem 1.1.6, we have to determine whether the $L_G(\lambda)$ given in Theorem 1.1.10 and Theorem 1.1.11 are symplectic, orthogonal, both, or neither. In the case where $p \neq 2$ this is easily done with Lemma 4.4.3 and Table 4.1. Consider then $p = 2$. If $\lambda \neq -w_0\lambda$, then $L_G(\lambda)$ is not self-dual and thus is not symplectic nor orthogonal. Suppose then that $\lambda = -w_0\lambda$. Then $L_G(\lambda)$ is self-dual and is symplectic by a result observed in [Fon74] (see Lemma 4.4.5). The question of whether $L_G(\lambda)$ is orthogonal is more subtle and in general an open problem. However, for the λ occurring in Table 1.3, the orthogonality of $L_G(\lambda)$ can be decided from the results in [Kor17]; we have included this information in Table 1.3 below[5].

As a first step to the solution of Problem 1.1.6, we note the following. It is well known that in $\mathrm{SL}(V)$ there is only one class of distinguished unipotent elements, the class of a full Jordan block (Lemma 2.2.2). The irreducible representations $\varphi : G \to \mathrm{GL}(V)$ where $\varphi(u)$ is a full Jordan block have been known for some time already.

[5]For many of the entries, this could also done using older results from the literature, see for example [GW95, Theorem 3.4] and [Gow97, Corollary 4.3].

Theorem 1.1.12 ([Sup95, Theorem 1.9])**.** *Let G be connected simple and let $\lambda \in X(T)^+$ be p-restricted. A unipotent element $u \in G$ acts on $L_G(\lambda)$ with a single Jordan block if and only if u is a regular unipotent element in G and one of the following holds:*

(i) $G = A_1$.

(ii) $G = A_l$, $G = B_l$ *or* $G = C_l$, *and* $\lambda = \omega_1$.

(iii) $G = A_l$ *and* $\lambda = \omega_l$.

(iv) $G = C_2$ *and* $\lambda = \omega_2$.

(v) $G = G_2$ *and* $\lambda = \omega_1$.

By Theorem 1.1.12, we can reduce the proof of Theorem 1.1.10 and Theorem 1.1.11 to the case of irreducible representations $\varphi : G \to \mathrm{GL}(V)$ where V is symplectic or orthogonal. Equivalently, we reduce to the case where V is self-dual as a G-module (Lemma 4.4.3). When $p \neq 2$, by the description of distinguished unipotent elements in classical groups (Proposition 2.3.4), the problem is then reduced to finding all distinguished unipotent elements $u \in G$ and self-dual irreducible G-modules V such that $\varphi(u)$ has all Jordan block sizes distinct, and either all block sizes are even or all block sizes are odd. When $p = 2$, Jordan block sizes do not determine whether a unipotent element is distinguished or not, but one knows that a distinguished unipotent element has all Jordan block sizes even with multiplicity ≤ 2 (Proposition 2.4.4 (ii)). Our strategy for Theorem 1.1.11 is then to determine the cases where $\varphi(u)$ has all Jordan block sizes even with multiplicity ≤ 2, and then determine among these when $\varphi(u)$ is distinguished.

G	λ	Class of u	u of order $> p$	$m_u(\lambda)$
A_l	ω_1	regular	any	$2l$
$A_l, l \geq 2$	$\omega_1 + \omega_l$	regular	Proposition 5.3.2 (ii)	$2l$
A_1	$0 \leq c \leq p-1$	regular	none	c
A_5	ω_3	regular	$p = 5$	6
B_l	ω_1	any	any	
B_3	101	regular	$p = 5$	12
B_3	002	regular	$p = 5$	12
B_3	300	regular	none	18
$B_l, l \geq 3$	$2\omega_1$	regular	Proposition 5.7.7 (b) (ii)	$4l$
$B_l, l \geq 3$	ω_2	regular	Proposition 5.5.5 (b) (ii)-(iii)	$4l-2$
$B_l, 3 \leq l \leq 8$	ω_n	regular	Proposition 5.10.1	$\frac{l(l+1)}{2}$
C_l	ω_1	any	any	
C_2	$b0,\ 1 \leq b \leq 5$	regular	$p = 3$	$3b$
C_2	$0b,\ 1 \leq b \leq 5$	regular	$p = 3$	$4b$
C_2	11	regular	$p = 3$	7
C_2	12	regular	none	11
C_2	21	regular	none	10
C_3	300	regular	none	15
C_3	ω_3	regular	$p = 3, 5$	9
C_4	ω_3	regular	none	15
C_4	ω_4	regular	none	16
C_5	ω_3	regular	none	21
C_5	ω_5	regular	none	25
$C_l, l \geq 3$	$2\omega_1$	regular	Proposition 5.7.11 (b) (ii)	$4l-2$
$C_l, l \geq 3$	ω_2	regular	Proposition 5.5.10 (b) (ii)-(iv)	$4l-4$
$D_l, l \geq 4$	ω_1	any		
D_6	ω_6	regular	$p = 3, 5, 7$	15
D_6	ω_6	$[3, 9]$	$p = 7$	11
D_8	ω_8	regular	$p = 11$	28
$D_l, l \geq 4$ even	$2\omega_1$	regular	Proposition 5.7.8 (b) (ii)	$4l-4$
$D_l, l \geq 4$ odd	ω_2	regular	Proposition 5.5.6 (b) (ii)-(iii)	$4l-6$

Table 1.1: $\operatorname{char} K \neq 2$: Distinguished actions for irreducible representations of simple G of classical type.

G	λ	Class of u	u of order $> p$	$m_u(\lambda)$
G_2	10	regular	$p = 3, 5$	6
G_2	01	regular	$p = 3, 5$	10
G_2	11	regular	none	16
G_2	20	regular	$p = 5$	12
G_2	02	regular	none	20
G_2	30	regular	none	18
F_4	ω_1	regular	$p = 5, 7, 11$	22
F_4	ω_4	regular	$p = 3, 5, 7, 11$	16
F_4	ω_4	$F_4(a_1)$	$p = 5, 7$	10
E_6	ω_2	regular	$p = 5, 7, 11$	22
E_7	ω_1	regular	$p = 11$	34
E_7	ω_7	regular	$p = 3, 5, 11, 13, 17$	27
E_7	ω_7	$E_7(a_1)$	$p = 5, 13$	21
E_7	ω_7	$E_7(a_2)$	$p = 5, 7, 11$	17
E_8	ω_8	regular	$p = 11, 13, 17, 23$	58
E_8	ω_8	$E_8(a_1)$	$p = 11$	46

Table 1.2: $\operatorname{char} K \neq 2$: Distinguished actions for irreducible representations of simple G of exceptional type.

G	λ	unipotent class	$L(\lambda)\downarrow K[u]$	orthogonal?
A_l, $l \geq 1$	ω_1	regular	$[l+1]$	not self-dual
A_l, $l \geq 1$	ω_l	regular	$[l+1]$	not self-dual
A_2	$\omega_1+\omega_2$	regular	$[4^2]$	yes
A_3	ω_2	regular	$[2,4]$	yes
A_4	$\omega_1+\omega_4$	regular	$[4^2,8^2]$	yes
B_l, C_l, $l \geq 2$	ω_1	any distinguished		no
B_2, C_2	ω_2	regular	$[4]$	no
B_2, C_2	ω_2	2_1^2	$[2^2]$	no
B_3, C_3	ω_2	regular	$[6,8]$	yes
B_3, C_3	ω_3	regular	$[2,6]$	yes
B_4, C_4	ω_4	regular	$[8^2]$	yes
B_4, C_4	ω_4	$2_1,6_1$	$[2^2,6^2]$	yes
B_5, C_5	ω_2	regular	$[6,8,14,16]$	yes
B_5, C_5	ω_5	regular	$[2,6,10,14]$	yes
B_6, C_6	ω_2	$2_1,10_1$	$[6,8,10^2,14,16]$	no
B_6, C_6	ω_6	$2_1,10_1$	$[2^2,6^2,10^2,14^2]$	yes
D_l, $l \geq 4$	ω_1	any distinguished		yes
D_4	ω_3	any distinguished		yes
D_4	ω_4	any distinguished		yes
D_6	ω_2	regular	$[6,8,10^2,14,16]$	no
D_6	ω_5	regular	$[2,6,10,14]$	yes
D_6	ω_6	regular	$[2,6,10,14]$	yes
G_2	ω_1	regular	$[6]$	no
G_2	ω_2	regular	$[6,8]$	yes
F_4	ω_4	regular	$[10,16]$	yes
F_4	ω_1	regular	$[10,16]$	yes
E_7	ω_7	regular	$[2,10,18,26]$	yes
E_7	ω_1	regular	$[8,10,16,18,22,26,32]$	no

Table 1.3: $\operatorname{char} K = 2$: Distinguished actions for irreducible representations of simple G.

1.2 A more general problem

The main problem of this thesis (Problem 1.1.2) is a special case of the following more general problem, which remains open.

Problem 1.2.1. *Let $u \in G$ a distinguished unipotent element. Classify all connected reductive subgroups X of G that contain u, up to G-conjugacy.*

In the final chapter of this thesis, we will present some partial results on Problem 1.2.1. In order to present these results, we will next describe the most natural strategy for solving Problem 1.2.1. A first step for solving Problem 1.2.1 would be to find the *maximal connected reductive* subgroups X of G that contain a distinguished unipotent element u. Then since u must also be distinguished in X, the strategy is to do the same thing with X and proceed in this way inductively down the subgroup lattice.

However, in the induction step one encounters a non-trivial problem which needs to be dealt with. Let $Y < G$ be a positive-dimensional connected reductive subgroup containing u. Then Y is contained in some *maximal closed connected* subgroup M of G. As a corollary of the Borel-Tits theorem, we know that M is parabolic or reductive. If M is connected reductive, then by our solution to Problem 1.1.2, we know the possibilities for M and could apply induction.

But if M is parabolic, it is not obvious what one should do here. In this case Y cannot be contained in a Levi factor of M, because u is a distinguished unipotent element. In the terminology due to Serre [Ser05], this means that Y is a non-G-completely reducible (non-G-cr) subgroup (Definition 6.1.1). Essentially, the obstacle arising here is that while we have a good understanding of what *reductive* maximal connected subgroups look like (by [LS04] and [Sei87]), we know little about subgroups which are *maximal connected reductive* but not maximal connected.

Fix u and G in Problem 1.2.1. With the remarks in the previous paragraphs in mind, a natural strategy for solving Problem 1.2.1 proceeds with the following steps.

Step I: Find all *reductive* maximal closed connected subgroups of G containing u.

Step II: Determine if a closed connected reductive subgroup of G containing u can be contained in a proper parabolic subgroup of G, and if so, find them.

Step III: Use the results in steps I and II to find all closed connected reductive subgroups of G containing u.

The main result of this thesis is the completion of Step I, i.e. the solution of Problem 1.1.2. It is easy to see (Lemma 6.1.2) that Step II is asking to classify, up to G-conjugacy, all non-G-cr connected reductive subgroups X of G that contain u. In previous literature, a result due to Testerman and Zalesski states that no such X exists if u is a regular unipotent element [TZ13]. One might hope that this would generalize to distinguished unipotent elements, but it turns out that this is not the case. Several examples are given in Chapter 6, see e.g. Example 6.3.3. However, the general impression is that such examples are quite rare. This is seen in the main result of Chapter 6 (Theorem 6.2.12), which gives a complete classification in the case where u has order p; the result is that there are only two examples, both of which occur in the case $(G, p) = (C_2, 2)$.

1.3 The structure of this thesis

The structure of the text is as follows. This introductory chapter will end with Section 1.4 below, where we fix (mostly standard) notation and terminology that will be used throughout the text. **Chapter 2** contains basic facts about conjugacy classes of unipotent elements in simple algebraic groups. Most of the results in Chapter 2 are well known, proofs are given for some results for which we were unable to find a reference.

Chapter 3 is concerned with the properties of unipotent linear maps on a finite-dimensional vector space. Most of the chapter concerns the Jordan decomposition of the tensor product, exterior square and symmetric square of unipotent linear maps. In characteristic zero this would be a trivial topic, but in our setting ($p > 0$) this becomes more difficult. We describe results from the literature which allow one to determine these decompositions, and then apply them to establish various facts which will be needed in the sequel. One useful result is Proposition 3.5.3, which determines when the symmetric square or the exterior square of a unipotent Jordan block has no repeated block sizes in its Jordan decomposition. Also important for later use will be lemmas 3.4.10, 3.4.11, 3.4.12, and 3.4.13, which describe the smallest Jordan block sizes occurring in the symmetric and exterior square of a unipotent matrix. In Section 3.6, we describe the conjugacy class of a unipotent element $u_1 \otimes u_2 \in \mathrm{Sp}(V_1) \otimes \mathrm{Sp}(V_2)$ in $\mathrm{Sp}(V_1 \otimes V_2)$ in some small cases when $p = 2$. Chapter 3 finishes with Section 3.7, where we will prove Theorem 1.1.4, one of our main results.

In **Chapter 4**, we consider the representation theory of simple algebraic groups that will be needed in the text. Most important result for our main problem is Proposition 4.6.8, which is key in our proof of Theorem 1.1.10 in the case where u has order p. Proposition 4.6.8 gives a classification of $\mathrm{SL}_2(K)$-modules on which a non-identity unipotent element u of $\mathrm{SL}_2(K)$ acts with ≤ 1 Jordan block of size p. An important consequence of this will be Proposition 4.6.10, which shows that a self-dual $\mathrm{SL}_2(K)$-module is semisimple if u acts on it with ≤ 1 Jordan block of size p.

Chapter 5 is devoted to the proof of two of our main results, Theorem 1.1.4 and Theorem 1.1.10. Here we begin with Section 5.1, which contains the main reduction for Theorem 1.1.10 in the case where u has order $> p$. Essentially, we show that if $u \in G$ has order $> p$, then it is enough to prove Theorem 1.1.10 for a small number of λ. Section 5.2 contains a similar reduction for Theorem 1.1.11, the main result when $p = 2$. Sections 5.3-5.12 contain the proof of Theorem 1.1.10 and Theorem 1.1.11 for various families of λ that we need to consider. Most of the work is in Section 5.5 and Section 5.7, where we establish Theorem 1.1.10 in the case where G is of classical type and $\lambda = \omega_2$ or $\lambda = 2\omega_1$. The results established in Chapter 3 are key in the proofs of our results. In Section 5.13, we prove Theorem 1.1.10 in the case where u has order p. The proof is based on Proposition 4.6.10 and Theorem 2.6.8, which allow us to apply the methods of Liebeck, Seitz and Testerman in [LST15]. The proofs of Theorem 1.1.4 and Theorem 1.1.10 are given in Section 5.14.

In **Chapter 6**, the final chapter of the thesis, we consider connected reductive subgroups that are non-G-cr and contain a distinguished unipotent element u of G. The main result is Theorem 6.2.12, which is gives a complete classification in the case where u has order p. In the case where u has order $> p$, we present some examples and partial results.

1.4 Notation and terminology

Throughout the whole text, let K be an algebraically closed field of characteristic $p > 0$. Unless otherwise mentioned, G denotes a connected semisimple algebraic group over K, and V will be a finite-dimensional vector space over K. If G is simple, then we say that p is *good for* G in the following cases:

- G is simple of type A_l; for all $p > 0$,
- G is simple of type B_l, C_l, or D_l; for $p > 2$,
- G is simple of type G_2, F_4, E_6, or E_7; for $p > 3$,
- G is simple of type E_8, for $p > 5$.

Otherwise we say that p is *bad for* G. We say that p is *very good* for G if p is good for G and in addition $p \nmid l+1$ if G is of type A_l. If G is connected semisimple, then we say that p is *good for* G if it is good for each simple factor of G, otherwise we say that p is *bad for* G.

All groups that we consider are linear algebraic groups over K, and by a subgroup we always mean a closed subgroup. For an algebraic group X, we denote its identity component by X°. For an element $u \in X$ of finite order, we denote the order of u by $|u|$.

All modules and representations that we consider will be finite-dimensional and rational. Throughout we will view G as its group of rational points over K, and often G will studied either as a Chevalley group constructed with the usual Chevalley construction (see e.g. [Ste68]), or as a classical group in its natural representation (i.e. $G = \mathrm{SL}(V)$, $G = \mathrm{Sp}(V)$ or $G = \mathrm{SO}(V)$). We will occasionally denote G by its type, so notation such as $G = C_l$ means that G is a simple algebraic group of type C_l.

Let $l = \operatorname{rank} G$. For G we will use the following notation, similarly to the notation used in [Jan03].

- T: A maximal torus of G.
- $X(T)$: Character group of T.
- Φ: Root system of G with respect to T. These are elements of $X(T)$.
- Δ: A system of simple roots for Φ.
- Φ^+: The set of positive roots in Φ, with respect to Δ.
- δ: The half-sum of positive roots, i.e. $\delta = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$.
- $\preceq$: The usual partial ordering on $X(T)$, i.e. $\mu \preceq \lambda$ if and only if $\lambda = \mu$, or $\lambda - \mu$ is a sum of positive roots.
- W: Weyl group of G with respect to T.

- $(-,-)$: W-invariant inner product on the vector space $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$.
- $\alpha^\vee$ (for $\alpha \in \Phi$): the dual root $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$.
- $\Phi^\vee$: The dual root system $\{\alpha^\vee : \alpha \in \Phi\}$.
- $\langle -,- \rangle$: defined as $\langle \lambda, \mu \rangle = \frac{2(\lambda,\mu)}{(\mu,\mu)}$ for all $\lambda, \mu \in X(T)$.
- $X(T)^+$: The set of dominant weights for G, with respect to Δ. This is the set of $\lambda \in X(T)$ which satisfy $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Delta$.
- $\operatorname{ch} V$: Character of a G-module V (an element of $\mathbb{Z}[X(T)]$).
- $m_V(\mu)$: Multiplicity of the weight $\mu \in X(T)$ in the G-module V.
- $\omega_1, \omega_2, \ldots, \omega_l \in X(T)^+$ fundamental dominant weights, with respect to Δ. We use the standard Bourbaki labeling of the simple roots, as given in [Hum72, 11.4, pg. 58].
- $m_u(\lambda)$ ($u \in G$ unipotent, $\lambda \in X(T)^+$): See Definition 2.7.2.
- $L(\lambda)$, $L_G(\lambda)$: Irreducible G-module with highest weight $\lambda \in X(T)^+$.
- $V(\lambda)$, $V_G(\lambda)$: Weyl module for G with highest weight $\lambda \in X(T)^+$.
- $T(\lambda)$, $T_G(\lambda)$: Tilting module for G with highest weight $\lambda \in X(T)^+$.
- $\operatorname{rad} V(\lambda)$: The unique maximal submodule of $V(\lambda)$.
- g^G: The G-conjugacy class of $g \in G$.
- $\mathscr{L}(G)$: Lie algebra of G.

We will use the notation $L_G(\lambda)$ instead of $L(\lambda)$ whenever we want to make clear that $L_G(\lambda)$ is a module for G (similarly $V_G(\lambda)$ and $T_G(\lambda)$).

For a dominant weight $\lambda \in X(T)^+$, we can write $\lambda = \sum_{i=1}^{l} m_i\omega_i$ where $m_i \in \mathbb{Z}_{\geq 0}$. We say that λ is *p-restricted* if $0 \leq m_i \leq p-1$ for all $1 \leq i \leq l$.

Let $u \in \mathrm{GL}(V)$ be unipotent. Suppose that u has Jordan block sizes $1 \leq d_1 < d_2 < \cdots < d_t$ and let $n_i \geq 1$ be the number of blocks of size d_i occurring in the Jordan form on u. We will often use the notation $[d_1^{n_1}, \ldots, d_t^{n_t}]$ to denote the Jordan form of u. Here if $n_i = 1$, we will write d_i instead of d_i^1. We will often say that u *acts on V with Jordan blocks* $[d_1^{n_1}, \ldots, d_t^{n_t}]$. We also say that u *acts on V with no repeated blocks* if $n_i = 1$ for all $1 \leq i \leq t$. If $t = 1$ and $n_1 = 1$, we say that u acts on V with a *single Jordan block* or as a *full Jordan block*.

It will often be convenient for us to describe the action of u on a representation in terms of $K[u]$-modules. Suppose that u has order $q = p^t$. Then there exist exactly q indecomposable $K[u]$-modules $V_1, V_2, \ldots, V_q$. Here $\dim V_i = i$ and u acts on V_i as a full Jordan block. We will denote by $r \cdot V_n$ the direct sum $V_n \oplus \cdots \oplus V_n$, where V_n occurs r times. If V is a $K[u]$-module with decomposition $V = n_1V_{d_1} \oplus \cdots \oplus n_tV_{d_t}$, where $1 \leq d_1 < d_2 < \cdots < d_t$ and $n_i \geq 1$, we will sometimes write this as $V = [d_1^{n_1}, \ldots, d_t^{n_t}]$. We will say that V has *no repeated blocks* if $n_i = 1$ for all $1 \leq i \leq t$.

A bilinear form b is *non-degenerate*, if its *radical* $\operatorname{rad} b = \{v \in V : b(v,w) = 0 \text{ for all } w \in V\}$ is zero. For a quadratic form $Q : V \to K$ on a vector space V, its *polarization* is the bilinear form b_Q defined by $b_Q(v,w) = Q(v+w) - Q(v) - Q(w)$ for all $v, w \in V$. We say that Q is *non-degenerate*, if its *radical* $\operatorname{rad} Q = \{v \in \operatorname{rad} b_Q : Q(v) = 0\}$ is zero.

For a representation V of G, a bilinear form $(-,-)$ is said to be G-invariant if $(gv, gw) = (v, w)$ for all $g \in G$ and $v, w \in V$. We say that V is *symplectic* if it has a non-degenerate G-invariant alternating bilinear form, and we say V is *orthogonal* if it has a non-degenerate G-invariant quadratic form.

For a vector space V over K, we will denote by $\mathrm{Sp}(V)$ the group of linear maps stabilizing a non-degenerate alternating bilinear form on V, and by $\mathrm{O}(V)$ the group of linear maps stabilizing a non-degenerate quadratic form on V. The identity component of $\mathrm{O}(V)$ will be denoted by $\mathrm{SO}(V)$. Note that if $p \neq 2$, then $\mathrm{SO}(V) = \mathrm{O}(V) \cap \mathrm{SL}(V)$. If $p = 2$, then $\mathrm{SO}(V)$ is the kernel of the Dickson invariant on $\mathrm{O}(V)$.

Given a morphism $\phi : G' \to G$ of algebraic groups, we can twist representations of G with ϕ. That is, if $\rho : G \to \mathrm{GL}(V)$ is a representation of G, then $\rho\phi$ is a representation of G'. We denote the corresponding G'-module by V^{ϕ}. For the Frobenius map $F : G \to G$, we will denote $V^{F^n} = V^{[n]}$.

If a representation V of G has composition series $V = V_1 \supset V_2 \supset \cdots \supset V_t \supset V_{t+1} = 0$ with composition factors $W_i \cong V_i/V_{i+1}$, we will occasionally denote this by $V = W_1/W_2/\cdots/W_t$.

We denote by ν_p the p-adic valuation on the integers, so for a nonzero integer $n \in \mathbb{Z}$ we define $\nu_p(n)$ to be the largest integer $k \geq 0$ such that p^k divides n.

Chapter 2

Unipotent elements in simple algebraic groups

2.1 Preliminaries

The purpose of this chapter is to list various basic facts about unipotent elements in G that will be needed in the sequel. As in the main problem of this thesis, our main interest is in *distinguished* unipotent elements (Definition 1.1.1). One particularly important class of distinguished unipotent elements is the class of *regular* unipotent elements, which will come up often.

Definition 2.1.1. Let $u \in G$ be a unipotent element. We say that $u \in G$ is *regular*, if $\dim C_G(u) = \operatorname{rank} G$.

It is well known that a regular unipotent element of G is distinguished [SS70, 1.14.(a)]. Furthermore, we know that there exists a unique conjugacy class of regular unipotent elements in G, and this conjugacy class is dense in the variety of all unipotent elements of G [SS70, Theorem 1.8]. In the sections that follow, we will describe the regular and distinguished unipotent conjugacy classes in groups of classical type.

We will also need the following basic lemma referred to in the introduction, which will allow us to establish our results without being concerned with the isogeny type of G.

Lemma 2.1.2. *Let $f : G_1 \to G_2$ be an isogeny between two simple algebraic groups G_1 and G_2. Then*

(i) *The map f restricts to a bijection between the unipotent varieties of G_1 and G_2, and f induces a bijection between the unipotent conjugacy classes of G_1 and G_2.*

(ii) *Let $u \in G_1$ be a unipotent element. Then u is distinguished in G_1 if and only if $f(u)$ is distinguished in G_2.*

(iii) *The map f induces a bijection between the set of closed connected reductive subgroups of G_1 and G_2 via $X \mapsto f(X)$.*

Proof. Claim (i) follows as in [Car85, Proposition 5.1.1]. Claim (ii) also follows with a standard argument that we give here for completeness. Suppose first that

u is centralized by a non-trivial torus $S < G_1$. Then since $\ker f$ is finite, it follows that $f(S)$ is a non-trivial torus centralizing $f(u)$.

For the other direction of (ii), suppose that $f(u)$ is centralized by a non-trivial torus $S' < G_2$. Since f is surjective, we have $S' = f(S)$ for $S = f^{-1}(S')^\circ$. Since S is connected and consists of semisimple elements, it is a torus. We proceed to show that S centralizes u. To this end, note first that since S' centralizes $f(u)$, it follows that $[s, u] \in \ker f$ for all $s \in S$. On the other hand, the set $[S, u]$ of all commutators $[s, u]$, $s \in S$, is irreducible, being the image of S under the rational map defined by $s \mapsto [s, u]$. Since the kernel $\ker f$ is finite, it follows that $[S, u] = 1$, so S is a non-trivial torus centralizing u.

For (iii), note first that images of reductive groups are reductive. Thus we have a map $X \mapsto f(X)$ between the closed connected reductive subgroups of G_1 and G_2. We show that this map is bijective. If $X, Y < G_1$ are closed connected reductive subgroups and $f(X) = f(Y)$, then $X \ker f = Y \ker f$. Since $\ker f$ is finite, we have $(X \ker f)^\circ = X$ and $(Y \ker f)^\circ = Y$, which gives $X = Y$. Therefore the map in question is injective. To show surjectivity, suppose that $Z < G_2$ is a closed connected reductive subgroup. Since f is surjective, we have $f(X) = Z$ for $X = f^{-1}(Z)^\circ$. What remains is to see that X is reductive, and this follows easily from the fact that $\ker f$ is finite. Indeed, if U is a non-trivial closed connected unipotent which is normal in X, then $f(U)$ would be a non-trivial closed connected unipotent subgroup which is normal in Z. □

To describe unipotent conjugacy classes for a simple algebraic group with given root system Φ, it follows from Lemma 2.1.2 (i) that it is enough to do this for some fixed isogeny type. For simple groups of classical type, in the sections that follow we will give a description of unipotent conjugacy classes in $\mathrm{SL}(V)$, $\mathrm{Sp}(V)$, and $\mathrm{SO}(V)$.

2.2 Unipotent elements in $\mathrm{SL}(V)$

Because we are working over an algebraically closed field, the description of the unipotent conjugacy classes in $\mathrm{SL}(V)$ is simply a matter of linear algebra, as shown by the following lemma which is well known.

Lemma 2.2.1. *Let $x, y \in \mathrm{GL}(V)$. If x and y are conjugate in $\mathrm{GL}(V)$, then x and y are conjugate in $\mathrm{SL}(V)$.*

Proof. Suppose that $y = g^{-1}xg$ for some $g \in \mathrm{GL}(V)$. Then for any scalar $c \in K$, we have $y = h^{-1}xh$ for $h = cg$. Since K is algebraically closed, we can choose a scalar c such that $\det(h) = 1$. □

Therefore in $\mathrm{SL}(V)$, the conjugacy class of a unipotent element is determined by its Jordan form. In other words, unipotent conjugacy classes in $\mathrm{SL}(V)$ can be labeled by partitions of $\dim V$.

In $\mathrm{SL}(V)$ there is only one conjugacy class of distinguished unipotent elements, and this is the class of regular unipotent elements. We record this in the following lemma. For a proof, see for example [LS12, Proposition 3.5].

Lemma 2.2.2. *Let $u \in \mathrm{SL}(V)$ be unipotent. Then the following statements are equivalent.*

(i) *u is distinguished in $\mathrm{SL}(V)$.*

(ii) *u is regular in* $\mathrm{SL}(V)$.

(iii) *u has only one Jordan block.*

2.3 Unipotent elements in classical groups ($p \neq 2$)

Assume that $p \neq 2$.

The correspondence between partitions and unipotent conjugacy classes holds also in other classical groups in odd characteristic. The following propositions are well known. See for example Theorem 3.1.(i), Theorem 3.1.(ii) and Proposition 3.5 in [LS12].

Proposition 2.3.1. *Suppose that $G = \mathrm{Sp}(V)$ or $G = \mathrm{O}(V)$. Then two unipotent elements of G are conjugate in G if and only if they have the same Jordan block structure.*

Proposition 2.3.2. *Let $u \in \mathrm{SL}(V)$ be unipotent. Suppose that u has Jordan block sizes $1 \leq d_1 < d_2 < \cdots < d_t$ and let $n_i \geq 1$ be the number of blocks of size d_i occurring in the Jordan form on u. Then*

(i) *A conjugate of u lies in $\mathrm{Sp}(V)$ if and only if n_i is even for all i such that d_i is odd.*

(ii) *A conjugate of u lies in $\mathrm{SO}(V)$ if and only if n_i is even for all i such that d_i is even.*

(iii) *Suppose that $u \in \mathrm{SO}(V)$. Then the conjugacy class of $u^{\mathrm{O}(V)}$ consists of a single $\mathrm{SO}(V)$-class, unless d_i is even for all i, in which case $u^{\mathrm{O}(V)}$ splits into two $\mathrm{SO}(V)$-classes.*

Proposition 2.3.3. *In a simple classical group G with natural module V, the Jordan block structure of a regular unipotent element $u \in G$ is as follows.*

(i) $G = \mathrm{SL}(V)$*: single Jordan block of size* $\dim V$.

(ii) $G = \mathrm{Sp}(V)$*: single Jordan block of size* $\dim V$.

(iii) $G = \mathrm{SO}(V)$, $\dim V$ *odd: single Jordan block of size* $\dim V$.

(iv) $G = \mathrm{SO}(V)$, $\dim V$ *even:* $[\dim V - 1, 1]$.

Proposition 2.3.4. *In a simple classical group G with natural module V, the Jordan block structure of a distinguished unipotent element $u \in G$ is as follows.*

(i) $G = \mathrm{SL}(V)$*: single Jordan block of size* $\dim V$.

(ii) $G = \mathrm{Sp}(V)$*:* $[d_1, d_2, \ldots, d_t]$ *where d_i are distinct even integers.*

(iii) $G = \mathrm{SO}(V)$*:* $[d_1, d_2, \ldots, d_t]$ *where d_i are distinct odd integers.*

2.4 Unipotent elements in classical groups ($p = 2$)

Assume that $p = 2$.

We will describe the classification of unipotent conjugacy classes in the classical groups $\mathrm{Sp}(V)$, $\mathrm{O}(V)$ and $\mathrm{SO}(V)$, where $\dim V$ is even. For $\dim V$ odd, there exists an isogeny $\mathrm{SO}(V) \to \mathrm{Sp}(V/V^{\perp})$, so with Lemma 2.1.2 we will also get a classification in this case.

For the rest of this section, let $G = \mathrm{Sp}(V)$, where $\dim V$ is even. Let $(-,-)$ be the G-invariant alternating bilinear form on V and let $Q : V \to K$ be a non-degenerate quadratic form which has polarization $(-,-)$. We denote by $\mathrm{O}(V)$ the group of linear maps on V that stabilize the quadratic form Q. Now $\mathrm{O}(V) < G$, and the following result shows that a classification of the unipotent conjugacy classes of G gives a corresponding classification for $\mathrm{O}(V)$.

Theorem 2.4.1. (i) *The group $\mathrm{O}(V)$ intersects every conjugacy class of G.*

(ii) *Let $g, g' \in \mathrm{O}(V)$. Then g and g' are conjugate in $\mathrm{O}(V)$ if and only if they are conjugate in G.*

(iii) *Let $u \in \mathrm{SO}(V)$ be a unipotent element. Then u is distinguished in G if and only if u is distinguished in $\mathrm{SO}(V)$.*

Proof. Statements (i) and (ii) follow from [Dye79, Theorem 4, Theorem 5]. Statement (iii) follows from [LS12, Lemma 6.15 (ii)]. □

We will be using the classification of unipotent conjugacy classes of G as given by Hesselink in [Hes79]. Roughly, the result of Hesselink states that the conjugacy class of a unipotent element of G is determined by its Jordan block sizes on V and some additional data associated with each Jordan block size. In particular, it turns out (in contrast to the odd characteristic case) that the Jordan block sizes are not enough to determine whether a unipotent element is distinguished or not. For example, suppose that $\dim V = 2d$ where d is even. Then it follows from the results described below that G has exactly two conjugacy classes of unipotent elements which act on V with two Jordan blocks of size d. One of the classes consists of distinguished unipotent elements, and the other one consists of non-distinguished unipotent elements.

We now describe the main result of Hesselink. Let $u \in G$ be unipotent and suppose that the Jordan block sizes of u acting on V are $1 \leq d(1) < d(2) < \cdots < d(t)$, with block size $d(i)$ occurring with multiplicity $n_i \geq 1$. Set $X = u - 1$. For $n \geq 0$, define a quadratic form $\alpha_n : V \to K$ by $\alpha_n(v) = (X^{n+1}v, X^n v)$ for all $v \in V$. Define the *index function* of V as the map $\chi_V : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$, where

$$\chi_V(m) = \min\{n \geq 0 : X^m v = 0 \text{ implies } \alpha_n(v) = 0\}$$

for all $m \geq 0$. Then by [Hes79, Theorem 3.8] the conjugacy class of u in G is uniquely determined by the $d(i)$, n_i and the function χ_V. In fact, the conjugacy class of u is determined by the values of χ_V on the $d(i)$.

Proposition 2.4.2 ([Hes79, 3.9])**.** *The conjugacy class of u in G is uniquely determined by the symbol $(d(1)^{n_1}_{\chi_V(d(1))}, d(2)^{n_2}_{\chi_V(d(2))}, \ldots, d(t)^{n_t}_{\chi_V(d(t))})$.*

To describe when u is a distinguished unipotent element in G, we will use the description of Liebeck and Seitz given in [LS12, Chapter 6].

Definition 2.4.3. Let W be a $K[u]$-module W with a u-invariant form $(-,-)$. We say that W is *orthogonally indecomposable*, if W cannot be written as a orthogonal direct sum of two non-zero $K[u]$-submodules.

Now as a $K[u]$-module, V decomposes into an orthogonal direct sum of orthogonally indecomposable modules. In [Hes79], Hesselink classifies the orthogonally indecomposable summands that can occur. They fall into two families which are labeled by integers, we denote them here by $V(m)$ with $m \geq 2$ even, and by $W(m)$ with $m \geq 1$.

We will not go into any detail in this section on how these indecomposables are defined or classified. For more details and for a construction of $V(m)$ and $W(m)$, see for example [LS12, Chapter 6.1]. For now it will be enough to know the following facts.

- The indecomposable $V(m)$ has dimension m and u acts on $V(m)$ with a single Jordan block of size m.
- The indecomposable $W(m)$ has dimension $2m$, and $W(m) = W_1 \oplus W_2$ as a $K[u]$-module, where W_i are totally singular and u acts on W_i with a single Jordan block of size m.

It turns out that the decomposition of V into an orthogonal direct sum of submodules of the form $V(m)$ and $W(m)$ determines the conjugacy class of u in G. More specifically, we have the the following:

Proposition 2.4.4 ([LS12, Proposition 6.1, Lemma 6.2, Proposition 6.22])**.** *Let $u \in G$ be unipotent. Then there is an orthogonal decomposition*

$$V \downarrow K[u] = \sum_{i=1}^{t} W(m_i)^{a_i} \oplus \sum_{j=1}^{s} V(2k_j)^{b_j}$$

with $b_j \leq 2$ for all j and $1 \leq m_1 < \cdots < m_t$, and $1 \leq k_1 < \cdots < k_s$. Furthermore, we have the following:

(i) *The summands occurring in the decomposition are unique and determine the conjugacy class of u in G.*

(ii) *The element u is distinguished in G if and only if $V \downarrow K[u] = \sum_{j=1}^{s} V(2k_j)^{b_j}$.*

(iii) *If $u \in \mathrm{O}(V)$, then $u \in \mathrm{SO}(V)$ if and only if $\sum_j b_j$ is even.*

(iv) *If $u \in \mathrm{SO}(V)$, then the conjugacy class $u^{\mathrm{O}(V)}$ consists of a single $\mathrm{SO}(V)$-class, unless $V \downarrow K[u] = \sum_{i=1}^{t} W(m_i)^{a_i}$, in which case $u^{\mathrm{O}(V)}$ splits into two $\mathrm{SO}(V)$-classes.*

(v) *The element u is regular in G if and only if $V \downarrow K[u] = V(2k)$.*

(vi) *If $u \in \mathrm{SO}(V)$, then u is regular in $\mathrm{SO}(V)$ if and only if $V \downarrow K[u] = V(2k) + V(2)$.*

Finally, we will explain how to translate between the descriptions of the conjugacy classes given in Proposition 2.4.2 and Proposition 2.4.4.

Lemma 2.4.5. *Let $u \in G$ be unipotent, and suppose that*

$$V \downarrow K[u] = \sum_{i=1}^{t} W(m_i)^{a_i} \oplus \sum_{j=1}^{s} V(2k_j)^{b_j}$$

with $b_j \leq 2$ for all j and $1 \leq m_1 < \cdots < m_t$, and $1 \leq k_1 < \cdots < k_s$. Let $1 \leq d(1) < d(2) < \cdots < d(t')$ be the Jordan block sizes of u acting on V. Then for all i, we have the following.

(i) *If $m = d(i)$ is odd, then $\chi_V(m) = \frac{m-1}{2}$.*

(ii) *If $m = d(i)$ is even, then $\chi_V(m) = \frac{m}{2}$ if $V(m)$ occurs as an orthogonal direct summand of $V \downarrow K[u]$.*

(iii) *If $m = d(i)$ is even, then $\chi_V(m) = \frac{m-2}{2}$ if $V(m)$ does not occur as an orthogonal direct summand of $V \downarrow K[u]$.*

Proof. Write $V \downarrow K[u] = W_1 \oplus \cdots \oplus W_r$ as an orthogonal direct sum of orthogonally indecomposable $K[u]$-modules, so now each summand is equal to some $W(m_i)$ $(1 \leq i \leq t)$ or $V(2k_j)$ $(1 \leq j \leq s)$.

Let $m = d(i)$. By [Hes79, 3.5], we have the following:

$$\chi_{V(m)}(m) = \frac{m}{2}, \qquad \text{if } m \text{ is even.}$$
$$\chi_{W(m)}(m) = \frac{m-2}{2}, \qquad \text{if } m \text{ is even.}$$
$$\chi_{W(m)}(m) = \frac{m-1}{2}, \qquad \text{if } m \text{ is odd.}$$

Furthermore, by [Hes79, 3.9 (b)] for all $1 \leq i \leq r$ we have $0 \leq \chi_{W_i}(m) \leq \frac{1}{2}m$, with equality if and only if m is even and $W_i = V(m)$. Now our claim about the value of $\chi_V(m)$ follows from the fact that by [Hes79, 3.7 (c)] we have $\chi_V(m) = \max_{1 \leq i \leq r}\{\chi_{W_i}(m)\}$. □

Following [Spa82, I.2.6, pg. 20] we define a map ε on the Jordan block sizes d of a unipotent element $u \in \mathrm{Sp}(V)$.

Definition 2.4.6. Let d be a Jordan block size of $u \in \mathrm{Sp}(V)$. We define $\varepsilon(d) \in \{0, 1\}$ as follows:

- If d is odd, then $\varepsilon(d) = 0$.
- If d is even, then $\varepsilon(d) = 0$ if $V(d)$ does not occur as an orthogonal direct summand of $V \downarrow K[u]$. (By Lemma 2.4.5, this is equivalent to $\chi_V(d) = \frac{d-2}{2}$).
- If d is even, then $\varepsilon(d) = 1$ if $V(d)$ occurs as an orthogonal direct summand of $V \downarrow K[u]$. (By Lemma 2.4.5, this is equivalent to $\chi_V(d) = \frac{d}{2}$).

As an immediate corollary of Theorem 2.4.2, Lemma 2.4.5, and Proposition 2.4.4, we get the following.

Corollary 2.4.7. *Let $u \in G$ be unipotent and suppose that the Jordan block sizes of u acting on V are $1 \leq d(1) < d(2) < \cdots < d(t)$, with block size $d(i)$ occurring with multiplicity $n_i \geq 1$. Then*

(i) *The conjugacy class of u in* $\mathrm{Sp}(V)$ *is uniquely determined by the symbol* $(d(1)^{n_1}_{\varepsilon(d(1))}, d(2)^{n_2}_{\varepsilon(d(2))}, \ldots, d(t)^{n_t}_{\varepsilon(d(t))})$.

(ii) *The element u is distinguished in* $\mathrm{Sp}(V)$ *if and only if* $n_i \leq 2$ *and* $\varepsilon(d(i)) = 1$ *for all i.*

In later sections, we will label the conjugacy classes in $\mathrm{Sp}(V)$ by the decomposition $V \downarrow K[u]$ given by Proposition 2.4.4, or by the symbols given by Corollary 2.4.7 (i). By Theorem 2.4.1, we can use the same labeling for unipotent conjugacy classes of $\mathrm{O}(V)$. Furthermore, by Proposition 2.4.4 (iv), for distinguished $\mathrm{SO}(V)$-classes we can use the same labeling without ambiguity. Note also that by Lemma 2.1.2 (i), we can (and will) also use this labeling for any simple group of type C_l or D_l.

To end this section, we give a lemma which is useful for computing the values $\varepsilon(d(i))$ defined above.

Lemma 2.4.8. *Let $u \in$ $\mathrm{Sp}(V)$ be unipotent and suppose that the Jordan block sizes of u acting on V are $1 \leq d(1) < d(2) < \cdots < d(t)$, with block size $d(i)$ occurring with multiplicity $n_i \geq 1$. Fix $m = d(i)$ and set $X = u - 1$. Then:*

(i) *If m is odd, then $\varepsilon(m) = 0$.*

(ii) *Assume that m is even. Let $v_1, \ldots, v_r$ be a basis of* $\mathrm{Ker}\, X^m$. *Then the following are equivalent:*

  (a) $\varepsilon(m) = 0$.

  (b) $(X^{m-1}v, v) = 0$ *for all* $v \in \mathrm{Ker}\, X^m$.

  (c) $(X^{m-1}v_i, v_i) = 0$ *for all* $1 \leq i \leq r$.

Proof. Claim (i) follows from the definition of ε.

For (ii), suppose that m is even. By Lemma 2.4.5, we have $\varepsilon(m) = 0$ if and only if $\chi_V(m) = \frac{m-2}{2}$. Since $\chi_V(m) \geq \frac{m-2}{2}$ by Lemma 2.4.5, it follows that $\varepsilon(m) = 0$ if and only if $\alpha_{\frac{m-2}{2}}(v) = 0$ for all $v \in \mathrm{Ker}\, X^m$. Thus to prove that (a) and (b) are equivalent, it will be enough to show that $(X^{m-1}v, v) = \alpha_{\frac{m}{2}-1}(v)$ for all $v \in \mathrm{Ker}\, X^m$.

To this end, let $v, w \in \mathrm{Ker}\, X^m$. It is a consequence of [Spa82, Lemme II.6.10, pg. 99] (with $e_i = X^{m-i}v$ and $f_j = X^{m-j}w$ there) that for all $1 \leq i, j \leq m-1$ such that $i + j = m$, we have

$$(X^{i-1}v, X^j w) + (X^i v, X^{j-1}w) = 0. \tag{*}$$

Applying (*) with $i = 1$, $i = 2$, ..., $i = m-1$, we get

$$\begin{array}{ll}
i = 1 & (v, X^{m-1}w) = (Xv, X^{m-2}w) \\
i = 2 & (Xv, X^{m-2}w) = (X^2v, X^{m-3}w) \\
\vdots & \vdots \\
i = k & (X^{k-1}v, X^{m-k}w) = (X^k v, X^{m-(k+1)}w) \\
\vdots & \vdots \\
i = m-1 & (X^{m-2}v, Xw) = (X^{m-1}v, w)
\end{array}$$

It follows that for all $v, w \in \operatorname{Ker} X^m$ we have $(X^{m-1}v, w) = (X^i v, X^{m-i-1}w)$ for all $1 \leq i \leq m-1$. With $i = m/2$, this shows that

$$(X^{m-1}v, v) = (X^{\frac{m}{2}}v, X^{\frac{m}{2}-1}v) = \alpha_{\frac{m}{2}-1}(v)$$

for all $v \in \operatorname{Ker} X^m$. Hence (a) and (b) are equivalent.

It is trivial that (b) implies (c). To show that (c) implies (b), fix a basis $v_1, \ldots, v_r$ of $\operatorname{Ker} X^m$. Applying (*) with $i = m-1$, we get $(X^{m-1}v_j, v_{j'}) + (X^{m-1}v_{j'}, v_j) = 0$ for all $1 \leq j, j' \leq r$. Thus for $v = \sum_{j=1}^r c_j v_j \in \operatorname{Ker} X^m$ we get $(X^{m-1}v, v) = \sum_{j=1}^r c_j^2 (X^{m-1}v_j, v_j)$. Hence $(X^{m-1}v, v) = 0$ if $(X^{m-1}v_j, v_j) = 0$ for all $1 \leq j \leq r$. □

Remark 2.4.9. Following the results above, it is straightforward to implement a computer program that determines the conjugacy class of $u \in G$, when given the matrix of u and the alternating G-invariant bilinear form $(-,-)$ with respect to some basis of V. Some of our computer calculations (see Section 2.9) are based on the use of such a program.

Indeed, calculating with ranks of powers of $X = u - 1$ allows one to find very quickly the Jordan block sizes of u acting on V (see Lemma 3.1.2). Then for each Jordan block size d of u, one needs to calculate $\varepsilon(d)$. This is done by applying Lemma 2.4.8. If d is odd, then $\varepsilon(d) = 0$. If d is even, we compute a basis $v_1, \ldots, v_r$ for $\operatorname{Ker} X^d$. Then by Lemma 2.4.8 (ii) we have $\varepsilon(d) = 0$ if and only if $(X^{d-1}v_i, v_i) = 0$ for all $1 \leq i \leq r$.

2.5 Explicit distinguished unipotent elements ($p \neq 2$)

Assume that $p \neq 2$.

The purpose of this section is to describe representatives for distinguished unipotent conjugacy classes in the classical groups $\operatorname{Sp}(V)$ and $\operatorname{SO}(V)$, in terms of their action on the natural module V.

Note first that it will be enough to construct unipotent elements acting on V with a single Jordan block. Indeed, consider the case $G = \operatorname{Sp}(V)$. Then a distinguished unipotent element $u \in G$ should act on V with Jordan block sizes $d_1, \ldots, d_t$ where d_i are distinct even integers such that $\sum_{i=1}^t d_i = \dim V$ (Proposition 2.3.4 (iii)). We can decompose V into a orthogonal direct sum $V = W_1 \oplus \cdots \oplus W_t$, where each W_i is a non-degenerate subspace and $\dim W_i = d_i$. Now if $u_i \in \operatorname{Sp}(W_i)$ acts W_i with a single Jordan block of size d_i, then $u = u_1 \oplus \cdots \oplus u_t \in \operatorname{Sp}(V)$ acts on V with Jordan blocks $[d_1, \ldots, d_t]$. In precisely the same way, we see that it will be enough to construct (for $\dim V$ odd) unipotent elements $u \in \operatorname{SO}(V)$ acting on V with a single Jordan block.

For the construction of unipotent elements with a single Jordan block, we proceed as in [Jan04, 1.7]. Let $d > 0$ and let V be a vector space of dimension d with basis $e_1, e_2, \ldots, e_d$. Define a bilinear form $(-,-)$ on V by

$$(e_i, e_j) = \begin{cases} (-1)^i & \text{if } i + j = d + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(-,-)$ is non-degenerate, and

$$(-,-) \text{ is } \begin{cases} \text{symmetric,} & \text{if } d \text{ is odd,} \\ \text{alternating,} & \text{if } d \text{ is even.} \end{cases}$$

Define a linear map $X : V \to V$ by $Xe_i = -e_{i-1}$, where we set $e_j = 0$ for $j < 0$.

Then X is skew with respect to the form $(-,-)$, that is, $(Xv, w)+(v, Xw) = 0$ for all $v, w \in V$. Now X is a nilpotent element, so the Cayley transform $u = (1-X)(1+X)^{-1}$ is a unipotent linear map which leaves the form $(-,-)$ invariant [Wey39, Lemma 2.10.A, pg. 57]. That is, $(uv, uw) = (v, w)$ for all $v, w \in V$. Furthermore, the Cayley transform of u is equal to $(1-u)(1+u)^{-1} = X$, so we have $(u-1)e_i = (u+1)e_{i-1}$ for all $1 \leq i \leq n$. From this it is straightforward to see that for all $1 \leq i \leq d$, we have

$$ue_i = e_i + 2\sum_{j<i} e_j$$

and that u acts on V with a single Jordan block.

Therefore the unipotent element u defined by $(u-1)e_i = (u+1)e_{i-1}$ works as a representative for the unipotent class of a single Jordan block. If d is odd, then $(-,-)$ is symmetric and $u \in \mathrm{SO}(V)$, where $\mathrm{SO}(V)$ is the stabilizer of $(-,-)$. If d is even, then $(-,-)$ is alternating and $u \in \mathrm{Sp}(V)$, where $\mathrm{Sp}(V)$ is the stabilizer of $(-,-)$.

2.6 Unipotent elements and labeled Dynkin diagrams (p good)

Assume that G is a connected semisimple algebraic group, and that p is good for G.

Definition 2.6.1. We say that a parabolic subgroup $P \leq G$ with Levi factor L and unipotent radical Q is *distinguished* if $\dim L = \dim Q/[Q, Q]$. If $P = P_J$ with $J \subseteq \Delta$, the *labeled Dynkin diagram associated with* P is defined by labeling $\alpha \in \Delta$ with 0 if $\alpha \in J$ and with 2 if $\alpha \notin J$.

Definition 2.6.2. Let P be a parabolic subgroup of G. We say that a unipotent element $u \in R_u(P)$ is a *Richardson element for* P if the P-conjugacy class of u is dense in $R_u(P)$.

In good characteristic, there is a uniform way of classifying the unipotent conjugacy classes in G. Roughly, this is done by establishing a correspondence between the unipotent elements of G and the Richardson elements of distinguished parabolic subgroups of Levi factors of G. In this section we will describe this correspondence for distinguished unipotent elements of G, and the correspondence between distinguished unipotent elements and labeled Dynkin diagrams. In general, we note that each unipotent conjugacy class (not just distinguished) has a labeled Dynkin diagram associated to it. But for the purposes of the present work, we will only need to know what the labeled Dynkin diagrams of distinguished unipotent elements look like.

The distinguished parabolic subgroups and their labelings were determined by Bala and Carter in [BC76a] and [BC76b]. The following theorem shows that in good characteristic there is a bijection between distinguished unipotent classes and distinguished parabolic subgroups[6].

Theorem 2.6.3 (Bala-Carter Theorem)**.** *Let $u \in G$ be a distinguished unipotent element. Then there exists a unique distinguished parabolic subgroup $P < G$ such that u is a Richardson element for P.*

In view of Theorem 2.6.3, the following definition makes sense.

Definition 2.6.4. Let $u \in G$ be a distinguished unipotent element. Let $P = P_J$, $J \subseteq \Delta$ be a distinguished parabolic corresponding to u in Theorem 2.6.3. Then the *labeled Dynkin diagram associated with* u is defined to be the labeled Dynkin diagram associated with P.

In good characteristic, the conjugacy classes of distinguished unipotent elements in classical groups ($\mathrm{SL}(V)$, $\mathrm{Sp}(V)$ and $\mathrm{SO}(V)$) are determined by Jordan block sizes (Propositions 2.3.1, 2.3.4 and 2.3.2). Using the Jordan block sizes, there is a straightforward method for computing the associated labeled Dynkin diagram. We record this in the following proposition, which is given as in [LS12, Theorem 3.18]. Recall that we use the standard Bourbaki labeling of the simple roots, as given in [Hum72, 11.4, pg. 58].

Proposition 2.6.5. *Let $G = \mathrm{SL}(V)$, $\mathrm{Sp}(V)$, or $\mathrm{SO}(V)$ and let $u \in G$ be a distinguished unipotent element. Suppose that u acts on V with Jordan block sizes $[d_1, \ldots, d_t]$, where $t \geq 1$ and $1 \leq d_1 < \cdots < d_t$. Then the labeled Dynkin diagram of u is given as follows.*

(i) *If $G = \mathrm{SL}(V)$ (type A), then $t = 1$, u is a regular unipotent and the labeling is $22\ldots2$.*

(ii) *If $G = \mathrm{Sp}(V)$ (type C), the labeling is given as follows. If $t = 1$ (i.e. u is regular), then the labeling is $22\ldots2$. If $t > 1$, then starting from the left hand side of the Dynkin diagram, begin with $(d_t - d_{t-1} - 2)/2$ labels 2; then $(d_{t-1} - d_{t-2})/2$ sequences 20; then $(d_{t-2} - d_{t-3})/2$ sequences 200; and so on, until we get to $d_1/2$ sequences $20\ldots0$ ($t-1$ zeros); and finally label 2 on the last node.*

(iii) *If $G = \mathrm{SO}(V)$ with $\dim V$ odd (type B), the labeling is given as follows. If $t = 1$ (i.e. u is regular), then the labeling is $22\ldots2$. If $t > 1$, then starting from the left hand side of the Dynkin diagram, begin with $(d_t - d_{t-1} - 2)/2$ labels 2; then $(d_{t-1} - d_{t-2})/2$ sequences 20; then $(d_{t-2} - d_{t-3})/2$ sequences 200; and so on, until we get to $(d_1 - 1)/2$ sequences $20\ldots0$ ($t-1$ zeros); and finally label the rest of the diagram with $20\ldots0$ ($(t-1)/2$ zeros).*

(iv) *If $G = \mathrm{SO}(V)$ with $\dim V$ even (type D), the labeling is given as follows. Starting from the left hand side of the Dynkin diagram, begin with $(d_t - d_{t-1} - 2)/2$ labels 2; then $(d_{t-1} - d_{t-2})/2$ sequences 20; then $(d_{t-2} - d_{t-3})/2$ sequences 200; and so on, until we get to $(d_1 - 1)/2$ sequences $20\ldots0$ ($t-1$ zeros); finally if $t > 2$ label the rest of the diagram with $20\ldots0$ ($t/2$ zeroes); if $t = 2$ label the last two nodes 22.*

[6] Theorem 2.6.3 was originally proven by Bala and Carter in the case where $p = 0$ or p large. In [Pom77] and [Pom80], Pommerening extended the results of Bala and Carter to good characteristic. The approach of Pommerening was based on case-by-case analysis. Later Premet [Pre03] gave the first uniform proof, which was simplified by Tsujii in [Tsu08].

Remark 2.6.6. According to Proposition 2.6.5, for classical groups the labeled Dynkin diagrams of distinguished unipotent elements begin with a string of 2s, then some number of 20s, then some number of 200s; and so on, until the last one or two nodes for which the pattern depends on the type of the group and the unipotent class. We call the string of 2s in the beginning the *initial string.*

For distinguished unipotent classes in exceptional groups, we name them with the usual Bala-Carter label of the associated distinguished parabolic subgroup, as given in [BC76b]. For each of these classes, we record the associated labeled Dynkin diagrams in the following proposition. See for example [Car85, 5.9, pg. 175-177] or [LS12, Table 2.2, pg. 25].

Proposition 2.6.7. *Let G be simple of exceptional type and $u \in G$ a distinguished unipotent element. If u is regular, then the labeled Dynkin diagram of u is $22\dots2$. If u is not regular, then the labeled Dynkin diagram of u is as in Table 2.1.*

Type	Unipotent class	Labeled diagram
G_2	$G_2(a_1)$	02
F_4	$F_4(a_1)$	2202
F_4	$F_4(a_2)$	0202
F_4	$F_4(a_3)$	0200
E_6	$E_6(a_1)$	222022
E_6	$E_6(a_3)$	200202
E_7	$E_7(a_1)$	2220222
E_7	$E_7(a_2)$	2220202
E_7	$E_7(a_3)$	2002022
E_7	$E_7(a_4)$	2002002
E_7	$E_7(a_5)$	0002002
E_8	$E_8(a_1)$	22202222
E_8	$E_8(a_2)$	22202022
E_8	$E_8(a_3)$	20020222
E_8	$E_8(a_4)$	20020202
E_8	$E_8(b_4)$	20020022
E_8	$E_8(a_5)$	20020020
E_8	$E_8(b_5)$	00020022
E_8	$E_8(a_6)$	00020020
E_8	$E_8(b_6)$	00020002
E_8	$E_8(a_7)$	00002000

Table 2.1: Labeled diagrams of non-regular distinguished unipotent elements for simple groups of exceptional type.

Following [LT99, pg. 7], we define the *labeled diagram* of a simple subgroup $A < G$ of type A_1 as follows. Let $A < G$ be a simple subgroup of type A_1. Now there exists a surjective morphism $\phi : \mathrm{SL}_2(K) \to A$ of algebraic groups. Set

$$T_A = \left\{ \phi \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} : \mu \in K^* \right\}.$$

By replacing A with a conjugate, we can assume that T_A is contained in the chosen maximal torus T of G. For each root $\alpha \in \Phi$, let c_α be the integer such that

$$\alpha\left(\phi\begin{pmatrix}\mu & 0\\ 0 & \mu^{-1}\end{pmatrix}\right) = c_\alpha.$$

One can now choose the base Δ in a way such that $c_\alpha \geq 0$ for all $\alpha \in \Phi^+$ [LT99, Proposition 2.3]. We then define a *labeled diagram* of A to be the Dynkin diagram of G with node $\alpha \in \Delta$ labeled by c_α.

The following theorem is a key tool for the proof of Theorem 1.1.10 for unipotent elements of order p.

Theorem 2.6.8 ([LT99, Theorem 4.2])**.** *Let $u \in G$ be a unipotent element of order p. Then there exists a subgroup $A < G$ of type A_1 such that $u \in A$ and a labeled diagram of A is that of u.*

Now let $u \in G$ be a unipotent element of order p and let $u \in A$, where $A < G$ is as in Theorem 2.6.8. In this setting, one can study the action of $u \in G$ on a representation V of G by considering the restriction $V \downarrow A$. Indeed, here knowledge of the labeled diagram of u tells how T-weights restrict to T_A-weights. Therefore one gets information on the composition factors of $V \downarrow A$, which can be used to gain information on the action of u on V. This is explored in more detail in Section 4.6 and Section 5.13.

2.7 Largest Jordan block size of a unipotent element

Let $u \in G$ be a unipotent element and let V be a G-module, afforded by $\varphi : G \to \mathrm{GL}(V)$. Denote by d_u the largest Jordan block size of $\varphi(u)$. The purpose of this section is to describe various upper bounds which are known for d_u. The motivation for this is the fact that if u acts on V as a distinguished unipotent element, then we get an upper bound on $\dim V$ that is a quadratic polynomial in d_u. This will be key in our solution of Problem 1.1.9, and will be explained in sections 5.1 and 5.2.

For basic upper bounds on d_u, the following elementary observation will be useful, as found for example in [Sup95, Lemma 2.2, Lemma 2.15].

Lemma 2.7.1. *Let $u \in G$ be a unipotent element. Then*

(i) $d_u \leq |u|$,

(ii) *If $u' \in G$ is a regular unipotent element, then $d_u \leq d_{u'}$.*

Proof. Let $f : V \to V$ be any unipotent linear map on V. For any $k \geq 1$, we have $(f-1)^{p^k} = f^{p^k} - 1$. Therefore $(f-1)^{p^\alpha} = 0$ for $|f| = p^\alpha$, and thus the largest Jordan block size of f is $\leq p^\alpha$. This gives (i).

For (ii), note that if $u' \in G$ is any unipotent element such that the conjugacy class u^G is contained in the Zariski closure of $(u')^G$, then $d_u \leq d_{u'}$ since $\{f \in \mathrm{GL}(V) : (f-1)^{d_{u'}} = 0\}$ is Zariski closed in $\mathrm{GL}(V)$. Thus (ii) follows from the fact that the regular unipotent class is dense in the unipotent variety of G. □

Definition 2.7.2. Suppose that p is good for G, let $u \in G$ be a unipotent element, and let $\lambda \in X(T)^+$. Write the dominant weight λ as $\lambda = \sum_{i=1}^{l} q_i\alpha_i$, where $q_i \in \mathbb{Q}^+$

for all $1 \leq i \leq l$ (this is possible for example by [Hum72, Section 13, Table 1]). Define

$$m_u(\lambda) = \sum_{i=1}^{l} d_i q_i,$$

where d_i is the label of α_i in the labeled Dynkin diagram of u.

If u is regular, then $m_u(\lambda) = 2\sum_{i=1}^{l} q_i$ in Definition 2.7.2. Note also that for any unipotent element $u \in G$, we have $m_u(\lambda + \mu) = m_u(\lambda) + m_u(\mu)$ for all $\lambda, \mu \in X(T)^+$.

For the rest of this section we give some properties of $m_u(\lambda)$, mostly in the case where u is a regular unipotent element.

Lemma 2.7.3. *Write $\lambda = \sum_{i=1}^{l} c_i\omega_i$, where ω_i are the fundamental dominant weights with the usual Bourbaki labeling. For a regular unipotent u, the values of $m_u(\lambda)$ for each irreducible root system are as in the table below.*

Φ	$m_u(\lambda)$
A_l	$\sum_{i=1}^{l} i(l+1-i)c_i$
B_l	$\sum_{i=1}^{l-1} i(2l+1-i)c_i + \frac{l(l+1)}{2}c_l$
C_l	$\sum_{i=1}^{l} i(2l-i)c_i$
D_l	$\sum_{i=1}^{l-2} i(2l-i-1)c_i + \frac{l(l-1)}{2}c_{l-1} + \frac{l(l-1)}{2}c_l$
G_2	$6c_1 + 10c_2$
F_4	$22c_1 + 42c_2 + 30c_3 + 16c_4$
E_6	$16c_1 + 22c_2 + 30c_3 + 42c_4 + 30c_5 + 16c_6$
E_7	$34c_1 + 49c_2 + 66c_3 + 96c_4 + 75c_5 + 52c_6 + 27c_7$
E_8	$92c_1 + 136c_2 + 182c_3 + 270c_4 + 220c_5 + 168c_6 + 114c_7 + 58c_8$

Proof. Write $\lambda = \sum_{i=1}^{l} c_i\omega_i$. Then $m_u(\lambda) = \sum_{i=1}^{l} c_i m_u(\omega_i)$ since $m_u(\lambda_1 + \lambda_2) = m_u(\lambda_1) + m_u(\lambda_2)$. Using the values of the q_i for the fundamental weights given in [Hum72, Section 13, Table 1], it is easy to compute $m_u(\omega_i)$ for each irreducible root system. We omit this computation. □

Lemma 2.7.4. *Let $u \in G$ be a regular unipotent element. Then u has order p if and only if $p \nmid m_u(\omega_i)$ for all $1 \leq i \leq l$.*

Proof. Now there exists a number $M_G \geq 0$, depending only on the type of G, such that a regular unipotent element $u \in G$ has order p if and only if $p \geq M_G$. For G of type A_l, B_l, C_l or D_l we have $M_G = l+1$, $M_G = 2l+1$, $M_G = 2l$ and $M_G = 2l-1$ respectively (Proposition 2.3.3).

If G has type A_l, we have $m_u(\omega_i) = i(l+1-i)$ by Lemma 2.7.3. Therefore $p \nmid m_u(\omega_i)$ for all $1 \leq i \leq l$ if and only if $p \nmid i$ for all $1 \leq i \leq l$, which is equivalent to $p \geq l+1$. If G has type B_l, C_l, or D_l, a similar application of Lemma 2.7.3 shows that $p \nmid m_u(\omega_i)$ for all $1 \leq i \leq l$ if and only if $p \geq 2l+1$, $p \geq 2l$ and $p \geq 2l-1$, respectively.

Suppose then that G is simple of exceptional type. If G has type G_2, F_4, E_6, E_7, or E_8, then we have $M_G = 7$, $M_G = 13$, $M_G = 13$, $M_G = 19$, and $M_G = 31$ respectively (Appendix A). For each exceptional type, using Lemma 2.7.3, one can easily verify that the prime factors occurring in $m_u(\omega_i)$ are precisely all the primes $< M_G$. Therefore $p \geq M_G$ if and only if $p \nmid m_u(\omega_i)$ for all $1 \leq i \leq \operatorname{rank} G$, as desired. For convenience we have listed the prime factors of the $m_u(\omega_i)$ in Table 2.1. □

	G_2	F_4	E_6	E_7	E_8
ω_1	2, 3	2, 11	2	2, 17	2, 23
ω_2	2, 5	2, 3, 7	2, 11	7	2, 17
ω_3		2, 3, 5	2, 3, 5	2, 3, 11	2, 7, 13
ω_4		2	2, 3, 7	2, 3	2, 3, 5
ω_5			2, 3, 5	3, 5	2, 5, 11
ω_6			2	2, 13	2, 3, 7
ω_7				3	2, 3, 19
ω_8					2, 29

Table 2.1: Prime factors occurring in $m_u(\omega_i)$, where u is a regular unipotent element in G of exceptional type.

The following lemma which has some useful implications for the representation theory of G. Recall that $\delta \in X(T)^+$ denotes the half-sum of the positive roots in Φ, and that $\delta = \sum_{i=1}^{l} \omega_i$.

Lemma 2.7.5. *Let $u \in G$ be a regular unipotent element. Then the inequality $m_u(\lambda) \geq \langle \lambda + \delta, \alpha \rangle - 1$ holds for all $\lambda \in X(T)^+$ and $\alpha \in \Phi^+$.*

Proof. Let $\lambda \in X(T)^+$ and $\alpha \in \Phi^+$. Write $\lambda = \sum_{i=1}^{l} c_i\omega_i$ and $\alpha = \sum_{i=1}^{l} k_i\alpha_i$, where c_i, k_i are nonnegative integers. Then

$$\langle \lambda + \delta, \alpha \rangle = \sum_{i=1}^{l} (c_i + 1)k_i \frac{2(\omega_i, \alpha_i)}{(\alpha, \alpha)} = \sum_{i=1}^{l} (c_i + 1)k_i \frac{(\alpha_i, \alpha_i)}{(\alpha, \alpha)} = \sum_{i=1}^{l} (c_i + 1)t_i$$

where $t_i = k_i \frac{(\alpha_i,\alpha_i)}{(\alpha,\alpha)}$. Note that $\alpha^\vee = \sum_{i=1}^{l} t_i\alpha_i^\vee$, so by the equalities above $\langle \lambda + \delta, \alpha \rangle$ gets its largest value when $\alpha^\vee$ is the highest root of the dual root system $\Phi^\vee$ with respect to the base $\Delta^\vee$ [Hum72, Lemma 10.4A]. In other words, $\langle \lambda + \delta, \alpha \rangle$ gets its largest value when α is the highest short root β of Φ with respect to the base Δ. Therefore it will be enough to show that $m_u(\lambda) \geq \langle \lambda + \delta, \beta \rangle - 1$.

According to [Ser94, Proposition 5], we have

$$\langle \lambda + \delta, \beta \rangle \leq 1 + \sum_{\alpha \in \Phi^+} \langle \lambda, \alpha \rangle.$$

Hence it will be enough to show that $\sum_{\alpha \in \Phi^+} \langle \lambda, \alpha \rangle = m_u(\lambda)$. To this end, write $\lambda = \sum_{i=1}^{l} q_i\alpha_i$, where $q_i \in \mathbb{Q}^+$. Then

$$\sum_{\alpha \in \Phi^+} \langle \lambda, \alpha \rangle = \sum_{i=1}^{l} q_i \sum_{\alpha \in \Phi^+} \langle \alpha_i, \alpha \rangle$$

so it will be enough to prove that $\sum_{\alpha\in\Phi^+}\langle\alpha_i,\alpha\rangle = 2$. That is, it suffices to prove that $\sum_{\alpha\in\Phi^+-\{\alpha_i\}}\langle\alpha_i,\alpha\rangle = 0$. Now α_i is a simple root, so the reflection σ_{α_i} permutes the positive roots other than α_i [Hum72, Lemma 10.2B]. Hence

$$\sum_{\alpha\in\Phi^+-\{\alpha_i\}}\langle\alpha_i,\alpha\rangle = \sum_{\alpha\in\Phi^+-\{\alpha_i\}}\langle\sigma_{\alpha_i}(\alpha_i),\sigma_{\alpha_i}(\alpha)\rangle = \sum_{\alpha\in\Phi^+-\{\alpha_i\}}\langle-\alpha_i,\alpha\rangle,$$

which gives $\sum_{\alpha\in\Phi^+-\{\alpha_i\}}\langle\alpha_i,\alpha\rangle = 0$. □

Now the following corollary of Lemma 2.7.5 is immediate from the Jantzen sum formula [Jan03, II.8.19].

Corollary 2.7.6. *Let $\lambda\in X(T)^+$ and let $u\in G$ be a regular unipotent element. If $p > m_u(\lambda)$, then $V_G(\lambda)$ is irreducible.*

We finish by describing a connection between $m_u(\lambda)$ and simple algebraic groups in characteristic 0. Let $G_{\mathbb{C}}$ be a simple algebraic group over $\mathbb{C}$ with the same root system Φ as G. Now the Bala-Carter classification of unipotent conjugacy classes and the results described in 2.6 are still valid for $G_{\mathbb{C}}$. Thus we can make the following definition.

Definition 2.7.7. Assume that p is good for G and let $u\in G$ be a unipotent element. We denote by $u_{\mathbb{C}}$ some representative $u_{\mathbb{C}}\in G_{\mathbb{C}}$ of a unipotent conjugacy class of $G_{\mathbb{C}}$ which has the same labeled diagram as u.

Furthermore, Theorem 2.6.8 holds for $G_{\mathbb{C}}$, so we can find a simple subgroup $X_{\mathbb{C}} < G_{\mathbb{C}}$ of type A_1 such that $u_{\mathbb{C}}\in X_{\mathbb{C}}$ and such that $X_{\mathbb{C}}$ has the same labeled diagram as $u_{\mathbb{C}}$.

The following theorem is a straightforward consequence of various results observed by Suprunenko [Sup09, Lemma 2.38, Lemma 2.39, Corollary 2.40].

Theorem 2.7.8. *Let $u\in G$ be a unipotent element and $\lambda\in X(T)^+$. Then u acts on $L_G(\lambda)$ with largest Jordan block of size at most $m_u(\lambda)+1$.*

Proof. Let $G_{\mathbb{C}}$, $u_{\mathbb{C}}$, $X_{\mathbb{C}}$ be as above. Now $L_{G_{\mathbb{C}}}(\lambda)\downarrow X_{\mathbb{C}}$ has highest weight $m_u(\lambda)$, so it follows that $u_{\mathbb{C}}$ acts on $L_{G_{\mathbb{C}}}(\lambda)$ with largest Jordan block of size $m_u(\lambda)+1$.

Now the claim follows from Lemma 2.38 in [Sup09], once we observe that Lemma 2.39 in [Sup09] holds also for exceptional groups; see for example [LS12, Table 13.3, pg. 172] and [LS12, 18.1, pg. 287]. □

We will say that the action of u on $V_G(\lambda)$ has the same *Jordan block sizes as the corresponding action in characteristic* 0, if the Jordan block sizes of u acting on $V_G(\lambda)$ are same as the Jordan block sizes of $u_{\mathbb{C}}$ acting on $V_{G_{\mathbb{C}}}(\lambda)$.

Lemma 2.7.9. *Let $u\in G$ be a unipotent element and let $\lambda\in X(T)^+$ be nonzero. If $p > m_u(\lambda)$, then the action of u on $V_G(\lambda)$ has the same Jordan block sizes as the corresponding action in characteristic* 0.

Proof. Keep the notation $G_{\mathbb{C}}$, $u_{\mathbb{C}}$, $X_{\mathbb{C}}$ as before. Assume that $p > m_u(\lambda)$. It follows now from Theorem 2.7.8 that u has order p. Thus by Theorem 2.6.8, we can find a subgroup $X < G$ of type A_1 containing u and having the same labeled diagram as u. The module $V_G(\lambda)\downarrow X$ has highest weight $m_u(\lambda)$. Since $p > m_u(\lambda)$, it follows from [AJL83, Lemma 2.2] that $V_G(\lambda)\downarrow X$ is semisimple.

Now note that $V_{G_{\mathbb{C}}}(\lambda) \downarrow X_{\mathbb{C}}$ and $V_G(\lambda) \downarrow X$ have the same character, so they have the same composition factors. Since both of them are semisimple and all composition factors of $V_G(\lambda) \downarrow X$ have highest weight strictly less p, the claim follows. □

2.8 Representatives for unipotent classes in Chevalley groups

For computer calculations, it is often convenient to consider a simple linear algebraic group G as a Chevalley group, constructed via the Chevalley construction as in [Ste68]. In this section we will describe how to construct representatives $u \in G$ for all unipotent classes of a Chevalley group G. For the purposes of this text, these representatives are needed only for the implementation of some of our computer calculations, which are described in 2.9.

We begin by recalling the Chevalley construction. Details can be found in [Ste68]. Let Φ be a root system with base Δ and set of positive roots Φ^+. Let $\mathfrak{g}_{\mathbb{C}}$ be a complex semisimple Lie algebra with root system Φ, and fix some Chevalley basis $\{X_\alpha : \alpha \in \Phi\} \cup \{H_\alpha : \alpha \in \Delta\}$ of $\mathfrak{g}_{\mathbb{C}}$. Now for all $\alpha, \beta \in \Phi$ with $\alpha + \beta \in \Phi$, we have $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$. We call $N_{\alpha,\beta}$ the *structure constants* of the Chevalley basis. Here we have $N_{\alpha,\beta} = \pm(k+1)$, where k is the largest integer such that $\beta - k\alpha \in \Phi$. Let $\mathscr{U}_{\mathbb{Z}}$ be the Kostant $\mathbb{Z}$-form with respect to this Chevalley basis. That is, $\mathscr{U}_{\mathbb{Z}}$ is the subring of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ generated by 1 and all $\frac{X_\alpha^k}{k!}$ for $\alpha \in \Phi$ and $k \geq 1$.

Let $V_{\mathbb{C}}$ be some irreducible representation of the Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Then one can find a $\mathscr{U}_{\mathbb{Z}}$-invariant lattice $V_{\mathbb{Z}}$ in $V_{\mathbb{C}}$. Define $V = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$. Since $V_{\mathbb{Z}}$ is invariant under $\mathscr{U}_{\mathbb{Z}}$, and since X_α act as nilpotent linear maps on $V_{\mathbb{C}}$, we can define $x_\alpha(c) = \exp(cX_\alpha) \in \mathrm{GL}(V)$ for all $\alpha \in \Phi$ and $c \in K$. Then a *Chevalley group over K induced by V* (which depends on the choice of $V_{\mathbb{C}}$) is the subgroup

$$G = \langle x_\alpha(c) : \alpha \in \Phi, c \in K \rangle$$

of $\mathrm{GL}(V)$.

One can show that G is a connected semisimple linear algebraic group with root system Φ. Furthermore, it is well known that any unipotent element of G is conjugate to an element of the form $\prod_{\alpha \in \Phi^+} x_\alpha(c_\alpha)$, where the product is taken with respect to some ordering of Φ^+. The following useful lemma is due to Steinberg.

Lemma 2.8.1 ([Ste65, Lemma 3.2 (c)]). *A unipotent element $\prod_{\alpha \in \Phi^+} x_\alpha(c_\alpha)$ of G is regular if and only if $c_\alpha \neq 0$ for all $\alpha \in \Delta$.*

Our goal is to find representatives of the form $\prod_{\alpha \in \Phi^+} x_\alpha(c_\alpha)$ for each unipotent conjugacy class of G. We will do this in this section, and moreover the representatives that we give will have the property that $c_\alpha \in \{-1, 0, 1\}$ for all $\alpha \in \Phi^+$. Note that by [Ste68, Corollary 5, pg. 44] and Lemma 2.1.2 (i), the conjugacy class of such a representative does not depend on the choice of the representation $V_{\mathbb{C}}$ we choose. However, the conjugacy class of an unipotent element given by the expression $\prod_{\alpha \in \Phi^+} x_\alpha(c_\alpha)$ might be ambiguous if the structure constants $N_{\alpha,\beta}$ are not specified. We explain next how we will choose the structure constants.

We define a total order (as in [Car72, 2.1]) on Φ in the following way. For $\alpha, \beta \in \Phi$, we define $\alpha \leq \beta$ if and only if $\alpha = \beta$, or $\beta - \alpha = \sum_{i=1}^{k} c_i \alpha_i$, for some

$1 \le k \le l$ such that $c_i \in \mathbb{Z}$ for all $1 \le i \le k$, and $c_k > 0$. As in [Car72, 4.2], we will say that a pair (α, β) is *special* if $\alpha, \beta \in \Phi^+$, $\alpha \le \beta$, and $\alpha + \beta \in \Phi$. A special pair (α, β) will be called *extraspecial*, if for any special pair (α', β') with $\alpha' + \beta' = \alpha + \beta$, we have $\alpha \le \alpha'$. As noted in [Car72, 4.2], each $\gamma \in \Phi^+ - \Delta$ corresponds to a unique extraspecial pair (α, β) with $\alpha + \beta = \gamma$, and this defines a bijection between $\Phi^+ - \Delta$ and the set of extraspecial pairs.

It is possible to show [Car72, Proposition 4.2.2] that the structure constants $N_{\alpha,\beta}$ of the Chevalley basis are determined by the values of $N_{\alpha,\beta}$ for extraspecial pairs (α, β). In what follows, we will always use a Chevalley basis such that the structure constants for extraspecial pairs are positive. Such a basis can always be found, since by [Car72, Proposition 4.2.2] the signs of the structure constants $N_{\alpha,\beta}$ for extraspecial pairs (α, β) can be chosen arbitrarily. For types B_l, C_l, and D_l, we will use a Chevalley basis described in [Jan73, 11, pg. 38]. For our purposes, the main reason to use structure constants which are positive for extraspecial pairs is that these are also the default structure constants used in MAGMA.

Definition 2.8.2. A *subsystem subgroup* of G is a connected semisimple subgroup of G which is normalized by a maximal torus of G.

We will see in this section that for most of the unipotent conjugacy classes of G, we can find a class representative in a proper subsystem subgroup of G; this will reduce the description of the unipotent class representatives to a relatively small number of cases. Typical examples of subsystem subgroups are given by *closed* subsets Ψ of Φ.

Definition 2.8.3. We say that $\Psi \subseteq \Phi$ is *closed*, if both of the following hold:

(i) $\Psi = -\Psi$.

(ii) For all $\alpha, \beta \in \Psi$ such that $\alpha + \beta \in \Phi$, we have $\alpha + \beta \in \Psi$.

It is well known (see for example [MT11, Theorem 13.6]) that for any closed subset $\Psi \subseteq \Phi$, we have a subsystem subgroup

$$\langle x_\alpha(c) : \alpha \in \Psi, c \in K \rangle$$

of G with root system Ψ.

Next we need to make some remarks about graph automorphisms and their action on unipotent conjugacy classes. The following result is proven in [Ste68, Corollary (b) of Theorem 29].

Lemma 2.8.4. *Assume that G is simply connected. Let $\sigma' : \Delta \to \Delta$ be a graph automorphism of the Dynkin diagram Δ. Then there exists an isomorphism $\sigma : G \to G$ and signs $\varepsilon_\alpha = \pm 1$ such that $\varepsilon_\alpha = 1$ for all $\alpha \in \pm\Delta$, and $\sigma(x_\alpha(c)) = x_{\sigma'(\alpha)}(\varepsilon_\alpha c)$ for all $\alpha \in \Phi$ and $c \in K$.*

For a graph automorphism $\sigma' : \Delta \to \Delta$ of the Dynkin diagram, we call the isomorphism σ in Lemma 2.8.4 the *graph automorphism of G induced by σ*. We know that each graph automorphism of Δ is either an involution or a triality graph automorphism of D_4. The following two lemmas describe the action of involutory graph automorphisms on unipotent conjugacy classes.

Lemma 2.8.5 ([LS12, Corollary 6]). *Let G be a simply connected algebraic group of type A_l or E_6 and let σ be a graph automorphism of G. Then σ fixes all unipotent classes of G.*

Lemma 2.8.6. *Let G be a semisimple algebraic group of type D_l ($l \geq 2$) with natural module V, and let σ be the graph automorphism of G induced by the graph automorphism swapping the two end nodes of the Dynkin diagram. Then σ fixes all unipotent classes of G, except the following ones:*

(i) $p \neq 2$*: For all c_i even such that $\sum_{i=1}^{t} c_i = l$, the automorphism σ swaps the two classes of unipotents u which satisfy $V \downarrow K[u] = \bigoplus_{i=1}^{t} 2 \cdot V_{c_i}$.*

(ii) $p = 2$*: For all c_i even such that $\sum_{i=1}^{t} c_i = l$, the automorphism σ swaps the two classes of unipotents u which satisfy $V \downarrow K[u] = \bigoplus_{i=1}^{t} W(c_i)$.*

Proof. For $l \neq 4$ this is [LS12, Corollary 6]. However, the same proof also works for $l = 4$. □

To complete the picture, one should still describe the action of a triality graph automorphism of D_4. We postpone a discussion of this to Section 2.10.

We now move on to describing the unipotent class representatives. For simple linear algebraic groups of exceptional type, explicit representatives for all unipotent conjugacy classes have been written down in [Sim13, Tables 3.1-3.9], where the representatives were found using [LS12] and computations done by Ross Lawther. We list these representatives for the distinguished unipotent classes in tables 2.1 - 2.5, where in some cases the representative given depends on the characteristic p of K. In the tables we denote $x_i(c) = x_{\alpha_i}(c)$ for all $1 \leq i \leq N$, where $N = |\Phi^+|$ and $\Phi^+ = \{\alpha_1, \ldots, \alpha_N\}$ is ordered with the total order $\leq$ described above. We have given the ordering explicitly in Appendix C. This ordering of the roots is the same one that is used in MAGMA. Note that the ordering of positive roots used in [Sim13] for type F_4 is different to this ordering. We also remark that in [Sim13] the structure constants used are those given in [GAP16], which are usually different from those used in MAGMA. However, we have verified that the representatives given in tables 2.1 - 2.5 work with both sets of structure constants.

For classical types, we will describe representatives for all unipotent conjugacy classes in the subsections that follow.

Class	p	Representative
G_2	any	$x_1(1)x_2(1)$
$G_2(a_1)$	any	$x_2(1)x_5(1)$
$(\tilde{A}_1)_3$	3	$x_1(1)x_5(1)$

Table 2.1: Representatives for distinguished unipotent classes of a Chevalley group of type G_2.

2.8.1 Type A_l, $l \geq 1$

Let $e_1, \ldots, e_l, e_{l+1}$ be a basis for a complex vector space $V_{\mathbb{C}}$, and let $V_{\mathbb{Z}}$ be the $\mathbb{Z}$-lattice spanned by this basis. Let $\mathfrak{sl}(V_{\mathbb{C}})$ be the Lie algebra formed by the linear endomorphisms of $V_{\mathbb{C}}$ with trace zero. Then $\mathfrak{sl}(V_{\mathbb{C}})$ is a semisimple Lie algebra of type A_l. Let $\mathfrak{h}$ be the Cartan subalgebra formed by the diagonal matrices in

Class	p	Representative
F_4	any	$x_1(1)x_2(1)x_3(1)x_4(1)$
$F_4(a_1)$	any	$x_1(1)x_2(1)x_6(1)x_7(1)$
$F_4(a_2)$	2	$x_4(1)x_7(1)x_5(1)x_9(1)$
	$\neq 2$	$x_4(1)x_2(1)x_8(1)x_9(1)$
$F_4(a_3)$	any	$x_2(1)x_5(1)x_9(1)x_{18}(1)$
$(C_3(a_1))_2$	2	$x_2(1)x_4(1)x_9(1)x_{20}(1)$
$(\tilde{A}_2A_1)_2$	2	$x_1(1)x_3(1)x_4(1)x_{16}(1)$

Table 2.2: Representatives for distinguished unipotent classes of a Chevalley group of type F_4.

Class	p	Representative
E_6	any	$x_1(1)x_2(1)x_3(1)x_4(1)x_5(1)x_6(1)$
$E_6(a_1)$	any	$x_1(1)x_2(1)x_9(1)x_{10}(1)x_5(1)x_6(1)$
$E_6(a_3)$	2	$x_1(1)x_8(1)x_9(1)x_{11}(1)x_{14}(1)x_{19}(1)$
	$\neq 2$	$x_{13}(1)x_1(1)x_{15}(1)x_6(1)x_{14}(1)x_4(1)$

Table 2.3: Representatives for distinguished unipotent classes of a Chevalley group of type E_6.

Class	p	Representative
E_7	any	$x_1(1)x_2(1)x_3(1)x_4(1)x_5(1)x_6(1)x_7(1)$
$E_7(a_1)$	any	$x_1(1)x_3(1)x_{10}(1)x_9(1)x_5(1)x_6(1)x_7(1)$
$E_7(a_2)$	any	$x_1(1)x_2(1)x_3(1)x_9(1)x_{11}(1)x_{12}(1)x_{13}(1)$
$E_7(a_3)$	2	$x_1(1)x_7(1)x_9(1)x_{10}(1)x_{12}(1)x_{16}(1)x_{22}(1)$
	$\neq 2$	$x_{15}(1)x_1(1)x_{17}(1)x_6(1)x_7(1)x_{16}(1)x_4(1)$
$E_7(a_4)$	2	$x_1(1)x_4(1)x_7(1)x_{10}(1)x_{22}(1)x_{23}(1)x_{24}(1)$
	$\neq 2$	$x_1(1)x_4(1)x_7(1)x_{13}(1)x_{15}(1)x_{16}(1)x_{17}(1)x_{23}(1)$
$E_7(a_5)$	3	$x_4(1)x_{15}(1)x_{16}(1)x_{19}(1)x_{20}(1)x_{21}(1)x_{29}(1)$
	$\neq 3$	$x_4(1)x_7(1)x_{20}(1)x_{21}(1)x_{22}(1)x_{23}(1)x_{24}(1)$

Table 2.4: Representatives for distinguished unipotent classes of a Chevalley group of type E_7.

$\mathfrak{sl}(V_\mathbb{C})$, with respect to the basis (e_i). For $1 \leq i \leq n$, define maps $\varepsilon_i : \mathfrak{h} \to \mathbb{C}$ by $\varepsilon_i(h) = h_i$, where h is a diagonal matrix with diagonal entries $(h_1, \ldots, h_l, h_{l+1})$. Now $\Phi = \{\varepsilon_i - \varepsilon_j : i \neq j\}$ is the root system for $\mathfrak{sl}(V_\mathbb{C})$ and $\Phi^+ = \{\varepsilon_i - \varepsilon_j : i < j\}$ is a system of positive roots. The base Δ of Φ corresponding to Φ^+ is $\Delta = \{\alpha_1, \ldots, \alpha_l\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for all $1 \leq i \leq l$.

For $1 \leq i, j \leq l+1$, let $E_{i,j}$ be the linear endomorphism on $V_\mathbb{C}$ such that $E_{i,j}(e_j) = e_i$ and $E_{i,j}(e_k) = 0$ for $k \neq j$. A Chevalley basis for $\mathfrak{sl}(V_\mathbb{C})$ with positive structure constants for extraspecial pairs is given by $X_{\varepsilon_i - \varepsilon_j} = E_{i,j}$ for all $i \neq j$. Let $\mathscr{U}_\mathbb{Z}$ be the Kostant $\mathbb{Z}$-form with respect to this Chevalley basis of $\mathfrak{sl}(V_\mathbb{C})$.

Note that $V_\mathbb{Z}$ is a $\mathscr{U}_\mathbb{Z}$-invariant lattice in $V_\mathbb{C}$. We define $V = V_\mathbb{Z} \otimes_\mathbb{Z} K$. Then the following lemma is well known, and amounts to the fact that $\mathrm{SL}(V)$ is generated by transvections.

Lemma 2.8.7. *The Chevalley group of type A_l induced by V is equal to $G = \mathrm{SL}(V)$.*

Class	p	Representative
E_8	any	$x_1(1)x_2(1)x_3(1)x_4(1)x_5(1)x_6(1)x_7(1)x_8(1)$
$E_8(a_1)$	any	$x_1(1)x_2(1)x_{10}(1)x_{11}(1)x_5(1)x_6(1)x_7(1)x_8(1)$
$E_8(a_2)$	any	$x_1(1)x_2(1)x_3(1)x_{10}(1)x_{12}(1)x_{13}(1)x_{14}(1)x_8(1)$
$E_8(a_3)$	2	$x_7(1)x_8(1)x_9(1)x_{10}(1)x_{11}(1)x_{12}(1)x_{13}(1)x_{19}(1)$
	$\neq 2$	$x_1(1)x_4(1)x_6(1)x_7(1)x_8(1)x_{17}(1)x_{18}(1)x_{19}(1)$
$E_8(a_4)$	2	$x_9(1)x_{10}(1)x_{11}(1)x_{12}(1)x_{13}(1)x_{14}(1)x_{15}(1)x_{19}(1)$
	$\neq 2$	$x_1(1)x_4(1)x_6(1)x_8(1)x_{17}(1)x_{18}(1)x_{19}(1)x_{21}(1)$
$E_8(b_4)$	2	$x_7(1)x_8(1)x_9(1)x_{17}(1)x_{18}(1)x_{19}(1)x_{20}(1)x_{21}(1)$
	$\neq 2$	$x_1(1)x_4(1)x_7(1)x_8(1)x_{14}(1)x_{17}(1)x_{19}(1)x_{26}(1)$
$E_8(a_5)$	2	$x_9(1)x_{14}(1)x_{15}(1)x_{17}(1)x_{18}(1)x_{19}(1)x_{20}(1)x_{26}(1)$
	$\neq 2$	$x_1(1)x_{12}(1)x_{14}(1)x_{15}(1)x_{17}(1)x_{21}(1)x_{26}(1)x_{27}(1)$
$E_8(b_5)$	3	$x_4(1)x_8(1)x_{17}(1)x_{18}(1)x_{21}(1)x_{23}(1)x_{24}(1)x_{33}(1)$
	$\neq 3$	$x_4(1)x_7(1)x_8(1)x_{18}(1)x_{19}(1)x_{23}(1)x_{31}(1)x_{33}(1)$
$E_8(a_6)$	3	$x_4(1)x_{12}(1)x_{14}(1)x_{15}(1)x_{23}(1)x_{25}(1)x_{26}(1)x_{31}(1)$
	$\neq 3$	$x_4(1)x_7(1)x_{18}(1)x_{19}(1)x_{23}(1)x_{29}(1)x_{31}(1)x_{33}(1)$
$E_8(b_6)$	2	$x_4(1)x_{15}(1)x_{18}(1)x_{19}(1)x_{23}(1)x_{31}(1)x_{33}(1)x_{39}(1)$
	3	$x_4(1)x_{17}(1)x_{12}(1)x_{29}(1)x_{30}(1)x_{31}(1)x_{34}(1)x_{35}(1)$
	$\neq 2, 3$	$x_4(1)x_8(1)x_{18}(1)x_{19}(1)x_{23}(1)x_{31}(1)x_{33}(1)x_{39}(-1)x_{41}(1)$
$E_8(a_7)$	any	$x_{30}(1)x_{33}(1)x_{35}(1)x_{36}(1)x_{40}(1)x_{45}(1)x_{53}(1)x_{58}(1)$
$(A_7)_3$	3	$x_1(1)x_3(1)x_4(1)x_5(1)x_6(1)x_7(1)x_8(1)x_{67}(1)$
$(D_7(a_1))_2$	2	$x_2(1)x_3(1)x_{10}(1)x_{12}(1)x_6(1)x_7(1)x_8(1)x_{83}(1)$
$(D_5A_2)_2$	2	$x_1(1)x_2(1)x_3(1)x_4(1)x_5(1)x_7(1)x_8(1)x_{51}(1)$

Table 2.5: Representatives for distinguished unipotent classes of a Chevalley group of type E_8.

By abuse of notation we will identify the basis $(e_i \otimes 1)$ of V with (e_i). Recall that by Lemma 2.2.1, the conjugacy class of a unipotent element $u \in G$ is determined by its Jordan block sizes. In other words, the unipotent conjugacy classes in G are labeled by symbols $[d_1, \ldots, d_t]$, where $1 \leq d_1 \leq \cdots \leq d_t$ and $\sum_{i=1}^{t} d_i = l + 1$. Set $k_1 = 1$ and $k_i = 1 + \sum_{j=1}^{i-1} d_j$ for $1 < i \leq t$. Let V_i be the subspace of V with basis $B_i = \{e_j : k_i \leq j \leq k_i + d_i - 1\}$. Now $V = V_1 \oplus \cdots \oplus V_t$, so we have a naturally embedded subgroup $\mathrm{SL}(V_1) \times \cdots \times \mathrm{SL}(V_t) < G$, where $\mathrm{SL}(V_i)$ is the subgroup of all $g \in G$ such that $g(V_i) = V_i$ and $ge_j = e_j$ for all $e_j \notin B_i$.

Note next that a representative of the unipotent conjugacy class labeled by $[d_1, \ldots, d_t]$ is given by $u = u_1 \cdots u_t$, where $u_i \in \mathrm{SL}(V_i)$ acts on V_i with a single Jordan block of size d_i. If $d_i = 1$, such an u_i is given by $u_i = 1$. Suppose then that $d_i > 1$. Now by Lemma 2.8.7, the subgroup $\mathrm{SL}(V_i)$ is a subsystem subgroup of $\mathrm{SL}(V)$ with root system $\{\pm(\varepsilon_j - \varepsilon_{j'}) : k_i \leq j < j' \leq k_i + d_i - 1\}$ of type $A_{d_i - 1}$, and a base $\{\alpha_j : k_i \leq j \leq k_i + d_i - 2\}$. Thus it follows from Lemma 2.2.2 and Lemma 2.8.1 that $u_i = \prod_{j=k_i}^{k_i+d_i-2} x_{\alpha_j}(1) \in \mathrm{SL}(V_i)$ acts on V_i with a single Jordan block of size d_i. Therefore we have the following result.

Lemma 2.8.8 (Type A). *Let $G = \mathrm{SL}(V)$ be a Chevalley group of type A_l as defined above, where $l \geq 1$. Let $1 \leq d_1 \leq \cdots \leq d_t$, where $\sum_{i=1}^{t} d_i = l + 1$. Set $k_1 = 1$ and $k_i = 1 + \sum_{j=1}^{i-1} d_j$ for $1 < i \leq t$. Define*

$$u_i = \begin{cases} 1, & \text{if } d_i = 1 \\ \prod_{j=k_i}^{k_i+d_i-2} x_{\alpha_j}(1), & \text{if } d_i > 1. \end{cases}$$

Then $u = u_1 \cdots u_t$ lies in the unipotent class of G labeled by $[d_1, \dots, d_t]$.

2.8.2 Type C_l, $l \geq 1$

α		$\alpha = \sum_{k=1}^{l} c_k \alpha_i$
$\varepsilon_i - \varepsilon_j$,	$1 \leq i < j \leq l$	$\sum_{i \leq k \leq j-1} \alpha_k$
$\varepsilon_i + \varepsilon_j$,	$1 \leq i < j \leq l$	$\sum_{i \leq k \leq j-1} \alpha_k + \sum_{j \leq k \leq l-1} 2\alpha_k + \alpha_l$
$2\varepsilon_i$,	$1 \leq i \leq l$	$\sum_{i \leq k \leq l-1} 2\alpha_k + \alpha_l$

Table 2.6: Type C_l, expressions for roots $\alpha \in \Phi^+$ in terms of base Δ.

Let $e_1, \dots, e_l, e_{-l}, \dots, e_{-1}$ be a basis for a complex vector space $V_{\mathbb{C}}$, and let $V_{\mathbb{Z}}$ be the $\mathbb{Z}$-lattice spanned by this basis. We define a non-degenerate alternating bilinear form $(-,-)$ on $V_{\mathbb{C}}$ by setting $(e_i, e_{-i}) = 1 = -(e_{-i}, e_i)$ and $(e_i, e_j) = 0$ for $i \neq -j$. Let $\mathfrak{sp}(V_{\mathbb{C}})$ be the Lie algebra consisting of the linear endomorphisms X of $V_{\mathbb{C}}$ satisfying $(Xv, w) + (v, Xw) = 0$ for all $v, w \in V_{\mathbb{C}}$. Then $\mathfrak{sp}(V_{\mathbb{C}})$ is a semisimple Lie algebra of type C_l. Let $\mathfrak{h}$ be the Cartan subalgebra formed by the diagonal matrices in $\mathfrak{sp}(V_{\mathbb{C}})$. Then $\mathfrak{h} = \{\mathrm{diag}(h_1, \dots, h_l, -h_l, \dots, -h_1) : h_i \in \mathbb{C}\}$. For $1 \leq i \leq l$, define maps $\varepsilon_i : \mathfrak{h} \to \mathbb{C}$ by $\varepsilon_i(h) = h_i$ where h is a diagonal matrix with diagonal entries $(h_1, \dots, h_l, -h_l, \dots, -h_1)$. Now $\Phi = \{\pm(\varepsilon_i \pm \varepsilon_j) : 1 \leq i < j \leq l\} \cup \{\pm 2\varepsilon_i : 1 \leq i \leq l\}$ is the root system for $\mathfrak{sp}(V_{\mathbb{C}})$, and $\Phi^+ = \{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq l\} \cup \{2\varepsilon_i : 1 \leq i \leq l\}$ is a system of positive roots. The base Δ of Φ corresponding to Φ^+ is $\Delta = \{\alpha_1, \dots, \alpha_l\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i < l$ and $\alpha_l = 2\varepsilon_l$. We give the expressions of roots $\alpha \in \Phi^+$ in terms of Δ in Table 2.6.

For all i, j let $E_{i,j}$ be the linear endomorphism on $V_{\mathbb{C}}$ such that $E_{i,j}(e_j) = e_i$ and $E_{i,j}(e_k) = 0$ for $k \neq j$. Then a Chevalley basis for $\mathfrak{sp}(V_{\mathbb{C}})$ with positive structure constants for extraspecial pairs is given by

$$
\begin{aligned}
X_{\varepsilon_i - \varepsilon_j} &= E_{i,j} - E_{-j,-i} && \text{for all } i \neq j,\\
X_{\varepsilon_i + \varepsilon_j} &= E_{j,-i} + E_{i,-j} && \text{for all } i \neq j,\\
X_{-(\varepsilon_i + \varepsilon_j)} &= E_{-j,i} + E_{-i,j} && \text{for all } i \neq j,\\
X_{2\varepsilon_i} &= E_{i,-i} && \text{for all } i,\\
X_{-2\varepsilon_i} &= E_{-i,i} && \text{for all } i.
\end{aligned}
$$

Note that $V_{\mathbb{Z}}$ is a $\mathscr{U}_{\mathbb{Z}}$-invariant lattice in $V_{\mathbb{C}}$. We define $V = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$. Now $(-,-)$ also gives a non-degenerate alternating bilinear form on V, and we have the following result.

Lemma 2.8.9 ([Ree57, 5])**.** *The Chevalley group of type C_l induced by V is equal to the group $G = \mathrm{Sp}(V)$ of invertible linear maps $V \to V$ preserving $(-,-)$.*

By abuse of notation we identify the basis $(e_i \otimes 1)$ of V with (e_i). Let $u \in G$ be a unipotent element.

Suppose first that $p \neq 2$. Then by Proposition 2.3.1, the conjugacy class of u in G is uniquely determined by the Jordan block sizes of u on V. Furthermore, each Jordan block of odd size must occur with even multiplicity by Proposition 2.3.2

(i). That is, the conjugacy class of u is uniquely determined by the decomposition

$$V \downarrow K[u] = \bigoplus_{i=1}^{t} 2 \cdot V_{c_i} \oplus \bigoplus_{j=1}^{s} V_{2d_j},$$

where $1 \leq c_1 \leq \cdots \leq c_t$ are odd, and $1 \leq d_1 \leq \cdots \leq d_s$, and $\sum_{i=1}^{t} c_i + \sum_{j=1}^{s} d_j = l$.

Consider next $p = 2$. It follows from Proposition 2.4.4 that the conjugacy class of u is uniquely determined by the decomposition

$$V \downarrow K[u] = W(c_1) + \cdots + W(c_t) + V(2d_1) + \cdots + V(2d_s)$$

into an orthogonal direct sum of $K[u]$-modules, where $1 \leq c_1 \leq \cdots \leq c_t$, $1 \leq d_1 \leq \cdots \leq d_s$, and each $V(2k)$ occurs at most twice as a summand (equivalently, $d_{i+2} > d_i$ for all i). Furthermore, since u acts on $W(c_i)$ with two Jordan blocks of size c_i and on $V(2d_j)$ with a single Jordan block of size $2d_j$, we have $\sum_{i=1}^{t} c_i + \sum_{j=1}^{s} d_j = l$.

Let p again be arbitrary, and let (c_i) and (d_j) be the two (possibly empty) sequences associated with $V \downarrow K[u]$ as above. Define $c_{t+i} = d_i$ for all $1 \leq i \leq s$. Set $k_1 = 1$, and $k_i = 1 + c_1 + \cdots + c_{i-1}$ for $1 < i \leq t+s$. For all $1 \leq i \leq t+s$, let V_i be the subspace of V spanned by $B_i = \{e_{\pm j} : k_i \leq j \leq k_i + c_i - 1\}$. Now V is an orthogonal direct sum $V = V_1 \oplus \cdots \oplus V_{t+s}$, so we have a naturally embedded subgroup $\mathrm{Sp}(V_1) \times \cdots \times \mathrm{Sp}(V_{t+s}) < G$, where $\mathrm{Sp}(V_i)$ is the subgroup of all $g \in G$ such that $g(V_i) = V_i$ and $ge_j = e_j$ for all $e_j \notin B_i$.

It follows then that a representative of the unipotent class determined by the two sequences (c_i) and (d_j) is given by $u_1 \cdots u_{t+s}$, where $u_i \in \mathrm{Sp}(V_i)$ and we have the following:

(C1) If $1 \leq i \leq t$, then u_i acts on V_i with two Jordan blocks of size c_i. Furthermore, if $p = 2$, then $V_i \downarrow K[u_i] = W(c_i)$.

(C2) If $t+1 \leq i \leq t+s$, then u_i acts on V_i with a single Jordan block of size $2d_i$.

Note that by Lemma 2.8.9, the subgroup $\mathrm{Sp}(V_i)$ is a subsystem subgroup of $\mathrm{Sp}(V)$ with root system $\Phi_i = \{\pm(\varepsilon_j \pm \varepsilon_{j'}) : k_i \leq j < j' \leq k_i + c_i - 1\} \cup \{\pm 2\varepsilon_j : k_i \leq j \leq k_i + c_i - 1\}$ of type C_{c_i}. Furthermore, the root system Φ_i has a base $\Delta_i = \{\alpha_j : k_i \leq j \leq k_i + c_i - 2\} \cup \{2\varepsilon_{k_i+c_i-1}\}$.

For $1 \leq i \leq t$, if $c_i = 1$, it is clear that (C1) is satisfied by $u_i = 1$. Suppose then that $c_i > 1$. Now it is a consequence of the next lemma that $u_i = \prod_{j=k_i}^{k_i+c_i-2} x_{\alpha_j}(1)$ satisfies (C1).

Lemma 2.8.10. *Let $G = \mathrm{Sp}(V)$ be a Chevalley group of type C_l as defined above, where $l \geq 2$. Set $u = x_{\alpha_1}(1) \cdots x_{\alpha_{l-1}}(1)$. Then u acts on V with two Jordan block sizes of l. Furthermore, if $p = 2$, we have $V \downarrow K[u] = W(l)$.*

Proof. Now by Lemma 2.8.1, the element u is a regular unipotent element of a Levi factor $L \cong \mathrm{SL}_l(K)$ of type A_{l-1}. For $p = 2$, it is shown in [LS12, 6.1] that we have $V \downarrow K[u] = W(l)$, as desired.

Suppose then that $p \neq 2$. As noted in [LS12, Proof of Corollary 3.6], we have $V \downarrow L \cong W \oplus W^*$, where W is the natural module of L. Now by Proposition 2.3.3 (i), a regular unipotent element of L acts on W (hence also on W^*) with a single Jordan block of size l, so the claim follows. □

Consider then $t+1 \leq i \leq t+s$. It follows from Proposition 2.3.3 and Proposition 2.4.4 (v) that (C2) is satisfied when we take u_i to be a regular unipotent element of $\mathrm{Sp}(V_i)$. Thus by Lemma 2.8.1, condition (C2) is satisfied when we pick $u_i = x_{2\varepsilon_{k_i+c_i-1}}(1)$ if $c_i = 1$, and $u_i = \prod_{j=k_i}^{k_i+c_i-2} x_{\alpha_j}(1) \cdot x_{2\varepsilon_{k_i+c_i-1}}(1)$ if $c_i > 1$.

Putting all of this together, we get the following two lemmas which describe representatives for the unipotent classes of G.

Lemma 2.8.11 (Type C ($p \neq 2$)). *Assume that $p \neq 2$. Let $G = \mathrm{Sp}(V)$ be a Chevalley group of type C_l as defined above, where $l \geq 1$. Let $1 \leq c_1 \leq \cdots \leq c_t$ be odd, let $1 \leq d_1 \leq \cdots \leq d_s$, where $\sum_{i=1}^t c_i + \sum_{j=1}^s d_j = l$. Set $c_{t+i} = d_i$ for all $1 \leq i \leq s$, and define $k_1 = 1$, and $k_i = 1 + c_1 + \cdots + c_{i-1}$ for $1 < i \leq s$. Define*

$$u_i = \begin{cases} 1, & \text{if } 1 \leq i \leq t \text{ and } c_i = 1, \\ \prod_{j=k_i}^{k_i+c_i-2} x_{\alpha_j}(1), & \text{if } 1 \leq i \leq t \text{ and } c_i > 1, \\ x_{2\varepsilon_{k_i+c_i-1}}(1), & \text{if } t+1 \leq i \leq t+s \text{ and } c_i = 1, \\ \prod_{j=k_i}^{k_i+c_i-2} x_{\alpha_j}(1) \cdot x_{2\varepsilon_{k_i+c_i-1}}(1), & \text{if } t+1 \leq i \leq t+s \text{ and } c_i > 1. \end{cases}$$

Then $u = u_1 \cdots u_{t+s}$ lies in the unipotent class of G determined by the decomposition $V \downarrow K[u] = \bigoplus_{i=1}^{t} 2 \cdot V_{c_i} \oplus \bigoplus_{j=1}^{s} V_{2d_j}$.

Lemma 2.8.12 (Type C ($p = 2$)). *Assume that $p = 2$. Let $G = \mathrm{Sp}(V)$ be a Chevalley group of type C_l as defined above, where $l \geq 1$. Let $1 \leq c_1 \leq \cdots \leq c_t$ and $1 \leq d_1 \leq \cdots \leq d_s$, where $d_{i+2} > d_i$ for all i, and $\sum_{i=1}^t c_i + \sum_{j=1}^s d_j = l$. Set $c_{t+i} = d_i$ for all $1 \leq i \leq s$, and define $k_1 = 1$, and $k_i = 1 + c_1 + \cdots + c_{i-1}$ for $1 < i \leq s$. Define*

$$u_i = \begin{cases} 1, & \text{if } 1 \leq i \leq t \text{ and } c_i = 1, \\ \prod_{j=k_i}^{k_i+c_i-2} x_{\alpha_j}(1), & \text{if } 1 \leq i \leq t \text{ and } c_i > 1, \\ x_{2\varepsilon_{k_i+c_i-1}}(1), & \text{if } t+1 \leq i \leq t+s \text{ and } c_i = 1, \\ \prod_{j=k_i}^{k_i+c_i-2} x_{\alpha_j}(1) \cdot x_{2\varepsilon_{k_i+c_i-1}}(1), & \text{if } t+1 \leq i \leq t+s \text{ and } c_i > 1. \end{cases}$$

Then $u = u_1 \cdots u_{t+s}$ lies in the unipotent class of G determined by the orthogonal decomposition $V \downarrow K[u] = W(c_1) + \cdots + W(c_t) + V(2d_1) + \cdots + V(2d_s)$.

2.8.3 Type D_l, $l \geq 2$

α	$\alpha = \sum_{k=1}^{l} c_k \alpha_i$
$\varepsilon_i - \varepsilon_j, \quad 1 \leq i < j \leq l$	$\sum_{i \leq k \leq j-1} \alpha_k$
$\varepsilon_i + \varepsilon_j, \quad 1 \leq i < j \leq l-1$	$\sum_{i \leq k \leq j-1} \alpha_k + \sum_{j \leq k \leq l-2} 2\alpha_k + \alpha_{l-1} + \alpha_l$
$\varepsilon_i + \varepsilon_l, \quad 1 \leq i \leq l-1$	$\sum_{i \leq k \leq l-2} \alpha_k + \alpha_l$

Table 2.7: Type D_l, expressions for roots $\alpha \in \Phi^+$ in terms of base Δ.

Let $e_1, \ldots, e_l, e_{-l}, \ldots, e_{-1}$ be a basis for a complex vector space $V_{\mathbb{C}}$, and let $V_{\mathbb{Z}}$ be the $\mathbb{Z}$-lattice spanned by this basis. We have a non-degenerate symmetric bilinear form $(-,-)$ on $V_{\mathbb{C}}$ defined by $(e_i, e_{-i}) = 1 = (e_{-i}, e_i)$ and $(e_i, e_j) = 0$ for $i \neq -j$. Let $\mathfrak{so}(V_{\mathbb{C}})$ be the Lie algebra formed by the linear endomorphisms X of $V_{\mathbb{C}}$ satisfying $(Xv, w) + (v, Xw) = 0$ for all $v, w \in V_{\mathbb{C}}$. Then $\mathfrak{so}(V_{\mathbb{C}})$ is a semisimple Lie algebra of type D_l. Note that here type D_2 is the same as type $A_1 \times A_1$, and D_3 is the same as type A_3.

Let $\mathfrak{h}$ be the Cartan subalgebra formed by the diagonal matrices in $\mathfrak{so}(V_{\mathbb{C}})$. Then $\mathfrak{h} = \{\mathrm{diag}(h_1, \ldots, h_l, -h_l, \ldots, -h_1) : h_i \in \mathbb{C}\}$. For $1 \leq i \leq l$, define maps $\varepsilon_i : \mathfrak{h} \to \mathbb{C}$ by $\varepsilon_i(h) = h_i$ where h is a diagonal matrix with diagonal entries $(h_1, \ldots, h_l, -h_l, \ldots, -h_1)$. Now $\Phi = \{\pm(\varepsilon_i \pm \varepsilon_j) : 1 \leq i < j \leq l\}$ is the root system of $\mathfrak{so}(V_{\mathbb{C}})$, and $\Phi^+ = \{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq l\}$ is a system of positive roots. Here the base Δ of Φ corresponding to Φ^+ is $\Delta = \{\alpha_1, \ldots, \alpha_l\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i < l$ and $\alpha_l = \varepsilon_{l-1} + \varepsilon_l$. We give the expressions of roots $\alpha \in \Phi^+$ in terms of Δ in Table 2.7.

As before, for all i, j let $E_{i,j}$ be the linear endomorphism on $V_{\mathbb{C}}$ such that $E_{i,j}(e_j) = e_i$ and $E_{i,j}(e_k) = 0$ for $k \neq j$. Then a Chevalley basis for $\mathfrak{so}(V_{\mathbb{C}})$ with positive structure constants for extraspecial pairs is given by

$$\begin{aligned} X_{\varepsilon_i - \varepsilon_j} &= E_{i,j} - E_{-j,-i} && \text{for all } i \neq j, \\ X_{\varepsilon_i + \varepsilon_j} &= E_{j,-i} - E_{i,-j} && \text{for all } i < j, \\ X_{-(\varepsilon_i + \varepsilon_j)} &= E_{-i,j} - E_{-j,i} && \text{for all } i < j. \end{aligned}$$

Now $V_{\mathbb{Z}}$ is a $\mathscr{U}_{\mathbb{Z}}$-invariant lattice in $V_{\mathbb{C}}$. We define $V = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$. By abuse of notation we identify the basis $(e_i \otimes 1)$ of V with (e_i).

Note that $(-,-)$ also defines a non-degenerate form on V. Furthermore, the quadratic form $q_{\mathbb{C}} : V_{\mathbb{C}} \to \mathbb{C}$ defined by $q_{\mathbb{C}}(v) = \frac{1}{2}(v, v)$ restricts to $q_{\mathbb{Z}} : V_{\mathbb{Z}} \to \mathbb{Z}$. Thus we have a quadratic form $q = q_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$ on V which has the form $(-,-)$ as its polarization. Let G be the Chevalley group of type D_l given by V. Then q is a non-degenerate G-invariant quadratic form on V, and we have the following result.

Lemma 2.8.13 ([Ree57, 6], [Hée84, 14.2])**.** *The Chevalley group of type D_l induced by V is equal to the group $G = \mathrm{SO}(V)$.*

Here by $\mathrm{SO}(V)$ we mean the identity component of $\mathrm{O}(V)$, where $\mathrm{O}(V)$ is the subgroup of all $g \in \mathrm{GL}(V)$ which satisfy $q(gv) = q(v)$ for all $v \in V$. Let $u \in G$ be a unipotent element.

Suppose first that $p \neq 2$. Considering the action of u on V, by Proposition 2.3.2 (ii) each Jordan block of even size must occur with even multiplicity. Furthermore, the dimension $\dim V$ is even, so there must be an even number of Jordan blocks of odd size. Thus we have a decomposition

$$V \downarrow K[u] = \bigoplus_{i=1}^{t} 2 \cdot V_{c_i} \oplus \bigoplus_{j=t+1}^{t+s} \left(V_{2d_j+1} \oplus V_{2d'_j+1}\right),$$

where $1 < c_1 \leq \cdots \leq c_t$ are even, and $0 \leq d_{t+1} \leq d'_{t+1} \leq \cdots \leq d_{t+s} \leq d'_{t+s}$, and $\sum_{i=1}^{t} c_i + \sum_{j=t+1}^{t+s}(d_j + d'_j + 1) = l$.

By Proposition 2.3.2, this decomposition uniquely determines the conjugacy class of u in G, except when $V \downarrow K[u] = \bigoplus_{i=1}^{t} 2 \cdot V_{c_i}$, in which case we have two classes with this decomposition. To find representatives for each of these classes, let $\tau : G \to G$ be the automorphism associated with the involutory graph automorphism of the Dynkin diagram which swaps the two end nodes. By Lemma

2.8.6 (i), if u is some unipotent element with $V \downarrow K[u] = \bigoplus_{i=1}^{t} 2 \cdot V_{c_i}$, then u and $\tau(u)$ are not conjugate. Therefore it will suffice to find some $u \in G$ with this decomposition, as then u and $\tau(u)$ give representatives for the two classes.

Consider next $p = 2$. By Proposition 2.4.4, we have that u acts on V with an even number of Jordan blocks, and thus there is an orthogonal decomposition

$$V \downarrow K[u] = \bigoplus_{i=1}^{t} W(c_i) \oplus \bigoplus_{j=t+1}^{t+s} \big(V(2d_j) + V(2d_j')\big)$$

where $1 \leq c_1 \leq \cdots \leq c_t$, and $1 \leq d_{t+1} \leq d'_{t+1} \leq \cdots \leq d_{t+s} \leq d'_{t+s}$, and $\sum_{i=1}^{t} c_i + \sum_{j=t+1}^{t+s}(d_j + d_j') = l$.

By Proposition 2.4.4 (iv), this decomposition uniquely determines the conjugacy class of u in G, except when $V \downarrow K[u] = \bigoplus_{i=1}^{t} W(c_i)$, in which case we have two classes with this decomposition. As in the $p \neq 2$ case, it will suffice to find just one u with such a decomposition. Then by Lemma 2.8.6 (ii) the elements u and $\tau(u)$ are representatives for the two conjugacy classes with this decomposition.

Let p again be arbitrary. Let (c_i), (d_j), and (d_j') be the (possibly empty) sequences associated with $V \downarrow K[u]$ as above. Now for $t + 1 \leq i \leq t + s$, set $c_i = d_i + d_i' + 1$ if $p \neq 2$ and $c_i = d_i + d_i'$ if $p = 2$. Set $k_1 = 1$ and $k_i = 1 + c_1 + \cdots + c_{i-1}$ for $1 < i \leq t + s$. For all $1 \leq i \leq t + s$, let V_i be the subspace of spanned by $B_i = \{e_{\pm j} : k_i \leq j \leq k_i + c_i - 1\}$. Now V is an orthogonal direct sum $V = V_1 \oplus \cdots \oplus V_{t+s}$, so we have a naturally embedded subgroup $\mathrm{SO}(V_1) \times \cdots \times \mathrm{SO}(V_{t+s})$, where $\mathrm{SO}(V_i)$ is the subgroup of all $g \in \mathrm{SO}(V)$ such that $ge_j = e_j$ for all $e_j \notin B_i$.

It follows then that for a $u \in G$ with $V \downarrow K[u]$ as given by the sequences (c_i), (d_j), and (d_j'), we can choose $u = u_1 \cdots u_{t+s}$, where $u_i \in \mathrm{SO}(V_i)$ and we have the following:

(D1) If $1 \leq i \leq t$, then u_i acts on V_i with two Jordan blocks of size c_i. Furthermore, if $p = 2$, then $V_i \downarrow K[u_i] = W(c_i)$.

(D2) If $t + 1 \leq i \leq t + s$ and $p \neq 2$, then u_i acts on V_i with Jordan blocks $[2d_i + 1, 2d_i' + 1]$.

(D3) If $t + 1 \leq i \leq t + s$ and $p = 2$, then $V_i \downarrow K[u_i] = V(2d_i) + V(2d_i')$.

Note that by Lemma 2.8.13, the subgroup $\mathrm{SO}(V_i)$ is a subsystem subgroup of $\mathrm{SO}(V)$ with root system $\Phi_i = \{\pm(\varepsilon_j \pm \varepsilon_{j'}) : k_i \leq j < j' \leq k_i + c_i - 1\}$ of type D_{c_i}. Furthermore, the root system Φ_i has a base $\Delta_i = \{\alpha_j : k_i \leq j \leq k_i + c_i - 2\} \cup \{\varepsilon_{k_i+c_i-2} + \varepsilon_{k_i+c_i-1}\}$.

Consider first u_i for $1 \leq i \leq t$. Now it is a consequence of the next lemma that (D1) holds for $u_i = \prod_{j=k_i}^{k_i+c_i-2} x_{\alpha_j}(1)$.

Lemma 2.8.14. *Let $G = \mathrm{SO}(V)$ be a Chevalley group of type D_l as defined above, where $l \geq 2$. Set $u = x_{\alpha_1}(1) \cdots x_{\alpha_{l-1}}(1)$. Then u acts on V with two Jordan block sizes of l. Furthermore, if $p = 2$, we have $V \downarrow K[u] = W(l)$.*

Proof. The claim follows with the same proof as Lemma 2.8.10. □

For u_i with $t+1 \leq i \leq t+s$, consider first $p \neq 2$. If $d_i = d_i' = 0$, condition (D2) holds for $u_i = 1$. If $d_i = 0$ and $d_i' > 0$, then by Proposition 2.3.3 condition (D2) holds when u_i is a regular unipotent element of $\mathrm{SO}(V_i)$. Thus by Lemma 2.8.1, we have (D2) when $u_i = \prod_{j=k_i}^{k_i+c_i-2} x_{\alpha_j}(1) \cdot x_{\varepsilon_{k_i+c_i-2}+\varepsilon_{k_i+c_i-1}}(1)$. If $d_i, d_i' > 0$, then by the next lemma, the condition (D2) holds when $u_i = v_i v_i'$, where

$$v_i = \prod_{j=k_i}^{k_i+d_i-2} x_{\alpha_j}(1) \cdot x_{\varepsilon_{k_i+d_i-1}-\varepsilon_{k_i+c_i-1}}(1) \cdot x_{\varepsilon_{k_i+d_i-1}+\varepsilon_{k_i+c_i-1}}(1)$$

and

$$v_i' = \prod_{j=k_i+d_i}^{k_i+c_i-2} x_{\alpha_j}(1) \cdot x_{\varepsilon_{k_i+c_i-2}+\varepsilon_{k_i+c_i-1}}(-1).$$

Lemma 2.8.15. *Assume that $p \neq 2$. Let $G = \mathrm{SO}(V)$ be a Chevalley group of type D_l as defined above, where $l \geq 3$. Let $e + f + 1 = l$, where $e, f > 0$. Set $u_1 = \prod_{i=1}^{e-1} x_{\alpha_i}(1) \cdot x_{\varepsilon_e - \varepsilon_l}(1) \cdot x_{\varepsilon_e+\varepsilon_l}(1)$ and $u_2 = \prod_{i=e+1}^{l-1} x_{\alpha_i}(1) \cdot x_{\alpha_l}(-1)$. Then the unipotent element $u = u_1 u_2$ acts on V with Jordan blocks $[2e+1, 2f+1]$.*

Proof. The claim is proven by Suprunenko in [Sup09, Lemma 2.24], where the expression with root subgroups has some different signs since the Chevalley basis used in [Sup09] has different structure constants than our Chevalley basis. We give an outline for the proof for completeness. We have $V = W \oplus W'$ an orthogonal direct sum, where $W = \langle e_{\pm 1}, \ldots, e_{\pm e}, e_l - e_{-l} \rangle$ and $W' = \langle e_{\pm(e+1)}, \ldots, e_{\pm(l-1)}, e_l + e_{-l} \rangle$. Thus we have a naturally embedded subgroup $\mathrm{SO}(W) \times \mathrm{SO}(W') < G$. Now $\mathrm{SO}(W)$ is generated by the root elements corresponding to simple roots and their negatives, and these are

$$\begin{aligned} &x_{\pm\alpha_i}(c) && c \in K \text{ and } 1 \leq i \leq e-1, \\ &x_{\varepsilon_e-\varepsilon_l}(c) x_{\varepsilon_e+\varepsilon_l}(c) && c \in K, \\ &x_{-\varepsilon_e+\varepsilon_l}(c) x_{-\varepsilon_e-\varepsilon_l}(c) && c \in K. \end{aligned}$$

The corresponding root elements for $\mathrm{SO}(W')$ are

$$\begin{aligned} &x_{\pm\alpha_i}(c) && c \in K \text{ and } e+1 \leq i \leq l-2, \\ &x_{\alpha_{l-1}}(c) x_{\alpha_l}(-c) && c \in K, \\ &x_{-\alpha_{l-1}}(c) x_{-\alpha_l}(-c) && c \in K. \end{aligned}$$

By Lemma 2.8.1 the element u_1 is a regular unipotent element of $\mathrm{SO}(W)$ and u_2 is a regular unipotent element of $\mathrm{SO}(W')$, so the claim follows from Proposition 2.3.3 (iii). ☐

Finally we explain how to choose u_i for $t+1 \leq i \leq t+s$ when $p = 2$. If $d_i = 1$, then by 2.4.4 (vi) we have (D3) when u_i is a regular unipotent element of $\mathrm{SO}(V_i)$. It follows then from Lemma 2.8.1 that we can choose $u_i = \prod_{j=k_i}^{k_i+c_i-2} x_{\alpha_j}(1) \cdot x_{\varepsilon_{k_i+c_i-2}+\varepsilon_{k_i+c_i-1}}(1)$ and (D3) will hold. Suppose then that $d_i > 1$, so now $d_i + d_i' \geq 4$. It follows from the next lemma (recall that we are assuming $d_i' \geq d_i$, so here $c_i - 2 \geq d_i' \geq \frac{c_i}{2}$) that (D3) holds when

$$u_i = \prod_{j=k_i}^{k_i+d_i'-2} x_{\alpha_j}(1) \cdot x_{\varepsilon_{k_i+d_i'-1}-\varepsilon_{k_i+c_i-1}}(1) \cdot x_{\varepsilon_{k_i+d_i'-1}+\varepsilon_{k_i+c_i-1}}(1) \cdot \prod_{j=k_i+d_i'}^{k_i+c_i-2} x_{\alpha_j}(1).$$

Lemma 2.8.16. *Assume that $p = 2$. Let $G = \mathrm{SO}(V)$ be a Chevalley group of type D_l as defined above, where $l \geq 4$. Let $l - 2 \geq e \geq \frac{l}{2}$. Then for the unipotent element $u = \prod_{i=1}^{e-1} x_{\alpha_i}(1) \cdot x_{\varepsilon_e-\varepsilon_l}(1) \cdot x_{\varepsilon_e+\varepsilon_l}(1) \cdot \prod_{i=e+1}^{l-1} x_{\alpha_i}(1)$, we have an orthogonal decomposition $V \downarrow K[u] = V(2e) + V(2l-2e)$.*

Proof. We compute explicitly the action of u on the basis $e_{\pm 1}, \ldots, e_{\pm l}$ to determine the decomposition $V \downarrow K[u]$. By looking at the Chevalley basis we are using, we see that $x_{\alpha_i}(1) = I + E_{i,i+1} + E_{-(i+1),-i}$ for all $1 \leq i \leq l-1$, where I is the identity and $E_{i,j}$ is defined as before. Furthermore, we see that $x_{\varepsilon_e - \varepsilon_l}(1) = I + E_{e,l} + E_{-l,-e}$ and $x_{\varepsilon_e + \varepsilon_l}(1) = I + E_{l,-e} + E_{e,-l}$. Now it is straightforward to see that u acts on the basis elements as follows:

$$\begin{aligned} e_1 &\mapsto e_1 \\ (2 \leq i \leq e)\ e_i &\mapsto e_i + e_{i-1} + \cdots + e_1 \\ (1 \leq i \leq e-1)\ e_{-i} &\mapsto e_{-i} + e_{-(i+1)} \\ e_{-e} &\mapsto e_l + e_{-l} + e_{-e} + e_e + e_{e-1} + \cdots + e_1 \end{aligned}$$

$$\begin{aligned} e_{e+1} &\mapsto e_{e+1} \\ (e+2 \leq i \leq l-1)\ e_i &\mapsto e_i + e_{i-1} + \cdots + e_{e+1} \\ e_l &\mapsto e_l + e_{l-1} + \cdots + e_1 \\ (e+1 \leq i \leq l-2)\ e_{-i} &\mapsto e_{-i} + e_{-(i+1)} \\ e_{-(l-1)} &\mapsto e_{-(l-1)} + e_{-l} + e_e + e_{e-1} + \cdots + e_1 \\ e_{-l} &\mapsto e_{-l} + e_e + e_{e-1} + \cdots + e_1 \end{aligned}$$

A calculation shows that the fixed point space of u has dimension 2, and that it is spanned by e_1 and e_{e+1}. Therefore the Jordan normal form of u has two Jordan blocks. To see that the Jordan block sizes are $2e$ and $2l - 2e$, it will be enough to show that $(u-1)^{2e} = 0$ and $(u-1)^{2e-1} \neq 0$, as then the largest Jordan block size in u is $2e$. To this end, a calculation shows that $(u-1)^{2e} e_i = 0$ for all the basis vectors e_i, and that

$$(u-1)^{2e-1}(e_{-1}) = \begin{cases} e_1, & \text{if } e > l/2, \\ e_{e+1} + e_1, & \text{if } e = l/2. \end{cases}$$

It follows then from Proposition 2.4.4 that either $V \downarrow K[u] = V(2e) + V(2e-2l)$ or $V \downarrow K[u] = W(2e)$. Our goal is to show that $V \downarrow K[u] = V(2e) + V(2e-2l)$. Now $(u-1)^{2e} = 0$, so by Lemma 2.4.8 this is equivalent to finding a $v \in V$ such that $((u-1)^{2e-1}v, v) \neq 0$. By the calculation done above, we can choose $v = e_{-1}$, as then $((u-1)^{2e-1}v, v) = (e_1, e_{-1}) \neq 0$. □

Now putting all of this together, we get the following two lemmas.

Lemma 2.8.17 (Type D ($p \neq 2$))**.** *Assume that $p \neq 2$. Let $G = \mathrm{SO}(V)$ be a Chevalley group of type D_l as defined above, where $l \geq 2$. Let $1 < c_1 \leq \cdots \leq c_t$ be even, let $0 \leq d_{t+1} \leq d'_{t+1} \leq \cdots \leq d_{t+s} \leq d'_{t+s}$ such that $\sum_{i=1}^{t} c_i + \sum_{j=t+1}^{t+s}(d_j + d'_j + 1) = l$. Set $c_i = d_i + d'_i + 1$ for $t+1 \leq i \leq t+s$, set $k_1 = 1$ and $k_i = 1 + c_1 + \cdots + c_{i-1}$ for $1 < i \leq t+s$. For $t+1 \leq i \leq t+s$ with $d_i, d'_i > 0$, set*

$$v_i = \prod_{j=k_i}^{k_i+d_i-2} x_{\alpha_j}(1) \cdot x_{\varepsilon_{k_i+d_i-1} - \varepsilon_{k_i+c_i-1}}(1) \cdot x_{\varepsilon_{k_i+d_i-1} + \varepsilon_{k_i+c_i-1}}(1).$$

For all $t+1 \leq i \leq t+s$ with $d'_i > 0$, set

$$v'_i = \prod_{j=k_i+d_i}^{k_i+c_i-2} x_{\alpha_j}(1) \cdot x_{\varepsilon_{k_i+c_i-2} + \varepsilon_{k_i+c_i-1}}(-1).$$

Define

$$u_i = \begin{cases} \prod_{j=k_i}^{k_i+c_i-2} x_{\alpha_j}(1), & \text{if } 1 \leq i \leq t. \\ 1, & \text{if } t+1 \leq i \leq t+s \text{ and } d_i = d_i' = 0. \\ v_i', & \text{if } t+1 \leq i \leq t+s \text{ and } d_i = 0, d_i' > 0. \\ v_i v_i' & \text{if } t+1 \leq i \leq t+s \text{ and } d_i, d_i' > 0. \end{cases}$$

Then $u = u_1 \cdots u_{t+s}$ *satisfies* $V \downarrow K[u] = \bigoplus_{i=1}^{t} 2 \cdot V_{c_i} \oplus \bigoplus_{j=t+1}^{t+s} \left(V_{2d_j+1} \oplus V_{2d_j'+1}\right)$.

Lemma 2.8.18 (Type D $(p = 2)$)**.** *Assume that* $p = 2$*. Let* $G = \mathrm{SO}(V)$ *be a Chevalley group of type* D_l *as defined above, where* $l \geq 2$*. Let* $1 \leq c_1 \leq \cdots \leq c_t$*, and* $1 \leq d_{t+1} \leq d_{t+1}' \leq \cdots \leq d_{t+s} \leq d_{t+s}'$*, such that* $\sum_{i=1}^{t} c_i + \sum_{j=t+1}^{t+s}(d_j + d_j') = l$*. Set* $c_i = d_i + d_i'$ *for* $t+1 \leq i \leq t+s$*, set* $k_1 = 1$ *and* $k_i = 1 + c_1 + \cdots + c_{i-1}$ *for* $1 < i \leq t+s$*. For* $t+1 \leq i \leq t+s$ *with* $d_i > 1$*, set*

$$v_i = \prod_{j=k_i}^{k_i+d_i'-2} x_{\alpha_j}(1) \cdot x_{\varepsilon_{k_i+d_i'-1} - \varepsilon_{k_i+c_i-1}}(1) \cdot x_{\varepsilon_{k_i+d_i'-1} + \varepsilon_{k_i+c_i-1}}(1).$$

For all $t+1 \leq i \leq t+s$ *with* $d_i' > 1$*, set*

$$v_i' = \prod_{j=k_i+d_i'}^{k_i+c_i-2} x_{\alpha_j}(1).$$

Define

$$u_i = \begin{cases} 1, & \text{if } 1 \leq i \leq t \text{ and } c_i = 1. \\ \prod_{j=k_i}^{k_i+c_i-2} x_{\alpha_j}(1), & \text{if } 1 \leq i \leq t \text{ and } c_i > 1. \\ v_i', & \text{if } t+1 \leq i \leq t+s \text{ and } d_i = 1. \\ v_i v_i', & \text{if } t+1 \leq i \leq t+s \text{ and } d_i > 1. \end{cases}$$

Then $u = u_1 \cdots u_{t+s}$ *satisfies* $V \downarrow K[u] = \bigoplus_{i=1}^{t} W(c_i) \oplus \bigoplus_{j=t+1}^{t+s} \left(V(2d_j) + V(2d_j')\right)$.

What remains is to describe representatives in the cases where we have two unipotent conjugacy classes with the same decomposition $V \downarrow K[u]$. For these cases, representatives are given by the next two lemmas.

Lemma 2.8.19 (Type D, split classes)**.** *Let* $G = \mathrm{SO}(V)$ *be a Chevalley group of type* D_l *as defined above, where* $l \geq 2$*. Let* $1 < c_1 \leq \cdots \leq c_t$ *be such that* c_i *is even for all* i *and* $\sum_{i=1}^{l} c_i = l$*. Set* $k_1 = 1$ *and* $k_i = 1 + c_1 + \cdots + c_{i-1}$ *for* $1 < i \leq t$*. For all* $1 \leq i \leq t$*, define* $u_i = \prod_{j=k_i}^{k_i+c_i-2} x_{\alpha_j}(1)$*. Set* $u_t' = x_{\alpha_l}(1)$ *if* $c_t = 2$ *and* $u_t' = \prod_{j=k_t}^{k_t+c_t-3} x_{\alpha_j}(1) x_{\alpha_l}(1)$ *if* $c_t > 2$*. Then:*

(i) *If* $p \neq 2$*, the unipotent elements* $u_1 \cdots u_{t-1} u_t$ *and* $u_1 \cdots u_{t-1} u_t'$ *are representatives for the two classes of unipotent elements* u *with* $V \downarrow K[u] = \bigoplus_{i=1}^{t} 2 \cdot V_{c_i}$*.*

(ii) *If* $p = 2$*, the unipotent elements* $u_1 \cdots u_{t-1} u_t$ *and* $u_1 \cdots u_{t-1} u_t'$ *are representatives for the two classes of unipotent elements* u *with* $V \downarrow K[u] = \bigoplus_{i=1}^{t} W(c_i)$*.*

Proof. For both claims, we see from Lemma 2.8.17 and Lemma 2.8.18 above that for $u = u_1 \cdots u_t$ we have $V \downarrow K[u] = \bigoplus_{i=1}^t 2 \cdot V_{c_i}$ if $p \neq 2$ and $V \downarrow K[u] = \bigoplus_{i=1}^t W(c_i)$ if $p = 2$. Furthermore, for the graph automorphism τ of G induced by the graph automorphism of Δ swapping the two end nodes, we have $\tau(x_{\alpha_i}(1)) = x_{\alpha_i}(1)$ for $1 \leq i \leq l-2$, and $\tau(x_{\alpha_{l-1}}(1)) = x_{\alpha_l}(1)$, $\tau(x_{\alpha_l}(1)) = x_{\alpha_{l-1}}(1)$. Thus $\tau(u) = u_1 \cdots u_{t-1}u_t'$, so now the claim follows from Lemma 2.8.6. □

2.8.4 Type B_l, $l \geq 2$ ($p = 2$)

Suppose that $p = 2$. Let Φ be a simple root system of type C_l and let G be a simply connected Chevalley group over K with this root system. Then the dual root system $\Phi^\vee$ is a simple root system of type B_l. Denote by $G^\vee$ the simply connected Chevalley group over K with root system $\Phi^\vee$, and denote by $\alpha^\vee$ the dual root corresponding to a root $\alpha \in \Phi$. Since $p = 2$, it follows from [Ste68, Theorem 28] that there is an isogeny $f : G \to G^\vee$ of algebraic groups satisfying

$$f(x_\alpha(c)) = \begin{cases} x_{\alpha^\vee}(c) & \text{if } \alpha \text{ is a long root,} \\ x_{\alpha^\vee}(c^2) & \text{if } \alpha \text{ is a short root.} \end{cases}$$

for all $\alpha \in \Phi$ and $c \in K$.

Now by Lemma 2.1.2 (i), the map f defines a bijection between the unipotent conjugacy classes of G and $G^\vee$. Furthermore, if Δ is a base for Φ, then $\Delta^\vee = \{\alpha^\vee : \alpha \in \Delta\}$ is a base for $\Phi^\vee$, and $(\Phi^\vee)^+ = \{\alpha^\vee : \alpha \in \Phi^+\}$ is the set of positive roots. Thus by using Lemma 2.8.12, we find representatives $\prod_{\alpha \in (\Phi^\vee)^+} x_{\alpha^\vee}(c_{\alpha^\vee})$ for unipotent conjugacy classes of $G^\vee$, as desired.

2.8.5 Type B_l, $l \geq 1$ ($p \neq 2$)

α		$\alpha = \sum_{k=1}^l c_k \alpha_i$
$\varepsilon_i - \varepsilon_j$,	$1 \leq i < j \leq l$	$\sum_{i \leq k \leq j-1} \alpha_k$
$\varepsilon_i + \varepsilon_j$,	$1 \leq i < j \leq l$	$\sum_{i \leq k \leq j-1} \alpha_k + \sum_{j \leq k \leq l} 2\alpha_k$
ε_i,	$1 \leq i \leq l$	$\sum_{i \leq k \leq l} \alpha_k$

Table 2.8: Type B_l, expressions for roots $\alpha \in \Phi^+$ in terms of base Δ.

Assume that $p \neq 2$. Let $e_1, \ldots, e_l, e_0, e_{-l}, \ldots, e_{-1}$ be a basis for a complex vector space $V_\mathbb{C}$, and let $V_\mathbb{Z}$ be the $\mathbb{Z}$-lattice spanned by this basis. We have a non-degenerate symmetric bilinear form $(-,-)$ on $V_\mathbb{C}$ defined by $(e_i, e_{-i}) = 1 = (e_{-i}, e_i)$, by $(e_0, e_0) = 2$, and $(e_i, e_j) = 0$ for $i \neq -j$. Let $\mathfrak{so}(V_\mathbb{C})$ be the Lie algebra formed by the linear endomorphisms X of $V_\mathbb{C}$ satisfying $(Xv, w) + (v, Xw) = 0$ for all $v, w \in V_\mathbb{C}$. Then $\mathfrak{so}(V_\mathbb{C})$ is a simple Lie algebra of type B_l. Let $\mathfrak{h}$ be the Cartan subalgebra formed by the diagonal matrices in $\mathfrak{so}(V_\mathbb{C})$. Then $\mathfrak{h} = \{\text{diag}(h_1, \ldots, h_l, 0, -h_l, \ldots, -h_1) : h_i \in \mathbb{C}\}$. For $1 \leq i \leq l$, define maps $\varepsilon_i : \mathfrak{h} \to \mathbb{C}$ by $\varepsilon_i(h) = h_i$ where $h = \text{diag}(h_1, \ldots, h_l, 0, -h_l, \ldots, -h_1)$. Now $\Phi = \{\pm(\varepsilon_i \pm \varepsilon_j) : 1 \leq i < j \leq l\} \cup \{\pm\varepsilon_i : 1 \leq i \leq l\}$ is the root system of $\mathfrak{so}(V_\mathbb{C})$, and $\Phi^+ = \{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq l\} \cup \{\varepsilon_i : 1 \leq i \leq l\}$ is a system of positive roots. The base Δ of Φ corresponding to Φ^+ is $\Delta = \{\alpha_1, \ldots, \alpha_l\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for

$1 \leq i < l$ and $\alpha_l = \varepsilon_l$. We give the expressions of roots $\alpha \in \Phi^+$ in terms of Δ in Table 2.8.

As before, for all i, j let $E_{i,j}$ be the linear endomorphism on $V_{\mathbb{C}}$ such that $E_{i,j}(e_j) = e_i$ and $E_{i,j}(e_k) = 0$ for $k \neq j$. Then a Chevalley basis for $\mathfrak{so}(V_{\mathbb{C}})$ with positive structure constants for extraspecial pairs is given by

$$\begin{aligned} X_{\varepsilon_i - \varepsilon_j} &= E_{i,j} - E_{-j,-i} && \text{for all } i \neq j, \\ X_{\varepsilon_i + \varepsilon_j} &= E_{j,-i} - E_{i,-j} && \text{for all } i < j, \\ X_{-(\varepsilon_i + \varepsilon_j)} &= E_{-i,j} - E_{-j,i} && \text{for all } i < j, \\ X_{\varepsilon_i} &= 2E_{i,0} - E_{0,-i} && \text{for all } i, \\ X_{-\varepsilon_i} &= E_{0,i} - 2E_{-i,0} && \text{for all } i. \end{aligned}$$

Now $V_{\mathbb{Z}}$ is a $\mathscr{U}_{\mathbb{Z}}$-invariant lattice in $V_{\mathbb{C}}$. We define $V = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$. Note that $(-,-)$ also defines a non-degenerate symmetric form on V. Then we have the following result.

Lemma 2.8.20 ([Ree57, 7])**.** *The Chevalley group of type B_l induced by V is equal to the group $G = \mathrm{SO}(V)$ of invertible linear maps $V \to V$ with determinant 1 that preserve $(-,-)$.*

By abuse of notation we identify the basis $(e_i \otimes 1)$ of V with (e_i). Let $u \in G$ be a unipotent element. Since $p \neq 2$, by Proposition 2.3.1 and Proposition 2.3.2 the conjugacy class of u in G is uniquely determined by the Jordan block sizes of u on V. By Proposition 2.3.2 (ii), we know that each Jordan block of even size must occur with even multiplicity. Furthermore, since $\dim V$ is odd, the total number of Jordan blocks of odd size must be odd. Thus the conjugacy class of u is uniquely determined by the decomposition

$$V \downarrow K[u] = \bigoplus_{i=1}^{t} 2 \cdot V_{c_i} \oplus \bigoplus_{j=1}^{s} \left(V_{2d_j+1} \oplus V_{2d'_j+1} \right) \oplus V_{2d_{s+1}+1},$$

where $1 \leq c_1 \leq \cdots \leq c_t$ are even, and $0 \leq d_1 \leq d'_1 \leq \cdots \leq d_s \leq d'_s \leq d_{s+1}$, and $\sum_{i=1}^{t} c_i + \sum_{j=1}^{s} (d_j + d'_j + 1) + d_{s+1} = l$.

Now let W_1 be the subspace of V spanned by $B_1 = \{e_{\pm j} : 1 \leq j \leq l - d_{s+1}\}$ and let W_2 be the subspace of V spanned by $B_2 = \{e_{\pm j} : l - d_{s+1} + 1 \leq j \leq l\} \cup \{e_0\}$. We have an orthogonal direct sum $V = W_1 \oplus W_2$, so there is a naturally embedded subgroup $\mathrm{SO}(W_1) \times \mathrm{SO}(W_2) < \mathrm{SO}(V)$, where $\mathrm{SO}(W_i)$ is the subgroup of all $g \in G$ such that $g(W_i) = W_i$ and $ge_j = e_j$ for all $e_j \notin B_i$.

It follows then that a representative for the unipotent class determined by the (possibly empty) sequences (c_i), (d_j), and (d'_j) is given by $u = u_1 u_2$, where $u_1 \in \mathrm{SO}(W_1)$ and

$$W_1 \downarrow K[u_1] = \bigoplus_{i=1}^{t} 2 \cdot V_{c_i} \oplus \bigoplus_{j=1}^{s} \left(V_{2d_j+1} \oplus V_{2d'_j+1} \right),$$

and where $u_2 \in \mathrm{SO}(W_2)$ and u_2 acts on W_2 with a single Jordan block of size $2d_{s+1} + 1$.

For finding u_1, note first that if $\dim W_1 \leq 2$, then we can choose $u_1 = 1$. Suppose then that $\dim W_1 > 2$ and write $\dim W_1 = 2k$. Then it is a consequence of Lemma 2.8.13 that $\mathrm{SO}(W_1)$ is a subsystem subgroup of type D_k with root system $\{\pm(\varepsilon_i \pm \varepsilon_j) : 1 \leq i < j \leq k\}$. The root system has a base $\{\varepsilon_i - \varepsilon_{i+1} : 1 \leq i \leq k-1\} \cup \{\varepsilon_{k-1} + \varepsilon_k\}$, with set of positive roots $\{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq k\}$. Now one can apply Lemma 2.8.17 to find a suitable $u_1 \in \mathrm{SO}(W_1)$ as a product of root elements.

We proceed to explain how to find u_2. If $d_{s+1} = 0$, then it is clear that we can choose $u_2 = 1$. Suppose then that $d_{s+1} > 0$. Now it is a consequence of Lemma 2.8.20 that $\mathrm{SO}(W_2)$ is a subsystem subgroup of G with root system $\{\pm(\varepsilon_i \pm \varepsilon_j) : l - d_{s+1} + 1 \leq i < j \leq l\} \cup \{\pm\varepsilon_i : l - d_{s+1} + 1 \leq i \leq l\}$ and a base $\{\alpha_i : l - d_{s+1} + 1 \leq i \leq l\}$. It follows from Proposition 2.3.3 and Lemma 2.8.1 that we can choose $u_2 = \prod_{j=l-d_{s+1}+1}^{l} x_{\alpha_j}(1)$.

2.9 Computation of Jordan block sizes in irreducible representations

For our solution of Problem 1.1.6, we have to answer the following question in some specific cases.

Problem 2.9.1. *Let $\varphi : G \to \mathrm{GL}(V)$ be a non-trivial rational irreducible representation of G. For a unipotent element $u \in G$, what are the Jordan block sizes of $\varphi(u)$?*

Since in characteristic $p = 2$ the Jordan block sizes do not determine when a unipotent element of a classical group ($\mathrm{Sp}(V)$ or $\mathrm{SO}(V)$) is distinguished (Section 2.4), we will also have to consider the following question.

Problem 2.9.2. *Assume that $p = 2$. Let $\varphi : G \to \mathrm{GL}(V)$ be a non-trivial rational irreducible representation of G with a non-degenerate G-invariant alternating bilinear form. For a unipotent element $u \in G$, what is the conjugacy class of $\varphi(u)$ in $\mathrm{Sp}(V)$?*

Recall that we are working over a field of positive characteristic. Thus in general Problem 2.9.1 and Problem 2.9.2 are out of reach by current methods, as even finding the dimensions of the rational irreducible representations is a major open problem. However, for fixed G, a fixed unipotent conjugacy class $\mathscr{C}$ of G, and V of small dimension, it is possible to calculate the Jordan block sizes of $\varphi(u)$ for $u \in \mathscr{C}$ and the orthogonal decomposition of $V \downarrow K[u]$ with a computer. We will proceed to explain the methods we have used to do this.

First note that using the methods from 2.8, we can find a representative $u = \prod_{\alpha \in \Phi^+} x_\alpha(c_\alpha)$ of $\mathscr{C}$, where $c_\alpha \in \mathbb{Z}$ for all $\alpha \in \Phi^+$. Let $\lambda \in X(T)^+$ be non-zero p-restricted dominant weight, and write $\lambda = \sum_{i=1}^{l} a_i \omega_i$, where $0 \leq a_i \leq p-1$. Our goal is to determine the decomposition $L_G(\lambda) \downarrow K[u]$.

Recall that the Weyl module $V_G(\lambda)$ has a unique maximal G-submodule M, and $V_G(\lambda)/M \cong L_G(\lambda)$. Furthermore, M is also the unique maximal $G(\mathbb{F}_p)$-module of $V_G(\lambda)$ by [Won72, Theorem 2D]. Since our representative u is in $G(\mathbb{F}_p)$, we do our computations with the finite group $G(\mathbb{F}_p)$.

The $G(\mathbb{F}_p)$-module $V_G(\lambda)$ can be constructed with the Chevalley construction. This can be done with MAGMA, which uses an implementation of the algorithm in [CMT04]. For example, the following MAGMA commands construct the representation $\mathrm{r} : G(\mathbb{F}_p) \to \mathrm{GL}(V_G(\omega_2))$ for $G(\mathbb{F}_p)$ of type A_3:

```
G := GroupOfLieType("A3", GF(p) : Isogeny := "SC");
r := HighestWeightRepresentation(G, [0,1,0]);
```

In particular, with respect to a fixed K-vector space basis of weight vectors (given by MAGMA), we can compute the matrix of $x_\alpha(c)$ acting on $V_G(\lambda)$ for all $\alpha \in \Phi$ and $c \in \mathbb{F}_p$. In the example above, we would get the matrix of $x_{\alpha_1+\alpha_2}(1)$ acting on $V_G(\lambda)$ with the following commands. Note that here $x_{\alpha_1+\alpha_2}(1) = x_{\alpha_4}(1)$ in the ordering of Φ^+ used in MAGMA, which is the total order $\leq$ defined in Section 2.8.

```
g := elt< G | <4,1>>;
r(g);
```

Since $V_G(\lambda)$ has a unique maximal submodule, it follows that $V_G(\lambda)^*$ has a simple socle, isomorphic to $L_G(\lambda)^*$ and generated by a maximal vector $f \in V_G(\lambda)^*$ of weight $-w_0(\lambda)$, which is unique up to a scalar. Then using the matrices giving the action of $x_\alpha(c)$ on $V_G(\lambda)$, we can compute a basis for $Gf \cong L_G(\lambda)^*$ and thus the matrix of $x_\alpha(c)$ acting on $L_G(\lambda)^*$ for any $\alpha \in \Phi$ and $c \in \mathbb{F}_p$. By using the expression of u as a product of the elements $x_\alpha(c)$, we then get the matrix of u acting on $L_G(\lambda)^*$. Now it is a simple matter of computing ranks of some matrices to find the Jordan block sizes of u acting on $L_G(\lambda)^*$ (see Lemma 3.1.2). These Jordan block sizes are the same as of u acting on $L_G(\lambda)$, giving us the Jordan block sizes occurring $L_G(\lambda) \downarrow K[u]$.

If $p \neq 2$ we now have all the information we need for our purposes. If $p = 2$ and $L_G(\lambda)$ is self-dual, we also need a way to determine the conjugacy class of the image of u in $\mathrm{Sp}(L_G(\lambda))$. To this end, one can use the MAGMA command **InvariantBilinearForms** to find a non-degenerate alternating bilinear form on $L_G(\lambda)$ invariant under the action of $x_\alpha(c)$ for all $\alpha \in \Phi$ and $c \in K$. It is then straightforward to apply Lemma 2.4.5 to determine the conjugacy class of u in $\mathrm{Sp}(L_G(\lambda))$ (see Remark 2.4.9).

2.10 Action of triality on unipotent conjugacy classes of D_4

In this section, we finish the discussion on the action of graph automorphisms given in Section 2.8 and give the action of a graph automorphism of $G = D_4$ induced by triality. We will do this by direct computation. For $p = 3$ the result is given in [LLS14, Proof of Lemma 3.2].

We begin by recalling the Chevalley construction from Section 2.8.3. Let the root system $\Phi = \{\pm(\varepsilon_i \pm \varepsilon_j) : 1 \leq i < j \leq 4\}$ of type D_4 be as in 2.8.3, with base $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq 3$, and $\alpha_4 = \varepsilon_3 + \varepsilon_4$. We choose a Chevalley basis $\{X_\alpha : \alpha \in \Phi\} \cup \{H_\alpha : \alpha \in \Delta\}$ as in Section 2.8.3.

Let G be a simply connected Chevalley group constructed with the Chevalley construction, using this Chevalley basis. Let $G' = \mathrm{SO}(V)$ be the Chevalley group constructed using $V_\mathbb{C}$ as in Section 2.8.3. Denote the root elements generating G' by $x'_\alpha(c)$, where $\alpha \in \Phi$ and $c \in K$. By [Ste68, Corollary 5, pg.44], we have a surjective morphism $\varphi : G \to G'$ such that $\varphi(x_\alpha(c)) = x'_\alpha(c)$ for all $\alpha \in \Phi$ and $c \in K$.

We will order the positive roots $\Phi^+ = \{\alpha_1, \ldots, \alpha_{12}\}$ with the total order $\leq$ defined in Section 2.8. We give the order explicitly in Table 2.1. One finds that the structure constants $N_{\alpha,\beta}$ for our Chevalley basis are as given in Table 2.2.

Now let $\sigma' : \Delta \to \Delta$ be the triality graph automorphism of Δ, which acts on the simple roots as follows.

$$\alpha_1 \mapsto \alpha_3$$
$$\alpha_2 \mapsto \alpha_2$$
$$\alpha_3 \mapsto \alpha_4$$
$$\alpha_4 \mapsto \alpha_1$$

Then by Lemma 2.8.4, there exists an isomorphism $\sigma : G \to G$ and signs $\varepsilon_\alpha = \pm 1$ such that $\varepsilon_\alpha = 1$ for all $\alpha \in \Delta$, and $\sigma(x_\alpha(c)) = x_{\sigma'(\alpha)}(\varepsilon_\alpha c)$ for all $\alpha \in \Phi$ and $c \in K$.

The signs ε_α for the isomorphism σ are uniquely determined, and can be found using the Chevalley commutator relations [Ste68, Corollary to Lemma 15] as follows. Since the root system D_4 is simply laced, the commutator relations have the simple form $[x_\alpha(c), x_\beta(d)] = x_{\alpha+\beta}(N_{\alpha,\beta}cd)$ for all $\alpha, \beta \in \Phi$ such that $\alpha + \beta \in \Phi$. Applying σ to both sides we get the relation $\varepsilon_\alpha \varepsilon_\beta N_{\sigma'(\alpha),\sigma'(\beta)} = \varepsilon_{\alpha+\beta} N_{\alpha,\beta}$. From this one can calculate that $\varepsilon_\alpha = 1$ for all α, except $\alpha = \pm(\alpha_1 + \alpha_2)$ and $\alpha = \pm(\alpha_2 + \alpha_4)$, for which $\varepsilon_\alpha = -1$.

Our goal next is to compute the action of σ on the unipotent conjugacy classes of G. The first step is to find representatives for all unipotent conjugacy classes. This can be done with the results of Section 2.8.3. We have listed representatives in Table 2.3 and Table 2.4. Note that there are some pairs of classes of unipotent elements u with same decomposition $V \downarrow K[u]$. To distinguish between them, in tables 2.3 and 2.4 we have labeled one of them with the symbol $(V \downarrow K[u])'$ and the other by $(V \downarrow K[u])''$.

Now using the signs ε_α determined above, we find $\sigma(u)$ as a product of root elements, for each unipotent class representative $u \in G$ in tables 2.3 and 2.4. Using this and the map $\varphi : G \to G'$ defined above, we find explicit matrices for $\varphi(\sigma(u))$, which allows one to compute the decomposition $V \downarrow K[\sigma(u)]$ (see Remark 2.4.9). The end result of this computation is that the orthogonal decompositions are as given in Table 2.5 and Table 2.6.

What still remains is to check, for the split classes, which image of σ gives the class $(V \downarrow K[u])'$, and which one gives $(V \downarrow K[u])''$. Since σ is an automorphism of order 3, the orbits of σ acting on the unipotent classes have size 1 or 3. From this fact and the decompositions $V \downarrow K[\sigma(u)]$ it is easy to deduce that the action of σ on the unipotent classes is precisely as given in Table 2.5 and Table 2.6.

Remark 2.10.1. There are also many other ways to determine the conjugacy class of $\sigma(u)$ without relying on the computation of $V \downarrow K[\sigma(u)]$. For example, it is immediate from Lemma 2.8.1 that the regular unipotent class is stabilized by σ.

We see in particular that in all cases all distinguished unipotent classes are stabilized by σ. Combining this observation with Lemma 2.8.5 and Lemma 2.8.6, we get the following corollary.

Proposition 2.10.2. *Let G be a simple algebraic group and let σ be a graph automorphism of G. Then*

(i) *σ fixes all distinguished unipotent classes of G.*

(ii) *Let V be a G-module and $u \in G$ a distinguished unipotent element. Then $V \downarrow K[u] \cong V^\sigma \downarrow K[u]$. In particular, the element u acts on V as a distinguished unipotent element if and only if u acts on V^σ as a distinguished unipotent element.*

$\alpha = \alpha_i$	$\alpha = \sum_{i=1}^4 k_i \alpha_i$	$\alpha = \varepsilon_i \pm \varepsilon_j$
α_1	α_1	$\varepsilon_1 - \varepsilon_2$
α_2	α_2	$\varepsilon_2 - \varepsilon_3$
α_3	α_3	$\varepsilon_3 - \varepsilon_4$
α_4	α_4	$\varepsilon_3 + \varepsilon_4$
α_5	$\alpha_1 + \alpha_2$	$\varepsilon_1 - \varepsilon_3$
α_6	$\alpha_2 + \alpha_3$	$\varepsilon_2 - \varepsilon_4$
α_7	$\alpha_2 + \alpha_4$	$\varepsilon_2 + \varepsilon_4$
α_8	$\alpha_1 + \alpha_2 + \alpha_3$	$\varepsilon_1 - \varepsilon_4$
α_9	$\alpha_1 + \alpha_2 + \alpha_4$	$\varepsilon_1 + \varepsilon_4$
α_{10}	$\alpha_2 + \alpha_3 + \alpha_4$	$\varepsilon_2 + \varepsilon_3$
α_{11}	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$	$\varepsilon_1 + \varepsilon_3$
α_{12}	$\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$	$\varepsilon_1 + \varepsilon_2$

Table 2.1: Positive roots of D_4.

	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}
α_1		1				1	1			1		
α_2	-1		1	1							1	
α_3		-1			-1		1		1			
α_4		-1			-1	1		1				
α_5			1	1						-1		
α_6	-1			-1					1			
α_7	-1		-1					1				
α_8				-1			-1					
α_9			-1			-1						
α_{10}	-1				1							
α_{11}		-1										
α_{12}												

Table 2.2: Structure constants N_{α_i,α_j} used for D_4, with empty entries indicating zero and roots ordered as in Table 2.1.

Unipotent class	Representative
$V(2)+V(6)$	$x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)$
$V(4)^2$	$x_{\alpha_1}(1)x_{\varepsilon_2-\varepsilon_4}(1)x_{\varepsilon_2+\varepsilon_4}(1)x_{\alpha_3}(1)$
$W(4)'$	$x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_3}(1)$
$W(4)''$	$x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_4}(1)$
$W(1)+V(2)+V(4)$	$x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\varepsilon_2+\varepsilon_3}(1)$
$W(1)+W(3)$	$x_{\alpha_1}(1)x_{\alpha_2}(1)$
$W(2)+V(2)^2$	$x_{\alpha_1}(1)x_{\varepsilon_1+\varepsilon_2}(1)x_{\alpha_3}(1)$
$(W(2)^2)'$	$x_{\alpha_1}(1)x_{\alpha_3}(1)$
$(W(2)^2)''$	$x_{\alpha_1}(1)x_{\alpha_4}(1)$
$V(2)^2+W(1)^2$	$x_{\alpha_1}(1)x_{\varepsilon_1+\varepsilon_2}(1)$
$W(2)+W(1)^2$	$x_{\alpha_1}(1)$

Table 2.3: Representatives for unipotent classes of $G = D_4$ when $p = 2$.

Unipotent class	Representative
$[1,7]$	$x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)$
$[3,5]$	$x_{\alpha_1}(1)x_{\varepsilon_2-\varepsilon_4}(1)x_{\varepsilon_2+\varepsilon_4}(1)x_{\alpha_3}(1)x_{\alpha_4}(-1)$
$[1^3,5]$	$x_{\alpha_1}(1)x_{\varepsilon_2-\varepsilon_4}(1)x_{\varepsilon_2+\varepsilon_4}(1)$
$[4^2]'$	$x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_3}(1)$
$[4^2]''$	$x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_4}(1)$
$[1^2,3^2]$	$x_{\alpha_1}(1)x_{\alpha_2}(1)$
$[1,2^2,3]$	$x_{\alpha_1}(1)x_{\varepsilon_1+\varepsilon_2}(1)x_{\alpha_3}(1)$
$[1^5,3]$	$x_{\alpha_1}(1)x_{\varepsilon_1+\varepsilon_2}(1)$
$[2^4]'$	$x_{\alpha_1}(1)x_{\alpha_3}(1)$
$[2^4]''$	$x_{\alpha_1}(1)x_{\alpha_4}(1)$
$[1^4,2^2]$	$x_{\alpha_1}(1)$

Table 2.4: Representatives for unipotent classes of $G = D_4$ when $p \neq 2$.

Unipotent class of u	Unipotent class of $\sigma(u)$
$V(2)+V(6)$	$V(2)+V(6)$
$V(4)^2$	$V(4)^2$
$W(4)'$	$W(1)+V(2)+V(4)$
$W(4)''$	$W(4)'$
$W(1)+V(2)+V(4)$	$W(4)''$
$W(1)+W(3)$	$W(1)+W(3)$
$W(2)+V(2)^2$	$W(2)+V(2)^2$
$(W(2)^2)'$	$W(1)^2+V(2)^2$
$(W(2)^2)''$	$(W(2)^2)'$
$W(1)^2+V(2)^2$	$(W(2)^2)''$
$W(2)+W(1)^2$	$W(2)+W(1)^2$

Table 2.5: Action of σ on unipotent classes of $G = D_4$ when $p = 2$.

Unipotent class of u	Unipotent class of $\sigma(u)$
$[1,7]$	$[1,7]$
$[3,5]$	$[3,5]$
$[1^3,5]$	$[4^2]''$
$[4^2]'$	$[1^3,5]$
$[4^2]''$	$[4^2]'$
$[1^2,3^2]$	$[1^2,3^2]$
$[1,2^2,3]$	$[1,2^2,3]$
$[1^5,3]$	$[2^4]''$
$[2^4]'$	$[1^5,3]$
$[2^4]''$	$[2^4]'$
$[1^4,2^2]$	$[1^4,2^2]$

Table 2.6: Action of σ on unipotent classes of $G = D_4$ when $p \neq 2$.

Chapter 3

Linear algebra

In this chapter, we give various lemmas about unipotent linear maps that will be needed in the sequel.

Let $f : V \to V$ be a linear map. If W is a subspace of V invariant under f, we will denote the restriction of f to W by f_W and the linear map induced on V/W by $f_{V/W}$.

3.1 Basic lemmas about unipotent elements

Definition 3.1.1. Let $u \in \mathrm{GL}(V)$ be unipotent. For all $m \geq 1$, we denote by $r_m(u)$ the number of Jordan blocks of size m in the Jordan decomposition of u.

Lemma 3.1.2. *Let $u \in \mathrm{GL}(V)$ be unipotent and denote $X = u - 1$. Then for all $m \geq 1$, we have*

$$r_m(u) = \operatorname{rank} X^{m+1} + \operatorname{rank} X^{m-1} - 2 \operatorname{rank} X^m$$

and

$$r_m(u) = 2 \dim \ker X^m - \dim \ker X^{m+1} - \dim \ker X^{m-1}.$$

Proof. The formulas follow from the fact that $\dim \ker X^m = \sum_{k=1}^{m} t_k$, where t_k is the number of Jordan blocks of size $\geq k$ [Jan04, 1.1]. □

Lemma 3.1.3. *Let u be a unipotent matrix of order p. Then the number Jordan blocks of size p in the Jordan decomposition of u is equal to* $\operatorname{rank}(u-1)^{p-1}$.

Proof. By Lemma 3.1.2, we have that the number of Jordan blocks of size p is equal to $\operatorname{rank}(u-1)^{p+1} + \operatorname{rank}(u-1)^{p-1} - 2\operatorname{rank}(u-1)^p$. The claim follows because $(u-1)^p = (u-1)^{p+1} = 0$. □

The next lemma is observed in [Sup01, (1), pg. 2585]. We include the proof (which is easy) for completeness.

Lemma 3.1.4. *Let u be a unipotent linear map on the vector space V and suppose that u has order p. Suppose that V has a filtration $V = W_1 \supseteq W_2 \supseteq \cdots \supseteq W_t \supseteq W_{t+1} = 0$ of $K[u]$-submodules. Then $r_p(u) \geq \sum_{i=1}^{t} r_p(u_{W_i/W_{i+1}})$.*

Proof. Let $X = u - 1$. Writing X as a block triangular matrix, we have

$$X^{p-1} = \begin{pmatrix} (X_{W_t})^{p-1} & * & * & \cdots & * \\ 0 & (X_{W_{t-1}/W_t})^{p-1} & * & \cdots & * \\ 0 & 0 & \ddots & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (X_{W_1/W_2})^{p-1} \end{pmatrix}$$

so $\operatorname{rank} X^{p-1} \geq \sum_{i=1}^{t} \operatorname{rank}(X_{W_i/W_{i+1}})^{p-1}$ and the claim follows from Lemma 3.1.3. □

3.2 Jordan blocks in subspaces and quotients

Some of the irreducible representations $L_G(\lambda)$ of G that we have to consider in the solution of Problem 1.1.6 are best understood as subquotients of certain indecomposable modules W. In many cases that are relevant to us, the module W will be uniserial of the form $L_G(0)/L_G(\lambda)/L_G(0)$. For example, if $G = \mathrm{SL}(V)$, then $W = V \otimes V^* = L_G(0)/L_G(\omega_1 + \omega_l)/L_G(0)$ (uniserial) if p divides $\dim V$. We have a decent understanding of what Jordan blocks sizes of $u \in G$ acting on W look like, and the idea is that we can use this information to study the Jordan block sizes of u acting on $L_G(\lambda)$.

We begin with two lemmas which describe how the Jordan block sizes change when moving from the whole space to a subspace of codimension one, or to a quotient by a one-dimensional subspace.

Lemma 3.2.1. *Let $u \in \mathrm{GL}(V)$ be unipotent and denote $X = u - 1$. Suppose that $W \subseteq V$ is a subspace invariant under u such that $\dim V/W = 1$. Let $m \geq 0$ be such that $\ker X^m \subseteq W$ and $\ker X^{m+1} \not\subseteq W$. Then*

(a) *if $m = 0$, we have*

- $r_1(u_W) = r_1(u) - 1$,
- $r_i(u_W) = r_i(u)$ *for all* $i \neq 1$.

(b) *if $m \geq 1$, we have*

- $r_{m+1}(u_W) = r_{m+1}(u) - 1$,
- $r_m(u_W) = r_m(u) + 1$,
- $r_i(u_W) = r_i(u)$ *for all* $i \neq m, m+1$.

Proof. Now $\ker X^i \subseteq W$ for all $0 \leq i \leq m$, which means that $\dim \ker X_W^i = \dim \ker X^i$ for all $0 \leq i \leq m$. By Lemma 3.1.2, this implies that for all $1 \leq i \leq m-1$, we have $r_i(u) = r_i(u_W)$.

Next note that $\ker X^i \not\subseteq W$ for all $i \geq m+1$, which means that $V/W = (\ker X^i + W)/W \cong W/\ker X^i \cap W$. Hence $\dim \ker X_W^i = \dim \ker X^i - 1$ for all $i \geq m+1$. Thus by Lemma 3.1.2, we have $r_m(u_W) = r_m(u) + 1$ if $m \geq 1$. Similarly $r_{m+1}(u_W) = r_{m+1}(u) - 1$ and $r_i(u_W) = r_i(u)$ for all $i \geq m+2$. □

Lemma 3.2.2. *Let $u \in \mathrm{GL}(V)$ be unipotent and denote $X = u - 1$. Suppose that $W \subseteq V$ is a subspace invariant under u such that $\dim W = 1$. Let $m \geq 0$ be such that $W \subseteq X^m(V)$ but $W \not\subseteq X^{m+1}(V)$. Then*

(a) *if* $m = 0$, *we have*

- $r_1(u_{V/W}) = r_1(u) - 1$,
- $r_i(u_{V/W}) = r_i(u)$ *for all* $i \neq 1$.

(b) *if* $m \geq 1$, *we have*

- $r_{m+1}(u_{V/W}) = r_{m+1}(u) - 1$,
- $r_m(u_{V/W}) = r_m(u) + 1$,
- $r_i(u_{V/W}) = r_i(u)$ *for all* $i \neq m, m+1$.

Proof. Now $X^i_{V/W}(V/W) = (X^i(V) + W)/W \cong X^i(V)/(W \cap X^i(V))$. Thus for $0 \leq i \leq m$ we have $\operatorname{rank} X^i_{V/W} = \operatorname{rank} X^i - 1$, because $W \subseteq X^i(V)$. Similarly we have $\operatorname{rank} X^i_{V/W} = \operatorname{rank} X^i$ for $i \geq m+1$, because $W \not\subseteq X^i(V)$ and thus $W \cap X^i(V) = 0$. Therefore by Lemma 3.1.2, we have $r_m(u_{V/W}) = r_m(u) + 1$ if $m \geq 1$. Similarly one sees $r_{m+1}(u_{V/W}) = r_{m+1}(u) - 1$ and $r_i(u_{V/W}) = r_i(u)$ for all $i \neq m, m+1$. □

In general, if $u \in \mathrm{GL}(V)$ is unipotent, then it is possible to find a filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ of u-invariant subspaces such that $\dim V_i/V_{i-1} = 1$ for all $1 \leq i \leq n$. For any such filtration, we can use Lemma 3.2.1 and Lemma 3.2.2 to describe, in terms of the Jordan block sizes of u on V, the possible Jordan block sizes of u acting on any subquotient V_i/V_j. We record one such description in the next lemma.

Lemma 3.2.3. *Let* $u \in \mathrm{GL}(V)$ *be unipotent and suppose that there exist* u*-invariant subspaces* $W' \subseteq W \subseteq V$ *such that* $\dim V/W = 1$ *and* $\dim W' = 1$. *Denote the image of* u *in* $\mathrm{GL}(W/W')$ *by* u'. *Then one of the following holds:*

(i) $r_1(u') = r_1(u) - 2$, *and* $r_k(u') = r_k(u)$ *for all* $k \neq 1$.

(ii) *There exists* $m \geq 2$ *such that* $r_m(u') = r_m(u) - 2$, $r_{m-1}(u') = r_{m-1}(u') + 2$, *and* $r_k(u') = r_k(u)$ *for all* $k \notin \{m, m-1\}$.

(iii) *There exists* $m \geq 3$ *such that* $r_m(u') = r_m(u) - 1$, $r_{m-2}(u') = r_{m-2}(u) + 1$, *and* $r_k(u') = r_k(u)$ *for all* $k \notin \{m, m-2\}$.

(iv) $r_2(u') = r_2(u) - 1$, *and* $r_k(u') = r_k(u)$ *for all* $k \neq 2$.

(v) *There exists* $m \geq n+2$, $n \geq 2$ *such that* $r_m(u') = r_m(u) - 1$, $r_{m-1}(u') = r_{m-1}(u) + 1$, $r_n(u') = r_n(u) - 1$, $r_{n-1}(u') = r_{n-1}(u) + 1$, *and* $r_k(u') = r_k(u)$ *for all* $k \notin \{m, m-1, n, n-1\}$.

(vi) *There exists* $m \geq 3$ *such that* $r_m(u') = r_m(u) - 1$, $r_{m-1}(u') = r_{m-1}(u) + 1$, $r_1(u') = r_1(u) - 1$, *and* $r_k(u') = r_k(u)$ *for all* $k \notin \{1, m, m-1\}$.

Proof. The claim follows by first applying Lemma 3.2.1 to u and W, and then Lemma 3.2.2 on u_W and W/W'. □

Definition 3.2.4. Suppose that $p \neq 2$. We will say that the action of a unipotent element $u \in \mathrm{GL}(V)$ on V is *inadmissible*, if at least one of the following holds:

- Some Jordan block of u of size > 1 has multiplicity ≥ 3.
- There are ≥ 2 Jordan block sizes of u with multiplicity ≥ 2.

Otherwise we will say that the action of u on V is *admissible*.

Definition 3.2.5. Suppose that $p = 2$. We will say that the action of a unipotent element $u \in \mathrm{GL}(V)$ on V is *inadmissible*, if at least one of the following holds:

- Some Jordan block size of u has multiplicity ≥ 4.
- There are ≥ 2 Jordan block sizes of u with multiplicity ≥ 3.
- There are ≥ 2 odd Jordan block sizes of u with multiplicity ≥ 2.
- Some Jordan block size of u has multiplicity ≥ 3, and u has a Jordan block of odd size.

Otherwise we will say that the action of u on V is *admissible*.

The point of making these two definitions is in the two lemmas below, which say that in certain cases if the action of u is "inadmissible", then u does not act as a distinguished unipotent element on a particular subquotient. These lemmas will be applied in Chapter 5 when we consider irreducible representations such as $L_{A_l}(\omega_1 + \omega_l)$ and $L_{C_l}(\omega_2)$, which can be constructed as such subquotients.

Lemma 3.2.6. *Suppose that $p \neq 2$. Let $u \in \mathrm{GL}(V)$ and suppose that there exist $K[u]$-submodules $W \subseteq W' \subseteq V$ such that $\dim W = 1$, $\dim V/W' = 1$ and that u leaves a non-degenerate symmetric form invariant on W'/W. If the action of u on V is inadmissible, then the image of u in $\mathrm{SO}(W'/W)$ is not a distinguished unipotent element.*

Proof. Denote the image of u in $\mathrm{SO}(W'/W)$ by u'. Suppose that the action of u on V is inadmissible. We will show that u' has a Jordan block size of multiplicity ≥ 2, and thus cannot be a distinguished unipotent element in $\mathrm{SO}(W'/W)$ (Proposition 2.3.4). To do this, we apply Lemma 3.2.3 and consider the different possibilities (i) - (vi) to the Jordan block sizes of u' given there. Below (i) - (vi) will always refer to the different cases of Lemma 3.2.3, and m and n will be the integers in cases (ii), (iii), (v), and (vi) of Lemma 3.2.3.

Suppose that the action of u on V is inadmissible. First, if u has some Jordan block of size > 1 with multiplicity ≥ 3, then in all cases (i) - (vi) it is clear that $r_k(u') \geq 2$ for some k.

The other possibility is that two distinct block sizes of u have multiplicity ≥ 2, say $r_{k_0}(u), r_{k_0'}(u) \geq 2$ for some $k_0 > k_0' \geq 1$. In cases (i), (ii), (iii) and (iv) it is clear that we have $r_{k_0}(u') \geq 2$ or $r_{k_0'}(u') \geq 2$.

In case (v), if we had $r_k(u') \leq 1$ for all k, then we must have $m = k_0$ and $n = k_0'$. Furthermore, now $r_m(u) = 2$, $r_n(u) = 2$, so $r_m(u') = r_n(u') = 1$. Since we are assuming that u' leaves a nondegenerate symmetric form invariant on W/W', the multiplicity of a block of even size must be even. Therefore both m and n are odd. But then $m-1$ and $n-1$ are even, so $r_{m-1}(u')$ and $r_{n-1}(u')$ are even. Since $r_{m-1}(u'), r_{n-1}(u')$ are ≥ 1 in this case, we have in fact $r_{m-1}(u'), r_{n-1}(u') \geq 2$.

Finally consider the possibility of case (vi). Similarly to case (v), if we had $r_k(u') \leq 1$ for all k, then we have $m = k_0$ and $1 = k_0'$, and $r_m(u) = 2$ and

$r_1(u) = 2$. Therefore $r_m(u') = 1$. Again since u' leaves a nondegenerate symmetric form invariant on W/W', any block of even size has even multiplicity, so it follows that m must be odd. Then $m-1$ is even, so because $r_{m-1}(u') \geq 1$ in this case, we have $r_{m-1}(u') \geq 2$. □

Lemma 3.2.7. *Suppose that $p = 2$. Let $u \in \mathrm{GL}(V)$ and suppose that there exist $K[u]$-submodules $W \subseteq W' \subseteq V$ such that $\dim W = 1$, $\dim V/W' = 1$ and that u leaves a non-degenerate alternating bilinear form invariant on W'/W. If the action of u on V is inadmissible, then the image of u in $\mathrm{Sp}(W'/W)$ is not a distinguished unipotent element.*

Proof. Denote the image of u in $\mathrm{Sp}(W'/W)$ by u'. Suppose that the action of u on V is inadmissible. We will show that u' has either a Jordan block of odd size, or a Jordan block of of multiplicity ≥ 3, and thus cannot be a distinguished unipotent element in $\mathrm{Sp}(W'/W)$ (Proposition 2.4.4). To do this, we apply Lemma 3.2.3 and consider the different possibilities (i) - (vi) to the Jordan block sizes of u' given there. Below (i) - (vi) will always refer to the different cases of Lemma 3.2.3, and m and n will be the integers in cases (ii), (iii), (v), and (vi) of Lemma 3.2.3.

Suppose that the action of u on V is inadmissible. Consider first the case where some block size of u has multiplicity ≥ 4, say $r_{k_0}(u) \geq 4$ for some $k_0 \geq 1$. In case (i) we have $r_{k_0}(u') \geq 4$ or $r_1(u') \geq 2$, and so u' is not distinguished in $\mathrm{Sp}(W'/W)$. Consider then case (ii). If $m \neq k_0$, then $r_{k_0}(u') \geq 4$ and thus u' is not distinguished in $\mathrm{Sp}(W'/W)$. If $m = k_0$, then $r_{k_0}(u') \geq 2$. So if k_0 is odd, then u' has odd block sizes and cannot be distinguished in $\mathrm{Sp}(W'/W)$. On the other hand if k_0 is even, then $r_{k_0-1}(u') \geq 2$ and again u' has odd block sizes. In cases (iii), (iv), (v) and (vi) it is clear that $r_{k_0}(u') \geq 3$ and thus u' is not distinguished in $\mathrm{Sp}(W'/W)$.

Suppose next that u has two distinct Jordan block sizes, each occurring with multiplicity ≥ 3, say $k_0' > k_0 \geq 1$ with $r_{k_0'}(u), r_{k_0}(u) \geq 3$. In all cases it is clear that $r_{k_0'}(u'), r_{k_0}(u) \geq 1$, so if k_0' or k_0 is odd then u' has odd block sizes and cannot be distinguished in $\mathrm{Sp}(W'/W)$. Suppose then that k_0' and k_0 are even. Then in cases (i), (ii), (iii), (iv) and (vi) it is clear that $r_{k_0'}(u') \geq 3$ or $r_{k_0}(u') \geq 3$ so u' cannot be distinguished in $\mathrm{Sp}(W'/W)$. In case (v) we also have $r_{k_0'}(u') \geq 3$ or $r_{k_0}(u') \geq 3$ unless $k_0' = m$ and $k_0 = n$. If $k_0' = m$ and $k_0 = n$, then $r_{k_0-1}(u') \geq 1$ and so u' has odd block sizes and cannot be distinguished in $\mathrm{Sp}(W'/W)$.

Consider next the case where u has ≥ 2 odd Jordan block sizes of u with multiplicity ≥ 2. That is, suppose that there exist $k_0' > k_0 \geq 1$ odd such that $r_{k_0'}(u) \geq 2$ and $r_{k_0}(u) \geq 2$. In all cases it is clear that $r_{k_0'}(u') \geq 1$ or $r_{k_0}(u') \geq 1$, so u' has odd block sizes and cannot be distinguished in $\mathrm{Sp}(W'/W)$.

Finally, suppose that some Jordan block size of u has multiplicity ≥ 3, and that u has a Jordan block of odd size. That is, assume that there exists $k_0', k_0 \geq 1$ such that k_0 is odd, $r_{k_0'}(u) \geq 3$ and $r_{k_0}(u) \geq 1$. Now if k_0' is odd, then $r_{k_0'}(u') \geq 1$, and so u' has an odd block size and cannot be distinguished in $\mathrm{Sp}(W'/W)$. Assume then that k_0' is even. In case (i) we have $r_{k_0'}(u') \geq 3$ and thus u' is not distinguished in $\mathrm{Sp}(W'/W)$. In cases (ii) and (vi) we have $r_{k_0'}(u') \geq 3$ if $k_0' \neq m$, and $r_{k_0'-1}(u') \geq 2$ if $k_0' = m$ (giving odd block sizes), so u' is not distinguished in $\mathrm{Sp}(W'/W)$. Similarly in case (v) $r_{k_0'}(u') \geq 3$ if $k_0' \neq m, n$, and $r_{k_0'-1}(u') \geq 2$ if $k_0' = m$ or $k_0' = n$ (giving odd block sizes), so u' is not distinguished in $\mathrm{Sp}(W'/W)$. In case (iii), we have $r_{k_0'}(u') \geq 3$ if $k_0' \neq m$ and $r_{k_0}(u') \geq 1$ of $k_0' = m$, so u' has an odd block size and cannot be distinguished in $\mathrm{Sp}(W'/W)$. In case (iv) we have $r_{k_0}(u') \geq 1$ and so u' has odd block sizes and cannot be distinguished in $\mathrm{Sp}(W'/W)$. □

Lemma 3.2.8. *Suppose that $p = 2$. Let $u \in \mathrm{GL}(V)$ and suppose that there exist $K[u]$-submodules W, W' of V such that $\dim W' = 1$, $V = W' \oplus W$ and that u leaves a non-degenerate alternating bilinear form invariant on W. If the action of u on V is inadmissible, then the image of u in $\mathrm{Sp}(W)$ is not a distinguished unipotent element.*

Proof. Now the Jordan blocks of the restriction u_W of u to W are the same as those of u acting on V, except that a Jordan block of size 1 is removed. If the action of u on V is inadmissible, then clearly u_W has a block of odd size or some block of multiplicity ≥ 3. □

3.3 Decomposition of tensor products

In this section, we give various results about the Jordan form of the tensor product of two unipotent matrices. To consider the Jordan decomposition of tensor products of unipotent matrices, it is convenient to express them in terms of the representation theory of a cyclic p-group. We will use the notation for indecomposable $K[u]$-modules as described in Section 1.4, where u is a unipotent linear map of order $q = p^\alpha$.

There is a large amount of literature concerning the decomposition of $V_m \otimes V_n$ into a direct sum of indecomposables, for example [Sri64], [Ral66], [McF79], [Nor95], [Hou03], and [Bar11]. There is no explicit formula for the decomposition of $V_m \otimes V_n$ as there is in characteristic 0, but there are various recursive descriptions which suffice for our purposes.

We will often study the decomposition of $V_m \otimes V_n$, $\wedge^2(V_n)$ and $S^2(V_n)$ using certain finite sequences of integers, which are defined in terms of m, n, and p. For these finite sequences of integers, the next definition gives various notation which will be convenient later.

Definition 3.3.1 ([Bar11], [GPX16, Definition 1])**.** Let $s = (a_1, \ldots, a_n)$ and $s' = (b_1, \ldots, b_m)$ be finite sequences of integers. The following notation will be defined.

(i) For all $k \in \mathbb{Z}$, set $s + k = (a_1 + k, \ldots, a_n + k)$.

(ii) Define $s \oplus s' = (a_1, \ldots, a_n, b_1, \ldots, b_m)$.

(iii) Define the *negative reverse* of s by $nr(s) = (-a_n, \ldots, -a_1)$.

(iv) Denote by $s_>$ the subsequence of positive terms in s, and by $s_<$ the subsequence of negative terms in s.

(v) The *k-multiple of* s is the sequence $(ka_1, \ldots, ka_1, \ldots, ka_n, \ldots ka_n)$ of length kn (each element and its multiplicity in the sequence is multiplied by k).

(vi) For $m \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geq 0}$, denote $(m : k) = (m, m, \ldots, m)$, where m occurs k times. Here $(m : 0)$ is the empty sequence $()$.

(vii) For $1 \leq k \leq n$, the kth term of the sequence s will be denoted by $s(k)$.

We will now describe one of the main results in [Bar11] which gives the decomposition of $V_m \otimes V_n$ into a direct sum of indecomposables. This information is contained in a sequence $s_p(m, n)$ defined below.

Definition 3.3.2 ([Bar15, Definition 1]). Let $0 \leq m \leq n$ be integers. The sequence $s_p(m,n)$ of integers is defined recursively as follows. Define $s_p(0,n) = (0 : n)$. Assume now that $0 < m \leq n$ and let $k \geq 0$ be the integer such that $p^k \leq n < p^{k+1}$. Write $n = bp^k + d$, where $0 < b < p$ and $0 \leq d < p^k$. Write $m = ap^k + c$ where $0 \leq a < p$ and $0 \leq c < p^k$. We define

$$s_p(m,n) = s_1 \oplus s_2 \oplus s_3,$$

where $s_3 = nr(s_1)$ and s_1 and s_2 are given by the following exhaustive list of cases.

Case (1): $m + n > p^{k+1}$. Then

$$s_1 = (p^{k+1} : m + n - p^{k+1})$$
$$s_2 = s_p(p^{k+1} - n, p^{k+1} - m)$$

Case (2): $m + n \leq p^{k+1}$ and $c + d > p^k$. Then

$$s_1 = ((a + b + 1)p^k : c + d - p^k)$$
$$s_2 = s_p((a + b + 1)p^k - n, (a + b + 1)p^k - m)$$

Case (3): $m + n \leq p^{k+1}$, $1 \leq c + d \leq p^k$, and $a > 0$. Then

$$s_1 = s_p(\min(c,d), \max(c,d)) + (a + b)p^k$$
$$s_2 = s_p((a + b)p^k - n, (a + b)p^k - m)$$

Case (4): $m + n \leq p^{k+1}$, $1 \leq c + d \leq p^k$, $a = 0$, and $d > 0$. Then

$$s_1 = s_p(m, bp^k - d)_< + 2bp^k$$
$$s_2 = (0 : n - m)$$

Case (5): $m + n \leq p^{k+1}$, $1 \leq c + d \leq p^k$, $a = 0$, and $d = 0$. Then

$$s_1 = (bp^k : m)$$
$$s_2 = (0 : bp^k - m)$$

Case (6): $m + n \leq p^{k+1}$, $c = 0$, and $d = 0$. Then

$$s_1 = ((a + b - 1)p^k : p^k)$$
$$s_2 = s_p((a - 1)p^k, (b - 1)p^k)$$

For more detail and examples on how to apply Definition 3.3.2, see [Bar11] and [Bar15].

Lemma 3.3.3 ([Bar11, Proposition 1]). *Assume that* $0 \leq m \leq n$. *Then*

(i) $s_p(m,n)$ *is a nonincreasing sequence of* $m + n$ *integers.*

(ii) $n - m + 1 \leq s_p(m,n)(l) \leq m + n - 1$ *when* $1 \leq l \leq m$.

(iii) $s_p(m,n)(m + n - l + 1) = -s_p(m,n)(l)$ *for* $1 \leq l \leq m + n$.

(iv) *$s_p(m,n)(l) > 0$ for $1 \leq l \leq m$, and $s_p(m,n)(l) < 0$ for $n+1 \leq l \leq m+n$, and $s_p(m,n)(l) = 0$ otherwise.*

Remark 3.3.4. For $0 < m \leq n$, in Definition 3.3.2 we set $s_p(m,n) = s_1 \oplus s_2 \oplus s_3$, where $s_3 = nr(s_1)$ and s_1 and s_2 are given depending on which of the cases (1)-(6) of Definition 3.3.2 hold for m and n.

According to Lemma 3.3.3 (i) the sequence $s_p(m,n)$ is nonincreasing, so it follows that $s_2(i) \leq s_1(j)$ for all i and j. In fact, something slightly stronger is true: we can show that $s_2(i) < s_1(j)$ for all i and j. We omit the full details, but this is easy to verify in each of the cases (1)-(6) of Definition 3.3.2, using the fact (Lemma 3.3.3) that for all $0 \leq m' \leq n'$, we have $-m' - n' + 1 \leq s_p(m',n')(k) \leq m' + n' - 1$ for all $1 \leq k \leq m' + n'$.

Now in terms of the sequence $s_p(m,n)$, the decomposition of $V_m \otimes V_n$ is given by the following result. Note that the statement makes sense in view of Lemma 3.3.3 (iv).

Theorem 3.3.5 ([Bar11, Theorem 1])**.** *Let $0 \leq m \leq n \leq q$. Then*

$$V_m \otimes V_n = \oplus_{k=1}^{m} V_{s_p(m,n)(k)}$$

We will need the following description of $V_m \otimes V_n$ which holds in the case where $1 \leq m, n \leq p$. This is a consequence of [Fei82, ch. VIII, Theorem 2.7], as noted in [Sup09, Lemma 2.8]. It could also be deduced from Theorem 3.3.5.

Lemma 3.3.6. *Let $1 \leq m \leq n \leq p$. Then*

$$V_m \otimes V_n \cong \oplus_{i=0}^{h-1} V_{n-m+2i+1} \oplus N \cdot V_p = \oplus_{i=1}^{h} V_{n-m+2h-2i+1} \oplus N \cdot V_p$$

where $h = \min\{m, p-n\}$ and $N = \max\{0, m+n-p\}$. In particular, $V_m \otimes V_p = m \cdot V_p$ for all $1 \leq m \leq p$.

The following corollary is an easy consequence of Lemma 3.3.6 and will be useful later. It is also a special case of [Bar15, Theorem 2].

Corollary 3.3.7. *Let $1 \leq m \leq n \leq p$. Then*

(i) *If $p \geq n+m-1$, then $V_n \otimes V_m \cong \oplus_{i=0}^{m-1} V_{n-m+2i+1} = \oplus_{i=1}^{m} V_{n+m-2i+1}$.*

(ii) *If $p < n+m-1$, then $V_n \otimes V_m$ has ≥ 2 Jordan blocks of size p.*

Proof. Let h and N be as in Lemma 3.3.6. If $p > n+m-1$, then $h = m$ and $N = 0$, so the claim follows immediately from Lemma 3.3.6. If $p = n+m-1$, then $h = m-1$ and $N = 1$, so by Lemma 3.3.6 we have

$$V_n \otimes V_m \cong \oplus_{i=0}^{m-2} V_{2i+1} \oplus V_p = \oplus_{i=0}^{m-1} V_{n-m+2i+1}$$

as claimed. Finally if $p < n+m-1$, then $N \geq 2$, so by Lemma 3.3.6 the tensor product $V_n \otimes V_m$ has ≥ 2 Jordan blocks of size p. □

The following theorem is due to Glasby, Praeger and Xia. It reduces finding the decomposition of $V_m \otimes V_n$ to the case where p does not divide $\gcd(m,n)$. In the case where $m \leq n \leq p$, this was proven by Renaud in [Ren79, Lemma 2.2].

Theorem 3.3.8 ([GPX16, Theorem 5]). *Let $0 < m \leq n \leq q$. Then for all $k \geq 0$, the sequence $s_p(p^k m, p^k n)$ is the p^k-multiple of $s_p(m,n)$.*

The following results are well known and could be deduced using Theorem 3.3.5. They also follow from [GPX16, Theorem 4, Table 1].

Lemma 3.3.9. *Let $n \geq 2$. Then*

$$V_n \otimes V_2 = \begin{cases} V_n \oplus V_n & \text{if } n \equiv 0 \mod p \\ V_{n-1} \oplus V_{n+1} & \text{if } n \not\equiv 0 \mod p \end{cases}$$

Lemma 3.3.10. *Assume that $p \neq 2$ and let $n \geq 3$. Then*

$$V_n \otimes V_3 = \begin{cases} V_n \oplus V_n \oplus V_n & \text{if } n \equiv 0 \mod p \\ V_{n-1} \oplus V_{n-1} \oplus V_{n+2} & \text{if } n \equiv 1 \mod p \\ V_{n-2} \oplus V_{n+1} \oplus V_{n+1} & \text{if } n \equiv -1 \mod p \\ V_{n-2} \oplus V_n \oplus V_{n+2} & \text{if } n \not\equiv 0, 1, -1 \mod p \end{cases}$$

Lemma 3.3.11. *Assume that $p = 2$ and let $n \geq 3$. Then*

$$V_n \otimes V_3 = \begin{cases} V_n \oplus V_n \oplus V_n & \text{if } n \equiv 0 \mod 4 \\ V_{n-1} \oplus V_{n-1} \oplus V_{n+2} & \text{if } n \equiv 1 \mod 4 \\ V_{n-2} \oplus V_n \oplus V_{n+2} & \text{if } n \equiv 2 \mod 4 \\ V_{n-2} \oplus V_{n+1} \oplus V_{n+1} & \text{if } n \equiv 3 \mod 4 \end{cases}$$

We present a result due to Barry which determines when $V_m \otimes V_n$ has no repeated blocks. For this we will need the following definition from [Bar15].

Definition 3.3.12. Assume that $p \neq 2$. We define the following sets of pairs of integers. Let

$$S = \{(k,d) : 1 < k \leq d \leq p+1-k\} \cup \{(k, p+k-1) : 1 < k \leq \frac{p+1}{2}\}.$$

For all integers $t \geq 2$, we define $S_t = (T_1 \setminus T_2) \cup T_3$, where

$$T_1 = \{(ip^{t-1} + \frac{p^{t-1} \pm 1}{2}, jp^{t-1} + \frac{p^{t-1} \pm 1}{2}) : 1 \leq i \leq \frac{p-1}{2}, i \leq j \leq p-i-1\}$$
$$T_2 = \{(ip^{t-1} + \frac{p^{t-1} + 1}{2}, ip^{t-1} + \frac{p^{t-1} - 1}{2}) : 1 \leq i \leq \frac{p-1}{2}\}$$
$$T_3 = \{(ip^{t-1} + \frac{p^{t-1} + 1}{2}, ip^{t-1} + \frac{p^{t-1} - 1}{2} + p^t) : 1 \leq i \leq \frac{p-1}{2}\}$$

We define the set $\mathscr{S}$ to be the set of all pairs (m,n) and (n,m) of integers such that $1 \leq m \leq n$ and one of the following conditions hold:

(i) $m = 1$.

(ii) $1 < m \leq p$, and $(m, n') \in S$, where n' is the unique integer such that $m \leq n' \leq p+m-1$ and $n' \equiv n \mod p$.

(iii) $p^{t-1} < m \leq p^t$ $(t \geq 2)$, and $(m, n') \in S_t$, where n' is the unique integer such that $m \leq n' \leq p^t + m - 1$ and $n' \equiv n \mod p^t$.

Using his results in [Bar11] (Theorem 3.3.5 above), Barry has shown the following.

Theorem 3.3.13 ([Bar15, Theorem 2, Theorem 3, Corollary 1])**.** *Assume that $p \neq 2$. Let $m, n \geq 1$. Then*

(i) *The tensor product $V_m \otimes V_n$ has no repeated blocks if and only if (m, n) is contained in the set $\mathscr{S}$ of Definition 3.3.12.*

(ii) *If $V_m \otimes V_n$ has no repeated blocks, then $V_n \otimes V_m \cong \oplus_{i=1}^{\min(m,n)} V_{n+m-2i+1}$.*

Theorem 3.3.14 ([Bar15, Theorem 1])**.** *Assume that $p = 2$. Let $1 \leq m \leq n$. Then*

(i) *The tensor product $V_m \otimes V_n$ has no repeated blocks if and only if one of the following conditions hold:*

- $m = 1$,
- $m = 2$ *and* $n \equiv 1 \mod 2$,
- $m = 3$ *and* $n \equiv 2 \mod 4$.

(ii) *If $V_m \otimes V_n$ has no repeated blocks, then $V_n \otimes V_m \cong \oplus_{i=1}^{\min(m,n)} V_{n+m-2i+1}$.*

Later in this chapter we will give a result similar to Theorem 3.3.13 for $\wedge^2(V_n)$ and $S^2(V_n)$ (Proposition 3.5.3).

With Theorem 3.3.13 and Theorem 3.3.14, one could actually give a description of all $K[u]$-modules V and W such that $V \otimes W$ has no repeated blocks. We finish this section by giving this description in a specific case which is relevant to Problem 1.1.2, namely the case where $p \neq 2$ and all the Jordan block sizes in V and W are of the same parity (see Lemma 3.3.17 below). At the end of this chapter, this result will be used in the proof of Theorem 1.1.4 in the tensor product subgroup case.

Lemma 3.3.15. *Assume that $p \neq 2$. Let $m \geq 1$ and $1 \leq n \leq n'$ be integers such that $n \equiv n' \mod 2$. Suppose that $V_m \otimes (V_n \oplus V_{n'})$ has no repeated blocks. Then $n' - n \geq 2m$, and in particular $n' > m$.*

Proof. First note that $V_m \otimes (V_n \oplus V_{n'}) = (V_m \otimes V_n) \oplus (V_m \otimes V_{n'})$, so it follows that $V_m \otimes V_n$ and $V_m \otimes V_{n'}$ have no repeated blocks.

Suppose that $n' \leq m$. In this case we have $V_m \otimes V_n = \oplus_{i=1}^{n} V_{m+n-2i+1}$ and $V_m \otimes V_{n'} = \oplus_{j=1}^{n'} V_{m+n'-2j+1}$ by Theorem 3.3.13 (ii). The largest block size occurring in $V_m \otimes V_n$ is $m + n - 1$. We claim that $V_m \otimes V_{n'}$ also has a block of size $m + n - 1$. For this, note that the block sizes occurring in $V_m \otimes V_{n'}$ are all the integers which lie in the interval $[m - n' + 1, m + n' - 1]$, and which are congruent to $m - n' + 1 \mod 2$. A straightforward verification shows that $m + n - 1$ is such an integer, so $V_m \otimes V_{n'}$ also has a block of size $m + n - 1$. Thus $V_m \otimes (V_n \oplus V_{n'})$ has ≥ 2 Jordan blocks of size $m + n - 1$, which is a contradiction.

Therefore we must have $n' > m$, so now $V_m \otimes V_n \cong \oplus_{i=1}^{\min(m,n)} V_{m+n-2i+1}$ and $V_m \otimes V_{n'} \cong \oplus_{j=1}^{m} V_{n'+m-2j+1}$ by Theorem 3.3.13 (ii). The Jordan block sizes of $V_m \otimes V_n$ consist of integers in the interval $[|m - n| + 1, m + n - 1]$ congruent to $n + m - 1 \mod 2$, while the Jordan block sizes of $V_m \otimes V_{n'}$ consist of integers in the interval $[n' - m + 1, n' + m - 1]$ congruent to $n' + m - 1 \mod 2$. Now

$n + m - 1 \equiv n' + m - 1 \mod 2$, so in order for $(V_m \otimes V_n) \oplus (V_m \otimes V_{n'})$ to have no repeated blocks, the two intervals must be disjoint. This is equivalent to $m + n - 1 < n' - m + 1$, which is equivalent to $n' - n \geq 2m$ since $n' \equiv n \mod 2$. This completes the proof of the lemma. □

Lemma 3.3.16. *Assume that $p \neq 2$. Let $m, m', n, n' \geq 1$ be integers such that $m \equiv m' \mod 2$ and $n \equiv n' \mod 2$. Then $(V_m \oplus V_{m'}) \otimes (V_n \oplus V_{n'})$ has repeated blocks.*

Proof. Without loss of generality, we can assume that $m \leq m'$ and $n \leq n'$. If $(V_m \oplus V_{m'}) \otimes (V_n \oplus V_{n'})$ has no repeated blocks, then the same is true for its direct summand $V_{m'} \otimes (V_n \oplus V_{n'})$, so by Lemma 3.3.15 we have $n' > m'$. On the other hand, now the direct summand $(V_m \oplus V_{m'}) \otimes V_{n'}$ also has no repeated blocks, so by Lemma 3.3.15 we have $m' > n'$, which is a contradiction. □

Lemma 3.3.17. *Assume that $p \neq 2$. Let V_1 and V_2 be $K[u]$-modules such that $1 < \dim V_1 \leq \dim V_2$, $V_1 \downarrow K[u] = \bigoplus_{i=1}^{s} V_{m_i}$, $V_2 \downarrow K[u] = \bigoplus_{j=1}^{t} V_{n_j}$, where $1 \leq m_1 \leq \cdots \leq m_s$ and $1 \leq n_1 \leq \cdots \leq n_t$ are integers such that $m_i \equiv m_{i'} \mod 2$ for all $1 \leq i, i' \leq s$ and $n_j \equiv n_{j'} \mod 2$ for all $1 \leq j, j' \leq t$.*

Suppose that $s > 1$ or $t > 1$. Then $V_1 \otimes V_2$ has no repeated block sizes if and only if the following conditions hold:

(i) *$s = 1$, so $V_1 \downarrow K[u] = V_m$ for $m = m_1$;*

(ii) *$n_i - n_{i-1} \geq 2m$ for all $2 \leq i \leq t$;*

(iii) *(m, n_i) is contained in the set $\mathscr{S}$ of Definition 3.3.12, for all $1 \leq i \leq t$.*

Furthermore, when (i), (ii), and (iii) hold, we have

$$V_1 \otimes V_2 = \bigoplus_{j=1}^{\min(m,n_1)} V_{n_1+m-2j+1} \oplus \bigoplus_{i=2}^{t} \bigoplus_{j=1}^{m} V_{n_i+m-2j+1}.$$

Proof. Suppose that the conditions (i), (ii), and (iii) hold, so $V_1 = V_m$ and $V_2 = \oplus_{j=1}^{t} V_{n_j}$, where $1 \leq n_1 < \cdots < n_t$. Note that we have $n_i > m$ for all $2 \leq i \leq t$ by (ii). It follows then from condition (iii) and Theorem 3.3.13 (ii) that

$$V_1 \otimes V_2 = \bigoplus_{j=1}^{\min(m,n_1)} V_{n_1+m-2j+1} \oplus \bigoplus_{i=2}^{t} \bigoplus_{j=1}^{m} V_{n_i+m-2j+1},$$

proving the last claim of the lemma. In order to show that $V_1 \otimes V_2$ has no repeated blocks, note that the Jordan block sizes occurring in $V_1 \otimes V_2$ are the integers congruent to $n_1 + m + 1 \mod 2$ which lie in one of the intervals $[|n_1 - m| + 1, n_1 + m - 1]$, $[n_i - m + 1, n_i + m - 1]$, $i \geq 2$. It follows from (ii) that these intervals are pairwise disjoint, so $V_1 \otimes V_2$ has no repeated blocks.

For the "only if" direction of the claim, suppose that $V_1 \otimes V_2$ has no repeated blocks. First of all, we claim that $t = 1$ or $s = 1$. Indeed, if $t, s > 1$, then $(V_{m_1} \oplus V_{m_2}) \otimes (V_{n_1} \oplus V_{n_2})$ is a direct summand $V_1 \otimes V_2$, so we have a contradiction by Lemma 3.3.16. Next we show that $s = 1$. If $s > 1$, then we must have $t = 1$. Now note that $(V_{m_1} \oplus V_{m_2}) \otimes V_{n_1}$ has no repeated blocks, being a direct summand of $V_1 \otimes V_2$. By Lemma 3.3.15 we have $m_2 > n_1$, but this contradicts the assumption

$\dim V_1 \leq \dim V_2 = n_1$. Therefore $s = 1$, in other words, condition (i) holds. Set $m = m_1$, so $V_1 = V_m$. Now

$$V_1 \otimes V_2 = \bigoplus_{i=1}^{t} V_m \otimes V_{n_i},$$

so it follows from Theorem 3.3.13 (i) that for all $1 \leq i \leq t$, the pair (m, n_i) is contained in the contained in the set $\mathscr{S}$ of Definition 3.3.12. In other words, condition (iii) holds.

Since $V_m \otimes (V_{n_1} \oplus V_{n_2})$ is a direct summand of $V_1 \otimes V_2$, by Lemma 3.3.15 we have $n_2 > m$. Thus by Theorem 3.3.13 (ii), we have

$$V_1 \otimes V_2 = \bigoplus_{j=1}^{\min(m,n_1)} V_{n_1+m-2j+1} \oplus \bigoplus_{i=2}^{t} \bigoplus_{j=1}^{m} V_{n_i+m-2j+1}.$$

In order for $V_1 \otimes V_2$ to have no repeated blocks, it follows as in the first paragraph that the intervals $[|n_1 - m| + 1, n_1 + m - 1]$, $[n_i - m + 1, n_i + m - 1]$, $i \geq 2$ must be pairwise disjoint. It is straightforward to verify that this is equivalent to condition (ii), which completes the proof of the lemma. □

3.4 Jordan decompositions in tensor squares ($p \neq 2$)

In this section, we make the following assumption.

Assume that $p \neq 2$.

The purpose of this section is to describe and apply the results of Barry in [Bar11], which give the decomposition of the $K[u]$-modules $\wedge^2(V_n)$ and $S^2(V_n)$ into a direct sum of indecomposables, where u is a unipotent linear map of order $q \geq n$.

Set $s_p(n) := s_p(n, n)$ for all $n \geq 0$. As a special case of Theorem 3.3.5, the positive terms of the sequence $s_p(n)$ give the decomposition of $V_n \otimes V_n$ into a direct sum of indecomposables. That is,

$$V_n \otimes V_n = \oplus_{k=1}^{n} V_{s_p(n)(k)}$$

for all $0 < n \leq q$.

It is easily checked from Definition 3.3.2 that the following lemma holds.

Lemma 3.4.1. *If $n = 0$, then $s_p(0) = (0 : 0) = ()$. Assume then that $n > 0$, and let $k \geq 0$ be such that $p^k \leq n < p^{k+1}$, and write $n = bp^k + d$, where $1 \leq b < p$ and $0 \leq d < p^k$. Then $s_p(n)$ is defined recursively as follows.*

Case (1): $2n > p^{k+1}$.

$$s_p(n) = (p^{k+1} : 2n - p^{k+1}) \oplus s_p(p^{k+1} - n) \oplus (-p^{k+1} : 2n - p^{k+1})$$

Case (2): $2n \leq p^{k+1}$ *and* $d \geq \frac{p^k+1}{2}$ *(in this case $k > 0$).*

$$s_p(n) = ((2b+1)p^k : 2d - p^k) \oplus s_p((2b+1)p^k - n) \oplus (-(2b+1)p^k : 2d - p^k)$$

Case (3): *$2n \leq p^{k+1}$ and $1 \leq d \leq \frac{p^k-1}{2}$ (in this case $k > 0$).*

$$s_p(n) = (s_p(d) + 2bp^k) \oplus s_p(2bp^k - n) \oplus (s_p(d) - 2bp^k)$$

Case (4): *$2n \leq p^{k+1}$ and $d = 0$.*

$$s_p(n) = ((2b-1)p^k : p^k) \oplus s_p((b-1)p^k) \oplus (-(2b-1)p^k : p^k)$$

The sequence $s_p(n)$ has the following properties, cf. Lemma 3.3.3.

Lemma 3.4.2. *Assume that $n > 0$. Then*

(i) *$s_p(n)$ is a nonincreasing sequence of $2n$ integers.*

(ii) *$1 \leq s_p(n)(l) \leq 2n - 1$ when $1 \leq l \leq n$.*

(iii) *$s_p(n)(2n + 1 - l) = -s_p(n)(l)$ for $1 \leq l \leq 2n$.*

(iv) *$s_p(n)(l) > 0$ for $1 \leq l \leq n$ and $s_p(n)(l) < 0$ for $n + 1 \leq l \leq 2n$.*

(v) *$s_p(n)(j)$ is odd for all $1 \leq j \leq 2n$.*

Proof. Statements (i), (ii), (iii) and (iv) follow from Lemma 3.3.3. We proceed to prove (v) by induction on n. If $n = 1$, then $s_p(n) = (1, -1)$. Suppose then that $n > 1$. Now $p^k \leq n < p^{k+1}$ for some $k \geq 0$, and $n = bp^k + d$ for $1 \leq b < p$ and $0 \leq d < p^k$. If n is as in case (1), (2) or (4) of Lemma 3.4.1, then the claim follows by induction (note that p^{k+1}, $(2b + 1)p^k$ and $(2b - 1)p^k$ are odd). In case (3) the claim also follows, since $s_p(d)+2bp^k$ and $s_p(2bp^k - n)$ are sequences of odd integers by induction. □

Note that the largest entry in $s_p(n)$, i.e., the largest Jordan block occurring in $V_n \otimes V_n$, is easily deduced from Lemma 3.4.1. We will also need to know the smallest Jordan block sizes occurring in $V_n \otimes V_n$. This information is given in the following lemma.

Lemma 3.4.3. *Let $n > 1$. Set $\alpha = \nu_p(n)$ and $\beta = \max\{\nu_p(n - 1), \nu_p(n + 1)\}$. Then the smallest entries of the sequence $s_p(n)_>$ are given as follows.*

(i) *If $\alpha > 0$, then $s_p(n)_> = (\ldots, p^\alpha, p^\alpha, \ldots, p^\alpha)$ where p^α occurs exactly p^α times in the sequence.*

(ii) *If $\alpha = 0$ and $\beta > 0$, then $s_p(n)_> = (\ldots, p^\beta, p^\beta, \ldots, p^\beta, 1)$ where p^β occurs exactly $p^\beta - 2$ times in the sequence.*

(iii) *If $\alpha = 0$ and $\beta = 0$, then $s_p(n)_> = (\ldots, 3, 1)$ where 3 occurs only once in the sequence.*

Proof. We proceed to prove the claim by induction on n. If $n = 2$, then it is easy to see from Lemma 3.4.1 that $s_p(n)_> = (3, 1)$ for all p and that the claim holds.

Suppose that $n > 2$ and let k be such that $p^k \leq n < p^{k+1}$. We first take care of two special cases, namely $n = p^{k+1} - 1$ and $n = p^k$. If $n = p^{k+1} - 1$, then $\alpha = 0$ and $\beta = k + 1$. Furthermore,

$$s_p(n) = (p^{k+1} : p^{k+1} - 2) \oplus (1, -1) \oplus (-p^{k+1} : p^{k+1} - 2)$$

by Lemma 3.4.1 (1), so the claim holds. For $n = p^k$, we have $s_p(n) = (p^k : p^k) \oplus (-p^k : p^k)$ by Lemma 3.4.1 (4), so the claim holds.

Suppose then that $p^k < n < p^{k+1} - 1$. By Lemma 3.4.1, we have

$$s_p(n) = s_1 \oplus s_p(n') \oplus nr(s_1),$$

where s_1 is a sequence of positive integers and in cases (1), (2), (3) and (4) of Lemma 3.4.1 we have $n' = p^{k+1} - n$, $n' = (2b+1)p^k - n$, $n' = 2bp^k - n$ and $n' = n - p^k$ respectively. Note that every entry of the sequence $s_p(n')$ is $<$ than any entry of the sequence s_1 (Remark 3.3.4). Thus by applying induction on $s_p(n')$, the claim for $s_p(n)_>$ follows once we verify that all of the following hold: $n' > 1$, $\nu_p(n') = \nu_p(n)$ and $\{\nu_p(n'+1), \nu_p(n'-1)\} = \{\nu_p(n+1), \nu_p(n-1)\}$.

To this end, using $p^k < n < p^{k+1} - 1$, an easy check in cases (1)-(4) of Lemma 3.4.1 shows that $n' > 1$. Next, note that we have $n' \equiv -n \mod p^k$, so $n' \pm 1 \equiv -(n \mp 1) \mod p^k$. Since $1 < n' < n < p^{k+1} - 1$, it follows that $\nu_p(n') = \nu_p(n)$ and $\nu_p(n \pm 1) = \nu_p(n \mp 1)$. □

In characteristic zero, the decomposition of the exterior and symmetric square of a unipotent matrix is easy to describe. This is seen from the following result, which is presumably well known, but we give a proof for completeness.

Proposition 3.4.4. *Let $u_{\mathbb{C}}$ be a unipotent Jordan block acting on a $\mathbb{C}$-vector space V of dimension $n > 1$.*

(i) *The Jordan block sizes of $u_{\mathbb{C}}$ acting on $S^2(V)$ are $[2n-1, 2n-5, \ldots, c]$, where $c = 1$ if n is odd and $c = 3$ if n is even.*

(ii) *The Jordan block sizes of $u_{\mathbb{C}}$ acting on $\wedge^2(V)$ are $[2n-3, 2n-7, \ldots, c]$, where $c = 3$ if n is odd and $c = 1$ if n is even.*

Proof. We apply the representation theory of $\mathrm{SL}_2(\mathbb{C})$. In an irreducible representation V of $\mathrm{SL}_2(\mathbb{C})$, a nonidentity unipotent element of $\mathrm{SL}_2(\mathbb{C})$ acts as a single Jordan block. Representations of $\mathrm{SL}_2(\mathbb{C})$ over $\mathbb{C}$ are semisimple, so then the Jordan block structure of $u_{\mathbb{C}}$ acting on $S^2(V)$ and $\wedge^2(V)$ is determined by the irreducible summands. These summands are determined by the weight multiplicities.

Fix the maximal torus formed by the diagonal matrices in $\mathrm{SL}_2(\mathbb{C})$. We can identify the weights with integers, and then in V we have a basis of weight vectors $e_1, e_2, \ldots, e_n$ where e_i has weight $n-2i+1$. Now $S^2(V)$ has a basis of weight vectors given by $e_i e_j$, with $1 \leq i \leq j \leq n$ where $e_i e_j$ has weight $2n-2(i+j-1)$. It follows that in $S^2(V)$ the highest weight $2n-2$ has multiplicity 1, the weight $2n-6$ has multiplicity 2, and generally $2n-4k+2$ has multiplicity k for $1 \leq k \leq \frac{n+1}{2}$. From this (i) follows.

Similarly in $\wedge^2(V)$ a basis of weight vectors is given by $e_i \wedge e_j$, with $1 \leq i < j \leq n$, and where $e_i \wedge e_j$ has weight $2n-2(i+j-1)$. The highest weight $2n-4$ has multiplicity 1, the weight $2n-8$ has multiplicity 2 and generally $2n-4k$ has multiplicity k for $1 \leq k \leq \frac{n}{2}$. From this (ii) follows. □

We continue with results in positive characteristic $p \neq 2$. The sequence $s_p(n)$ defined above can used to be describe the decomposition of $S^2(V_n)$ and $\wedge^2(V_n)$, as seen in [Bar11][7].

[7]Note that Theorem 2 in [Bar11] has a typo: case (3) is missing the assumption $2n \leq (2b+1)p^k \leq p^{k+1}$, and in the decomposition of $S^2(V_n)$ the direct sum should run from $j = 1$ to $j = d$.

Theorem 3.4.5 ([Bar11, Corollary 3]). *Let $0 < n \leq q$. Then*

$$S^2(V_n) = \bigoplus_{k=1}^{\lceil n/2 \rceil} V_{s_p(n)(2k-1)} \text{ and } \wedge^2 (V_n) = \bigoplus_{k=1}^{\lfloor n/2 \rfloor} V_{s_p(n)(2k)}.$$

Theorem 3.4.6 ([Bar11, Theorem 2]). *Let $0 < n \leq q$. Now $p^k \leq n < p^{k+1}$ for some $k \geq 0$, and $n = bp^k + d$ for $1 \leq b < p$ and $0 \leq d < p^k$. Then*

(1) *If $2n > p^{k+1}$, then*

$$\wedge^2(V_n) = (n - \frac{p^{k+1}+1}{2})V_{p^{k+1}} \oplus S^2(V_{p^{k+1}-n})$$

and

$$S^2(V_n) = (n - \frac{p^{k+1}-1}{2})V_{p^{k+1}} \oplus \wedge^2(V_{p^{k+1}-n})$$

(2) *If $2n \leq p^{k+1}$ and $d \geq \frac{p^k+1}{2}$, then*

$$\wedge^2(V_n) = (n - \frac{(2b+1)p^k+1}{2})V_{(2b+1)p^k} \oplus S^2(V_{(2b+1)p^k-n})$$

and

$$S^2(V_n) = (n - \frac{(2b+1)p^k-1}{2})V_{(2b+1)p^k} \oplus \wedge^2(V_{(2b+1)p^k-n})$$

(3) *If $2n \leq p^{k+1}$ and $1 \leq d \leq \frac{p^k-1}{2}$, then*

$$\wedge^2(V_n) = \bigoplus_{j=1}^{d} V_{s_p(d)(2j)+2bp^k} \oplus \wedge^2(V_{2bp^k-n})$$

and

$$S^2(V_n) = \bigoplus_{j=1}^{d} V_{s_p(d)(2j-1)+2bp^k} \oplus S^2(V_{2bp^k-n})$$

(4) *If $2n \leq p^{k+1}$ and $d = 0$, then*

$$\wedge^2(V_n) = \frac{p^k-1}{2}V_{(2b-1)p^k} \oplus S^2(V_{(b-1)p^k})$$

and

$$S^2(V_n) = \frac{p^k+1}{2}V_{(2b-1)p^k} \oplus \wedge^2(V_{(b-1)p^k})$$

Remark 3.4.7. In Theorem 3.4.6, we have formulated the cases differently than Barry in [Bar11]. However, it is easy to see that (1), (2), (3) and (4) in Theorem 3.4.6 are equivalent to

(1)' $2n > p^{k+1}$,

(2)' $1 \leq b < p$, $0 < d < p^k$ and $(2b+1)p^k < 2n \leq p^{k+1}$,

(3)' $1 \leq b < p$, $0 < d < p^k$ and $2bp^k < 2n \leq (2b+1)p^k \leq p^{k+1}$

(4)' $n = bp^k$ and $2b < p$

respectively. The cases (1)', (2)', (3)', and (4)' are the cases given in [Bar11, Theorem 2].

Proposition 3.4.8. *Suppose that a block size t has multiplicity ≥ 2 in $\wedge^2(V_n)$ or $S^2(V_n)$. Then t is a multiple of p.*

Proof. It is straightforward to prove by induction on n, using Lemma 3.4.1, that if some value $t > 0$ occurs twice in the sequence $s_p(n)$, then t is a multiple of p. This is also a consequence of [GPX15, Theorem 4]. In any case, from this fact and Theorem 3.4.5, the claim follows. □

The following lemma is elementary, see for example [FH91, B.1, pg. 473].

Lemma 3.4.9. *Let V and W be KG-modules for any group G. Then we have the following isomorphisms of KG-modules.*

(i) $S^2(V \oplus W) \cong S^2(V) \oplus (V \otimes W) \oplus S^2(W)$,

(ii) $\wedge^2(V \oplus W) \cong \wedge^2(V) \oplus (V \otimes W) \oplus \wedge^2(W)$

Lemma 3.4.10. *Let $n > 1$, let $\alpha = \nu_p(n)$, and let $\beta = \max\{\nu_p(n-1), \nu_p(n+1)\}$. Let m be the smallest block size occurring in $\wedge^2(V_n)$. Then the following hold:*

(i) *If $\alpha = 0$ and n is even, then $m = 1$. Furthermore, V_m has multiplicity 1 in $\wedge^2(V_n)$.*

(ii) *If $\alpha = 0$, n is odd, and $\beta = 0$, then $m = 3$. Furthermore, V_m has multiplicity 1 in $\wedge^2(V_n)$.*

(iii) *If $\alpha = 0$, n is odd, and $\beta > 0$, then $m = p^\beta$. Furthermore, V_m has multiplicity $\frac{p^\beta-1}{2}$ in $\wedge^2(V_n)$.*

(iv) *If $\alpha > 0$, then $m = p^\alpha$. Furthermore, V_m has multiplicity $\frac{p^\alpha+1}{2}$ in $\wedge^2(V_n)$ if n is even and multiplicity $\frac{p^\alpha-1}{2}$ if n is odd.*

Proof. This is a straightforward consequence of Theorem 3.4.5 and Lemma 3.4.3. □

Lemma 3.4.11. *Let $V = V_{d_1} \oplus \cdots \oplus V_{d_t}$, where $t \geq 1$ and $d_i \geq 1$ for all i. Let $\alpha = \nu_p(\gcd(d_1, \ldots, d_t))$. If $\alpha > 0$, then the smallest block size occurring in $\wedge^2(V)$ is p^α.*

Proof. Suppose that $\alpha > 0$, and for all i set $\alpha_i = \nu_p(d_i)$. By Lemma 3.4.9 we have

$$\wedge^2(V) \cong \bigoplus_{i=1}^{t} \wedge^2(V_{d_i}) \oplus \bigoplus_{1 \leq i < j \leq t} V_{d_i} \otimes V_{d_j}$$

as $K[u]$-modules. Now p^α divides d_i and d_j for all $i \leq j$, so by Theorem 3.3.8 we have the equality

$$\min s_p(d_i, d_j)_> = p^\alpha \cdot \min s_p(\frac{d_i}{p^\alpha}, \frac{d_j}{p^\alpha})_>$$

which is always $\geq p^{\alpha}$.

Therefore by Theorem 3.3.5, in $V_{d_i} \otimes V_{d_j}$ the smallest Jordan block has size $\geq p^{\alpha}$. By Lemma 3.4.10 the smallest block size occurring in $\wedge^2(V_{d_i})$ is $p^{\alpha_i} \geq p^{\alpha}$. Thus each block size occurring in the $K[u]$-module $\wedge^2(V)$ is $\geq p^{\alpha}$. Furthermore, a block of size p^{α} occurs in $\wedge^2(V)$ since $\alpha = \alpha_i$ for some i, which proves the claim. □

Lemma 3.4.12. *Let $n > 1$, let $\alpha = \nu_p(n)$, and let $\beta = \max\{\nu_p(n-1), \nu_p(n+1)\}$. Let m be the smallest block size occurring in $S^2(V_n)$. Then the following hold:*

(i) *If $\alpha = 0$ and n is odd, then $m = 1$. Furthermore, V_m has multiplicity 1 in $S^2(V_n)$.*

(ii) *If $\alpha = 0$, n is even, and $\beta = 0$, then $m = 3$. Furthermore, V_m has multiplicity 1 in $S^2(V_n)$.*

(iii) *If $\alpha = 0$, n is even, and $\beta > 0$, then $m = p^{\beta}$. Furthermore, V_m has multiplicity $\frac{p^{\beta}-1}{2}$ in $S^2(V_n)$.*

(iv) *If $\alpha > 0$, then $m = p^{\alpha}$. Furthermore, V_m has multiplicity $\frac{p^{\alpha}-1}{2}$ in $S^2(V_n)$ if n is even and multiplicity $\frac{p^{\alpha}+1}{2}$ if n is odd.*

Proof. This is a straightforward consequence of Theorem 3.4.5 and Lemma 3.4.3. □

Lemma 3.4.13. *Let $V = V_{d_1} \oplus \cdots \oplus V_{d_t}$, where $t \geq 1$ and $d_i \geq 1$ for all i. Let $\alpha = \nu_p(\gcd(d_1, \ldots, d_t))$. If $\alpha > 0$, then the smallest block size occurring in $S^2(V)$ is p^{α}.*

Proof. The same proof as in Lemma 3.4.11 works: just replace $\wedge^2$ with S^2 and apply Lemma 3.4.12 instead of Lemma 3.4.10. □

3.5 Symmetric and exterior squares with no repeated blocks ($p \neq 2$)

As in 3.4, we retain the notation from Section 3.3 and make the following assumption.

Assume that $p \neq 2$.

In this section we determine all $n > 1$ such that $S^2(V_n)$ or $\wedge^2(V_n)$ have no repeated blocks. First, in view of Proposition 3.4.4, we make the following definitions.

Definition 3.5.1. We say that the decomposition of $S^2(V_n)$ is *as in characteristic* 0 if

$$S^2(V_n) = V_{2n-1} \oplus V_{2n-5} \oplus \cdots \oplus V_c$$

where $c = 1$ if n is odd and $c = 3$ if n is even.

Definition 3.5.2. We say that the decomposition of $\wedge^2(V_n)$ is *as in characteristic* 0 if

$$\wedge^2(V_n) = V_{2n-3} \oplus V_{2n-7} \oplus \cdots \oplus V_c,$$

where $c = 3$ if n is odd and $c = 1$ if n is even.

The main result of this section is the following proposition.

Proposition 3.5.3. *Let $n > 1$, and $V = \wedge^2(V_n)$ or $V = S^2(V_n)$. The decomposition of V has no repeated blocks precisely in the following cases:*

(i) *$p \geq 2n-1$ for $V = S^2(V_n)$,*

(ii) *$p \geq 2n-3$ for $V = \wedge^2(V_n)$,*

(iii) *$n = p + \frac{p-3}{2}$ for $V = \wedge^2(V_n)$,*

(iv) *$n = bp^k + \frac{p^k \pm 1}{2}$ for some $k \geq 1$, $0 \leq b \leq \frac{p-1}{2}$, for $V = \wedge^2(V_n)$ and $V = S^2(V_n)$.*

In all of the cases above, the decomposition of V is as in characteristic 0. Furthermore, if there are repeated blocks in V, then some block of size > 1 has multiplicity ≥ 2.

The proof is essentially an application of Theorem 3.4.6. The fact that some block of size > 1 has multiplicity ≥ 2 when there are repeated blocks follows from our proof, but also from Proposition 3.4.8. Similarly the fact that the decomposition is as in characteristic 0 when there are no repeated blocks follows from the proof, but it is also possible to deduce a priori that this is the case; we omit the discussion of how this could be done.

We begin by a series of lemmas which will be needed in the proof.

Lemma 3.5.4. *Let $p \geq n$.*

(i) *If $p \geq 2n-1$, then the decomposition of $S^2(V_n)$ is as in characteristic 0. If $n \leq p < 2n-1$, then $S^2(V_n)$ has ≥ 2 blocks of size p.*

(ii) *If $p \geq 2n-3$, then the decomposition of $\wedge^2(V_n)$ is as in characteristic 0. If $n \leq p < 2n-3$, then $\wedge^2(V_n)$ has ≥ 2 blocks of size p.*

Proof. If $p \geq 2n-1$, then it is immediate from Corollary 3.3.7 that $s_p(n)_> = (2n-2j+1)_{j=1}^n$, so by Theorem 3.4.5 both $S^2(V_n)$ and $\wedge^2(V_n)$ are as in characteristic 0.

If $p = 2n-3$, then by Lemma 3.3.6 we have $s_p(n)_> = (p,p,p) \oplus (2n-2j+1)_{j=3}^{n-3}$. Now it follows from Theorem 3.4.5 that $\wedge^2(V_n)$ is as in characteristic 0 and $S^2(V_n)$ has 2 blocks of size p.

Suppose then that $n \leq p < 2n-3$. In this case it follows from Lemma 3.3.6 that $V_n \otimes V_n$ has ≥ 5 blocks of size p. In particular, the first four terms of $s_p(n)_>$ are equal to p, and thus by Theorem 3.4.5 both $\wedge^2(V_n)$ and $S^2(V_n)$ have ≥ 2 blocks of size p. □

Lemma 3.5.5. *Let $n = bp^k + \frac{p^k \pm 1}{2}$, where $0 \leq b \leq \frac{p-1}{2}$ and $k \geq 1$. Then $s_p(n) = (2n-2j+1)_{j=1}^{2n}$.*

Proof. In the notation of Definition 3.3.12, we have $(n,n) \in \mathscr{S}$ since $(n,n) \in T_1 \setminus T_2$, so it follows from Theorem 3.3.13 (ii) that $s_p(n)_> = (2n-2j+1)_{j=1}^n$. Now the claim follows from Lemma 3.4.2 (iii). □

Lemma 3.5.6. *Let $n = p + \frac{p-3}{2}$. Then the decomposition of $\wedge^2(V_n)$ is as in characteristic 0, but $S^2(V_n)$ has ≥ 2 blocks of size p.*

Proof. If $p = 3$, then the result follows from Lemma 3.5.4. Assume then that $p > 3$. By Theorem 3.4.6 (3), we get (here $d = \frac{p-3}{2}$)

$$\begin{aligned}\wedge^2(V_n) &= \oplus_{j=1}^d V_{s_p(d)(2j)+2p} \oplus \wedge^2(V_{\frac{p+3}{2}}) \\ &= \oplus_{j=1}^d V_{2n-4j+1} \oplus \wedge^2(V_{\frac{p+3}{2}}).\end{aligned}$$

Here the last equality follows since $p \geq 2d-1$ and thus $s_p(d) = (2d-2j+1)_{j=1}^{2d}$ by Corollary 3.3.7. Now by Lemma 3.5.4 the decomposition of $\wedge^2(V_{\frac{p+3}{2}})$ is as in characteristic 0, so the decomposition of $\wedge^2(V_n)$ is also as in characteristic 0.

For the symmetric square, applying Theorem 3.4.6 (2) gives

$$S^2(V_n) = \oplus_{j=1}^d V_{s_p(d)(2j-1)+2p} \oplus S^2(V_{\frac{p+3}{2}}).$$

Now $S^2(V_{\frac{p+3}{2}})$ has ≥ 2 blocks of size p by Lemma 3.5.4, as claimed. □

Lemma 3.5.7. *Let $n = bp^k + \frac{p^k \pm 1}{2}$, where $k \geq 1$ and $0 \leq b \leq \frac{p-1}{2}$. Then the decompositions of $\wedge^2(V_n)$ and $S^2(V_n)$ are as in characteristic 0.*

Proof. According to Lemma 3.5.5, we have $s_p(n) = (2n - 2j + 1)_{j=1}^{2n}$. Thus the claim follows immediately from Theorem 3.4.5. □

We are now ready to prove Proposition 3.5.3.

Proof of Proposition 3.5.3. The fact that in cases (i)-(iv) the decompositions are as in characteristic 0 follows from lemmas 3.5.4, 3.5.6 and 3.5.7. To prove that they are the only cases where there are no repeated blocks, we proceed by induction on n (case $n = 1$ is obvious).

If $p \geq n$, then Proposition 3.5.3 follows from Lemma 3.5.4. Assume then that $p < n$. Then there exists $k \geq 1$ such that $p^k \leq n < p^{k+1}$ and $n = bp^k + d$, where $1 \leq b < p$ and $0 \leq d < p^k$. We go through the different possibilities given in Theorem 3.4.6.

Case (1): $2n > p^{k+1}$.
For the symmetric square, by Theorem 3.4.6 we have

$$S^2(V_n) = (n - \frac{p^{k+1}-1}{2})V_{p^{k+1}} \oplus \wedge^2(V_{p^{k+1}-n}),$$

which means that if $S^2(V_n)$ has no repeated blocks, then $n - \frac{p^{k+1}-1}{2} \leq 1$. Since $2n > p^{k+1}$, it follows that $2n = p^{k+1} + 1$, so $n = \frac{p^{k+1}+1}{2}$ (case (iv) in 3.5.3).

For the exterior square, we see similarly with Theorem 3.4.6 that if $\wedge^2(V_n)$ has no repeated blocks, then $n - \frac{p^{k+1}+1}{2} \leq 1$. Since $2n > p^{k+1}$, it folllows that $n = \frac{p^{k+1}+1}{2}$ (case (iv) in 3.5.3) or $n = \frac{p^{k+1}+3}{2}$. If $n = \frac{p^{k+1}+3}{2}$, then by Theorem 3.4.6 we have

$$\wedge^2(V_n) = V_{p^{k+1}} \oplus S^2(V_{\frac{p^{k+1}-3}{2}}).$$

We are assuming that $\wedge^2(V_n)$ has no repeated blocks, so the same must be true for $S^2(V_{\frac{p^{k+1}-3}{2}})$ as well. Since $\frac{p^{k+1}-3}{2}$ is not of the form given in 3.5.3 (iv), by induction it must be of the form given in 3.5.3 (i). That is, we have $p \geq 2 \cdot \frac{p^{k+1}-3}{2} - 1 = p^{k+1} - 4$. But this is not possible, since $p \geq 3$ and $k \geq 1$ gives $p^{k+1} - 4 \geq p^2 - 4 > p$.

In the rest of the cases we have $2n \leq p^{k+1}$, and we note that this implies $b \leq \frac{p-1}{2}$.

Case (2): $2n \leq p^{k+1}$ and $d \geq \frac{p^k+1}{2}$.
For the symmetric square, by Theorem 3.4.6

$$S^2(V_n) = (n - \frac{(2b+1)p^k - 1}{2})V_{(2b+1)p^k} \oplus \wedge^2(V_{(2b+1)p^k - n}),$$

which means that if $S^2(V_n)$ has no repeated blocks, then $n - \frac{(2b+1)p^k-1}{2} \leq 1$. Since $n = bp^k + d$, this gives $d \leq \frac{p^k+1}{2}$, so in fact $d = \frac{p^k+1}{2}$. We have $b \leq \frac{p-1}{2}$, so we are in case (iv) of 3.5.3.

For the exterior square, we see similarly with Theorem 3.4.6 that if $\wedge^2(V_n)$ has no repeated blocks, then $n - \frac{(2b+1)p^k+1}{2} \leq 1$. Since $n = bp^k + d$, this gives $d \leq \frac{p^k+3}{2}$, and thus $d = \frac{p^k+1}{2}$ or $d = \frac{p^k+3}{2}$. If $d = \frac{p^k+1}{2}$, then we are in case (iv) of 3.5.3 since $b \leq \frac{p-1}{2}$. Suppose then that $d = \frac{p^k+3}{2}$. In this case $(2b+1)p^k - n = n - 3$, so $\wedge^2(V_n) = V_{2n-3} \oplus S^2(V_{n-3})$ by Theorem 3.4.6. Because $n - 3 = bp^k + \frac{p^k-3}{2}$ is not of the form given in 3.5.3 (iv), by applying induction on $S^2(V_{n-3})$ it follows that $p \geq 2(n-3) - 1 = 2n - 7 = (2b+1)p^k - 4$. But this is not possible, since $p \geq 3$, $k \geq 1$, and $b \geq 1$ gives $(2b+1)p^k - 4 \geq 3p - 4 > p$.

Case (3): $2n \leq p^{k+1}$ and $1 \leq d \leq \frac{p^k-1}{2}$.
We consider first $S^2(V_n)$ and $\wedge^2(V_n)$ in the case where $b = 1$.

When $b = 1$, for the symmetric square we have

$$S^2(V_n) = \oplus_{j=1}^{d} V_{s_p(d)(2j-1)+2p^k} \oplus S^2(V_{p^k-d})$$

by Theorem 3.4.6. Assuming that $S^2(V_n)$ has no repeated blocks, by the decomposition above the same must be true for $S^2(V_{p^k-d})$. Thus by applying induction on $p^k - d$, one of the following must hold:

- Case (i) of 3.5.3: $p \geq 2(p^k - d) - 1$.
- Case (iv) of 3.5.3: $p^k - d = cp^{k-1} + \frac{p^{k-1} \pm 1}{2}$, where $0 \leq c \leq \frac{p-1}{2}$ and $k > 1$.

Suppose first that $S^2(V_{p^k-d})$ is as in case (i) of 3.5.3. If $k > 1$, we have $2(p^k - d) - 1 \geq p^k > p$. Thus $k = 1$ and $p \geq 2(p-d) - 1$. This gives $d \geq \frac{p-1}{2}$, so $d = \frac{p-1}{2}$ and we are in case (iv) of 3.5.3. If $S^2(V_{p^k-d})$ is as in case (iv) of 3.5.3, then $p^k - d = cp^{k-1} + \frac{p^{k-1} \pm 1}{2}$, where $0 \leq c \leq \frac{p-1}{2}$ and $k > 1$. In particular $p^k - d \leq \frac{p^k+1}{2}$, so $d \geq \frac{p^k-1}{2}$ and therefore $d = \frac{p^k-1}{2}$. In other words, n is as in case (iv) of 3.5.3.

When $b = 1$, for the exterior square

$$\wedge^2(V_n) = \oplus_{j=1}^{d} V_{s_p(d)(2j)+2p^k} \oplus \wedge^2(V_{p^k-d})$$

by Theorem 3.4.6. Assuming that $\wedge^2(V_n)$ has no repeated blocks, by the decomposition above the same must be true for $\wedge^2(V_{p^k-d})$, so we can apply induction on $p^k - d$. Thus one of the following must hold:

- Case (ii) of 3.5.3: $p \geq 2(p^k - d) - 3$.
- Case (iii) of 3.5.3: $p^k - d = p + \frac{p-3}{2}$.
- Case (iv) of 3.5.3: $p^k - d = cp^{k-1} + \frac{p^{k-1} \pm 1}{2}$, where $0 \leq c \leq \frac{p-1}{2}$ and $k > 1$.

If we are in case (ii) of 3.5.3 for $\wedge^2(V_{p^k-d})$, then $p \geq 2(p^k - d) - 3 \geq p^k - 2$, so $2 \geq p^k - p$. This forces $k = 1$, so $p \geq 2(p - d) - 3$ and $d \geq \frac{p-3}{2}$. Hence $d = \frac{p-1}{2}$ or $d = \frac{p-3}{2}$, so either $n = p + \frac{p-1}{2}$ (case (iv) of 3.5.3) or $n = p + \frac{p-3}{2}$ (case (iii) of 3.5.3). If we are in case (iii) of 3.5.3 for $\wedge^2(V_{p^k-d})$, then $p^k - d = p + \frac{p-3}{2}$. It is clear in this situation that $k > 1$. Furthermore, we must have $k = 2$ since $p^k - d \geq \frac{p^k+1}{2}$ and $p + \frac{p-3}{2} < p^2$. Thus $d = p^2 - \frac{3(p-1)}{2} \geq \frac{p^2}{2}$, contradicting the fact that $d < \frac{p^2}{2}$. If we are in case (iv) of 3.5.3 for $p^k - d$, then as in the case of the symmetric square, we get $p^k - d \leq \frac{p^k+1}{2}$. This implies $d \leq \frac{p^k-1}{2}$, so $d = \frac{p^k-1}{2}$ and n is as in case (iv) of 3.5.3.

Next we consider the case where $b > 1$. For the symmetric square, we have

$$S^2(V_n) = \oplus_{j=1}^{d} V_{s_p(d)(2j-1)+2bp^k} \oplus S^2(V_{bp^k-d})$$

by Theorem 3.4.6. Assuming that $S^2(V_n)$ has no repeated blocks, the same must be true for $S^2(V_{bp^k-d})$ as well. Now $p < 2(bp^k - d) - 1$, so by induction it follows $bp^k - d$ must be as in case (iv) of 3.5.3. Since $bp^k - d = (b-1)p^k + (p^k - d)$, this means that $p^k - d = \frac{p^k \pm 1}{2}$, which gives $d = \frac{p^k \mp 1}{2}$. Thus n is also as in case (iv) of 3.5.3.

For the exterior square, we have

$$\wedge^2(V_n) = \oplus_{j=1}^{d} V_{s_p(d)(2j)+2bp^k} \oplus \wedge^2(V_{bp^k-d})$$

by Theorem 3.4.6. Assuming that $\wedge^2(V_n)$ has no repeated blocks, by induction the same is true for $\wedge^2(V_{bp^k-d})$. Now $p < 2(bp^k - d) - 3$, so by induction it follows $bp^k - d$ must be as in case (iii) or case (iv) of 3.5.3. If $\wedge^2(V_{bp^k-d})$ is as in case (iv), we see as in the previous paragraph for the symmetric square that n is as in case (iv) of 3.5.3. If $bp^k - d = (b-1)p^k + (p^k - d)$ is as in case (iii) of 3.5.3, it follows that $b = 2$, $k = 1$ and $p - d = \frac{p-3}{2}$. But then $d = \frac{p+3}{2}$, which is in contradiction with $d \leq \frac{p-1}{2}$.

Case (4): $2n \leq p^{k+1}$ and $d = 0$.
For the symmetric square, in this case

$$S^2(V_n) = \frac{p^k + 1}{2} V_{(2b-1)p^k} \oplus \wedge^2(V_{(b-1)p^k})$$

by Theorem 3.4.6. Now $\frac{p^k+1}{2} \geq 2$, so $S^2(V_n)$ has ≥ 2 blocks of size $(2b-1)p^k$.

For the exterior square,

$$\wedge^2(V_n) = \frac{p^k - 1}{2} V_{(2b-1)p^k} \oplus S^2(V_{(b-1)p^k})$$

by Theorem 3.4.6. Now if $\wedge^2(V_n)$ has no repeated blocks, then $\frac{p^k-1}{2} \leq 1$. Therefore $p^k \leq 3$, which forces $p = 3$ and $k = 1$. Since $b \leq \frac{p-1}{2} = 1$, we have $b = 1$. Therefore $n = 3$ and we are in case (ii) of 3.5.3. □

3.6 Decomposing tensor products of form modules

Assume that $p = 2$.

Let $u \in \mathrm{SL}(V)$ be a unipotent element. Suppose that V_1 and V_2 are $K[u]$-modules equipped with non-degenerate u-invariant alternating bilinear forms β_1 and β_2, respectively. Then there is a *product form* $\beta = \beta_1 \otimes \beta_2$ on the tensor product $V_1 \otimes V_2$, defined by $\beta(v \otimes w, v' \otimes w') = \beta_1(v, v')\beta_2(w, w')$ for all $v, v' \in V_1$ and $w, w' \in V_2$. The purpose of this section is to give, in some small cases, the decomposition of $V(2m) \otimes V(2l)$ with respect to the product form into orthogonally indecomposable (Definition 2.4.3) summands. Although methods for finding the Jordan block sizes of $V(2m) \otimes V(2l)$ can be found in the literature (see e.g. Theorem 3.3.5), it seems that in general the decomposition of this $K[u]$-module into orthogonally indecomposable summands is not known. As the main result of this section (Lemmas 3.6.2 - 3.6.4, Table 3.6) we will give the decompositions for $m \in \{1, 2, 3\}$. This will allow us to decide when tensor product subgroups of classical groups contain distinguished unipotent elements (Proposition 3.6.6).

To begin, we will give an explicit construction of $V(2m)$ and $V(2l)$. Let $e_1, \ldots, e_m, e_{-m}, \ldots, e_{-1}$ be a basis for a vector space W over K, and define an alternating bilinear form on W by $(e_i, e_j) = 1$ if $i = -j$, and 0 otherwise. We will define the action of u on W as follows:

$$\begin{aligned} ue_1 &= e_1 \\ ue_i &= e_i + e_{i-1} + \cdots + e_1 \text{ for all } 2 \leq i \leq m \\ ue_{-m} &= e_{-m} + e_m + e_{m-1} + \cdots + e_1 \\ ue_{-i} &= e_{-i} + e_{-(i+1)} \text{ for all } 1 \leq i \leq m-1 \end{aligned}$$

Then the form $(-, -)$ is non-degenerate, u-invariant, and $W \downarrow K[u] = V(2m)$. For $V(2l)$, we use the same construction, but with different notation for convenience. We denote a basis of $V(2l)$ by $f_1, \ldots, f_{2l}$, with alternating bilinear form defined by $(f_i, f_j) = 1$ if $i+j = 2l+1$ and 0 otherwise. The action of u on $V(2l)$ is defined as above:

$$\begin{aligned} uf_1 &= f_1 \\ uf_i &= f_i + f_{i-1} + \cdots + f_1 \text{ for all } 2 \leq i \leq l+1 \\ uf_i &= f_i + f_{i-1} \text{ for all } l+1 < i \leq 2l \end{aligned}$$

Denote by X the element $u - 1$ of $K[u]$. For determining the decomposition of a $K[u]$-module with a u-invariant form into orthogonally indecomposable summands, we will need some knowledge about the action of the powers of X. For the situations that we are about to consider, we will give the action of X^k explicitly. First, recall that the action of u on the tensor product $V_1 \otimes V_2$ of $K[u]$-modules V_i is defined by $u \cdot (v_1 \otimes v_2) = uv_1 \otimes uv_2$ for all $v_i \in V_i$. Therefore if $k = 2^\alpha$ and $u^k v_1 = v_1$, it follows from the identity $X^k = u^k - 1$ that $X^k \cdot (v_1 \otimes v_2) = v_1 \otimes X^k v_2$. In particular, this holds for all $v_i \in V_i$ if $2^\alpha \geq \dim V_1$. It follows then that it will be enough to compute the action of X^k for $1 \leq k < 2^\alpha$, which will give formulas for the action of X^k on $V_1 \otimes V_2$ which depend on $k \mod 2^\alpha$.

For example, consider the action of u on $V(2) \otimes V(2l)$ as defined above. First of all, it is easy to see that for all $v \in V(2l)$ we have $X \cdot (e_1 \otimes v) = e_1 \otimes Xv$ and $X \cdot (e_{-1} \otimes v) = (e_{-1} + e_1) \otimes Xv + e_1 \otimes v$. Furthermore, now u^2 acts trivially on $V(2)$, so $X^2 \cdot (e_i \otimes v) = e_i \otimes X^2 v$ for all $v \in V(2l)$ and $i = 1, -1$. It follows then that we get the formulae of Table 3.1 which hold for all $v \in V(2l)$. In the same way, one computes formulae for the action of X^k on $V(4) \otimes V(2l)$ and $V(6) \otimes V(2l)$. We have given these in Table 3.2 and Table 3.3 for later use.

It will also be useful for us to know the action of X^k on $V(2l)$ explicitly. This can be determined using the next lemma.

Lemma 3.6.1. *Consider $V(2l)$ with basis $f_1, \dots, f_{2l}$ and the action of u on $V(2l)$ defined as above. Then we have the following:*

(i) *$X^k f_j = 0$ for all $j \leq k$.*

(ii) *$X^k f_j \in \langle f_1, \dots, f_{j-k} \rangle$ for all $j > k$.*

(iii) *Let $0 \leq d \leq l$. Then $X^d f_j = 0$ for all $j \leq d$. For $d < j \leq l+1$, we have $X^d f_j = \sum_{t=1}^{j-d} \mu_t^{(j)} f_t$, where $\mu_t^{(j)} = \binom{j-1-t}{j-d-t}$.*

(iv) *Let $k \leq l+1$. Then $X^{2l-k} f_j = 0$ for all $j \leq 2l-k$. For $2l-k < j \leq 2l$, we have $X^{2l-k} f_j = \sum_{t=1}^{k+j-2l} \lambda_t^{(j)} f_t$, where $\lambda_t^{(j)} = \binom{l-t}{j+k-2l-t}$.*

(v) *Let $k \leq l+1$ and let $\alpha \geq 1$ be such that $2^\alpha \geq k$. Then the coefficients $\lambda_t^{(j)}$ in (iv) are determined by the value of l modulo 2^α.*

Proof. The claims (i) and (ii) are immediate from the fact that $Xf_1 = 0$ and $Xf_j \in \langle f_1, \dots, f_{j-1} \rangle$ for all $1 < j \leq 2l$.

For (iii), we begin by considering the action of X on the X-invariant subspace $\langle f_1, \dots, f_{l+1} \rangle$. The action of X on this subspace with respect to the basis $f_1, \dots, f_{l+1}$ is given by following matrix.

$$A = \begin{pmatrix} 0 & 1 & \cdots & \cdots & 1 \\ & 0 & 1 & \cdots & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & 1 \\ & & & & 0 \end{pmatrix}$$

Now note that in (iii), we have $\mu_t^{(j)} = \mu_{t-1}^{(j-1)}$. Thus one observes that it is enough to show that for all $1 \leq d \leq l$, the matrix A^d has zero entries, except in the $(l+1-d) \times (l+1-d)$ upper right corner which is upper triangular of the form

$$\begin{pmatrix} a_1 & a_2 & \cdots & \cdots & a_{l+1-d} \\ & a_1 & a_2 & \cdots & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & a_2 \\ & & & & a_1 \end{pmatrix}$$

where $a_s = \binom{s+d-2}{s-1}$ for all $1 \leq s \leq l+1-d$. We proceed to prove this claim by induction on d. For $d = 1$ this is clear. Suppose then that the claim holds for some $1 \leq d < l$. By multiplying A^d with A, one calculates that $A^{d+1} = A \cdot A^d$ has zero entries, except for the $(l-d)\times(l-d)$ upper right corner which is upper triangular of the form

$$\begin{pmatrix} a'_1 & a'_2 & \cdots & \cdots & a'_{l-d} \\ & a'_1 & a'_2 & \cdots & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & a'_2 \\ & & & & a'_1 \end{pmatrix}$$

where $a'_f = a_1 + \cdots + a_f$ for all $1 \leq f \leq l-d$. Now

$$\sum_{s=1}^{f} \binom{s+d-2}{s-1} = \sum_{s=0}^{f-1} \binom{d-1+s}{s} = \binom{d+f-1}{f-1}$$

by a standard combinatorial identity ("hockey-stick identity")[8], so it follows that $a'_f = \binom{f+d-1}{f-1}$. Hence the claim holds for $d+1$ as well, so (iii) follows by induction.

For claim (iv), the fact that $X^{2l-k} f_j = 0$ for all $j \leq 2l-k$ follows from (i). Suppose then that $2l-k < j \leq 2l$. First, if $j < l+1$, then $2l-k \leq l-1$ which implies that $k = l+1$ and $j = l$. Therefore $2l-k = l-1$, and the claim is easily verified by applying (iii) with $d = l-1$. Consider then $j \geq l+1$. In this case $X^{j-(l+1)} f_j = f_{l+1}$, so $X^{2l-k} f_j = X^{3l-k-j+1} \cdot X^{j-(l+1)} f_j = X^{3l-k-j+1} f_{l+1}$. Now it is immediate from (iii) that $X^{3l-k-j+1} f_{l+1} = \sum_{t=1}^{k+j-2t} \lambda_t^{(j)} f_t$ with $\lambda_t^{(j)} = \binom{l-t}{j+k-2l-t}$, proving (iv).

Finally for claim (v), note that for $\lambda_t^{(j)} = \binom{l-t}{j+k-2l-t}$ we have $\lambda_t^{(j)} = \lambda_{t-1}^{(j-1)}$, so it follows from (iv) that the matrix of X^{2l-k} with respect to the basis $f_1, \ldots, f_{2l}$ has zero entries, except in the $k \times k$ upper right corner which has the upper triangular form

$$\begin{pmatrix} b_k & b_{k-1} & \cdots & \cdots & b_1 \\ & b_k & b_{k-1} & \cdots & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & b_{k-1} \\ & & & & b_k \end{pmatrix}$$

where $b_t = \binom{l-t}{k-t}$ for all $1 \leq t \leq k$. To prove (v), it suffices to show for all $1 \leq t \leq k$ that $b_t \mod 2$ depends only on $l \mod 2^\alpha$. In other words, we should show that if l' is such that $l' \equiv l \mod 2^\alpha$, then $\binom{l'-t}{k-t} \equiv \binom{l-t}{k-t} \mod 2$ for all $1 \leq k \leq l+1$ and $1 \leq t \leq k$. This congruence is a straightforward consequence of Lucas' theorem[9]. □

In Table 3.4 and Table 3.5 we have given the action of X^{2l-k} on $V(2l)$ for $1 \leq k \leq l+1$ with $1 \leq k \leq 6$. These tables were found by a computer calculation, using Lemma 3.6.1 (iv) and (v).

---

[8] The identity is a common exercise and follows easily from the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, see e.g. [Knu97, 1.2.6, (10)].

[9] Lucas' theorem [Luc78, Section XXI] states that for two non-negative integers a and b with expansions $a = \sum_{i\geq 0} a_i p^i$ and $b = \sum_{i \geq 0} b_i p^i$ in prime base p, we have the congruence $\binom{a}{b} \equiv \prod_{i\geq 0} \binom{a_i}{b_i} \mod p$. Here we set $\binom{m}{n} = 0$ if $m < n$. For a short proof, see for example [Fin47].

$e_i \otimes v$	$X^k \cdot (e_i \otimes v)$	
$e_1 \otimes v$	$e_1 \otimes X^k v$	
$e_{-1} \otimes v$	$e_{-1} \otimes X^k v$,	if $k \equiv 0 \mod 2$
	$(e_{-1} + e_1) \otimes X^k v + e_1 \otimes X^{k-1} v$,	if $k \equiv 1 \mod 2$

Table 3.1: Action of X^k on $V(2) \otimes V(2l)$.

$e_i \otimes v$	$X^k \cdot (e_i \otimes v)$	
$e_1 \otimes v$	$e_1 \otimes X^k v$	
$e_2 \otimes v$	$e_2 \otimes X^k v$,	if $k \equiv 0 \mod 2$
	$(e_2 + e_1) \otimes X^k v + e_1 \otimes X^{k-1} v$,	if $k \equiv 1 \mod 2$
$e_{-2} \otimes v$	$e_{-2} \otimes X^k v$,	if $k \equiv 0 \mod 4$
	$(e_{-2} + e_2 + e_1) \otimes X^k v + (e_2 + e_1) \otimes X^{k-1} v$,	if $k \equiv 1 \mod 4$
	$(e_{-2} + e_1) \otimes X^k v + e_1 \otimes X^{k-2} v$,	if $k \equiv 2 \mod 4$
	$(e_{-2} + e_2) \otimes X^k v + (e_2 + e_1) \otimes X^{k-1} v + e_1 \otimes X^{k-2} v$,	if $k \equiv 3 \mod 4$
$e_{-1} \otimes v$	$e_{-1} \otimes X^k v$,	if $k \equiv 0 \mod 4$
	$(e_{-1} + e_{-2}) \otimes X^k v + e_{-2} \otimes X^{k-1} v$,	if $k \equiv 1 \mod 4$
	$(e_{-1} + e_2 + e_1) \otimes X^k v + (e_2 + e_1) \otimes X^{k-2} v$,	if $k \equiv 2 \mod 4$
	$(e_{-1} + e_{-2} + e_2) \otimes X^k v + (e_{-2} + e_1) \otimes X^{k-1} v + e_2 \otimes X^{k-2} v + e_1 \otimes X^{k-3} v$,	if $k \equiv 3 \mod 4$

Table 3.2: Action of X^k on $V(4) \otimes V(2l)$.

$V(2m) \otimes V(2l)$	$V(2m) \otimes V(2l) \downarrow K[u]$	
$V(2) \otimes V(2l)$, $l \geq 1$	$W(2l)$,	if $l \equiv 0 \mod 2$
	$V(2l)^2$,	if $l \equiv 1 \mod 2$
$V(4) \otimes V(2l)$, $l \geq 2$	$W(2l)^2$,	if $l \equiv 0 \mod 4$
	$W(2l-2) + W(2l+2)$,	if $l \equiv 1 \mod 4$
	$W(2l) + V(2l)^2$,	if $l \equiv 2 \mod 4$
	$W(2l-2) + W(2l+2)$,	if $l \equiv 3 \mod 4$
$V(6) \otimes V(2l)$, $l \geq 3$	$W(2l)^3$,	if $l \equiv 0 \mod 4$
	$W(2l-2)^2 + V(2l+4)^2$,	if $l \equiv 1 \mod 4$
	$W(2l-4) + W(2l) + W(2l+4)$,	if $l \equiv 2 \mod 4$
	$V(2l-4)^2 + W(2l+2)^2$,	if $l \equiv 3 \mod 4$

Table 3.6: Decomposition of $V(2m) \otimes V(2l)$ for $1 \leq m \leq 3$.

$e_i \otimes v$	$X^k \cdot (e_i \otimes v)$	
$e_1 \otimes v$	$e_1 \otimes X^k v$	
$e_2 \otimes v$	$e_2 \otimes X^k v,$	if $k \equiv 0 \mod 2$
	$(e_2 + e_1) \otimes X^k v + e_1 \otimes X^{k-1} v,$	if $k \equiv 1 \mod 2$
$e_3 \otimes v$	$e_3 \otimes X^k v,$	if $k \equiv 0 \mod 4$
	$(e_3 + e_2 + e_1) \otimes X^k v + (e_2 + e_1) \otimes X^{k-1} v,$	if $k \equiv 1 \mod 4$
	$(e_3 + e_1) \otimes X^k v + e_1 \otimes X^{k-2} v,$	if $k \equiv 2 \mod 4$
	$(e_3 + e_2) \otimes X^k v + (e_2 + e_1) \otimes X^{k-1} v + e_1 \otimes X^{k-2} v,$	if $k \equiv 3 \mod 4$
$e_{-3} \otimes v$	$e_{-3} \otimes X^k v,$	if $k \equiv 0 \mod 4$
	$(e_{-3} + e_3 + e_2 + e_1) \otimes X^k v + (e_3 + e_2 + e_1) \otimes X^{k-1} v,$	if $k \equiv 1 \mod 4$
	$(e_{-3} + e_2) \otimes X^k v + e_2 \otimes X^{k-2} v,$	if $k \equiv 2 \mod 4$
	$(e_{-3} + e_3) \otimes X^k v + (e_3 + e_2) \otimes X^{k-1} v + (e_2 + e_1) \otimes X^{k-2} v + e_1 \otimes X^{k-3} v$	if $k \equiv 3 \mod 4$
$e_{-2} \otimes v$	$e_{-2} \otimes X^k v,$	if $k \equiv 0 \mod 8$
	$(e_{-2} + e_{-3}) \otimes X^k v + e_{-3} \otimes X^{k-1} v,$	if $k \equiv 1 \mod 8$
	$(e_{-2} + e_3 + e_2 + e_1) \otimes X^k v + (e_3 + e_2 + e_1) \otimes X^{k-2} v,$	if $k \equiv 2 \mod 8$
	$(e_{-2} + e_{-3} + e_3 + e_1) \otimes X^k v + (e_{-3} + e_2) \otimes X^{k-1} v + (e_3 + e_1) \otimes X^{k-2} v + e_2 \otimes X^{k-3} v,$	if $k \equiv 3 \mod 8$
	$(e_{-2} + e_1) \otimes X^k v + e_1 \otimes X^{k-4} v,$	if $k \equiv 4 \mod 8$
	$(e_{-2} + e_{-3} + e_1) \otimes X^k v + e_{-3} \otimes X^{k-1} v + e_1 \otimes X^{k-4} v,$	if $k \equiv 5 \mod 8$
	$(e_{-2} + e_3 + e_2) \otimes X^k v + (e_3 + e_2 + e_1) \otimes X^{k-2} v + e_1 \otimes X^{k-4} v,$	if $k \equiv 6 \mod 8$
	$(e_{-2} + e_{-3} + e_3) \otimes X^k v + (e_{-3} + e_2) \otimes X^{k-1} v + (e_3 + e_1) \otimes X^{k-2} v + e_2 \otimes X^{k-3} v + e_1 \otimes X^{k-4} v,$	if $k \equiv 7 \mod 8$
$e_{-1} \otimes v$	$e_{-1} \otimes X^k v,$	if $k \equiv 0 \mod 8$
	$(e_{-1} + e_{-2}) \otimes X^k v + e_{-2} \otimes X^{k-1} v,$	if $k \equiv 1 \mod 8$
	$(e_{-1} + e_{-3}) \otimes X^k v + e_{-3} \otimes X^{k-2} v,$	if $k \equiv 2 \mod 8$
	$(e_{-1} + e_{-2} + e_{-3} + e_3 + e_2 + e_1) \otimes X^k v + (e_{-2} + e_3 + e_2 + e_1) \otimes X^{k-1} v + (e_{-3} + e_3 + e_2 + e_1) \otimes X^{k-2} v + (e_3 + e_2 + e_1) \otimes X^{k-3} v,$	if $k \equiv 3 \mod 8$
	$(e_{-1} + e_2) \otimes X^k v + e_2 \otimes X^{k-4} v,$	if $k \equiv 4 \mod 8$
	$(e_{-1} + e_{-2} + e_2 + e_1) \otimes X^k v + (e_{-2} + e_1) \otimes X^{k-1} v + (e_2 + e_1) \otimes X^{k-4} v + e_1 \otimes X^{k-5} v,$	if $k \equiv 5 \mod 8$
	$(e_{-1} + e_{-3} + e_2) \otimes X^k v + e_{-3} \otimes X^{k-2} v + e_2 \otimes X^{k-4} v,$	if $k \equiv 6 \mod 8$
	$(e_{-1} + e_{-2} + e_{-3} + e_3) \otimes X^k v + (e_{-2} + e_3 + e_2) \otimes X^{k-1} v + (e_{-3} + e_3 + e_2 + e_1) \otimes X^{k-2} v + (e_3 + e_2 + e_1) \otimes X^{k-3} v + (e_2 + e_1) \otimes X^{k-4} v + e_1 \otimes X^{k-5} v,$	if $k \equiv 7 \mod 8$

Table 3.3: Action of X^k on $V(6) \otimes V(2l)$.

X^{2l-1} $(l \geq 1)$	(1)	
X^{2l-2} $(l \geq 1)$	$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ $l \equiv 0 \mod 2$	$\begin{pmatrix} 1 & 0 \\ & 1 \end{pmatrix}$ $l \equiv 1 \mod 2$
X^{2l-3} $(l \geq 2)$	$\begin{pmatrix} 1 & 1 & 1 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$ $l \equiv 0 \mod 4$	$\begin{pmatrix} 1 & 0 & 0 \\ & 1 & 1 \\ & & 1 \end{pmatrix}$ $l \equiv 1 \mod 4$
	$\begin{pmatrix} 1 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$ $l \equiv 2 \mod 4$	$\begin{pmatrix} 1 & 0 & 1 \\ & 1 & 1 \\ & & 1 \end{pmatrix}$ $l \equiv 3 \mod 4$
X^{2l-4} $(l \geq 3)$	$\begin{pmatrix} 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$ $l \equiv 0 \mod 4$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ & 1 & 1 & 1 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$ $l \equiv 1 \mod 4$
	$\begin{pmatrix} 1 & 1 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$ $l \equiv 2 \mod 4$	$\begin{pmatrix} 1 & 0 & 1 & 0 \\ & 1 & 1 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$ $l \equiv 3 \mod 4$

Table 3.4: Action of X^{2l-k} on $V(2l)$ for $1 \leq k \leq 4$. Here the matrices represent the upper right corner of the matrix X^{2l-k} with respect to the basis $f_1, \dots, f_{2l}$; the remaining entries in the matrix of X^{2l-k} are zero.

Lemma 3.6.2. *Let $l \geq 1$. Then the decomposition of $V(2) \otimes V(2l)$ with respect to the product form is as given in Table 3.6.*

Proof. First of all, note that by Lemma 3.3.9 we have $V_2 \otimes V_{2l} = V_{2l} \oplus V_{2l}$, so the Jordan block sizes in $V(2) \otimes V(2l)$ are as claimed. We next determine the decomposition of $V(2) \otimes V(2l)$ into orthogonally indecomposable summands. Note for the orthogonal decomposition, our claim is that $\varepsilon(2l) = 0$ if $l \equiv 0 \mod 2$, and $\varepsilon(2l) = 1$ if $l \equiv 1 \mod 2$, see Definition 2.4.6. We proceed to prove this using Lemma 2.4.8.

By Lemma 2.4.8, we have $\varepsilon(2l) = 1$ if and only if $(X^{2l-1}(e_i \otimes f_j), e_i \otimes f_j) \neq 0$ for some basis vector $e_i \otimes f_j$ of $V(2) \otimes V(2l)$. We thus proceed to compute $(X^{2l-1}(e_i \otimes f_j), e_i \otimes f_j)$ for all e_i and f_j.

Using Table 3.1, we see that, for all $v \in V(2l)$,

$$\begin{aligned} (X^{2l-1} \cdot (e_1 \otimes v), e_1 \otimes v) &= 0, \text{ and} \\ (X^{2l-1} \cdot (e_{-1} \otimes v), e_{-1} \otimes v) &= (X^{2l-1}v, v) + (X^{2l-2}v, v). \end{aligned}$$

X^{2l-5} ($l \geq 4$)	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 0 \\ & & 1 & 1 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}$ $l \equiv 0 \mod 8$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 0 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}$ $l \equiv 1 \mod 8$
	$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & 1 & 1 & 1 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}$ $l \equiv 2 \mod 8$	$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}$ $l \equiv 3 \mod 8$
	$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ & 1 & 0 & 1 & 0 \\ & & 1 & 1 & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}$ $l \equiv 4 \mod 8$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 1 \\ & & 1 & 0 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}$ $l \equiv 5 \mod 8$
	$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ & 1 & 0 & 0 & 0 \\ & & 1 & 1 & 1 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}$ $l \equiv 6 \mod 8$	$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}$ $l \equiv 7 \mod 8$
X^{2l-6} ($l \geq 5$)	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 0 & 1 \\ & & 1 & 1 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix}$ $l \equiv 0 \mod 8$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 1 \\ & & 1 & 0 & 1 & 0 \\ & & & 1 & 1 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}$ $l \equiv 1 \mod 8$
	$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 & 0 \\ & & 1 & 1 & 1 & 1 \\ & & & 1 & 0 & 1 \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix}$ $l \equiv 2 \mod 8$	$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 1 & 1 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}$ $l \equiv 3 \mod 8$
	$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 0 & 1 & 0 & 0 \\ & & 1 & 1 & 0 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix}$ $l \equiv 4 \mod 8$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \\ & & 1 & 0 & 1 & 0 \\ & & & 1 & 1 & 0 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}$ $l \equiv 5 \mod 8$
	$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ & 1 & 0 & 0 & 0 & 1 \\ & & 1 & 1 & 1 & 1 \\ & & & 1 & 0 & 1 \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix}$ $l \equiv 6 \mod 8$	$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 0 & 0 & 1 \\ & & 1 & 0 & 0 & 0 \\ & & & 1 & 1 & 1 \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix}$ $l \equiv 7 \mod 8$

Table 3.5: Action of X^{2l-k} on $V(2l)$ for $5 \leq k \leq 6$. Here the matrices represent the upper right corner of the matrix X^{2l-k} with respect to the basis $f_1, \dots, f_{2l}$; the remaining entries in the matrix of X^{2l-k} are zero.

Furthermore, with Table 3.4, one finds that

$$(*)\begin{cases}(X^{2l-1}f_j,f_j)+(X^{2l-2}f_j,f_j)=0 \text{ for all } j\leq 2l-1.\\[2ex] (X^{2l-1}f_{2l},f_{2l})+(X^{2l-2}f_{2l},f_{2l})=\begin{cases}0, & \text{if } l\equiv 0 \mod 2.\\ 1, & \text{if } l\equiv 1 \mod 2.\end{cases}\end{cases}$$

From (*) we conclude that $\varepsilon(2l)=0$ if $l\equiv 0 \mod 2$, and $\varepsilon(2l)=1$ if $l\equiv 1 \mod 2$, finishing the proof of the lemma. □

Lemma 3.6.3. *Let $l\geq 2$. Then the decomposition of $V(4)\otimes V(2l)$ with respect to the product form is as given in Table 3.6.*

Proof. As a consequence of Theorem 3.3.5, Theorem 3.3.8 and Lemma 3.3.9, we have

$$V_4\otimes V_{2l}=\begin{cases}4\cdot V_{2l} & \text{if } l\equiv 0 \mod 2\\ 2\cdot V_{2l-2}\oplus 2\cdot V_{2l+2} & \text{if } l\equiv 1 \mod 2\end{cases}$$

so the Jordan block sizes in $V(4)\otimes V(2l)$ are as claimed.

We prove next that the decomposition of $V(4)\otimes V(2l)$ into orthogonally indecomposable summands is as claimed.

We consider first $l\equiv 0 \mod 2$. Note that here the claim is that $\varepsilon(2l)=0$ if $l\equiv 0 \mod 4$ and $\varepsilon(2l)=1$ if $l\equiv 2 \mod 4$, see Definition 2.4.6. Now $V_4\otimes V_{2l}=4\cdot V_{2l}$, so by Lemma 2.4.8, it will be enough to show that $(X^{2l-1}(e_i\otimes f_j),e_i\otimes f_j)=0$ for all i and j if and only if $l\equiv 0 \mod 4$. Since $2l-1\equiv 3 \mod 4$, with Table 3.2 we see that

$$\begin{aligned}(X^{2l-1}\cdot(e_1\otimes v),e_1\otimes v)&=0\\ (X^{2l-1}\cdot(e_2\otimes v),e_2\otimes v)&=0\\ (X^{2l-1}\cdot(e_{-2}\otimes v),e_{-2}\otimes v)&=(X^{2l-1}v,v)+(X^{2l-2}v,v)\\ (X^{2l-1}\cdot(e_{-1}\otimes v),e_{-1}\otimes v)&=(X^{2l-2}v,v)+(X^{2l-4}v,v)\end{aligned}$$

for all $v\in V(2l)$. Now if $l=2$, one computes with Table 3.4 that $X^{2l-2}f_{2l}=f_1+f_2$ and $X^{2l-4}f_{2l}=f_{2l}$, so $(X^{2l-1}\cdot(e_{-1}\otimes f_{2l}),e_{-1}\otimes f_{2l})=1$ and thus $\varepsilon(2l)=1$, as claimed.

Suppose then that $l>2$. Since $l\equiv 0 \mod 2$, it follows from (*) in the proof of Lemma 3.6.2 that we have $(X^{2l-1}\cdot(e_{-2}\otimes v),e_{-2}\otimes v)=(X^{2l-1}v,v)+(X^{2l-2}v,v)=0$ for all $v\in V(2l)$. Furthermore, with Table 3.4 one computes that $(X^{2l-1}\cdot(e_{-1}\otimes f_j),e_{-1}\otimes f_j)=(X^{2l-2}f_j,f_j)+(X^{2l-4}f_j,f_j)=0$ for all $j\leq 2l-1$. Finally, with Table 3.4 we see that

$$\begin{aligned}(X^{2l-1}\cdot(e_{-1}\otimes f_{2l}),e_{-1}\otimes f_{2l})&=(X^{2l-2}f_{2l},f_{2l})+(X^{2l-4}f_{2l},f_{2l})\\ &=\begin{cases}0, & \text{if } l\equiv 0 \mod 4.\\ 1, & \text{if } l\equiv 2 \mod 4.\end{cases}\end{aligned}$$

Hence $\varepsilon(2l)=0$ if $l\equiv 0 \mod 4$ and $\varepsilon(2l)=1$ if $l\equiv 2 \mod 4$. This completes the proof of the claim in the case where $l\equiv 0 \mod 2$.

Next we consider $l\equiv 1 \mod 2$. Here the claim is that $\varepsilon(2l+2)=0$ and $\varepsilon(2l-2)=0$ (Definition 2.4.6). To show that $\varepsilon(2l+2)=0$, by Lemma 2.4.8 it will be enough to show that $(X^{2l+1} \cdot (e_i \otimes f_j), e_i \otimes f_j) = 0$ for all i and j. Now $2l+1 \equiv 3 \mod 4$, so with Table 3.2 one gets

$$\begin{aligned}(X^{2l+1} \cdot (e_1 \otimes v), e_1 \otimes v) &= 0\\ (X^{2l+1} \cdot (e_2 \otimes v), e_2 \otimes v) &= 0\\ (X^{2l+1} \cdot (e_{-2} \otimes v), e_{-2} \otimes v) &= (X^{2l+1}v, v) + (X^{2l}v, v) = 0\\ (X^{2l+1} \cdot (e_{-1} \otimes v), e_{-1} \otimes v) &= (X^{2l}v, v) + (X^{2l-2}v, v) = (X^{2l-2}v, v)\end{aligned}$$

for all $v \in V(2l)$. Since $l \equiv 1 \mod 2$, we have $(X^{2l-2}f_j, f_j) = 0$ for all j (Table 3.4), so it follows that $\varepsilon(2l+2) = 0$.

To show that $\varepsilon(2l-2) = 0$, by Lemma 2.4.8 it will be enough to demonstrate that $(X^{2l-3} \cdot \gamma, \gamma) = 0$ for all γ in a basis for the kernel of X^{2l-2} acting on $V(4) \otimes V(2l)$. Now $2l-2 \equiv 0 \mod 4$, so by Table 3.2 we have $X^{2l-2} \cdot (v \otimes w) = v \otimes X^{2l-2}w$ for all $v \in V(4)$, $w \in V(2l)$. Hence a basis for the kernel of X^{2l-2} acting on $V(4) \otimes V(2l)$ is given by $e_i \otimes f_j$, where $j \leq 2l-2$. Now $2l-3 \equiv 3 \mod 4$, so with Table 3.2 we find that

$$\begin{aligned}(X^{2l-3} \cdot (e_1 \otimes v), e_1 \otimes v) &= 0\\ (X^{2l-3} \cdot (e_2 \otimes v), e_2 \otimes v) &= 0\\ (X^{2l-3} \cdot (e_{-2} \otimes v), e_{-2} \otimes v) &= (X^{2l-3}v, v) + (X^{2l-4}v, v)\\ (X^{2l-3} \cdot (e_{-1} \otimes v), e_{-1} \otimes v) &= (X^{2l-4}v, v) + (X^{2l-6}v, v)\end{aligned}$$

for all $v \in V(2l)$.

With Table 3.4, it is straightforward to verify that for $l \equiv 1 \mod 2$, we have $(X^{2l-3} \cdot (e_{-2} \otimes f_j), e_{-2} \otimes f_j) = (X^{2l-3}f_j, f_j) + (X^{2l-4}f_j, f_j) = 0$ for all $j \leq 2l-2$. Similarly one verifies for all $l \equiv 1 \mod 2$ with $l > 3$ that $(X^{2l-3} \cdot (e_{-1} \otimes f_j), e_{-1} \otimes f_j) = (X^{2l-4}f_j, f_j) + (X^{2l-6}f_j, f_j) = 0$ for all $j \leq 2l-2$. Finally, for $l = 3$ a calculation shows that $(X^{2l-4}f_j, f_j) + (X^{2l-6}f_j, f_j) = 0$ for all $j \leq 2l-2$, completing the proof of the claim. □

Lemma 3.6.4. *Let $l \geq 3$. Then the decomposition of $V(6) \otimes V(2l)$ with respect to the product form is as given in Table 3.6.*

Proof. As a consequence of Theorem 3.3.8 and Lemma 3.3.11, we have

$$V_6 \otimes V_{2l} = \begin{cases} 6 \cdot V_{2l} & \text{if } l \equiv 0 \mod 4\\ 4 \cdot V_{2l-2} \oplus 2 \cdot V_{2l+4} & \text{if } l \equiv 1 \mod 4\\ 2 \cdot V_{2l-4} \oplus 2 \cdot V_{2l} \oplus 2 \cdot V_{2l+4} & \text{if } l \equiv 2 \mod 4\\ 2 \cdot V_{2l-4} \oplus 4 \cdot V_{2l+4} & \text{if } l \equiv 3 \mod 4\end{cases}$$

so the Jordan block sizes in $V(6) \otimes V(2l)$ are as claimed.

We prove next that the decomposition of $V(6) \otimes V(2l)$ into orthogonally indecomposable summands is as claimed. We consider the different possibilities for $l \mod 4$.

$l \equiv 0 \mod 4$**:**
Here the claim is that $\varepsilon(2l) = 0$ (Definition 2.4.6). By Lemma 2.4.8, it will be enough to show that $(X^{2l-1}(e_i \otimes f_j), e_i \otimes f_j) = 0$ for all e_i and f_j. Since $2l-1 \equiv 7$

mod 8, with Table 3.3 one gets

$$\begin{aligned}
(X^{2l-1}\cdot(e_1\otimes v), e_1\otimes v) &= 0\\
(X^{2l-1}\cdot(e_2\otimes v), e_2\otimes v) &= 0\\
(X^{2l-1}\cdot(e_3\otimes v), e_3\otimes v) &= 0\\
(X^{2l-1}\cdot(e_{-3}\otimes v), e_{-3}\otimes v) &= (X^{2l-1}v,v)+(X^{2l-2}v,v)\\
(X^{2l-1}\cdot(e_{-2}\otimes v), e_{-2}\otimes v) &= (X^{2l-2}v,v)+(X^{2l-4}v,v)\\
(X^{2l-1}\cdot(e_{-1}\otimes v), e_{-1}\otimes v) &= (X^{2l-3}v,v)+(X^{2l-4}v,v)\\
&\quad +(X^{2l-5}v,v)+(X^{2l-6}v,v)
\end{aligned}$$

for all $v\in V(2l)$. Since $l\equiv 0 \mod 2$, it follows from (*) in the proof of Lemma 3.6.2 that we have $(X^{2l-1}f_j,f_j)+(X^{2l-2}f_j,f_j)=0$ for all j. Furthermore, one computes with Table 3.4 and Table 3.5 that for all j we have $(X^{2l-2}f_j,f_j)+(X^{2l-4}f_j,f_j)=0$ and $(X^{2l-3}f_j,f_j)+(X^{2l-4}f_j,f_j)+(X^{2l-5}f_j,f_j)+(X^{2l-6}f_j,f_j)=0$. Hence $\varepsilon(2l)=0$, as claimed.

$l\equiv 1 \mod 4$:
In this case, the claim is that $\varepsilon(2l+4)=1$ and $\varepsilon(2l-2)=0$ (Definition 2.4.6). To show that $\varepsilon(2l+4)=1$, by Lemma 2.4.8 it will suffice to show that $(X^{2l+3}(e_i\otimes f_j), e_i\otimes f_j)\neq 0$ for some $e_i\otimes f_j$. Now $2l+3\equiv 5 \mod 8$, so with Table 3.3 one finds that $(X^{2l+3}(e_{-1}\otimes v), e_{-1}\otimes v)=(X^{2l-1}v,v)+(X^{2l-2}v,v)$. Now for $v=f_{2l}$ we have $X^{2l-1}v=f_1$ and $X^{2l-2}v=f_2$ (Table 3.4), so it follows that $(X^{2l+3}(e_{-1}\otimes f_{2l}), e_{-1}\otimes f_{2l})=1$. Therefore $\varepsilon(2l+4)=1$.

To prove that $\varepsilon(2l-2)=0$, by Lemma 2.4.8 we should show that $(X^{2l-1}\cdot\gamma,\gamma)=0$ for all $\gamma\in V(6)\otimes V(2l)$ in a basis for the kernel of X^{2l-2} acting on $V(6)\otimes V(2l)$. Now $2l-2\equiv 0 \mod 8$, so by Table 3.3 we have $X^{2l-2}\cdot(v\otimes w)=v\otimes X^{2l-2}w$ for all $v\in V(6)$, $w\in V(2l)$. Hence a basis for the kernel of X^{2l-2} acting on $V(6)\otimes V(2l)$ is given by $e_i\otimes f_j$, where $j\leq 2l-2$. Since $2l-3\equiv 7 \mod 8$, with Table 3.3 one finds:

$$\begin{aligned}
(X^{2l-3}\cdot(e_1\otimes v), e_1\otimes v) &= 0\\
(X^{2l-3}\cdot(e_2\otimes v), e_2\otimes v) &= 0\\
(X^{2l-3}\cdot(e_3\otimes v), e_3\otimes v) &= 0\\
(X^{2l-3}\cdot(e_{-3}\otimes v), e_{-3}\otimes v) &= (X^{2l-3}v,v)+(X^{2l-4}v,v)\\
(X^{2l-3}\cdot(e_{-2}\otimes v), e_{-2}\otimes v) &= (X^{2l-4}v,v)+(X^{2l-6}v,v)\\
(X^{2l-3}\cdot(e_{-1}\otimes v), e_{-1}\otimes v) &= (X^{2l-5}v,v)+(X^{2l-6}v,v)\\
&\quad +(X^{2l-7}v,v)+(X^{2l-8}v,v)
\end{aligned}$$

Using Table 3.4 and Table 3.5, one verifies for all $j\leq 2l-2$ that we have $(X^{2l-3}f_j,f_j)+(X^{2l-4}f_j,f_j)=0$ and $(X^{2l-4}f_j,f_j)+(X^{2l-6}f_j,f_j)=0$. We also claim that for all $j\leq 2l-2$ we have $(X^{2l-5}f_j,f_j)+(X^{2l-6}f_j,f_j)+(X^{2l-7}f_j,f_j)+(X^{2l-8}f_j,f_j)=0$. If $j\leq l$, then this is clear, since we have (X^kf_j,f_j) for any $k\geq 0$ by Lemma 3.6.1 (i) and (ii). If $l<j\leq 2l-2$, then we can write $f_j=X^2f_{j+2}$ and verify using Table 3.4 and Table 3.5 that $(X^{2l-5}f_j,f_j)+(X^{2l-6}f_j,f_j)+(X^{2l-7}f_j,f_j)+(X^{2l-8}f_j,f_j)=(X^{2l-3}f_{j+2},f_j)+(X^{2l-4}f_{j+2},f_j)+(X^{2l-5}f_{j+2},f_j)+(X^{2l-6}f_{j+2},f_j)=0$. It follows therefore that $\varepsilon(2l-2)=0$.

$l\equiv 2 \mod 4$:

In this case, the claim is that $\varepsilon(2l+4)=0$, $\varepsilon(2l)=0$, and $\varepsilon(2l-4)=0$ (Definition 2.4.6). To show that $\varepsilon(2l+4)=0$, by Lemma 2.4.8 it will be enough to show that $(X^{2l+3}(e_i\otimes f_j), e_i\otimes f_j)=0$ for all e_i and f_j. Now $2l+3\equiv 7 \mod 8$, so with Table 3.3 we get

$$\begin{aligned}
(X^{2l+3}\cdot(e_1\otimes v), e_1\otimes v) &= 0\\
(X^{2l+3}\cdot(e_2\otimes v), e_2\otimes v) &= 0\\
(X^{2l+3}\cdot(e_3\otimes v), e_3\otimes v) &= 0\\
(X^{2l+3}\cdot(e_{-3}\otimes v), e_{-3}\otimes v) &= (X^{2l+3}v,v)+(X^{2l+2}v,v)=0\\
(X^{2l+3}\cdot(e_{-2}\otimes v), e_{-2}\otimes v) &= (X^{2l+2}v,v)+(X^{2l}v,v)=0\\
(X^{2l+3}\cdot(e_{-1}\otimes v), e_{-1}\otimes v) &= (X^{2l+1}v,v)+(X^{2l}v,v)+(X^{2l-1}v,v)+(X^{2l-2}v,v)\\
&= (X^{2l-1}v,v)+(X^{2l-2}v,v)
\end{aligned}$$

for all $v\in V(2l)$. Since $l\equiv 0 \mod 2$, it follows from (*) in the proof of Lemma 3.6.2 that we have $(X^{2l-1}f_j,f_j)+(X^{2l-2}f_j,f_j)=0$ for all j. Hence $\varepsilon(2l-4)=0$.

For showing that $\varepsilon(2l)=0$, by Lemma 2.4.8 it will suffice to show that $(X^{2l-1}v,v)=0$ for all $v\in V(6)\otimes V(2l)$ in a basis of the kernel of X^{2l} acting on $V(6)\otimes V(2l)$. Now $2l\equiv 4 \mod 8$, so with Table 3.3 one verifies that

$$\begin{aligned}
X^{2l}\cdot(e_i\otimes v) &= 0 \ (i=1,2,3,-3)\\
X^{2l}\cdot(e_{-2}\otimes v) &= e_1\otimes X^{2l-4}v\\
X^{2l}\cdot(e_{-1}\otimes v) &= e_2\otimes X^{2l-4}v
\end{aligned}$$

for all $v\in V(2l)$. Hence a basis for the kernel of X^{2l} acting on $V(6)\otimes V(2l)$ is given by the $e_i\otimes f_j$ such that $1\le j\le 2l-4$ if $i=-2$ or $i=-1$. Since $2l-1\equiv 3 \mod 8$, with Table 3.3 we get

$$\begin{aligned}
(X^{2l-1}\cdot(e_1\otimes v), e_1\otimes v) &= 0\\
(X^{2l-1}\cdot(e_2\otimes v), e_2\otimes v) &= 0\\
(X^{2l-1}\cdot(e_3\otimes v), e_3\otimes v) &= 0\\
(X^{2l-1}\cdot(e_{-3}\otimes v), e_{-3}\otimes v) &= (X^{2l-1}v,v)+(X^{2l-2}v,v)\\
(X^{2l-1}\cdot(e_{-2}\otimes v), e_{-2}\otimes v) &= (X^{2l-2}v,v)+(X^{2l-4}v,v)\\
(X^{2l-1}\cdot(e_{-1}\otimes v), e_{-1}\otimes v) &= (X^{2l-1}v,v)+(X^{2l-2}v,v)\\
&\quad +(X^{2l-3}v,v)+(X^{2l-4}v,v)
\end{aligned}$$

for all $v\in V(2l)$. Now $l\equiv 0 \mod 2$, so it follows from (*) in the proof of Lemma 3.6.2 that we have $(X^{2l-1}f_j,f_j)+(X^{2l-2}f_j,f_j)=0$ for all f_j. Furthermore, using Table 3.4 and Table 3.5, it is straightforward to verify that $(X^{2l-2}f_j,f_j)+(X^{2l-4}f_j,f_j)=0$ and $(X^{2l-3}f_j,f_j)+(X^{2l-4}f_j,f_j)+(X^{2l-5}f_j,f_j)+(X^{2l-6}f_j,f_j)=0$ for all $j\le 2l-4$. It follows then that $\varepsilon(2l)=0$.

What remains in this case is to show that $\varepsilon(2l-4)=0$. By Lemma 2.4.8, it will suffice to show that $(X^{2l-5}v,v)=0$ for all v in a basis of the kernel of X^{2l-4} acting on $V(6)\otimes V(2l)$. Now $2l-4\equiv 0 \mod 8$, so by Table 3.3 we have $X^{2l-4}\cdot(v\otimes w)=v\otimes X^{2l-4}w$ for all $v\in V(6)$, $w\in V(2l)$. Hence a basis for the kernel of X^{2l-4} acting on $V(6)\otimes V(2l)$ is given by $e_i\otimes f_j$, where $j\le 2l-4$. Now

$2l-5 \equiv 7 \mod 8$, so with Table 3.3 we get

$$\begin{aligned}
(X^{2l-5} \cdot (e_1 \otimes v), e_1 \otimes v) &= 0 \\
(X^{2l-5} \cdot (e_2 \otimes v), e_2 \otimes v) &= 0 \\
(X^{2l-5} \cdot (e_3 \otimes v), e_3 \otimes v) &= 0 \\
(X^{2l-5} \cdot (e_{-3} \otimes v), e_{-3} \otimes v) &= (X^{2l-5}v, v) + (X^{2l-6}v, v) \\
(X^{2l-5} \cdot (e_{-2} \otimes v), e_{-2} \otimes v) &= (X^{2l-6}v, v) + (X^{2l-8}v, v) \\
(X^{2l-5} \cdot (e_{-1} \otimes v), e_{-1} \otimes v) &= (X^{2l-7}v, v) + (X^{2l-8}v, v) \\
&\quad + (X^{2l-9}v, v) + (X^{2l-10}v, v)
\end{aligned}$$

With Table 3.5, it is easy to verify that $(X^{2l-5}f_j, f_j)+(X^{2l-6}f_j, f_j) = 0$ for all $j \leq 2l-4$. We claim that $(X^{2l-6}f_j, f_j)+(X^{2l-8}f_j, f_j) = 0$ for all $j \leq 2l-4$. If $j \leq l$, this is clear, since we have $(X^k f_j, f_j)$ for any $k \geq 0$ by Lemma 3.6.1 (i) and (ii). If $l < j \leq 2l-4$, then we can write $f_j = X^4 f_{j+4}$ and use Table 3.4 to verify that $(X^{2l-6}f_j, f_j) + (X^{2l-8}f_j, f_j) = (X^{2l-2}f_{j+4}, f_j) + (X^{2l-4}f_{j+4}, f_j) = 0$. Similarly, one checks that $(X^{2l-7}f_j, f_j) + (X^{2l-8}f_j, f_j) + (X^{2l-9}f_j, v) + (X^{2l-10}f_j, f_j) = 0$ for all $j \leq 2l-4$. Therefore $\varepsilon(2l-4) = 0$, as claimed.

$l \equiv 3 \mod 4$**:**
In this case, the claim is that $\varepsilon(2l+2) = 0$ and $\varepsilon(2l-4) = 1$ (Definition 2.4.6). To show that $\varepsilon(2l+2) = 0$, by Lemma 2.4.8 it will suffice to show that $(X^{2l+1}(e_i \otimes f_j), e_i \otimes f_j)) = 0$ for all e_i and f_j. Now $2l+1 \equiv 7 \mod 8$, so with Table 3.3 one gets

$$\begin{aligned}
(X^{2l+1} \cdot (e_1 \otimes v), e_1 \otimes v) &= 0 \\
(X^{2l+1} \cdot (e_2 \otimes v), e_2 \otimes v) &= 0 \\
(X^{2l+1} \cdot (e_3 \otimes v), e_3 \otimes v) &= 0 \\
(X^{2l+1} \cdot (e_{-3} \otimes v), e_{-3} \otimes v) &= (X^{2l+1}v, v) + (X^{2l}v, v) = 0 \\
(X^{2l+1} \cdot (e_{-2} \otimes v), e_{-2} \otimes v) &= (X^{2l}v, v) + (X^{2l-2}v, v) = (X^{2l-2}v, v) \\
(X^{2l+1} \cdot (e_{-1} \otimes v), e_{-1} \otimes v) &= (X^{2l-1}v, v) + (X^{2l-2}v, v) \\
&\quad + (X^{2l-3}v, v) + (X^{2l-4}v, v)
\end{aligned}$$

for all $v \in V(2l)$. Now using Table 3.4, it is straightforward to verify for all j that $(X^{2l-2}f_j, f_j) = 0$ and $(X^{2l-1}f_j, f_j)+(X^{2l-2}f_j, f_j)+(X^{2l-3}f_j, f_j)+(X^{2l-4}f_j, f_j) = 0$. Thus $\varepsilon(2l+2) = 0$.

Finally, we show that $\varepsilon(2l-4) = 1$ by finding a $\gamma \in V(6) \otimes V(2l)$ such that γ is in the kernel of X^{2l-4} acting on $V(6) \otimes V(2l)$, and such that $(X^{2l-5}\gamma, \gamma) = 1$ (Lemma 2.4.8). To this end, define

$$\gamma = e_2 \otimes f_{2l} + e_{-3} \otimes (f_{2l-2} + f_{2l-4}) + e_{-1} \otimes f_{2l-4}.$$

Now $2l-4 \equiv 2 \mod 8$, so with Table 3.3 and Table 3.4 one computes

$$\begin{aligned}
X^{2l-4} \cdot (e_2 \otimes f_{2l}) &= e_2 \otimes f_4 \\
X^{2l-4} \cdot (e_{-3} \otimes (f_{2l-2} + f_{2l-4})) &= e_{-3} \otimes f_2 + e_2 \otimes f_4 \\
X^{2l-4} \cdot (e_{-1} \otimes f_{2l-4}) &= e_{-3} \otimes f_2
\end{aligned}$$

and thus $X^{2l-4}\cdot\gamma = 0$. If $l = 3$, a calculation shows that $(X^{2l-5}\cdot\gamma,\gamma) = (X\cdot\gamma,\gamma) = 1$. Suppose then that $l > 3$. With Table 3.3 and Table 3.5, one computes that

$$\begin{aligned} X^{2l-5}\cdot(e_2\otimes f_{2l}) &= (e_2+e_1)\otimes(\delta f_1+f_4+f_5) \\ &\quad + e_1\otimes(\delta f_2+f_4+f_6) \\ X^{2l-5}\cdot(e_{-3}\otimes(f_{2l-2}+f_{2l-4})) &= (e_{-3}+e_3+e_2+e_1)\otimes(f_2+f_3) \\ &\quad + (e_3+e_2+e_1)\otimes(f_2+f_4) \\ X^{2l-5}\cdot(e_{-1}\otimes f_{2l-4}) &= (e_{-1}+e_{-2})\otimes f_1 + e_{-2}\otimes f_2 \end{aligned}$$

where $\delta = 0$ if $l \equiv 3 \mod 8$ and $\delta = 1$ if $l \equiv 7 \mod 8$. Now it is straightforward to verify that $(X^{2l-5}\cdot\gamma,\gamma) = 1$. □

We can now apply the decompositions given in 3.6 to determine when maximal tensor product subgroups contain distinguished unipotent elements.

Lemma 3.6.5. *Let $1 < m \leq n$ be even. Suppose that in the $K[u]$-module $V_m\otimes V_n$ all block sizes have multiplicity ≤ 2. Then $m \leq 6$.*

Proof. Write $m = 2k$ and $n = 2k'$. By Theorem 3.3.5 and Theorem 3.3.8, we have

$$V_m\otimes V_n = 2\cdot V_{2k_1}\oplus\cdots\oplus 2\cdot V_{2k_t},$$

where $V_k\otimes V_k' = V_{k_1}\oplus\cdots\oplus V_{k_t}$. Thus if in $V_m\otimes V_n$ all block sizes have multiplicity ≤ 2, it follows that in $V_k\otimes V_{k'}$ all block sizes have multiplicity ≤ 1. By Theorem 3.3.14, this implies that $k \leq 3$, hence $m \leq 6$, as claimed. □

Proposition 3.6.6. *Let V_1 and V_2 be vector spaces equipped with non-degenerate alternating bilinear forms β_1 and β_2, respectively. Assume that $1 < \dim V_1 \leq \dim V_2$. Let $u_i \in \mathrm{Sp}(V_i)$ be a unipotent element. Then with respect to the product form $\beta_1\otimes\beta_2$ on $V_1\otimes V_2$, the unipotent element $u = u_1\otimes u_2 \in \mathrm{Sp}(V_1)\otimes\mathrm{Sp}(V_2)$ is a distinguished unipotent element of $\mathrm{Sp}(V_1\otimes V_2)$ if and only if the following hold:*

- $\dim V_1 = 2$*, and* $V_1\downarrow K[u_1] = V(2)$ *(in other words, $u_1 \in \mathrm{Sp}(V_1)$ is a regular unipotent element).*
- *We have an orthogonal decomposition $V_2\downarrow K[u_2] = V(2k_1)+\cdots+V(2k_t)$, where $t \geq 1$, $1 \leq k_1 < \cdots < k_t$, and k_i is odd for all $1 \leq i \leq t$.*

Furthermore, when these two conditions hold, we have $V_1\otimes V_2\downarrow K[u] = V(2k_1)^2 + \cdots + V(2k_t)^2$.

Proof. Suppose that $u = u_1\otimes u_2$ is a distinguished unipotent element of $\mathrm{Sp}(V_1\otimes V_2)$. First we claim that u_i is a distinguished unipotent element in $\mathrm{Sp}(V_i)$ for $i = 1, 2$. Indeed, now $u_1\otimes u_2 = \varphi(u_1,u_2)$, where φ is the homomorphism $\mathrm{Sp}(V_1)\times\mathrm{Sp}(V_2)\to\mathrm{Sp}(V_1\otimes V_2)$ of algebraic groups defined by $(g_1,g_2)\mapsto g_1\otimes g_2$. It follows then as in Lemma 2.1.2 (ii) that (u_1,u_2) cannot be centralized by a nontrivial torus in $\mathrm{Sp}(V_1)\times\mathrm{Sp}(V_2)$, hence that u_i is a distinguished unipotent element in $\mathrm{Sp}(V_i)$ for $i = 1, 2$.

We first consider the case where V_i are orthogonally indecomposable as $K[u_i]$-modules. Since u_i is distinguished in $\mathrm{Sp}(V_i)$, by Proposition 2.4.4 we have $V_1\downarrow K[u_1] = V(2m)$ and $V_2\downarrow K[u_2] = V(2n)$, where $1 \leq m \leq n$.

Since $u_1 \otimes u_2$ is a distinguished unipotent element of $\mathrm{Sp}(V_1 \otimes V_2)$, it follows from Proposition 2.4.4 that all Jordan block sizes in the $K[u]$-module $V_{2m} \otimes V_{2n}$ have multiplicity ≤ 2. By Lemma 3.6.5, it follows that $m \leq 3$. For $m = 1$, $m = 2$, and $m = 3$, one checks from Table 3.6 that $u_1 \otimes u_2$ is a distinguished unipotent element (as described in Proposition 2.4.4) if and only if $m = 1$ and n is odd. In this case, we have $V_1 \downarrow K[u] = V(2)$ and $V_2 \downarrow K[u] = V(2n)$, and $V_1 \otimes V_2 = V(2n)^2$ by Table 3.6. This proves the claim in the case where V_1 and V_2 are both orthogonally indecomposable.

Consider then the case where we have orthogonal decompositions $V_1 = \oplus_{i=1}^{s} W_i$ and $V_2 = \oplus_{j=1}^{t} Z_j$ where W_i and Z_j are orthogonally indecomposable $K[u]$-modules for all i and j. This gives an orthogonal decomposition $V_1 \otimes V_2 = \oplus_{i,j} (W_i \otimes Z_j)$ of $K[u]$-modules. Since $u_1 \otimes u_2$ is a distinguished unipotent element of $\mathrm{Sp}(V_1 \otimes V_2)$, it follows that for all i and j, the $K[u]$-module $W_i \otimes Z_j$ corresponds to a distinguished unipotent element of $\mathrm{Sp}(W_i \otimes Z_j)$. From the indecomposable case we have treated before, it follows that for all i, j one of the following holds:

- $W_i \downarrow K[u] = V(2)$, and $Z_j \downarrow K[u] = V(2k)$ for some $k \geq 1$ odd;
- $W_i \downarrow K[u] = V(2k)$ for some $k > 1$ odd, and $Z_j \downarrow K[u] = V(2)$.

Note that in both cases we have $W_i \otimes Z_j \downarrow K[u] = V(2k)^2$ by the indecomposable case. We show now that the second possibility cannot occur. Suppose that for some i, we have $W_i \downarrow K[u] = V(2k)$ for $k > 1$ odd. Then it follows that we have $Z_j \downarrow K[u] = V(2)$ for all j. Thus $t = 1$, as otherwise $V_1 \otimes V_2 \downarrow K[u]$ would have ≥ 4 Jordan blocks of size $2k$. But then $\dim V_1 \geq \dim W_i > \dim V_2 = 2$, contradicting our assumption $\dim V_1 \leq \dim V_2$. Therefore $W_i \downarrow K[u] = V(2)$ for all i, and for all j we have $Z_j \downarrow K[u] = V(2k_j)$ where $k_j \geq 1$ is odd. Now we have $s = 1$, as otherwise $(V_1 \otimes V_2) \downarrow K[u]$ would have ≥ 4 Jordan blocks of size $2k_j$ for all j.

Thus $V_1 \downarrow K[u] = V(2)$, and $V_1 \otimes V_2 = \oplus_{j=1}^{t} V(2k_j)^2$. Again because each Jordan block size has multiplicity ≤ 2 in $V_1 \otimes V_2$, each k_j must be distinct. By ordering the summands suitably, we have $1 \leq k_1 < \cdots < k_t$, so V_1 and V_2 are as claimed. This completes the proof of the proposition. $\square$

3.7 Proof of Theorem 1.1.4

In this section, we put results from previous sections together and prove Theorem 1.1.4. For convenience, we recall the statement of Theorem 1.1.4:

Theorem 1.1.4. *Let $G = \mathrm{SL}(V)$ ($\dim V \geq 2$), $G = \mathrm{Sp}(V)$ ($\dim V \geq 2$), or $G = \mathrm{SO}(V)$ ($\dim V \geq 5$), where V is a finite-dimensional vector space over K. Fix a distinguished unipotent element $u \in G$. Let X be one of the maximal closed connected subgroup of G given in (a) - (f) of Theorem 1.1.3. Then the cases where X contains a G-conjugate of u are precisely the following:*

(a) *$G = \mathrm{SL}(V)$, $X = \mathrm{Sp}(V)$ ($\dim V$ even) or $X = \mathrm{SO}(V)$ ($p \neq 2$ and $\dim V$ odd), and u is a regular unipotent element.*

(b) *$G = \mathrm{Sp}(V)$, $p = 2$, $\dim V > 2$, $X = \mathrm{SO}(V)$, and the number of Jordan blocks of u is even.*

(c) $G = \mathrm{SO}(V)$, $p = 2$, $\dim V$ *is even,* X *is the stabilizer of a nonsingular* 1*-space, and* u *has a Jordan block of size* 2*.*

(d) X *is a maximal parabolic subgroup.*

(e) $G = \mathrm{Sp}(V)$ *or* $G = \mathrm{SO}(V)$, $V = W \oplus W^{\perp}$ *where* W *is a non-degenerate subspace of* V, $X = \mathrm{stab}_G(W)^{\circ}$, *and* $V \downarrow K[u] \cong \bigoplus_{i=1}^{t} V_{d_i} \oplus \bigoplus_{j=1}^{s} V_{d'_j}$ *for integers* $d_i, d'_j \geq 1$ *such that the following conditions hold:*

- $\dim W = \sum_{i=1}^{t} d_i$.
- *If* $p = 2$ *and* $G = \mathrm{SO}(V)$, *then* $t \equiv 0 \mod 2$.

(f) $V = V_1 \otimes V_2$ *with* $\dim V_1 \leq \dim V_2$ *and one of the following holds:*

(i) $G = \mathrm{SO}(V)$, $p = 2$, $\dim V_1 = 2$, $X = \mathrm{Sp}(V_1) \otimes \mathrm{Sp}(V_2)$, *and the orthogonal decomposition* $V \downarrow K[u]$ *(Proposition 2.4.4) is equal to* $V(2d_1)^2 + \cdots + V(2d_t)^2$ *for some* $1 \leq d_1 < \cdots < d_t$ *such that* d_i *is odd for all* i*.*

(ii) $G = \mathrm{Sp}(V)$ *or* $G = \mathrm{SO}(V)$, $p \neq 2$, X *is as in Theorem 1.1.3 (ii), (iii) or (iv); and for* $m = \dim V_1$ *and* $n = \dim V_2$ *one of the following conditions hold:*

- *The pair* (m, n) *is contained in the set* $\mathscr{S}$ *of Definition 3.3.12, and* $V \downarrow K[u] \cong \bigoplus_{i=1}^{m} V_{m+n-2i+1}$. *(In this case* $V \downarrow K[u] \cong V_m \otimes V_n$*).*
- *There exist integers* $1 \leq n_1 < n_2 < \cdots < n_t$ *such that* $\sum_{i=1}^{t} n_i = n$, $n_i \equiv n_{i'} \mod 2$ *for all* $1 \leq i, i' \leq t$, $n_i - n_{i-1} \geq 2m$ *for all* $2 \leq i \leq t$, *the pair* (m, n_i) *is contained in the set* $\mathscr{S}$ *of Definition 3.3.12 for all* $1 \leq i \leq t$, *and*

$$V \downarrow K[u] \cong \bigoplus_{j=1}^{\min(m,n_1)} V_{n_1+m-2j+1} \oplus \bigoplus_{i=2}^{t} \bigoplus_{j=1}^{m} V_{n_i+m-2j+1}.$$

(In this case $V \downarrow K[u] \cong V_m \otimes (V_{n_1} \oplus V_{n_2} \oplus \cdots \oplus V_{n_t})$*).*

Proof. We begin with cases (a) - (d), and here we will see that claim follows from results which are more or less well known. For case (a), we know that the only distinguished unipotent elements of $\mathrm{SL}(V)$ are those with only one Jordan block (Lemma 2.2.2), so the claim follows from Proposition 2.3.3 (ii)-(iii) and Proposition 2.4.4 (v). Case (b) is a consequence of Proposition 2.4.4 (iii), and case (c) follows from [LS12, 6.8]. For case (d), recall that every element of G is contained in some Borel subgroup. Hence any parabolic subgroup of G intersects every conjugacy class of G. In particular, a maximal parabolic subgroup of G intersects every unipotent conjugacy class of G.

For case (e), suppose that $G = \mathrm{Sp}(V)$ or $G = \mathrm{SO}(V)$. If $V = W \oplus W^{\perp}$ and $u \in \mathrm{stab}_G(W)^{\circ}$, then it is very obvious that $V \downarrow K[u] \cong \bigoplus_{i=1}^{t} V_{d_i} \oplus \bigoplus_{j=1}^{s} V_{d'_j}$, where $W \downarrow K[u] \cong \bigoplus_{i=1}^{t} V_{d_i}$ and $W^{\perp} \downarrow K[u] \cong \bigoplus_{j=1}^{s} V_{d'_j}$; thus $\dim W = \sum_{i=1}^{t} d_i$ in this case. If $p = 2$ and $G = \mathrm{SO}(V)$, then $\mathrm{stab}_G(W)^{\circ} = \mathrm{SO}(W) \times \mathrm{SO}(W^{\perp})$. Since the restriction of u to W is contained in $\mathrm{SO}(W)$, it follows from Proposition 2.4.4 (iii) that $t \equiv 0 \mod 2$.

Conversely for (e), let $V \downarrow K[u] \cong \bigoplus_{i=1}^{t} V_{d_i} \oplus \bigoplus_{j=1}^{s} V_{d'_j}$, and assume that $t \equiv 0 \mod 2$ if $p = 2$ and $G = \mathrm{SO}(V)$. By [LS12, Proposition 3.5] for $p \neq 2$ and

Proposition 2.4.4 for $p = 2$, it is possible to write the decomposition of $V \downarrow K[u]$ as an orthogonal direct sum

$$V \downarrow K[u] = V_1 \oplus \cdots \oplus V_t \oplus V_1' \oplus \cdots \oplus V_s',$$

where $V_i \cong V_{d_i}$ and $V_j' \cong V_{d_j'}$ for all $1 \leq i \leq t$ and $1 \leq j \leq s$. Then for $Z = \bigoplus_{i=1}^{t} V_i$ we have $Z^{\perp} = \bigoplus_{j=1}^{s} V_j'$, so $u \in \operatorname{stab}_G(Z)$ for $V = Z \oplus Z^{\perp}$ with $\dim Z = \sum_{i=1}^{t} d_i$. In fact, we have $u \in \operatorname{stab}_G(Z)^{\circ}$, when $p \neq 2$ or $G = \operatorname{Sp}(V)$ this is true for any unipotent element of $\operatorname{stab}_G(Z)$; when $p = 2$ and $G = \operatorname{SO}(V)$, this follows from the assumption $t \equiv 0 \mod 2$ and Proposition 2.4.4 (iii). Finally, any other orthogonal decomposition $V = W \oplus W^{\perp}$ with $\dim W = \dim Z$ is conjugate to the decomposition $V = Z \oplus Z^{\perp}$; i.e. there exists some $g \in G$ such that $g(Z) = W$. Then $gug^{-1} \in g \operatorname{stab}_G(Z)^{\circ} g^{-1} = \operatorname{stab}_G(W)^{\circ}$, as desired.

What remains is to deal with the stabilizers of tensor decompositions in Theorem 1.1.3 (f). In this case, sufficency follows from Proposition 3.6.6 and Lemma 3.3.17. We show that the conditions listed in our claim are necessary. For this, suppose that $V = V_1 \otimes V_2$, where $1 < \dim V_1 \leq \dim V_2$. Let X be one of the subgroups in Theorem 1.1.3 (f) and suppose that $u \in X$. Now $X = X_1 \otimes X_2$, where for all $i = 1, 2$ we have $X_i = \operatorname{SL}(V_i)$, $X_i = \operatorname{Sp}(V_i)$, or $X_i = \operatorname{SO}(V_i)$. Since u is a distinguished unipotent element, it follows as in the beginning of the proof of Proposition 3.6.6 that $u = u_1 \otimes u_2$, where u_i is a distinguished unipotent element of X_i.

We now consider different possibilities for the subgroups X in Theorem 1.1.3 (f). First suppose that $G = \operatorname{SL}(V)$, so now $X_i = \operatorname{SL}(V_i)$ for $i = 1, 2$. Since u_i is distinguished in X_i, we have that u_i acts on V_i with a single Jordan block (Lemma 2.2.2). Similarly u acts on V with a single Jordan block. This is a contradiction, since it is well known that the tensor product of any two Jordan blocks of size > 1 has ≥ 2 Jordan blocks (see e.g. Theorem 3.3.5 or [SS97, Lemma 1.5]).

Suppose then that $G = \operatorname{Sp}(V)$ or $G = \operatorname{SO}(V)$. If $p = 2$, then the only case in Theorem 1.1.3 (f) that applies is the one where $G = \operatorname{SO}(V)$ and $X = \operatorname{Sp}(V_1) \otimes \operatorname{Sp}(V_2)$. In this case the claim follows from Proposition 3.6.6. Consider then $p \neq 2$, so for all $i = 1, 2$ we have $X_i = \operatorname{Sp}(V_i)$ or $X_i = \operatorname{SO}(V_i)$. Since u_i is distinguished in X_i, it follows from Proposition 2.3.2 that $V_1 \downarrow K[u_1] = \bigoplus_{i=1}^{s} V_{m_i}$ and $V_2 \downarrow K[u_2] = \bigoplus_{j=1}^{t} V_{n_j}$, where $1 \leq m_1 < \cdots < m_t$ and $1 \leq n_1 < \cdots < n_s$ are integers such that $m_i \equiv m_{i'} \mod 2$ and $n_j \equiv n_{j'} \mod 2$ for all $1 \leq i, i' \leq s$ and $1 \leq j, j' \leq t$. Since u is distinguished in G, the tensor product $V_1 \otimes V_2$ has no repeated block sizes. Now the claim is immediate from Lemma 3.3.17. This completes the proof of the theorem. □

Chapter 4

Some representation theory

The purpose of this chapter is to list some results on the representation theory of simple algebraic groups for later reference. Most important will be the results on representations of $\mathrm{SL}_2(K)$ (Section 4.2 and Section 4.6), which will be key in the proof of Theorem 1.1.10 for unipotent elements of order p.

4.1 Weyl modules and tilting modules

Recall the notation $L_G(\lambda)$ for the irreducible G-module with highest weight $\lambda \in X(T)^+$, and $V_G(\lambda)$ for the *Weyl module* of G with highest weight λ.

Proposition 4.1.1. *Let $\lambda, \mu \in X(T)^+$ such that $\mu \not\succ \lambda$. Then if $0 \to L_G(\mu) \to E \to L_G(\lambda) \to 0$ is a nonsplit extension of $L_G(\lambda)$ by $L_G(\mu)$, we have $E \cong V_G(\lambda)/W$ for some submodule W of $V_G(\lambda)$.*

Proof. This follows from [Jan03, Lemma II.2.13, II.2.14]. □

Definition 4.1.2. Let V be a G-module. A filtration $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_t = V$ of submodules V_i of V is called a *Weyl filtration*, if for all $1 \leq i \leq t$ we have $V_i/V_{i-1} \cong V_G(\lambda_i)$ for some $\lambda_i \in X(T)^+$. We say that V is a *tilting module*, if both V and V^* have a Weyl filtration.

The following facts about tilting modules will be useful in the sequel.

Theorem 4.1.3 ([Jan03, Corollary II.E.2], [Mat90])**.** *Let V_1 and V_2 be G-modules. Then*

(i) *(Donkin) The direct sum $V_1 \oplus V_2$ is a tilting module if and only if both V_1 and V_2 are tilting modules.*

(ii) *(Mathieu, Donkin) If V_1 and V_2 are tilting modules, then the tensor product $V_1 \otimes V_2$ is tilting.*

Theorem 4.1.4 ([Jan03, Lemma II.E.3, Lemma II.E.5, Proposition II.E.6])**.** *Let $\lambda \in X(T)^+$. Then:*

(i) *(Ringel) Up to isomorphism, there is a unique indecomposable tilting module $T_G(\lambda)$ with $\dim T_G(\lambda)_\lambda = 1$ and such that $T_G(\lambda)_\mu \neq 0$ implies $\mu \preceq \lambda$.*

(ii) *If Q is a tilting module with $Q_\lambda \neq 0$ such that λ is maximal among the weights of Q, then $T_G(\lambda)$ is a direct summand of Q.*

(iii) *If Q is a tilting module, there exist unique integers $n(\nu) \geq 0$, almost all 0, such that*

$$Q \cong \bigoplus_{\nu \in X(T)^+} T_G(\nu)^{n(\nu)}.$$

We call $T_G(\lambda)$ the *indecomposable tilting module with highest weight* λ. The composition factors of $T_G(\lambda)$ are not known in general. For G of type A_1, we have the following result which gives a recursive description of the indecomposable tilting modules.

Theorem 4.1.5 ([Sei00, Lemma 2.3], [Don93, Example 2, pg. 47])**.** *Assume that $G = \mathrm{SL}_2(K)$ and let $c \in \mathbb{Z}_{\geq 0}$ be a dominant weight. Then:*

(i) *If $0 \leq c \leq p-1$, the indecomposable tilting module $T_G(c)$ is irreducible, so $T_G(c) = L_G(c)$.*

(ii) *If $p \leq c \leq 2p-2$, the indecomposable tilting module $T_G(c)$ is uniserial of dimension $2p$, and $T_G(c) = L_G(2p-2-c)/L_G(c)/L_G(2p-2-c)$.*

(iii) *(Donkin) If $c > 2p-2$, then $T_G(c) \cong T_G(p-1+r) \otimes T_G(s)^{[1]}$, where $s \geq 1$ and $0 \leq r \leq p-1$ are such that $c = sp + (p-1+r)$.*

4.2 Irreducible representations of $\mathrm{SL}_2(K)$

In this subsection, we give some basic results about irreducible representations of $\mathrm{SL}_2(K)$. Throughout we will identify the weights of $\mathrm{SL}_2(K)$ with $\mathbb{Z}$, and the dominant weights with $\mathbb{Z}_{\geq 0}$. We begin with the following lemma which is well known and follows from Steinberg's tensor product theorem.

Lemma 4.2.1. *Let $\lambda \in \mathbb{Z}_{\geq 0}$ be a dominant weight of $G = \mathrm{SL}_2(K)$. Write $\lambda = \sum_{i \geq 0} \lambda_i p^i$, where $0 \leq \lambda_i \leq p-1$. Then the weights occurring in the irreducible module $L_G(\lambda)$ are precisely the ones of the form*

$$\sum_{i \geq 0} (\lambda_i - 2k_i) p^i$$

where $0 \leq k_i \leq \lambda_i$. Furthermore, each weight space of $L_G(\lambda)$ is one-dimensional.

For the rest of this section, we will give some technical lemmas about the action of a unipotent element u of $\mathrm{SL}_2(K)$ on irreducible representations of $\mathrm{SL}_2(K)$. These will be used later in Section 5.13. The first lemma is well known.

Lemma 4.2.2. *Let $\lambda \in \mathbb{Z}_{\geq 0}$ be a non-zero weight of $G = \mathrm{SL}_2(K)$ such that $0 \leq \lambda \leq p-1$. Then a non-identity unipotent element $u \in G$ acts on $L_G(\lambda)$ with a single Jordan block of size $\lambda + 1$.*

Lemma 4.2.3. *Let $\lambda \in \mathbb{Z}_{\geq 0}$ be a non-zero weight of $G = \mathrm{SL}_2(K)$. Then a non-identity unipotent element $u \in G$ acts on $L_G(\lambda)$ with Jordan blocks of distinct sizes if and only if $\lambda = ap^k + bp^l$, where $0 \leq k < l$, $0 \leq a, b \leq p-1$ and $a + b \leq p-1$. In this case u acts on $L_G(\lambda)$ with Jordan blocks*

$$[a+b+1, a+b-1, \ldots, a+b-2d+1]$$

where $d = \min\{a, b\}$.

Proof. Write λ in base p, say $\lambda = \sum_{i=0}^{t} k_i p^i$, where $0 \leq k_i \leq p-1$. Now by Steinberg's tensor product theorem, we have

$$L_G(\lambda) \cong \bigotimes_{i=0}^{t} L_G(k_i)^{[i]}.$$

Since u acts on $L_G(k_i)$ with a single Jordan block of size $k_i + 1$ (Lemma 4.2.2), it follows that

$$L_G(\lambda) \downarrow K[u] \cong \bigotimes_{i=0}^{t} V_{k_i+1}.$$

It is now easy to see that to prove the claim, it is enough to show that the following facts hold:

(1) If $1 \leq d_1, d_2 \leq p$ and $d_1 + d_2 > p+1$, then $V_{d_1} \otimes V_{d_2}$ has repeated blocks.

(2) If $1 \leq d_1 \leq d_2 \leq p$ and $d_1 + d_2 \leq p+1$, we have $V_{d_1} \otimes V_{d_2} \cong \oplus_{i=0}^{d_1-1} V_{d_1+d_2-2i-1}$.

(3) If $s > 2$ and $1 < d_1 \leq d_2 \leq \cdots \leq d_s \leq p$, then the tensor product $\otimes_{i=1}^{s} V_{d_i}$ has repeated blocks.

The claims (1) and (2) are Corollary 3.3.7 (i) and (ii), respectively. What remains is to show that $\otimes_{i=1}^{s} V_{d_i}$ has repeated blocks if $s \geq 3$ and $1 < d_1 \leq d_2 \leq \cdots \leq d_s \leq p$. Clearly it will be enough to do this for $s = 3$. For the sake of contradiction, suppose then that $V_{d_1} \otimes V_{d_2} \otimes V_{d_3}$ has no repeated blocks for some $1 < d_1 \leq d_2 \leq d_3 \leq p$. Then $V_{d_1} \otimes V_{d_2}$ has no repeated blocks, so applying (1) and (2) to $V_{d_1} \otimes V_{d_2}$ we get

$$V_{d_1} \otimes V_{d_2} \otimes V_{d_3} \cong (V_{d_1+d_2-1} \otimes V_{d_3}) \oplus (V_{d_1+d_2-3} \otimes V_{d_3}) \oplus \cdots$$

Now $V_{d_1+d_2-1} \otimes V_{d_3}$ and $V_{d_1+d_2-3} \otimes V_{d_3}$ also have no repeated blocks, so again applying (1) and (2) to these tensor products gives

$$V_{d_1+d_2-1} \otimes V_{d_3} \cong V_{d_1+d_2+d_3-2} \oplus V_{d_1+d_2+d_3-4} \oplus \cdots$$

and

$$V_{d_1+d_2-3} \otimes V_{d_3} \cong V_{d_1+d_2+d_3-4} \oplus \cdots$$

Therefore the tensor product $V_{d_1} \otimes V_{d_2} \otimes V_{d_3}$ has ≥ 2 Jordan blocks of size $d_1 + d_2 + d_3 - 4$, contradiction. □

Lemma 4.2.4. *Assume that* $p \geq 3$. *Let* $r \geq p$. *Suppose that a nonidentity unipotent element* u *of* $\mathrm{SL}_2(K)$ *acts on the module* $L_G(r) \oplus L_G(r-2)$ *with no repeated blocks. Then* $r = p+1$, $r = p$, $r = 2p$ *or* $r = p + p^l$ *with* $l \geq 2$.

Proof. By applying Lemma 4.2.3 to the module $L_G(r)$, it follows that $r = ap^k + bp^l$, where $0 < a \leq p-1$, $0 \leq k < l$, $0 \leq b \leq p-1$, and $a + b \leq p-1$. If $k \geq 2$, then $r-2$ is written in base p as

$$r - 2 = (p-2) + (p-1)p + \cdots + (p-1)p^{k-1} + (a-1)p^k + bp^l,$$

so u acts on $L_G(r-2)$ with repeated blocks, by Lemma 4.2.3. Therefore we must have $k = 0$ or $k = 1$.

Suppose that $k = 0$, so now $r = a + bp^l$, where $0 < a \leq p - 1$, $l \geq 1$, and $0 < b \leq p - 1$. If $a \geq 2$, then $r - 2 = (a - 2) + bp^l$ in base p, and so by Lemma 4.2.3 we have

$$L_G(r) \downarrow K[u] = [a + b + 1, a + b - 1, \ldots]$$
$$L_G(r - 2) \downarrow K[u] = [a + b - 1, \ldots]$$

so u acts on $L_G(r) \oplus L_G(r-2)$ with two Jordan blocks of size $a+b-1$, contradiction. Therefore we must have $a = 1$. Now if $l \geq 2$, then $r - 2$ is written in base p as

$$r - 2 = (p - 1) + (p - 1)p + \cdots + (p - 1)p^{l-1} + (b - 1)p^l,$$

so u acts on $L_G(r - 2)$ with repeated blocks by Lemma 4.2.3. Therefore $l = 1$, so $r = 1 + bp$ and $r - 2 = (p - 1) + (b - 1)p$ in base p. Since u acts on $L_G(r - 2)$ with no repeated blocks, we have $b = 1$ by Lemma 4.2.3. That is, we have $r = p + 1$, as claimed.

Next we consider the case where $k = 1$. Now $r = ap + bp^l$, where $0 < a \leq p - 1$, $l \geq 2$, and $0 \leq b \leq p - 1$. Then $r - 2$ is written in base p as $r - 2 = (p - 2) + (a - 1)p + bp^l$. Since u acts on $L_G(r - 2)$ with no repeated blocks, it follows from Lemma 4.2.3 that either $b = 0$ and $a \leq 2$, or $a = 1$ and $b \leq 1$. In other words, we have $r = p$, $r = 2p$, or $r = p + p^l$ for $l \geq 2$. This completes the proof of the claim. □

Lemma 4.2.5. *Assume that $p \geq 5$. Let $r \geq p$. Suppose that a nonidentity unipotent element u of $\mathrm{SL}_2(K)$ acts on the module $L_G(r) \oplus L_G(r - 4)$ with no repeated blocks. Then one of the following holds:*

(i) $p \mid r$.

(ii) $r = p + 1$, $r = p + 2$, $r = p + 3$, $r = 2p + 1$, $r = 2p + 2$, *or* $r = 1 + 3p$.

(iii) $r = a + p^l$ *for some* $4 \leq a \leq p - 2$ *and* $l \geq 1$.

Proof. By applying Lemma 4.2.3 to the module $L_G(r)$, it follows that $r = ap^k + bp^l$, where $0 < a \leq p - 1$, $0 \leq k < l$, $0 \leq b \leq p - 1$, and $a + b \leq p - 1$. If $k > 0$, then we are in case (i) of the claim. Suppose then that $k = 0$, so now $r = a + bp^l$, where $0 < a \leq p - 1$, $l > 0$, and $0 < b \leq p - 1$. Note that here $b > 0$ since we are assuming that $r \geq p$.

If $a \geq 4$, then $r - 4$ is written in base p as $r - 4 = (a - 4) + bp^l$. If $b > 1$, then it follows from Lemma 4.2.3 that

$$L_G(r) \downarrow K[u] = [a + b + 1, a + b - 1, a + b - 3, \ldots]$$

and

$$L_G(r - 4) \downarrow K[u] = [a + b - 3, \ldots].$$

Thus $L_G(r) \oplus L_G(r-4) \downarrow K[u]$ has two Jordan blocks of size $a+b-3$, contradiction. Therefore $b = 1$, so $r = a + p^l$, where $a \geq 4$. Since $a + b \leq p - 1$, we have $4 \leq a \leq p - 2$, so r is as in case (iii) of the claim.

Consider then $a < 4$. If $l > 1$, then $r - 4$ is written in base p as

$$r - 4 = (p - 4 + a) + (p - 1)p + \cdots + (p - 1)p^{l-1} + (b - 1)p^l,$$

so by Lemma 4.2.3 the element u acts on $L_G(r-4)$ with repeated blocks, contradiction. Therefore $l=1$, so now $r = a + bp^l$, where $1 \leq a \leq 3$. Furthermore, since $r-4$ is written in base p as $r-4 = (p-4+a)+(b-1)p$, it follows from Lemma 4.2.3 that $(p-4+a)+(b-1) \leq p-1$. This gives $a+b \leq 4$, and since $1 \leq a \leq 3$ and $1 \leq b \leq p-1$, the only possibilities for r are those given in case (ii) of the claim. □

Lemma 4.2.6. *Assume that $p \geq 5$. Let $r \geq p$. Suppose that a nonidentity unipotent element u of* $\mathrm{SL}_2(k)$ *acts on the module $L_G(r) \oplus L_G(r-2) \oplus L_G(r-4)$ with no repeated blocks. Then $r = p$ or $r = p+1$.*

Proof. According to Lemma 4.2.4, we must have $r = p$, $r = p+1$, $r = 2p$, or $r = p + p^l$ with $l \geq 2$.

If $r = 2p$, then $r-2 = (p-2)+p$ and $r-4 = (p-4)+p$, so by Lemma 4.2.3 we have $L_G(r-2) \downarrow K[u] = [p-2, p]$ and $L_G(r-4) \downarrow K[u] = [p-4, p-2]$. Therefore $L_G(r-2) \oplus L_G(r-4) \downarrow K[u]$ has two blocks of size $p-2$. If $r = p + p^l$ with $l \geq 2$, the same argument shows that $L_G(r-2) \oplus L_G(r-4) \downarrow K[u]$ has two blocks of size $p-2$. Therefore the only possibilities are $r = p$ and $r = p+1$, as desired. □

4.3 Weyl group and weight orbits

Recall that W denotes the Weyl group of G. Fix a base $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ for the root system Φ of G, with respect to the maximal torus T. We will denote the reflection in W corresponding to a root $\alpha \in \Phi$ by $\sigma_\alpha \in W$.

Let V be any G-module. Then we have a decomposition

$$V = \bigoplus_{\mu \in X(T)} V_\mu,$$

where V_μ denotes the T-weight space for weight $\mu \in X(T)$. Now W acts on $X(T)$, and $X(T)^+$ is a fundamental domain for this action [Hum72, Lemma 10.3B]. Thus we have a decomposition

$$V = \bigoplus_{\mu \in X(T)^+} \bigoplus_{w \in W} V_{w\mu}.$$

This implies that $\dim V \geq |W\mu|$ if $\mu \in X(T)^+$ and $V_\mu \neq 0$, and this basic inequality will often be useful.

To describe the sizes of the orbits, we have the following lemma which follows from [Hum72, Lemma 10.3B].

Lemma 4.3.1. *Let $\lambda \in X(T)^+$ and write $\lambda = \sum_{i=1}^{l} a_i \omega_i$, where $a_i \geq 0$. Then* $\mathrm{Stab}_W(\lambda)$ *is generated by all σ_{α_i} such that $a_i = 0$.*

Since the Weyl group is always generated by simple reflections, from Lemma 4.3.1 we see that the W-stabilizer of $\lambda = \sum_{i=1}^{l} a_i \omega_i$ is the Weyl group of the semisimple Lie algebra with Dynkin diagram $\Delta' = \{\alpha_i : a_i = 0\}$. This makes the computation of $|W\lambda| = |W|/|\mathrm{Stab}_W(\lambda)|$ easy, since the orders of Weyl groups of indecomposable root systems are known (see for instance [Hum72, Table 12.1]).

4.4 Invariant forms on G-modules

In this section, we list some basic results on the existence of G-invariant bilinear forms on G-modules. We begin with the following definitions.

Definition 4.4.1. Let V be a G-module. The following notation will be defined.

(i) $\mathrm{Bil}(V)$ is the vector space of all bilinear forms on V.

(ii) For $\beta \in \mathrm{Bil}(V)$ and $g \in G$, define $\beta^g \in \mathrm{Bil}(V)$ by $\beta^g(v,w) = \beta(g^{-1}v, g^{-1}w)$.

(iii) $\mathrm{Bil}(V)^G$ is the vector space of all G-invariant bilinear forms on V, in other words, the space of all $\beta \in \mathrm{Bil}(V)$ such that $\beta^g = \beta$.

Definition 4.4.2. For $\lambda \in X(T)^+$, we define $d(\lambda) = \sum_{\alpha>0} \langle \lambda, \alpha \rangle$.

As seen in the next lemma and other lemmas below (Lemma 4.4.9, Lemma 4.4.10), the quantity $d(\lambda)$ in Definition 4.4.2 will often be useful for describing when a G-module has a G-invariant alternating or symmetric bilinear form.

Lemma 4.4.3 ([Ste68, Lemma 78, Lemma 79])**.** *Let $\lambda \in X(T)^+$. Then:*

(i) *$L_G(\lambda)^* \cong L_G(-w_0\lambda)$, where w_0 is the longest element in the Weyl group. In particular, $L_G(\lambda)$ is self-dual if and only if $\lambda = -w_0\lambda$.*

(ii) *Assume that $\lambda = -w_0\lambda$ and $p \neq 2$. Then $L_G(\lambda)$ has a non-degenerate G-invariant bilinear form b, unique up to a scalar. Furthermore, the form b is symmetric if $d(\lambda) \equiv 0 \mod 2$, and alternating if $d(\lambda) \equiv 1 \mod 2$.*

Corollary 4.4.4. *Assume that $p \neq 2$. Let $G = \mathrm{SL}_2(K)$ and fix a dominant weight $\lambda \in \mathbb{Z}_{\geq 0}$ of G. Then $L_G(\lambda)$ is self-dual, and $L_G(\lambda)$ is symplectic if λ is odd and orthogonal if λ is even.*

By Lemma 4.4.3, in characteristic $p \neq 2$ deciding whether an irreducible module is symplectic or orthogonal is a straightforward computation with roots and weights. Let $\lambda \in X(T)^+$ be a dominant weight and write $\lambda = \sum_{i=1}^{l} m_i\omega_i$, where $m_i \in \mathbb{Z}_{\geq 0}$. In Table 4.1, we give the value of $d(\lambda) \mod 2$ (when $\lambda = -w_0(\lambda)$) for each simple type, in terms of the coefficients m_i.

In characteristic 2, it turns out that each nontrivial, irreducible self-dual module is symplectic, as shown by the following lemma.

Lemma 4.4.5 (Fong, [Fon74])**.** *Assume that* $\mathrm{char}\, K = 2$. *Let V be a nontrivial, irreducible self-dual representation of a group G. Then V has non-degenerate G-invariant bilinear form b, unique up to a scalar. Furthermore, b is alternating.*

Lemma 4.4.5 shows that when $p = 2$, the image of any irreducible self-dual representation of G lies in $\mathrm{Sp}(V)$. Note that in the proof of Lemma 4.4.5 nothing about algebraic groups is needed, so this holds in a much more general setting. Another lemma which is well known and also works in greater generality is the following.

Lemma 4.4.6. *Let V be a G-module. Then $V \oplus V^*$ has a non-degenerate G-invariant alternating bilinear form, and a non-degenerate G-invariant symmetric bilinear form.*

Root system	When is $\lambda = -w_0(\lambda)$?	$d(\lambda) \mod 2$ when $\lambda = -w_0(\lambda)$
A_l $(l \geq 1)$	iff $m_i = m_{l-i+1}$ for all i	0, when l is even $\frac{l+1}{2} \cdot m_{\frac{l+1}{2}}$, when l is odd
B_l $(l \geq 2)$	always	0, when $l \equiv 0, 3 \mod 4$ m_l, when $l \equiv 1, 2 \mod 4$.
C_l $(l \geq 2)$	always	$m_1 + m_3 + m_5 + \cdots$
D_l $(l \geq 4)$	l even: always l odd: iff $m_l = m_{l-1}$	0, when $l \not\equiv 2 \mod 4$ $m_l + m_{l-1}$, when $l \equiv 2 \mod 4$.
G_2	always	0
F_4	always	0
E_6	iff $m_1 = m_6$ and $m_3 = m_5$	0
E_7	always	$m_2 + m_5 + m_7$
E_8	always	0

Table 4.1: Values of $d(\lambda)$ modulo 2 for a weight $\lambda = \sum_{i=1}^{l} m_i \omega_i$

Proof. Define a bilinear form on $V \oplus V^*$ by $\langle v + f, v' + f' \rangle = f'(v) + cf(v')$, where $c \in K$ is some fixed non-zero scalar. It is straightforward to verify that $\langle -, - \rangle$ is a non-degenerate G-invariant bilinear form. Furthermore, the form $\langle -, - \rangle$ is symmetric if we choose $c = 1$ and alternating if we choose $c = -1$. □

The following two lemmas are well known. For proofs, see for example [Bou59, §1, Définition 12, pg. 30], [McG05], and [McG02].

Lemma 4.4.7. *Suppose that a G-module V has a non-degenerate G-invariant alternating bilinear form $(-, -)$. Then*

(i) *For all $1 \leq k \leq \dim V$, the exterior power $\wedge^k(V)$ admits a non-degenerate G-invariant bilinear form, which is alternating if k is odd and symmetric if k is even.*

(ii) *If $p = 0$ or $p > k$, then the symmetric power $S^k(V)$ admits a non-degenerate G-invariant bilinear form, which is alternating if k is odd and symmetric if k is even.*

Lemma 4.4.8. *Suppose that a G-module V has a non-degenerate G-invariant symmetric form $(-, -)$. Then*

(i) *For all $1 \leq k \leq \dim V$, the exterior power $\wedge^k(V)$ admits a non-degenerate G-invariant symmetric form.*

(ii) *If $p = 0$ or $p > k$, then the symmetric power $S^k(V)$ admits a non-degenerate G-invariant symmetric form.*

The following lemma describes G-invariant bilinear forms on Weyl modules. It is most likely well known, see for example [Jan73, Satz 8] for a similar result.

Lemma 4.4.9. *Let $\lambda \in X(T)^+$ and suppose that $\lambda = -w_0\lambda$, where w_0 is the longest element in the Weyl group. Then:*

(i) *The Weyl module $V_G(\lambda)$ has a non-zero G-invariant bilinear form b, unique up to a scalar.*

(ii) *The radical* $\operatorname{rad} b$ *is the unique maximal submodule of $V_G(\lambda)$.*

(iii) *Assume that $p \neq 2$. Then b is symmetric if $d(\lambda) \equiv 0 \mod 2$ and alternating if $d(\lambda) \equiv 1 \mod 2$.*

(iv) *Assume that $p = 2$. Then b is alternating.*

Proof. We know that there exists a surjective G-morphism $\pi : V_G(\lambda) \to L_G(\lambda)$, which has the unique maximal submodule of $V_G(\lambda)$ as its kernel. It follows from [GN16, Lemma 4.3] that the map π induces an isomorphism $\operatorname{Bil}(L_G(\lambda))^G \to \operatorname{Bil}(V_G(\lambda))^G$. Under this isomorphism, a bilinear form $b \in \operatorname{Bil}(L_G(\lambda))^G$ maps to $b_\pi \in \operatorname{Bil}(V_G(\lambda))^G$, where $b_\pi(v, v') = b(\pi(v), \pi(v'))$ for all $v, v' \in V_G(\lambda)$.

Now Lemma 4.4.3 implies that $\operatorname{Bil}(L_G(\lambda))^G$ has dimension 1, so claim (i) follows. For claim (ii), let $b \in \operatorname{Bil}(L_G(\lambda))^G$ be non-zero, so that $b_\pi \in \operatorname{Bil}(V_G(\lambda))^G$ is non-zero. It is clear that the radical $\operatorname{rad} b_\pi$ contains the kernel of π, which is the unique maximal submodule of $V_G(\lambda)$. Since $\operatorname{rad} b_\pi$ is a G-submodule of $V_G(\lambda)$ and since b_π is non-zero, it follows that $\operatorname{rad} b_\pi = \ker \pi$. This proves claim (ii).

Claim (iii) follows from the fact that b_π is symmetric if and only if b is symmetric, and the fact that b_π is alternating if and only if b is alternating. Claim (iv) follows similarly, using Lemma 4.4.5. □

We consider next G-invariant bilinear forms on tilting modules.

Lemma 4.4.10. *Let $\lambda \in X(T)^+$. Then:*

(i) *$T_G(\lambda)^* \cong T_G(-w_0\lambda)$, where w_0 is the longest element in the Weyl group. In particular, $T_G(\lambda)$ is self-dual if and only if $\lambda = -w_0\lambda$.*

(ii) *Assume that $\lambda = -w_0\lambda$ and $p \neq 2$. Then $T_G(\lambda)$ admits a non-degenerate G-invariant symmetric or alternating bilinear form b. Furthermore, any such form b is symmetric if $d(\lambda) = 0 \mod 2$ and alternating if $d(\lambda) = 1 \mod 2$.*

Proof. Claim (i) is [Jan03, Remark II.E.6].

For claim (ii), assume that $\lambda = -w_0\lambda$ and $p \neq 2$. Now $T_G(\lambda) \cong T_G(\lambda)^*$ by (i). Furthermore, since $T_G(\lambda)$ is indecomposable, the ring $\operatorname{End}_G(T_G(\lambda))$ is a local ring. It follows then from [QSSS76, Lemma 2.1] or [Wil76, Satz 2.11 (a)] that there exists a non-degenerate G-invariant bilinear form b on $T_G(\lambda)$ such that b is symmetric or alternating.

For the other part of claim (ii), let $W \subseteq T_G(\lambda)$ be a G-submodule such that $W \cong V_G(\lambda)$, see [Jan03, II.E.4]. We show first that W cannot be totally isotropic with respect to b. For this, note that $W_\lambda = 1$. Thus if $W \subseteq W^\perp$, then by $V/W^\perp \cong W^*$ we get $\dim T_G(\lambda)_\lambda \geq 2$, contradicting $\dim T_G(\lambda)_\lambda = 1$ (Theorem 4.1.4 (i)). Therefore the restriction of b_W of b to W is non-zero. This implies that b is symmetric if and only if b_W is symmetric, and b is alternating if and only if b_W is alternating. Now the claim follows from Lemma 4.4.9 (iii). □

Corollary 4.4.11. *Assume that $p \neq 2$. Let $G = \mathrm{SL}_2(K)$ and $n \in \mathbb{Z}_{\geq 0}$. Then the indecomposable tilting module $T_G(n)$ has a non-degenerate G-invariant bilinear form, which is symmetric if n is even and alternating if n is odd.*

Lemma 4.4.12. *Assume that $p = 2$. Let $G = \mathrm{SL}_2(K)$ and $n \in \mathbb{Z}_{>0}$. Then the indecomposable tilting module $T_G(n)$ has a non-degenerate G-invariant alternating bilinear form.*

Proof. If $n = 1$, then $T_G(n) = L_G(1)$ (Theorem 4.1.5 (i)) and the claim follows from Lemma 4.4.5. For $n = 2$, we have $T_G(n) = L_G(0)/L_G(2)/L_G(0)$ uniserial of dimension 4 by Theorem 4.1.5. The tensor product $L_G(1) \otimes L_G(1)$ is a tilting module (Theorem 4.1.3) with highest weight 2, so it follows that $L_G(1) \otimes L_G(1) = T_G(2)$ (Theorem 4.1.4). Since $L_G(1)$ has a non-degenerate G-invariant alternating bilinear form, the same is true for the tensor product $L_G(1) \otimes L_G(1)$ (see for example [KL90, 4.4, pg. 126-127]).

For $n > 2$, we will prove the claim by induction on n. Suppose that $T_G(n')$ has a non-degenerate G-invariant alternating bilinear form for all $0 < n' < n$. If $n = 2k$, then $T_G(n) \cong T_G(2) \otimes T_G(k-1)^{[1]}$ by Theorem 4.1.5 (iii). Applying induction to $T_G(2)$ and $T_G(k-1)$, it follows that the tensor product $T_G(2) \otimes T_G(k-1)^{[1]}$ has a non-degenerate G-invariant alternating bilinear form (again, for example by [KL90, 4.4, pg. 126-127]). If $n = 2k+1$, the claim follows with the same argument, since $T_G(n) \cong T_G(1) \otimes T_G(k-1)^{[1]}$ by Theorem 4.1.5 (iii). □

Lemma 4.4.13. *Let $\lambda \in X(T)^+$ such that $\lambda = -w_0\lambda$. Let V be a uniserial G-module such that $V = L_G(0)/L_G(\lambda)/L_G(0)$, and suppose that V admits a non-degenerate G-invariant alternating bilinear form β. Then any non-degenerate G-invariant alternating bilinear form on V is a scalar multiple of β.*

Proof. Let γ be some non-degenerate G-invariant alternating bilinear form on V. It is straightforward to see (for example [Wil76, Satz 2.3]) that there exists a unique G-isomorphism $\varphi \in \mathrm{End}_G(V)$ such that $\gamma(v, v') = \beta(\varphi(v), v')$ for all $v, v' \in V$.

By [GN16, Example 4.5], we have $\dim \mathrm{End}_G(V) = 2$, and $\mathrm{End}_G(V)$ has a basis consisting of the identity map $1 : V \to V$ and a map $N : V \to V$ such that N is nilpotent and $N(V)$ is the unique 1-dimensional G-submodule of V. Then by replacing γ with a scalar multiple if necessary, we can assume that $\varphi = 1 + \lambda \cdot N$ for some $\lambda \in K$. Let $z \in N(V)$ be a non-zero vector, so $N(V) = \langle z \rangle$. Since V is indecomposable, it follows that $\langle z \rangle \subseteq \langle z \rangle^{\perp}$ with respect to the bilinear form β. Furthermore, we have $\mathrm{Ker}\, N = \langle z \rangle^{\perp}$ since V is uniserial. Thus we can find a $w \in V$ such that $w \notin \langle z \rangle^{\perp}$ and $N(w) = z$. Since β and γ are both alternating, we have $\gamma(w, w) = \beta(w, w) + \lambda \cdot \beta(Nw, w) = \lambda \cdot \beta(z, w) = 0$. Since $w \notin \langle z \rangle^{\perp}$, we have $\beta(z, w) \neq 0$ and so $\lambda = 0$. Therefore $\gamma = \beta$, and the lemma follows. □

The following lemmas are useful in some cases for classifying subgroups of classical groups up to conjugacy. Below for a bilinear form β on a vector space V, we denote by $\mathscr{I}(V, \beta)$ the group consisting of all $g \in \mathrm{GL}(V)$ such that $\beta(gv, gw) = \beta(v, w)$ for all $v, w \in V$ (equivalently, $\beta^g = \beta$, see Definition 4.4.1).

Lemma 4.4.14. *Let $\rho : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(V)$ be two representations of G. Set $\rho(G) = X$ and $\psi(G) = Y$. Suppose that there exists a non-degenerate bilinear form β on V such that $X, Y < \mathscr{I}(V, \beta)$. Assume that β is alternating or symmetric, and assume that $p \neq 2$ if β is symmetric. Suppose that up to scalar multiples, the form β is the unique non-degenerate X-invariant alternating or* *symmetric bilinear form on V. Then if ρ and ψ are equivalent, the subgroups X and Y are conjugate in $\mathscr{I}(V,\beta)$.*

Proof. Suppose that ρ and ψ are equivalent. Then there exists a $f \in \mathrm{GL}(V)$ such that $fYf^{-1} = X$. Since the form β is Y-invariant, the form β^f is invariant under $fYf^{-1} = X$. By uniqueness of β, it follows that $\beta^f = \lambda\beta$ for some non-zero scalar $\lambda \in K$. Now $\beta^g = \beta$ for $g = \sqrt{\lambda}f$. Hence $g \in \mathscr{I}(V,\beta)$, and it is clear that $gYg^{-1} = X$. □

Lemma 4.4.15. *Assume that $p \neq 2$. Let $\rho : G \to \mathrm{GL}(V)$ and $\psi : G \to \mathrm{GL}(V)$ be two equivalent representations of G such that the associated G-module V is indecomposable. Let $X = \rho(G)$ and $Y = \psi(G)$. Suppose that $X, Y < \mathscr{I}(V,\beta)$, where β is a non-degenerate alternating or symmetric bilinear form. Then X and Y are conjugate in $\mathscr{I}(V,\beta)$.*

Proof. Since the G-modules associated with ρ and ψ are isomorphic, this follows from the general result [Wil76, Satz 3.12] (alternatively, [QSSS76, Korollar 3.5]). □

4.5 Weight multiplicities in $V_G(\lambda)$ and $L_G(\lambda)$

The purpose of this subsection is to list some well known results about weight multiplicities in Weyl modules and irreducible modules of G.

Let $(-,-)$ be the usual inner product defined on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$, normalized so that long roots in Φ have norm 1. For $\lambda, \mu \in X(T)^+$, we will define

$$d(\lambda,\mu) = (\lambda+\rho, \lambda+\rho) - (\mu+\rho, \mu+\rho)$$

where ρ is the half-sum of positive roots. The following useful result is a consequence of the strong linkage principle.

Proposition 4.5.1 (Linkage principle)**.** *Assume that $p > 2$, and that $p > 3$ if G is of type G_2. Let $\lambda, \mu \in X(T)^+$. If $L_G(\mu)$ occurs as a composition factor of $V_G(\lambda)$, then p divides the integer $2 \cdot d(\lambda,\mu)$.*

Proof. See [Sei87, 6.2]. □

In Table 4.1 and Table 4.2, we have given the value of $d(\lambda,\mu)$ for some particular $\lambda \succ \mu$. In the tables we have also included the weight multiplicity $m_{V_G(\lambda)}(\mu)$ of μ in $V_G(\lambda)$, which can be computed with Freudenthal's formula [Hum72, Theorem 22.3]. This information will be used later in Section 5.13. The following two lemmas will also be useful. In the first lemma, recall that we are using the Bourbaki labeling of the Dynkin diagrams, so in root systems of rank 2 the simple short root is α_1.

Lemma 4.5.2. *Suppose that G has rank 2. Let $\lambda = c_1\omega_1 + c_2\omega_2 \in X(T)^+$, where $c_1c_2 \neq 0$. For $\mu = \lambda - \alpha_1 - \alpha_2$, we have $0 < m_{L_G(\lambda)}(\mu) \leq 2$. Furthermore, $m_{L_G(\lambda)}(\mu) = 1$ if and only if one of the following holds:*

(i) *G has type A_2 and $p \mid 1 + c_1 + c_2$,*

(ii) *G has type C_2 and $p \mid 2 + c_1 + 2c_2$,*

(iii) *G has type G_2 and $p \mid 3 + c_1 + 3c_2$.*

Proof. This is [Tes88, 1.35]. □

Lemma 4.5.3. *Suppose that G is of type A_l and $\lambda = c_1\omega_1 + c_l\omega_l$. Let $\mu = \lambda - \alpha_1 - \alpha_2 - \cdots - \alpha_l$. Then $m_{V_G(\lambda)}(\mu) = l$, and*

$$m_{L_G(\lambda)}(\mu) = \begin{cases} l-1 & \text{if } p \mid c_1 + c_l + l - 1, \\ l & \text{if } p \nmid c_1 + c_l + l - 1. \end{cases}$$

Proof. This a special case of [Sei87, 8.6]. □

The following basic lemma will be useful occasionally. We omit the proof which is easy.

Lemma 4.5.4. *Let $\lambda, \mu \in X(T)^+$. Suppose that for all $\lambda \succ \mu' \succeq \mu$, the Weyl module $V_G(\lambda)$ does not have $L_G(\mu')$ as a composition factor. Then $m_{L_G(\lambda)}(\mu) = m_{V_G(\lambda)}(\mu)$.*

We know that if $\lambda \in X(T)^+$, then the set of weights in $V_G(\lambda)$ is *saturated*. That is, if μ is a weight in $V_G(\lambda)$, then $\mu - i\alpha$ is also a weight for all $\alpha \in \Phi$ and i between 0 and $\langle \mu, \alpha \rangle$. With some mild assumptions on the characteristic, the same is true for $L_G(\lambda)$ as well if λ is p-restricted.

Theorem 4.5.5 (Premet [Pre87])**.** *Suppose that G is simple and (G, p) is not one of $(B_n, 2)$, $(C_n, 2)$, $(F_4, 2)$ or $(G_2, 3)$. Then for any p-restricted $\lambda \in X(T)^+$, the set of weights occurring in $V_G(\lambda)$ and the set of weights occurring in $L_G(\lambda)$ are equal. In particular, the set of weights in $L_G(\lambda)$ is saturated.*

For the rest of this section, we will describe how some weight multiplicities $m_{L_G(\lambda)}(\mu)$ can be found by computing within an irreducible representation of some Levi factor of G. To explain this well known technique, we have to establish some notation.

For $\alpha \in \Phi$, let U_α be the root subgroup associated with α. Fix some base Δ of the root system Φ of G. Now for any subset $J \subset \Delta$, denote $\Phi_J = \mathbb{Z}J \cap \Phi$. Then the *standard parabolic subgroup* P_J associated with J is defined as $P_J = L_J Q_J$, where

$$Q_J = \langle U_\alpha : \alpha \in \Phi^+ \setminus \Phi_J \rangle$$

and

$$L_J = \langle T, U_\alpha : \alpha \in \Phi_J \rangle.$$

Here Q_J is the unipotent radical of P_J, and P_J is a semidirect product $L_J \ltimes Q_J$ of algebraic groups (Levi decomposition).

Let $L_J' = [L_J, L_J]$ be the commutator subgroup of L_J. Then L_J' is a connected semisimple algebraic group with maximal torus $T_J = T \cap L_J'$. For a weight $\lambda \in X(T)^+$, we will denote the restriction of λ to T_J by λ'. Then $\Phi_J' = \{\alpha' : \alpha \in \Phi_J\}$ is a root system of L_J' with base $\Delta' = \{\alpha' : \alpha \in J\}$. With this base Δ', the fundamental dominant weights are ω_i' for $\alpha_i \in J$.

Now the relation between weight multiplicities in $L_G(\lambda)$ and in $L_{L_J'}(\lambda')$ is given by the corollary of the next proposition. Below we write $L_G(\lambda)^{Q_J}$ for the fixed point subspace of Q_J on $L_G(\lambda)$.

Proposition 4.5.6 (Smith [Smi82]). *Let $J \subseteq \Delta$ and $\lambda = \sum_{i=1}^{l} c_i\omega_i \in X(T)^+$. Let P_J be the parabolic subgroup determined by J with Levi decomposition $P_J = L_J \ltimes Q_J$. Then*

$$L_G(\lambda)^{Q_J} = \langle V_\mu : \mu = \lambda - \sum_{\alpha_i \in J} k_i\alpha_i, \text{ where } k_i \in \mathbb{Z}_{\geq 0}\rangle$$

and

$$L_G(\lambda)^{Q_J} \downarrow L_J' \cong L_{L_J'}(\lambda')$$

where $\lambda' = \sum_{\alpha_i \in J} c_i\omega_i'$.

Corollary 4.5.7. *Let $J \subseteq \Delta$ and $\lambda = \sum_{\alpha_i \in J} c_i\omega_i \in X(T)^+$. Let P_J be the parabolic subgroup determined by J with Levi decomposition $P_J = L_J \ltimes Q_J$. Then $m_{L_G(\lambda)}(\mu) = m_{L_{L_J'}(\lambda')}(\mu')$ for all $\mu = \lambda - \sum_{\alpha_i \in J} k_i\alpha_i$, where $k_i \in \mathbb{Z}_{\geq 0}$.*

b	$\mu \in X(T)^+$	$d(\lambda, \mu)$	$m_{V(\lambda)}(\mu)$
$b \geq 1$	$\lambda - \alpha_1 - \alpha_2$	$b+1$	1
$b \geq 3$	$\lambda - \alpha_1 - 2\alpha_2$	$2b$	1
$b \geq 2$	$\lambda - 2\alpha_1 - 2\alpha_2$	$2(3b+2)$	2

Table 4.1: Type G_2: some weight multiplicities $m_{V(\lambda)}(\mu)$ for $\lambda = b\omega_2$.

b	$\mu \in X(T)^+$	$d(\lambda, \mu)$	$m_{V(\lambda)}(\mu)$
$b \geq 1$	$\lambda - \alpha_1 - \alpha_2$	$b+1$	1
$b \geq 3$	$\lambda - \alpha_1 - 2\alpha_2$	$2b$	1
$b \geq 5$	$\lambda - \alpha_1 - 3\alpha_2$	$3(b-1)$	1
$b \geq 2$	$\lambda - 2\alpha_1 - 2\alpha_2$	$2(b+1)$	2
$b \geq 4$	$\lambda - 2\alpha_1 - 3\alpha_2$	$2(b-4)$	2
$b \geq 7$	$\lambda - \alpha_1 - 4\alpha_2$	$6(b-3)$	2

Table 4.2: Type C_2: some weight multiplicities $m_{V(\lambda)}(\mu)$ for $\lambda = b\omega_2$.

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4.6 Representations of $\mathrm{SL}_2(K)$

In this section, let G be the algebraic group $\mathrm{SL}_2(K)$ with natural module E. Fix also a nonidentity unipotent element $u \in G$. Throughout we will identify the weights of a maximal torus of G with $\mathbb{Z}$, and the dominant weights with $\mathbb{Z}_{\geq 0}$.

The main purpose of this section is to classify indecomposable G-modules V where u acts on V with at most one Jordan block of size p. One consequence of this is a criterion for a representation of G to be semisimple. Specifically, we prove that a self-dual G-module V must be semisimple if u acts on V with at most one Jordan block of size p (Proposition 4.6.10). In Section 5.13, this result will be the starting point for classifying irreducible representations of simple algebraic groups in which a unipotent element of order p acts as a distinguished unipotent element.

We begin by considering Jordan block sizes in Weyl modules and tilting modules of G.

Lemma 4.6.1. *Let $m = qp + r$, where $q \geq 0$ and $0 \leq r < p$. Then u acts on the Weyl module $V_G(m)$ with Jordan blocks $[p, p, \ldots, p, r+1]$, where p occurs q times.*

Proof. The Weyl module $V_G(m)$ is isomorphic to $S^m(E)^*$, see for example [Jan03, II.2.16]. Therefore it suffices to compute the Jordan blocks of u acting on the symmetric power $S^m(E)$.

Now there exists a basis x, y of E such that $ux = x$ and $uy = x + y$. Let $x^m, x^{m-1}y, \ldots, y^m$ be the basis induced on $S^m(E)$. Then

$$u(x^{m-k}y^k) = x^{m-k}(x+y)^k = \sum_{i=0}^{k} \binom{k}{i} x^{m-i}y^i,$$

so with respect to this basis, the matrix of u acting on $S^m(E)$ is the upper triangular Pascal matrix $P = (\binom{i-1}{j-1})_{0 \leq i,j \leq m}$. Here we define $\binom{i}{j} = 0$ if $i < j$. The Jordan form of the transpose of P is computed for example in [Cal02], and from this result we find that P has q Jordan blocks of size p and one Jordan block of size $r+1$, as claimed. □

Lemma 4.6.2. *Let $c \geq p-1$. Then $T_G(c) \downarrow K[u] = N \cdot V_p$, where $N = \frac{\dim T_G(c)}{p}$. In particular, $T_G(c) \downarrow K[u] = V_p \oplus V_p$ if $p \leq c \leq 2p-2$.*

Proof. If $c = p-1$, then the claim follows from Theorem 4.1.5 (i) and Lemma 4.2.2. If $p \leq c \leq 2p-2$, the claim follows from [Sei00, Lemma 2.3], see [McN02, Proposition 5].

Consider then the case where $c > 2p-2$. By Theorem 4.1.5 (iii), we have

$$T_G(c) \cong T_G(p-1+r) \otimes T_G(s)^{[1]},$$

where $s \geq 1$ and $0 \leq r \leq p-1$ are such that $c = sp + (p-1+r)$. Since $p-1 \leq p-1+r \leq 2p-2$, by the previous paragraph $T_G(s)^{[1]} \downarrow K[u] = N' \cdot V_p$ for $N' = \frac{\dim T_G(s)}{p}$. Therefore $T_G(c) \downarrow K[u] \cong N \cdot V_p$ for $N = \frac{\dim T_G(c)}{p}$ by Lemma 3.3.6. □

For studying representations of G which are not semisimple the starting point is the following result, originally due to Cline [AJL83, Corollary 3.9].

Theorem 4.6.3 (Cline). *Let $\lambda = \sum_{i \geq 0} \lambda_i p^i$ and $\mu = \sum_{i \geq 0} \mu_i p^i$ be weights of G, where $0 \leq \lambda_i, \mu_i < p$. Then $\mathrm{Ext}^1_G(L_G(\lambda), L_G(\mu)) \neq 0$ precisely when there exists a $k \geq \nu_p(\lambda+1)$ such that $\mu_i = \lambda_i$ for all $i \neq k, k+1$ and $\mu_k = p - 2 - \lambda_k$, $\mu_{k+1} = \lambda_{k+1} \pm 1$. In the case where $\mathrm{Ext}^1_G(L_G(\lambda), L_G(\mu)) \neq 0$, we have $\mathrm{Ext}^1_G(L_G(\lambda), L_G(\mu)) \cong K$.*

Note that if there exists a nonsplit extension of $L_G(\lambda)$ by $L_G(\mu)$, it is unique up to isomorphism because $\mathrm{Ext}^1_G(L_G(\lambda), L_G(\mu)) \cong K$ by Theorem 4.6.3.

We will use the following result of Serre to construct some indecomposable representations of G. The result is not specific to algebraic groups, and actually holds for any group H and finite-dimensional representations V and W of H over K.

Theorem 4.6.4 ([Ser97]). *Let V and W be G-modules. If $V \otimes W$ is semisimple, then V is semisimple if $p \nmid \dim W$.*

Proposition 4.6.5. *Let V be a G-module which is a nonsplit extension of $L_G(\lambda)$ by $L_G(\mu)$. Then u acts on V with at least one Jordan block of size p. If u acts on V with precisely one Jordan block of size p, then there exists $p \leq c \leq 2p-2$ and $l \geq 0$ such that one of the following holds:*

(i) $\lambda = cp^l$, $\mu = (2p-2-c)p^l$ *and* $V \cong V_G(c)^{[l]}$.

(ii) $\lambda = (2p-2-c)p^l$, $\mu = cp^l$ *and* $V \cong (V_G(c)^*)^{[l]}$.

Moreover, if V, λ, μ, and c are as in case (i) or case (ii), then V is a nonsplit extension of $L_G(\lambda)$ by $L_G(\mu)$, and u acts on V with Jordan blocks $[p, c-p+1]$.

Proof. We begin by considering the claims about $V_G(c)^{[l]}$ and $(V_G(c)^*)^{[l]}$, where $p \leq c \leq 2p-2$ and $l \geq 0$. It is well known (and easily seen by considering the weights in $V_G(c)$) that $V_G(c)$ is a nonsplit extension

$$0 \to L_G(2p-2-c) \to V_G(c) \to L_G(c) \to 0.$$

Since the irreducible representations of G are self-dual, we see that $V_G(c)^*$ is a nonsplit extension

$$0 \to L_G(c) \to V_G(c)^* \to L_G(2p-2-c) \to 0.$$

Let $\lambda = cp^l$ and $\mu = (2p-2-c)p^l$, where $l \geq 0$. By taking a Frobenius twist, we see that for all $l \geq 0$ the module $V_G(c)^{[l]}$ is a non-split extension of $L_G(\lambda)$ by $L_G(\mu)$, and its dual $(V_G(c)^*)^{[l]}$ is a non-split extension of $L_G(\mu)$ by $L_G(\lambda)$. Furthermore, by Lemma 4.6.1 the unipotent element $u \in G$ acts on $V_G(c)$ with Jordan blocks $[p, c-p+1]$. The Jordan block sizes are not changed by taking a dual or a Frobenius twist, so for all $l \geq 0$ the element u acts on both $V_G(c)^{[l]}$ and $(V_G(c)^*)^{[l]}$ with Jordan blocks $[p, c-p+1]$. This completes the proof that the properties of $V_G(c)^{[l]}$ and $(V_G(c)^*)^{[l]}$ are as claimed.

Now let $\lambda, \mu \in \mathbb{Z}_{\geq 0}$ be arbitrary weights, and let V be a nonsplit extension

$$0 \to L_G(\mu) \to V \to L_G(\lambda) \to 0.$$

Assume that u acts on V with at most one Jordan block of size p. We note first that to prove the proposition, it will be enough to show that there exist $p \leq c \leq 2p-2$ and $l \geq 0$ such that λ and μ are as in case (i) or (ii) of the claim. Indeed, if this holds, then since $\operatorname{Ext}^1_G(L_G(\lambda), L_G(\mu)) \cong K$ (Theorem 4.6.3), we must have $V \cong V_G(c)^{[l]}$ or $V \cong (V_G(c)^*)^{[l]}$. Furthermore, as seen in the first paragraph, then u acts on V with Jordan blocks $[p, c-p+1]$, in particular with exactly one Jordan block of size p.

We proceed to show that λ and μ are as claimed. Write $\lambda = \sum_{i \geq 0} \lambda_i p^i$, where $0 \leq \lambda_i \leq p-1$ for all i. Recall that by the Steinberg tensor product theorem, we have

$$L_G(\lambda) \cong \bigotimes_{i \geq 0} L_G(\lambda_i)^{[i]}$$

Consider first the case where $\lambda_l = p-1$ for some l. Then u acts on $L_G(\lambda_l)^{[l]}$ with a single Jordan block of size p. Now the tensor product of a Jordan block of size p with any Jordan block of size $c \leq p$ consists of c Jordan blocks of size p (Lemma 3.3.6), so it follows that $L_G(\lambda) \downarrow K[u] = N \cdot V_p$, where $N = \frac{\dim L_G(\lambda)}{p}$.

Since u acts on V with at most one Jordan block of size p, by Lemma 3.1.4 the action of u on $L_G(\lambda)$ can have at most one Jordan block of size p. It follows then that $N = 1$, so $\lambda = (p-1)p^l$. Since V is a nonsplit extension, by Theorem 4.6.3 the weight μ must be $(p-2)p^{l-1} + (p-2)p^l$ or $(p-1)p^l + (p-2)p^k + p^{k+1}$ for some $k \neq l, l-1$. By Lemma 3.1.4 the action of u on $L_G(\mu)$ has no Jordan blocks of size p. With Lemma 3.3.6, one finds that this happens only if $p = 2$ and $\mu = (p-2)p^{l-1} + (p-2)p^l = 0$. Then λ and μ are as in case (i) of the claim.

Thus we can assume that $0 \leq \lambda_i \leq p-2$ for all i. Write $\mu = \sum_{i \geq 0} \mu_i p^i$, where $0 \leq \mu_i \leq p-1$ for all i. According to Theorem 4.6.3, there exists an l such that we have $\mu_i = \lambda_i$ for $i \neq l, l+1$, and $\mu_l = p - 2 - \lambda_l$, $\mu_{l+1} = \lambda_{l+1} \pm 1$. We can write $\lambda = \zeta + \lambda_l p^l + \lambda_{l+1} p^{l+1}$ and $\mu = \zeta + \mu_l p^l + \mu_{l+1} p^{l+1}$. By the Steinberg tensor product theorem, we have

$$L_G(\lambda) \cong L_G(\zeta) \otimes L_G(\lambda_l + \lambda_{l+1} p)^{[l]}$$
$$L_G(\mu) \cong L_G(\zeta) \otimes L_G(\mu_l + \mu_{l+1} p)^{[l]}$$

Note that here p does not divide the dimension of $L_G(\zeta)$, because we are assuming that $\lambda_i < p - 1$ for all i. Furthermore, by Theorem 4.6.3 there exists a G-module W which is a nonsplit extension

$$0 \to L_G(\lambda_l + \lambda_{l+1} p) \to W \to L_G(\mu_l + \mu_{l+1} p) \to 0.$$

Therefore by Theorem 4.6.4, the module $L_G(\zeta) \otimes W^{[l]}$ is a nonsplit extension of $L_G(\lambda)$ by $L_G(\mu)$. Hence $L_G(\zeta) \otimes W^{[l]} \cong V$, because a nonsplit extension of $L_G(\lambda)$ by $L_G(\mu)$ is unique by Theorem 4.6.3.

Consider first the case where $\zeta = 0$. Here $V \cong W^{[l]}$, so without loss of generality we may assume that $l = 0$. Write $\lambda = c + dp$ and $\mu = (p-2-c) + (d \pm 1)p$, where $0 \leq c, d \leq p-2$ and $d > 0$ if $\mu = (p-2-c) + (d-1)p$.

We consider $\lambda = c + dp$ and $\mu = (p-2-c) + (d-1)p$. Here $\dim V = \dim L_G(\lambda) + \dim L_G(\mu) = \lambda + 1 = \dim V_G(\lambda)$. Now V is a nonsplit extension of $L_G(\lambda)$ by $L_G(\mu)$ and $\lambda \succ \mu$, so by Proposition 4.1.1 we must have $V \cong V_G(\lambda)$. By Lemma 4.6.1, the action of u on V has Jordan blocks $[p, p, \ldots, p, c+1]$, where p occurs d times. Therefore u acts on V with at most one Jordan block of size p if and only if $d = 1$, that is, when $\lambda = c + p$ and $\mu = (p-2-c)$. Then λ and μ are as in case (i) of the claim.

Next consider the case where $\lambda = c + dp$ and $\mu = (p-2-c) + (d+1)p$. In this case $\dim V = \dim L_G(\lambda) + \dim L_G(\mu) = \mu + 1 = \dim V_G(\mu)$. Because V^* is a nonsplit extension of $L_G(\mu)$ by $L_G(\lambda)$ and $\mu \succ \lambda$, by Proposition 4.1.1 we have $V^* \cong V_G(\mu)$. By Lemma 4.6.1, the action of u on V has Jordan blocks $[p, p, \ldots, p, p-(c+1)]$, where p occurs $d+1$ times. In this case u acts with at most one Jordan block of size p if and only if $d = 0$, that is, when $\lambda = c$ and $\mu = (p-2-c) + p$. Then λ and μ are as in case (ii) of the claim.

Finally we consider the possibility that $\zeta \neq 0$. Since W is a nonsplit extension of two irreducible modules, it follows from the $\zeta = 0$ case that u acts on W with at least one Jordan block of size p. Now the tensor product of a Jordan block of size p with any Jordan block of size $c \leq p$ consists of c Jordan blocks of size p (Lemma 3.3.6). Therefore if $\zeta \neq 0$, then u acts on V with more than one Jordan block of size p, contradiction. □

Lemma 4.6.6. *Let $p \leq c \leq 2p-2$. Then for all $l \geq 0$:*

(i) $\operatorname{Ext}^1_G(V_G(c)^{[l]}, L_G(c)^{[l]}) = 0.$

(ii) $\operatorname{Ext}^1_G(V_G(c)^{[l]}, L_G(2p-2-c)^{[l]}) = 0.$

(iii) $\operatorname{Ext}^1_G(L_G(c)^{[l]}, V_G(c)^{[l]}) = 0.$

(iv) $\operatorname{Ext}^1_G(L_G(2p-2-c)^{[l]}, V_G(c)^{[l]}) \cong K.$

Furthermore, every nonsplit extension of $L_G(2p-2-c)^{[l]}$ *by* $V_G(c)^{[l]}$ *is isomorphic to* $T_G(c)^{[l]}$.

Proof. Set $c' = 2p-2-c$. We note first that the last claim of the lemma follows from (iv), once we show that $T_G(c)^{[l]}$ is a nonsplit extension of $L_G(c')^{[l]}$ by $V_G(c)^{[l]}$. To this end, by [Sei00, Lemma 2.3 (b)] the tilting module $T_G(c)$ is a nonsplit extension of $L_G(c')$ by $V_G(c)$. The extension stays nonsplit after taking a Frobenius twist, so $T_G(c)^{[l]}$ is a nonsplit extension of $L_G(c')^{[l]}$ by $V_G(c)^{[l]}$.

For claims (i) - (iv), we prove them first in the case where $l = 0$. In this case, claims (i) and (ii) follow from the fact $\operatorname{Ext}^1_G(V_G(\lambda), L_G(\mu)) = 0$ for any $\mu \leq \lambda$ [Jan03, II.2.14]. For (iii) and (iv), recall that there is an exact sequence

$$0 \to L_G(c') \to V_G(c) \to L_G(c) \to 0.$$

Applying the functor $\operatorname{Hom}_G(L_G(d), -)$ gives a long exact sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}_G(L_G(d), L_G(c')) \to \operatorname{Hom}_G(L_G(d), V_G(c)) \to \operatorname{Hom}_G(L_G(d), L_G(c)) \\ &\to \operatorname{Ext}^1_G(L_G(d), L_G(c')) \to \operatorname{Ext}^1_G(L_G(d), V_G(c)) \to \operatorname{Ext}^1_G(L_G(d), L_G(c)) \to \cdots \end{aligned}$$

Considering this long exact sequence with $d = c$, we get an exact sequence

$$0 \to K \to K \to \operatorname{Ext}^1_G(L_G(c), V_G(c)) \to 0$$

since $\operatorname{Ext}^1_G(L_G(c), L_G(c')) \cong K$ and $\operatorname{Ext}^1_G(L_G(c), L_G(c)) = 0$ by Theorem 4.6.3 and [Jan03, II.2.12], respectively. This proves (iii).

With $d = 2p-2-c$, we get an exact sequence

$$0 \to \operatorname{Ext}^1_G(L_G(c'), V_G(c)) \to K.$$

Therefore $\dim \operatorname{Ext}^1_G(L_G(c'), V_G(c)) \leq 1$. Thus to prove (iv), it will be enough to show that there exists some nonsplit extension of $L_G(c')$ by $V_G(c)$. For this, we have already noted in the beginning of the proof that $T_G(c)$ is such an extension.

We consider then claims (i) - (iv) for $l > 0$. If $p \neq 2$, then the claims follow from the case $l = 0$, since $\operatorname{Ext}^1_G(X^{[l]}, Y^{[l]}) \cong \operatorname{Ext}^1_G(X, Y)$ for all G-modules X and Y [Jan03, II.10.17]. Suppose then that $p = 2$. Note that then we must have $c = 2$, and $c' = 0$. By [Jan03, Proposition II.10.16, Remark 12.2], for any G-modules X and Y there exists a short exact sequence

$$\begin{aligned} 0 \to \operatorname{Ext}^1_G(X, Y) &\to \operatorname{Ext}^1_G(X^{[l]}, Y^{[l]}) \\ &\to \operatorname{Hom}_G(X, L_G(1) \otimes Y) \to 0. \end{aligned}$$

Thus claims (i) - (iv) will follow from the case $l = 0$ once we show that $\operatorname{Hom}_G(X, L_G(1) \otimes Y) = 0$ in the following cases:

(i)′ $X = V_G(2)$ and $Y = L_G(2)$,

(ii)′ $X = V_G(2)$ and $Y = L_G(0)$,

(iii)′ $X = L_G(2)$ and $Y = V_G(2)$,

(iv)′ $X = L_G(0)$ and $Y = V_G(2)$.

In all cases (i)′ - (iv)′, it is straightforward to see that X and $L_G(1) \otimes Y$ have no composition factors in common. Thus $\mathrm{Hom}_G(X, L_G(1) \otimes Y) = 0$, as claimed. □

Lemma 4.6.7. *Let W be a nonsplit extension of two irreducible G-modules, and let Z be an irreducible G-module. If V is a nonsplit extension of Z by W, then V is indecomposable.*[10]

Proof. We can assume that $W \subseteq V$ and $V/W \cong Z$. Let $W' \subseteq W$ be the unique irreducible submodule of W. If V is not indecomposable, then there exists an irreducible submodule $Z' \subseteq V$ such that $Z' \neq W'$. Then $Z' \cap W = 0$, so $V = W \oplus Z'$. Therefore $V \cong W \oplus Z$, which is impossible when V is a nonsplit extension of Z by W. □

Proposition 4.6.8. *Let V be an indecomposable G-module. Then one of the following holds:*

(i) *u acts on V with ≥ 2 Jordan blocks of size p.*

(ii) *V is irreducible.*

(iii) *V is isomorphic to a Frobenius twist of $V_G(c)$ or $V_G(c)^*$, where $p \leq c \leq 2p-2$. Furthermore, u acts on V with Jordan blocks $[p, c-p+1]$.*

Proof. Let V be a counterexample of minimal dimension to the claim. Then V is not irreducible and u acts on V with ≤ 1 Jordan block of size p. Note also that by Lemma 3.1.4, the element u acts on any subquotient of V with ≤ 1 Jordan block of size p. Therefore any proper subquotient of V must be as in (ii) or (iii) of the claim.

Since V is not irreducible, there exists a subquotient Q of V which is a nonsplit extension of two irreducible G-modules. By Proposition 4.6.5, the subquotient Q is isomorphic to $V_G(c)^{[l]}$ or $(V_G(c)^*)^{[l]}$ for some $l \geq 0$ and $p \leq c \leq 2p-2$.

We are assuming that V is a counterexample, so there must be a subquotient Q' of V which is a nonsplit extension of Q and some irreducible module Z. By Lemma 4.6.7, the subquotient Q' is indecomposable, and so by the minimality of V we actually have $Q' = V$. By replacing V with V^* if necessary, this reduces us to the situation where V is a nonsplit extension of $V_G(c)^{[l]}$ and some irreducible module $L_G(d)$.

There must be a subquotient of V which is a nonsplit extension of $L_G(d)$ with $L_G(c)^{[l]}$ or $L_G(2p-2-c)^{[l]}$. Therefore by Proposition 4.6.5, it follows that $L_G(d) = L_G(c)^{[l]}$ or $L_G(d) = L_G(2p-2-c)^{[l]}$, respectively. Thus $V \cong T_G(c)^{[l]}$ by Lemma 4.6.6. This gives us a contradiction, because u acts on $T_G(c)$ with two Jordan blocks of size p by Lemma 4.6.2. □

Corollary 4.6.9. *Let V be any representation of G. Suppose that u acts on V with all Jordan blocks of size $< p$. Then V is semisimple.*

[10] Of course, the statement of the lemma and the proof given here work in a much more general setting.

Proof. By Proposition 4.6.8, the only indecomposable G-modules on which u acts with all Jordan blocks of size $< p$ are the irreducible ones. □

Proposition 4.6.10. *Let V be a self-dual representation of G. Suppose that u acts on V with at most* 1 *Jordan block of size p. Then V is semisimple.*

Proof. Write $V = W_1 \oplus \cdots \oplus W_t$, where W_i are indecomposable G-modules. Suppose that $W_i \not\cong W_i^*$ for some i. Now irreducible G-modules are self-dual, so W_i is not irreducible and thus u acts on W_i with exactly 1 Jordan block of size p (propositions 4.6.5 and 4.6.8). On the other hand, $V \cong V^* \cong W_1^* \oplus \cdots \oplus W_t^*$, so we have $W_j \cong W_i^*$ for some $i \neq j$ since the indecomposable summands are unique by the Krull-Schmidt theorem. But then $W_i \oplus W_i^*$ is a summand of V, which is a contradiction since u acts on $W_i \oplus W_i^*$ with two Jordan blocks of size p.

Thus $W_i \cong W_i^*$ for all i, and so by Proposition 4.6.8 each W_i must be irreducible, since $V_G(c) \not\cong V_G(c)^*$ for $p \leq c \leq 2p - 2$. □

Chapter 5

Irreducible overgroups of distinguished unipotent elements

The purpose of this chapter is to prove Theorem 1.1.10 and Theorem 1.1.11. That is, we classify all pairs (G, u), where $u \in G$ is a unipotent element and G is a simple subgroup of $\mathscr{I}(V) = \mathrm{SL}(V)$, $\mathscr{I}(V) = \mathrm{Sp}(V)$, or $\mathscr{I}(V) = \mathrm{SO}(V)$ such that

- V is irreducible and tensor-indecomposable as a G-module,
- u is a distinguished unipotent element of $\mathscr{I}(V)$.

When u has order $> p$, the basic idea of the proof is to first rule out most of the irreducible representations of G with an elementary argument described in Section 5.1-5.2, and then deal with the remaining representations case-by-case (Section 5.3-5.12). In the case of unipotent elements u order p (Section 5.13), we first reduce to the case where p is good for G. Then u is contained in a subgroup $A < G$ of type A_1 such that the labeled diagram of A is that of u (Theorem 2.6.8). With Proposition 4.6.10, one can then use the representation theory of $\mathrm{SL}_2(K)$ and methods from [LST15] to complete the proof.

5.1 Reduction, unipotent elements of order $> p$ $(p \neq 2)$

Assume that $p \neq 2$.

In this section, we reduce the proof of Theorem 1.1.10 in the case where u has order $> p$ to a small number of λ to consider. The reduction is based on the following lemma, Premet's theorem (Theorem 4.5.5), and results of Lübeck given in [Lüb01] and [Lüb17].

Lemma 5.1.1. *Let $\varphi : G \to \mathrm{GL}(V)$ a representation of G. For a unipotent element $u \in G$, let d_u be the size of the largest Jordan block of $\varphi(u)$. Then*

(i) *If $u \in G$ is unipotent with order p^α, then $d_u \leq p^\alpha$.*

(ii) *If $u' \in G$ is a regular unipotent element, then $d_u \leq d_{u'}$ for all unipotent elements $u \in G$.*

(iii) *If $V = L(\lambda)$ is irreducible and $u \in G$ is a unipotent element, then $d_u \leq m_u(\lambda) + 1$ (Definition 2.7.2).*

(iv) *If a unipotent element $u \in G$ acts on V as a distinguished unipotent element, then* $\dim V \leq \frac{(d_u+1)^2}{4}$ *if d_u is odd and* $\dim V \leq \frac{d_u(d_u+2)}{4}$ *if d_u is even. In any case if u acts on V as a distinguished unipotent element, then* $\dim V \leq \frac{(d_u+1)^2}{4}$.

Proof. Claims (i) and (ii) are given in Lemma 2.7.1, and claim (iii) is Theorem 2.7.8.

For (iv), we know that if u acts on V as a distinguished unipotent element, then Jordan block sizes of $\varphi(u)$ are distinct and all odd, or distinct and all even. Therefore for $d = d_u$, we have

$$\dim V \leq d + (d-2) + \cdots = \begin{cases} \frac{d(d+2)}{4} & \text{if } d \text{ is even,} \\ \frac{(d+1)^2}{4} & \text{if } d \text{ is odd.} \end{cases}$$

so the claim follows since $\frac{d(d+2)}{4} < \frac{(d+1)^2}{4}$. □

Remark 5.1.2. Note that Lemma 5.1.1 (iv) holds under the slightly weaker assumption that all Jordan block sizes of u acting on V are distinct and either all even or all odd.

Remark 5.1.3. This section is concerned with elements of order $> p$, but note that Lemma 5.1.1 also makes sense when u has order p. We will occasionally apply it in this situation as well. However, when u has order $> p$ and G is of classical type, we can get an upper bound on the order of u which only depends on rank G and not on p. This will be a key fact that we will use in subsections 5.1.1, 5.1.2 and 5.1.3.

We give the reduction for each simple type in the subsections that follow.

5.1.1 Type A_l

In this subsection, assume that G is simple of type A_l. The only distinguished unipotent class in G is the regular unipotent class, which acts with a single Jordan block of size $l+1$ on the natural module of G (Proposition 2.3.4). Therefore it will be enough to consider actions of some regular unipotent element $u \in G$ (Lemma 1.1.8). In this section we are only considering distinguished unipotent elements of order $> p$. Hence for the rest of this subsection, we make the following assumption:

Assume that $p \leq l$.

Note that our assumption implies that $l \geq 2$. Our reduction for type A_l is given by the results which follow.

Proposition 5.1.4. *Assume that $2 \leq l \leq 14$. Let $u \in G$ be a regular unipotent element and let $\lambda \in X(T)^+$ be a nonzero and p-restricted. If u acts on $L(\lambda)$ as a distinguished unipotent element, then one of the following holds.*

(i) $\lambda = \omega_1$ *or* $\lambda = \omega_l$.

(ii) $\lambda = \omega_1 + \omega_l$.

(iii) *l is odd and* $\lambda = \omega_{\frac{l+1}{2}}$.

(iv) $p = 3$, $l = 3$ *and* $\lambda = 2\omega_2$.

Proof. If $L(\lambda)$ is not self-dual, then we are done by Proposition 1.1.12. Assume then that $L(\lambda)$ is self-dual and suppose that u acts on $L(\lambda)$ as a distinguished unipotent element.

The regular unipotent element u acts on the natural module of G with a single Jordan block of size $l+1$ (Proposition 2.3.4). Therefore the order of u is equal to $M_{l,p} := p^{s+1}$, where $s \geq 1$ is such that $p^s < l+1 \leq p^{s+1}$. Therefore

$$\dim L(\lambda) \leq \frac{(M_{l,p}+1)^2}{4}$$

by Lemma 5.1.1. For all $2 \leq l \leq 14$ and $p \leq l$, checking self-dual irreducible representations of dimension $\leq \frac{(M_{l,p}+1)^2}{4}$ in the tables given in [Lüb17], we end up with representations given by (ii), (iii) and (iv) in our claim. □

Lemma 5.1.5. *If $p = 3$ and $l = 3$, then a regular unipotent element $u \in G$ does not act as a distinguished unipotent element on $L(2\omega_2)$.*

Proof. A computer calculation with MAGMA (Section 2.9) shows that in this case u acts on $L(2\omega_2)$ with Jordan blocks $[5^2, 9]$; hence not as a distinguished unipotent element[11]. □

Lemma 5.1.6. *Assume that $l \geq 15$. Let $\mu = \sum_{i=1}^{l} a_i\omega_i \in X(T)^+$. If one of the following statements hold, then $|W\mu| > \frac{(l^2+1)^2}{4}$.*

(i) $a_i \neq 0$, *where* $l = 2i-1$.

(ii) $a_i a_{l+1-i} \neq 0$ *for some* $3 \leq i \leq \lfloor \frac{l}{2} \rfloor$.

(iii) $a_1 a_2 a_{l-1} a_l \neq 0$.

Proof. Write $f(l) = \frac{(l^2+1)^2}{4}$. In what follows the W-orbit sizes are computed as described in Section 4.3.

(i): Suppose that $a_i \neq 0$ for $l = 2i-1$. Now $|W\mu| \geq |W(\omega_i)| = \binom{l+1}{i} = \binom{2i}{i}$, so it will be enough to show that $\binom{2i}{i} > f(l) = f(2i-1)$ for all $l \geq 15$, equivalently for all $i \geq 8$.

Now for the central binomial coefficient $\binom{2i}{i}$, we have the well known estimate $\binom{2i}{i} \geq 2^i$. This estimate can be seen by induction on i, using the identity $\binom{2i}{i} = 2 \cdot (2 - \frac{1}{i})\binom{2(i-1)}{i-1}$. One can verify that $2^i > f(l) = f(2i-1)$ for all $i \geq 19$, so $\binom{2i}{i} > f(l) = f(2i-1)$ if $i \geq 19$. For $8 \leq i \leq 18$, the inequality $\binom{2i}{i} > f(2i-1)$ is seen by a calculation.

[11] For a computer-free proof, one can proceed as follows. It is possible to show that for the natural module V of G, we have $S^2(\wedge^2(V)) = L(0)/L(2\omega_2)/L(0)$ (uniserial). Furthermore, it is easy to compute with Theorem 3.4.5 that for $p = 3$, we have $S^2(\wedge^2(V_4)) = [1^2, 5^2, 9]$. Thus the action of u on $S^2(\wedge^2(V))$ is inadmissible (Definition 3.2.4), so it follows then from Lemma 3.2.6 that u does not act as a distinguished unipotent element on $L(2\omega_2)$.

(ii): Suppose that $a_i a_{l+1-i} \neq 0$, where $3 \leq i \leq \lfloor \frac{l}{2} \rfloor$. Now $|W\mu| \geq |W(\omega_i + \omega_{l+1-i})| = \binom{l+1}{2i}\binom{2i}{i}$. For $i = 3$ and $i = 4$, one can verify that the inequality $\binom{l+1}{2i}\binom{2i}{i} > f(l)$ holds for all $l \geq 9$.

Consider then $i \geq 5$. Note that $|W\mu| \geq |W(\omega_i)| = \binom{l+1}{i}$. Since $5 \leq i \leq \lfloor \frac{l}{2} \rfloor$, we have $\binom{l+1}{i} \geq \binom{l+1}{5}$. One can verify that the inequality $\binom{l+1}{5} > f(l)$ holds for all $l \geq 35$, so it follows that $|W\mu| > f(l)$ for all $l \geq 35$. Finally for $15 \leq l \leq 34$, the inequality $\binom{l+1}{2i}\binom{2i}{i} > f(l)$ can be verified by calculation.

(iii): In this case $|W\mu| \geq |W(\omega_1 + \omega_2 + \omega_{l-1} + \omega_l)| = (l+1)l(l-1)(l-2)$. Thus the claim follows from the single variable polynomial inequality $(l+1)l(l-1)(l-2) > f(l)$, which holds for all $l \geq 4$.

□

Proposition 5.1.7. *Assume that $l \geq 15$. Let $u \in G$ be a regular unipotent element and let $\lambda \in X(T)^+$ be nonzero and p-restricted. If u acts on $L(\lambda)$ as a distinguished unipotent element, then one of the following holds.*

(i) $\lambda = \omega_1$ *or* $\lambda = \omega_l$.

(ii) $\lambda = \omega_1 + \omega_l$.

Proof. If $L(\lambda)$ is not self-dual, then we are done by Proposition 1.1.12. Assume then that $L(\lambda)$ is self-dual (that is, $\lambda = -w_0\lambda$) and suppose that u acts on $L(\lambda)$ as a distinguished unipotent element.

The regular unipotent element u acts on the natural module of G with a single Jordan block of size $l+1$ (Proposition 2.3.4). Therefore the order of u is equal to p^{s+1}, where $s \geq 1$ is such that $p^s < l+1 \leq p^{s+1}$. Now $p^{s+1} = p \cdot p^s \leq l^2$ since $p^s \leq l$, and thus

$$\dim L(\lambda) \leq \frac{(l^2+1)^2}{4} \qquad (*)$$

by Lemma 5.1.1. Furthermore, we also have

$$\dim L(\lambda) \leq \frac{(m_u(\lambda)+2)^2}{4} \qquad (**)$$

by Lemma 5.1.1.

Write $\lambda = \sum_{i=1}^{l} a_i\omega_i$, where $0 \leq a_i \leq p-1$ for all i. Since $\lambda = -w_0\lambda$, by Table 4.1 we have $a_i = a_{l+1-i}$ for all i. We now consider various possibilities for the coefficients a_i to rule out everything except (i) and (ii) in the claim. In all cases, this will be done by showing that there exists a weight $\mu = \sum_{i=1}^{l} a_i'\omega_i \in X(T)^+$ such that $\mu \preceq \lambda$ and $|W\mu| > \frac{(l^2+1)^2}{4}$ or $|W\mu| > \frac{(m_u(\lambda)+2)^2}{4}$. Then by Premet's theorem (Theorem 4.5.5) we have $\dim L_G(\lambda) \geq |W\mu|$, so this will contradict (*) or (**). In what follows some specific W-orbit sizes of weights are computed using 4.3.

Case (1): l odd and $a_i \neq 0$ for $i = \frac{l+1}{2}$:

In this case we can choose $\lambda = \mu$, since then $|W\mu| > \frac{(l^2+1)^2}{4}$ by Lemma 5.1.6 (i).

Case (2): $a_i \neq 0$, where $3 \leq i \leq \lfloor \frac{l}{2} \rfloor$:

Here we can also choose $\lambda = \mu$, since $a_i = a_{l+1-i}$ and thus $|W\mu| > \frac{(l^2+1)^2}{4}$ by Lemma 5.1.6 (ii).

Case (3): $a_1 a_2 \neq 0$:

Now $a_1 = a_l$ and $a_2 = a_{l-1}$, so similarly to cases (1) and (2), this follows from Lemma 5.1.6 (iii).

By the arguments given for cases (1)-(3), we are left to consider $\lambda = b\omega_1 + b\omega_l$ and $\lambda = b\omega_2 + b\omega_{l-1}$ for $1 \leq b \leq p-1$. We consider the various possibilities to conclude that $\lambda = \omega_1 + \omega_l$, which completes the proof.

Case (4): $\lambda = b\omega_2 + b\omega_{l-1}$, where $b \geq 2$:

In this case, for $\mu = \lambda - \alpha_2 - \alpha_{l-1}$ we have $\mu \in X(T)^+$ such that $a'_3 a'_{l-2} \neq 0$, so $|W\mu| > \frac{(l^2+1)^2}{4}$ by Lemma 5.1.6 (ii).

Case (5): $\lambda = \omega_2 + \omega_{l-1}$:

Here $m_u(\lambda) = 4l - 4$ by Lemma 2.7.3. For $\mu = \lambda$, one can verify that the polynomial inequality $|W\mu| = \frac{(l+1)!}{2!2!(l-3)!} > \frac{(m_u(\lambda)+2)^2}{4}$ holds for all $l \geq 5$.

Case (6): $\lambda = b\omega_1 + b\omega_l$, where $b \geq 3$:

In this case, for $\mu = \lambda - \alpha_1 - \alpha_l$ we have $\mu \in X(T)^+$ such that $a'_1 a'_2 a'_{l-1} a'_l \neq 0$, so $|W\mu| > \frac{(l^2+1)^2}{4}$ by Lemma 5.1.6 (iii).

Case (7): $\lambda = 2\omega_1 + 2\omega_l$.

Here $m_u(\lambda) = 4l$ by Lemma 2.7.3. Now for $\mu = \lambda - \alpha_1 - \alpha_l = \omega_2 + \omega_{l-1}$ one can easily verify that the polynomial inequality $|W\mu| = \frac{(l+1)!}{2!2!(l-3)!} > \frac{(m_u(\lambda)+2)^2}{4}$ holds for all $l \geq 6$.

□

We summarize our reduction for type A in the following proposition, which is an immediate consequence of Proposition 5.1.4, Lemma 5.1.5 and Proposition 5.1.7.

Proposition 5.1.8. *Let $u \in G$ be a regular unipotent element and let $\lambda \in X(T)^+$ be nonzero and p-restricted. Assume that u has order $> p$. If u acts on $L_G(\lambda)$ as a distinguished unipotent element, then one of the following holds.*

(i) $\lambda = \omega_1$ *or* $\lambda = \omega_l$.

(ii) $\lambda = \omega_1 + \omega_l$.

(iii) l *is odd,* $3 \leq l \leq 13$, *and* $\lambda = \omega_{\frac{l+1}{2}}$.

Note that in case (i) of Proposition 5.1.8 a regular unipotent element acts as a distinguished unipotent element. Cases (ii) and (iii) of Proposition 5.1.8 will be dealt with in sections 5.3 and 5.12, respectively.

5.1.2 Types B_l and C_l

Let G be a simple algebraic group of type B_l or C_l. We know that G has unipotent elements of order $> p$ if and only if a regular unipotent element $u \in G$ has order $> p$. For type B_l and type C_l, a regular unipotent element has order $> p$ if and only if $p < 2l + 1$ and $p < 2l$ respectively. Now p is odd, so this is equivalent to $p \leq 2l - 1$. Therefore we will make the following assumption for the rest of this subsection.

Assume that $p \leq 2l - 1$.

Note that a regular unipotent of G has order p^{s+1}, where $s \geq 1$ is such that $p^s < 2l + 1 \leq p^{s+1}$. In particular, we have $p^{s+1} = p \cdot p^s \leq 2l \cdot 2l = 4l^2$.

Thus if some unipotent element $u \in G$ acts on a representation V as a distinguished unipotent element, then

$$\dim V \leq \frac{4l^2(4l^2 + 2)}{4} = 4l^4 + 2l^2 \tag{*}$$

by Lemma 5.1.1. If $V = L_G(\lambda)$ and $u \in G$ is a regular unipotent element, then any unipotent element of G acts on V with largest block of size $\leq m_u(\lambda) + 1$ by Lemma 5.1.1. Thus if some unipotent element of G acts on V as a distinguished unipotent element, then

$$\dim V \leq \frac{(m_u(\lambda) + 2)^2}{4} \tag{**}$$

by Lemma 5.1.1. Similarly to the proof of Proposition 5.1.7, our reduction is based on applying the bound (*) on $V = L_G(\lambda)$ when $\lambda = \sum_{i=1}^{l} a_i \omega_i$ with $m_u(\lambda)$ large, and then applying (**) when $m_u(\lambda)$ is small.

Lemma 5.1.9. *Assume that* $3 \leq l \leq 9$ *and* $G = B_l$. *Let* $\lambda \in X(T)^+$ *be nonzero* p*-restricted. If some unipotent element of* G *acts on* $L_G(\lambda)$ *as a distinguished unipotent element, then one of the following holds.*

(i) $\lambda = \omega_i$ *for some* $1 \leq i \leq l$.

(ii) $\lambda = 2\omega_1$.

(iii) $\lambda = 3\omega_1$.

(iv) $p = 5$, $l = 3$, *and* $\lambda = 2\omega_3$, $\lambda = \omega_1 + \omega_3$ *or* $\lambda = \omega_2 + \omega_3$.

Lemma 5.1.10. *Assume that* $2 \leq l \leq 9$ *and* $G = C_l$. *Let* $\lambda \in X(T)^+$ *be nonzero* p*-restricted. If some unipotent element of* G *acts on* $L_G(\lambda)$ *as a distinguished unipotent element, then one of the following holds.*

(i) $\lambda = \omega_i$ *for some* $1 \leq i \leq l$.

(ii) $\lambda = 2\omega_1$.

(iii) $\lambda = 3\omega_1$.

(iv) $p = 3$, $l = 2$, *and* $\lambda = 2\omega_2$, $\lambda = \omega_1 + \omega_2$, *or* $\lambda = 2\omega_1 + \omega_2$.

(v) $p = 5$, $l = 3$, *and* $\lambda = \omega_2 + \omega_3$ *or* $\lambda = 2\omega_3$.

Proof of Lemma 5.1.9 and Lemma 5.1.10. Suppose that some unipotent element of G acts on $L_G(\lambda)$ as a distinguished unipotent element. Let $u \in G$ be a regular unipotent element. Then it follows from Lemma 5.1.1 that $\dim L_G(\lambda) \leq M$, where $M = \frac{(|u|+1)^2}{4}$. For $2 \leq l \leq 9$, we list the largest possible value of $|u|$ and M in Table 5.1 (recall the assumption $p \leq 2l-1$).

Now the irreducible representations $L_G(\lambda)$ such that $\dim L_G(\lambda) \leq M$ are found in [Lüb17]. In tables 5.2 - 5.5, we list all λ such that $\lambda \neq 2\omega_1, 3\omega_1, \omega_i$ and $\dim L_G(\lambda) \leq M$. We have also included the minimal possible value of $\dim L_G(\lambda)$.

One can verify that for most λ occurring in the tables, we have $\dim L_G(\lambda) > \frac{(m_u(\lambda)+2)^2}{4}$, which contradicts (**). The cases in the tables for which $\dim L_G(\lambda) \leq \frac{(m_u(\lambda)+2)^2}{4}$ is a possibility are:

Case (1): $G = C_2$, with $\lambda \in \{2\omega_2, \omega_1+\omega_2, 2\omega_1+\omega_2\}$.

Case (2): $G = C_3$, with $\lambda \in \{\omega_1+\omega_2, \omega_1+\omega_3, \omega_2+\omega_3, 2\omega_3\}$.

Case (3): $G = B_3$, with $\lambda \in \{2\omega_3, \omega_1+\omega_3, \omega_1+\omega_2, \omega_2+\omega_3\}$.

What remains is to verify the claim of the two lemmas for these cases. For $G = C_2$ (case (1)), the assumption $p \leq 2l-1$ implies $p = 3$, so case (1) is case (iv) of the claim.

For $G = C_3$ (case (2)), we have $p = 3$ or $p = 5$. If $p = 3$, then u has order 3^2, and $\dim L_G(\lambda) > 25 = \frac{(3^2+1)^2}{4}$ for all λ occurring in Table 5.2. Thus by Lemma 5.1.1, no unipotent element of G acts on $L_G(\lambda)$ as a distinguished unipotent element when $p = 3$. To verify the claim of Lemma 5.1.10 for $p = 5$, we should show that no unipotent element of G acts on $L_G(\lambda)$ as a distinguished unipotent element when $\lambda \in \{\omega_1+\omega_2, \omega_1+\omega_3\}$. Using [Lüb01], one can verify for $p = 5$ that $\dim L_G(\omega_1+\omega_2) = 64$ and $\dim L_G(\omega_1+\omega_3) = 70$, so $\dim L_G(\lambda) > \frac{(m_u(\lambda)+2)^2}{4}$ for $\lambda \in \{\omega_1+\omega_2, \omega_1+\omega_3\}$. Thus the claim follows from Lemma 5.1.1.

Finally we have to consider $G = B_3$ (case (3)). Now $p \leq 2l-1$ implies that $p = 3$ or $p = 5$. If $p = 3$, then u has order 3^2, and $\dim L_G(\lambda) > 25 = \frac{(3^2+1)^2}{4}$ for all λ occurring in Table 5.4. Thus by Lemma 5.1.1, no unipotent element of G acts on $L_G(\lambda)$ as a distinguished unipotent element when $p = 3$, as desired. What remains is to show that for $p = 5$, no unipotent element of G acts as a distinguished unipotent element on $L_G(\omega_1+\omega_2)$. For this the claim follows again from Lemma 5.1.1: we have $\dim L_G(\omega_1+\omega_2) = 105$ when $p > 3$ by [Lüb01], so $\dim L_G(\omega_1+\omega_2) > \frac{(m_u(\lambda)+2)^2}{4}$ when $p = 5$.

□

Lemma 5.1.11. *In the following cases no unipotent element $u \in G$ of order $> p$ acts on $L_G(\lambda)$ as a distinguished unipotent element:*

(i) $G = C_2$, $p = 3$, *and* $\lambda = 2\omega_1+\omega_2$.

(ii) $G = C_3$, $p = 5$, *and* $\lambda \in \{\omega_2+\omega_3, 2\omega_3\}$.

(iii) $G = B_3$, $p = 5$, *and* $\lambda = \omega_2+\omega_3$.

G	Largest possible value of $\lvert u\rvert$	Largest possible value of M
B_2, C_2	3^2	25
B_3, C_3	5^2	169
B_4, C_4	7^2	625
B_5, C_5	7^2	625
B_6, C_6	11^2	3721
B_7, C_7	13^2	7225
B_8, C_8	13^2	7225
B_9, C_9	17^2	21025

Table 5.1: For $G = B_l$ and $G = C_l$ $(2 \leq l \leq 9)$, largest possible order $|u|$ of a regular unipotent element u of order $> p$, and largest possible value of $M = \frac{(|u|+1)^2}{4}$.

Proof. In all of these cases, the only unipotent elements in G with order $> p$ are the regular ones. For a regular unipotent element $u \in G$, we have computed the decompositions $L_G(\lambda) \downarrow K[u]$ in cases (i) - (iii) with a computer program implemented in MAGMA (Section 2.9)[12]. The claim follows from these decompositions, which are given in Table 5.6. □

Lemma 5.1.12. *Assume that $l \geq 10$. Let $\mu = \sum_{i=1}^{l} a_i\omega_i \in X(T)^+$. If one of the following statements hold, then $|W\mu| > 4l^4 + 2l^2$.*

(i) *$a_i \neq 0$ for some $i \geq 19$.*

(ii) *$a_ia_j \neq 0$ for some $1 \leq i < j \leq l-1$ such that $j \geq 5$.*

(iii) *$a_ia_l \neq 0$ for some $2 \leq i \leq l-2$.*

Proof. Write $f(l) = 4l^4 + 2l^2$. In what follows the W-orbit sizes are computed using 4.3.

(i): Now $|W\mu| \geq |W(\omega_i)| = 2^i\binom{l}{i}$, so it will be enough to show that $2^i\binom{l}{i} > f(l)$ for all $l \geq i \geq 19$. We show first that this inequality holds for $i = l$, that is, we show that $2^l > f(l)$ for all $l \geq 19$. We do this by induction on l. For $l = 19$ this is a calculation, and for $l > 19$ by induction we get $2^l = 2 \cdot 2^{l-1} > 2 \cdot f(l-1) \geq f(l)$, where the last inequality is an easily verified single variable polynomial inequality.

We now proceed to prove that $2^i\binom{l}{i} > f(l)$ for all $l \geq i \geq 19$ by induction on i. For $i = 19$, one can verify the inequality by calculation. Consider then $i > 19$. The case $i = l$ was dealt with in the previous paragraph, so suppose that $i < l$. In this case

$$2^i\binom{l}{i} = 2^i\left(\binom{l-1}{i} + \binom{l-1}{i-1}\right) > 2^i\binom{l-1}{i-1} > 2 \cdot f(l-1)$$

by induction. Furthermore, $2 \cdot f(l-1) \geq f(l)$ as mentioned in the previous paragraph, so $2^i\binom{l}{i} > f(l)$, as claimed.

[12]One could also give a computer-free proof for many of the entries in Table 5.6, similarly to Footnote 11.

G	λ	Minimal possible value of $\dim L_G(\lambda)$	$m_u(\lambda)$
C_2	$2\omega_2$	14	8
	$\omega_1+\omega_2$	16	7
	$2\omega_1+\omega_2$	25	10
C_3	$\omega_1+\omega_2$	50	13
	$\omega_1+\omega_3$	57	14
	$\omega_2+\omega_3$	62	17
	$2\omega_3$	63	18
	$2\omega_2$	90	16
	$4\omega_1$	126	20
C_4	$\omega_1+\omega_2$	112	19
	$\omega_1+\omega_4$	240	23
	$2\omega_2$	266	24
	$\omega_1+\omega_3$	279	22
	$\omega_3+\omega_4$	312	31
	$2\omega_4$	313	32
	$4\omega_1$	330	28
	$\omega_2+\omega_3$	504	27
	$\omega_2+\omega_4$	513	28
	$2\omega_1+\omega_2$	558	26
C_5	$\omega_1+\omega_2$	210	25
	$2\omega_2$	615	32
C_6	$\omega_1+\omega_2$	352	31
	$2\omega_2$	1221	40
	$4\omega_1$	1365	44
	$\omega_1+\omega_3$	1924	38
	$2\omega_1+\omega_2$	2847	42
	$\omega_2+\omega_3$	3432	47
	$\omega_1+\omega_5$	3638	46
	$\omega_1+\omega_6$	3652	47
C_7	$\omega_1+\omega_2$	546	37
	$2\omega_2$	2184	48
	$4\omega_1$	2380	52
	$\omega_1+\omega_3$	3795	46
	$2\omega_1+\omega_2$	5355	50
	$\omega_2+\omega_3$	7098	57

Table 5.2: For G of type C_l, $2 \leq l \leq 7$ and $2 < p < 2l$: irreducible p-restricted representations $L_G(\lambda)$ of dimension $\leq M$, where M is as in Table 5.1 and $\lambda \neq \omega_i, 2\omega_1, 3\omega_1$.

G	λ	Minimal possible value of $\dim L_G(\lambda)$	$m_u(\lambda)$
C_8	$\omega_1+\omega_2$	800	43
	$2\omega_2$	3620	56
	$4\omega_1$	3876	60
	$\omega_1+\omega_3$	6749	54
C_9	$\omega_1+\omega_2$	1122	49
	$2\omega_2$	5661	64
	$4\omega_1$	5985	68
	$\omega_1+\omega_3$	11154	62
	$2\omega_1+\omega_2$	14193	66

Table 5.3: For G of type C_l, $8 \leq l \leq 9$ and $2 < p < 2l$: irreducible p-restricted representations $L_G(\lambda)$ of dimension $\leq M$, where M is as in Table 5.1 and $\lambda \neq \omega_i, 2\omega_1, 3\omega_1$.

G	λ	Minimal possible value of $\dim L_G(\lambda)$	$m_u(\lambda)$
B_3	$2\omega_3$	35	12
	$\omega_1+\omega_3$	48	12
	$\omega_1+\omega_2$	63	16
	$\omega_2+\omega_3$	64	16
	$3\omega_3$	104	18
	$2\omega_1+\omega_3$	120	18
	$2\omega_2$	132	20
	$\omega_1+2\omega_3$	168	18
B_4	$\omega_1+\omega_4$	112	18
	$2\omega_4$	126	20
	$\omega_1+\omega_2$	147	22
	$\omega_2+\omega_4$	304	24
	$\omega_3+\omega_4$	336	28
	$2\omega_2$	369	28
	$4\omega_1$	450	32
	$3\omega_4$	544	30
	$\omega_1+\omega_3$	558	26
	$2\omega_1+\omega_4$	576	26
	$\omega_2+\omega_3$	579	32
B_5	$\omega_1+\omega_2$	264	28
	$\omega_1+\omega_5$	320	25
	$2\omega_5$	462	30

Table 5.4: For G of type B_l, $3 \leq l \leq 5$ and $2 < p < 2l+1$: irreducible p-restricted representations $L_G(\lambda)$ of dimension $\leq M$, where M is as in Table 5.1 and $\lambda \neq \omega_i, 2\omega_1, 3\omega_1$.

G	λ	Minimal possible value of $\dim L_G(\lambda)$	$m_u(\lambda)$
B_6	$\omega_1+\omega_2$	416	34
	$\omega_1+\omega_6$	768	33
	$2\omega_2$	1559	44
	$2\omega_6$	1716	42
	$4\omega_1$	1728	48
	$\omega_1+\omega_3$	2847	42
	$\omega_2+\omega_6$	3392	43
B_7	$\omega_1+\omega_2$	650	40
	$\omega_1+\omega_7$	1664	42
	$2\omega_2$	2715	52
	$4\omega_1$	2940	56
	$\omega_1+\omega_3$	5250	50
	$2\omega_7$	6435	56
	$2\omega_1+\omega_2$	6798	54
B_8	$\omega_1+\omega_2$	935	46
	$\omega_1+\omega_8$	4096	52
	$2\omega_2$	4251	60
	$4\omega_1$	4540	64
B_9	$\omega_1+\omega_2$	1273	52
	$2\omega_2$	6763	68
	$4\omega_1$	7124	72
	$\omega_1+\omega_9$	9216	63
	$\omega_1+\omega_3$	14193	66
	$2\omega_1+\omega_2$	17424	70

Table 5.5: For G of type B_l, $6 \leq l \leq 9$ and $2 < p < 2l+1$: irreducible p-restricted representations $L_G(\lambda)$ of dimension $\leq M$, where M is as in Table 5.1 and $\lambda \neq \omega_i, 2\omega_1, 3\omega_1$.

G	λ	$L_G(\lambda) \downarrow K[u]$
C_2 ($p=3$)	$2\omega_2$	$[5,9]$
	$\omega_1+\omega_2$	$[2,6,8]$
	$2\omega_1+\omega_2$	$[7,9^2]$
C_3 ($p=5$)	$\omega_2+\omega_3$	$[10^2,12,15^2]$
	$2\omega_3$	$[10^2,13,15^2]$
B_3 ($p=5$)	$2\omega_3$	$[1,5,7,9,13]$
	$\omega_1+\omega_3$	$[3,5,7,9,11,13]$
	$\omega_2+\omega_3$	$[5,7,10^2,15,17]$

Table 5.6: For $G=C_2$, $G=C_3$, and $G=B_3$; actions of a regular unipotent element $u \in G$ on some small irreducible representations $L_G(\lambda)$.

(ii): Now $|W\mu| \geq |W(\omega_j)|$, so the claim follows from (i) if $j \geq 19$. Consider then $1 \leq i < j \leq l-1$ such that $5 \leq j \leq 18$. Since $|W\mu| \geq |W(\omega_i + \omega_j)| = 2^j \binom{l}{j}\binom{j}{i}$, it will be enough to verify that $2^j \binom{l}{j}\binom{j}{i} > f(l)$ for all $l \geq 10$. For all $1 \leq i < j \leq l-1$ with $5 \leq j \leq 18$ (finitely many cases to check), this polynomial inequality is straightforward to verify by a calculation.

(iii): As in (ii), the claim follows from (i) if $i \geq 19$. Consider then $2 \leq i \leq 18$, where $i \leq l-2$. Similarly to (ii), we have now have finitely many i to check, so a calculation shows that $|W\mu| \geq |W(\omega_i + \omega_l)| = 2^l \binom{l}{i} > f(l)$ holds for all $l \geq 10$.

□

Lemma 5.1.13. *Assume that $l \geq 10$. Let $u \in G$ be a distinguished unipotent element and let $\lambda \in X(T)^+$ be nonzero and p-restricted. If u acts on $L(\lambda)$ as a distinguished unipotent element, then $\lambda = 2\omega_1$ or $\lambda = \omega_i$ for some $1 \leq i \leq l$.*

Proof. Write $\lambda = \sum_{i=1}^{l} a_i \omega_i$, where $0 \leq a_i \leq p-1$. Set $n = 2l+1$ if G has type B_l and $n = 2l$ if G has type C_l. Suppose that some unipotent element u of G acts on $L_G(\lambda)$ as a distinguished unipotent element.

Our strategy is to rule out the various possibilities for the coefficients a_i case by case and eventually conclude that either $\lambda = \omega_i$ or $\lambda = 2\omega_1$ (cf. Proposition 5.1.7). Most cases will be ruled out by showing that there exists a weight $\mu = \sum_{i=1}^{l} a_i' \omega_i \in X(T)^+$ such that $\mu \preceq \lambda$ and $|W\mu| > 4l^4 + 2l^2$ or $|W\mu| > \frac{(m_u(\lambda)+2)^2}{4}$. Then by Premet's theorem (Theorem 4.5.5) we have $\dim L_G(\lambda) \geq |W\mu|$, so this will contradict (*) or (**). Throughout we apply 4.3 to compute the W-orbit sizes $|W\mu|$.

Case (1): $a_i a_l \neq 0$ for $1 \leq i \leq l-1$:

In this case for $i = l-1$, we set $\mu = \lambda - \alpha_{l-1} - \alpha_l$ so we have $\mu \in X(T)^+$ with $a_{l-2}' a_l' \neq 0$ if G has type B_l and $a_{l-2}' a_{l-1}' \neq 0$ if G has type C_l. If $2 \leq i \leq l-2$, we set $\mu = \lambda$. If $i = 1$ and $a_1 \geq 2$, we set $\mu = \lambda - \alpha_1$ and then we have $\mu \in X(T)^+$ with $a_2' a_l' \neq 0$. If $a_l \geq 2$, then we set $\mu = \lambda - \alpha_l$ and we have $\mu \in X(T)^+$ with $a_1' a_{l-1}' \neq 0$. Now in these cases we have $|W\mu| > 4l^4 + 2l^2$ by Lemma 5.1.12 (ii) or (iii).

What remains is the case where $a_1 = 1$ and $a_l = 1$. By the previous cases, we can assume that $\lambda = \omega_1 + \omega_l$. If G is of type C_l, then $\lambda \succ \mu$ for $\mu = \omega_1 + \omega_{l-2} \in X(T)^+$, and $|W\mu| > 4l^4 + 2l^2$ by Lemma 5.1.12 (ii). If G is of type B_l, then $m_u(\lambda) = 2l + \frac{l(l+1)}{2}$. It is easy to verify that for $\mu = \lambda$, we have $|W\mu| = 2^l l > \frac{(m_u(\lambda)+2)^2}{4}$ for all $l \geq 6$.

Case (2): $a_i a_j \neq 0$ for $1 \leq i < j \leq l-1$ with $j \geq 5$:

Here $|W\lambda| > 4l^4 + 2l^2$ by Lemma 5.1.12 (ii), so we can pick $\mu = \lambda$.

Case (3): $a_i a_4 \neq 0$ for $1 \leq i \leq 3$:

In this case if $i = 3$, we set $\mu = \lambda - \alpha_3 - \alpha_4$ and we have $\mu \in X(T)^+$ with $a_2' a_5' \neq 0$. If $i = 2$, we set $\mu = \lambda - \alpha_2 - \alpha_3 - \alpha_4$ and we have $\mu \in X(T)^+$ with $a_1' a_5' \neq 0$. If $a_1 \geq 2$ or $a_4 \geq 2$, then we set $\mu = \lambda - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$ and we have $\mu \in X(T)^+$ with $a_1' a_5' \neq 0$ or $a_4' a_5' \neq 0$. Thus in these cases $|W\mu| > 4l^4 + 2l^2$ by Lemma 5.1.12 (ii).

What remains is the case where $a_1 = 1$, $a_4 = 1$. By the previous cases treated, we can assume that $\lambda = \omega_1 + \omega_4$. In this case we find a weight $\mu \in X(T)^+$ with $\lambda \succ \mu$ and $|W\mu| > \frac{(m_u(\lambda)+2)^2}{4}$ in Table 5.7.

Case (4): $a_2a_3 \neq 0$:

In this case if $a_1 \geq 1$, $a_2 \geq 2$, or $a_3 \geq 2$, then for $\mu = \lambda - \alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4$ we have $\mu \in X(T)^+$ and $a_1'a_5' \neq 0$, $a_2'a_5' \neq 0$, or $a_3'a_5' \neq 0$, respectively. Thus if $a_1 \geq 1$, $a_2 \geq 2$, or $a_3 \geq 2$, then by Lemma 5.1.12 (ii) we have $|W\mu| > 4l^4 + 2l^2$.

What remains is the possibility that $a_1 = 0$, $a_2 = 1$, and $a_3 = 1$. By the previous cases, we can assume that $\lambda = \omega_2 + \omega_3$. In this case we find a weight $\mu \in X(T)^+$ with $\lambda \succ \mu$ and $|W\mu| > \frac{(m_u(\lambda)+2)^2}{4}$ in Table 5.7.

Case (5): $a_1a_3 \neq 0$:

In this case if $a_1 \geq 3$, then for $\mu = \lambda - 2\alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4$ we have $\mu \in X(T)^+$ and $a_1'a_5' \neq 0$. If $a_3 \geq 2$, then for $\mu = \lambda - \alpha_2 - 2\alpha_3 - \alpha_4$ we have $\mu \in X(T)^+$ and $a_1'a_5' \neq 0$. Thus if $a_1 \geq 3$ or $a_3 \geq 2$, then by Lemma 5.1.12 (ii) we have $|W\mu| > 4l^4 + 2l^2$.

Consider then $a_1 \leq 2$ and $a_3 = 1$. By the previous cases, we can assume that $\lambda = a_1\omega_1 + \omega_3$. In this case we find a weight $\mu \in X(T)^+$ with $\lambda \succ \mu$ and $|W\mu| > \frac{(m_u(\lambda)+2)^2}{4}$ in Table 5.7.

Case (6): $a_1a_2 \neq 0$:

If $a_1 \geq 4$, then for $\mu = \lambda - 3\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$ we have $\mu \in X(T)^+$ with $a_1'a_5' \neq 0$. If $a_2 \geq 3$, then for $\mu = \lambda - 2\alpha_1 - 4\alpha_2 - 3\alpha_3 - 2\alpha_4 - \alpha_5$ we have $\mu \in X(T)^+$ with $a_1'a_6' \neq 0$. Thus if $a_1 \geq 4$ or $a_2 \geq 3$, then by Lemma 5.1.12 (ii) we have $|W\mu| > 4l^4 + 2l^2$.

Consider then $a_1 \leq 3$ and $a_2 \leq 2$. By the previous cases, we can assume that $\lambda = a_1\omega_1 + a_2\omega_2$. In this case we find a weight $\mu \in X(T)^+$ with $\lambda \succ \mu$ and $|W\mu| > \frac{(m_u(\lambda)+2)^2}{4}$ in Table 5.7.

By the arguments given for cases (1)-(6), we are left to consider $\lambda = b\omega_i$ with $1 \leq b \leq p-1$. We consider the various possibilities to conclude that if $b > 1$, then $b = 2$ and $i = 1$, which completes the proof.

Case (7): $\lambda = b\omega_l$, where $b \geq 2$:

In this case if $b \geq 3$, then for $\mu = \lambda - 2\alpha_l - \alpha_{l-1}$ we have $\mu \in X(T)^+$. Furthermore, we have $a_{l-2}'a_l' \neq 0$ if G has type B_l and $a_{l-2}'a_{l-1}' \neq 0$ if G has type C_l. Thus $|W\mu| > 4l^4 + 2l^2$ by Lemma 5.1.12 (ii) or (iii).

Consider then $b = 2$, so now $\lambda = 2\omega_l$. If G has type C_l, then $\lambda \succ \mu$ for $\mu = \omega_{l-2} + \omega_l \in X(T)^+$, and $|W\mu| > 4l^4 + 2l^2$ by Lemma 5.1.12 (iii). Suppose then that G has type B_l. In this situation we need to consider more than one orbit of weights. First of all, we have $m_u(\lambda) = 2\frac{l(l+1)}{2} = l(l+1)$. Furthermore, it is easy to see that $\lambda \succ 0$ and $\lambda \succ \omega_i$ for all $1 \leq i \leq l-1$. Now $|W\omega_i| = 2^i\binom{l}{i}$, so in this situation $\dim L_G(2\omega_l) \geq \sum_{i=0}^{l} 2^i\binom{l}{i} = 3^l$. One can verify that $\dim L_G(2\omega_l) \geq 3^l > \frac{(m_u(\lambda)+2)^2}{4}$ for all $l \geq 6$, which contradicts (**).

Case (8): $\lambda = b\omega_i$, where $1 \leq i \leq l-1$ and $b \geq 2$:

In this case if $4 \leq i \leq l-1$, we set $\mu = \lambda - \alpha_i$ and we have $\mu \in X(T)^+$ with $a'_{i-1}a'_{i+1} \neq 0$. If $i = 3$, we set $\mu = \lambda - \alpha_2 - 2\alpha_3 - \alpha_4$ and we have $\mu \in X(T)^+$ with $a'_1a'_5 \neq 0$. If $i = 2$ and $b \geq 3$, we set $\mu = \lambda - \alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$ and we have $\mu \in X(T)^+$ with $a'_1a'_5 \neq 0$. If $i = 1$ and $b \geq 6$, then for $\mu = \lambda - 4\alpha_1 - 3\alpha_2 - 2\alpha_3 - \alpha_4$ we have $\mu \in X(T)^+$ with $a'_1a'_5 \neq 0$. In these cases we have $|W\mu| > 4l^4 + 2l^2$.by Lemma 5.1.12 (ii) or (iii).

What remain are the cases $\lambda = 3\omega_2$ and $\lambda = b\omega_1$ with $3 \leq b \leq 5$. For $\lambda = 3\omega_2$ and $\lambda = b\omega_1$ with $4 \leq b \leq 5$, we find a weight $\mu \in X(T)^+$ with $\lambda \succ \mu$ and $|W\mu| > \frac{(m_u(\lambda)+2)^2}{4}$ in Table 5.7. Finally for $\lambda = 3\omega_1$, we note first that $m_u(\lambda) = 3n-3$. Furthermore, $\lambda \succ \mu$ for $\mu = \omega_3$. Now one verifies that $\dim L_G(3\omega_1) \geq |W\lambda| + |W\mu| = 2l + 2^3\binom{l}{3} > \frac{(m_u(\lambda)+2)^2}{4}$ for all $l \geq 10$, which contradicts (**).

□

λ	$\mu \prec \lambda$	$\lambda - \mu$	$\lvert W\mu\rvert$	$m_u(\lambda)$
$\omega_1 + \omega_4$	ω_5	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$	$2^5\binom{l}{5}$	$5n-17$
$\omega_2 + \omega_3$	ω_5	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$	$2^5\binom{l}{5}$	$5n-13$
$2\omega_1 + \omega_3$	ω_5	$2\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$	$2^5\binom{l}{5}$	$5n-11$
$\omega_1 + \omega_3$	ω_4	$\alpha_1 + \alpha_2 + \alpha_3$	$2^4\binom{l}{4}$	$4n-10$
$3\omega_1 + 2\omega_2$	$\omega_1 + \omega_6$	$3\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5$	$2^6\binom{l}{6} \cdot 6$	$7n-11$
$2\omega_1 + 2\omega_2$	$\omega_1 + \omega_5$	$2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4$	$2^5\binom{l}{5} \cdot 5$	$6n-10$
$\omega_1 + 2\omega_2$	ω_5	$2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4$	$2^5\binom{l}{5}$	$5n-9$
$3\omega_1 + \omega_2$	ω_5	$3\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4$	$2^5\binom{l}{5}$	$5n-7$
$2\omega_1 + \omega_2$	ω_4	$2\alpha_1 + 2\alpha_2 + \alpha_3$	$2^4\binom{l}{4}$	$4n-6$
$\omega_1 + \omega_2$	ω_3	$\alpha_1 + \alpha_2$	$2^3\binom{l}{3}$	$3n-5$
$2\omega_2$	ω_4	$\alpha_1 + 2\alpha_2 + \alpha_3$	$2^4\binom{l}{4}$	$4n-8$
$5\omega_1$	ω_5	$4\alpha_1 + 3\alpha_3 + 2\alpha_2 + \alpha_1$	$2^5\binom{l}{5}$	$5n-5$
$4\omega_1$	ω_4	$3\alpha_1 + 2\alpha_2 + \alpha_3$	$2^4\binom{l}{4}$	$4n-4$

Table 5.7: Type B_l ($n = 2l+1$) and type C_l ($n = 2l$): For some specific $\lambda \in X(T)^+$, weights $\mu \prec \lambda$ such that $\mu \in X(T)^+$ and $|W\mu| > \frac{(m_u(\lambda)+2)^2}{4}$ for all $l \geq 10$.

We summarize our reduction for type B and C in the following proposition, which is a corollary of Lemma 5.1.9, Lemma 5.1.10, Lemma 5.1.11, and Lemma 5.1.13.

Proposition 5.1.14. *Suppose that G is simple of type B_l ($l \geq 3$) or type C_l ($l \geq 2$). Let $\lambda \in X(T)^+$ be nonzero and p-restricted. If some unipotent element*

$u \in G$ of order $> p$ acts on $L_G(\lambda)$ as a distinguished unipotent element, then one of the following holds.

(i) $\lambda = \omega_i$ *for some* $1 \leq i \leq l$.

(ii) $\lambda = 2\omega_1$.

(iii) $\lambda = 3\omega_1$.

(iv) $G = C_2$, *u is a regular unipotent element,* $p = 3$, *and* $\lambda \in \{\omega_1 + \omega_2, 2\omega_2\}$.

(v) $G = B_3$, *u is a regular unipotent element,* $p = 5$, *and* $\lambda \in \{2\omega_3, \omega_1 + \omega_3\}$.

Cases (i), (ii), and (iii) of Proposition 5.1.14 will be dealt with in sections 5.12, 5.7, and 5.9 respectively. Note that by Table 5.6 in cases (iv) and (v) the element u acts on $L_G(\lambda)$ as a distinguished unipotent element, and that we have recorded these examples in Table 1.1.

5.1.3 Type D_l

Let G be a simple algebraic group of type D_l. In this subsection, we give the reduction for G using the same arguments as in section 5.1.2.

Now G has unipotent elements of order $> p$ if and only if a regular unipotent element of G has order $> p$. Furthermore, since G has type D_l, a regular unipotent element has order $> p$ if and only if $p < 2l - 1$. Thus we will make the following assumption for the rest of this subsection.

Assume that $p < 2l - 1$.

Then a regular unipotent element of G has order p^{s+1}, where $s \geq 1$ is such that $p^s < 2l - 1 \leq p^{s+1}$. As in 5.1.2, we see that $p^{s+1} \leq 4l^2$. Therefore if some unipotent element of G acts on a representation V as a distinguished unipotent element, the inequalities (*) and (**) from subsection 5.1.2 must hold.

The following lemma gives our reduction for small l (cf. Lemma 5.1.4, Lemma 5.1.9, Lemma 5.1.10).

Lemma 5.1.15. *Assume that* $4 \leq l \leq 9$. *Let* $\lambda \in X(T)^+$ *be nonzero p-restricted. If some unipotent element of G acts on* $L_G(\lambda)$ *as a distinguished unipotent element, then one of the following holds.*

(i) $\lambda = \omega_i$ *for some* $1 \leq i \leq l$.

(ii) $\lambda = 2\omega_1$.

Proof. Suppose that some unipotent element of G acts on $L_G(\lambda)$ as a distinguished unipotent element. Let $u \in G$ be a regular unipotent element. Then it follows from Lemma 5.1.1 that $\dim L_G(\lambda) \leq M$, where $M = \frac{(|u|+1)^2}{4}$. For $4 \leq l \leq 9$, we list the largest possible value of $|u|$ and M in Table 5.8 (recall the assumption $p < 2l-1$).

Now the irreducible p-restricted representations $L_G(\lambda)$ such that $\dim L_G(\lambda) \leq M$ are found in the tables of [Lüb01]. In Table 5.9, we list (based on [Lüb01]) all λ such that $\lambda \neq 2\omega_1, \omega_i$ and $\dim L_G(\lambda) \leq M$. We have also included the value of $m_u(\lambda)$ in the tables, which is easily calculated with the table in Lemma 2.7.3.

In any case, one can verify that for all λ occurring in Table 5.9, we have $\dim L_G(\lambda) > \frac{(m_u(\lambda)+2)^2}{4}$, which would contradict (**). This implies the claim. ☐

G	Largest possible value of $\lvert u \rvert$	Largest possible value of M
D_4	5^2	169
D_5	7^2	625
D_6	7^2	625
D_7	11^2	3721
D_8	13^2	7225
D_9	13^2	7225

Table 5.8: For $G = D_l$ $(4 \leq l \leq 9)$, largest possible order $|u|$ of a regular unipotent element u of order $> p$, and largest possible value of $M = \frac{(|u|+1)^2}{4}$.

Lemma 5.1.16. *Assume that $l \geq 10$. Let $\mu = \sum_{i=1}^{l} a_i\omega_i \in X(T)^+$. If one of the following statements hold, then $|W\mu| > 4l^4 + 2l^2$.*

(i) $a_i \neq 0$ *for some* $i \geq 20$.

(ii) $a_ia_j \neq 0$ *for some* $1 \leq i < j \leq l-2$ *such that* $j \geq 5$.

(iii) $a_ia_l \neq 0$ *or* $a_ia_{l-1} \neq 0$ *for some* $3 \leq i \leq l-3$.

(iv) $a_1a_la_{l-1} \neq 0$ *or* $a_2a_la_{l-1} \neq 0$.

(v) $a_1a_2a_{l-1} \neq 0$ *or* $a_1a_2a_l \neq 0$.

Proof. Write $f(l) = 4l^4 + 2l^2$. In what follows the W-orbit sizes are computed using 4.3.

(i): If $i \leq l-2$, then the value of $|W\omega_i|$ is the same as for type B_l and C_l, so the claim follows Lemma 5.1.12 (ii). For $i = l-1$ and $i = l$, we have $|W\mu| \geq |W\omega_i| = 2^{l-1}$, so it will be enough to show that $2^{l-1} > f(l)$ for all $l \geq 20$. This follows with the exact same proof as the one given for the inequality $2^l > f(l)$ in Lemma 5.1.12 (ii).

(ii): Now the value of $|W(\omega_i + \omega_j)|$ is the same as for type B_l and C_l, so the claim follows from Lemma 5.1.12 (ii).

(iii): If $l \geq 20$, the claim follows from (i). For $10 \leq l \leq 19$, one can verify by a computer calculation that $|W\mu| \geq |W(\omega_i + \omega_l)| = 2^{l-1}\binom{l}{i} > f(l)$ for all $3 \leq i \leq l-3$.

(iv), (v): As in (iii), the claim follows from (i) if $l \geq 20$, and is easy to verify manually for $10 \leq l \leq 19$.

□

Lemma 5.1.17. *Assume that $l \geq 10$. Let $u \in G$ be a distinguished unipotent element and let $\lambda \in X(T)^+$ be nonzero and p-restricted. If u acts on $L(\lambda)$ as a distinguished unipotent element, then $\lambda = 2\omega_1$ or $\lambda = \omega_i$ for some $1 \leq i \leq l$.*

Proof. Write $\lambda = \sum_{i=1}^{l} a_i\omega_i$, where $0 \leq a_i \leq p-1$. Suppose that some unipotent element u of G acts on $L_G(\lambda)$ as a distinguished unipotent element. We proceed to rule out various possibilities for a_i as in the proofs of 5.1.7 and 5.1.13 to show the claim. That is, in most cases we find a contradiction with (*) or (**) by

G	λ	Minimal possible value of $\dim L_G(\lambda)$	$m_u(\lambda)$
D_4	$\omega_1+\omega_4$	56	12
	$3\omega_1$	104	18
	$\omega_1+\omega_2$	104	16
	$\omega_1+2\omega_4$	168	18
D_5	$2\omega_5$	126	20
	$\omega_1+\omega_5$	128	18
	$\omega_1+\omega_2$	190	22
	$3\omega_1$	210	24
	$\omega_4+\omega_5$	210	20
	$\omega_2+\omega_5$	544	24
	$3\omega_5$	544	30
	$2\omega_2$	559	28
	$2\omega_1+\omega_5$	576	26
	$4\omega_1$	606	32
D_6	$\omega_1+\omega_6$	320	25
	$3\omega_1$	340	30
	$\omega_1+\omega_2$	340	28
	$2\omega_6$	462	30
D_7	$\omega_1+\omega_2$	532	34
	$3\omega_1$	546	36
	$\omega_1+\omega_7$	768	33
	$2\omega_7$	1716	42
	$2\omega_2$	1975	44
	$4\omega_1$	2275	48
	$\omega_6+\omega_7$	3003	42
D_8	$\omega_1+\omega_2$	768	40
	$3\omega_1$	800	42
	$\omega_1+\omega_8$	1920	42
	$2\omega_2$	3483	52
	$4\omega_1$	3605	56
	$2\omega_8$	6435	56
	$\omega_1+\omega_3$	6900	50
D_9	$3\omega_1$	1104	48
	$\omega_1+\omega_2$	1104	46
	$\omega_1+\omega_9$	4096	52
	$2\omega_2$	5490	60
	$4\omega_1$	5644	64

Table 5.9: For G of type D_l, $4 \leq l \leq 9$ and $2 < p < 2l-1$: irreducible p-restricted representations $L_G(\lambda)$ of dimension $\leq M$, where M is as in Table 5.8 and $\lambda \neq 2\omega_1, \omega_i$.

finding a $\mu = \sum_{i=1}^{l} a_i'\omega_i \in X(T)^+$ such that $\mu \preceq \lambda$, and $|W\mu| > 4l^4 + 2l^2$ or $|W\mu| > \frac{(m_u(\lambda)+2)^2}{4}$. Throughout we apply 4.3 to compute the W-orbit sizes $|W\mu|$.

Note that if $a_l' = 0$ and $a_{l-1}' = 0$, then the Weyl orbit size $|W\mu|$ is the same as for the corresponding weight (one with the same coefficients a_i') for type B_l or C_l. From this fact, we will see in several cases that the same arguments as those given in the proof of Lemma 5.1.13 work.

Case (1): $a_i a_{l-1} \neq 0$ or $a_i a_l \neq 0$ for some $3 \leq i \leq l-3$:

Here $|W\lambda| > 4l^4 + 2l^2$ by Lemma 5.1.16 (iii), so we can pick $\mu = \lambda$.

Case (2): $a_{l-2}a_{l-1} \neq 0$ or $a_{l-2}a_l \neq 0$:

By applying a graph automorphism on λ, it is enough to consider $a_{l-2}a_{l-1} \neq 0$. In this case for $\mu = \lambda - \alpha_{l-2} - \alpha_{l-1}$ we have $\mu \in X(T)^+$ and $a_{l-3}'a_l' \neq 0$, so $|W\mu| > 4l^4 + 2l^2$ by Lemma 5.1.16 (iii).

Case (3): $a_{l-1}a_l \neq 0$:

If $a_1 a_{l-1} a_l \neq 0$ or $a_2 a_{l-1} a_l \neq 0$, then $|W\lambda| > 4l^4 + 2l^2$ by Lemma 5.1.16 (iv) and we can pick $\mu = \lambda$. Otherwise by case (1) and (2) we can assume that $\lambda = a_{l-1}\omega_{l-1} + a_l\omega_l$. If $a_{l-1} \geq 2$ or $a_l \geq 2$, then for $\mu = \lambda - \alpha_{l-2} - \alpha_{l-1} - \alpha_l$ we have $a_{l-1}'a_{l-3}' \neq 0$ or $a_l'a_{l-3}' \neq 0$, so $|W\mu| > 4l^4 + 2l^2$ by Lemma 5.1.16 (iii).

Consider then $a_{l-1} = 1$, $a_l = 1$, so now $\lambda = \omega_{l-1} + \omega_l$. In this case $m_u(\lambda) = l(l-1)$ and one can verify that $|W\lambda| = 2^{l-1}l > \frac{(m_u(\lambda)+2)^2}{4}$ for all $l \geq 8$.

Case (4): $a_2 a_{l-1} \neq 0$ or $a_2 a_l \neq 0$:

By applying a graph automorphism on λ, it is enough to consider $a_2 a_{l-1} \neq 0$. If $a_1 a_2 a_{l-1} \neq 0$, then $|W\lambda| > 4l^4 + 2l^2$ by Lemma 5.1.16 (v), so we can pick $\mu = \lambda$. Thus we can assume $a_1 = 0$, and by cases (1), (2) and (3) we can assume $\lambda = a_2\omega_2 + a_{l-1}\omega_{l-1}$.

If $a_{l-1} \geq 2$, then for $\mu = \lambda - \alpha_{l-1}$ we have $\mu \in X(T)^+$ and $a_2'a_{l-2}' \neq 0$, so $|W\mu| > 4l^4 + 2l^2$ by Lemma 5.1.16 (ii). If $a_2 \geq 2$, then for $\mu = \lambda - \alpha_2$ we have $\mu \in X(T)^+$ and $a_3'a_{l-1}' \neq 0$, so $|W\mu| > 4l^4 + 2l^2$ by Lemma 5.1.16 (iii).

Consider then $a_2 = 1$, $a_{l-1} = 1$, so now $\lambda = \omega_2 + \omega_{l-1}$. In this case $m_u(\lambda) = 2(2l-3) + \frac{l(l-1)}{2}$ and one can verify that $|W\lambda| = 2^{l-2}l(l-1) > \frac{(m_u(\lambda)+2)^2}{4}$ for all $l \geq 6$.

Case (5): $a_1 a_{l-1} \neq 0$ or $a_1 a_l \neq 0$:

By applying a graph automorphism on λ, it is enough to consider $a_1 a_{l-1} \neq 0$. By cases (1), (2), (3) and (4) we may assume that $\lambda = a_1\omega_1 + a_{l-1}\omega_{l-1}$. If $a_1 \geq 3$, then for $\mu = \lambda - 2\alpha_1 - \alpha_2$ we have $\mu \in X(T)^+$ and $a_3'a_{l-1}' \neq 0$, so $|W\mu| > 4l^4 + 2l^2$ by Lemma 5.1.16 (iii). If $a_{l-1} \geq 2$, then for $\mu = \lambda - \alpha_{l-1}$ we have $\mu \in X(T)^+$ and $a_1'a_{l-2}' \neq 0$, so $|W\mu| > 4l^4 + 2l^2$ by Lemma 5.1.16 (ii).

Consider then $a_1 \leq 2$ and $a_{l-1} = 1$, so now $\lambda = \omega_1 + \omega_{l-1}$ or $\lambda = 2\omega_1 + \omega_{l-1}$. In this situation $m_u(\lambda) \leq m_u(2\omega_1 + \omega_{l-1}) = 2(2l-2) + \frac{l(l-1)}{2}$

and one verifies that $|W\lambda| = 2^{l-1}l > \frac{(m_u(2\omega_1+\omega_{l-1})+2)^2}{4} \geq \frac{(m_u(\lambda)+2)^2}{4}$ for all $l \geq 8$.

Cases (6)-(10): $a_ia_j \neq 0$ for some $1 \leq i < j \leq l-2$ with $j \geq 5$; $a_ia_4 \neq 0$ for $1 \leq i \leq 3$; $a_2a_3 \neq 0$; $a_1a_3 \neq 0$; or $a_1a_2 \neq 0$:

In these cases, the exact same arguments as those given in cases (2)-(6) in the proof of Lemma 5.1.13 work.

By the arguments given for cases (1)-(10), we are left to consider $\lambda = b\omega_i$ with $1 \leq b \leq p-1$. We consider the various possibilities to conclude that if $b > 1$, then $b = 2$ and $i = 1$, which completes the proof.

Case (11): $\lambda = b\omega_{l-1}$ or $\lambda = b\omega_l$, where $b \geq 2$:

By applying a graph automorphism on λ, it is enough to consider $\lambda = b\omega_{l-1}$. If $b \geq 3$, then for $\mu = \lambda - 2\alpha_{l-1} - \alpha_{l-2}$ we have $\mu \in X(T)^+$ and $a'_{l-3}a'_l \neq 0$, so $|W\mu| > 4l^4 + 2l^2$ by Lemma 5.1.16 (iii).

Consider then $b = 2$, so $\lambda = 2\omega_{l-1}$. If l is even, then $\lambda \succ \mu$ for all $\mu = \omega_i \in X(T)^+$ with $1 \leq i \leq l-2$ even. Furthermore, a standard binomial identity gives

$$\sum_{\substack{1 \leq i \leq l-2 \\ i \text{ even}}} |W\omega_i| = \sum_{\substack{1 \leq i \leq l-2 \\ i \text{ even}}} 2^i \binom{l}{i} = \frac{3^l+1}{2} - 2^l.$$

Similarly, if l is odd, then $\lambda \succ \mu$ for all $\mu = \omega_i \in X(T)^+$ with $1 \leq i \leq l-2$ odd. Again a standard binomial identity gives

$$\sum_{\substack{1 \leq i \leq l-2 \\ i \text{ odd}}} |W\omega_i| = \sum_{\substack{1 \leq i \leq l-2 \\ i \text{ odd}}} 2^i \binom{l}{i} = \frac{3^l+1}{2} - 2^l.$$

Therefore it follows that $\dim L_G(\lambda) \geq \frac{3^l+1}{2} - 2^l$. Now $m_u(\lambda) = l(l-1)$ and one verifies that $\frac{3^l+1}{2} - 2^l > \frac{(m_u(\lambda)+2)^2}{4}$ for all $l \geq 6$.

Case (12): $\lambda = b\omega_i$, where $1 \leq i \leq l-2$ and $b \geq 2$:

In this case, the exact same arguments as those given in case (8) in the proof of Lemma 5.1.13 work.

□

We summarize our reduction for type D in the following proposition, which is immediate from Lemma 5.1.17 and Lemma 5.1.15.

Proposition 5.1.18. *Let $G = D_l$, where $l \geq 4$. Let $\lambda \in X(T)^+$ be nonzero p-restricted. If some unipotent element $u \in G$ of order $> p$ acts on $L_G(\lambda)$ as a distinguished unipotent element, then one of the following holds.*

(i) $\lambda = \omega_i$ *for some* $1 \leq i \leq l$.

(ii) $\lambda = 2\omega_1$.

Cases (i) and (ii) of Proposition 5.1.18 will be dealt with in sections 5.12 and 5.7, respectively.

5.1.4 Type G_2

We will prove the following proposition, which gives the claim of Theorem 1.1.10 for $G = G_2$ in the case where u has order $> p$ (see Table 1.2).

Proposition 5.1.19. *Let $G = G_2$ and $\lambda \in X(T)^+$ be nonzero p-restricted. A unipotent element $u \in G$ of order $> p$ acts on $L_G(\lambda)$ as a distinguished unipotent element if and only if u is regular and one of the following holds:*

(i) $\lambda = \omega_1$ *or* $\lambda = \omega_2$.

(ii) $p = 5$ *and* $\lambda = 2\omega_1$.

(iii) $p = 5$ *and* $\lambda = \omega_1 + 2\omega_2$.

Proof. In $G = G_2$, there are unipotent elements of order $> p$ if and only if $p \leq 5$ (see Appendix A). Furthermore, when $2 < p \leq 5$, only the regular unipotent element has order $> p$. Therefore it will be enough to only consider regular unipotent elements of G.

Let $u \in G$ be a regular unipotent element and let $\lambda \in X(T)^+$ be p-restricted. If $p = 3$, then u has order 3^2. Thus if u acts on $L_G(\lambda)$ as a distinguished unipotent element, then $\dim L_G(\lambda) \leq \frac{(3^2+1)^2}{4} = 25$ by Lemma 5.1.1. By [Lüb01], this implies that $\lambda = \omega_1$ or $\lambda = \omega_2$. In both cases u acts on $L_G(\lambda)$ with a single Jordan block of size 7 (Proposition 1.1.12).

Suppose then that $p = 5$. Now u has order 5^2, so if u acts on $L_G(\lambda)$ as a distinguished unipotent element, then $\dim L_G(\lambda) \leq \frac{(5^2+1)^2}{4} = 169$ by Lemma 5.1.1. It follows from the results in [Lüb01] that λ occurs in Table 5.10, where we have also given the decomposition of $L_G(\lambda) \downarrow K[u]$. This data was obtained by a calculation with MAGMA (Section 2.9). We see that u acts as a distinguished unipotent element precisely in the cases $\lambda = \omega_1$, $\lambda = \omega_2$, $\lambda = 2\omega_1$, and $\lambda = \omega_1 + 2\omega_2$. □

λ	$\dim L_G(\lambda)$	$L_G(\lambda) \downarrow K[u]$
ω_1	7	$[\mathbf{7}]$
ω_2	14	$[\mathbf{3}, \mathbf{11}]$
$2\omega_1$	27	$[\mathbf{5}, \mathbf{9}, \mathbf{13}]$
$\omega_1 + \omega_2$	64	$[5, 7, 10^2, 15, 17]$
$3\omega_1$	77	$[5^2, 10^2, 13, 15, 19]$
$2\omega_2$	77	$[1, 5, 10^2, 15^2, 21]$
$\omega_1 + 2\omega_2$	97	$[\mathbf{13}, \mathbf{15}, \mathbf{21}, \mathbf{23}, \mathbf{25}]$

Table 5.10: Action of a regular unipotent element u of $G = G_2$ on some small irreducible representations for $p = 5$.

5.1.5 Types F_4, E_6, E_7, and E_8

Suppose that G is simple of type F_4, E_6, E_7, or E_8. In this situation our reduction is essentially an application of Lemma 5.1.1 and results due to Lübeck, cf. Lemma 5.1.9, Lemma 5.1.10, and Lemma 5.1.15.

Lemma 5.1.20. *Suppose that G is simple of type F_4, E_6, E_7, or E_8. Let $\lambda \in X(T)^+$ be p-restricted. If some unipotent element of G with order $> p$ acts on $L_G(\lambda)$ as a distinguished unipotent element, then $\lambda = \omega_i$.*

Proof. Let $u \in G$ be a regular unipotent element. If some unipotent element $u \in G$ acts on $L_G(\lambda)$ as a distinguished unipotent element, then by Lemma 5.1.1 we have $\dim L_G(\lambda) \leq M$, where $M = \frac{(|u|+1)^2}{4}$. We give the maximal possible value of $|u|$ and M (when u has order $> p$) for G of type F_4, E_6, E_7, and E_8 in Table 5.11.

We have listed the $\lambda \neq \omega_i$ such that $\dim L_G(\lambda) \leq M$ in tables 5.12 - 5.15, based on [Lüb17]. We have also included the value of $m_u(\lambda)$, which was computed using Lemma 2.7.3. One can verify that

$$\dim L_G(\lambda) > \frac{(m_u(\lambda)+2)^2}{4}$$

for all λ in tables 5.12 - 5.15, so by Lemma 5.1.1 for these λ no unipotent element of G acts on $L_G(\lambda)$ as a distinguished unipotent element, as desired. □

G	Largest possible value of $\|u\|$	Largest possible value of M
F_4	11^2	3721
E_6	11^2	3721
E_7	17^2	21025
E_8	29^2	177241

Table 5.11: For $G = F_4$ and $G = E_l$, largest possible order $|u|$ of a regular unipotent element u of order $> p$, and largest possible value of $M = \frac{(|u|+1)^2}{4}$.

λ	Minimal possible value of $\dim L_G(\lambda)$	$m_u(\lambda)$
$2\omega_4$	298	32
$2\omega_1$	755	44
$\omega_1+\omega_4$	1053	38
$\omega_3+\omega_4$	2404	46
$3\omega_4$	2651	48

Table 5.12: Type F_4, irreducible representations $L_G(\lambda)$ of dimension ≤ 3721, with $\lambda \neq \omega_i$ and $2 < p \leq 11$.

λ	Minimal possible value of $\dim L_G(\lambda)$	$m_u(\lambda)$
$2\omega_1$	324	32
$\omega_1+\omega_6$	572	32
$\omega_1+\omega_2$	1377	38
$\omega_1+\omega_3$	2404	46
$2\omega_2$	2430	44
$3\omega_1$	3002	48

Table 5.13: Type E_6, irreducible representations $L_G(\lambda)$ of dimension ≤ 3721 up to graph automorphism, with $\lambda \neq \omega_i$ and $2 < p \leq 11$.

λ	Minimal possible value of $\dim L_G(\lambda)$	$m_u(\lambda)$
$2\omega_7$	1330	54
$\omega_1+\omega_7$	5568	61
$2\omega_1$	5832	68
$3\omega_7$	18752	81

Table 5.14: Type E_7, irreducible representations $L_G(\lambda)$ of dimension ≤ 21025, with $\lambda \neq \omega_i$ and $2 < p \leq 17$.

λ	Minimal possible value of $\dim L_G(\lambda)$	$m_u(\lambda)$
$2\omega_8$	23125	116

Table 5.15: Type E_8, irreducible representations $L_G(\lambda)$ of dimension ≤ 177241, with $\lambda \neq \omega_i$ and $2 < p \leq 29$.

5.2 Reduction ($p = 2$)

Assume that $p = 2$.

In this section, we reduce the proof of Theorem 1.1.11 to a small number of λ to consider. The reduction is based on the following elementary observation and the results of Lübeck [Lüb01] on small-dimensional irreducible representations of G.

Lemma 5.2.1. *Let $u \in G$ be a unipotent element and $\varphi : G \to \mathrm{GL}(V)$ a representation of G. Let d be the size of the largest Jordan block of $\varphi(u)$. If u acts on V as a distinguished unipotent element, then $\dim V \leq \frac{d(d+2)}{2}$.*

Proof. We know that if u acts on V as a distinguished unipotent element, then either $\varphi(u)$ is a single Jordan block, or all Jordan block sizes of $\varphi(u)$ are even with multiplicity ≤ 2 (Lemma 2.4.4 (ii)). If $\varphi(u)$ is a single Jordan block, then $\dim V = d \leq \frac{d(d+2)}{2}$. If all Jordan blocks of $\varphi(u)$ are even with multiplicity ≤ 2, then

$$\dim V \leq d + d + (d-2) + (d-2) + \cdots + 2 + 2 = \frac{d(d+2)}{2}$$

as desired. □

We give the reduction for each simple type in the subsections that follow.

5.2.1 Type A_l

In this subsection, assume that G is simple of type A_l, $l \geq 1$.

Proposition 5.2.2. *Let $u \in G$ be regular and let $\lambda \in X(T)^+$ be nonzero 2-restricted. If u acts on $L(\lambda)$ as a distinguished unipotent element, then one of the following holds.*

(i) $\lambda = \omega_1$ *or* $\lambda = \omega_l$.

(ii) $l \geq 2$ *and* $\lambda = \omega_1 + \omega_l$.

(iii) $l = 5$ *and* $\lambda = \omega_3$.

(iv) $l = 3$ *and* $\lambda = \omega_2$.

Proof. If $l = 1$, there is nothing to do, so we will assume that $l \geq 2$. By Proposition 1.1.12 we can assume that $L(\lambda)$ is self-dual, that is, $\lambda = -w_0\lambda$. Suppose that u acts on $L(\lambda)$ as a distinguished unipotent element.

Now u acts on the natural module of G with a single Jordan block of size $l+1$. Therefore the order of u is equal to $M_l := 2^{s+1}$, where $2^s < l+1 \leq 2^{s+1}$. Then in any representation of G, the action of u has largest Jordan block of size $\leq 2^{s+1} = 2 \cdot 2^s \leq 2l$. Thus by Lemma 5.2.1 we have $\dim L(\lambda) \leq \frac{M_l(M_l+2)}{2} \leq \frac{2l(2l+2)}{2} = 2l^2 + 2l$.

Now $2l^2 + 2l < \frac{l^3}{8}$ if $l \geq 17$, so by [Lüb01, Theorem 5.1] we have $\lambda = \omega_1 + \omega_l$ if $l \geq 17$. In the cases where $2 \leq l \leq 16$, checking irreducible representations of dimension $\leq \frac{M_l(M_l+2)}{2}$ in the tables given in [Lüb01], we find that the only self-dual ones besides $L(\omega_1 + \omega_l)$ are $L(\omega_3)$ in the case $l = 5$, and $L(\omega_2)$ in the case $l = 3$. □

Lemma 5.2.3. *Let $u \in G$ be regular. Then*

(i) *For $l = 5$, we have the orthogonal decomposition $L_G(\omega_3) \downarrow K[u] = V(2)^2 + W(8)$ (Proposition 2.4.4) with respect to any non-degenerate G-invariant alternating bilinear form on $L_G(\omega_3)$. In particular, the element u does not act as a distinguished unipotent element on $L_G(\omega_3)$.*

(ii) *For $l = 3$, we have the orthogonal decomposition $L_G(\omega_2) \downarrow K[u] = V(2) + V(4)$ (Proposition 2.4.4) with respect to any non-degenerate G-invariant alternating bilinear form on $L_G(\omega_2)$. In particular, the element u acts as a distinguished unipotent element on $L_G(\omega_2)$.*

Proof. Both (i) and (ii) can be verified by a computation with MAGMA (Section 2.9). Claim (ii) also follows from the fact that the image of the representation $\rho : \mathrm{SL}_4(K) \to \mathrm{GL}(L_G(\omega_2))$ is equal to $\mathrm{SO}_4(K)$, and that a regular unipotent element of $\mathrm{SO}_4(K)$ has decomposition $V(2) + V(4)$ on the natural module (Proposition 2.4.4 (vi)). □

Therefore for type A_l, the claim of Theorem 1.1.11 is reduced to the case $\lambda = \omega_1 + \omega_l$, which will be considered in Section 5.4.

5.2.2 Types B_l, C_l and D_l

In this subsection, assume that G is of type B_l ($l \geq 2$), type C_l ($l \geq 2$), or of type D_l ($l \geq 4$).

Proposition 5.2.4. *Let $u \in G$ be distinguished and let $\lambda \in X(T)^+$ be 2-restricted. If u acts on $L(\lambda)$ as a distinguished unipotent element, then one of the following holds.*

(i) $\lambda = \omega_1$ *or* $\lambda = \omega_2$.

(ii) $G = B_l$ *or* $G = C_l$, *and* $\lambda = \omega_l$ *where* $3 \leq l \leq 7$, *or* $l = 9$.

(iii) $G = D_l$ *and* $\lambda = \omega_l$ *or* $\lambda = \omega_{l-1}$, *where* $l = 4$, $l = 6$, $l = 8$, *or* $l = 10$.

(iv) $l = 5$ *and* $\lambda = \omega_3$.

Proof. Now u acts on the natural representation of G with largest Jordan block of size $\leq 2l$. So if $2^s < 2l \leq 2^{s+1}$, the element u must have order $\leq 2^{s+1} = 2 \cdot 2^s \leq 4l$. Therefore by Lemma 5.2.1, we have $\dim L(\lambda) \leq \frac{2^{s+1}(2^{s+1}+2)}{2} \leq \frac{4l(4l+2)}{2} = 8l^2 + 4l$.

Now $8l^2 + 4l < l^3$ if $l \geq 9$, so by [Lüb01, Theorem 5.1] we have $\lambda = \omega_1$ or $\lambda = \omega_2$ if $l \geq 12$. In the cases where $2 \leq l \leq 11$, checking irreducible, self-dual representations of dimension $\leq \frac{2^{s+1}(2^{s+1}+2)}{2}$ given in [Lüb01], we find that the the only ones besides ω_1 and ω_2 are those in (ii), (iii) and (iv) of the claim. □

Lemma 5.2.5. *Suppose that G is of type C_5 and let $u \in G$ be a distinguished unipotent element. Then u does not act as a distinguished unipotent element on $L(\omega_3)$.*

Proof. We have $\dim L(\omega_3) = 100$ for example by [Lüb01]. Thus if u has order $\leq 2^3$, then by Lemma 5.2.1 the action of u is not distinguished on $L(\omega_3)$ (because $\frac{2^3(2^3+2)}{2} = 40 < 100$). The only distinguished unipotent element of G with order $> 2^3$ is the regular unipotent element. A computation with MAGMA (Section 2.9) shows that the regular unipotent acts on $L(\omega_3)$ with Jordan blocks $[6^2, 10, 14, 16^4]$, and thus the action is not distinguished. □

Therefore for types B_l, C_l and D_l, the claim of Theorem 1.1.11 is reduced to the cases $\lambda = \omega_2$ and $\lambda = \omega_l$ ($l \leq 10$), which will be dealt with in Section 5.6 and Section 5.11, respectively.

5.2.3 Exceptional types

In this subsection we assume that G is of exceptional type.

Proposition 5.2.6. *Let $u \in G$ be distinguished and let $\lambda \in X(T)^+$ be 2-restricted. If u acts on $L(\lambda)$ as a distinguished unipotent element, then one of the following holds.*

(i) *$G = G_2$ and $\lambda = \omega_1$ or $\lambda = \omega_2$.*

(ii) *$G = F_4$ and $\lambda = \omega_1$ or $\lambda = \omega_4$.*

(iii) *$G = E_6$ and $\lambda = \omega_2$.*

(iv) *$G = E_7$ and $\lambda = \omega_1$ or $\lambda = \omega_7$.*

(v) *$G = E_8$ and $\lambda = \omega_8$.*

Proof. As in the proof of Proposition 5.2.2, we can assume that $L(\lambda)$ is self-dual.

When G is of type G_2, F_4, E_6, E_7 or E_8, then all distinguished unipotent elements of G have order at most 2^3, 2^4, 2^4, 2^5, and 2^5 respectively (see tables in Appendix A). Then by Lemma 5.2.1, the dimension of $L(\lambda)$ is at most 40, 144, 144, 544, and 544 respectively. Going through the tables given in [Lüb01], one finds that the irreducible self-dual 2-restricted modules of dimension at most the bound are those in (i), (ii), (iii), (iv), and (v) respectively. □

For groups of exceptional type, Proposition 5.2.6 reduces the claim of Theorem 1.1.11 to a small number of λ to consider. These representations are treated in Appendix B.

5.3 Representation $L_G(\omega_1 + \omega_l)$ for G of type A_l $(p \neq 2)$

Assume that $p \neq 2$ and that G is of type A_l $(l \geq 2)$.

Let V be the natural module of G and set $n = l + 1 = \dim V$.

The purpose of this section is to determine when a unipotent element $u \in G$ acts as a distinguished unipotent element on $L(\omega_1 + \omega_l)$.

It is well known that we can find $L(\omega_1 + \omega_l)$ as a subquotient of $V \otimes V^*$, as shown by the following lemma which also holds when $p = 2$. For a proof, one can apply [Sei87, Lemma 8.6] as in [McN98, Proposition 4.6.10 (a)].

Lemma 5.3.1. *As G-modules, we have*

$$V \otimes V^* \cong \begin{cases} L(\omega_1 + \omega_l) \oplus L(0) & \text{if } p \mid n \\ L(0)/L(\omega_1 + \omega_l)/L(0) \ (\text{uniserial}) & \text{if } p \nmid n \end{cases}$$

We will now determine when a regular unipotent $u \in G$ acts as a distinguished unipotent element on $L(\omega_1 + \omega_l)$.

Proposition 5.3.2. *Let $u \in G$ be regular. Then u acts on $L(\omega_1 + \omega_l)$ as a distinguished unipotent element if and only if one of the following holds:*

(i) $p \geq 2n - 1$,

(ii) $n = bp^k + \frac{p^k \pm 1}{2}$ *for some* $k \geq 1$ *and* $0 \leq b \leq \frac{p-1}{2}$.

In all of these cases, u acts on $L(\omega_1 + \omega_l)$ with Jordan blocks $[2n - 1, 2n - 3, \ldots, 3]$.

Proof. Recall the notation for $K[u]$-modules from Section 1.4. As a $K[u]$-module, we have $V \cong V_n$ and so $V \otimes V^* \cong V_n \otimes V_n$.

Suppose first that $p \mid n$, so now $V \otimes V^* \cong L(0)/L(\omega_1 + \omega_l)/L(0)$ as a G-module. By Theorem 3.3.5 and Lemma 3.4.3, the unipotent element u acts on $V \otimes V^*$ with p^α Jordan blocks of size p^α, where $\alpha = \nu_p(n)$. Thus the action of u on $V \otimes V^*$ is inadmissible (Definition 3.2.4), so by Lemma 3.2.6 the unipotent element u does not act on $L(\omega_1 + \omega_l)$ as a distinguished unipotent element.

Therefore it will be enough to consider the case where $p \nmid n$. Now $V \otimes V^* \cong L(\omega_1 + \omega_l) \oplus L(0)$ as a G-module. Since in $V_n \otimes V_n$ block size 1 occurs with multiplicity 1 (Lemma 3.4.3), it follows that u acts on $L(\omega_1 + \omega_l)$ as a distinguished unipotent element if and only if $V_n \otimes V_n$ has no repeated blocks. Since $p \neq 2$, we have $V_n \otimes V_n \cong \wedge^2(V_n) \oplus S^2(V_n)$ as a $K[u]$-module, so the claim follows from Proposition 3.5.3 and Proposition 3.4.4. □

5.4 Representation $L_G(\omega_1 + \omega_l)$ for G of type A_l $(p = 2)$

Assume that $p = 2$ and that G is of type A_l $(l \geq 2)$.

Let V be the natural module of G.

The purpose of this section is to determine when a unipotent element $u \in G$ acts as a distinguished unipotent element on $L(\omega_1 + \omega_l)$. The answer is given by Proposition 5.4.4; we have also recorded this in Table 1.3.

As a special case of Lemma 5.3.1, we have the following.

Lemma 5.4.1. *As G-modules, we have*

$$V \otimes V^* \cong \begin{cases} L(\omega_1 + \omega_l) \oplus L(0) & \textit{if } \dim V \textit{ is odd} \\ L(0)/L(\omega_1 + \omega_l)/L(0) \textit{ (uniserial)} & \textit{if } \dim V \textit{ is even} \end{cases}$$

We will also be applying the following theorem, which tells how a tensor square $V_n \otimes V_n$ splits into a sum of indecomposables. It could be easily deduced with the main result of [Bar11] (Theorem 3.3.5), but it is also given in [GL06, Corollary 3].

Theorem 5.4.2. *Let $n = 2^k + s$, where $k \geq 0$ and $0 \leq s < 2^k$. Then*

(i) *If $s = 0$, then $V_n \otimes V_n = n \cdot V_n$.*

(ii) *If $s > 0$, then $V_n \otimes V_n = (V_{n-2s} \otimes V_{n-2s}) \oplus 2s \cdot V_{2^{k+1}}$.*

Lemma 5.4.3. *All Jordan blocks in $V_n \otimes V_n$ have multiplicity ≤ 3 if and only if $n = 1, 2, 3$ or $n = 5$.*

Proof. If $n \leq 5$, we can easily decompose $V_n \otimes V_n$ using Theorem 5.4.2 and verify the claim. We have given these decompositions in Table 5.1 for convenience.

Suppose then that $n > 5$ and write $n = 2^k + s$, where $k \geq 2$ and $0 \leq s < 2^k$. If $s = 0$, then by Theorem 5.4.2 (i) the tensor square $V_n \otimes V_n$ has $n \geq 4$ blocks of size n. Consider then $0 < s < 2^k$, so now

$$V_n \otimes V_n = (V_{n-2s} \otimes V_{n-2s}) \oplus 2s \cdot V_{2^{k+1}}$$

by Theorem 5.4.2 (ii). Therefore if $s \geq 2$, then $V_n \otimes V_n$ has ≥ 4 blocks of size 2^{k+1}. What remains is the case where $s = 1$. In this case, we have $n - 2s = 2^k - 1 = 2^{k-1} + (2^{k-1} - 1)$ and thus by theorem 5.4.2 (ii) the tensor square $V_{n-2s} \otimes V_{n-2s}$ has $2(2^{k-1} - 1) = 2^k - 2 \geq 4$ blocks of size 2^k (we have $k \geq 3$ since we are assuming $n > 5$). □

Proposition 5.4.4. *Suppose that $G = A_l$, $l \geq 2$ and let $u \in G$ be a regular unipotent element. Then u acts on $L(\omega_1 + \omega_l)$ as a distinguished unipotent element if and only if $l = 2$ or $l = 4$.*

Proof. We know from lemma 5.4.1 that as a G-module, $V \otimes V^* = L(\omega_1 + \omega_l) \oplus L(0)$ if $2 \nmid l + 1$ and $V \otimes V^* = L(0)/L(\omega_1 + \omega_l)/L(0)$ if $2 \mid l + 1$.

Now u acts on V with a single Jordan block of size $l + 1$. Thus if $l = 3$ or if $l \geq 5$, then by Lemma 5.4.3 the action of u on $V \otimes V^*$ has some Jordan block with multiplicity ≥ 4. Hence the action of u on $V \otimes V^*$ is inadmissible (Definition 3.2.5), so it follows from Lemma 3.2.7 and Lemma 3.2.8 that u does not act on $L(\omega_1 + \omega_l)$ as a distinguished unipotent element in this case.

What remains is to consider $l = 2$ and $l = 4$. For these cases, a computation with MAGMA (Section 2.9) shows the following.

- If $l = 2$, then we have the orthogonal decomposition (Proposition 2.4.4) $L(\omega_1 + \omega_2) \downarrow K[u] = V(4)^2$.

- If $l = 4$, then we have the orthogonal decomposition (Proposition 2.4.4) $L(\omega_1 + \omega_4) \downarrow K[u] = V(4)^2 + V(8)^2$.

Therefore in both of these cases u acts on $L(\omega_1 + \omega_l)$ as a distinguished unipotent element (Proposition 2.4.4), as claimed. □

n	$V_n \otimes V_n$
1	$[1]$
2	$[2^2]$
3	$[1, 4^2]$
4	$[4^4]$
5	$[1, 4^2, 8^2]$

Table 5.1: Decomposition of $V_n \otimes V_n$ for $1 \leq n \leq 5$.

5.5 Representation $L_G(\omega_2)$ for G of classical type ($p \neq 2$)

Assume that $p \neq 2$.

Suppose that G is of type B_l, C_l or D_l with natural module V. In this section, we will determine when a unipotent element $u \in G$ acts on $L(\omega_2)$ as a distinguished unipotent element. We will show that this can happen only when u is a regular unipotent element. Furthermore, with the unique exception of type C_3 with $p = 3$, we will see that u acts on $L(\omega_2)$ as a distinguished unipotent element only if it acts on $\wedge^2(V)$ with no repeated blocks.

5.5.1 Construction of $L_G(\omega_2)$

In this subsection, we will describe the well known construction of $L_G(\omega_2)$ for $G = \mathrm{Sp}(V)$ (type C_l) and $G = \mathrm{SO}(V)$ (type B_l or D_l). For types B_l and D_l this is easy, as seen in the following lemma.

Lemma 5.5.1. *Let* $G = \mathrm{SO}(V)$. *Then* $\wedge^2(V) \cong L_G(\omega_2)$.

Proof. This is a consequence of [Sei87, 8.1 (a), 8.1 (b)], see for example [McN98, Proposition 4.2.2]. □

For the rest of this subsection, let $G = \mathrm{Sp}(V)$, where $\dim V = 2l$ ($l \geq 2$). Let $(-,-)$ be a G-invariant alternating bilinear form on V. The following lemma shows that the exterior square $\wedge^2(V)$ is not irreducible, but we can find the representation $L_G(\omega_2)$ as a subquotient of $\wedge^2(V)$.

Lemma 5.5.2. *As G-modules, we have*

$$\wedge^2(V) \cong \begin{cases} L(\omega_2) \oplus L(0) & \textit{if } p \nmid l, \\ L(0)/L(\omega_2)/L(0) \ \textit{(uniserial)} & \textit{if } p \mid l. \end{cases}$$

Proof. See for example [McN98, Lemma 4.8.2]. □

We now proceed to describe the submodule structure of $\wedge^2(V)$ explicitly. Fix a symplectic basis $e_1, \ldots, e_{2l}$ of V such that

$$(e_i, e_j) = \begin{cases} (-1)^i & \text{if } i + j = 2l + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Define $\gamma \in \wedge^2(V)$ by

$$\gamma = \sum_{i=1}^{l} (-1)^i e_i \wedge e_{2l+1-i}.$$

Then γ spans the unique one-dimensional G-submodule of $\wedge^2(V)$, as seen by the following lemma.

Lemma 5.5.3. *The element γ is fixed by the action of G on $\wedge^2(V)$.*

Proof. The form $(-,-)$ induces a G-module isomorphism $V \to V^*$ defined by $v \mapsto (v, -)$. This in turn induces an isomorphism $\psi : \wedge^2(V) \to \wedge^2(V^*)$ of G-modules.

Let $\mathrm{Alt}(V)$ be the space of alternating bilinear forms on V. Then $\mathrm{Alt}(V)$ is a G-module with the action defined by $(g \cdot \beta)(v, w) = \beta(g^{-1}v, g^{-1}w)$ for all $\beta \in \mathrm{Alt}(V)$, $g \in G$, and $v, w \in V$. Now there is an isomorphism $\chi : \wedge^2(V^*) \to \mathrm{Alt}(V)$ of G-modules, defined by $\chi(f \wedge f')(v, w) = f(v)f'(w) - f(w)f'(v)$ for all $f, f' \in V^*$ and $v, w \in V$.

Therefore it is enough to show that the alternating bilinear form $\chi\psi(\gamma) = \beta$ is fixed by the action of G. That is, we should show that β is a G-invariant alternating bilinear form on V. To this end, a straightforward calculation on the basis elements e_i shows that $\beta(v, w) = (v, w)$ for all $v, w \in V$. □

We define the linear map $\varphi : \wedge^2(V) \to K$ by $\varphi(v \wedge w) = (v, w)$ for all $v, w \in V$; it is easily seen that φ is a surjective morphism of G-modules.

Now we can use the G-submodules $\ker \varphi$ and $\langle \gamma \rangle$ to describe the submodule structure of $\wedge^2(V)$. Note that we have $\varphi(\gamma) = l$. Therefore if $p \nmid l$, then $\gamma \notin \ker \varphi$ and so $\wedge^2(V) = \ker \varphi \oplus \langle \gamma \rangle$ as a G-module. In this case $\ker \varphi \cong L(\omega_2)$ by Lemma 5.5.2. If $p \mid l$, then $\gamma \in \ker \varphi$, and thus $\ker \varphi / \langle \gamma \rangle \cong L(\omega_2)$ by Lemma 5.5.2.

5.5.2 Types B_l and D_l

Lemma 5.5.4. *Let $G = \mathrm{SO}(V)$ and let $u \in G$ be a unipotent. Suppose that u is a distinguished unipotent element of G, so as a $K[u]$-module $V = V_{d_1} \oplus \cdots \oplus V_{d_t}$, where the d_i are distinct and odd (Proposition 2.3.4). Assume that $t > 1$. Then u acts on $\wedge^2(V)$ with no repeated blocks if and only if $t = 2$, and $V = V_d \oplus V_1$, where $\wedge^2(V_d)$ has no repeated blocks and $d \equiv 1 \mod 4$.*

Proof. Suppose that u acts on $\wedge^2(V)$ with no repeated blocks. Now as a $K[u]$-module, $\wedge^2(V)$ has $\wedge^2(V_{d_1}) \oplus \cdots \oplus \wedge^2(V_{d_t})$ as a direct summand by Lemma 3.4.9. Therefore u acts on $\wedge^2(V_{d_i})$ with no repeated blocks for all i, and by Proposition 3.5.3, each $\wedge^2(V_{d_i})$ decomposes as in characteristic 0. In particular, since all d_i are odd, for all $d_i > 1$ we have a block of size 3 in $\wedge^2(V_{d_i})$ (Proposition 3.4.4). Thus $d_i > 1$ for at most one i, which implies $t = 2$ and $V = V_d \oplus V_1$ for some odd $d > 1$. By Lemma 3.4.9, we have

$$\wedge^2(V) = \wedge^2(V_d) \oplus V_d$$

as a $K[u]$-module. Since $\wedge^2(V_d)$ has no repeated blocks, it decomposes as in characteristic 0 (Proposition 3.5.3). In particular, by Proposition 3.4.4, the $K[u]$-module $\wedge^2(V_d)$ has V_d as a summand if and only if $2d-3 \equiv d \mod 4$, which happens if and only if $d \equiv 3 \mod 4$. Therefore u acts on $\wedge^2(V)$ with no repeated blocks if and only if $d \equiv 1 \mod 4$. □

Proposition 5.5.5 (Type B_l). *Let G be the orthogonal group* $\mathrm{SO}(V)$, *where* $\dim V = 2l+1$ *($l \geq 2$). Then*

(a) *A non-regular unipotent element of G does not act as a distinguished unipotent element in the representation $L(\omega_2)$.*

(b) *A regular unipotent element of G acts as a distinguished unipotent element in $L(\omega_2)$ if and only if one of the following holds:*

 (i) $p \geq 4l-1$,

 (ii) $2l+1 = bp^k + \frac{p^k \pm 1}{2}$, *where* $k \geq 1$ *and* $0 \leq b \leq \frac{p-1}{2}$,

 (iii) $2l+1 = p + \frac{p-3}{2}$,

Proof. Recall that $\wedge^2(V) \cong L(\omega_2)$ (Lemma 5.5.1). Let $u \in G$ be a distinguished unipotent element.

If u is not regular, then the fact that $\dim V$ is odd implies that the action of u on V has ≥ 3 Jordan blocks (Proposition 2.3.2). Then by Lemma 5.5.4 the action of u on $\wedge^2(V)$ has repeated blocks, and thus the action is not distinguished, proving (a).

Suppose that u is regular, so now $V \downarrow K[u] = V_{2l+1}$. Then u acts on $L(\omega_2)$ as a distinguished unipotent element if and only if u acts on $\wedge^2(V)$ with no repeated blocks, so the claim follows from Proposition 3.5.3. □

Proposition 5.5.6 (Type D_l). *Let G be the orthogonal group* $\mathrm{SO}(V)$, *where* $\dim V = 2l$ *($l \geq 4$). Then*

(a) *A non-regular unipotent element of G does not act as a distinguished unipotent element in the representation $L(\omega_2)$.*

(b) *A regular unipotent element of G acts as a distinguished unipotent element in $L(\omega_2)$ if and only if l is odd and one of the following holds:*

 (i) $p \geq 4l-5$,

 (ii) $2l-1 = bp^k + \frac{p^k \pm 1}{2}$, *where* $k \geq 1$ *and* $0 \leq b \leq \frac{p-1}{2}$,

 (iii) $2l-1 = p + \frac{p-3}{2}$,

Proof. Recall that $\wedge^2(V) \cong L(\omega_2)$ (Lemma 5.5.1). Let $u \in G$ be a distinguished unipotent element. If u is not regular, then u acts on V with ≥ 2 Jordan blocks of size > 1 (Proposition 2.3.3). Then by Lemma 5.5.4, the action of u on $\wedge^2(V)$ has repeated blocks and so the action is not distinguished, which proves (a).

Suppose next that u is regular, so $V \downarrow K[u] = V_{2l-1} \oplus V_1$ (Proposition 2.3.3). In this case the claim follows from Lemma 5.5.4 and Proposition 3.5.3, since u acts on $L(\omega_2)$ as a distinguished unipotent element if and only if u acts on $\wedge^2(V)$ with no repeated blocks. □

5.5.3 Type C_l

Lemma 5.5.7. *Let $G = \mathrm{Sp}(V)$ and let $u \in G$ be a unipotent. Suppose that u is a distinguished unipotent element of G, so as a $K[u]$-module $V = V_{d_1} \oplus \cdots \oplus V_{d_t}$, where $0 < d_1 < \cdots < d_t$ are even (Proposition 2.3.4). If $t > 1$, then u acts on $\wedge^2(V)$ with some block of size > 1 having multiplicity ≥ 2.*

Proof. It follows from Lemma 3.4.9 that $\wedge^2(V) \downarrow K[u]$ has $\wedge^2(V_{d_1} \oplus V_{d_2})$ as a direct summand. Thus it will be enough to prove the claim in the case where $t = 2$, say $V = V_m \oplus V_n$ with $0 < m < n$ even. Suppose that u acts on $\wedge^2(V)$ such that all blocks of size > 1 have multiplicity ≤ 1. Now $\wedge^2(V) \downarrow K[u] = \wedge^2(V_m) \oplus (V_m \otimes V_n) \oplus \wedge^2(V_n)$, so by Proposition 3.5.3 the action of u on $\wedge^2(V_m)$ and $\wedge^2(V_n)$ is as in characteristic 0.

Note that since m and n are both even, we have $2m - 3 \equiv 2n - 3 \equiv 1 \mod 4$, so every block in $\wedge^2(V_m)$ must occur in $\wedge^2(V_n)$ by Proposition 3.4.4. In particular if $m \geq 4$, then both $\wedge^2(V_m)$ and $\wedge^2(V_n)$ have a block of size 5. Thus $\wedge^2(V)$ has ≥ 2 blocks of size 5, contradiction.

If $m = 2$, then we have $\wedge^2(V) = \wedge^2(V_n) \oplus (V_n \otimes V_2) \oplus V_1$. Now by Lemma 3.3.9 we have $V_n \otimes V_2 = V_{n-1} \oplus V_{n+1}$ if $p \nmid n$ and $V_n \otimes V_2 = V_n \oplus V_n$ if $p \mid n$. We have $n \geq 4$, so $n + 1 \leq 2n - 3$ and therefore either V_{n-1} or V_{n+1} occurs in $\wedge^2(V_n)$. Thus either a block of size $n - 1$, n or $n + 1$ has multiplicity 2 in $\wedge^2(V)$, contradiction. □

Lemma 5.5.8. *Let $G = \mathrm{Sp}(V)$ and let $u \in G$ unipotent. Suppose that u is a distinguished unipotent element of G, so as a $K[u]$-module $V = V_{d_1} \oplus \cdots \oplus V_{d_t}$, where the d_i are distinct and even (Proposition 2.3.4). Assume that $p \mid \dim V$, so $\wedge^2(V) = L(0)/L(\omega_2)/L(0)$ as a G-module (Lemma 5.5.2). Let u_0 be the image of u in $\mathrm{SL}(\wedge^2(V))$ and u_0'' the image of u in $\mathrm{SL}(L(\omega_2))$. Let $\alpha = \nu_p(\gcd(d_1, \ldots, d_t))$. Then for the Jordan block sizes $r_m(u_0'')$ (Definition 3.1.1) the following hold:*

(a) *If $\alpha = 0$, then $r_1(u_0'') = r_1(u_0) - 2$ and $r_m(u_0'') = r_m(u_0)$ for all $m > 1$.*

(b) *If $\alpha > 0$, then one of the following holds:*

  (i) *$r_{p^\alpha}(u_0'') = r_{p^\alpha}(u_0) - 1$, $r_{p^\alpha - 2}(u_0'') = 1$, and $r_m(u_0'') = r_m(u_0)$ for all $m \neq p^\alpha, p^\alpha - 2$.*

  (ii) *$r_{p^\alpha}(u_0'') = r_{p^\alpha}(u_0) - 2$, $r_{p^\alpha - 1}(u_0'') = 2$, and $r_m(u_0'') = r_m(u_0)$ for all $m \neq p^\alpha, p^\alpha - 1$.*

Proof. We begin by constructing u as in Section 2.5. Let $V = W_1 \oplus \cdots \oplus W_t$, an orthogonal direct sum, with $\dim W_i = d_i$. For each i, let $e_1^{(i)}, \ldots, e_{d_i}^{(i)}$ be a basis for W_i such that

$$(e_x^{(i)}, e_y^{(i)}) = \begin{cases} (-1)^x & \text{if } x + y = d_i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Set $e_x = 0$ for all $x \leq 0$. Now define a linear map $u : V \to V$ by $(u - 1)e_x^{(i)} = (u + 1)e_{x-1}^{(i)}$ for all $1 \leq i \leq t$ and $1 \leq x \leq d_i$. Then $u \in \mathrm{Sp}(V)$ (see Section 2.5) and $V \downarrow K[u] \cong V_{d_1} \oplus \cdots \oplus V_{d_t}$ with $W_i \cong V_{d_i}$.

Let $\varphi : \wedge^2(V) \to K$ and $\gamma \in \wedge^2(V)$ be as in subsection 5.5.1, so now $\ker \varphi / \langle \gamma \rangle \cong L(\omega_2)$ as G-modules.

Without loss of generality, assume that $\alpha = \nu_p(d_1)$. Set $d = d_1$ and $e_x = e_x^{(1)}$ for all $1 \leq x \leq d$. Denote the restriction of u_0 to $\ker \varphi$ by u_0'. We can and will consider u_0'' to be the map induced by u_0 on $\ker \varphi / \langle \gamma \rangle$.

We consider first the case where $\alpha = 0$. Now

$$\gamma_1 = \sum_{j=1}^{d/2} (-1)^j e_j \wedge e_{d-j+1}$$

is a fixed point for u_0 by Lemma 5.5.3. Furthermore, we have $\varphi(\gamma_1) = d/2$, so $\gamma_1 \notin \ker \varphi$ since p does not divide d. Thus $\ker(u_0 - 1) \not\subseteq \ker \varphi$, so by Lemma 3.2.1 we have $r_1(u_0') = r_1(u_0) - 1$ and $r_m(u_0') = r_m(u_0)$ for all $m > 1$.

Since $\wedge^2(V)$ admits a non-degenerate G-invariant symmetric form (Lemma 4.4.7), it follows from Proposition 2.3.2 (ii) that $r_m(u_0) = r_m(u_0')$ is even if $m > 0$ is even. On the other hand, we also have a non-degenerate G-invariant symmetric form on $\ker \varphi / \langle \gamma \rangle \cong L(\omega_2)$ (see Table 4.1), so $r_m(u_0'')$ is even if $m > 0$ is even, as well. Thus when we look at the Jordan blocks of u_0'' in terms of those of u_0' using Lemma 3.2.2, we see that the situation in Lemma 3.2.2 (b) cannot occur. Indeed, otherwise $r_m(u_0'')$ would be odd for some even $m > 0$. Thus we have $r_1(u_0'') = r_1(u_0') - 1 = r_1(u_0) - 2$ and $r_m(u_0'') = r_m(u_0') = r_m(u_0)$ for all $m > 1$, as desired.

Consider then the case where $\alpha > 0$. Write $d = p^\alpha k$, where p does not divide k. Note that in this case the smallest Jordan block size of u acting on $\wedge^2(V)$ is p^α by Lemma 3.4.11.

We will show next that $\ker(u_0 - 1)^{p^\alpha} \not\subseteq \ker \varphi$. Since $(u_0 - 1)^{p^\alpha} = u_0^{p^\alpha} - 1$, this is equivalent to finding a fixed point for $u_0^{p^\alpha}$ outside of $\ker \varphi$.

Since $(u-1)e_x = (u+1)e_{x-1}$, it follows that $(u-1)^m e_x = (u+1)^m e_{x-m}$ for all $m \geq 1$. In particular, $(u-1)^{p^\alpha} e_x = (u+1)^{p^\alpha} e_{x-p^\alpha}$, so

$$(u^{p^\alpha} - 1)e_x = (u^{p^\alpha} + 1)e_{x-p^\alpha}$$

for all x.

Therefore, the subspace W of V with symplectic basis

$$e_1, e_{1+p^\alpha}, \ldots, e_{(k-1)p^\alpha+1}, e_{p^\alpha}, e_{2p^\alpha}, \ldots, e_{kp^\alpha}$$

is a non-degenerate u^{p^α}-invariant subspace. Then by Lemma 5.5.3, the element $\gamma' \in \wedge^2(V)$ defined by

$$\gamma' = \sum_{j=0}^{k-1} (-1)^{j+1} e_{1+jp^\alpha} \wedge e_{(k-j)p^\alpha}$$

is a fixed point for $u_0^{p^\alpha}$. Furthermore, now $\varphi(\gamma') = k$ so $\gamma' \notin \ker \varphi$.

We have seen that $\ker(u_0 - 1)^{p^\alpha} \not\subseteq \ker \varphi$ and that the smallest Jordan block of u_0 has size p^α, so it follows from Lemma 3.2.1 that $r_{p^\alpha}(u_0') = r_{p^\alpha}(u_0) - 1$, $r_{p^\alpha - 1}(u_0') = 1$ and $r_m(u_0') = r_m(u_0)$ for all $m \neq p^\alpha, p^\alpha - 1$. Because $\ker \varphi / \langle \gamma \rangle \cong L(\omega_2)$ has a G-invariant symmetric form, it follows that $r_{p^\alpha - 1}(u_0'')$ must be even. Thus when we look at the Jordan blocks of u_0'' in terms of those of u_0' using Lemma 3.2.2, we see that we must have $m = p^\alpha - 1$ or $m = p^\alpha - 2$ in Lemma 3.2.2 for $r_{p^\alpha - 1}(u_0'')$ to be even. This proves the claim, since these two possibilities correspond to the situations (i) and (ii) given in case (b) of our claim. □

Remark 5.5.9. In case (b) of Lemma 5.5.8, both cases (i) and (ii) can occur. For example, consider the case where $p = 3$ and $G = \mathrm{Sp}(V)$ with $\dim V = 18$ (type C_9). Here we have $\wedge^2(V) = L(0)/L(\omega_2)/L(0)$.

For $u \in G$ with $V \downarrow K[u] = V_{18}$ (regular unipotent), one computes that

$$\wedge^2(V) \downarrow K[u] = [9^5, 27^4]$$
$$L(\omega_2) \downarrow K[u] = [7, 9^4, 27^4]$$

so the blocks are given as in Lemma 5.5.8 (b) (i). For $u \in G$ with $V \downarrow K[u] = V_6 \oplus V_{12}$, one computes that

$$\wedge^2(V) \downarrow K[u] = [3^4, 9^5, 15^5, 21]$$
$$L(\omega_2) \downarrow K[u] = [2^2, 3^2, 9^5, 15^5, 21]$$

so here the blocks are given as in Lemma 5.5.8 (b) (ii).

Although it is not necessary for the solution of our main problem, it would be interesting to find a way to determine which of the cases (i) or (ii) occur in Lemma 5.5.8 (b). One could also try to generalize Lemma 5.5.8 for non-distinguished unipotent elements. For example, in our proof of Lemma 5.5.8 we never used the fact that the block sizes d_i are distinct, and indeed Lemma 5.5.8 holds for all $u \in \mathrm{Sp}(V)$ which have all Jordan block sizes even.

Proposition 5.5.10 (Type C_l)**.** *Let G be the symplectic group* $\mathrm{Sp}(V)$*, where* $\dim V = 2l$ *($l \geq 2$). Then*

(a) *A non-regular unipotent element of G does not act as a distinguished unipotent element in the representation $L(\omega_2)$.*

(b) *A regular unipotent element of G acts as a distinguished unipotent element in $L(\omega_2)$ if and only if one of the following holds:*

  (i) $p \geq 4l - 3$,

  (ii) $2l = bp^k + \frac{p^k \pm 1}{2}$, *where* $k \geq 1$ *and* $0 \leq b \leq \frac{p-1}{2}$,

  (iii) $2l = p + \frac{p-3}{2}$,

  (iv) $p = 3$, $l = 3$.

Proof. We consider the claims (a) and (b) first in the case where $p \nmid l$. In this case $\wedge^2(V) = L(\omega_2) \oplus L(0)$ (Lemma 5.5.2), so by Lemma 5.5.7 a non-regular unipotent $u \in G$ acts on $\wedge^2(V)$, thus on $L(\omega_2)$ with some block of size > 1 having multiplicity ≥ 2. This proves (a). If $u \in G$ is regular, then $V \downarrow K[u] = V_{2l}$ and it follows from Proposition 3.5.3 that u acts on $L(\omega_2)$ as a distinguished unipotent element if and only if it acts on $\wedge^2(V)$ with no repeated blocks. Thus (b) follows from Proposition 3.5.3.

Consider then $p \mid l$, so now $\wedge^2(V) = L(0)/L(\omega_2)/L(0)$ as a G-module. Let $u \in G$ be a distinguished unipotent element, say $V \downarrow K[u] = V_{d_1} \oplus \cdots \oplus V_{d_t}$ where d_i are distinct and even. Set $\alpha = \nu_p(\gcd(d_1, \ldots, d_t))$.

Suppose first that u is not regular, i.e. that $t > 1$. If $\alpha = 0$, then it follows from Lemma 5.5.7 and Lemma 5.5.8 that u acts on $L(\omega_2)$ with some Jordan block of size > 1 having multiplicity ≥ 2. Consider then the case where $\alpha > 0$. For all i, set $\alpha_i = \nu_p(d_i)$. It follows from Lemma 3.4.10 that u acts on $\wedge^2(V_{d_i})$ with smallest

Jordan block size p^{α_i}, which occurs with multiplicity $\frac{p^{\alpha_i}+1}{2} \geq 2$. Since $\wedge^2(V)$ has $\wedge^2(V_{d_1}) \oplus \cdots \oplus \wedge^2(V_{d_t})$ as a direct summand by Lemma 3.4.9, and since $t > 1$, it follows that the action of u on $\wedge^2(V)$ is inadmissible (Definition 3.2.4). By Lemma 3.2.6, the action of u on $L(\omega_2)$ is not distinguished. This proves (a).

For (b), suppose that u is regular, say $V \downarrow K[u] = V_d$ where $d = \dim V$. It follows from Lemma 3.4.10 that u acts on $\wedge^2(V)$ with smallest Jordan block size p^α, which occurs with multiplicity $\frac{p^\alpha+1}{2}$. Therefore if $p > 3$ or $\alpha > 1$, the action of u on $\wedge^2(V)$ is inadmissible and thus by Lemma 3.2.6, the action of u on $L(\omega_2)$ is not distinguished.

Suppose then that $p = 3$ and $\alpha = 1$, say $d = 3k$ where 3 does not divide k. Now by Theorem 3.3.8, we have $s_p(d)_> = (3\lambda_1, 3\lambda_1, 3\lambda_1, \ldots, 3\lambda_k, 3\lambda_k, 3\lambda_k)$ where $s_p(k)_> = (\lambda_1, \ldots, \lambda_k)$. Note that $\lambda_k = 1$ and $\lambda_{k'} > 1$ for all $k' < k$ by Lemma 3.4.3.

Since k is even, it follows from Theorem 3.4.5 that if $k > 2$, then u acts on $\wedge^2(V)$ with ≥ 2 blocks of size $3\lambda_k$ and ≥ 2 blocks of size $3\lambda_{k-2}$. Once again the action of u on $\wedge^2(V)$ is inadmissible, so the action of u on $L(\omega_2)$ is not distinguished by Lemma 3.2.6.

Therefore the only possibility left is that $k = 2$, and this is precisely the case (b) (iv) in our claim. One can compute, either directly or e.g. with MAGMA (Section 2.9), that in this case we have

$$\wedge^2(V) \downarrow K[u] = [3, 3, 9]$$
$$L(\omega_2) \downarrow K[u] = [1, 3, 9]$$

so the action of u on $L(\omega_2)$ is distinguished. This completes the proof of the proposition. □

Remark 5.5.11. Let $G = \mathrm{Sp}(V)$. In Proposition 5.5.10 (b) (i)-(iii), it is clear that $p \nmid l$, and thus $V(\omega_2) = L(\omega_2)$ and $\wedge^2(V) \cong L(\omega_2) \oplus L(0)$ by Lemma 5.5.2. Therefore it follows from Proposition 5.5.10 that if some unipotent element $u \in G$ acts on $L(\omega_2)$ as a distinguished unipotent element, then $V(\omega_2) = L(\omega_2)$ unless $p = 3$, $l = 3$, and u is a regular unipotent element of G. This reflects a more general phenomenon that is seen in Theorem 1.1.10. Looking at the p-restricted highest weights λ that occur in the statement of Theorem 1.1.10, one can observe that there are very few cases where $V(\lambda)$ is not irreducible.

5.6 Representation $L_G(\omega_2)$ for G of type C_l ($p = 2$)

Assume that $p = 2$.

Suppose that G is of type C_l ($l \geq 2$) and let V be the natural module of G. In this section, we determine when a unipotent element $u \in G$ acts as a distinguished unipotent element on $L(\omega_2)$. The answer is given in Proposition 5.6.7 below and is also recorded in Table 1.3.

As in subsection 5.5.1, we can find the representation $L(\omega_2)$ as a subquotient of the exterior square $\wedge^2(V)$. Lemma 5.5.2 also holds in characteristic two (e.g. by [McN98, Lemma 4.8.2]) and becomes the following.

Lemma 5.6.1. *As G-modules, we have*

$$\wedge^2(V) \cong \begin{cases} L(\omega_2) \oplus L(0) & \text{if } l \text{ is odd} \\ L(0)/L(\omega_2)/L(0) \text{ (uniserial)} & \text{if } l \text{ is even} \end{cases}$$

Let $u \in G$ be a distinguished unipotent element and suppose that u has order $2^{\alpha+1}$. We retain the notation for $K[u]$-modules from Section 5.3. We will apply the following result, which tells us how the exterior square of an indecomposable $K[u]$-module splits into a sum of indecomposables. It was proven by Gow and Laffey in [GL06] and it is a special case of a more general result due to Himstedt and Symonds [HS14, Theorem 1.1].

Theorem 5.6.2. *Let $0 < s \leq 2^{n-1}$ and $n \geq 1$. Then*

$$\wedge^2(V_{2^{n-1}+s}) = \wedge^2(V_{2^{n-1}-s}) \oplus V_{2^n-s} \oplus (s-1)V_{2^n}$$

Remark 5.6.3. In Theorem 5.6.2, we interpret $V_0 = 0$, so $s = 2^{n-1}$ gives $\wedge^2(V_{2^n}) = (2^{n-1}-1)V_{2^n} \oplus V_{2^{n-1}}$.

Lemma 5.6.4. *Suppose that all blocks in $\wedge^2(V_r)$ have multiplicity ≤ 3. Then $r \leq 20$.*

Proof. Write $r = 2^{n-1} + s$, where $n \geq 1$ and $0 < s \leq 2^{n-1}$. Now

$$\wedge^2(V_{2^{n-1}+s}) = \wedge^2(V_{2^{n-1}-s}) \oplus V_{2^n-s} \oplus (s-1)V_{2^n}$$

so $s - 1 \leq 3$ and so $s \leq 4$. On the other hand, $2^{n-1} - s = 2^{n-2} + 2^{n-2} - s$ so applying $2^{n-2} - s - 1 \leq 3$ and so $2^{n-2} \leq s + 4 \leq 8$. Therefore $n \leq 5$ and thus $r = 2^{n-1} + s \leq 2^4 + 4 = 20$. □

Using Theorem 5.6.2, we can easily decompose $\wedge^2(V_r)$ for $1 \leq r \leq 20$ (see Table 5.2) and get the following corollary.

Corollary 5.6.5. *Suppose that all blocks in $\wedge^2(V_r)$ have multiplicity ≤ 3, and that at most one block has multiplicity 3. Then $r \leq 12$.*

In the proof of the next lemma, we will make use of Table 5.3, which gives the decomposition of $V_r \otimes V_s$ for $2 \leq r \leq s \leq 12$ where r and s are even. The data in table 5.3 can be computed with a computer program (for example MAGMA), or by hand using recursive formulae as described for example in [Bar11, Theorem 1] or [GL06, Corollary 3].

Lemma 5.6.6. *Let $V = V_{r_1} \oplus \cdots \oplus V_{r_t}$ be a $K[u]$-module, where $2 \leq r_1 \leq r_2 \leq \cdots \leq r_t$ are even. Then the action of u on $\wedge^2(V)$ is inadmissible (Definition 3.2.5), unless one of the following holds.*

(i) *$t = 1$ and $V = V_r$, where $r \leq 12$.*

(ii) *$t = 2$ and $V = V_2 \oplus V_2$ or $V = V_2 \oplus V_{10}$.*

Proof. If $V = V_r$ and the action of u on $\wedge^2(V)$ is admissible, then by Corollary 5.6.5 we have $r \leq 12$.

Consider then $V = V_{r_1} \oplus V_{r_2}$, where $2 \leq r_1 \leq r_2$ are even and suppose that the action of u on $\wedge^2(V)$ is admissible (Definition 3.2.5). Now $\wedge^2(V) \cong \wedge^2(V_{r_1}) \oplus (V_{r_1} \otimes V_{r_2}) \oplus \wedge^2(V_{r_2})$, so by (i) we must have $r_1, r_2 \leq 12$. Furthermore, now the action of u on $V_{r_1} \otimes V_{r_2}$ is admissible, so by Table 5.3 the pair (r_1, r_2) is one of the following: $(2,2)$, $(2,4)$, $(2,6)$, $(2,8)$, $(2,10)$, $(2,12)$, $(4,6)$, $(4,10)$, $(6,12)$. Computing $\wedge^2(V_{r_1} \oplus V_{r_2})$ in these cases using Table 5.2 and Table 5.3, one gets the following decompositions.

$$\begin{aligned}
\wedge^2(V_2 \oplus V_2) &= [1^2, 2^2]\\
\wedge^2(V_2 \oplus V_4) &= [1, 2, 4^3]\\
\wedge^2(V_2 \oplus V_6) &= [1^2, 6^3, 8]\\
\wedge^2(V_2 \oplus V_8) &= [1, 4, 8^5]\\
\wedge^2(V_2 \oplus V_{10}) &= [1^2, 6, 8, 10^2, 14, 16]\\
\wedge^2(V_2 \oplus V_{12}) &= [1, 2, 4, 12^3, 16^3]\\
\wedge^2(V_4 \oplus V_6) &= [1, 2, 4^3, 6, 8^3]\\
\wedge^2(V_4 \oplus V_{10}) &= [1, 2, 4, 6, 8^3, 12^2, 14, 16]\\
\wedge^2(V_6 \oplus V_{12}) &= [1, 2, 4, 6, 8^3, 12^3, 16^5]
\end{aligned}$$

Therefore (r_1, r_2) must be $(2, 2)$ or $(2, 10)$.

Finally, suppose that $V = V_{r_1} \oplus V_{r_2} \oplus \cdots \oplus V_{r_t}$ where $2 \leq r_1 \leq r_2 \leq \cdots \leq r_t$ are even and $t \geq 3$. Note that $\wedge^2(V)$ has $\wedge^2(V_{r_1} \oplus V_{r_2})$ and $\wedge^2(V_{r_2} \oplus V_{r_3})$ as a direct summand. Thus if the action of u on $\wedge^2(V)$ is admissible, then the action of u on each of $\wedge^2(V_{r_1} \oplus V_{r_2})$ and $\wedge^2(V_{r_2} \oplus V_{r_3})$ is also admissible. By (ii), this implies that $V_{r_1} \oplus V_{r_2} \oplus V_{r_3} = V_2 \oplus V_2 \oplus V_{10}$ or $V_2 \oplus V_2 \oplus V_2$. A computation shows that

$$\begin{aligned}
\wedge^2(V_2 \oplus V_2 \oplus V_2) &= [1^3, 2^6]\\
\wedge^2(V_2 \oplus V_2 \oplus V_{10}) &= [1^3, 2^2, 6, 8, 10^4, 14, 16]
\end{aligned}$$

and then since $\wedge^2(V_{r_1} \oplus V_{r_2} \oplus V_{r_3})$ is a direct summand of $\wedge^2(V)$, it follows that the action of u on $\wedge^2(V)$ is inadmissible. □

Proposition 5.6.7. *The action of u on $L(\omega_2)$ is distinguished if and only if l and u occur in Table 5.1.*

Proof. Suppose that the action of u on $L(\omega_2)$ is distinguished. Now by lemmas 5.6.1, 5.6.6, 3.2.7 and 3.2.8 one of the following holds:

- $2 \leq l \leq 6$ and u is regular.
- $l = 2$ and u is in class (2_1^2).
- $l = 6$ and u is in class $(2_1, 10_1)$.

If $l = 4$ and u is regular, then a computer calculation shows that the action of u on $L(\omega_2)$ has Jordan blocks $[2, 8^3]$, so the action is not distinguished. If $l = 6$ and u is regular, then the action of u on $L(\omega_2)$ has Jordan blocks $[4, 12, 16^3]$, so here too the action is not distinguished.

The remaining cases are those in Table 5.1. In these cases, one can verify by a computation with MAGMA (Section 2.9) that the Jordan blocks are as in the claim, and that u acts as a distinguished unipotent element on $L(\omega_2)$. □

l	Class of u	Jordan blocks of u acting on $L(\omega_2)$
2	regular	$[4]$
2	(2_1^2)	$[2^2]$
3	regular	$[6,8]$
5	regular	$[6,8,14,16]$
6	$(2_1,10_1)$	$[6,8,10^2,14,16]$

Table 5.1: Cases where a unipotent element u of $G = C_l$ acts on $L(\omega_2)$ as a distinguished unipotent element.

r	$\wedge^2(V_r)$	r	$\wedge^2(V_r)$
1	0	11	$[3,7,13,16^2]$
2	$[1]$	12	$[2,4,12,16^3]$
3	$[3]$	13	$[3,11,16^4]$
4	$[2,4]$	14	$[1,10,16^5]$
5	$[3,7]$	15	$[9,16^6]$
6	$[1,6,8]$	16	$[8,16^7]$
7	$[5,8^2]$	17	$[9,16^6,31]$
8	$[4,8^3]$	18	$[1,10,16^5,30,32]$
9	$[5,8^2,15]$	19	$[3,11,16^4,29,32^2]$
10	$[1,6,8,14,16]$	20	$[2,4,12,16^3,28,32^3]$

Table 5.2: Decomposition of $\wedge^2(V_r)$ for $1 \leq r \leq 20$

r \ s	2	4	6	8	10	12
2	$[2^2]$	$[4^2]$	$[6^2]$	$[8^2]$	$[10^2]$	$[12^2]$
4		$[4^4]$	$[4^2,8^2]$	$[8^4]$	$[8^2,12^2]$	$[12^4]$
6			$[2^2,8^4]$	$[8^6]$	$[8^4,14^2]$	$[8^2,12^2,16^2]$
8				$[8^8]$	$[8^6,16^2]$	$[8^4,16^4]$
10					$[2^2,8^4,16^4]$	$[4^2,8^2,16^6]$
12						$[4^4,16^8]$

Table 5.3: Decomposition of $V_r \otimes V_s$ for $2 \leq r, s \leq 12$ even.

5.7 Representation $L_G(2\omega_1)$ for G of classical type ($p \neq 2$)

Assume that $p \neq 2$.

Suppose that G is of type B_l, C_l or D_l with natural module V. In this section, we will determine when a unipotent element $u \in G$ acts on $L(2\omega_1)$ as a distinguished unipotent element. We will show that this can happen only when u is a regular unipotent element. Furthermore, we will also see that u acts on $L(2\omega_1)$ as a distinguished unipotent element only if it acts on $S^2(V)$ with no repeated blocks. The results and proofs given are very similar to those found in Section 5.5.

5.7.1 Construction of $L_G(2\omega_1)$

In this subsection, we will describe a construction of $L_G(2\omega_1)$ for $G = \mathrm{Sp}(V)$ (type C_l) and $G = \mathrm{SO}(V)$ (type B_l or D_l). For type C_l this is easy, as seen in the following lemma.

Lemma 5.7.1. *Let $G = \mathrm{Sp}(V)$. Then $S^2(V) \cong L_G(2\omega_1)$.*

Proof. See for example [McN98, Proposition 4.2.2]. □

For the rest of this subsection, let $G = \mathrm{SO}(V)$ with $\dim V = n$, and denote the symmetric form on V by $(-,-)$. In this situation the following lemma shows that the symmetric square $S^2(V)$ is not irreducible, but we can find the representation $L_G(2\omega_1)$ as a subquotient of $S^2(V)$ (cf. Lemma 5.5.2).

Lemma 5.7.2. *As G-modules, we have*

$$S^2(V) \cong \begin{cases} L(2\omega_1) \oplus L(0) & \text{if } p \nmid n, \\ L(0)/L(2\omega_1)/L(0) \text{ (uniserial)} & \text{if } p \mid n. \end{cases}$$

Proof. See for example [McN98, Lemma 4.7.3]. □

We will now consider the submodule structure of $S^2(V)$ more explicitly, similarly to the description of $\wedge^2(V)$ in 5.5.1.

Fix a basis $e_1, \ldots, e_n$ of V such that

$$(e_i, e_j) = \begin{cases} (-1)^{\min\{i,j\}} & \text{if } i + j = n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this form is exactly the same as the one given in Section 2.5 if n is odd.

If $n = 2l + 1$ (type B_l), we define $\gamma \in S^2(V)$ to be the vector

$$\left(\sum_{i=1}^{l} (-1)^i e_i e_{n+1-i} \right) + \frac{1}{2}(-1)^{l+1} e_{l+1}^2.$$

Similarly if $n = 2l$ (type D_l), we define $\gamma \in S^2(V)$ to be the vector

$$\gamma = \sum_{i=1}^{l} (-1)^i e_i e_{n+1-i}.$$

The following lemma shows that γ is fixed by the action of G.

Lemma 5.7.3. *The element γ is fixed by the action of G on $S^2(V)$.*

Proof. (cf. Lemma 5.5.3) The form $(-,-)$ on V induces a G-module isomorphism $V \to V^*$ defined by $v \mapsto (v,-)$. This in turn induces an isomorphism $\psi : S^2(V) \to S^2(V^*)$ of G-modules.

Let $\mathrm{Quad}(V)$ be the vector space of quadratic forms on V. Then $\mathrm{Quad}(V)$ is a G-module with the action defined by $(g \cdot Q)(v) = Q(g^{-1}v)$ for all $Q \in \mathrm{Quad}(V)$, $g \in G$ and $v \in V$. Now there is an isomorphism $\chi : S^2(V^*) \to \mathrm{Quad}(V)$ of G-modules, defined by $\chi(ff')(v) = f(v)f'(v)$ for all $f, f' \in V^*$ and $v \in V$.

Therefore it is enough to show that $\chi\psi(\gamma) = Q$ is fixed by the action of G. That is, we should show that Q is a G-invariant quadratic form. To this end, a straightforward calculation on the basis elements e_i shows that $Q(v) = \frac{1}{2}(v,v)$ for all $v \in V$. □

Define a linear map $\varphi : S^2(V) \to K$ by $\varphi(vw) = (v, w)$ for all $v, w \in V$. Then φ is a surjective morphism of G-modules. As in 5.5.1, we will use $\ker\varphi$ and γ to describe the submodule structure of $S^2(V)$.

Note that we have $\varphi(\gamma) = n/2$. Therefore if $p \nmid n$, then $\gamma \notin \ker\varphi$ and so $S^2(V) = \ker\varphi \oplus \langle\gamma\rangle$. In this case $\ker\varphi \cong L(2\omega_1)$ by Lemma 5.7.2. If $p \mid n$, then $\gamma \in \ker\varphi$, and thus $\ker\varphi/\langle\gamma\rangle \cong L(2\omega_1)$ by Lemma 5.7.2.

5.7.2 Types B_l and D_l

Lemma 5.7.4. *Let $G = \mathrm{SO}(V)$ and let $u \in G$ be a unipotent. Suppose that u is a distinguished unipotent element of G, so as a $K[u]$-module $V = V_{d_1} \oplus \cdots \oplus V_{d_t}$, where $0 < d_1 < \cdots < d_t$ are odd (Proposition 2.3.4). If $t > 1$, then one of the following holds:*

(i) *The element u acts on $S^2(V)$ with some block of size > 1 having multiplicity ≥ 2,*

(ii) *As a $K[u]$-module $V = V_1 \oplus V_n$, where $n \equiv 3 \mod 4$ and $S^2(V_n)$ has no repeated blocks.*

Proof. (cf. Lemma 5.5.7) It follows from Lemma 3.4.9 that $S^2(V) \downarrow K[u]$ has $S^2(V_{d_i} \oplus V_{d_j})$ as a direct summand for any $i \neq j$. Therefore it will be enough to prove the lemma in the case where $t = 2$, say $V = V_m \oplus V_n$ with $0 < m < n$ odd.

Suppose that u acts on $S^2(V)$ such that all blocks of size > 1 have multiplicity ≤ 1. Now $S^2(V) \downarrow K[u] = S^2(V_m) \oplus (V_m \otimes V_n) \oplus S^2(V_n)$, so by Proposition 3.5.3 the action of u on $S^2(V_m)$ and $S^2(V_n)$ is as in characteristic 0. Since m and n are both odd, we have $2m - 1 \equiv 2n - 1 \mod 4$, so every block in $S^2(V_m)$ must occur in $S^2(V_n)$ by Proposition 3.4.4. Therefore $m = 1$, as otherwise both $S^2(V_m)$ and $S^2(V_n)$ would have a block of size 5.

Now $S^2(V) \downarrow K[u] = V_1 \oplus V_n \oplus S^2(V_n)$. Furthermore, a block of size n occurs in $S^2(V_n)$ if and only if $n \equiv 2n - 1 \mod 4$, which happens if and only if $n \equiv 1 \mod 4$. Hence we must have $n \equiv 3 \mod 4$, and so (ii) of the lemma holds. □

Lemma 5.7.5. *Let $G = \mathrm{SO}(V)$ and let $u \in G$ unipotent. Suppose that u is a distinguished unipotent element of G, so as a $K[u]$-module $V = V_{d_1} \oplus \cdots \oplus V_{d_t}$, where $0 < d_1 < \cdots < d_t$ are odd (Proposition 2.3.4). Assume that $p \mid \dim V$, so $S^2(V) = L(0)/L(2\omega_1)/L(0)$ as a G-module. Let u_0 be the image of u in $\mathrm{SL}(S^2(V))$ and u_0'' the image of u in $\mathrm{SL}(L(2\omega_1))$. Let $\alpha = \nu_p(\gcd(d_1, \ldots, d_t))$. Then for the Jordan block sizes $r_m(u_0'')$ (Definition 3.1.1) the following hold:*

(a) *If $\alpha = 0$, then $r_1(u_0'') = r_1(u_0) - 2$ and $r_m(u_0'') = r_m(u_0)$ for all $m > 1$.*

(b) *If $\alpha > 0$, then one of the following holds:*

  (i) *$r_{p^\alpha}(u_0'') = r_{p^\alpha}(u_0) - 1$, $r_{p^\alpha - 2}(u_0'') = 1$, and $r_m(u_0'') = r_m(u_0)$ for all $m \neq p^\alpha, p^\alpha - 2$.*

  (ii) *$r_{p^\alpha}(u_0'') = r_{p^\alpha}(u_0) - 2$, $r_{p^\alpha - 1}(u_0'') = 2$, and $r_m(u_0'') = r_m(u_0)$ for all $m \neq p^\alpha, p^\alpha - 1$.*

Proof. (cf. Lemma 5.5.8) We begin by constructing u as in Section 2.5. Let $V = W_1 \oplus \cdots \oplus W_t$ as an orthogonal direct sum, with $\dim W_i = d_i$. For all i, let $e_1^{(i)}, \ldots, e_{d_i}^{(i)}$ be a basis for W_i such that

$$(e_x^{(i)}, e_y^{(i)}) = \begin{cases} (-1)^x & \text{if } x + y = d_i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Set $e_x = 0$ for all $x \leq 0$. Now define a linear map $u : V \to V$ by $(u-1)e_x^{(i)} = (u+1)e_{x-1}^{(i)}$. Then $u \in \mathrm{SO}(V)$ (see Section 2.5) and $V \downarrow K[u] \cong V_{d_1} \oplus \cdots \oplus V_{d_t}$ with $W_i \cong V_{d_i}$.

Let $\varphi : S^2(V) \to K$ be as in 5.5.1. Let $\gamma \in S^2(V)$ be a generator for the one-dimensional G-submodule of $S^2(V)$, which exists by Lemma 5.7.2. We have $\ker \varphi / \langle \gamma \rangle \cong L(2\omega_1)$.

Without loss of generality, assume that $\alpha = \nu_p(d_1)$. Set $d = d_1$ and $e_x = e_x^{(1)}$ for all $1 \leq x \leq d$. Denote the restriction of u_0 to $\ker \varphi$ by u_0'. We can and will consider u_0'' to be the map induced by u_0 on $\ker \varphi / \langle \gamma \rangle$.

We consider first the case where $\alpha = 0$. Write $d = 2f + 1$. Now

$$\gamma_1 = \left(\sum_{j=1}^{f} (-1)^j e_j e_{d-j+1} \right) + \frac{1}{2}(-1)^{f+1} e_{f+1}^2$$

is a fixed point for u_0 by Lemma 5.7.3. Furthermore, we have $\varphi(\gamma_1) = d/2$, so $\gamma_1 \notin \ker \varphi$ since p does not divide d. Thus $\ker(u_0 - 1) \not\subseteq \ker \varphi$, so by Lemma 3.2.1 we have $r_1(u_0') = r_1(u_0) - 1$ and $r_m(u_0') = r_m(u_0)$ for all $m > 1$. Since we have a G-invariant symmetric form on $S^2(V)$ (Lemma 4.4.8) and $L_G(2\omega_1)$ (Table 4.1), it follows as in the proof of Lemma 5.5.8 (fifth paragraph) that $r_1(u_0'') = r_1(u_0) - 2$ and $r_m(u_0'') = r_m(u_0)$ for all $m > 1$, as desired.

Consider then the case where $\alpha > 0$. Write $d = p^\alpha k$, where p does not divide k. Note that in this case the smallest Jordan block size of u acting on $S^2(V)$ is p^α by Lemma 3.4.13.

We will show next that $\ker(u_0 - 1)^{p^\alpha} \not\subseteq \ker \varphi$. Since $(u_0 - 1)^{p^\alpha} = u_0^{p^\alpha} - 1$, this is equivalent to finding a fixed point for $u_0^{p^\alpha}$ outside of $\ker \varphi$.

Since $(u-1)e_x = (u+1)e_{x-1}$, it follows that $(u-1)^m e_x = (u+1)^m e_{x-k}$ for all $m \geq 1$. In particular, $(u-1)^{p^\alpha} e_x = (u+1)^{p^\alpha} e_{x-p^\alpha}$, so

$$(u^{p^\alpha} - 1)e_x = (u^{p^\alpha} + 1)e_{x-p^\alpha}$$

for all x.

Therefore the subspace W of V with basis

$$e_1, e_{1+p^\alpha}, \ldots, e_{(k-1)p^\alpha+1}, e_{p^\alpha}, e_{2p^\alpha}, \ldots, e_{kp^\alpha}$$

is a non-degenerate u^{p^α}-invariant subspace. By Lemma 5.7.3, the element $\gamma' \in S^2(V)$ defined by

$$\gamma' = \sum_{j=0}^{k-1} (-1)^{j+1} e_{1+jp^\alpha} e_{(k-j)p^\alpha}$$

is a fixed point for $u_0^{p^\alpha}$. Now $\varphi(\gamma') = k$, so $\gamma' \notin \ker \varphi$ since p does not divide k.

We have shown that $\ker(u_0 - 1)^{p^\alpha} \not\subseteq \ker \varphi$ and that the smallest Jordan block size of u_0 is p^α. Since we have a G-invariant symmetric form on $S^2(V)$ and $L_G(2\omega_1)$, the claim follows by arguing as in the end of the proof of Lemma 5.5.8. □

Remark 5.7.6. (cf. Remark 5.5.9) In case (b) of Lemma 5.7.5, both cases (i) and (ii) can occur. For example, consider the case where $p = 5$ and $G = \mathrm{SO}(V)$ with $\dim V = 5$ (type B_2). Here we have $S^2(V) = L(0)/L(2\omega_1)/L(0)$.

For $u \in G$ with $V \downarrow K[u] = V_5$ (regular unipotent), one computes that

$$S^2(V) \downarrow K[u] = [5^3]$$
$$L(2\omega_1) \downarrow K[u] = [3, 5^2]$$

so the blocks are given as in Lemma 5.5.8 (b) (i).

For an example of Lemma 5.5.8 (b) (ii), consider the case where $p = 3$ and $G = \mathrm{SO}(V)$ with $\dim V = 18$ (type D_9). Once again $S^2(V) = L(0)/L(2\omega_1)/L(0)$.

For a distinguished unipotent $u \in G$ with $V \downarrow K[u] = V_3 \oplus V_{15}$, one computes that

$$S^2(V) \downarrow K[u] = [3^4, 9, 15^5, 21, 27^2]$$
$$L(2\omega_1) \downarrow K[u] = [2^2, 3^2, 9, 15^5, 21, 27^2]$$

so the blocks are given as in Lemma 5.5.8 (b) (ii).

The following two propositions determine when for $G = \mathrm{SO}(V)$ a unipotent element $u \in G$ acts on $L_G(2\omega_1)$ as a distinguished unipotent element. They will be proven simultaneously.

Proposition 5.7.7 (Type B_l)**.** *Let G be the orthogonal group $\mathrm{SO}(V)$, where $\dim V = 2l + 1$ ($l \geq 2$). Then*

(a) *A non-regular unipotent element of G does not act as a distinguished element in the representation $L(2\omega_1)$.*

(b) *A regular unipotent element of G acts as a distinguished unipotent element in $L(2\omega_1)$ if and only if one of the following holds:*

  (i) $p \geq 4l + 1$,

  (ii) $2l + 1 = bp^k + \frac{p^k \pm 1}{2}$, *where $k \geq 1$ and $0 \leq b \leq \frac{p-1}{2}$.*

Proposition 5.7.8 (Type D_l)**.** *Let G be the orthogonal group $\mathrm{SO}(V)$, where $\dim V = 2l$ ($l \geq 4$). Then*

(a) *A non-regular unipotent element of G does not act as a distinguished element in the representation $L(2\omega_1)$.*

(b) *A regular unipotent element of G acts as a distinguished unipotent element in $L(2\omega_1)$ if and only if l is even and one of the following holds:*

  (i) $p \geq 4l - 3$,

  (ii) $2l - 1 = bp^k + \frac{p^k \pm 1}{2}$, *where $k \geq 1$ and $0 \leq b \leq \frac{p-1}{2}$.*

Proof of Proposition 5.7.7 and Proposition 5.7.8. (cf. Proposition 5.5.10) Consider first the case where $p \nmid \dim V$, so $S^2(V) = L(2\omega_1) \oplus L(0)$ (Lemma 5.7.2). Note that a nonregular distinguished unipotent element of G acts on V with ≥ 2 blocks of size > 1 (Proposition 2.3.4). Therefore by Lemma 5.7.4, a non-regular unipotent $u \in G$ acts on $S^2(V)$, hence on $L(2\omega_1)$ with some block of size > 1 having multiplicity ≥ 2. This proves (a).

For (b), still assuming $p \nmid \dim V$, let $u \in G$ be a regular unipotent element. Suppose first that $\dim V = 2l + 1$, so $V \downarrow K[u] = V_{2l+1}$ (Proposition 2.3.3). It follows from Proposition 3.5.3 that u acts on $L(2\omega_1)$ as a distinguished unipotent element if and only if it acts on $S^2(V)$ with no repeated blocks, and thus (b) follows from Proposition 3.5.3. Consider then $\dim V = 2l$, so $V \downarrow K[u] = V_{2l-1} \oplus V_1$ (Proposition 2.3.3). In this case, by Lemma 5.7.4 the action of u on $L(2\omega_1)$ is distinguished if and only if u acts on $S^2(V_{2l-1})$ with no repeated blocks and $2l - 1 \equiv 3 \mod 4$. That is, if and only if u acts on $S^2(V_{2l-1})$ with no repeated blocks and l is even. Once again (b) follows from Proposition 3.5.3.

Consider then the case where $p \mid \dim V$, so $S^2(V) = L(0)/L(2\omega_1)/L(0)$ (Lemma 5.7.2). Note that in the proposition being proven, we have $p \nmid \dim V$ in (i)-(ii) of (b); therefore we prove next that no unipotent element of G acts as a distinguished unipotent element on $L(2\omega_1)$. Let $u \in G$ be a distinguished unipotent element, say $V \downarrow K[u] = V_{d_1} \oplus \cdots \oplus V_{d_t}$ where d_i are distinct and odd. Set $\alpha = \nu_p(\gcd(d_1, \ldots, d_t))$.

Suppose first that $\alpha = 0$. Note that in this case $t > 1$. We will show that u acts on $S^2(V)$ with some Jordan block of size > 1 having multiplicity ≥ 2. Then it follows from Lemma 5.7.5 that u acts on $L(2\omega_1)$ with some Jordan block of size > 1 having multiplicity ≥ 2. Hence by Lemma 5.7.4 we can assume that $\dim V = 2l$, $V \downarrow K[u] = V_{2l-1} \oplus V_1$, and $S^2(V_{2l-1})$ has no repeated blocks. Since p divides $\dim V = 2l$, it follows from Proposition 3.5.3 that

$$2l - 1 = bp^k + \frac{p^k \pm 1}{2},$$

for some $k \geq 1$ and $0 \leq b \leq \frac{p-1}{2}$. If $2l - 1 = bp^k + \frac{p^k - 1}{2}$, then $2l = bp^k + \frac{p^k+1}{2}$. But then the fact that $p \mid 2l$ implies that $p \mid p^k + 1$, contradiction. Therefore $2l - 1 = bp^k + \frac{p^k+1}{2}$, and so $2l = bp^k + \frac{p^k+3}{2}$. Since $p \mid 2l$, it follows that $p \mid p^k + 3$ and hence $p = 3$. Because $b = 0$ or $b = 1$ in this case, we can assume that $2l - 1 = \frac{3^k+1}{2}$ for some $k \geq 1$. Therefore $2l = \frac{3^k+3}{2}$, and so $3^k + 3 \equiv 0 \mod 4$, which means that k must be even. But then $3^k + 3 \equiv 4 \mod 8$, so it follows that $l = \frac{3^k+3}{4}$ is odd. Thus $2l - 1 \equiv 1 \mod 4$, and by Lemma 5.7.4 the element u acts on $S^2(V)$ with some Jordan block of size > 1 having multiplicity ≥ 2 (namely, block size $2l - 1$ occurs with multiplicity two).

Finally consider the case where $\alpha > 0$. Suppose first that $t > 1$. Set $\alpha_i = \nu_p(d_i)$. It follows from Lemma 3.4.12 that u acts on $S^2(V_{d_i})$ with smallest Jordan block size p^{α_i}, which occurs with multiplicity $\frac{p^{\alpha_i}+1}{2} \geq 2$. Since $S^2(V)$ has $S^2(V_{d_1}) \oplus \cdots \oplus S^2(V_{d_t})$ as a direct summand by Lemma 3.4.9, and since $t > 1$, it follows that the action of u on $S^2(V)$ is inadmissible (Definition 3.2.4). By Lemma 3.2.6, the action of u on $L(2\omega_1)$ is not distinguished.

What remains is the case where $\alpha > 0$ and $t = 1$. In this case $\dim V = 2l+1 = d$, where $\nu_p(d) = \alpha$, and $V \downarrow K[u] = V_d$. It follows from Lemma 3.4.12 that u acts on $S^2(V)$ with smallest Jordan block size p^α, which occurs with multiplicity $\frac{p^\alpha+1}{2}$. Therefore if $p > 3$ or $\alpha > 1$, the action of u on $S^2(V)$ is inadmissible and thus by Lemma 3.2.6, the action of u on $L(2\omega_1)$ is not distinguished.

Suppose then that $p = 3$ and $\alpha = 1$, say $d = 3k$ where 3 does not divide k. Now by Theorem 3.3.8, we have $s_p(d)_> = (3\lambda_1, 3\lambda_1, 3\lambda_1, \ldots, 3\lambda_k, 3\lambda_k, 3\lambda_k)$ where $s_p(k)_> = (\lambda_1, \ldots, \lambda_k)$. Note that $\lambda_k = 1$ and $\lambda_{k'} > 1$ for all $k' < k$ by Lemma 3.4.3.

Since d is odd, and $l \geq 2$, we have $k \geq 3$. It follows from Theorem 3.4.5 that u acts on $S^2(V)$ with ≥ 2 blocks of size $3\lambda_k$ and ≥ 2 blocks of size $3\lambda_{k-2}$. Once again the action of u on $S^2(V)$ is inadmissible, so the action of u on $L(2\omega_1)$ is not distinguished by Lemma 3.2.6. This completes the proof of the proposition. □

Remark 5.7.9. (cf. Remark 5.5.11) Let $G = \mathrm{SO}(V)$. It is clear in (b) (i) - (ii) of Proposition 5.7.7 and Proposition 5.7.8 that we have $p \nmid l$. Thus it follows from Proposition 5.7.7 and Proposition 5.7.8 that if some unipotent element $u \in G$ acts on $L(\omega_2)$ as a distinguished unipotent element, then $V(\omega_2) = L(\omega_2)$ and $S^2(V) \cong L(\omega_2) \oplus L(0)$.

5.7.3 Type C_l

Lemma 5.7.10. *Let $G = \mathrm{Sp}(V)$ and let $u \in G$ be unipotent. Suppose that u is a distinguished unipotent element of G, so as a $K[u]$-module $V = V_{d_1} \oplus \cdots \oplus V_{d_t}$, where $0 < d_1 < \cdots < d_t$ are even (Proposition 2.3.4). If $t > 1$, then u acts on $S^2(V)$ with repeated blocks.*

Proof. Suppose that $t > 1$ and that u acts on $S^2(V)$ with no repeated blocks. Now as a $K[u]$-module, $S^2(V)$ has $S^2(V_{d_1}) \oplus \cdots \oplus S^2(V_{d_t})$ as a direct summand by Lemma 3.4.9. Therefore u acts on $S^2(V_{d_i})$ with no repeated blocks for all i, and by Proposition 3.5.3, each $S^2(V_{d_i})$ decomposes as in characteristic 0. In particular, since all d_i are even, for all i we have a block of size 3 in $S^2(V_{d_i})$ (Proposition 3.4.4). Therefore u acts on $S^2(V)$ with ≥ 2 Jordan blocks of size 3, contradiction. □

Proposition 5.7.11 (Type C_l). *Let G be the symplectic group $\mathrm{Sp}(V)$, where $\dim V = 2l$ $(l \geq 2)$. Then*

(a) *A non-regular unipotent element of G does not act as a distinguished element in the representation $L(2\omega_1)$.*

(b) *A regular unipotent element of G acts as a distinguished unipotent element in $L(2\omega_1)$ if and only if one of the following holds:*

  (i) $p \geq 4l - 1$,

  (ii) $2l = bp^k + \frac{p^k \pm 1}{2}$, *where $k \geq 1$ and $0 \leq b \leq \frac{p-1}{2}$.*

Proof. Recall that $S^2(V) \cong L(2\omega_1)$ (Lemma 5.7.1). If $u \in G$ is a distinguished unipotent element that is not regular, then u acts on $S^2(V)$ with repeated blocks by Lemma 5.7.10, and thus the action of u on $L(2\omega_1)$ is not distinguished. This proves (a).

For (b), let $u \in G$ be a regular unipotent element, so $V \downarrow K[u] = V_{2l}$ (Proposition 2.3.3). Then u acts on $L(2\omega_1)$ as a distinguished unipotent element if and only if u acts on $S^2(V)$ with no repeated blocks. Thus (b) follows from Proposition 3.5.3. □

5.8 Representation $L_G(\omega_3)$ for G of classical type ($p \neq 2$)

Assume that $p \neq 2$.

Suppose that G is of classical type with $\operatorname{rank} G \geq 3$. In this section, we will determine when a unipotent element $u \in G$ acts on $L(\omega_3)$ as a distinguished unipotent element. The answer is given by the following proposition. In the statement we have excluded $G = D_4$, since in this case $L(\omega_3)$ is a twist of $L(\omega_1)$ by triality (Proposition 2.10.2 (ii)).

Proposition 5.8.1. *Suppose that G is simple of type A_l ($l \geq 3$), B_l ($l \geq 3$), C_l ($l \geq 3$) or D_l ($l \geq 5$). A unipotent element $u \in G$ acts on the irreducible representation $L(\omega_3)$ as a distinguished unipotent element if and only if u is a regular unipotent element and G, p are in Table 5.1.*

G	u of order $> p$	u of order p
A_3	$p = 3$	$p \geq 5$
A_5	$p = 5$	$p \geq 11$
C_3	$p = 3, 5$	$p \geq 11$
C_4	none	$p \geq 17$
C_5	none	$p \geq 23$
B_3	$p = 5$	$p \geq 13$

Table 5.1: $u \in G$ regular and acts on $L(\omega_3)$ as a distinguished unipotent element

For the proof, we will use the construction of $L(\omega_3)$ as a subquotient of $\wedge^3(V)$, where V is the natural module for G. Thus in some cases we will then have to compute the decomposition of the $K[u]$-mdoule $\wedge^3(V_d)$. These computations were done with MAGMA and they are contained in tables at the end of this section.

5.8.1 Types A_l, B_l, and D_l

For G of type A_l, B_l and D_l, the claim in Proposition 5.8.1 is really a claim about the exterior cube of a unipotent matrix, as seen from the next proposition.

Proposition 5.8.2. *Let $G = \mathrm{SL}(V)$ or $G = \mathrm{SO}(V)$ with $\operatorname{rank} G \geq 3$. Then $L(\omega_3) \cong \wedge^3(V)$.*

Proof. This is a consequence of [Sei87, 8.1], as noted in [McN98, Proposition 4.2.2]. □

Now let $u \in \mathrm{GL}(V)$ be a unipotent element of order q and recall the notation V_1, V_2, …, V_q from Section 1.4 for indecomposable $K[u]$-modules.

Lemma 5.8.3. *Suppose that u acts on $\wedge^3(V_d)$ such that all Jordan block sizes are distinct and all even or all odd. Then $d \leq 8$.*

Proof. We can consider u as a regular unipotent element of $G = A_{d-1}$, such that $V = V_d$ is the natural module of G. By Proposition 5.8.2 and Lemma 5.1.1 (iii), we know that the largest Jordan block size of u acting on $\wedge^3(V_d)$ is $\leq m_u(\omega_3) + 1 = 3d - 8$ (Lemma 2.7.3). One can verify that the polynomial inequality $\dim \wedge^3(V_d) = \binom{d}{3} > \frac{(m_u(\omega_3)+2)^2}{4}$ holds for all $d \geq 12$. Thus by Lemma 5.1.1 (iv), if u acts on $\wedge^3(V_d)$ such that all Jordan block sizes are distinct and all even or all odd, then $d < 12$.

For $9 \leq d \leq 11$, the claim is immediately seen from the decompositions given in Table 5.2. □

Note that when G is of type A_l, the irreducible module $L(\omega_3)$ is self-dual if and only if $l = 5$ (Table 4.1). Therefore for type A_l the claim of Proposition 5.8.1 follows from Theorem 1.1.12, Lemma 5.8.3 and Table 5.2. In particular, we find that a regular unipotent element of $G = A_l$ acts on $L(\omega_3)$ as a distinguished unipotent element if and only if $l = 3$, or $l = 5$ and $p \geq 5$; these examples have also been recorded in Table 1.1.

For G of type B_l or D_l, the claim of Proposition 5.8.1 follows from Proposition 5.8.2 and the following lemma.

Lemma 5.8.4. *Let $G = \mathrm{SO}(V)$, where $\dim V \geq 3$. Let $u \in G$ be a distinguished unipotent element, so as a $K[u]$-module $V = V_{d_1} \oplus \cdots \oplus V_{d_t}$, where the d_i are distinct and odd (Proposition 2.3.4). Then u acts on $\wedge^3(V)$ as a distinguished unipotent element if and only if one of the following holds:*

(i) *$V = V_3$, or $V = V_1 \oplus V_3$, for any p;*

(ii) *$V = V_5$, for $p = 3$ or $p \geq 7$;*

(iii) *$V = V_7$, for $p = 5$ or $p \geq 13$.*

Proof. If $t = 1$, then the claim follows immediately from Lemma 5.8.3 and the data in Table 5.2. Suppose then that $t > 1$.

For any $K[u]$-modules W and W', we have an isomorphism

$$\wedge^3(W \oplus W') \cong \wedge^3(W) \oplus (\wedge^2(W) \otimes W') \oplus (W \otimes \wedge^2(W')) \oplus \wedge^3(W') \qquad (*)$$

of $K[u]$-modules (this is an elementary fact, see for example [FH91, B.1, pg. 473]). It follows from (*) that $\wedge^3(V_{d_i} \oplus V_{d_j})$ is a direct summand of $\wedge^3(V)$ for all $i \neq j$. Therefore it is enough to prove the lemma in the case where $t = 2$, say $V = V_m \oplus V_n$ with $0 < m < n$ odd.

Suppose that u acts on $\wedge^3(V)$ as a distinguished unipotent element. We will show that $V = V_1 \oplus V_3$, which will prove the "only if" part of the lemma. Now $\wedge^3(V)$ has no repeated blocks, so it follows from (*) that $\wedge^3(V_s)$ and $\wedge^2(V_s)$ have no repeated blocks for all $s \in \{m, n\}$. Therefore by Lemma 5.8.3, we have $m, n \in \{1, 3, 5, 7\}$. By Table 5.2 and Proposition 3.5.3, if $\wedge^3(V_n)$ or $\wedge^2(V_n)$ has no repeated blocks for $n \in \{1, 3, 5, 7\}$, then their decomposition is as in the following table.

n	$\wedge^2(V_n)$	$\wedge^3(V_n)$
1	0	0
3	$[3]$	$[1]$
5	$[3, 7]$	$[3, 7]$
7	$[3, 7, 11]$	$[1, 5, 7, 9, 13]$

Consider first $1 < m < n$. If $(m, n) \neq (3, 5)$, we see from the table above that $\wedge^3(V_n)$ and $\wedge^3(V_m)$ have block sizes in common, so it follows from (*) that u acts on $\wedge^3(V)$ with repeated blocks, contradiction. For $(m, n) = (3, 5)$, we have by (*) and the table above that

$$\wedge^3(V_3 \oplus V_5) \cong V_1 \oplus (V_3 \otimes V_5) \oplus (V_3 \otimes (V_3 \oplus V_7)) \oplus (V_3 \oplus V_7).$$

However, then $V_3 \otimes V_5$ has no repeated block sizes, so by Lemma 3.3.10 we have $V_3 \otimes V_5 \cong V_3 \oplus V_5 \oplus V_7$. But then $\wedge^3(V_3 \oplus V_5)$ has at least two blocks of size 3, contradiction.

Thus we must have $m = 1$, and then $\wedge^3(V) \cong \wedge^3(V_n) \oplus \wedge^2(V_n)$ by (*). If $n \in \{5, 7\}$, then it follows from the table above that $\wedge^3(V_n)$ and $\wedge^2(V_n)$ have block sizes in common. Hence we must have $n = 3$, so $V = V_1 \oplus V_3$, as claimed.

Finally for the other direction of the lemma, for $V = V_1 \oplus V_3$, we have $\wedge^3(V) \cong \wedge^3(V_3) \oplus \wedge^2(V_3)$ by (*). It is clear that $\wedge^3(V_3) \cong V_1$ for all p, and $\wedge^2(V_3) \cong V_3$ for all p (e.g. by Proposition 3.5.3). Hence $\wedge^3(V_3 \oplus V_1) \cong V_3 \oplus V_1$ for all p. This completes the proof of the lemma. □

5.8.2 Type C_l

In this subsection, let $G = \mathrm{Sp}(V)$, where $\dim V = 2l$ for some $l \geq 3$. We can find $L(\omega_3)$ as a subquotient of $\wedge^3(V)$, as seen by the following lemma.

Lemma 5.8.5. *As a G-module, we have*

$$\wedge^3(V) \cong \begin{cases} L(\omega_3) \oplus L(\omega_1) & \text{if } l \not\equiv 1 \mod p, \\ L(\omega_1)/L(\omega_3)/L(\omega_1) \text{ (uniserial)} & \text{if } l \equiv 1 \mod p. \end{cases}$$

In particular, we have $\dim L(\omega_3) \geq \binom{2l}{3} - 4l$.

Proof. This is well known, see for example [McN98, Lemma 4.8.2]. □

Lemma 5.8.6. *Suppose that $u \in G$ acts on $L(\omega_3)$ as a distinguished unipotent element. Then u is regular and $3 \leq l \leq 5$.*

Proof. Now for a regular unipotent element $u \in G$ we have $m_u(\omega_3) = 3(2l - 3) = 6l - 9$ by Lemma 2.7.3, and thus by Lemma 5.8.5 we have

$$\dim L(\omega_3) \geq \binom{2l}{3} - 4l > \frac{(m_u(\omega_3) + 2)^2}{4}$$

if $l \geq 7$. Therefore by Lemma 5.1.1, no unipotent element of G acts on $L(\omega_3)$ as a distinguished unipotent element if $l \geq 7$.

Consider then $3 \leq l \leq 6$. In these cases for all non-regular distinguished $u \in G$, one can easily compute their labeled Dynkin diagram using Proposition 2.6.5, and then compute $m_u(\omega_3)$ using the fact that

$$\omega_3 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + 3\alpha_{l-1} + \frac{3}{2}\alpha_l.$$

For convenience we have listed this information in Table 5.3. One verifies that

$$\dim L(\omega_3) < \frac{(m_u(\omega_3) + 2)^2}{4}$$

in all cases, so it follows from Lemma 5.1.1 that a non-regular u does not act on $L(\omega_3)$ as a distinguished unipotent element.

Finally consider case where u is regular and $l = 6$, so now $V \downarrow K[u] = V_{12}$. By Lemma 5.8.5, in this case $\dim L(\omega_3) = 208$ if $p \neq 5$, and $\dim L(\omega_3) = 196$ if $p = 5$. If $p = 5$, then u has order 5^2. But then $\dim L(\omega_3) = 196$ and $\frac{(5^2+1)^2}{4} = 169$, so by Lemma 5.1.1 the element u does not act as a distinguished unipotent element on $L(\omega_3)$. Suppose then that $p \neq 5$. According to Lemma 5.8.5, we have $\wedge^3(V) \cong L(\omega_3) \oplus V$. Now computing the action of u on $L(\omega_3)$ is a matter of computing $\wedge^3(V_{12})$, and this is given in Table 5.2. One sees immediately from the table that in all cases some block of size $\neq 12$ has multiplicity ≥ 2 in $\wedge^3(V_{12})$, and thus u does not act as a distinguished unipotent element on $L(\omega_3)$. □

We can now prove Proposition 5.8.1 for G of type C_l. Suppose that $u \in G$ is a distinguished unipotent element and that u acts on $L(\omega_3)$ as a distinguished unipotent element. By Lemma 5.8.6, the element u is a regular unipotent element and $3 \leq l \leq 5$, so $V \downarrow K[u] = V_{2l}$.

If $l \equiv 1 \mod p$, then we have $l = 4$, $p = 3$, and $\dim L(\omega_3) = 40$ by Lemma 5.8.5. In this case u has order 3^2 and $\frac{(3^2+1)^2}{4} = 25 < 40$, so by Lemma 5.1.1 the action of u on $L(\omega_3)$ is not distinguished.

If $l \not\equiv 1 \mod p$, then $\wedge^3(V) \cong L(\omega_3) \oplus V$ by Lemma 5.8.5. Now taking the decomposition of $\wedge^3(V_{2l})$ for $3 \leq l \leq 5$ found in Table 5.2, removing a block of size $2l$ gives the decomposition of $L(\omega_3) \downarrow K[u]$. The result of Proposition 5.8.1 is then easily verified.

5.8.3 Computations

Let $G = \mathrm{SL}(V)$, where $\dim V = d \geq 4$. Fix a regular unipotent element $u \in G$, so now $V \downarrow K[u] = V_d$. In Table 5.2, we have given the decomposition of the $K[u]$-module $\wedge^3(V_d)$ into indecomposable summands. This table was generated by a computer calculation, as follows.

We know that $\wedge^3(V) \cong L(\omega_3)$, and by Lemma 2.7.3 we have $m_u(\omega_3) = 3d-9$. It follows then from Lemma 2.7.9 that for all $p > 3d-9$, the decomposition of $\wedge^3(V_d)$ is the same. Thus for any given d, we can find all the possible decompositions of $\wedge^3(V_d)$ with a finite computation: first we compute $\wedge^3(V_d)$ for all $2 < p \leq 3d-9$, and then for a single prime $p > 3d-9$. For all of the entries in Table 5.2, these computations can be quickly done with the aid of a computer program.

d	$\wedge^3(V_d)$	p	d	$\wedge^3(V_d)$	p
4	$[4]$	$p \geq 3$	10	$[2, 9^6, 10, 18^3]$	$p = 3$
				$[10^8, 20^2]$	$p = 5$
5	$[3, 7]$	$p = 3$		$[4, 7^2, 10^2, 12, 14, 16, 18, 22]$	$p = 7$
	$[5^2]$	$p = 5$		$[10, 11^{10}]$	$p = 11$
	$[3, 7]$	$p \geq 7$		$[6, 10, 13^8]$	$p = 13$
				$[4, 6, 8, 10^2, 14, 17^4]$	$p = 17$
6	$[6^2, 8]$	$p = 3$		$[4, 6, 8, 10^2, 12, 14, 18, 19^2]$	$p = 19$
	$[4, 6, 10]$	$p = 5$		$[4, 6, 8, 10^2, 12, 14, 16, 18, 22]$	$p \geq 23$
	$[6, 7^2]$	$p = 7$			
	$[4, 6, 10]$	$p \geq 11$	11	$[1, 7, 9^4, 12^2, 18^4, 25]$	$p = 3$
				$[1, 5, 9, 10^4, 15^3, 19, 21, 25]$	$p = 5$
7	$[1, 7, 9^3]$	$p = 3$		$[1, 7^3, 9, 11, 13, 14^2, 17, 19, 21, 25]$	$p = 7$
	$[1, 5, 7, 9, 13]$	$p = 5$		$[11^{15}]$	$p = 11$
	$[7^5]$	$p = 7$		$[9, 13^{12}]$	$p = 13$
	$[1, 5, 7, 11^2]$	$p = 11$		$[1, 5, 7, 9, 11, 13, 17^7]$	$p = 17$
	$[1, 5, 7, 9, 13]$	$p \geq 13$		$[1, 5, 7, 9^2, 11, 13, 15, 19^5]$	$p = 19$
				$[1, 5, 7, 9^2, 11, 13^2, 15, 17, 19, 23^2]$	$p = 23$
8	$[2, 9^6]$	$p = 3$		$[1, 5, 7, 9^2, 11, 13^2, 15, 17, 19, 21, 25]$	$p \geq 29$
	$[5^2, 8, 10, 12, 16]$	$p = 5$			
	$[6, 7^2, 8, 14^2]$	$p = 7$	12	$[6^2, 8, 12^6, 18^3, 24^2, 26]$	$p = 3$
	$[4, 8, 11^4]$	$p = 11$		$[4, 6, 8, 10^2, 12^2, 15^2, 16, 18, 20, 24, 25^2]$	$p = 5$
	$[4, 6, 8, 12, 13^2]$	$p = 13$		$[7^4, 12, 14^5, 16, 21^2, 24, 28]$	$p = 7$
	$[4, 6, 8, 10, 12, 16]$	$p \geq 17$		$[10, 11^{10}, 12, 22^4]$	$p = 11$
				$[12, 13^{16}]$	$p = 13$
9	$[3, 9^9]$	$p = 3$		$[4, 8, 10, 12, 16, 17^{10}]$	$p = 17$
	$[5^2, 10^4, 15, 19]$	$p = 5$		$[4, 6, 8, 10, 12^2, 16, 19^8]$	$p = 19$
	$[3, 7^2, 9, 11, 14^2, 19]$	$p = 7$		$[4, 6, 8, 10^2, 12^2, 14, 16^2, 20, 23^4]$	$p = 23$
	$[7, 11^7]$	$p = 11$		$[4, 6, 8, 10^2, 12^2, 14, 16^2, 18, 20, 22, 24, 28]$	$p \geq 29$
	$[3, 7, 9, 13^5]$	$p = 13$			
	$[3, 7^2, 9, 11, 13, 17^2]$	$p = 17$			
	$[3, 7^2, 9, 11, 13, 15, 19]$	$p \geq 19$			

Table 5.2: Decomposition of $\wedge^3(V_d)$ for $4 \leq d \leq 12$.

G	u	Dynkin diagram of u	$m_u(\omega_3)$	$\dim L(\omega_3)$
C_6	$[2,10]$	222202	21	208 ($p \neq 5$), 196 ($p = 5$)
C_6	$[4,8]$	220202	15	208 ($p \neq 5$), 196 ($p = 5$)
C_6	$[2,4,6]$	202002	11	208 ($p \neq 5$), 196 ($p = 5$)
C_5	$[2,8]$	22202	15	110
C_5	$[4,6]$	20202	11	110
C_4	$[2,6]$	2202	9	48 ($p \neq 3$), 40 ($p = 3$)
C_3	$[2,4]$	202	5	14

Table 5.3: Values of $m_u(\omega_3)$ for non-regular distinguished unipotent elements u in C_l, where $3 \leq l \leq 6$.

5.9 Representation $L_G(3\omega_1)$ for G of classical type ($p \neq 2,3$)

Assume that $p \neq 2,3$.

Suppose that G is simple of classical type. In this section, we will determine when a unipotent element $u \in G$ acts on $L(3\omega_1)$ as a distinguished unipotent element. The method used will be similar to that found in previous sections, in particular Section 5.8.

The answer is given by the following proposition. Note that for type A_l the representation $L(3\omega_1)$ is not self-dual if $l \geq 2$, so by Theorem 1.1.12 we only need to consider G of type B_l, C_l or D_l.

Proposition 5.9.1. *Suppose that G is a simple algebraic group of type B_l ($l \geq 2$), C_l ($l \geq 2$) or D_l ($l \geq 4$). A unipotent element u acts on the irreducible representation $L(3\omega_1)$ as a distinguished unipotent element if and only if u is a regular unipotent element and G, p are as in Table 5.1.*

G	u of order $> p$	u of order p
C_2	none	$p \geq 11$
C_3	none	$p \geq 17$
B_2	none	$p \geq 13$
B_3	none	$p \geq 19$

Table 5.1: $u \in G$ regular and acts on $L(3\omega_1)$ as a distinguished unipotent element

The proof of Proposition 5.9.1 will be given in the next two subsections.

5.9.1 Types B_l and D_l

In this subsection, let $G = \mathrm{SO}(V)$, where $\dim V = 2l+1$ ($l \geq 2$) or $\dim V = 2l$ ($l \geq 4$). We can find the irreducible module $L(3\omega_1)$ as a subquotient of the symmetric cube $S^3(V)$, as seen in the following lemma which is well known.

Lemma 5.9.2. *As a G-module, we have*

$$S^3(V) \cong \begin{cases} L(3\omega_1) \oplus L(\omega_1) & \text{if } p \mid \dim V + 2, \\ L(\omega_1)/L(3\omega_1)/L(\omega_1) \text{ (uniserial)} & \text{if } p \nmid \dim V + 2. \end{cases}$$

In particular, we have $\dim L(3\omega_1) \geq \binom{\dim V+2}{3} - 2\dim V$.

Proof. See [McN98, Proposition 4.7.4]. □

Lemma 5.9.3. *Let $u \in G$ be a distinguished unipotent element. If u acts on $L(3\omega_1)$ as a distinguished unipotent element, then u is regular and $\dim V = 5$, $\dim V = 7$ or $\dim V = 9$.*

Proof. Write $n = \dim V$. Let $u \in G$ be a regular unipotent element. By Lemma 2.7.3 we have

$$m_u(3\omega_1) = 3m_u(\omega_1) = \begin{cases} 3(n-1) & \text{if } n \text{ is odd,} \\ 3(n-2) & \text{if } n \text{ is even.} \end{cases}$$

Furthermore, by Lemma 5.9.2, we have $\dim L(3\omega_1) \geq \binom{n+2}{3} - 2n$. Now

$$\dim L(3\omega_1) \geq \binom{n+2}{3} - 2n > \frac{(m_u(3\omega_1)+2)^2}{4} \qquad (*)$$

if $n = 8$ or $n \geq 10$, so it follows from Lemma 5.1.1 that no unipotent element of G acts on $L(3\omega_1)$ as a distinguished unipotent element if $n = 8$ or $n \geq 10$.

If $n \leq 7$, then all distinguished unipotent elements of G are regular by Proposition 2.3.3. Note that we are assuming $n \geq 8$ if n is odd, so we are done in this case. What remains is to consider the case where $n = 9$. If $u \in G$ is a distinguished unipotent element that is not regular, then by Proposition 2.3.3 the element u lies in the unipotent class labeled by $[1, 3, 5]$. In this case $m_u(3\omega_1) = 3m_u(\omega_1) = 12$, and $\dim L(3\omega_1) \geq 147$ by Lemma 5.9.2, hence inequality (*) holds. By Lemma 5.1.1, the element u does not act on $L(3\omega_1)$ as a distinguished unipotent element. □

By Lemma 5.9.3, what remains is to verify Proposition 5.9.1 in the cases where $u \in G$ is a regular unipotent element and $\dim V = 5$, $\dim V = 7$ or $\dim V = 9$. If $p \nmid \dim V + 2$, then $S^3(V) \cong L(3\omega_1) \oplus V$ by Lemma 5.9.2 and then the claim follows from the data in Table 5.2. Suppose then that $p \mid \dim V + 2$. Then either $\dim V = 5$ and $p = 7$, or $\dim V = 9$ and $p = 11$. In both cases u has order p and $\dim L(3\omega_1) > \frac{(p+1)^2}{4}$, so u does not act as a distinguished unipotent element on $L(3\omega_1)$ by Lemma 5.1.1. This completes the proof of Proposition 5.9.1 for G of type B_l and D_l.

5.9.2 Type C_l

In this subsection, let $G = \mathrm{Sp}(V)$, where $\dim V = 2l$ ($l \geq 2$). In this case we have $L(3\omega_1) \cong S^3(V)$ (see e.g. [Sei87, 1.14 and 8.1 (c)] or [McN98, Proposition 4.2.2 (h)]), so $\dim L(3\omega_1) = \binom{2l+2}{3}$.

Lemma 5.9.4. *Let $u \in G$ be a unipotent. If u acts on $L(3\omega_1)$ as a distinguished unipotent element, then u is regular and $l \leq 4$.*

Proof. Let $u \in G$ be a regular unipotent element. Then $m_u(3\omega_1) = 3m_u(\omega_1) = 3(2l-1) = 6l-3$ by Lemma 2.7.3. Thus we have

$$\dim L(3\omega_1) = \binom{2l+2}{3} > \frac{(m_u(3\omega_1)+2)^2}{4} \qquad (*)$$

for all $l \geq 5$. It follows then from Lemma 5.1.1 that no unipotent element of G acts on $L(3\omega_1)$ as a distinguished unipotent element if $l \geq 5$.

Suppose then that $l \leq 4$ and let $u \in G$ be a distinguished unipotent element. In this case, it is straightforward to verify that the inequality (*) holds if $u \in G$ is not a regular unipotent element. Indeed, if $l = 2$ then there are no non-regular distinguished unipotent elements. If $l = 3$, then u is in class labeled by $[2, 4]$ and $\dim L(3\omega_1) = 56$, $m_u(3\omega_1) = 9$. If $l = 4$, then u is in class labeled by $[2, 6]$ and $\dim L(3\omega_1) = 120$, $m_u(3\omega_1) = 15$. Therefore by Lemma 5.1.1, a non-regular unipotent element of G does not act on $L(3\omega_1)$ as a distinguished unipotent element. □

The claim of Proposition 5.9.1 for G of type C_l follows now from Lemma 5.9.4 and the information given in Table 5.2.

5.9.3 Computations

Let $G = \mathrm{SL}(V)$, where $\dim V = d \geq 4$. Fix a regular unipotent element $u \in G$, so now $V \downarrow K[u] = V_d$. In Table 5.2, we have given the decomposition of the $K[u]$-module $S^3(V_d)$ into indecomposable summands. This table was generated by a computer calculation, similarly to Table 5.2. Here we use the fact that $S^3(V) \cong L(3\omega_1)$ for $p > 3$, and $m_u(3\omega_1) = 3d - 3$. It follows as in subsection 5.8.3 that it will suffice to compute $S^3(V_d)$ for all $2 < p \leq 3d - 3$, and for a single prime $p > 3d - 3$.

d	$S^3(V_d)$	p
4	$[2,3,6,9]$	$p=3$
	$[5^4]$	$p=5$
	$[6,7^2]$	$p=7$
	$[4,6,10]$	$p\geq 11$
5	$[3,5,9^3]$	$p=3$
	$[5^7]$	$p=5$
	$[7^5]$	$p=7$
	$[1,5,7,11^2]$	$p=11$
	$[1,5,7,9,13]$	$p\geq 13$
6	$[3,8,9^5]$	$p=3$
	$[5^4,10^2,16]$	$p=5$
	$[7^8]$	$p=7$
	$[4,8,11^4]$	$p=11$
	$[4,6,8,12,13^2]$	$p=13$
	$[4,6,8,10,12,16]$	$p\geq 17$
7	$[3,9^9]$	$p=3$
	$[5^2,7,10^2,13,15,19]$	$p=5$
	$[7^{12}]$	$p=7$
	$[7,11^7]$	$p=11$
	$[3,7,9,13^5]$	$p=13$
	$[3,7^2,9,11,13,17^2]$	$p=17$
	$[3,7^2,9,11,13,15,19]$	$p\geq 19$
8	$[3,9^{13}]$	$p=3$
	$[4,6,8,10^2,12,15^2,20^2]$	$p=5$
	$[7^8,14^3,22]$	$p=7$
	$[10,11^{10}]$	$p=11$
	$[6,10,13^8]$	$p=13$
	$[4,6,8,10^2,14,17^4]$	$p=17$
	$[4,6,8,10^2,12,14,18,19^2]$	$p=19$
	$[4,6,8,10^2,12,14,16,18,22]$	$p\geq 23$
9	$[3,9^{18}]$	$p=3$
	$[1,5,9,10^4,15^2,20^4]$	$p=5$
	$[7^5,9,14^4,19,21,25]$	$p=7$
	$[11^{15}]$	$p=11$
	$[9,13^{12}]$	$p=13$
	$[1,5,7,9,11,13,17^7]$	$p=17$
	$[1,5,7,9^2,11,13,15,19^5]$	$p=19$
	$[1,5,7,9^2,11,13^2,15,17,19,23^2]$	$p=23$
	$[1,5,7,9^2,11,13^2,15,17,19,21,25]$	$p\geq 29$
10	$[4,9^{13},18^4,27]$	$p=3$
	$[10^8,20^7]$	$p=5$
	$[6,7^2,8,10,12,14^3,16,18,21^2,24,28]$	$p=7$
	$[11^{20}]$	$p=11$
	$[12,13^{16}]$	$p=13$
	$[4,8,10,12,16,17^{10}]$	$p=17$
	$[4,6,8,10,12^2,16,19^8]$	$p=19$
	$[4,6,8,10^2,12^2,14,16^2,20,23^4]$	$p=23$
	$[4,6,8,10^2,12^2,14,16^2,18,20,22,24,28]$	$p\geq 29$

Table 5.2: Decomposition of $S^3(V_d)$ for $4 \leq d \leq 10$.

5.10 Spin representations ($p \neq 2$)

Assume that $p \neq 2$.

Let G be simple of type B_l ($l \geq 3$) or type D_l ($l \geq 4$). In this section, we determine when a unipotent element u of G acts as a distinguished unipotent element on the irreducible representation $L_G(\omega_l)$.

In fact, we do a bit more: we determine when a distinguished unipotent element u of G acts on $L_G(\omega_l)$ such that all Jordan block sizes are distinct and all even or all odd (recall that $L_{D_l}(\omega_l)$ is not self-dual if l is odd). The precise result is the following proposition.

Proposition 5.10.1. *Suppose that G is simple of type B_l ($l \geq 3$) or D_l ($l \geq 4$). A distinguished unipotent element $u \in G$ acts on the irreducible representation $L_G(\omega_l)$ such that all Jordan block sizes are distinct and all even or all odd if and only if G and p are as in Table 5.1.*

G	Class of u	u of order $> p$	u of order p
D_4	regular	$p = 3, 5$	$p \geq 7$
D_4	$[5, 3]$	$p = 3$	$p \geq 5$
D_5	regular	$p = 3, 5, 7$	$p \geq 11$
D_5	$[7, 3]$	$p = 3, 5$	$p \geq 11$
D_6	regular	$p = 3, 5, 7$	$p \geq 17$
D_6	$[9, 3]$	$p = 7$	$p \geq 13$
D_7	regular	$p = 3, 5$	$p \geq 23$
D_7	$[11, 3]$	$p = 7$	$p \geq 17$
D_8	regular	$p = 11$	$p \geq 29$
D_9	regular	none	$p \geq 37$
B_3	regular	$p = 3, 5$	$p \geq 7$
B_4	regular	$p = 3, 5, 7$	$p \geq 11$
B_5	regular	$p = 3, 5, 7$	$p \geq 17$
B_6	regular	$p = 3, 5$	$p \geq 23$
B_7	regular	$p = 11$	$p \geq 29$
B_8	regular	none	$p \geq 37$

Table 5.1: For $G = B_l$ and $G = D_l$: all cases where a distinguished unipotent element $u \in G$ acts on $L_G(\omega_l)$ such that all Jordan block sizes are distinct and all even or all odd.

Lemma 5.10.2. *Let $G = D_l$, $l \geq 5$ and let $u \in G$ be a distinguished unipotent element. If u acts on $L_G(\omega_l)$ such that all Jordan block sizes are distinct and all even or all odd, then one of the following holds.*

(i) *The element u is a regular unipotent element and $l \leq 10$.*

(ii) *The element u lies in the class $[2l-3, 3]$ of G and $l \leq 8$.*

Proof. Let $u \in G$ be a regular unipotent element. Then $m_u(\omega_l) = \frac{l(l-1)}{2}$ by Lemma 2.7.3. From this one can verify that

$$\dim L_G(\omega_l) = 2^{l-1} > \frac{(m_u(\omega_l)+2)^2}{4} \qquad (*)$$

for all $l \geq 11$. Thus if some unipotent element of G acts on $L(\omega_l)$ such that all Jordan block sizes are distinct and all even or all odd, we have $l \leq 10$ by Lemma 5.1.1 (see Remark 5.1.2).

Suppose then that $l \leq 10$ and let $u \in G$ be a distinguished unipotent element. In this situation, it is straightforward to verify from Table 5.2 that the inequality (*) holds, except when we are in case (i) or (ii) of the claim. Thus the claim follows again from Lemma 5.1.1. □

With Lemma 5.10.2, the claim of Proposition 5.10.1 for G of type D_l, $l \geq 5$, follows from the data given in Table 5.5 and Table 5.4. The Jordan block sizes given in these tables were computed with MAGMA (Section 2.9). In the case where $l = 4$, Proposition 5.10.1 for $G = D_l$ follows from Proposition 2.10.2 (i) since $L_G(\omega_4)$ is a twist of the natural representation by a triality graph automorphism.

For G of type B_l, we can reduce our computations to the case of type D_l with the following lemma.

Lemma 5.10.3 ([Sei87, Table 1, IV_1]). *Let $l \geq 4$ and consider $B_{l-1} < D_l$ naturally embedded. Then $L_{D_l}(\omega_l) \downarrow B_{l-1} \cong L_{B_{l-1}}(\omega_{l-1})$.*

Note that B_{l-1} naturally embedded in D_l contains a regular unipotent element of D_l. Thus Proposition 5.10.1 for G of type B_l follows from the next lemma, combined with Lemma 5.10.3, and Proposition 5.10.1 for type D_l.

Lemma 5.10.4. *Let $G = B_l$, $l \geq 3$ and let $u \in G$ be a distinguished unipotent element. If u acts on $L_G(\omega_l)$ such that all Jordan block sizes are distinct and all even or all odd, then u is a regular unipotent element and $l \leq 9$.*

Proof. We proceed as in Lemma 5.10.2. For a regular unipotent element $u \in G$, we have $m_u(\omega_l) = \frac{l(l+1)}{2}$ by Lemma 2.7.3. From this we see that

$$\dim L_G(\omega_l) = 2^l > \frac{(m_u(\omega_l) + 2)^2}{4} \tag{*}$$

for all $l \geq 10$. Thus if some unipotent element of G acts on $L(\omega_l)$ such that all Jordan block sizes are distinct and all even or all odd, we have $l \leq 9$ by Lemma 5.1.1.

Suppose then that $l \leq 9$ and let $u \in G$ be a distinguished unipotent element. As in Lemma 5.10.2, one verifies from Table 5.3 that the inequality (*) holds for all non-regular distinguished unipotent elements of G. Thus the claim follows from Lemma 5.1.1. □

G	u	Labeled diagram of u	$m_u(\omega_l)$	$\dim L_G(\omega_l)$
D_{10}	$[19,1]$	2222222222	45	512
	$[17,3]$	2222222022	37	512
	$[15,5]$	2222202022	31	512
	$[13,7]$	2220202022	27	512
	$[11,9]$	2020202022	25	512
	$[11,5,3,1]$	2220200200	19	512
	$[9,7,3,1]$	2020200200	17	512
D_9	$[17,1]$	222222222	36	256
	$[15,3]$	222222022	29	256
	$[13,5]$	222202022	24	256
	$[11,7]$	220202022	21	256
	$[9,5,3,1]$	220200200	14	256
D_8	$[15,1]$	22222222	28	128
	$[13,3]$	22222022	22	128
	$[11,5]$	22202022	18	128
	$[9,7]$	20202022	16	128
	$[7,5,3,1]$	20200200	10	128
D_7	$[13,1]$	2222222	21	64
	$[11,3]$	2222022	16	64
	$[9,5]$	2202022	13	64
D_6	$[11,1]$	222222	15	32
	$[9,3]$	222022	11	32
	$[7,5]$	202022	9	32
D_5	$[9,1]$	22222	10	16
	$[7,3]$	22022	7	16

Table 5.2: Values of $m_u(\omega_l)$ for distinguished unipotent elements u in D_l, where $5 \leq l \leq 10$.

G	u	Labeled diagram of u	$m_u(\omega_l)$	$\dim L_G(\omega_l)$
B_9	$[19]$	222222222	45	512
	$[15,3,1]$	222222020	29	512
	$[13,5,1]$	222202020	24	512
	$[11,7,1]$	220202020	21	512
	$[11,5,3]$	222020020	19	512
	$[9,7,3]$	202020020	17	512
B_8	$[17]$	22222222	36	256
	$[13,3,1]$	22222020	22	256
	$[11,5,1]$	22202020	18	256
	$[9,7,1]$	20202020	16	256
	$[9,5,3]$	22020020	14	256
B_7	$[15]$	2222222	28	128
	$[11,3,1]$	2222020	16	128
	$[9,5,1]$	2202020	13	128
	$[7,5,3]$	2020020	10	128
B_6	$[13]$	222222	21	64
	$[9,3,1]$	222020	11	64
	$[7,5,1]$	202020	9	64
B_5	$[11]$	22222	15	32
	$[7,3,1]$	22020	7	32
B_4	$[9]$	2222	10	16
	$[5,3,1]$	2020	4	16
B_3	$[7]$	222	6	8

Table 5.3: Values of $m_u(\omega_l)$ for distinguished unipotent elements u in B_l, where $3 \leq l \leq 9$.

G	$L_G(\omega_l) \downarrow K[u]$	p
D_{10}	$[4, 9^4, 18^4, 22, 27^{14}]$	$p = 3$
	$[10^2, 12, 15^2, 25^{18}]$	$p = 5$
	$[6, 7^2, 14^5, 16, 21^4, 22, 28^4, 32, 35^2, 40, 46]$	$p = 7$
	$[10, 11^6, 14, 22^8, 28, 33^4, 40, 46]$	$p = 11$
	$[4, 6, 10, 13^4, 16, 18^2, 20, 22^2, 26^3, 28, 30, 32, 34, 36, 40, 46]$	$p = 13$
	$[10, 17^{14}, 26, 34^7]$	$p = 17$
	$[18, 19^{26}]$	$p = 19$
	$[4, 10, 16, 22, 23^{20}]$	$p = 23$
	$[4, 6, 10^2, 14, 16^2, 18, 20, 22, 28, 29^{12}]$	$p = 29$
	$[4, 6, 10^2, 12, 14, 16, 18^2, 20, 22, 24, 28, 31^{10}]$	$p = 31$
	$[4, 6, 10^2, 12, 14, 16^2, 18^2, 20, 22^2, 24, 26, 28, 30, 32, 36, 37^4]$	$p = 37$
	$[4, 6, 10^2, 12, 14, 16^2, 18^2, 20, 22^2, 24, 26, 28^2, 30, 32, 34, 40, 41^2]$	$p = 41$
	$[4, 6, 10^2, 12, 14, 16^2, 18^2, 20, 22^2, 24, 26, 28^2, 30, 32, 34, 36, 43^2]$	$p = 43$
	$[4, 6, 10^2, 12, 14, 16^2, 18^2, 20, 22^2, 24, 26, 28^2, 30, 32, 34, 36, 40, 46]$	$p \geq 47$
D_9	$[9^4, 13, 18^4, 27^5]$	$p = 3$
	$[1, 7, 10^2, 15, 17, 21, 25^7]$	$p = 5$
	$[1, 7, 9, 11, 14^2, 17, 19, 21, 23, 25, 27, 31, 37]$	$p = 7$
	$[1, 7, 11^3, 15, 17, 19, 22^2, 25, 27, 31, 37]$	$p = 11$
	$[1, 7, 9, 13^3, 17, 19, 21, 23, 26^2, 31, 37]$	$p = 13$
	$[1, 17^{15}]$	$p = 17$
	$[9, 19^{13}]$	$p = 19$
	$[1, 7, 11, 13, 17, 23^9]$	$p = 23$
	$[1, 7, 9, 11, 13, 15, 17, 19, 23, 25, 29^4]$	$p = 29$
	$[1, 7, 9, 11, 13, 15, 17, 19, 21, 23, 27, 31^3]$	$p = 31$
	$[1, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 31, 37]$	$p \geq 37$
D_8	$[7, 9, 12^2, 18^2, 25, 27]$	$p = 3$
	$[10^4, 20^2, 23, 25]$	$p = 5$
	$[7^2, 14^2, 15, 21^2, 29]$	$p = 7$
	$[5, 9, 11, 15, 17, 19, 23, 29]$	$p = 11$
	$[5, 13^4, 19, 26^2]$	$p = 13$
	$[9, 17^7]$	$p = 17$
	$[5, 11, 17, 19^5]$	$p = 19$
	$[5, 9, 11, 15, 19, 23^3]$	$p = 23$
	$[5, 9, 11, 15, 17, 19, 23, 29]$	$p \geq 29$
D_7	$[4, 10, 12, 16, 22]$	$p = 3$
	$[4, 10, 12, 16, 22]$	$p = 5$
	$[7^2, 14^2, 22]$	$p = 7$
	$[4, 11^2, 16, 22]$	$p = 11$
	$[12, 13^4]$	$p = 13$
	$[4, 10, 16, 17^2]$	$p = 17$
	$[4, 10, 12, 19^2]$	$p = 19$
	$[4, 10, 12, 16, 22]$	$p \geq 23$
D_6	$[6, 10, 16]$	$p = 3$
	$[6, 10, 16]$	$p = 5$
	$[6, 10, 16]$	$p = 7$
	$[10, 11^2]$	$p = 11$
	$[6, 13^2]$	$p = 13$
	$[6, 10, 16]$	$p \geq 17$
D_5	$[7, 9]$	$p = 3$
	$[5, 11]$	$p = 5$
	$[5, 11]$	$p = 7$
	$[5, 11]$	$p \geq 11$

Table 5.4: Action of a regular unipotent element u of $G = D_l$ on $L_G(\omega_l)$, where $5 \leq l \leq 10$.

G	$L_G(\omega_l) \downarrow K[u]$	p
D_8	$[3, 5, 9, 11, 12^2, 15, 17, 21, 23]$	$p = 3$
	$[3, 5, 10^2, 11, 13, 15, 17, 21, 23]$	$p = 5$
	$[7^4, 14^4, 21, 23]$	$p = 7$
	$[3, 5, 11^4, 15, 17, 22^2]$	$p = 11$
	$[11, 13^9]$	$p = 13$
	$[3, 5, 9, 11, 15, 17^5]$	$p = 17$
	$[3, 5, 9, 11^2, 13, 19^4]$	$p = 19$
	$[3, 5, 9, 11^2, 13, 15, 17, 21, 23]$	$p \geq 23$
D_7	$[6^2, 9, 11, 15, 17]$	$p = 3$
	$[5, 7, 10^2, 15, 17]$	$p = 5$
	$[5, 7, 9, 11, 15, 17]$	$p = 7$
	$[9, 11^5]$	$p = 11$
	$[5, 7, 13^4]$	$p = 13$
	$[5, 7, 9, 11, 15, 17]$	$p \geq 17$

G	$K[u] \downarrow L_G(\omega_l)$	p
D_6	$[6, 8, 9^2]$	$p = 3$
	$[5^2, 10, 12]$	$p = 5$
	$[4, 6, 10, 12]$	$p = 7$
	$[4, 6, 11^2]$	$p = 11$
	$[4, 6, 10, 12]$	$p \geq 13$
D_5	$[2, 6, 8]$	$p = 3$
	$[2, 6, 8]$	$p = 5$
	$[2, 7^2]$	$p = 7$
	$[2, 6, 8]$	$p \geq 11$

Table 5.5: Action of a unipotent element u in conjugacy class $[2l-3, 3]$ of $G = D_l$ on $L_G(\omega_l)$, where $5 \leq l \leq 8$.

5.11 Spin representations ($p = 2$)

Assume that $p = 2$.

In this section we will determine for G of type C_l ($l \geq 3$) and D_l ($l \geq 4$) when a unipotent element $u \in G$ acts as a distinguished unipotent element on the irreducible representation $L_G(\omega_l)$ (and also $L_G(\omega_{l-1})$ if G has type D_l). The answer is given by Proposition 5.11.3 and Proposition 5.11.4 below; it is also recorded in Table 1.3.

Since we are in characteristic two, we can and we will consider $D_l < C_l$ as the subsystem subgroup generated by the short root subgroups. The following lemma allows us to reduce our computations to the case where G is of type C_l.

Lemma 5.11.1. *$L_{C_l}(\omega_l) \downarrow D_l \cong L_{D_l}(\omega_l) \oplus L_{D_l}(\omega_{l-1})$.*

Proof. This is a consequence of [For96, Theorem 3.3, U_6 in Table II]. □

Corollary 5.11.2. *Let $l \geq 4$ be even. Let $u \in D_l < C_l$ be a distinguished unipotent element. Suppose that the image of u lies in the conjugacy class labeled by*

$$(d(1)^{n_1}_{\varepsilon(d(1))}, d(2)^{n_2}_{\varepsilon(d(2))}, \ldots, d(t)^{n_t}_{\varepsilon(d(t))})$$

of $\mathrm{Sp}(L_{C_l}(\omega_l))$ (Corollary 2.4.7). Then the image of u in $\mathrm{Sp}(L_{D_l}(\omega_l))$, and the image of u in $\mathrm{Sp}(L_{D_l}(\omega_{l-1}))$ lies in the conjugacy class labeled by

$$(d(1)^{n_1/2}_{\varepsilon(d(1))}, d(2)^{n_2/2}_{\varepsilon(d(2))}, \ldots, d(t)^{n_t/2}_{\varepsilon(d(t))}).$$

Proof. By Lemma 5.11.1, the module $L_{C_l}(\omega_l)$ has a D_l-submodule W isomorphic to $L_{D_l}(\omega_l)$. Note that $W \cong W^*$ because l is even (see Table 4.1). We claim that W is a non-degenerate subspace with respect to any non-degenerate C_l-invariant form on $L_{C_l}(\omega_l)$. If not, then $W^\perp \cap W \neq 0$, so $W \cap W^\perp = W$ and $W \subseteq W^\perp$ since W is irreducible as a D_l-module. But then $V/W^\perp \cong W^* \cong W$ and so W occurs twice as a D_l-composition factor of $L_{C_l}(\omega_l)$, contradicting Lemma 5.11.1.

Let $\tau : D_l \to D_l$ be the usual automorphism of D_l induced by the graph automorphism swapping the two end nodes of the Dynkin diagram of type D_l. Now $L_{D_l}(\omega_{l-1})^\tau \cong L_{D_l}(\omega_l)$, so it follows from Proposition 2.10.2 (ii) that the conjugacy class of the image of u in $\mathrm{Sp}(L_{D_l}(\omega_l))$ and $\mathrm{Sp}(L_{D_l}(\omega_{l-1}))$ is the same. Thus by Lemma 5.11.1 each n_i is even, and the image of u in $\mathrm{Sp}(L_{D_l}(\omega_l))$ and $\mathrm{Sp}(L_{D_l}(\omega_{l-1}))$ lies in the conjugacy class labeled by

$$(d(1)^{n_1/2}_{\epsilon_1}, d(2)^{n_2/2}_{\epsilon_2}, \ldots, d(t)^{n_t/2}_{\epsilon_t})$$

for some $\epsilon_i \in \{0, 1\}$.

Finally, what we did in the first paragraph shows that $L_{C_l}(\omega_l) \downarrow D_l = L_{D_l}(\omega_l) \oplus L_{D_l}(\omega_{l-1})$ as an orthogonal direct sum. Therefore we can conclude that as a $K[u]$-module, $L_{C_l}(\omega_l)$ has $V(d(i))$ as an orthogonal direct summand if and only if $L_{D_l}(\omega_l)$ has $V(d(i))$ as an orthogonal direct summand. By Lemma 2.4.5, we have $\epsilon_i = \varepsilon(d(i))$ for all i.

□

Proposition 5.11.3. *Suppose that G is of type C_l ($l \geq 3$) and let $u \in G$ be a distinguished unipotent element. Then u acts on $L(\omega_l)$ as a distinguished unipotent element if and only if u occurs as one of the boldface entries in Table 5.1.*

Proof. Suppose that u acts on $L(\omega_l)$ as a distinguished unipotent element. By Proposition 5.2.4, we have $3 \leq l \leq 7$ or $l = 9$. For these cases, the orthogonal decomposition of $L(\omega_l) \downarrow K[u]$ can be computed with MAGMA (Section 2.9), and the decompositions are given in Table 5.1 and Table 5.2 (using the labeling given in Corollary 2.4.7); one can easily verify the claim from the tables. □

Proposition 5.11.4. *Suppose that G is of type D_l ($l \geq 4$) and let $u \in G$ be a distinguished unipotent element. Then u acts on $L(\omega_l)$ (or $L(\omega_{l-1})$) as a distinguished unipotent element if and only if $l = 4$ and u is in class $(2_1, 6_1)$ or (4_1^2), or $l = 6$ and u is in class $(2_1, 10_1)$.*

Proof. Suppose that u acts on $L(\omega_l)$ or $L(\omega_{l-1})$ as a distinguished unipotent element. Now $l \in \{4, 6, 8, 10\}$ by Proposition 5.2.4. With Corollary 5.11.2 and Table 5.1 - 5.3, one can verify that the proposition holds. We illustrate this with an example and leave the rest to the reader. Consider a regular unipotent element u of G when $l = 10$. Now u lies in the unipotent conjugacy class of C_{10} labeled by $(2_1, 18_1)$, so we find in Table 5.3 that its image in $\mathrm{Sp}(L_{C_{10}}(\omega_{10}))$ lies in the unipotent class labeled by

$$(2_1^2, 6_1^2, 10_1^2, 14_1^2, 16_0^{24}, 18_1^2, 22_1^2, 26_1^2, 30_1^2, 32_0^{12}).$$

Then by Corollary 5.11.2, the image of u in $\mathrm{Sp}(L_G(\omega_{10}))$ lies in the unipotent class labeled by

$$(2_1^1, 6_1, 10_1, 14_1, 16_0^{12}, 18_1, 22_1, 26_1, 30_1, 32_0^6).$$

In particular, we find that u does not act as a distinguished unipotent element on $L(\omega_{10})$. □

Remark 5.11.5. In our proofs we have relied on tables 5.1 - 5.3, which were computed with MAGMA (Section 2.9). We note that most of the entries are not necessary for our proof, but we have included them for completeness. For example, when $G = C_9$, a non-regular distinguished unipotent element $u \in G$ has order at most 2^4; therefore $\frac{(|u|+1)^2}{4} < 2^9 = \dim L_G(\omega_9)$ and so u does not act on $L_G(\omega_9)$ as a distinguished unipotent element by Lemma 5.2.1. Hence for $G = C_9$, we actually only have to compute the action of a regular unipotent element $u \in G$ on $L_G(\omega_9)$. For $G = C_l$ with $3 \leq l \leq 7$, similar arguments can rule out most distinguished unipotent classes without computing their action on $L_G(\omega_l)$. Similar remarks also apply in the case where $G = D_l$.

Furthermore, when $G = C_l$, the computations given in tables 5.1 - 5.3 can be reduced to the case where $u \in G$ a regular unipotent element. For this, suppose that $G = \mathrm{Sp}(V)$. If $u \in G$ is a distinguished unipotent element which is not regular, then there is an orthogonal decomposition $V = V_1 \oplus \cdots \oplus V_t$ such that u is contained in $\mathrm{Sp}(V_1) \times \cdots \times \mathrm{Sp}(V_t)$ and acts on each V_i with a single Jordan block. Set $\dim V_i = 2d_i$, so now $\mathrm{Sp}(V_i)$ is simple of type C_{d_i}. We can write $u = u_1 \cdots u_t$, where $u_i \in \mathrm{Sp}(V_i)$ is a regular unipotent element of $\mathrm{Sp}(V_i)$. It follows from [Sei87, Theorem 4.1] that the restriction of $L_G(\omega_l)$ to $\mathrm{Sp}(V_1) \times \cdots \times \mathrm{Sp}(V_t)$ is irreducible, and isomorphic to the outer tensor product

$$L_{C_{d_1}}(\omega_{d_1}) \otimes \cdots \otimes L_{C_{d_t}}(\omega_{d_t}).$$

Thus if we know the orthogonal decompositions $L_{C_{d_i}}(\omega_{d_i}) \downarrow K[u_i]$ for all $1 \leq i \leq t$, by computing the orthogonal decomposition of their tensor product (see Section 3.6) we can find the orthogonal decomposition of $L_G(\omega_l) \downarrow K[u]$.

l	Class of u	Action of u on $L(\omega_l)$
3	**regular**	$\mathbf{2_1, 6_1}$
3	$(2_1, 4_1)$	4_0^2
4	**regular**	$\mathbf{8_1^2}$
4	$(\mathbf{2_1, 6_1})$	$\mathbf{2_1^2, 6_1^2}$
4	(4_1^2)	4_1^4
4	$(2_1^2, 4_1)$	4_0^4
5	**regular**	$\mathbf{2_1, 6_1, 10_1, 14_1}$
5	$(2_1, 8_1)$	8_0^4
5	$(4_1, 6_1)$	$4_0^4, 8_0^2$
5	$(2_1^2, 6_1)$	$2_1^4, 6_1^4$
5	$(2_1, 4_1^2)$	4_0^8
6	regular	$4_1^2, 12_1^2, 16_0^2$
6	$(\mathbf{2_1, 10_1})$	$\mathbf{2_1^2, 6_1^2, 10_1^2, 14_1^2}$
6	$(4_1, 8_1)$	8_0^8
6	$(2_1^2, 8_1)$	8_0^8
6	(6_1^2)	$2_1^4, 6_1^4, 8_0^4$
6	$(2_1, 4_1, 6_1)$	$4_0^8, 8_0^4$
6	$(2_1^2, 4_1^2)$	4_0^{16}
7	regular	$2_1, 6_1, 10_1, 14_1, 16_0^6$
7	$(2_1, 12_1)$	$4_0^4, 12_0^4, 16_0^4$
7	$(4_1, 10_1)$	$4_0^4, 8_0^4, 12_0^4, 16_0^2$
7	$(2_1^2, 10_1)$	$2_1^4, 6_1^4, 10_1^4, 14_1^4$
7	$(6_1, 8_1)$	8_0^{16}
7	$(2_1, 4_1, 8_1)$	8_0^{16}
7	$(2_1, 6_1^2)$	$2_1^8, 6_1^8, 8_0^8$
7	$(4_1^2, 6_1)$	$4_0^{16}, 8_0^8$
7	$(2_1^2, 4_1, 6_1)$	$4_0^{16}, 8_0^8$

Table 5.1: For $G = C_l$ with $3 \leq l \leq 7$, actions of distinguished unipotent elements $u \in G$ on $L_G(\omega_l)$.

l	Conjugacy class	Action of u on $L_G(\omega_l)$
8	(16_1)	16_1^{16}
8	$(2_1, 14_1)$	$2_1^2, 6_1^2, 10_1^2, 14_1^2, 16_0^{12}$
8	$(4_1, 12_1)$	$4_1^8, 12_1^8, 16_0^8$
8	$(2_1^2, 12_1)$	$4_0^8, 12_0^8, 16_0^8$
8	$(6_1, 10_1)$	$2_1^4, 6_1^4, 8_0^8, 10_1^4, 14_1^4, 16_0^4$
8	$(2_1, 4_1, 10_1)$	$4_0^8, 8_0^8, 12_0^8, 16_0^4$
8	(8_1^2)	8_1^{32}
8	$(2_1, 6_1, 8_1)$	8_0^{32}
8	$(4_1^2, 8_1)$	8_0^{32}
8	$(2_1^2, 4_1, 8_1)$	8_0^{32}
8	$(4_1, 6_1^2)$	$4_0^{16}, 8_0^{24}$
8	$(2_1^2, 6_1^2)$	$2_1^{16}, 6_1^{16}, 8_0^{16}$
8	$(2_1, 4_1^2, 6_1)$	$4_0^{32}, 8_0^{16}$
9	(18_1)	$2_1, 6_1, 10_1, 14_1, 16_0^{12}, 18_1, 22_1, 26_1, 30_1, 32_0^6$
9	$(2_1, 16_1)$	16_0^{32}
9	$(4_1, 14_1)$	$4_0^4, 8_0^4, 12_0^4, 16_0^{26}$
9	$(2_1^2, 14_1)$	$2_1^4, 6_1^4, 10_1^4, 14_1^4, 16_0^{24}$
9	$(6_1, 12_1)$	$4_0^8, 8_0^8, 12_0^8, 16_0^{20}$
9	$(2_1, 4_1, 12_1)$	$4_0^{16}, 12_0^{16}, 16_0^{16}$
9	$(8_1, 10_1)$	$8_0^{32}, 16_0^{16}$
9	$(2_1, 6_1, 10_1)$	$2_1^8, 6_1^8, 8_0^{16}, 10_1^8, 14_1^8, 16_0^8$
9	$(4_1^2, 10_1)$	$4_0^{16}, 8_0^{16}, 12_0^{16}, 16_0^8$
9	$(2_1^2, 4_1, 10_1)$	$4_0^{16}, 8_0^{16}, 12_0^{16}, 16_0^8$
9	$(2_1, 8_1^2)$	8_0^{64}
9	$(4_1, 6_1, 8_1)$	8_0^{64}
9	$(2_1^2, 6_1, 8_1)$	8_0^{64}
9	$(2_1, 4_1^2, 8_1)$	8_0^{64}
9	$(2_1, 4_1, 6_1^2)$	$4_0^{32}, 8_0^{48}$
9	$(2_1^2, 4_1^2, 6_1)$	$4_0^{64}, 8_0^{32}$

Table 5.2: For $G = C_l$ with $l \in \{8, 9\}$, actions of distinguished unipotent elements $u \in G$ on $L_G(\omega_l)$.

Conjugacy class	Action of u on $L_G(\omega_{10})$
(20_1)	$4_1^4, 12_1^4, 16_0^8, 20_1^4, 28_1^4, 32_0^{20}$
$(2_1, 18_1)$	$2_1^2, 6_1^2, 10_1^2, 14_1^2, 16_0^{24}, 18_1^2, 22_1^2, 26_1^2, 30_1^2, 32_0^{12}$
$(4_1, 16_1)$	16_0^{64}
$(2_1^2, 16_1)$	16_0^{64}
$(6_1, 14_1)$	$2_1^4, 6_1^4, 8_0^8, 10_1^4, 14_1^4, 16_0^{52}$
$(2_1, 4_1, 14_1)$	$4_0^8, 8_0^8, 12_0^8, 16_0^{52}$
$(8_1, 12_1)$	$8_0^{32}, 16_0^{48}$
$(2_1, 6_1, 12_1)$	$4_0^{16}, 8_0^{16}, 12_0^{16}, 16_0^{40}$
$(4_1^2, 12_1)$	$4_1^{32}, 12_1^{32}, 16_0^{32}$
$(2_1^2, 4_1, 12_1)$	$4_0^{32}, 12_0^{32}, 16_0^{32}$
(10_1^2)	$2_1^8, 6_1^8, 8_0^{16}, 10_1^8, 14_1^8, 16_0^{40}$
$(2_1, 8_1, 10_1)$	$8_0^{64}, 16_0^{32}$
$(4_1, 6_1, 10_1)$	$4_0^{16}, 8_0^{48}, 12_0^{16}, 16_0^{24}$
$(2_1^2, 6_1, 10_1)$	$2_1^{16}, 6_1^{16}, 8_0^{32}, 10_1^{16}, 14_1^{16}, 16_0^{16}$
$(2_1, 4_1^2, 10_1)$	$4_0^{32}, 8_0^{32}, 12_0^{32}, 16_0^{16}$
$(4_1, 8_1^2)$	8_0^{128}
$(2_1^2, 8_1^2)$	8_0^{128}
$(6_1^2, 8_1)$	8_0^{128}
$(2_1, 4_1, 6_1, 8_1)$	8_0^{128}
$(2_1^2, 4_1^2, 8_1)$	8_0^{128}
$(4_1^2, 6_1^2)$	$4_0^{64}, 8_0^{96}$
$(2_1^2, 4_1, 6_1^2)$	$4_0^{64}, 8_0^{96}$

Table 5.3: For $G = C_{10}$, actions of distinguished unipotent elements $u \in G$ on $L_G(\omega_{10})$.

5.12 Fundamental irreducible representations ($p \neq 2$)

Assume that $p \neq 2$.

In this section we determine when a unipotent element $u \in G$ of a simple algebraic group G acts as a distinguished unipotent element on a irreducible representation of fundamental highest weight ω_i. The result is given by the following proposition.

Proposition 5.12.1. *Assume that $p \neq 2$. Let G be a simple algebraic group of rank $l \geq 2$ and let $u \in G$ be a distinguished unipotent element. Then u acts on $L(\omega_i)$ $(1 \leq i \leq l)$ as a distinguished unipotent element if and only if G, u, p and ω_i are as in Table 5.1 or Table 5.2, where λ is given up to graph automorphism of G.*

Remark 5.12.2. Let $u \in G$ be a distinguished unipotent element of order p and let $\lambda = \omega_i$. Then Proposition 5.12.1 implies the following statement. The unipotent element u acts on $L_G(\lambda)$ as a distinguished unipotent element if and only if G, u, and λ occur in Table 5.1 or Table 5.2 (up to a graph automorphism applied to λ), and $p > m_u(\lambda)$. In Section 5.13, we will generalize this fact (Proposition 5.13.1).

We will prove Proposition 5.12.1 for each simple type in the subsections that follow.

5.12.1 Type A_l, B_l and D_l

Suppose that G is of type A_l, B_l or D_l. For the representation $L(\omega_l)$ of B_l and D_l, the claim of Proposition 5.12.1 follows from Proposition 5.10.1. Then since the half-spin representation $L(\omega_{l-1})$ of D_l is a twist of $L(\omega_l)$ by a graph automorphism, by Proposition 2.10.2 (ii) the claim follows also for the representation $L(\omega_{l-1})$ of D_l. Suppose then that $L_G(\omega_i)$ is not a spin or a half-spin representation. Then it is well known (see for example [Sei87, 8.1] or [McN98, Proposition 4.2.2]) that $L_G(\omega_i) \cong V_G(\omega_i) \cong \wedge^i(V)$, where V is the natural representation of G. For $L_G(\omega_1)$ the claim of Proposition 5.12.1 is clear. For $L(\omega_2)$ and $L(\omega_3)$, Proposition 5.12.1 follows from 5.5.5, 5.5.6 and 5.8.1.

We proceed next to show that when $i \geq 4$, no unipotent element of G acts on $\wedge^i(V)$ as a distinguished unipotent element. Let $n = \dim V$. Since we are interested in self-dual representations, for type A_l we can assume $i = \frac{l+1}{2}$, so l is odd. Since we are assuming that $L_G(\omega_i)$ is not a spin representation, for type B_l we have $i \leq l-1 = \frac{n-3}{2}$ and for type D_l we have $i \leq l-2 = \frac{n}{2} - 2$. In any case, we have $n \geq 2i$.

Let $u' \in G$ be a regular unipotent element. According to Lemma 5.1.1 and Lemma 2.7.3, the unipotent element u acts on $\wedge^i(V)$ with largest Jordan block of size $\leq m_{u'}(\omega_i) + 1 \leq i(n-i) + 1$. One can show that

$$\dim \wedge^i(V) = \binom{n}{i} > \frac{(i(n-i)+2)^2}{4}$$

if $n \geq 2i$ and $i \geq 5$ or if $i = 4$ and $n > 8$, so by Lemma 5.1.1 we are done in these cases.

What remains is to consider the case where $i = 4$ and $n = 8$. Since we are assuming that ω_i is not a spin representation for D_l, we have that $G = A_7$. In this

λ	G	u of order $> p$	u of order p	$m_u(\lambda)$
ω_1	A_l, B_l, C_l, D_l	any	any	
ω_3	A_5	$p=5$	$p \geq 11$	9
ω_2	B_l, $l \geq 3$	prop. 5.5.5	prop. 5.5.5	$4l-2$
ω_l	B_l, $3 \leq l \leq 8$	prop. 5.10.1	prop. 5.10.1	$\frac{l(l+1)}{2}$
ω_2	C_l	prop. 5.5.10	prop. 5.5.10	$4l-4$
ω_3	C_3	$p=3,5$	$p \geq 11$	9
ω_3	C_4	none	$p \geq 17$	15
ω_3	C_5	none	$p \geq 23$	21
ω_4	C_4	none	$p \geq 17$	16
ω_5	C_5	none	$p \geq 29$	25
ω_2	D_l	prop. 5.5.6	prop. 5.5.6	$4l-6$
ω_l	D_6,D_8	prop. 5.10.1	prop. 5.10.1	$\frac{l(l-1)}{2}$
ω_1	G_2	$p=3,5$	$p \geq 7$	6
ω_2	G_2	$p=3,5$	$p \geq 11$	10
ω_1	F_4	$p=5,7,11$	$p \geq 23$	22
ω_4	F_4	$p=3,5,7,11$	$p \geq 17$	16
ω_2	E_6	$p=5,7,11$	$p \geq 23$	22
ω_1	E_7	$p=11$	$p \geq 37$	34
ω_7	E_7	$p=3,5,11,13,17$	$p \geq 29$	27
ω_8	E_8	$p=11,13,17,23$	$p \geq 59$	58

Table 5.1: $p \neq 2$: Cases where a regular unipotent element $u \in G$ acts on $L_G(\omega_i)$ as a distinguished unipotent element.

λ	G	Conjugacy class of u	u of order $> p$	u of order p	$m_u(\lambda)$
ω_1	B_l, C_l, D_l	any distinguished	any	any	
ω_6	D_6	$[9,3]$	$p=7$	$p \geq 13$	11
ω_4	F_4	$F_4(a_1)$	$p=5,7$	$p \geq 11$	10
ω_7	E_7	$E_7(a_1)$	$p=5,13$	$p \geq 23$	21
ω_7	E_7	$E_7(a_2)$	$p=5,7,11$	$p \geq 19$	17
ω_8	E_8	$E_8(a_1)$	$p=11$	$p \geq 47$	46

Table 5.2: $p \neq 2$: Cases where a non-regular distinguished unipotent element $u \in G$ acts on $L_G(\omega_i)$ as a distinguished unipotent element.

case the only distinguished unipotent elements $u \in G$ are the regular ones, which act on V with a single Jordan block of size 8. A computation with MAGMA shows that u does not act on $\wedge^4(V)$ as a distinguished unipotent element; for convenience we have listed the decomposition of $\wedge^4(V) \downarrow K[u]$ for $p \geq 3$ in Table 5.3.

p	Decomposition of $\wedge^4(V_8)$
$p = 3$	$[7, 9^7]$
$p = 5$	$[1, 5^2, 9, 10^2, 13, 17]$
$p = 7$	$[1, 7^4, 13, 14^2]$
$p = 11$	$[1, 5, 9, 11^5]$
$p = 11$	$[1, 5^2, 9, 11, 13^3]$
$p \geq 17$	$[1, 5^2, 9^2, 11, 13, 17]$

Table 5.3: Decomposition of $\wedge^4(V_8)$ as a $K[u]$-module.

5.12.2 Type C_l

Suppose that G is a simple algebraic group of type C_l. To prove Proposition 5.12.1, without loss of generality we can assume that $G = \mathrm{Sp}(V)$, where $\dim V = 2l$. Here $V = L(\omega_1) = V(\omega_1)$, so for ω_1 the claim is obvious. For $L(\omega_2)$ and $L(\omega_3)$, the claim of Proposition 5.12.1 follows from Proposition 5.5.10 and Proposition 5.8.1, respectively.

For ω_i, $i \geq 4$ we will apply results due to Premet and Suprunenko, which give a recursive formula for the dimension of $L(\omega_i)$. To state their results, we need to first make some definitions. Let $a, b \in \mathbb{Z}_{\geq 0}$ and write $a = \sum_{i \geq 0} a_i p^i$ and $b = \sum_{i \geq 0} b_i p^i$ for the expansions of a and b in base p. We say that a *contains* b *to base* p if for all $i \geq 0$ we have $b_i = a_i$ or $b_i = 0$. [13]

For $r \geq 1$, we define $J_p(r)$ to be the set of integers $0 \leq j \leq r$ such that $j \equiv r$ mod 2 and $l - j + 1$ contains $\frac{r-j}{2}$ to base p. Furthermore, set

$$f_0 = 1$$

and

$$f_k = \dim L(\omega_k)$$

for all $k \geq 1$. Now [PS83, Theorem 2] implies the following.

Theorem 5.12.3. *Let $1 \leq r \leq l$. Then in the Weyl module $V(\omega_r)$, each composition factor has multiplicity 1, and the set of composition factors is*

$$\{L(\omega_j) : j \in J_p(r)\}.$$

Furthermore, the integers f_r ($r \geq 2$) satisfy the following recurrence relation.

$$f_r = \binom{2l}{r} - \binom{2l}{r-2} - \sum_{j \in J_p(r) - \{r\}} f_j$$

Note that the recursive formula above makes computing the values of f_r very easy. As a corollary (see Corollary 2 in [PS83]), we have the following which can also be deduced from the results of Gow in [Gow98].

[13] Note that in [PS83] there is a typo, the definition on pg. 1313, line 9 should say "for every $i = 0, 1, \ldots, n$ .."

Corollary 5.12.4. *If $p > l$, then $f_r = \binom{2l}{r} - \binom{2l}{r-2}$ for all $1 \leq r \leq l$.*

With the recursive formula we also get a lower bound for f_r.

Corollary 5.12.5. *Let $r \geq 4$. Then $f_r \geq \binom{2l}{r} - 2\binom{2l}{r-2} + 1$.*

Proof. Note that we have $\dim V(\omega_j) = \binom{2l}{j} - \binom{2l}{j-2}$ [Bou75, Ch. VIII, 13.3, pg. 203], so $f_j \leq \binom{2l}{j} - \binom{2l}{j-2}$. Since $J_p(r)$ is a subset of the nonnegative integers $r, r-2, r-4, \ldots$, the recursive formula of Theorem 5.12.3 gives

$$\begin{aligned} f_r &\geq \binom{2l}{r} - \binom{2l}{r-2} - f_{r-2} - f_{r-4} - \cdots \\ &\geq \binom{2l}{r} - \binom{2l}{r-2} - \left(\binom{2l}{r-2} - \binom{2l}{r-4}\right) - \left(\binom{2l}{r-4} - \binom{2l}{r-6}\right) - \cdots \\ &= \binom{2l}{r} - 2\binom{2l}{r-2} + C \end{aligned}$$

where $C = f_0 = 1$ if r is even and $C = f_1 = 2l$ if r is odd. In any case, we get the desired inequality. □

For $i = 4$ and $i = 5$, the claim of Proposition 5.12.1 for G will be proven in the next two lemmas.

Lemma 5.12.6. *Proposition 5.12.1 holds for $G = C_l$, $l \geq 4$, and $i = 4$.*

Proof. Let $u \in G$ be a regular unipotent element, so now $m_u(\omega_4) = 4(2l-4) = 8l-16$ by Lemma 2.7.3. By Corollary 5.12.5 we have $\dim L_G(\omega_4) \geq \binom{2l}{4} - 2\binom{2l}{2} + 1$. One can verify that the polynomial inequality $\binom{2l}{4} - 2\binom{2l}{2} + 1 > \frac{(m_u(\omega_4)+2)^2}{4}$ holds or all $l \geq 6$. Thus by Lemma 5.1.1, no unipotent element of G acts on $L_G(\omega_4)$ as a distinguished unipotent element if $l \geq 6$.

For $l = 4$ and $l = 5$, one can compute $m_{u'}(\omega_4)$ for all non-regular distinguished unipotent elements $u' \in G$. This is a straightforward application of Proposition 2.6.5, and the expression of ω_4 as a sum of simple roots [Hum72, 13, Table 1]. We have listed this information, along with $\dim L_G(\omega_4)$ (computed using Theorem 5.12.3) in Table 5.4. One verifies that

$$\dim L_G(\omega_4) > \frac{(m_{u'}(\omega_4) + 2)^2}{4}$$

in all cases, so it follows from Lemma 5.1.1 that a non-regular unipotent element $u' \in G$ does not act on $L_G(\omega_4)$ as a distinguished unipotent element.

Now we still need to consider the action of a regular unipotent element on $L_G(\omega_4)$ in the cases where $l = 4$ and $l = 5$. Using MAGMA (Section 2.9), we have computed these actions for all $p \geq 3$ and listed them in Table 5.6. One sees that for $l = 4$, a regular unipotent element acts on $L_G(\omega_4)$ as a distinguished unipotent element if and only if $p \geq 17$. Furthermore, for $l = 5$, the regular unipotent does not act on $L_G(\omega_4)$ as a distinguished unipotent element, as claimed. □

Lemma 5.12.7. *Proposition 5.12.1 holds for $G = C_l$ and $i = 5$.*

Proof. We proceed as in the proof of Lemma 5.12.6. Let $u \in G$ be a regular unipotent element, so now $m_u(\omega_5) = 5(2l-5) = 10l - 25$ by Lemma 2.7.3. By Corollary 5.12.5 we have $\dim L_G(\omega_5) \geq \binom{2l}{5} - 2\binom{2l}{3} + 1$, and one verifies that the polynomial inequality $\binom{2l}{5} - 2\binom{2l}{3} + 1 > \frac{(m_u(\omega_5)+2)^2}{4}$ holds or all $l \geq 6$. Thus by Lemma 5.1.1, no unipotent element of G acts on $L_G(\omega_5)$ as a distinguished unipotent element if $l \geq 6$.

What remains is to consider the case where $l = 5$. As in the second paragraph of the proof of Lemma 5.12.6, one verifies that

$$\dim L_G(\omega_5) > \frac{(m_{u'}(\omega_5) + 2)^2}{4}$$

for all non-regular distinguished unipotent elements $u' \in G$; for convenience we have listed the values of $m_{u'}(\omega_5)$ and $L_G(\omega_5)$ in Table 5.5. Thus no non-regular unipotent element of G acts on $L_G(\omega_5)$ as a distinguished unipotent element. Using MAGMA (Section 2.9) we have computed the action of a regular unipotent on $L_G(\omega_5)$ for $G = C_5$. The Jordan block sizes are given in Table 5.7, and the claim of Proposition 5.12.1 follows from this. □

To finish the proof of Proposition 5.12.1 for G, what remains is to consider the case where $i \geq 6$ and to show that in this case no unipotent element of G acts on $L_G(\omega_i)$ as a distinguished unipotent element. We proceed to do this.

Let $u \in G$ be a regular unipotent element, so now $m_u(\omega_i) = i(2l-i)$ by Lemma 2.7.3.

We start by considering $6 \leq i \leq 18$. First, if $i \leq l-3$, by calculation one can verify the polynomial inequality $\binom{2l}{i} - 2\binom{2l}{i-2} + 1 > \frac{(m_u(\omega_i)+2)^2}{4}$ for all $6 \leq i \leq 18$. Therefore by Corollary 5.12.5 and Lemma 5.1.1, no unipotent element of G acts on $L_G(\omega_i)$ as a distinguished unipotent element. For $6 \leq i \leq 18$ and $i = l$, $i = l-1$, $i = l-2$ with the formula of Theorem 5.12.3 one can compute $f_i = \dim L_G(\omega_i)$ and check that $f_i > \frac{(m_u(\omega_i)+2)^2}{4}$, so we are again done by Lemma 5.1.1.

What remains is to consider $i \geq 19$. If $p > l$, then $f_i = \binom{2l}{i} - \binom{2l}{i-2}$ by Corollary 5.12.4. One can verify that $\binom{2l}{i} - \binom{2l}{i-2} > \frac{(m_u(\omega_i)+2)^2}{4}$ for $l \geq i \geq 6$, so by Lemma 5.1.1 no unipotent element of G acts on $L(\omega_i)$ as a distinguished unipotent element in this case. Suppose then that $p \leq l$. By the argument given in the beginning of Section 5.1.2, we see that it will be enough to show that $\dim L(\omega_i) > 4l^4 + 2l^2$. Now $\dim L(\omega_i) \geq |W\omega_i|$ where W is the Weyl group, so $\dim L(\omega_i) > 4l^4 + 2l^2$ follows from Lemma 5.1.12 (i). This completes the proof of Proposition 5.12.1 for $G = C_l$.

G	u	Dynkin diagram of u	$m_u(\omega_4)$	$\dim L(\omega_4)$
C_5	$[2,8]$	22202	16	121 ($p = 3$), 165 ($p > 3$)
C_5	$[4,6]$	20202	12	121 ($p = 3$), 165 ($p > 3$)
C_4	$[2,6]$	2202	10	41 ($p = 3$), 42 ($p > 3$)

Table 5.4: Values of $m_u(\omega_4)$ for non-regular distinguished unipotent elements u in C_4 and C_5.

5.12.3 Type G_2

In this subsection we prove Proposition 5.12.1 for G simple of type G_2.

Since $p \neq 2$, we have $V_G(\omega_1) = L_G(\omega_1)$. Now the computations given in [Law95, Table 1] prove the claim of Proposition 5.12.1 for $L_G(\omega_1)$.

G	u	Dynkin diagram of u	$m_u(\omega_5)$	$\dim L(\omega_5)$
C_5	$[2,8]$	22202	16	122 ($p=3$), 132 ($p>3$)
C_5	$[4,6]$	20202	12	122 ($p=3$), 132 ($p>3$)

Table 5.5: Values of $m_u(\omega_5)$ for non-regular distinguished unipotent elements u in C_5.

G	p	$L_G(\omega_4) \downarrow K[u]$
C_4	$p=3$	$[5,9^4]$
	$p=5$	$[5,10^2,17]$
	$p=7$	$[7^2,14^2]$
	$p=11$	$[9,11^3]$
	$p=13$	$[5,11,13^2]$
	$p\geq 17$	$[5,9,11,17]$
C_5	$p=3$	$[9^4,13,18^4]$
	$p=5$	$[5^7,15^7,25]$
	$p=7$	$[1,7^3,9,11,13,14^2,17,19,21,25]$
	$p=11$	$[11^{15}]$
	$p=13$	$[9,13^{12}]$
	$p=17$	$[1,5,7,9,11,13,17^7]$
	$p=19$	$[1,5,7,9^2,11,13,15,19^5]$
	$p=23$	$[1,5,7,9^2,11,13^2,15,17,19,23^2]$
	$p\geq 29$	$[1,5,7,9^2,11,13^2,15,17,19,21,25]$

Table 5.6: Action of a regular unipotent $u \in G$ on $L_G(\omega_4)$, for $G=C_4$, $G=C_5$.

p	$L_G(\omega_5) \downarrow K[u]$
$p=3$	$[9^4,14,18^4]$
$p=5$	$[10^7,20^2,22]$
$p=7$	$[2,7^2,10,14^3,18,20,26]$
$p=11$	$[11^{12}]$
$p=13$	$[2,13^{10}]$
$p=17$	$[2,6,10,12,17^6]$
$p=19$	$[2,6,8,10,14,16,19^4]$
$p=23$	$[2,6,8,10,12,14,16,18,23^2]$
$p\geq 29$	$[2,6,8,10,12,14,16,18,20,26]$

Table 5.7: Action of a regular unipotent $u \in G$ on $L_G(\omega_5)$, for $G=C_5$.

For $L_G(\omega_2)$ and $p \neq 3$, we have $L_G(\omega_2) \cong \mathscr{L}(G)$ and the claim follows from [Law95, Table 2]. If $p = 3$, then a computation with MAGMA (Section 2.9) proves that the only distinguished action on $L_G(\omega_2)$ occurs for the regular unipotent element of G, which acts on $L_G(\omega_2)$ with a single Jordan block of size 7. For a different argument, one could also argue by using the exceptional isogeny $G \to G$ which exists when $p = 3$.

5.12.4 Type F_4

Let G be simple of type F_4. We proceed to prove Proposition 5.12.1 for G.

Since $p \neq 2$, we have that $V_G(\omega_1) \cong L_G(\omega_1) \cong \mathscr{L}(G)$. Now the computations given in [Law95, Table 4] prove the claim of Proposition 5.12.1 for $L_G(\omega_1)$.

For the fundamental weight ω_2, consider first $p = 3$. In this case the regular unipotent element of G has order 3^3 (see Appendix A), and $\dim L_G(\omega_2) = 1222$ by [Lüb01]. Since $1222 > \frac{(3^3+1)^2}{4}$, it follows from Lemma 5.1.1 that no unipotent element of G acts on $L_G(\omega_2)$ as a distinguished unipotent element. Suppose then that $p > 3$. By Lemma 2.7.3, we have $m_u(\omega_2) = 42$ for a regular unipotent element $u \in G$. Now $\dim L_G(\omega_2) = 1274$ by [Lüb01] and $1222 > \frac{(42+2)^2}{4}$, so by Lemma 5.1.1 no unipotent element of G acts on $L_G(\omega_2)$ as a distinguished unipotent element.

Consider next the fundamental weight ω_3. If $p > 3$, then $\dim L_G(\omega_3) = 273$ by [Lüb01] and $m_u(\omega_3) = 30$ for a regular unipotent u by Lemma 2.7.3. Since $273 > 272 = \frac{(30+2)^2}{4}$, by Lemma 5.1.1 no unipotent element of G acts on $L_G(\omega_3)$ as a distinguished unipotent element. If $p = 3$, then $\dim L(\omega_3) = 196$. The non-regular unipotent elements of G have order 3^2 in this case (Appendix A), and $196 > \frac{(3^2+1)^2}{4}$, so by Lemma 5.1.1 it is enough to consider a regular unipotent element u. A computation with MAGMA (Section 2.9) shows that when $p = 3$, a regular unipotent element acts on $L_G(\omega_3)$ with Jordan blocks $[3, 9^6, 18^4, 19, 21, 27]$, hence not as a distinguished unipotent element.

For $L_G(\omega_4)$, the claim of Proposition 5.12.1 follows from [Law95, Table 3] if $p > 3$ since $V_G(\omega_4) = L_G(\omega_4)$ in this case. For $p = 3$, a computation with MAGMA (Section 2.9) shows that the distinguished unipotent classes F_4, $F_4(a_1)$, $F_4(a_2)$, and $F_4(a_3)$ act on $L_G(\omega_4)$ with Jordan blocks $[1, 9, 15]$, $[7, 9^2]$, $[1, 3, 6^2, 9]$, and $[1^2, 3^3, 5^3]$ respectively. Note that here the only distinguished action occurs for the regular unipotent class F_4.

5.12.5 Type E_6, E_7 and E_8

Let G be simple of type E_6, E_7 or E_8 and let $\lambda = \omega_i$. Set $m = m_u(\omega_i)$ for a regular unipotent $u \in G$, and $d = \dim L(\omega_i)$. Then with Lemma 2.7.3 and the results of Lübeck in [Lüb01], we can compute values of m and a lower bound for d, which we list in Table 5.8.

By Lemma 5.1.1, we know that no unipotent element of G acts on $L(\omega_i)$ as a distinguished unipotent element if $d > \frac{(m+2)^2}{4}$. By calculating with the values in Table 5.8, we find that $d > \frac{(m+2)^2}{4}$ except when (G, ω_i) is (E_6, ω_1), (E_6, ω_2), (E_6, ω_6), (E_7, ω_1), (E_7, ω_7) or (E_8, ω_8). Since it is enough to consider self-dual representations by Theorem 1.1.12, it will be enough to consider (E_6, ω_2), (E_7, ω_1), (E_7, ω_7) and (E_8, ω_8). Now except for the case $G = E_6$, $p = 3$, $\lambda = \omega_2$, in these cases the claim of Proposition 5.12.1 follows from the computations of Lawther in [Law95].

λ	E_6	E_7	E_8
ω_1	$d = 27$, $m = 16$	$d = 133$, $m = 34$	$d = 3875$, $m = 92$
ω_2	$d \geq 77$, $m = 22$	$d \geq 856$, $m = 49$	$d > 100000$, $m = 136$
ω_3	$d = 352$, $m = 30$	$d \geq 8512$, $m = 66$	$d > 100000$, $m = 182$
ω_4	$d \geq 2771$, $m = 42$	$d > 100000$, $m = 96$	$d > 100000$, $m = 270$
ω_5	$d = 352$, $m = 30$	$d \geq 25896$, $m = 75$	$d > 100000$, $m = 220$
ω_6	$d = 27$, $m = 16$	$d \geq 1538$, $m = 52$	$d > 100000$, $m = 168$
ω_7		$d = 56$, $m = 27$	$d \geq 30132$, $m = 114$
ω_8			$d = 248$, $m = 58$

Table 5.8: Values of $m = m_u(\omega_i)$ and $d = \dim L_G(\omega_i)$ for u a regular unipotent element of $G = E_n$.

What remains is to consider the case where $G = E_6$, $p = 3$, and $\lambda = \omega_2$. In this case, a computation with MAGMA (Section 2.9) shows that the distinguished unipotent classes E_6, $E_6(a_1)$ and $E_6(a_3)$ act on $L_G(\omega_2)$ with Jordan block sizes $[1, 9^3, 15^2, 19]$, $[5, 9^8]$, and $[1, 3^3, 6^4, 7, 9^4]$ respectively. Therefore no unipotent class of E_6 acts on $L_G(\omega_2)$ as a distinguished unipotent element when $p = 3$.

5.13 Unipotent elements of order p ($p \neq 2$)

Assume that $p \neq 2$.

In this section we prove Theorem 1.1.10 in the case where u has order p. Specifically, the result proven in this section is the following.

Theorem 5.13.1. *Let $u \in G$ be a distinguished unipotent element of order p, and let $\lambda \in X(T)^+$ be a nonzero, p-restricted highest weight. Then u acts on $L_G(\lambda)$ as a distinguished unipotent element if and only if $p > m_u(\lambda)$ and G, λ and u are as in the Table 5.1 or Table 5.2, with λ given up to graph automorphism of A_n or D_n.*

Remark 5.13.2. Note that the λ occurring in Table 5.1 and Table 5.2 are precisely the ones that occur in the main result of [LST15]. Furthermore, the condition $p > m_u(\lambda)$ implies that the action of u on $V_G(\lambda)$ is the same as the corresponding action in characteristic 0 (Lemma 2.7.9). Therefore when u has order p, the result of Theorem 1.1.10 is similar to the characteristic 0 result of [LST15].

As seen in the previous sections, the situation is less uniform when u has order $> p$. In this setting things are unpredictable, and it is more difficult to determine when exactly the examples in [LST15] work in characteristic p. For unipotent elements of order $> p$, we also have a unique example of a λ that corresponds to a distinguished action of unipotent element, but does not occur in the main result of [LST15] (see Theorem 1.1.10 (iii)).

Remark 5.13.3. In [Sup03, Предложение 4] and [Sup05, Предложение 2], Suprunenko has announced results which have the consequence that if $u \in G$ is a unipotent element of order p, $\lambda \in X(T)^+$, and $2 < p \leq m_u(\lambda)$, then u acts on $L_G(\lambda)$ with ≥ 2 Jordan blocks of size p. Assuming this result, Theorem 5.13.1 for the case where u is a regular unipotent element becomes an immediate consequence of Corollary 2.7.6, Lemma 2.7.9, and the main result of [LST15].

G	λ	Conjugacy class of u	$m_u(\lambda)$
A_n	ω_1	regular	n
$A_n, n \geq 2$	$\omega_1 + \omega_n$	regular	$2n$
A_1	any $0 \leq c \leq p-1$	regular	c
A_5	ω_3	regular	9
B_n	ω_1	any	
B_3	$\omega_1 + \omega_3$	regular	12
B_3	$2\omega_3$	regular	12
B_3	$3\omega_1$	regular	18
$B_n, n \geq 3$	$2\omega_1$	regular	$4n$
$B_n, n \geq 3$	ω_2	regular	$4n-2$
$B_n, 3 \leq n \leq 8$	ω_n	regular	$\frac{n(n+1)}{2}$
C_n	ω_1	any	
C_2	$b\omega_1$ for $1 \leq b \leq 5$	regular	$3b$
C_2	$b\omega_2$ for $1 \leq b \leq 5$	regular	$4b$
C_2	$\omega_1 + \omega_2$	regular	7
C_2	$\omega_1 + 2\omega_2$	regular	11
C_2	$2\omega_1 + \omega_2$	regular	10
C_3	$3\omega_1$	regular	15
C_3	ω_3	regular	9
C_4	ω_3	regular	15
C_5	ω_3	regular	21
C_4	ω_4	regular	16
C_5	ω_5	regular	25
$C_n, n \geq 3$	$2\omega_1$	regular	$4n-2$
$C_n, n \geq 3$	ω_2	regular	$4n-4$
D_n	ω_1	any	
D_6	ω_6	regular	15
D_6	ω_6	$[9,3]$	11
D_8	ω_8	regular	28
$D_n, n \geq 4$ even	$2\omega_1$	regular	$4n-4$
$D_n, n \geq 4$ odd	ω_2	regular	$4n-6$

Table 5.1: G of classical type

G	λ	Conjugacy class of u	$m_u(\lambda)$
G_2	ω_1	regular	6
G_2	ω_2	regular	10
G_2	$\omega_1+\omega_2$	regular	16
G_2	$2\omega_1$	regular	12
G_2	$2\omega_2$	regular	20
G_2	$3\omega_1$	regular	18
F_4	ω_1	regular	22
F_4	ω_4	regular	16
F_4	ω_4	$F_4(a_1)$	10
E_6	ω_2	regular	22
E_7	ω_1	regular	34
E_7	ω_7	regular	27
E_7	ω_7	$E_7(a_1)$	21
E_7	ω_7	$E_7(a_2)$	17
E_8	ω_8	regular	58
E_8	ω_8	$E_8(a_1)$	46

Table 5.2: G of exceptional type

However, since the full proof of Suprunenko's results remain unpublished after her 2003 and 2005 announcements [Sup03] and [Sup05], we have decided to give a proof of Theorem 5.13.1 that does not rely on these results. Our proof of Theorem 5.13.1 is based on applying Proposition 4.6.10, which allows us to use the methods of Liebeck, Seitz and Testerman from [LST15]. In some cases we have to give different proofs, but the general steps of the proof follow [LST15].

The proof of Theorem 5.13.1 will be given in what follows. We begin by taking care of some special cases.

Lemma 5.13.4. *Theorem 5.13.1 holds if* $\lambda \neq -w_0(\lambda)$.

Proof. If $\lambda \neq -w_0(\lambda)$, then $L_G(\lambda)$ is not self-dual, and so u acts on $L_G(\lambda)$ as a distinguished unipotent element if and only if u acts on $L_G(\lambda)$ with a single Jordan block. Thus the claim follows from Proposition 1.1.12. □

Lemma 5.13.5. *Theorem 5.13.1 holds if* $\lambda = \omega_i$.

Proof. This follows from Proposition 5.12.1, see Remark 5.12.2. □

Lemma 5.13.6. *Theorem 5.13.1 holds if* G *is of type* A_1.

Proof. This follows from the fact that for all $0 \leq c \leq p-1$, a nonidentity unipotent element $u \in G$ acts on $L_G(c)$ with a single Jordan block of size $c+1$ (Lemma 4.2.2). □

Lemma 5.13.7. *Theorem 5.13.1 holds if* $p = 3$.

Proof. Let $p = 3$, let $u \in G$ be a distinguished unipotent element of order 3 and let $V = L_G(\lambda)$ be a nontrivial, p-restricted irreducible representation of G. Looking at the cases where $3 > m_u(\lambda)$ in Table 5.1 and Table 5.2, we see that the claim of Theorem 5.13.1 is that u acts on V as a distinguished unipotent element if and only if u is a regular unipotent element and (G, λ) is (A_1, c), (A_2, ω_1), or (A_2, ω_2).

Sufficiency follows from Lemma 5.13.6 and Lemma 5.13.5. Suppose then that u acts on V as a distinguished unipotent element. Since u has order 3, we have $\dim V \leq 4$ by Lemma 5.1.1. If G is of exceptional type, then by [Lüb01] any nontrivial irreducible representation has dimension ≥ 7. Thus G must be of classical type. Because u is distinguished of order 3, the largest Jordan block of u in the natural representation has size ≤ 3. Therefore G is either of type A_1 or A_2 and u is a regular unipotent element. For type A_1 the claim follows from Lemma 5.13.6. For type A_2, by [Lüb01] the only nontrivial, p-restricted irreducible representations of dimension ≤ 4 are the natural representation $L_G(\omega_1)$ and its dual $L_G(\omega_2)$. This proves the claim.

□

Lemma 5.13.8. *Theorem 5.13.1 holds if* $p = 5$.

Proof. Let $p = 5$, let $u \in G$ be a distinguished unipotent element of order 5 and let $V = L_G(\lambda)$ be a nontrivial, p-restricted irreducible representation of G. If $\lambda = \omega_i$, the result of Theorem 5.13.1 follows by Lemma 5.13.5. Assume then that $\lambda \neq \omega_i$. Now looking at the cases where $5 > m_u(\lambda)$ and $\lambda \neq \omega_i$ in Table 5.1 and Table 5.2, we see that the claim of Theorem 5.13.1 is that u acts on V as a distinguished unipotent element if and only if (G, λ) is (A_1, c) or $(A_2, \omega_1 + \omega_2)$.

Sufficency follows from Lemma 5.13.6 and Proposition 5.3.2. For the other direction, suppose that u acts on V as a distinguished unipotent element. Since u has order 5, we have $\dim V \leq 9$ by Lemma 5.1.1. If G is of exceptional type, then by [Lüb01] any nontrivial, p-restricted irreducible representation $\neq L_G(\omega_i)$ has dimension > 9. Therefore G must be of classical type. Because u has order 5, the largest Jordan block of u in the natural representation has size ≤ 5. Therefore it follows that G is of type A_l ($l \leq 4$), type C_2, type C_3, type B_4 or type D_4. Since we are assuming $\lambda \neq \omega_i$, by [Lüb01] the only case where we have a nontrivial, p-restricted representation of dimension ≤ 9 is when $G = A_1$ or $G = A_2$ and $\lambda = \omega_1 + \omega_2$. This completes the proof of the claim. □

It follows from the lemmas above that for the rest of this section, we can and will make the following assumptions.

- $p \geq 7$ (by Lemma 5.13.7 and Lemma 5.13.8)
- $\operatorname{rank} G \geq 2$ (by Lemma 5.13.6)
- $L_G(\lambda)$ is self-dual, ie. $\lambda = -w_0(\lambda)$ (by Lemma 5.13.4)

These assumptions will have some implications which we will apply throughout the proof. The first assumption implies that the characteristic p is good for G, so we can apply facts from Section 2.6 here. Thus we can assume that u is contained in a subgroup $A < G$ of type A_1, such that a maximal torus $T_A < A$ is contained in the fixed maximal torus T of G. Furthermore, we can assume that T_A gives us the labeled Dynkin diagram associated with u (Theorem 2.6.8). Therefore we can use the labeled diagram associated with u to determine how T-weights restrict to

T_A-weights. Indeed, write $\lambda \in X(T)$ as $\lambda = \sum_{i=1}^{l} q_i \alpha_i$, where α_i are the simple roots and $q_i \in \mathbb{Q}^+$ for all i. Identifying the T_A-weights with integers in the usual way, we see that λ restricts to the T_A-weight $\sum 2q_i \in \mathbb{Z}$, where the sum runs over the i such that α_i has label 2 in the labeled Dynkin diagram associated with u.

For the rest of this section we will denote $V = L_G(\lambda)$. According to Proposition 4.6.10, under our assumptions we also have the following.

- If u acts on V as a distinguished unipotent element, then $V \downarrow A$ is semisimple.

Throughout we shall also denote $r = m_u(\lambda)$, which is the T_A-weight given by the restriction of the T-weight λ (Definition 2.7.2). Now r is the highest weight of $V \downarrow A$, since all T-weights of V are of the form $\lambda - \sum k_i \alpha_i$, where $k_i \in \mathbb{Z}_{\geq 0}$. Furthermore, the labeled Dynkin diagram of u has all labels equal to 0 or 2, so it follows that all T_A-weights of $V \downarrow A$ are of the form $r - 2c$ for some $c \in \mathbb{Z}_{\geq 0}$.

In this section, we will often write T-weights of the form $\lambda - k_1\alpha_1 - k_2\alpha_2 - \cdots$ as $\lambda - 1^{k_1}2^{k_2}\cdots$. For example, we write $\lambda - \alpha_1 = \lambda - 1$ and $\lambda - 2\alpha_2 - \alpha_3 = \lambda - 2^2 3$. For a T-weight μ, we will write μ^{s_i} for its image under the reflection s_i corresponding to the simple root α_i.

Note also that since we are working in good characteristic, Premet's theorem (Theorem 4.5.5) applies and so the set of weights in $L_G(\lambda)$ is saturated. We will often use this fact in this section without mentioning it explicitly.

Now under our assumptions we will be able to prove the following lemma from [LST15] in our setting (positive characteristic), and essentially the rest of proof of Theorem 5.13.1 is based on applying it. The proof of the lemma is essentially the same as the one given in [LST15].

Lemma 5.13.9 ([LST15, Lemma 2.2]). *Write $\lambda = \sum c_i \omega_i$. Suppose that u acts on V as a distinguished unipotent element. Then*

(i) *If c_i is nonzero, then α_i has label* 2.

(ii) *At most two values of c_i can be nonzero.*

(iii) *For $c \geq 0$, the weight $r - 2c$ occurs with multiplicity at most $c + 1$.*

(iv) *If $r - 2$ has multiplicity* 1*, then for all $c \geq 1$ the weight $r - 2c$ has multiplicity at most c.*

Proof. As noted before, under our assumptions the fact that u acts on V as a distinguished unipotent implies that $V \downarrow A$ is semisimple. Then it follows that each composition factor of $V \downarrow A$ occurs with multiplicity one. Furthermore, the simple summands that occur in $V \downarrow A$ are determined by the weights of A that occur in $V \downarrow A$. Recall also (Lemma 4.2.1) that all weights of a simple module $L(c)$ have multiplicity 1 and that they are contained in $\{c, c-2, \cdots, -(c-2), -c\}$. We now proceed to use these facts to prove the claims (i) - (iv).

(i) If $c_i \neq 0$ and α_i has label 0, then the weight r of $V \downarrow A$ is afforded by weights λ and $\lambda - i$ of V. But since r is the highest weight in $V \downarrow A$, it follows that $L(r)$ must occur at least twice as a composition factor of $V \downarrow A$, which is a contradiction.

(ii) Suppose that i, j, k are distinct indices such that $c_i, c_j, c_k \neq 0$. Now (i) implies that the nodes α_i, α_j and α_k have label 2. Thus the weight $r-2$ is afforded by $\lambda - i$, $\lambda - j$ and $\lambda - k$. Since all the weights of $V \downarrow A$ are $\leq r$, this can only happen if either $L(r)$ or $L(r-2)$ occurs twice as a composition factor of $V \downarrow A$. As in (i), this is a contradiction.

(iii) Suppose that the weight $r - 2c$ occurs with multiplicity $\geq c+2$ in $V \downarrow A$. Since $r-2c$ can only occur in $L(s)$ for $s \geq r-2c$, it follows that one of $L(r)$, $L(r-2)$, ..., $L(r-2c)$ must occur at least twice as a summand of $V \downarrow A$, which is a contradiction.

(iv) We may assume that $c \geq 2$. Suppose that the weight $r - 2c$ occurs with multiplicity $\geq c+1$ in $V \downarrow A$. Since each simple summand of $V \downarrow A$ has multiplicity one, it follows that each of $L(r)$, $L(r-2)$, ..., $L(r-2c)$ must occur as a summand of $V \downarrow A$. Furthermore, $r-2c$ occurs as a weight in each of the modules $L(r)$, $L(r-2)$, ..., $L(r-2c)$. In particular, $r-2c$ occurs as a weight in $L(r)$.

Now because we are assuming that $r-2$ has multiplicity 1, it follows that $r-2$ does not occur as a weight of $L(r)$, and therefore $r \geq p$. Since $L(r) \oplus L(r-2) \oplus L(r-4)$ occurs as a direct summand of $V \downarrow A$, it follows by Lemma 4.2.6 that $r = p$ or $r = p+1$. But then $r-2c$ does not occur as a weight of $L(r)$, contradiction.

□

Remark 5.13.10. Lemma 5.13.9 (iv) corresponds to [LST15, Lemma 2.2 (iii)], but note that a somewhat different proof is needed in positive characteristic. This is also true for many other lemmas in this section that are analogous to lemmas in [LST15]. For example, compare Lemma 5.13.23 and [LST15, Lemma 4.5 (i)].

Now according to Lemma 5.13.9 (ii) and Lemma 5.13.5, it will be enough to prove Theorem 5.13.1 in the the following cases.

Case (I): $\lambda = c_i\omega_i + c_j\omega_j$, where $i \neq j$ and $c_i, c_j \geq 1$.

Case (II): $\lambda = b\omega_i$, where $b \geq 2$.

When u is a regular unipotent (that is, when all nodes α_i have label 2), we will prove Theorem 5.13.1 in case (I) in this subsection (Proposition 5.13.16). Case (II) for a regular unipotent element will be treated in subsection 5.13.1. The proofs for a distinguished, non-regular unipotent element u will be given in subsection 5.13.2. We will now proceed to prove some general facts about the two cases (I) and (II).

Lemma 5.13.11. *Let $\lambda = b\omega_i$, where $b \geq 1$. Assume that u acts on V as a distinguished unipotent element. Then if all nodes adjacent to α_i have label 2, the weight $r-2$ of A occurs with multiplicity 1.*

Proof. Note that α_i has label 2 by Lemma 5.13.9 (i), so $\lambda - i$ affords $r-2$. If $r-2$ does not have multiplicity 1 in $V \downarrow A$, then it would have to be afforded by some $\lambda - ij$, where α_j is a node adjacent to α_i. In this situation the node α_j would have to have label 0. □

Lemma 5.13.12 ([LST15, Lemma 2.3]). *Let $\lambda = b\omega_i$, where $b \geq 2$. Suppose that u acts on V as a distinguished unipotent element. Then*

(i) *The node α_i is an end node with label 2.*

(ii) *The node adjacent to α_i has label 2.*

(iii) *The weight $r-2$ of $V \downarrow A$ has multiplicity 1.*

Proof. (i) We know that α_i has label 2 by Lemma 5.13.9 (i). The fact that α_i is an end node follows with the same proof as in [LST15, Lemma 2.3 (i)].

(ii) If G has rank 2, then we know from the description of the labeled Dynkin diagrams (Section 2.6) that all nodes have label 2, unless $G = G_2$ and u lies in class $G_2(a_1)$. In this case α_1 has label 0 and α_2 has label 2, so we must have $\lambda = b\omega_2$. But then $r-2$ is afforded by $\lambda - 2$, $\lambda - 12$ and $\lambda - 1^2 2$, contradicting Lemma 5.13.9 (iii). If G has rank at least 3, then the claim follows with the same proof as in [LST15, Lemma 2.3 (ii)].

(iii) This follows immediately from (i), (ii) and Lemma 5.13.11.

□

Lemma 5.13.13. *Let $\lambda = b\omega_i$, where $b \geq 2$ and assume that u is a regular unipotent element. Suppose u acts on V as a distinguished unipotent element. Then $r = 2+p$, $r = 3+p$, $r = 2+2p$ or $r = a+p^k$ with $4 \leq a \leq p-2$ and $k \geq 1$.*

Proof. Now by Lemma 5.13.12, the node α_i is an end node, the node α_j adjacent to α_i has label 2, and the weight $r-2$ of $V \downarrow A$ occurs with multiplicity 1. Thus the weight $r-4$ has multiplicity 2, being afforded by the weights $\lambda - i^2$ and $\lambda - ij$.

Since u is a regular unipotent element and λ is p-restricted, it follows from Lemma 2.7.4 that $p \nmid r$. Therefore the weight $r-2$ occurs in $L(r)$. Thus $L(r-2)$ does not occur as a summand of $V \downarrow A$, so $L(r) \oplus L(r-4)$ must occur as a summand of $V \downarrow A$. Furthermore, $r-4$ must occur as a weight of $L(r)$. Now the claim follows from Lemma 4.2.5. □

Lemma 5.13.14 ([LST15, Lemma 2.6]). *Let $\lambda = c_i\omega_i + c_j\omega_j$, where $i \neq j$ and $c_i, c_j \geq 1$. Suppose that u acts on V as a distinguished unipotent element. Then*

(i) *Nodes adjoining α_i and α_j have label 2.*

(ii) *If $c_i > 1$ or $c_j > 1$, then α_i and α_j are adjacent.*

(iii) *Either α_i or α_j is an end node.*

(iv) *If all nodes have label 2 and α_i and α_j are not adjacent, then α_i and α_j are both end nodes.*

Proof. (i) This follows as in [LST15, Lemma 2.6 (ii)].

(ii) This follows with the argument given in the beginning of the proof of [LST15, Lemma 2.6 (iii)].

(iii) We give a proof following [LST15, Lemma 2.6 (iv)]. Suppose that α_i and α_j are not end nodes, so now $\operatorname{rank} G \geq 3$. Then there exists a node $\alpha_t \neq \alpha_j$ adjacent to α_i and a node $\alpha_s \neq \alpha_i$ adjacent to α_j. By (i), these nodes must have label 2, so $\lambda - ij$, $\lambda - it$ and $\lambda - js$ afford the weight $r - 4$. Now $c_i = c_j = 1$, as otherwise $\lambda - i^2$ or $\lambda - j^2$ would also afford the weight $r - 4$, contradicting Lemma 5.13.9 (iii). Similarly α_i and α_j must be adjacent, as otherwise there exists a node $\alpha_u \neq \alpha_t$ adjacent to α_i, and then $\lambda - iu$ affords the weight $r - 4$.

We will show next that $\lambda - ij$ has multiplicity 2, which contradicts Lemma 5.13.9 (iii) and finishes the proof. Now for a Levi factor $L = A_2$, $L = B_2$ and $L = C_2$, in the representation $L_L(\omega_1 + \omega_2)$ the weight $\omega_1 + \omega_2 - \alpha_1 - \alpha_2$ has multiplicity 2 by Lemma 4.5.2, since we are assuming $p \geq 7$. Thus by Corollary 4.5.7 the weight $\lambda - ij$ has multiplicity 2 in V.

(iv) This follows with the same argument as [LST15, Lemma 2.6 (vi)].

□

Lemma 5.13.15. *Let $\lambda = c_i\omega_i + c_j\omega_j$, where $i \neq j$ and $c_i, c_j \geq 1$. If u acts on V as a distinguished unipotent element, then $p > r$.*

Proof. Suppose that $r \geq p$ and that u acts on V as a distinguished unipotent element. Now the weight $r - 2$ has multiplicity 2 in $V \downarrow A$ since it is afforded by $\lambda - i$ and $\lambda - j$. Therefore $L(r-2) \oplus L(r)$ occurs as a summand of $V \downarrow A$, so by Lemma 4.2.4 we have $r = p$, $r = p+1$, $r = 2p$, or $r = p + p^l$ for some $l \geq 2$. On the other hand, now $r-2$ must occur as a weight of $L(r)$, so it follows that $r = p+1$. In this case $r - 4$ does not occur as a weight of $L(r)$. Therefore the weight $r - 4$ has multiplicity ≤ 2 in $V \downarrow A$, as otherwise $L(r-2)$ or $L(r-4)$ would occur twice as a summand. The rest of our proof is based on exploiting this fact.

Recall that by Lemma 5.13.14 (i) any node adjacent to α_i and α_j must have label 2. We claim that α_i and α_j are adjacent. If this is not the case, then there exists a node $\alpha_t \neq \alpha_j$ adjacent to α_i and a node $\alpha_s \neq \alpha_i$ adjacent to α_j. Then $\lambda - it$, $\lambda - ij$ and $\lambda - js$ afford $r - 4$, which is a contradiction since $r - 4$ has multiplicity ≤ 2 in $V \downarrow A$.

Thus α_i and α_j must be adjacent, and by Lemma 5.13.14 (iii) either α_i or α_j is an end node. Suppose first that G has rank at least 3. Without loss of generality assume that α_i is an end node. Then there exists a node $\alpha_k \neq \alpha_i$ adjacent to α_j, and then $\lambda - ij$ and $\lambda - jk$ afford the weight $r-4$. Thus we must have $c_i = c_j = 1$, as otherwise $\lambda - i^2$ or $\lambda - j^2$ would also afford the weight $r - 4$. As in the proof of Lemma 5.13.14 (iii), we can use Corollary 4.5.7 to conclude that $\lambda - ij$ has multiplicity 2 in V, giving a contradiction.

Suppose then that G has rank 2, so $\lambda = c_1\omega_1 + c_2\omega_2$. We will show that $\lambda - 12$ has multiplicity 2, which will show that $c_1 = c_2 = 1$ as otherwise $\lambda - 1^2$ or $\lambda - 2^2$ would also afford the weight $r - 4$. If G is of type G_2, then $r = 6c_1 + 10c_2$ by Lemma 2.7.3. Since $r = p + 1$, this implies $p > c_1 + 3c_2 + 3$, so by Lemma 4.5.2 the weight $\lambda - 12$ has multiplicity 2. The same argument works also for $G = A_2$ and $G = C_2$ to show that $\lambda - 12$ has multiplicity 2.

Finally, what remains is the case where G has rank 2 and $\lambda = \omega_1 + \omega_2$. If G has type A_2, then $r = 4$, so $p = 3$, but we are assuming that $p \geq 7$. For G of type C_2 or G_2, we have $r = 7$ and $r = 16$, respectively. In these cases $r = p + 1$ is not a possibility. □

Using Lemma 5.13.15 and the main result of [LST15], we can now prove Theorem 5.13.1 in the case where u is a regular unipotent element and $\lambda = c_i\omega_i + c_j\omega_j$ with $i \neq j$ and $c_i, c_j \geq 1$.

Proposition 5.13.16. *Theorem 5.13.1 holds when u is a regular unipotent element and $\lambda = c_i\omega_i + c_j\omega_j$, where $i \neq j$ and $c_i, c_j \geq 1$.*

Proof. Let $u \in G$ be a regular unipotent element and let $\lambda = c_i\omega_i + c_j\omega_j$, where $i \neq j$ and $c_i, c_j \geq 1$. Suppose that u acts on $L_G(\lambda)$ as distinguished unipotent element. By Lemma 5.13.15, we have $p > r$, so by Corollary 2.7.6 we have $V_G(\lambda) = L_G(\lambda)$. Furthermore, by Lemma 2.7.9 the action of u on $V_G(\lambda)$ has the same Jordan block sizes as the corresponding action in characteristic 0. Thus the claim of Theorem 5.13.1 follows from [LST15, Theorem 1]. □

Lemma 5.13.17 ([LST15, Lemma 2.6])**.** *Let $\lambda = c_i\omega_i + c_j\omega_j$, where $i \neq j$ and $c_i, c_j \geq 1$. Suppose that u acts on V as a distinguished unipotent element. Then*

(i) *If α_i and α_j are adjacent, then $\lambda - ij$ has multiplicity* 2.

(ii) *Either $c_i = 1$ or $c_j = 1$.*

(iii) *If $c_i > 1$ or $c_j > 1$, then G has rank* 2.

Proof. (i) According to Lemma 5.13.15, we have $p > m_u(\lambda) = 6c_1 + 10c_2$. Thus if G has type G_2, then it follows from Lemma 4.5.2 that the weight $\lambda - ij$ has multiplicity 2. Suppose then that G does not have type G_2. We always have $m_u(\omega_i) \geq 2$, so it follows that $m_u(\lambda) \geq 1 + c_i + c_j$, $1 + 2c_i + c_j$, $1 + c_i + 2c_j$. Then for $L = A_2$, $L = B_2$ and $L = C_2$, in the representation $L_L(c_1\omega_1 + c_2\omega_2)$, the weight $\omega_1 + \omega_2 - \alpha_1 - \alpha_2$ has multiplicity 2 by Lemma 4.5.2. Thus by Corollary 4.5.7, the weight $\lambda - ij$ has multiplicity 2 in V.

(ii) Using (i), the claim follows with the argument given in the end of the proof of [LST15, Lemma 2.6 (iii)].

(iii) Again with (i), this follows with the same proof as [LST15, Lemma 2.6 (v)]. □

5.13.1 Case where u is regular and $\lambda = b\omega_i$, $b \geq 2$

In this section we will handle the case where $\lambda = b\omega_i$, $b \geq 2$. The result in this case is the following proposition (cf. [LST15, Proposition 4.1]), which we will prove using lemmas given later in this section. This then finishes the proof of Theorem 5.13.1 in the case where u is a regular unipotent element.

Proposition 5.13.18. *Suppose that $\lambda = b\omega_i$, where $b \geq 2$. Then a regular unipotent element $u \in G$ acts on V as a distinguished unipotent element if and only if $p > m_u(\lambda)$ and G and λ are as in the table below, with λ given up to graph automorphism of D_4.*

For the proof, recall that we can assume that α_i is an end node (Lemma 5.13.12 (i)). Note also that now if u acts on V as a distinguished unipotent element, by Lemma 5.13.12 (iii) the weight $r - 2$ of $V \downarrow A$ has multiplicity 1. Then by Lemma 5.13.9 (iv), for all $c \geq 1$ the weight $r - 2c$ of $V \downarrow A$ must occur with multiplicity $\leq c$. We will use these facts in the proofs of the lemmas that follow. In this subsection u will always be a regular unipotent element of G.

λ	G
$2\omega_1$	B_n, C_n, D_n (n even), G_2
$3\omega_1$	C_2, C_3, B_3, G_2
$b\omega_1$, $b \leq 5$	C_2
$b\omega_2$, $b \leq 5$	C_2
$2\omega_3$	B_3
$2\omega_2$	G_2

Lemma 5.13.19. *Suppose that $G = B_n$ ($n \geq 3$), $G = C_n$ ($n \geq 3$) or $G = D_n$ ($n \geq 4$) and $\lambda = b\omega_1$, where $b \geq 3$. If u acts on V as a distinguished unipotent element, then $b = 3$.*

Proof. We proceed as in the end of the proof of [LST15, Lemma 4.4]. Suppose that $b \geq 4$. We show that the weight $r - 8$ of A occurs with multiplicity ≥ 5, which will give a contradiction by Lemma 5.13.9 (iv). If $n \geq 4$, then $r - 8$ is afforded by $\lambda - 1^4$, $\lambda - 1^3 2$, $\lambda - 1^2 2^2$, $\lambda - 1^2 23$ and $\lambda - 1234$. If $n = 3$, replacing the last of these weights by $\lambda - 123^2$ (type B_3) or $\lambda - 12^2 3$ (type C_3) gives us again a list of 5 weights affording $r - 8$. □

Lemma 5.13.20. *Suppose that $G = B_n$ with $n \geq 3$ and $\lambda = b\omega_n$ with $b \geq 2$. If u acts on V as a distinguished unipotent element, then $n = 3$ and $b = 2$.*

Proof. We proceed similarly to [LST15, Lemma 4.3]. Now we have $L_{B_2}(b\omega_2) = V_{B_2}(b\omega_2) = S^b(V_{B_2}(\omega_2))$ (see e.g. [Sei87, 1.14 and 8.1 (c)] or [McN98, Proposition 4.2.2.(h)]). Hence in $L_{B_2}(b\omega_2)$ the weight $b\omega_2 - \alpha_1 - 2\alpha_2$ occurs with multiplicity 2, so by using 4.5.7 we see that the weight $\lambda - n^2(n-1)$ has multiplicity 2 in $V \downarrow A$. So if $b \geq 3$, the weight $r - 6$ has then multiplicity at least 4, since it is afforded by $\lambda - n^2(n-1)$, $\lambda - n^3$ and $\lambda - n(n-1)(n-2)$, contradicting Lemma 5.13.9 (iv). Therefore $b = 2$.

If $n \geq 4$, then $r - 8$ has multiplicity at least 5, since it is afforded by $\lambda - n^2(n-1)(n-2)$, $\lambda - n^2(n-1)^2$, $\lambda - n(n-1)(n-2)(n-3)$ and $\lambda - n^3(n-1)$. This is again a contradiction by Lemma 5.13.9 (iv). Therefore we must have $b = 2$ and $n = 3$. □

Lemma 5.13.21. *If $G = C_n$ with $n \geq 3$ and $\lambda = b\omega_n$ with $b \geq 2$, then u does not act on V as a distinguished unipotent element.*

Proof. We proceed again similarly to [LST15, Lemma 4.3]. Suppose that u acts on V as a distinguished unipotent element. If $b \geq 3$, then the weight $r - 6$ of A is afforded by $\lambda - (n-2)(n-1)n$, $\lambda - (n-1)n^2$, $\lambda - n^3$ and $\lambda - (n-1)^2 n$, contradicting Lemma 5.13.9 (iv).

Suppose then that $b = 2$. In this case if $n \geq 5$, the weight $r - 12$ has multiplicity at least 7, since it is afforded by the following weights:

$$\lambda - n^2(n-1)^4$$
$$\lambda - n^2(n-1)^3(n-2)$$
$$\lambda - n^2(n-1)^2(n-2)^2$$
$$\lambda - n^2(n-1)^2(n-2)(n-3)$$
$$\lambda - n^2(n-1)(n-2)(n-3)(n-4)$$
$$\lambda - n(n-1)^2(n-2)^2(n-3)$$
$$\lambda - n(n-1)^2(n-2)(n-3)(n-4)$$

This again contradicts 5.13.9 (iv).

Consider $n = 4$ and $\lambda = 2\omega_4$, in which case $m_u(\lambda) = 32$ by Lemma 2.7.3. Since u is regular of order p we have $p \geq 2n = 8$, so $p > 7$. Then according to [Lüb01], we have $\dim V = 594 > 289 = \frac{(32+2)^2}{4}$, so Lemma 5.1.1 gives a contradiction.

In the case $n = 3$ a computation with MAGMA (Section 2.9), given in Table 5.3, shows that a regular unipotent element does not act on $V = L_G(2\omega_3)$ as a distinguished unipotent element. □

p	$L_G(2\omega_3) \downarrow K[u]$
$p = 3$	$[3, 9^9]$
$p = 5$	$[10^2, 13, 15^2]$
$p = 7$	$[7^{12}]$
$p = 11$	$[7, 11^7]$
$p = 13$	$[3, 7, 9, 13^5]$
$p = 17$	$[3, 7^2, 9, 11, 13, 17^2]$
$p \geq 19$	$[3, 7^2, 9, 11, 13, 15, 19]$

Table 5.3: Action of a regular unipotent $u \in G$ on $L_G(2\omega_3)$, for $G = C_3$.

Lemma 5.13.22. *If $G = C_2$ and $\lambda = b\omega_1$ with $b \geq 6$, then u does not act on V as a distinguished unipotent element.*

Proof. In this case we have $V = L_G(\lambda) = V_G(\lambda) = S^b(E)$ (see e.g. [Sei87, 1.14 and 8.1 (c)] or [McN98, Proposition 4.2.2.(h)]), where E is the natural module for G. Suppose that u acts on $V \downarrow A$ as a distinguished unipotent element. The weights that occur in E are m, $m-2$, $m-4$ and $m-6$, where $m = 3$. Now the weight $r - 12$ has multiplicity at least 7, because it is afforded by symmetric powers of vectors with weights as follows (each tuple below has length b):

$$(m, \ldots, m, m, m, m, m, m-6, m-6)$$
$$(m, \ldots, m, m, m, m, m-2, m-4, m-6)$$
$$(m, \ldots, m, m, m, m-2, m-2, m-2, m-6)$$
$$(m, \ldots, m, m, m, m, m-4, m-4, m-4)$$
$$(m, \ldots, m, m, m, m-2, m-2, m-4, m-4)$$
$$(m, \ldots, m, m, m-2, m-2, m-2, m-2, m-4)$$
$$(m, \ldots, m, m-2, m-2, m-2, m-2, m-2, m-2)$$

This is a contradiction by Lemma 5.13.9 (iv). □

Lemma 5.13.23. *If $G = C_2$ and $\lambda = b\omega_2$ with $b \geq 6$, then u does not act on V as a distinguished unipotent element.*

Proof. Suppose that u acts on V as a distinguished unipotent element. The results of Lübeck [Lüb01] imply that $\dim L_G(b\omega_2) \geq 85$ for $b \geq 6$, so we can assume that $p > 17$ by Lemma 5.1.1.

If $b = 6$, then a computation with MAGMA (Section 2.9), given in Table 5.4, shows that u acts on V with repeated blocks. Assume then that $b \geq 7$. Let $r = m_u(\lambda)$, so here $r = 4b$ by Lemma 2.7.3.

Now the weight $r-8$ is afforded by the weights $\lambda - 2^4$, $\lambda - 12^3$ and $\mu = \lambda - 1^2 2^2$. One can verify that all weights $\lambda \succeq \mu' \succ \mu$ have multiplicity 1 in $V_G(\lambda)$ (see Table 4.2). Hence by Lemma 4.5.4, the weight μ occurs with multiplicity 2 in V unless $L_G(\mu)$ is a composition factor of $V_G(\lambda)$.

Consider first the case where $L_G(\mu)$ is a composition factor of $V_G(\lambda)$. Now by Proposition 4.5.1 and Table 4.2, we have that p divides $b+1$. Since λ is p-restricted, we have $b = p - 1$. Then $r = 4b = 4p - 4 = 3p + (p - 4)$, which is a contradiction by Lemma 5.13.13.

Thus $L_G(\mu)$ does not occur as a composition factor of $V_G(\lambda)$, and so μ occurs with multiplicity 2 in $L_G(\lambda)$. We proceed to show next that $r - 10$ has multiplicity at least 6 in $V \downarrow A$, which will contradict Lemma 5.13.9 (iv). First of all, the weights $\nu_1 = \lambda - 12^4$ and $\nu_2 = \lambda - 1^2 2^3$ afford the weight $r - 10$. Furthermore, by Table 4.2, the weights ν_1 and ν_2 have multiplicity 2 in $V_G(\lambda)$. One can verify that all weights $\lambda \succeq \mu' \succ \nu_i$ except for $\mu' = \mu$ have multiplicity 1 in $V_G(\lambda)$ (see Table 4.2). Then because $L_G(\mu)$ does not occur as a composition factor of $V_G(\lambda)$, it follows from Lemma 4.5.4 that ν_i has multiplicity 2 in $L_G(\lambda)$ if $L_G(\nu_i)$ does not occur as a composition factor of $V_G(\lambda)$. Now by Proposition 4.5.1 and Table 4.2, if $L_G(\nu_1)$ (respectively $L_G(\nu_2)$) occurs as a composition factor of $V_G(\lambda)$, then p divides $b - 3$ (respectively $b - 4$). Because $b \leq p - 1$, it follows that p does not divide $b - 3$ nor $b - 4$, so we can conclude that ν_1 and ν_2 both have multiplicity 2 in $L_G(\lambda)$. Hence $r - 10$ occurs in $V \downarrow A$ with multiplicity at least 6, being afforded by $\lambda - 2^5$, $\lambda - 1^3 2^2$, ν_1, and ν_2. □

p	$L_G(6\omega_2) \downarrow K[u]$
$p = 7$	$[7^{20}]$
$p = 11$	$[1, 3, 5, 7, 11^{10}]$
$p = 13$	$[1, 3, 7, 9, 13^5]$
$p = 17$	$[1, 7, 13, 17^7]$
$p = 19$	$[1, 7, 9, 13, 15, 19^5]$
$p = 23$	$[1, 7, 9, 13^2, 15, 17, 19, 23^2]$
$p \geq 29$	$[1, 7, 9, 13^2, 15, 17, 19, 21, 25]$

Table 5.4: Action of a regular unipotent $u \in G$ on $L_G(6\omega_2)$, for $G = C_2$.

Lemma 5.13.24. *Suppose that $G = D_n$ and $\lambda = b\omega_n$ or $\lambda = b\omega_{n-1}$, where $b \geq 2$. If u acts on V as a distinguished unipotent element, then $n = 4$ and $b = 2$.*

Proof. This follows exactly as in the end of the proof of [LST15, Lemma 4.3]. □

Lemma 5.13.25. *If $G = G_2$ and $\lambda = b\omega_1$ with $b \geq 4$, then u does not act on V as a distinguished unipotent element.*

Proof. ([LST15, Lemma 4.5 (ii)]) The representation $L_G(\omega_1)$ gives an embedding $G < B_3$, and now we have $L_{B_3}(b\omega_1) \downarrow G \cong V$ by [Sei87, Table 1, III_1]. Since under this embedding G_2 contains a regular unipotent element of B_3 (Theorem 1.1.12), the claim follows from Lemma 5.13.19. □

Lemma 5.13.26. *If $G = G_2$ and $\lambda = b\omega_2$ with $b \geq 3$, then u does not act on V as a distinguished unipotent element.*

Proof. Here $r = 10b$ by Lemma 2.7.3.

Suppose that u acts on V as a distinguished unipotent element. Note first that by looking at the results of Lübeck in [Lüb01], we have $\dim V \geq 148$ under our assumptions. It follows then from Lemma 5.1.1 that $p > 23$, as otherwise $\dim V > \frac{(p+1)^2}{4}$.

Consider first $b = 3$. In this case $p > 23$ and [Lüb01] imply that $\dim V = 273 > 256 = \frac{(r+2)^2}{4}$, which is a contradiction by Lemma 5.1.1. Therefore $b \geq 4$.

Now the weight $r-8$ is afforded by $\lambda - 2^4$, $\lambda - 12^3$, $\lambda - 1^32$ and $\mu = \lambda - 1^22^2$. Note that μ occurs with multiplicity 2 in $V_G(\lambda)$ (Table 4.1). Furthermore, one can verify that all weights $\lambda \succeq \mu' \succ \mu$ have multiplicity 1 in $V_G(\lambda)$ (see Table 4.1). Thus if $L_G(\mu)$ does not occur as a composition factor of $V_G(\lambda)$, it follows from Lemma 4.5.4 that μ occurs with multiplicity 2 in V, contradicting Lemma 5.13.9 (iv).

Hence $L_G(\mu)$ must occur as a composition factor of $V_G(\lambda)$. By Proposition 4.5.1 and Table 4.1, this implies that p divides $3b+2$. Now $b \leq p-1$, so $3b+2 = p$ or $3b+2 = 2p$. Thus $r = 10b = \frac{10p-20}{3}$ or $r = \frac{20p-20}{3}$, which is a contradiction by Lemma 5.13.13. □

Lemma 5.13.27. *If $G = F_4$ and $\lambda = b\omega_i$ with $b \geq 2$, then u does not act on V as a distinguished unipotent element.*

Proof. Suppose that u acts on $V \downarrow A$ as a distinguished unipotent element. As in the results above, recall that by Lemma 5.13.12 the node α_i is an end node, and that the weight $r-2$ of A has multiplicity 1. If $b \geq 3$, then as in the proof of [LST15, Lemma 4.7], we see that the weight $r-8$ of $V \downarrow A$ occurs with multiplicity at least 5, contradicting Lemma 5.13.9 (iv).

Consider then the cases where $b = 2$. If $\lambda = 2\omega_1$, then $r = 44$ by Lemma 2.7.3. By [Lüb01], we have $\dim V \geq 755 > 529 = \frac{(r+2)^2}{4}$, which is a contradiction by Lemma 5.1.1. Finally if $\lambda = 2\omega_4$, then $r = 32$ by Lemma 2.7.3. Again by [Lüb01], we have $\dim V \geq 298 > 289 = \frac{(r+2)^2}{4}$, which is again a contradiction by Lemma 5.1.1. □

Lemma 5.13.28. *If $G = E_6$, $G = E_7$ or $G = E_8$ and $\lambda = b\omega_i$ with $b \geq 2$, then u does not act on V as a distinguished unipotent element.*

Proof. Suppose that u acts on $V \downarrow A$ as a distinguished unipotent element. Recall that by Lemma 5.13.12 the node α_i is an end node, and that the weight $r-2$ of A has multiplicity 1.

Consider first the case where $i = 1$. If $b = 2$, then by Lemma 2.7.3 we have $r = 32$, $r = 68$ and $r = 184$ when $G = E_6$, $G = E_7$, or $G = E_8$ respectively. Furthermore, it follows from [Lüb01] that $\dim V = 351$, $\dim V \geq 7370$, and $\dim V > 100000$ respectively. In any case, we see that $\dim V > \frac{(r+2)^2}{4}$, which is a contradiction by Lemma 5.1.1. If $b \geq 3$, then the weight $r-8$ of $V \downarrow A$ is afforded by $\lambda - 1^2 3^2$, $\lambda - 1^2 34$, $\lambda - 1234$ and $\lambda - 1345$, contradicting Lemma 5.13.9 (iv).

Thus we must have $i > 1$. If $i = 2$, then the weight $r-8$ is afforded by $\lambda - 2^2 4^2$, $\lambda - 2^2 43$, $\lambda - 2^2 45$, $\lambda - 2431$ and $\lambda - 2435$, which again contradicts Lemma 5.13.9 (iv).

Therefore α_i must be the last node in the Dynkin diagram of G, say $\lambda = b\omega_n$ where $G = E_n$. In this situation if $b = 2$, then by Lemma 2.7.3 we have $r = 32$, $r = 54$ and $r = 118$ when $G = E_6$, $G = E_7$, and $G = E_8$ respectively. Furthermore, it follows from [Lüb01] that $\dim V = 351$, $\dim V = 1463$, and $\dim V > 100000$ respectively. As earlier in the proof, we see that $\dim V > \frac{(r+2)^2}{4}$, which is a contradiction by Lemma 5.1.1.

Hence $b \geq 3$. In this case the weight $r - 12$ has multiplicity at least 7, being afforded by $\lambda - n^3(n-1)^3$, $\lambda - n^3(n-1)^2(n-2)$, $\lambda - n^3(n-1)(n-2)(n-3)$, $\lambda - n^2(n-1)^2(n-2)^2$, $\lambda - n^2(n-1)^2(n-2)(n-3)$, $\lambda - n^2(n-1)(n-2)(n-3)(n-4)$, and $\lambda - n(n-1)(n-2)(n-3)(n-4)(n-5)$. This is a contradiction by Lemma 5.13.9 (iv). □

We now summarize the lemmas above and give the proof of Proposition 5.13.18.

Proof of Proposition 5.13.18. Suppose that $\lambda = b\omega_i$, where $b \geq 2$. Assume that a regular unipotent element $u \in G$ acts on $V = L_G(\lambda)$ as a distinguished unipotent element. It follows from Lemma 5.13.12 (i) that α_i is an end node. Therefore if $G = A_n$, then the claim of Proposition 5.13.18 follows from Lemma 5.13.4.

If $G = B_n$ with $n \geq 3$, then it follows from Lemma 5.13.19 and Lemma 5.13.20 that $\lambda = 2\omega_1$, $\lambda = 3\omega_1$, or $n = 3$ and $\lambda = 2\omega_3$. Furthermore, for these cases the claim of Proposition 5.13.18 follows from Proposition 5.7.7 (b) (ii), Proposition 5.9.1, and Table 5.5, respectively. Here the decompositions given in Table 5.5 were computed with MAGMA (Section 2.9).

If $G = B_2 = C_2$, then it follows from Lemma 5.13.22 and Lemma 5.13.23 that $\lambda = b\omega_i$ with $b \leq 5$. For $b \leq 3$ the claim of Proposition 5.13.18 follows from Proposition 5.7.11 (b) (ii) (for $i = 1$, $b = 2$), Proposition 5.7.7 (b) (ii) (for $i = 2$, $b = 2$), and Proposition 5.9.1 (for $b = 3$). For $b = 4$ and $b = 5$, the claim of Proposition 5.13.18 follows from Table 5.6, which was computed with MAGMA (Section 2.9).

If $G = C_n$ with $n \geq 3$, then it follows from Lemma 5.13.19 and Lemma 5.13.21 that $\lambda = 2\omega_1$ or $\lambda = 3\omega_1$. For these cases the claim of Proposition 5.13.18 follows from Proposition 5.7.11 (b) (ii) and Proposition 5.9.1, respectively.

If $G = D_n$ with $n \geq 4$, then it follows from Lemma 5.13.19 and Lemma 5.13.24 that $\lambda = 2\omega_1$, $\lambda = 3\omega_1$, or $n = 4$ and $\lambda \in \{2\omega_3, 2\omega_4\}$. For these cases the claim of Proposition 5.13.18 follows from Proposition 5.7.8 (b) (ii), Proposition 5.9.1, and Proposition 2.10.2 (ii), respectively.

If $G = G_2$, then it follows from Lemma 5.13.25 and Lemma 5.13.26 that $\lambda \in \{2\omega_1, 3\omega_1, 2\omega_2\}$. In these cases the claim is a consequence of Table 5.7, which was computed with MAGMA (Section 2.9).

Finally if $G = F_4$, $G = E_6$, $G = E_7$, and $G = E_8$, we have a contradiction by Lemma 5.13.27 or Lemma 5.13.28. □

p	$L_G(2\omega_3) \downarrow K[u]$
$p = 3$	$[1, 7, 9^3]$
$p = 5$	$[1, 5, 7, 9, 13]$
$p = 7$	$[7^5]$
$p = 11$	$[1, 5, 7, 11^2]$
$p \geq 13$	$[1, 5, 7, 9, 13]$

Table 5.5: Action of a regular unipotent $u \in G$ on $L_G(2\omega_3)$, for $G = B_3$.

λ	p	$L_G(\lambda) \downarrow K[u]$
$4\omega_1$	$p = 5$	$[5^7]$
	$p = 7$	$[7^5]$
	$p = 11$	$[1, 5, 7, 11^2]$
	$p \geq 13$	$[1, 5, 7, 9, 13]$
$5\omega_1$	$p = 7$	$[7^8]$
	$p = 11$	$[4, 8, 11^4]$
	$p = 13$	$[4, 6, 8, 12, 13^2]$
	$p \geq 17$	$[4, 6, 8, 10, 12, 16]$
$4\omega_2$	$p = 5$	$[5^{11}]$
	$p = 7$	$[5, 7^7]$
	$p = 11$	$[11^5]$
	$p = 13$	$[5, 11, 13^3]$
	$p \geq 17$	$[5, 9, 11, 13, 17]$
$5\omega_2$	$p = 7$	$[7^{13}]$
	$p = 11$	$[3, 5, 9, 11^4]$
	$p = 13$	$[13^7]$
	$p = 17$	$[5, 9, 11, 15, 17^3]$
	$p = 19$	$[5, 9, 11, 13, 15, 19^2]$
	$p \geq 23$	$[5, 9, 11, 13, 15, 17, 21]$

Table 5.6: For $G = C_2$ and $u \in G$ a regular unipotent element, action of u on $L_G(\lambda)$ for $\lambda = b\omega_i$ with $i \in \{1, 2\}$ and $b \in \{4, 5\}$.

5.13.2 Case where u is not regular

In this section, we keep our setup as before, but assume throughout that u is a distinguished unipotent element that is not regular. The result in this case is the following proposition, which completes the proof of Theorem 5.13.1.

Proposition 5.13.29. *A non-regular unipotent element u of order p acts as a distinguished unipotent element on V if and only if $p > m_u(\lambda)$ and G and λ are as in the table below, with λ given up to graph automorphism of D_n.*

λ	p	$L_G(\lambda) \downarrow K[u]$
$2\omega_1$	$p = 3$	$[9^3]$
	$p = 5$	$[5, 9, 13]$
	$p = 7$	$[5, 7^3]$
	$p = 11$	$[5, 11^2]$
	$p \geq 13$	$[5, 9, 13]$
$3\omega_1$	$p = 5$	$[5^2, 10^2, 13, 15, 19]$
	$p = 7$	$[7^{11}]$
	$p = 11$	$[11^7]$
	$p = 13$	$[3, 9, 13^5]$
	$p = 17$	$[3, 7, 9, 11, 13, 17^2]$
	$p \geq 19$	$[3, 7, 9, 11, 13, 15, 19]$
$2\omega_2$	$p = 3$	$[9^3]$
	$p = 5$	$[1, 5, 10^2, 15^2, 21]$
	$p = 7$	$[7^{11}]$
	$p = 11$	$[11^7]$
	$p = 13$	$[1, 11, 13^5]$
	$p = 17$	$[1, 5, 9, 11, 17^3]$
	$p = 19$	$[1, 5, 9, 11, 13, 19^2]$
	$p \geq 23$	$[1, 5, 9, 11, 13, 17, 21]$

Table 5.7: For $G = G_2$ and $u \in G$ a regular unipotent element, action of u on $L_G(\lambda)$ for $\lambda = 2\omega_1$, $\lambda = 3\omega_1$, and $\lambda = 2\omega_2$.

λ	G	class of u
ω_1	B_n, C_n, D_n	any
ω_6	D_6	$[3, 9]$
ω_4	F_4	$F_4(a_1)$
ω_7	E_7	$E_7(a_1)$, $E_7(a_2)$
ω_8	E_8	$E_8(a_1)$

To prove Proposition 5.13.29, it will be enough to show in each case that u does not act as a distinguished unipotent element on V when $\lambda \neq \omega_i$. After that is done, the result follows from Lemma 5.13.5.

We begin by considering the case where G is of classical type. Because we are assuming that u is a non-regular distinguished unipotent element, we have $\operatorname{rank} G \geq 3$ and G is not of type A_n.

Lemma 5.13.30. *Suppose that $G = B_n$, $G = C_n$ or $G = D_n$ and assume that u acts on V as a distinguished unipotent element. Then*

(i) $\lambda = b\omega_i$ *for some* i.

(ii) *If* $\lambda = b\omega_i$ *with* $b \geq 2$, *then* $i = 1$.

Proof. (i): We argue as in [LST15, Lemma 6.2 (iii)]. Suppose that $\lambda \neq b\omega_i$. Then by Lemma 5.13.9 (ii) we have $\lambda = c_i\omega_i + c_j\omega_j$ where $i \neq j$ and $c_i, c_j \geq 1$. According to 5.13.14 (i), nodes adjoining α_i and α_j have label 2, so according to Proposition 2.6.5 the nodes α_i and α_j occur in the initial string of 2's. Now since $\operatorname{rank} G \geq 3$, by Lemma 5.13.17 (iii) we have $c_i = c_j = 1$, and Lemma 5.13.14 (iii) shows that either α_i or α_j is an end node. Because α_i and α_j occur in the initial string, without loss of generality we can assume $i = 1$ so $\lambda = \omega_1 + \omega_j$ for some $j > 1$.

If $j > 2$, then there exists a node $\alpha_k \neq \alpha_i$ adjacent to α_j. Since α_i and α_j occur in the initial string and not all labels are 2, the node α_j is not an end node. Therefore there exist distinct nodes α_s, $\alpha_t \neq \alpha_i$ which are adjacent to α_j. Then the weight $r - 4$ of $V \downarrow A$ is afforded by $\lambda - 12$, $\lambda - 1j$, $\lambda - tj$ and $\lambda - sj$, which is a contradiction by Lemma 5.13.9 (iv).

Thus $\lambda = \omega_1 + \omega_2$, and so the labeled Dynkin diagram has at least three 2's in the initial string. If the labeling starts with $2220\ldots$, then $r-4$ is afforded by the weights $\lambda - 12$, $\lambda - 23$ and $\lambda - 234$. Here $\lambda - 12$ has multiplicity 2 by Lemma 5.13.17 (i), so this contradicts Lemma 5.13.9 (iv). On the other hand, if the labeling starts with $2222\ldots$, then $r - 6$ is afforded by $\lambda - 123$, $\lambda - 234$, $\lambda - 1^22$ and $\lambda - 12^2$. Here $\lambda - 123 = (\lambda - 12)^{s_3}$ has multiplicity 2 by Lemma 5.13.17 (i), again a contradiction by Lemma 5.13.9 (iv).

(ii): This follows with the same argument as [LST15, Lemma 6.2 (iv)].

□

Lemma 5.13.31. *Suppose that $G = B_n$, $G = C_n$ or $G = D_n$. If $\lambda = b\omega_1$ with $b \geq 2$, then u does not act on V as a distinguished unipotent element.*

Proof. If $b = 2$, the claim follows from Proposition 5.5.10 (a), Proposition 5.5.5 (a) and Proposition 5.5.6 (a). If $b = 3$, then the claim follows from Proposition 5.9.1.

Suppose then that $b \geq 4$ and assume that u acts on $V \downarrow A$ as a distinguished unipotent element. According to Lemma 5.13.12, the nodes α_1 and α_2 have label 2, so $n \geq 3$. Furthermore, by Lemma 5.13.12 (iii) the weight $r - 2$ of A has multiplicity 1. Suppose that α_3 has label 0. In this case $r-4$ is afforded by $\lambda - 1^2$, $\lambda - 12$ and $\lambda - 123$, contradicting Lemma 5.13.9 (v). Therefore α_3 has label 2, and so $n \geq 4$. Suppose that α_4 has label 0. Then $r - 6$ is afforded by $\lambda - 1^3$, $\lambda - 1^22$, $\lambda - 123$ and $\lambda - 1234$, again a contradiction by Lemma 5.13.9 (v). Therefore α_4 has label 2. Now $r - 8$ is afforded by $\lambda - 1^4$, $\lambda - 1^32$, $\lambda - 1^22^2$, $\lambda - 1^223$ and $\lambda - 1234$, once again contradicting Lemma 5.13.9 (v) and completing our proof. □

This completes the proof of Proposition 5.13.29 for G of classical type. If G is of type B_n, C_n, or D_n and u acts on $V = L_G(\lambda)$ as a distinguished unipotent element, then $\lambda = \omega_i$ by Lemma 5.13.30 and Lemma 5.13.31.

We now move on to consider the exceptional groups.

Lemma 5.13.32. *Let $G = G_2$. If u acts on V as a distinguished unipotent element, then $\lambda = \omega_i$.*

Proof. Suppose that $\lambda \neq \omega_i$. Since u is not regular, it cannot act on V as a distinguished unipotent element in view of Lemma 5.13.9 (i) and Lemma 5.13.12 (ii). □

Lemma 5.13.33. *Let $G = F_4$. Then if u acts on V as a distinguished unipotent element, we have $\lambda = \omega_i$.*

Proof. We proceed similarly to [LST15, Lemma 6.7]. Suppose that u acts on $V \downarrow A$ as a distinguished unipotent element. Consider first the possibility that $\lambda = c_i\omega_i + c_j\omega_j$, where $i \neq j$ and $c_i, c_j \geq 1$. Then by Lemma 5.13.14, the nodes α_i and α_j and nodes adjacent to them have label 2. But by looking at the possible labelings given in Proposition 2.6.7, this means that all labels of the Dynkin diagram are 2, contrary to our assumption that u is not regular.

Therefore $\lambda = b\omega_i$ for some i. If $b \geq 2$, then by Lemma 5.13.12 the node α_i is an end node. Furthermore, α_i and the node adjacent to it have label 2. By Proposition 2.6.7, this means that $i = 1$ and the labeled Dynkin diagram of u is 2202. Now the weight $r - 4$ is afforded by $\lambda - 1^2$, $\lambda - 12$ and $\lambda - 123$, contradicting Lemma 5.13.9 (v), since $r - 2$ has multiplicity 1 by Lemma 5.13.12. Thus $\lambda = \omega_i$. □

Lemma 5.13.34. *Let $G = E_6$, $G = E_7$ or $G = E_8$. If u acts on V as a distinguished unipotent element, then $\lambda = \omega_i$.*

Proof. Assume that $G = E_n$ and that u acts on $V \downarrow A$ as a distinguished unipotent element.

Suppose first that $c_i, c_j \geq 1$ for some $i \neq j$. Then $\lambda = c_i\omega_i + c_j\omega_j$ by Lemma 5.13.9 (ii). Now the arguments given in the beginning of the proof of [LST15, Lemma 6.8] show that we must have $\lambda = \omega_{n-1} + \omega_n$. Since we are assuming $p \geq 7$, by 4.5.7 and 4.5.3 the weight $\lambda - n(n-1)$ occurs with multiplicity 2. Then the argument given in [LST15, Proof of Lemma 6.8, paragraph 2] shows that we have a contradiction with Lemma 5.13.9 (iii).

Now consider $\lambda = b\omega_i$. According to Lemma 5.13.12, the node α_i is an end node with label 2, the node adjacent to α_i has label 2, and the weight $r - 2$ of $V \downarrow A$ occurs with multiplicity 1. Now the arguments given in the proof of [LST15, Proof of Lemma 6.8, paragraphs 3-4] show that we have a contradiction with Lemma 5.13.9 (iv) if $b > 1$. Therefore $\lambda = \omega_i$.

□

With Lemma 5.13.32, Lemma 5.13.33, and Lemma 5.13.34, the claim of Proposition 5.13.29 for G of exceptional type follows. This also completes the proof of Theorem 5.13.1.

5.14 Proof of Theorem 1.1.10 and Theorem 1.1.11

In this section we will put results from previous sections together and prove Theorem 1.1.10 and Theorem 1.1.11.

Let $u \in G$ be a unipotent element, and let $\lambda \in X(T)^+$ be a non-zero p-restricted dominant weight.

We assume first that $p \neq 2$ and prove Theorem 1.1.10. Suppose that u acts on $L_G(\lambda)$ as a distinguished unipotent element. It follows from Lemma 1.1.8 that u is a distinguished unipotent element. If u has order p, then Theorem 1.1.10 is given by Theorem 5.13.1. Suppose then that u has order $> p$. In this case, for the different types, Theorem 1.1.10 is a consequence of the following results:

- $G = A_l$ ($l \geq 1$): Here the only distinguished unipotent class is the regular one (Lemma 2.2.2), so u is a regular unipotent element. It follows from Proposition 5.1.4 and Lemma 5.1.5 that $\lambda = \omega_i$ or $\lambda = \omega_1 + \omega_l$. In these cases the claim of Theorem 1.1.10 follows from Proposition 5.12.1 and Proposition 5.3.2, respectively.

- $G = B_l$ or $G = C_l$ ($l \geq 2$): In this case, it follows from Proposition 5.1.14 that either $\lambda = \omega_i$, $\lambda = 2\omega_1$, $\lambda = 3\omega_1$, Proposition 5.1.14 (iv) holds, or Proposition 5.1.14 (v) holds. Now the claim of Theorem 1.1.10 follows from Proposition 5.12.1, Proposition 5.7.7, Proposition 5.7.11, Proposition 5.9.1, and Table 5.6.

- $G = D_l$ ($l \geq 4$): Here it follows from Proposition 5.1.18 that $\lambda = \omega_i$ or $\lambda = 2\omega_1$. In these cases the claim of Theorem 1.1.10 follows from Proposition 5.12.1 and Proposition 5.7.8, respectively.

- $G = G_2$: In this case, the claim of Theorem 1.1.10 is given by Proposition 5.1.19.

- $G = F_4$, $G = E_6$, $G = E_7$, or $G = E_8$: In this case, Theorem 1.1.10 follows from Lemma 5.1.20 and Proposition 5.12.1.

This completes the proof of Theorem 1.1.10.

We assume next that $p = 2$ and proceed to prove Theorem 1.1.11. Suppose that u acts on $L_G(\lambda)$ as a distinguished unipotent element. It follows from Lemma 1.1.8 that u is a distinguished unipotent element. For each of the different types, Theorem 1.1.11 is given by the following results:

- $G = A_l$ ($l \geq 1$): Here u must be a regular unipotent element (Lemma 2.2.2). It follows from Proposition 5.2.2 and Lemma 5.2.2 (i) that $\lambda = \omega_1$, $\lambda = \omega_l$, $\lambda = \omega_1 + \omega_l$, or $l = 3$ and $\lambda = \omega_2$. In these cases, the claim follows from Theorem 1.1.12, Proposition 5.4.4, and Lemma 5.2.2 (ii).

- $G = C_l$ ($l \geq 2$): It follows from Proposition 5.2.4 that $\lambda = \omega_1$, $\lambda = \omega_2$, $\lambda = \omega_l$, or $l = 5$ and $\lambda = \omega_3$. For $\lambda = \omega_1$ the claim is obvious since $L_G(\omega_1)$ is the natural representation, and for rest of the cases the claim follows from Proposition 5.6.7, Proposition 5.11.3, and Lemma 5.2.5.

- $G = B_l$ ($l \geq 2$): It follows from Proposition 5.2.4 that $\lambda = \omega_i$. There exists an exceptional isogeny $\varphi : B_l \to C_l$ [Ste68, Theorem 28], and we have $L_{C_l}(\omega_i)^\varphi \cong L_{B_l}(\omega_i)$ if $1 \leq i \leq l-1$, and $L_{C_l}(\omega_l)^\varphi \cong L_{B_l}(\omega_l)^{[1]}$. Thus in this case, the claim of Theorem 1.1.11 follows from the result for type C_l, which we have already proven.

- $G = D_l$ ($l \geq 4$): It follows from Proposition 5.2.4 that $\lambda = \omega_i$. In the case where $\lambda = \omega_{l-1}$ or $\lambda = \omega_l$, the result is given by Proposition 5.11.4. Suppose then that $1 \leq i \leq l-2$. Note that for a simple group H of type C_l, we have $G < H$ as the subsystem subgroup generated by short root subgroups, and $L_{C_l}(\omega_i) \downarrow G \cong L_G(\omega_i)$ for $1 \leq i \leq l-2$ [Sei87, Theorem 4.1]. Thus the claim follows from the result for type C_l.

- G simple of exceptional type: In this case, the claim is given by Proposition 5.2.6 and the tables in Appendix B.

Chapter 6

Non-G-completely reducible overgroups

Let G be a simple linear algebraic group and let $u \in G$ be a distinguished unipotent element. In this section, we study connected reductive subgroups X of G containing u such that X is contained in some proper parabolic subgroup of G. Testerman and Zalesski [TZ13] have shown that no such X exist if u is a regular unipotent element. We will find that their result does not generalize to distinguished unipotent elements and we will give several counterexamples in this section. However, we are able to give a complete list of X contained in a proper parabolic subgroup in the case where u has order p (Theorem 6.2.12). In the case where u has order $> p$, we have partial results. Furthermore, the general impression from the examples and results we have is that such subgroups X are quite rare and finding a complete list of them should eventually be possible.

6.1 Preliminaries on G-complete reducibility

We begin by making a basic useful observation. Let X be a connected reductive subgroup of G containing u, and assume that X is contained in a proper parabolic subgroup P of G. Since every Levi factor of P is a centralizer of some non-trivial torus, and since u is a distinguished unipotent element, it follows that X cannot be contained in any Levi factor of P. In the terminology of the next definition due to Serre, we say that X is a *non-G-completely reducible* subgroup of G.

Definition 6.1.1 (Serre, [Ser03]). Let H be a closed subgroup of G. We say that H is *G-completely reducible* (G-cr), if whenever H is contained in a parabolic subgroup P of G, it is contained in a Levi factor of P. Otherwise we say that H is *non-G-completely reducible* (non-G-cr). If H is not contained in any proper parabolic subgroup of G, we say that H is *G-irreducible* (G-ir).

We record the observation made above in the next lemma.

Lemma 6.1.2. *Let $X < G$ be a reductive subgroup of G. Suppose that X contains a distinguished unipotent element of G. Then X is G-ir or X is non-G-cr.*

Below we will list some well known properties of G-cr and G-ir subgroups that will be needed in the sequel. As seen from the next theorem, for classical groups the concept of G-cr (G-ir) subgroups can be seen as a generalization of semisimplicity (irreducibility) in representation theory[14].

Theorem 6.1.3 ([Ser05, 3.2.2]). *Let $G = \mathrm{SL}(V)$, $G = \mathrm{Sp}(V)$, or $G = \mathrm{SO}(V)$. Assume that $p > 2$ if $G = \mathrm{Sp}(V)$ or $G = \mathrm{SO}(V)$. Then a closed subgroup $H < G$ is G-cr if and only if $V \downarrow H$ is semisimple.*

The following two results describe parabolic subgroups and G-ir subgroups in simple classical groups.

Theorem 6.1.4. *Let $G = \mathrm{SL}(V)$, $G = \mathrm{Sp}(V)$, or $G = \mathrm{SO}(V)$. Any parabolic subgroup of G is of the form $\mathrm{Stab}_G(W_1 \subset \cdots \subset W_t)$, where $W_1 \subset \cdots \subset W_t$ is a flag of subspaces such that for all $1 \leq i \leq t$, the subspace W_i is totally isotropic if $G = \mathrm{Sp}(V)$, and totally singular if $G = \mathrm{SO}(V)$.*

Proof. This is well known, for a proof, see for example [MT11, Proposition 12.13]. □

Theorem 6.1.5 ([LS96, pg. 32-33]). *Let $G = \mathrm{SL}(V)$, $G = \mathrm{Sp}(V)$, or $G = \mathrm{SO}(V)$. Let $X < G$ be a connected reductive subgroup. Then X is G-ir if and only if one of the following holds:*

(i) *$G = \mathrm{SL}(V)$ and $V \downarrow X$ is irreducible;*

(ii) *$G = \mathrm{Sp}(V)$ or $G = \mathrm{SO}(V)$, and $V \downarrow X = V_1 \perp \cdots \perp V_t$ (orthogonal direct sum), where:*

- *For all $1 \leq i \leq t$, the V_i are non-degenerate subspaces and irreducible X-modules,*
- *For all $1 \leq i, j \leq t$ with $i \neq j$, we have $V_i \ncong V_j$ as X-modules.*

(iii) *$G = \mathrm{SO}(V)$, $p = 2$, $\dim V$ is even, and $V \downarrow X = V_1 \perp V_2 \perp \cdots \perp V_t$ (orthgonal direct sum), where:*

- *$V_1 \downarrow X$ is non-degenerate, the subgroup X stabilizes a nonsingular 1-space R of V_1, and the image of the representation $X \to \mathrm{SO}(R^{\perp})$ is $\mathrm{SO}(R^{\perp})$-ir.*
- *For all $2 \leq i \leq t$, the V_i are non-degenerate subspaces and irreducible X-modules,*
- *For all $2 \leq i, j \leq t$ with $i \neq j$, we have $V_i \ncong V_j$ as X-modules.*

6.2 Unipotent elements of order p

In this subsection, we consider unipotent elements of order p that are contained in a non-G-cr subgroup. As an application of results in Section 4.6 and recent work of Litterick and Thomas in [LT], we describe all unipotent elements that can be contained in some non-G-cr subgroup of type A_1 when p is good for G. We will also show that except for two known examples which occur in the case $(G, p) = (C_2, 2)$,

[14]For exceptional groups, reductive non-G-completely reducible subgroups only occur in small characteristic [LS96, Theorem 1].

any connected reductive subgroup containing a distinguished unipotent element of order p must be G-ir (Theorem 6.2.12).

We begin by considering the situation where p is good for G. Let $G_{\mathbb{C}}$ be a simple algebraic group over $\mathbb{C}$ with the same root system Φ as G. In good characteristic, recall (Definition 2.7.7) that for any unipotent element $u \in G$, we can define a unipotent element $u_{\mathbb{C}} \in G_{\mathbb{C}}$ which has the same labeled diagram as u.

Definition 6.2.1. Assume that p is good for G. Let $u \in G$ be a unipotent element. For $u_{\mathbb{C}} \in G_{\mathbb{C}}$, we define $N(u_{\mathbb{C}})$ to be the largest Jordan block size of $u_{\mathbb{C}}$ acting on the adjoint representation of $G_{\mathbb{C}}$.

With the "order formula" proven by Testerman in [Tes95], Lawther has shown the following result.

Theorem 6.2.2 ([Law95, Theorem 1])**.** *Assume that p is good for G. Let $u \in G$ be a unipotent element. Then u has order p if and only if $N(u_{\mathbb{C}}) \leq 2p-1$.*

We now proceed to describe for each simple type when equality $N(u_{\mathbb{C}}) = 2p-1$ holds. It turns out that with one exception (regular unipotent elements in G of type A_{p-1}), equality holds if and only if the unipotent element is contained in some non-G-cr subgroup of type A_1.

Lemma 6.2.3 (Type A_l, $l \geq 2$)**.** *Let $G = \mathrm{SL}(V)$ and assume that $\dim V \geq 3$. Let $u \in G$ be a unipotent element with $V \downarrow K[u] = V_{d_1} \oplus \cdots \oplus V_{d_t}$, where $d_1 \geq d_2 \geq \cdots \geq d_t > 0$. Then*

(i) *$N(u_{\mathbb{C}}) = 2d_1 - 1$.*

(ii) *$N(u_{\mathbb{C}}) = 2p-1$ if and only if $d_1 = p$.*

(iii) *u is contained in a non-G-cr subgroup $X < G$ of type A_1 if and only if $d_1 = p$ and $t > 1$.*

Proof. Using the fact that the highest root in type A_l is equal to $\omega_1 + \omega_l$, claim (i) can be seen from [Sup09, Proposition 1.5, Algorithm 1.6]. Claim (ii) is immediate from (i).

We consider claim (iii). For the "only if" part, suppose that u is contained in a non-G-cr subgroup $X < G$ of type A_1. Then u must have order p, so $d_1 \leq p$. If $d_1 < p$, then by Corollary 4.6.9 the restriction $V \downarrow X$ is semisimple. By Theorem 6.1.3 the subgroup X is G-cr, contradiction. Therefore $d_1 = p$. Now if $t = 1$, then V has dimension p and thus $V \downarrow X$ must be semisimple by [Jan97, (A)], again a contradiction by Theorem 6.1.3. Thus $t > 1$, giving the claim.

For the "if" part of claim (iii), suppose that $d_1 = p$ and $t > 1$. Let $X = \mathrm{SL}_2(K)$ and fix a non-identity unipotent element $v \in X$. Consider first the case where $d_2 = p$. By Lemma 4.6.2, for the indecomposable tilting X-module $T_X(p)$ we have $T_X(p) \downarrow K[v] = V_p \oplus V_p$. Furthermore, by Lemma 4.2.2 we have $L_X(d_i - 1) \downarrow K[u] = V_{d_i}$ for all i. Thus we can identify V with the X-module

$$T_X(p) \oplus \bigoplus_{i=3}^{t} L_X(d_i - 1),$$

which gives an embedding $X < \mathrm{SL}(V)$ such that $V \downarrow K[v] = V_{d_1} \oplus \cdots \oplus V_{d_t}$. Now v and u are conjugate in G (Lemma 2.2.1), so by replacing X with a conjugate, we can assume that $u = v$. Now $V \downarrow X$ is not semisimple since $T_X(p)$ is not semisimple (Theorem 4.1.5 (ii)), so by Theorem 6.1.3 the subgroup X is a non-G-cr subgroup of type A_1 containing u.

Consider next the case where $d_2 < p$. It follows from Lemma 4.6.1 that $V_X(p+d_2-1) \downarrow K[u] = V_p \oplus V_{d_2}$. As before, by Lemma 4.2.2 we have $L_X(d_i-1) \downarrow K[u] = V_{d_i}$ for all i. We can identify V with the X-module

$$V_X(p+d_2-1) \oplus \bigoplus_{i=3}^{t} L_X(d_i-1),$$

which gives an embedding $X < \mathrm{SL}(V)$ such that $V \downarrow K[v] = V_{d_1} \oplus \cdots \oplus V_{d_t}$. As in the previous paragraph, we can assume that $v = u$. It follows from Theorem 6.1.3 that X is a non-G-cr subgroup of type A_1 containing u since $V_X(p+d_2-1)$ is not semisimple (Proposition 4.6.5). □

Lemma 6.2.4 (Type C_l, $l \geq 2$). *Let $G = \mathrm{Sp}(V)$ and assume that $p > 2$ and $\dim V \geq 4$. Let $u \in G$ be a unipotent element with $V \downarrow K[u] = V_{d_1} \oplus \cdots \oplus V_{d_t}$, where $d_1 \geq d_2 \geq \cdots \geq d_t > 0$. Then*

(i) $N(u_{\mathbb{C}}) = 2d_1 - 1$.

(ii) $N(u_{\mathbb{C}}) = 2p - 1$ *if and only if* $d_1 = p$.

(iii) *u is contained in a non-G-cr subgroup $X < G$ of type A_1 if and only if $d_1 = p$.*

Proof. Claim (i), claim (ii), and the "only if" part of claim (iii) follow exactly as in the proof of Lemma 6.2.3.

For the "if" part of claim (iii), suppose that $d_1 = p$. Let $X = \mathrm{SL}_2(K)$ and fix a non-identity unipotent element $v \in X$. Now for unipotent elements in $\mathrm{Sp}(V)$, odd Jordan block sizes must have even multiplicity (Proposition 2.3.2), so $d_2 = p$. Furthermore, we can write

$$V \downarrow K[u] = V_p \oplus V_p \oplus \bigoplus_{i=1}^{r} 2 \cdot V_{2a_i+1} \oplus \bigoplus_{j=1}^{s} V_{2b_j}$$

where $a_i \geq 0$ for all i, where $b_j \geq 1$ for all j, and $2 + 2r + s = t$. We proceed next to realize each of these summands as X-modules with non-degenerate X-invariant alternating bilinear forms.

It follows from Corollary 4.4.11 that the tilting X-module $T_X(p)$ has a non-degenerate X-invariant alternating bilinear form, and from Lemma 4.6.2 that $T_X(p) \downarrow K[v] = V_p \oplus V_p$. For all $1 \leq i \leq r$, it follows from Lemma 4.4.6 that the X-module $L_X(2a_i) \oplus L_X(2a_i)^*$ has a non-degenerate X-invariant alternating bilinear form, and from Lemma 4.2.2 that $L_X(2a_i) \oplus L_X(2a_i)^* \downarrow K[v] = 2 \cdot V_{2a_i+1}$. For all $1 \leq j \leq s$, it follows from Corollary 4.4.4 that the X-module $L_X(2b_j - 1)$ has a non-degenerate X-invariant alternating bilinear form, and from Lemma 4.2.2 that $L_X(2b_j - 1) \downarrow K[v] = V_{2b_j}$. Hence the X-module

$$W = T_X(p) \oplus \bigoplus_{i=1}^{r} (L_X(2a_i) \oplus L_X(2a_i)^*) \oplus \bigoplus_{j=1}^{s} L_X(2b_j - 1)$$

has a non-degenerate X-invariant alternating bilinear form and $W \downarrow K[v] = V_{d_1} \oplus \cdots \oplus V_{d_t}$. Thus by identifying V with W, we get an embedding $X < \mathrm{Sp}(V)$ such that $V \downarrow K[v] = V_{d_1} \oplus \cdots \oplus V_{d_t}$. By Proposition 2.3.1, the elements u and v are conjugate in G, so by replacing X with a conjugate we may assume that $u = v$. Now $V \downarrow X$ is not semisimple since $T_X(p)$ is not semisimple (Theorem 4.1.5 (ii)), so by Theorem 6.1.3 the subgroup X is a non-G-cr subgroup of type A_1 containing u. □

Lemma 6.2.5 (Type B_l ($l \geq 3$) and type D_l ($l \geq 4$)). *Let $G = \mathrm{SO}(V)$ and assume that $p > 2$ and $\dim V \geq 7$. Let $u \in G$ be a unipotent element with $V \downarrow K[u] = V_{d_1} \oplus \cdots \oplus V_{d_t}$, where $d_1 \geq d_2 \geq \cdots \geq d_t > 0$. Then*

(i) $N(u_{\mathbb{C}}) = \begin{cases} 2d_1 - 1, & \text{if } d_1 = d_2. \\ 2d_1 - 2, & \text{if } d_2 = d_1 - 1. \\ 2d_1 - 3, & \text{if } d_2 \leq d_1 - 2. \end{cases}$

(ii) $N(u_{\mathbb{C}}) = 2p - 1$ *if and only if* $d_1 = d_2 = p$.

(iii) *u is contained in a non-G-cr subgroup $X < G$ of type A_1 if and only if $d_1 = d_2 = p$.*

Proof. Using the fact that the highest root in type B_l ($l \geq 3$) and type D_l ($l \geq 4$) is equal to ω_2, claim (i) follows with [Sup09, Proposition 1.5, Algorithm 1.6]. For claim (ii), it is immediate from (i) that $d_1 = d_2 = p$ implies $N(u_{\mathbb{C}}) = 2p - 1$. For the other direction of (ii), suppose that $N(u_{\mathbb{C}}) = 2p - 1$. If $d_2 = d_1 - 1$, then by (i) we have $N(u_{\mathbb{C}}) = 2d_1 - 2$, contradiction since $2p - 1$ is odd. If $d_2 \leq d_1 - 2$, then by (i) we have $N(u_{\mathbb{C}}) = 2d_1 - 3 = 2p - 1$, so $d_1 = p + 1$. But then d_1 is even, which is a contradiction since even Jordan block sizes must have even multiplicity by Proposition 2.3.2. Therefore we must have $d_1 = d_2$. Now it is immediate from (i) and $N(u_{\mathbb{C}}) = 2p - 1$ that $d_1 = d_2 = p$.

For the "only if" part of claim (iii), suppose that u is contained in a non-G-cr subgroup $X < G$ of type A_1. If $d_1 < p$ or $d_2 < p$, it follows that u acts on V with ≤ 1 Jordan block of size p. Since $X < \mathrm{SO}(V)$, the restriction $V \downarrow X$ is self-dual, and thus $V \downarrow X$ is semisimple by Proposition 4.6.10. Then by Theorem 6.1.3 the subgroup X is G-cr, contradiction.

For the "if" part of claim (iii), suppose that $d_1 = d_2 = p$. Let $X = \mathrm{SL}_2(K)$ and fix a non-identity unipotent element $v \in X$. By Proposition 2.3.2, the even Jordan block sizes of u have even multiplicity, so we can write

$$V \downarrow K[u] = V_p \oplus V_p \oplus \bigoplus_{i=1}^{r} 2 \cdot V_{2a_i} \oplus \bigoplus_{j=1}^{s} V_{2b_j+1}$$

where $a_i \geq 1$ for all i and $b_j \geq 0$ for all j, and $2 + 2r + s = t$. Similarly to the proof of Lemma 6.2.4, we proceed to construct these summands as suitable X-modules with non-degenerate symmetric bilinear forms.

It follows from Corollary 4.4.11 that the tilting X-module $T_X(p+1)$ has a non-degenerate X-invariant symmetric bilinear form, and from Lemma 4.6.2 that $T_X(p+1) \downarrow K[v] = V_p \oplus V_p$. For all $1 \leq i \leq r$, it follows from Lemma 4.4.6 that the X-module $L_X(2a_i - 1) \oplus L_X(2a_i - 1)^*$ has a non-degenerate X-invariant symmetric bilinear form, and from Lemma 4.2.2 that $L_X(2a_i - 1) \oplus L_X(2a_i - 1)^* \downarrow K[v] = 2 \cdot V_{2a_i}$. For all $1 \leq j \leq s$, it follows from Corollary 4.4.4 that the X-module $L_X(2b_j)$

has a non-degenerate X-invariant symmetric bilinear form, and from Lemma 4.2.2 that $L_X(2b_j) \downarrow K[v] = V_{2b_j+1}$. Hence the X-module

$$W = T_X(p+1) \oplus \bigoplus_{i=1}^{r} (L_X(2a_i) \oplus L_X(2a_i)^*) \oplus \bigoplus_{j=1}^{s} L_X(2b_j - 1)$$

has a non-degenerate X-invariant symmetric bilinear form, and $W \downarrow K[v] = V_{d_1} \oplus \cdots \oplus V_{d_t}$. By identifying V with W, and replacing X with $\mathrm{PGL}_2(K)$ if the representation $\rho : X \to \mathrm{GL}(W)$ has nontrivial kernel, we get an embedding $X < \mathrm{SO}(V)$ such that $V \downarrow K[v] = V_{d_1} \oplus \cdots \oplus V_{d_t}$. By Proposition 2.3.1, the unipotent elements u and v are conjugate in the full orthogonal group $\mathrm{O}(V)$, so by replacing X with a conjugate we can assume that $u = v$. Now $V \downarrow X$ is not semisimple since $T_X(p+1)$ is not semisimple (Theorem 4.1.5 (ii)), so by Theorem 6.1.3 the subgroup X is a non-G-cr subgroup of type A_1 containing u. □

Lemma 6.2.6. *Let G be a simple algebraic group of exceptional type and assume that p is good for G. Let $u \in G$ be a unipotent element of order p. Then u is contained in a non-G-cr subgroup $X < G$ of type A_1 precisely in the following cases:*

(i) *$G = E_6$, $p = 5$, and u is in unipotent class A_4 or A_4A_1.*

(ii) *$G = E_7$, $p = 5$, and u is in unipotent class A_4, A_4A_1, or A_4A_2.*

(iii) *$G = E_7$, $p = 7$, and u is in unipotent class A_6.*

(iv) *$G = E_8$, $p = 7$, and u is in unipotent class A_6 or A_6A_1.*

Proof. Here we will rely heavily on the work of Litterick and Thomas in [LT]. The main result of [LT] gives a complete list of non-G-cr subgroups $X < G$ of type A_1, up to G-conjugacy. Thus to prove our claim, it will be enough to check for each X which conjugacy class of unipotent elements of order p it intersects. Note that there is only one such conjugacy class, since in X all non-identity unipotent elements are conjugate.

For each non-G-cr subgroup X, Litterick and Thomas give the X-module structure of the restriction of the adjoint representation of G, see [LT, Table 11 - Table 16]. Using results on the action of a non-identity unipotent element $u \in X$ on X-modules (example given below), this allows us to compute the Jordan block sizes of a non-identity unipotent element $u \in X$ on the adjoint representation. Then by [Law95, Theorem 2], we can use the tables in [Law95] to identify the precise conjugacy class of u in G. Doing this straightforward[15] computation for each non-G-cr subgroup of type A_1 given in [LT], one finds that they can only contain unipotent elements listed in (i) - (iv), and that all of the unipotent elements in (i) - (iv) are contained in some non-G-cr subgroup of type A_1.

We give one example of how the computation is done, all the other computations use similar methods. Let $p = 5$ and $G = E_6$. Fix non-negative integers r and s such that $rs = 0$. We consider an A_1 subgroup X of G, which is embedded into a maximal rank subgroup of type A_1A_5 via the representations $L_X(1)^{[r]}$ (for A_1 factor) and $V_X(5)^{[s]}$ (for A_5 factor). According to [LT], the subgroup X is

[15]But perhaps tedious, since there are 26 entries in the tables of [LT] to check.

non-G-cr, and by [LT, Table 11] restriction of the adjoint representation of G to X decomposes into a direct sum

$$L_X(1)^{[r]}\otimes L_X(9)^{[s]}+L_X(1)^{[r]}\otimes T_X(5)^{[s]}+L_X(2)^{[r]}+T_X(10)^{[r]}+T_X(6)^{[s]}+L_X(4)^{[s]}.$$

Let $u \in X$ be a non-identity unipotent element of X. For any X-module V, the Jordan block sizes of u acting on V and any Frobenius twist of V are equal. Thus it will suffice to compute the Jordan block sizes of u on acting the X-module

$$L_X(1)\otimes L_X(9)+L_X(1)\otimes T_X(5)+L_X(2)+T_X(10)+T_X(6)+L_X(4).$$

We proceed to find the $K[u]$-module decomposition for each of the summands.

- $L_X(1)\otimes L_X(9)$: By Steinberg's tensor product theorem, we have $L_X(9) \cong L_X(1)^{[1]}\otimes L_X(4)$. Then by Lemma 4.2.2, we have $L_X(1)\otimes L_X(9)\downarrow K[u] \cong V_2\otimes V_2\otimes V_5$, which decomposes to $4\cdot V_5$ by Lemma 3.3.6. Thus $L_X(1)\otimes L_X(9)\downarrow K[u] = 4\cdot V_5$.
- $L_X(1)\otimes T_X(5)$: Now $L_X(1)\downarrow K[u] = V_2$ by Lemma 4.2.2, and $T_X(6)\downarrow K[u] = 2\cdot V_5$ by Lemma 4.6.2. Hence $L_X(1)\otimes T_X(5)\downarrow K[u] = 4\cdot V_5$ by Lemma 3.3.6.
- $L_X(2)$: Here $L_X(2)\downarrow K[u] = V_3$ by Lemma 4.2.2.
- $T_X(10)$: With Theorem 4.1.5, one computes that $T_X(10)$ has dimension 20, so by Lemma 4.6.2 we have $T_X(10)\downarrow K[u] = 4\cdot V_5$.
- $T_X(6)$: Here $T_X(6)\downarrow K[u] = 2\cdot V_5$ by Lemma 4.6.2.
- $L_X(4)$: Here $L_X(4)\downarrow K[u] = V_5$ by Lemma 4.2.2.

It follows then that $\mathscr{L}(G)\downarrow K[u] = [3, 5^{15}]$. By [Law95, Theorem 2, Table 6], the unipotent element u lies in the conjugacy class A_4A_1 of G. □

If p is good for G, then using lemmas 6.2.3 - 6.2.6 we can now describe unipotent elements that are contained in a non-G-cr subgroup of order p in terms of $N(u_{\mathbb{C}})$.

Proposition 6.2.7. *Let G be a simple algebraic group and assume that p is good for G. Let $u \in G$ be a unipotent element. Then $N(u_{\mathbb{C}}) = 2p-1$ if and only if one of the following holds:*

(i) *G is of type A_{p-1} and $u \in G$ is a regular unipotent element,*

(ii) *u is contained in a non-G-cr subgroup $X < G$ of type A_1.*

Proof. Suppose that $N(u_{\mathbb{C}}) = 2p-1$. If G is simple of classical type, the claim follows from (ii) and (iii) of lemmas 6.2.3 - 6.2.5. If G is simple of exceptional type, it follows from [Law95, pg. 4128] that the conjugacy class of u is given in one of (i) - (iv) of Lemma 6.2.6.

For the other direction, if (i) holds, then we have $N(u_{\mathbb{C}}) = 2p-1$ by Lemma 6.2.3 (i). Suppose then that (ii) holds. If G is simple of classical type, we have $N(u_{\mathbb{C}}) = 2p-1$ by (ii) and (iii) of lemmas 6.2.3 - 6.2.5. If G is simple of exceptional type, we have $N(u_{\mathbb{C}}) = 2p-1$ by Lemma 6.2.6 and [Law95, pg. 4128]. □

As an immediate corollary, we have the following.

Corollary 6.2.8. *Let G be a simple algebraic group and assume that p is very good for G. Let $u \in G$ be a unipotent element. Then u is contained in a non-G-cr subgroup $X < G$ of type A_1 if and only if $N(u_{\mathbb{C}}) = 2p - 1$.*

Our proof of Proposition 6.2.7 and Corollary 6.2.8 is heavily based on case-by-case checking. It would be interesting to see if there is a general way to prove the result.

We now move on to consider non-G-cr subgroups containing distinguished unipotent elements of order p. The following result shows that no such subgroups exist if p is good for G.

Theorem 6.2.9. *Let G be a simple algebraic group and assume that p is good for G. Let $u \in G$ be a distinguished unipotent element of order p. If $X < G$ is a connected reductive subgroup of G containing u, then X is G-ir.*

Proof. Let X be a connected reductive subgroup of G containing u. We consider first the case where X is simple of type A_1. By Lemma 6.1.2, it will be enough to show that X is G-cr. If G is simple of exceptional type, it is immediate from Lemma 6.2.6 that X is G-cr. If G is simple of classical type, one can easily see that X is G-cr by using (iii) in lemmas 6.2.3 - 6.2.5, and the description of distinguished unipotent classes of G in terms of Jordan block sizes (Proposition 2.3.4).

Consider then the general case where X is a connected reductive subgroup of G containing u. Since u is not centralized by a nontrivial torus, the same must be true for X, so it follows that X is semisimple. Write $X = X_1 \cdots X_t$, where the X_i are simple and commute pairwise. If $p \neq 3$ or if no X_i is of type G_2, it follows from [Tes95, Theorem 0.1] and [PST00, Theorem 5.1] that there exists a connected simple subgroup $X' < X$ of type A_1 such that $u \in X'$. It follows from the previous paragraph that X' is G-ir, so X must be G-ir as well.

Suppose then that $p = 3$ and that some X_i is of type G_2. Since p is good for G, it follows that G is simple of classical type. Let V be the natural module for G. Now u is distinguished of order 3, so it follows from Lemma 5.1.1 that $\dim V \leq 4$. Since every non-trivial representation of a simple algebraic group of type G_2 has dimension > 4 (see e.g. [Lüb01]), it follows that no X_i can be simple of type G_2, contradiction. □

What remains then is to consider distinguished unipotent elements of order p in bad characteristic. There are only two cases where such elements exist (type C_2 for $p = 2$, and type G_2 for $p = 3$, see proof of Theorem 6.2.12). The next two lemmas will deal with type C_2 for $p = 2$.

Lemma 6.2.10. *Assume that $p = 2$ and let $G = \mathrm{SL}_2(K)$. Let V be a G-module such that $V \downarrow K[u] = [2, 2]$ for a non-identity unipotent element $u \in G$. Then one of the following holds:*

(i) *V is irreducible and isomorphic to $L_G(1)^{[n]} \otimes L_G(1)^{[m]}$, where $0 \leq n < m$.*

(ii) *$V \cong L_G(1)^{[n]} \oplus L_G(1)^{[m]}$, where $0 \leq n \leq m$.*

(iii) *$V \cong T_G(2)^{[n]} \cong L_G(1)^{[n]} \otimes L_G(1)^{[n]}$, where $n \geq 0$.*

Proof. If V is irreducible, then the fact that V has dimension 4 implies that $V \cong L_G(1)^{[n]} \otimes L_G(1)^{[m]}$ for some $0 \leq n < m$, so V is as in (i). Suppose then that V is not irreducible. Then the possibilities for the composition factors of V are

(1) Two composition factors: $L_G(1)^{[n]}$ and $L_G(1)^{[m]}$ for some $0 \leq n \leq m$.

(2) Three composition factors: $L_G(0)$ (twice), and $L_G(1)^{[n]}$ for some $n \geq 0$.

In case (1), we have $V \cong L_G(1)^{[n]} \oplus L_G(1)^{[m]}$ since $\operatorname{Ext}^1_G(L_G(1)^{[n]}, L_G(1)^{[m]}) = 0 = \operatorname{Ext}^1_G(L_G(1)^{[m]}, L_G(1)^{[n]})$ by Theorem 4.6.3. Thus V is as in (ii).

Consider then case (2), where V has composition factors $L_G(0)$, $L_G(0)$, and $L_G(1)^{[n]}$ for some $n \geq 0$. If $n = 0$, then $V \cong L_G(0) \oplus L_G(0) \oplus L_G(1)$, since $\operatorname{Ext}^1_G(L_G(0), L_G(0)) = 0$ and $\operatorname{Ext}^1_G(L_G(1), L_G(0)) = 0 = \operatorname{Ext}^1_G(L_G(0), L_G(1))$ by Theorem 4.6.3. In this case $V \downarrow K[u] = [1, 1, 2]$, contradicting the assumption that $V \downarrow K[u] = [2, 2]$. Therefore V has composition factors $L_G(0)$, $L_G(0)$, and $L_G(1)^{[n+1]} = L_G(2)^{[n]}$ for some $n \geq 0$. We show next that V must be indecomposable. Suppose that $V = W \oplus W'$ where W, W' are proper non-zero G-submodules of V. Since $\operatorname{Ext}^1_G(L_G(0), L_G(0)) = 0$, it follows that either W or W' has $L_G(0)$ as a direct summand. But then u acts on V with at least one Jordan block of size 1, contradicting the assumption that $V \downarrow K[u] = [2, 2]$. Therefore V is indecomposable.

Since V is indecomposable and not irreducible, there must be a subquotient Q of V which is a non-split extension between two irreducible G-modules. Now Q is a proper subquotient of V, so u acts on Q with exactly one Jordan block of size 2, and thus the subquotient Q must be isomorphic to $V_G(2)^{[n]}$ or $(V_G(2)^*)^{[n]}$ (Proposition 4.6.8). By replacing V with V^* if necessary, we may assume that Q is isomorphic to $V_G(2)^{[n]}$. Since V is indecomposable, it follows that V is a nonsplit extension of $V_G(2)^{[n]}$ and $L_G(0)$. By Lemma 4.6.6, it follows that $V \cong T_G(2)^{[n]}$. Finally, as noted in the proof of Lemma 4.4.12, we have $L_G(1) \otimes L_G(1) = T_G(2)$. Therefore $L_G(1)^{[n]} \otimes L_G(1)^{[n]} \cong T_G(2)^{[n]}$. This completes the proof of the lemma. □

Lemma 6.2.11. *Assume that $p = 2$. Let $G = \operatorname{Sp}(V)$, where $\dim V = 4$, so G is simple of type C_2. Fix a distinguished unipotent element $u \in G$ of order p, so $V \downarrow K[u] = V(2)^2$ (Proposition 2.4.4). If $X < G$ is connected reductive and $u \in X$, then X is G-ir unless one of the following holds:*

(i) *X is simple of type A_1, embedded into G via $L_X(1) \perp L_X(1)$ (orthogonal direct sum).*

(ii) *X is simple of type A_1, embedded into G via $L_X(1) \otimes L_X(1) \cong T_X(2)$.*

Furthermore, subgroups $X < G$ in (i) and (ii) exist, contain a conjugate of u, are contained in a proper parabolic subgroup, and the conditions (i) and (ii) determine X up to conjugacy in G.

Proof. Suppose that $X < G$ is connected reductive and $u \in X$. We show that either X is G-ir, or one of (i) or (ii) holds.

If X is normalized by a maximal torus of G, then X is G-cr by [BMR05, Proposition 3.20]. Since u is distinguished, it follows from Lemma 6.1.2 that X is G-ir.

Suppose then that X is not normalized by any maximal torus of G. Since G has rank 2, it follows that X is simple of type A_1. Now there exists a rational representation $\rho : \operatorname{SL}_2(K) \to \operatorname{SL}(V)$ such that $\rho(\operatorname{SL}_2(K)) = X$. If V is an irreducible X-module, then by Theorem 6.1.5 the subgroup X is G-ir. Thus we may assume that V is non-irreducible. Then by Lemma 6.2.10, as an $\operatorname{SL}_2(K)$-module V must be isomorphic to either

(1) $L_{\mathrm{SL}_2(K)}(1)^{[n]} \oplus L_{\mathrm{SL}_2(K)}(1)^{[m]}$, where $0 \leq n \leq m$; or

(2) $T_{\mathrm{SL}_2(K)}(2)^{[n]} \cong L_{\mathrm{SL}_2(K)}(1)^{[n]} \otimes L_{\mathrm{SL}_2(K)}(1)^{[n]}$, where $n \geq 0$.

In case (1) we have $\rho = \rho' \circ F^n$, where F is the usual Frobenius endomorphism and $\rho' : \mathrm{SL}_2(K) \to \mathrm{SL}(V)$ is a rational representation such that the corresponding $\mathrm{SL}_2(K)$-module V is isomorphic to $L_{\mathrm{SL}_2(K)}(1) \oplus L_{\mathrm{SL}_2(K)}(1)^{[m-n]}$. Similarly in case (2), we have $\rho = \rho' \circ F^n$, where F is the usual Frobenius endomorphism and $\rho' : \mathrm{SL}_2(K) \to \mathrm{SL}(V)$ is a rational representation such that the corresponding $\mathrm{SL}_2(K)$-module V is isomorphic to $T_{\mathrm{SL}_2(K)}(2)$. Since applying a Frobenius twist does not change the image of the representation ρ', it follows that we may assume that as an $\mathrm{SL}_2(K)$-module, V is isomorphic to either

$(1)'$ $L_{\mathrm{SL}_2(K)}(1) \oplus L_{\mathrm{SL}_2(K)}(1)^{[n]}$, where $n \geq 0$; or

$(2)'$ $T_{\mathrm{SL}_2(K)}(2)$.

In case $(2)'$ we have X as in case (ii) of the claim. Consider then case $(1)'$. Here if $n > 0$, it follows from Theorem 6.1.5 (ii) that X is G-ir. Suppose then that $n = 0$, so $V \cong L_{\mathrm{SL}_2(K)}(1) \oplus L_{\mathrm{SL}_2(K)}(1)$. Let W be a X-submodule of V such that $W \cong L_{\mathrm{SL}_2(K)}(1)$. If W is totally singular, then X is a Levi factor of type A_1 and a non-identity unipotent element $u \in X$ satisfies $V \downarrow K[u] = W(2)$ [LS12, 6.1]. This contradicts the assumption $V \downarrow K[u] = V(2)^2$. Therefore W is not totally singular, and thus it must be non-degenerate, since W is irreducible as an X-module. Thus $V = W \oplus W'$ as an orthogonal direct sum of X-modules, where $W \cong L_{\mathrm{SL}_2(K)}(1) \cong W'$. Thus X is as in case (i) of the claim.

We consider the existence and uniqueness claims for the subgroups X in (i) and (ii). We begin by considering subgroups X in (i). Now the $\mathrm{SL}_2(K)$-module $L_{\mathrm{SL}_2(K)}(1)$ has a non-degenerate $\mathrm{SL}_2(K)$-invariant alternating bilinear form by Lemma 4.4.5. Therefore it is clear that we can find a representation $\rho : \mathrm{SL}_2(K) \to G$ with $V = V_1 \perp V_2$ (orthogonal direct sum) such that $V_1 \cong L_{\mathrm{SL}_2(K)}(1) \cong V_2$, so we can choose $X = \rho(\mathrm{SL}_2(K))$.

We show next that such an X is the unique such subgroup up to G-conjugacy. For this, let $Y = \rho'(\mathrm{SL}_2(K))$ for some representation $\rho' : \mathrm{SL}_2(K) \to G$ such that $V = W_1 \perp W_2$, where W_1 and W_2 are Y-modules such that $W_1 \cong L_{\mathrm{SL}_2(K)}(1) \cong W_2$. It is well known that the orthogonal direct sum decompositions $V_1 \perp V_2$ and $W_1 \perp W_2$ are conjugate under the action of G, in other words, there exists $f \in G$ such that $f(W_1) = V_1$ and $f(W_2) = V_2$. Then fYf^{-1} stabilizes V_1 and V_2, so by replacing Y with fYf^{-1}, we may assume that $W_1 = V_1$ and $W_2 = V_2$. Now for $i = 1, 2$, the restrictions of ρ and ρ' to V_i are both isomorphic to $L_{\mathrm{SL}_2(K)}(1)$ as $\mathrm{SL}_2(K)$-modules, so there exists $x_i \in \mathrm{SL}(V_i)$ such that $x_i \rho'(g) x_i^{-1} = \rho(g)$ for all $g \in \mathrm{SL}_2(K)$. Since $\dim V_i = 2$, we have $\mathrm{SL}(V_i) = \mathrm{Sp}(V_i)$, so for $x = x_1 \oplus x_2$ we have $x \in G$ and $xYx^{-1} = X$. Thus X is unique up to G-conjugacy.

Next, note that since a non-identity unipotent element $u \in X$ acts on the module $L_{\mathrm{SL}_2(K)}(1)$ with a single Jordan block of size 2, it is clear that $V \downarrow K[u] = V(2)^2$. Then for subgroups X in (i), what still remains is to show that X is contained in a proper parabolic subgroup of G. To this end, let W, W' be X-submodules of V such that $V = W \perp W'$ with $W \cong L(1) \cong W'$. Let $(-,-)$ be the non-degenerate alternating bilinear form used to define G. There exists an X-equivariant isometry $f : W \to W'$, in other words, an isomorphism of X-modules such that $(f(w_1), f(w_2)) = (w_1, w_2)$ for all $w_i \in W$. Then it is straightforward to check that $Z = \{w + f(w) : w \in W\}$ is a totally singular X-submodule of V with $Z \cong L_{\mathrm{SL}_2(K)}(1)$. By Theorem 6.1.4, the subgroup X lies in a proper parabolic subgroup of G. This completes the proof of the claims for (i).

For the subgroup X in (ii), note that by Lemma 4.4.12 the tilting $\mathrm{SL}_2(K)$-module $T_{\mathrm{SL}_2(K)}(2)$ has a non-degenerate $\mathrm{SL}_2(K)$-invariant alternating bilinear form. Therefore there exists a representation $\rho : \mathrm{SL}_2(K) \to G$, with $V \downarrow X \cong T_{\mathrm{SL}_2(K)}(2)$ for $X = \rho(\mathrm{SL}_2(K))$. Now by Lemma 4.4.13, a non-degenerate $\mathrm{SL}_2(K)$-invariant alternating bilinear form on $T_{\mathrm{SL}_2(K)}(2)$ is unique up to scalar multiples; thus it follows that a subgroup $X < G$ of type A_1 with $V \downarrow X \cong T_{\mathrm{SL}_2(K)}(2)$ is unique up to G-conjugacy (Lemma 4.4.14). Next note that $L_{\mathrm{SL}_2(K)}(1)$ has a non-degenerate $\mathrm{SL}_2(K)$-invariant alternating bilinear form by Lemma 4.4.5, so by uniqueness of the non-degenerate $\mathrm{SL}_2(K)$-invariant alternating bilinear form on $T_{\mathrm{SL}_2(K)}(2)$, we may choose the form on $T_{\mathrm{SL}_2(K)}(2)$ to be the product form on $L_{\mathrm{SL}_2(K)}(1) \otimes L_{\mathrm{SL}_2(K)}(1) = T_{\mathrm{SL}_2(K)}(2)$. Now it follows from Table 3.6 that for a non-identity unipotent element $u \in X$, we have $T_{\mathrm{SL}_2(K)}(2) \downarrow K[u] \cong V(2) \otimes V(2) \cong V(2)^2$. Finally to show that X is contained in a parabolic subgroup, note that $V \cong T_{\mathrm{SL}_2(K)}(2)$ has a 1-dimensional X-submodule W (Theorem 4.1.5 (ii)). This submodule is totally isotropic since any 1-dimensional subspace of V is, so X is contained in a proper parabolic subgroup of G (Theorem 6.1.4). This completes the proof of the claims for (ii). □

We are now ready to prove the main result of this section.

Theorem 6.2.12. *Let G be a simple algebraic group and let $u \in G$ be a distinguished unipotent element of order p. Let $X < G$ be a connected reductive subgroup of G containing u. If X is contained in a proper parabolic subgroup of G, then $p = 2$, G is simple of type C_2, and X is as in Lemma 6.2.11.*

Proof. Suppose that X is contained in a proper parabolic subgroup of G. Since u is a distinguished unipotent element, X must be non-G-cr (Lemma 6.1.2). Then by Theorem 6.2.9 we have that p is bad for G.

Consider first the case where G is simple of exceptional type. Looking at the Tables in Appendix A, the fact that p is bad for G and the fact that u is a distinguished unipotent element of order p implies that $G = G_2$, $p = 3$, and u is in unipotent class $G_2(a_1)$ or $(\tilde{A}_1)_3$. However, in this case it follows from [Ste10] that every connected reductive subgroup G is G-cr, contradicting the fact that X is non-G-cr.

Suppose then that G is simple of classical type. Since p is bad for G, we have $p = 2$ and G is simple of type B_l $(l \geq 3)$, C_l $(l \geq 2)$, or D_l $(l \geq 4)$. Let $V = L(\omega_1)$ be the natural irreducible representation of G. Now u is a distinguished unipotent element of order p, so u acts on V with all Jordan block sizes even of size ≤ 2 (Proposition 2.4.4). Therefore $\dim V \leq 4$, so $l \leq 2$. Therefore G is simple of type C_2, and in this case the claim follows from Lemma 6.2.11. □

6.3 Unipotent elements of order $> p$

In this subsection, we consider distinguished unipotent elements of order $> p$ that are contained in a non-G-cr subgroup. Unlike in the previous subsection where we considered unipotent elements of order p, we only have partial results.

Exceptional types

We begin by considering the case of simple groups of exceptional type. Relying on the results of Litterick and Thomas in [LT], we get the following.

Theorem 6.3.1. *Let G be a simple algebraic group of exceptional type and assume that p is good for G. Let $u \in G$ be a distinguished unipotent element. If $X < G$ is a connected reductive subgroup of G containing u, then X is G-ir.*

Proof. Let X be a connected reductive subgroup of G containing u. Suppose that X is not G-ir. Since u is a distinguished unipotent element, it follows that X is non-G-cr (Lemma 6.1.2). Now according to [LT, Theorem 1-4], one of the following must hold:

- $p = 5$, and X is semisimple of type A_1, A_1A_1, or A_1A_2.
- $p = 7$, and X is semisimple of type A_1, G_2, A_1A_1, or A_1G_2.

We can see that in all of the cases above, unipotent elements of X have order p. Indeed, this is true for simple algebraic groups of type A_1 for all p, for type A_2 for all $p \geq 3$, and for type G_2 for all $p \geq 7$ (Appendix A). It follows then from Theorem 6.2.9 that X is G-ir. □

For type G_2, we can give the following result as an easy corollary of [Ste10].

Theorem 6.3.2. *Let G be a simple algebraic group of type G_2. Let $u \in G$ be a distinguished unipotent element. If $X < G$ is a connected reductive subgroup of G containing u, then X is G-ir.*

Proof. Let $X < G$ be a connected reductive subgroup of G containing u. If X is not G-ir, then the fact that u is distinguished implies that X is non-G-cr. It follows from [Ste10] that $p = 2$ and X is simple of type A_1. This is a contradiction, since now every distinguished unipotent element of G has order > 2 (Appendix A), and every non-identity unipotent element of X has order 2. □

Let G be a simple algebraic group of exceptional type. By the two theorems above, for classifying connected reductive non-G-cr overgroups of distinguished unipotent elements of G, what still remains is the case where G is simple of type F_4, E_6, E_7, or E_8 in bad characteristic. In this case the connected reductive non-G-cr subgroups of G are not known in general, so more work is still needed.

Classical types

We finish by discussing the situation for simple algebraic groups G of classical type. Here our work is still in progress, so we will mostly just give some examples of simple non-G-cr subgroups containing a distinguished unipotent element.

For this problem, the basic reductions such as Lemma 5.2.1 and Lemma 5.1.1 used in the irreducible case still work. With them, one can try to reduce to a small number of possible composition factors that can occur, and then study extensions between them to find a solution. In many cases we have a solution, and at least in the case where X is simple, a complete solution should be doable in future work. When we have a connected semisimple algebraic group $X = X_1 \cdots X_t$, one should try to limit the number and the types of simple factors X_i that can occur. This is again something which will be studied in future work.

Example 6.3.3. Assume that $p = 3$. We will show the following:

(i) For $G = \mathrm{SO}(V)$ with $\dim V = 27$, up to G-conjugacy there exists a unique $X < G$ simple of type F_4 such that $V \downarrow X \cong T_X(\omega_4)$;

(ii) Such an X is non-G-cr;

(iii) A regular unipotent element $u \in X$ lies in the distinguished unipotent class $[3, 9, 15]$ of G.

Let X be a simple algebraic group of type F_4. We begin by showing that $T_X(\omega_4) = L_X(0)/L_X(\omega_4)/L_X(0)$ and that $T_X(\omega_4)$ is uniserial.

We use the usual embedding of X into a simply connected group Y of type E_6. That is, we consider X as the centralizer of the involutory graph automorphism of Y induced by the nontrivial automorphism of the Dynkin diagram of Y.

Since in type E_6 the fundamental highest weight ω_1 is minuscule, we have $L_Y(\omega_1) = T_Y(\omega_1)$. According to [vdK01, Theorem 20], the restriction of every tilting module of Y to X is a tilting module for X. In particular, the restriction $L_Y(\omega_1) \downarrow X$ is tilting for X. For the character of this restriction, we have

$$\mathrm{ch}\, L_Y(\omega_1) \downarrow X = \mathrm{ch}\, V_X(\omega_4) + \mathrm{ch}\, V_X(0)$$

by [LS96, Table 8.7]. Thus $L_Y(\omega_1) \downarrow X$ has $T_X(\omega_4)$ as a direct summand (Theorem 4.1.4 (ii)). Furthermore, since $p = 3$, we have $V_X(\omega_4) = L_X(\omega_4)/L_X(0)$ (see e.g. [Lüb17]). Now since $T_X(\omega_4)$ must have $V_X(\omega_4)$ as a submodule and $V_X(\omega_4)^*$ as a homomorphic image [Jan03, II.E.4], we conclude that

$$L_Y(\omega_1) \downarrow X = T_X(\omega_4) = L_X(0)/L_X(\omega_4)/L_X(0)$$

and that $T_X(\omega_4)$ is uniserial.

By Lemma 4.4.10, we have a non-degenerate X-invariant symmetric bilinear form on $T_X(\omega_4)$. Now (i) follows from Lemma 4.4.15. For claim (ii), note that $T_X(\omega_4)$ is an indecomposable and non-irreducible X-module, so it follows that X is non-G-cr in $G = \mathrm{SO}(T_X(\omega_1))$. For (iii), let $u \in X$ be a regular unipotent element. From the way X is constructed as the stabilizer of a graph automorphism of Y, it is clear that the subgroup X contains a regular unipotent element of Y. Hence it follows that u is also a regular unipotent element of Y, see e.g. [TZ13, Lemma 2.1]. According to [Law95, Table 5], we have $L_Y(\omega_1) \downarrow K[u] = [3, 9, 15]$, hence $T_X(\omega_4) \downarrow K[u] = [3, 9, 15]$.

We will omit the proof, but for simple algebraic groups of exceptional type, it turns out that Example 6.3.3 is the only example in odd characteristic[16].

Theorem 6.3.4. *Assume that $p > 2$. Let X be a simple algebraic group of exceptional type. Suppose that $X < G$, where G is a simple algebraic group of classical type with natural module V. If X contains a distinguished unipotent element of G, then X is G-ir, unless $p = 3$, $X = F_4$, and $V \downarrow X = T_X(\omega_4)$.*

We will now give some further examples in characteristic $p = 2$.

[16]For X simple of classical type, so far we have not found any examples in odd characteristic, and based on partial results and examples we conjecture that there are none. In other words, we conjecture that Theorem 6.3.4 holds for any simple algebraic group X, not just X of exceptional type.

Example 6.3.5. (cf. Example 6.3.3) Assume that $p = 2$. We will show that the following hold:

(i) For $G = \mathrm{Sp}(V)$ with $\dim V = 8$, up to G-conjugacy we have a unique $X < G$ simple of type G_2 such that $V \downarrow X \cong T_X(\omega_1)$;

(ii) Such an X is non-G-cr.

(iii) A regular unipotent element $u \in X$ lies in the distinguished unipotent class $V(2) + V(6)$ of G.

Let X be a simple algebraic group of type G_2. It is well known that we have an embedding of X into a simply connected group Y of type D_4, by considering X as the centralizer of a suitable triality automorphism. Let V be the natural module for Y. We have that $V \downarrow X$ is tilting by [Bru98, Proposition 3.3 (vi)], so as in Example 6.3.3, one finds that $V \downarrow X = T_X(\omega_1)$ and that $T_X(\omega_1) = L_X(0)/L_X(\omega_1)/L_X(0)$ is uniserial.

Since V has a non-degenerate Y-invariant alternating bilinear form, we get an embedding $X < G$ for $G = \mathrm{Sp}(V)$. By Lemma 4.4.13 and Lemma 4.4.14, a subgroup $X < G$ of type G_2 with $V \downarrow X \cong T_X(\omega_1)$ is uniquely determined up to G-conjugacy, proving (i).

Claim (ii) follows from the fact that V is an indecomposable, non-irreducible X-module. For (iii), let $u \in X$ be a regular unipotent element. Now u is also a regular unipotent element of Y, so we have the orthogonal decomposition $V \downarrow K[u] = V(2) + V(6)$ (Proposition 2.4.4 (vi)), and so u is a distinguished unipotent element of G (Proposition 2.4.4 (ii)).

Example 6.3.6. We give two other examples similar to Example 6.3.3 and Example 6.3.5, without giving all the details. Let $p = 2$. Then one can show that the following hold[17]:

(i) For G simply connected of type C_l ($l \geq 2$) with natural module V, up to G-conjugacy we have a unique $X < G$ simple of type B_{l-1} such that $V \downarrow X \cong T_X(\omega_1)$.

(ii) For G simply connected of type C_{67} with natural module V, up to G-conjugacy we have a unique $X < G$ simple of type E_7 such that $V \downarrow X \cong T_X(\omega_1)$.

Furthermore, in both (i) and (ii) one finds that X is non-G-cr and a regular unipotent element of X is distinguished in G. In case (i) a regular unipotent element of X is in the unipotent conjugacy class $V(2) + V(2l - 2)$ of G, and in case (ii) in the unipotent conjugacy class $V(2) + V(8) + V(10) + V(16) + V(18) + V(22) + V(26) + V(32)$ of G.

We finish with the following more involved example for $X = G_2$.

[17] We omit the full details, but to see this, it is enough to show that in both (i) and (ii), we have that $T_X(\omega_1) = L_X(0)/L_X(\omega_1)/L_X(0)$ is uniserial and has a non-degenerate X-invariant alternating bilinear form. Then (i) and (ii) follow from Lemma 4.4.13 and Lemma 4.4.14.

Example 6.3.7. Assume that $p = 2$. For $G = \mathrm{Sp}_{14}(K)$, we will describe a family of non-G-cr simple subgroups of type G_2, parametrized by $\mathbb{Z}_{\geq 0}$ and $K \setminus \{0,1\}$, such that the subgroups are pairwise non-conjugate and each of them contain a distinguished unipotent element of G.

Let X be a simple algebraic group of type G_2 and let $u \in X$ be a regular unipotent element. Let $n \geq 1$ be an integer.

It is well known (and straightforward to compute) that the Weyl module $V(\omega_1)$ for X is a nonsplit extension

$$0 \to K \to V(\omega_1) \to L(\omega_1) \to 0. \tag{*}$$

Applying the functor $\mathrm{Hom}_X(L(\omega_1)^{[n]}, -)$ to (*) gives an exact sequence

$$0 \to \mathrm{Ext}^1_X(L(\omega_1)^{[n]}, K) \to \mathrm{Ext}^1_X(L(\omega_1)^{[n]}, V(\omega_1)) \to \mathrm{Ext}^1_X(L(\omega_1)^{[n]}, L(\omega_1)).$$

It follows from the main result of [Sin92] that $H^1(X, L(\omega_1) \otimes L(\omega_1)^{[n]}) = 0$, so $\mathrm{Ext}^1_X(L(\omega_1)^{[n]}, L(\omega_1)) = 0$. Similarly $\mathrm{Ext}^1_X(L(\omega_1)^{[n]}, K) \cong H^1(X, L(\omega_1)^{[n]}) \cong K$ by [Sin92]. Plugging this into the exact sequence above, we get the isomorphism $\mathrm{Ext}^1_X(L(\omega_1)^{[n]}, V(\omega_1)) \cong K$. Thus up to isomorphism, there is a unique X-module W which is a nonsplit extension

$$0 \to V(\omega_1) \to W \to L(\omega_1)^{[n]} \to 0.$$

This short exact sequence gives the long exact sequence in cohomology

$$\begin{aligned} 0 &\to H^1(X, V(\omega_1)) \to H^1(X, W) \to H^1(X, L(\omega_1)^{[n]}) \\ &\to H^2(X, V(\omega_1)) \to \cdots \end{aligned} \tag{**}$$

We will show next using (**) that $H^1(X, W) \cong K^2$. For this, first note that on the short exact sequence (*) the long exact sequence in cohomology gives $H^i(X, L(\omega_1)) \cong H^i(X, V(\omega_1))$ for all $i \geq 1$, since $H^i(X, K) = 0$ for all $i \geq 1$ by [Jan03, II.4.13]. Therefore we have $H^1(X, V(\omega_1)) \cong K$, since $H^1(X, L(\omega_1)) \cong K$ by the main result of [Sin92]. Furthermore, we have $H^1(X, L(\omega_1)^{[n]}) \cong K$ as noted in the previous paragraph. Plugging this into (**), we see that to show $H^1(X, W) \cong K^2$, it will be enough to show that $H^2(X, V(\omega_1)) = 0$. To this end, for the short exact sequence

$$0 \to L(\omega_1) \to V(\omega_1)^* \to K \to 0,$$

the long exact sequence in cohomology gives $H^i(X, L(\omega_1)) = 0$ for all $i \geq 2$, since $H^i(X, K) = 0$ and $H^i(X, V(\omega_1)^*) = 0$ for all $i \geq 1$ by [Jan03, II.4.13]. Thus in fact $H^i(X, V(\omega_1)) = 0$ for all $i \geq 2$.

For $c \in H^1(X, W) \cong \mathrm{Ext}^1_X(K, W)$, denote by V_c a representative

$$0 \to W \to V_c \to K \to 0$$

of the equivalence class of extensions corresponding to c. We omit the proof, but with explicit computations one can show that as a K-vector space, the first cohomology group $H^1(X, W)$ has a basis $\{a, b\}$ with the following properties:

(i) Each nonsplit extension $0 \to W \to V \to K \to 0$ is isomorphic to exactly one of the modules in the set $\{V_a\} \cup \{V_{a+\lambda \cdot b} : \lambda \in K\}$.

(ii) V_a and V_b are not self-dual.

(iii) For all $\lambda \in K \setminus \{0\}$, the module $V_{a+\lambda\cdot b}$ is self-dual, and has a non-degenerate X-invariant alternating bilinear form which is unique up to a scalar.

(iv) $V_a \downarrow K[u] = [2, 6^2]$ and $V_b \downarrow K[u] = [2, 6^2]$.

(v) For all $\lambda \in K \setminus \{0\}$, with respect to any non-degenerate X-invariant alternating bilinear form on $V_{a+\lambda\cdot b}$, we have the orthogonal decomposition (Proposition 2.4.4)

$$V_{a+\lambda\cdot b} \downarrow K[u] = \begin{cases} V(2) + V(6)^2, & \text{if } \lambda \in K \setminus \{0,1\}. \\ W(1) + V(6)^2, & \text{if } \lambda = 1. \end{cases}$$

Now let $G = \mathrm{Sp}(V)$ be a simple algebraic group of type C_7, so $\dim V = 14$. By (iii) above, for all $\lambda \in K \setminus \{0\}$ there exists a subgroup $X_{n,\lambda} < G$ such that $X_{n,\lambda}$ is simple of type G_2 and $V \downarrow X_{n,\lambda} \cong V_{a+\lambda\cdot b}$. Note that by (iii) and Lemma 4.4.14, such an $X_{n,\lambda}$ is uniquely determined up to G-conjugacy. Since $V_{a+\lambda\cdot b}$ is indecomposable and not irreducible as an $X_{n,\lambda}$-module, it follows that $X_{n,\lambda}$ is non-G-cr. If $\lambda \in K \setminus \{0,1\}$, it follows from (v) that the regular unipotent element of $X_{n,\lambda}$ is contained in the distinguished unipotent class $V(2) + V(6)^2$ of G.

In conclusion, what we have is that

$$\{X_{n,\lambda} : n \in \mathbb{Z}_{\geq 1}, \lambda \in K \setminus \{0,1\}\}$$

is an infinite family of pairwise non-conjugate, non-G-cr subgroups of G, each of which is simple of type G_2, and each of which intersects the distinguished unipotent conjugacy class $V(2) + V(6)^2$ of G.

Appendix A

Orders of distinguished unipotent elements

Here we list the orders of distinguished unipotent elements in exceptional groups in characteristic $p > 0$. All of them have order p, except the ones listed in the tables below. In the following, an empty entry means that the element has order p. The orders can be found by using the results of Lawther in [Law95] and [Law98].

Unipotent class	2	3	5
G_2	2^3	3^2	5^2
$G_2(a_1)$	2^2		
$(\tilde{A}_1)_3$	-	3	-

Table A.1: Type G_2

Unipotent class	2	3	5	7	11
F_4	2^4	3^3	5^2	7^2	11^2
$F_4(a_1)$	2^3	3^2	5^2	7^2	
$F_4(a_2)$	2^3	3^2	5^2		
$F_4(a_3)$	2^2	3^2			
$(C_3(a_1))_2$	2^2	-	-	-	-
$(\tilde{A}_2A_1)_2$	2^2	-	-	-	-

Table A.2: Type F_4

Unipotent class	2	3	5	7	11
E_6	2^4	3^3	5^2	7^2	11^2
$E_6(a_1)$	2^4	3^2	5^2	7^2	
$E_6(a_3)$	2^3	3^2	5^2		

Table A.3: Type E_6

Unipotent class	2	3	5	7	11	13	17
E_7	2^5	3^3	5^2	7^2	11^2	13^2	17^2
$E_7(a_1)$	2^4	3^3	5^2	7^2	11^2	13^2	
$E_7(a_2)$	2^4	3^3	5^2	7^2	11^2		
$E_7(a_3)$	2^4	3^3	5^2	7^2			
$E_7(a_4)$	2^3	3^2	5^2	7^2			
$E_7(a_5)$	2^3	3^2	5^2				

Table A.4: Type E_7

Unipotent class	2	3	5	7	11	13	17	19	23	29
E_8	2^5	3^4	5^3	7^2	11^2	13^2	17^2	19^2	23^2	29^2
$E_8(a_1)$	2^5	3^3	5^2	7^2	11^2	13^2	17^2	19^2	23^2	
$E_8(a_2)$	2^5	3^3	5^2	7^2	11^2	13^2	17^2	19^2		
$E_8(a_3)$	2^5	3^3	5^2	7^2	11^2	13^2	17^2			
$E_8(a_4)$	2^4	3^3	5^2	7^2	11^2	13^2				
$E_8(b_4)$	2^4	3^3	5^2	7^2	11^2	13^2				
$E_8(a_5)$	2^4	3^3	5^2	7^2	11^2					
$E_8(b_5)$	2^4	3^3	5^2	7^2	11^2					
$E_8(a_6)$	2^4	3^3	5^2	7^2						
$E_8(b_6)$	2^4	3^2	5^2	7^2						
$E_8(a_7)$	2^3	3^2	5^2							
$(A_7)_3$	-	3^2	-	-	-	-	-	-	-	-
$(D_7(a_1))_2$	2^4	-	-	-	-	-	-	-	-	-
$(D_5A_2)_2$	2^3	-	-	-	-	-	-	-	-	-

Table A.5: Type E_8

Appendix B

Actions in some irreducible representations ($p = 2$)

Assume that $p = 2$*.*

Let G be a simple algebraic group of exceptional type and let $\varphi : G \to \mathrm{GL}(V)$ be a self-dual irreducible representation of G. Then V has a non-degenerate, G-invariant symplectic form (Lemma 4.4.5), and thus $\varphi(G)$ lies in $\mathrm{Sp}(V)$. In the tables below, we give for each unipotent $u \in G$ the conjugacy class of $\varphi(u)$ in $\mathrm{Sp}(V)$ for some small self-dual irreducible representations of G. Specifically, we give the conjugacy classes in the cases where

- $G = G_2$ and $V = L(\omega_1)$, $V = L(\omega_2)$.
- $G = F_4$ and $V = L(\omega_1)$, $V = L(\omega_4)$.
- $G = E_6$ and $V = L(\omega_2)$.
- $G = E_7$ and $V = L(\omega_1)$, $V = L(\omega_7)$.
- $G = E_8$ and $V = L(\omega_8)$.

For these representations, the Jordan blocks of $\varphi(u)$ were already given in arbitrary characteristic by Lawther in [Law95] [Law98].

Note that when $G = F_4$, there exists an exceptional isogeny $\tau : G \to G$ [Ste68, Theorem 28]. Then any irreducible rational G-module induces another irreducible G-module V^τ by twisting with τ, and in particular $L_G(\omega_1)^\tau \cong L_G(\omega_4)$. Thus the entries in tables B.3 and B.4 are almost the same, except some of the classes are swapped by τ.

In the tables below, we label the conjugacy classes in $\mathrm{Sp}(V)$ as in corollary 2.4.7 (i). The computations were done with MAGMA (Section 2.9). In the tables we have bolded the cases where the action is distinguished. Note that there are a few cases where all Jordan block sizes are even and occur with multiplicity ≤ 2, but where the action is not distinguished (classes $E_7(a_1)$, $E_7(a_2)$ and $E_7(a_3)$ for $G = E_7$, $V = L(\omega_7)$)

Class of u	action on $L_{G_2}(\omega_1)$
$\mathbf{G_2}$	$\mathbf{6_1}$
$G_2(a_1)$	3_0^2
$\widetilde{A}_1$	2_1^3
A_1	$1_0^2, 2_0^2$

Table B.1: $G = G_2$ and $V = L(\omega_1)$

Class of u	action on $L_{G_2}(\omega_2)$
$\mathbf{G_2}$	$\mathbf{6_1, 8_1}$
$G_2(a_1)$	$3_1^2, 4_1^2$
$\widetilde{A}_1$	$1_0^2, 2_1^6$
A_1	$1_0^2, 2_1^6$

Table B.2: $G = G_2$ and $V = L(\omega_2)$

Class of u	action on $L_{F_4}(\omega_4)$	Class of u	action on $L_{F_4}(\omega_4)$
$\mathbf{F_4}$	$\mathbf{10_1, 16_1}$	$A_2\tilde{A}_1$	$2_1^6, 3_0^2, 4_0^2$
$F_4(a_1)$	$2_1, 8_1^3$	$(B_2)_2$	$1_0^4, 2_1, 4_1^5$
$F_4(a_2)$	$5_0^2, 8_1^2$	B_2	$1_0^4, 2_1, 4_1^5$
$F_4(a_3)$	$1_0^2, 4_1^6$	$\tilde{A}_2$	$3_0^6, 4_1^2$
C_3	$6_1^3, 8_1$	A_2	$1_0^8, 3_0^6$
B_3	$1_0^2, 2_1^3, 6_1^3$	$A_1\tilde{A}_1$	$1_0^2, 2_1^{12}$
$(C_3(a_1))_2$	$2_1^3, 4_1^5$	$(\tilde{A}_1)_2$	$1_0^6, 2_1^{10}$
$C_3(a_1)$	$2_1^3, 4_1^5$	$\tilde{A}_1$	$1_0^6, 2_1^{10}$
$(\tilde{A}_2A_1)_2$	$2_0^2, 3_0^2, 4_1^4$	A_1	$1_0^{14}, 2_0^6$
$\tilde{A}_2A_1$	$2_0^2, 3_0^2, 4_1^4$		

Table B.3: $G = F_4$ and $V = L(\omega_4)$

Class of u	action on $L_{F_4}(\omega_1)$	Class of u	action on $L_{F_4}(\omega_1)$
$\mathbf{F_4}$	$\mathbf{10_1, 16_1}$	$A_2\tilde{A}_1$	$2_0^2, 3_0^2, 4_1^4$
$F_4(a_1)$	$2_1, 8_1^3$	$(B_2)_2$	$2_1^3, 4_1^5$
$F_4(a_2)$	$5_0^2, 8_1^2$	B_2	$1_0^4, 2_1, 4_1^5$
$F_4(a_3)$	$1_0^2, 4_1^6$	$\tilde{A}_2$	$1_0^8, 3_0^6$
C_3	$1_0^2, 2_1^3, 6_1^3$	A_2	$3_0^6, 4_1^2$
B_3	$6_1^3, 8_1$	$A_1\tilde{A}_1$	$1_0^2, 2_1^{12}$
$(C_3(a_1))_2$	$2_1^3, 4_1^5$	$(\tilde{A}_1)_2$	$1_0^6, 2_1^{10}$
$C_3(a_1)$	$1_0^4, 2_1, 4_1^5$	$\tilde{A}_1$	$1_0^{14}, 2_0^6$
$(\tilde{A}_2A_1)_2$	$2_0^2, 3_0^2, 4_1^4$	A_1	$1_0^6, 2_1^{10}$
$\tilde{A}_2A_1$	$2_1^6, 3_0^2, 4_0^2$		

Table B.4: $G = F_4$ and $V = L(\omega_1)$

Class of u	action on $L_{E_6}(\omega_2)$	Class of u	action on $L_{E_6}(\omega_2)$
E_6	$6_1, 8_1, 16_1^4$	A_4	$1_0^4, 3_0^4, 4_1^2, 5_0^2, 7_0^4, 8_1^2$
$E_6(a_1)$	$4_1^2, 8_1^2, 11_0^2, 16_1^2$	A_3A_1	$1_0^2, 2_1^6, 4_1^{16}$
$E_6(a_3)$	$3_0^2, 4_1^2, 8_1^8$	$A_2A_1^2$	$2_1^{16}, 3_0^2, 4_1^{10}$
D_5	$6_1, 8_1^9$	A_2^2	$1_0^{14}, 4_1^{16}$
$D_5(a_1)$	$4_1^8, 6_1, 8_1^5$	A_3	$1_0^6, 2_0^4, 4_1^{16}$
A_5	$1_0^2, 2_1^6, 8_1^8$	A_2A_1	$1_0^8, 2_1^{10}, 3_0^6, 4_1^8$
A_4A_1	$2_1^4, 4_1^4, 5_0^2, 6_0^2, 8_1^4$	A_1^3	$1_0^2, 2_1^{38}$
$A_2^2A_1$	$1_0^2, 2_1^6, 4_1^{16}$	A_2	$1_0^{16}, 3_0^{18}, 4_1^2$
D_4	$2_0^8, 6_1^9, 8_1$	A_1^2	$1_0^{14}, 2_1^{32}$
$D_4(a_1)$	$3_0^2, 4_1^{18}$	A_1	$1_0^{34}, 2_1^{22}$

Table B.5: $G = E_6$ and $V = L(\omega_2)$

Class of u	action on $L_{E_7}(\omega_7)$	Class of u	action on $L_{E_7}(\omega_7)$
$\mathbf{E_7}$	$\mathbf{2_1, 10_1, 18_1, 26_1}$	D_4A_1	$2_1^{10}, 6_1^6$
$E_7(a_1)$	$10_1, 14_1, 16_0^2$	A_4A_1	$1_0^2, 2_0^2, 3_0^2, 4_0^2, 5_0^2, 6_0^2, 7_0^2$
$E_7(a_2)$	$2_1^2, 10_1^2, 16_0^2$	A_3A_2	$2_0^2, 3_0^4, 4_0^{10}$
$E_7(a_3)$	$6_1^2, 8_0^2, 14_1^2$	$(A_3A_2)_2$	$2_0^4, 4_0^{12}$
$E_7(a_4)$	$4_0^2, 8_0^6$	$A_3A_1^2$	$2_1^8, 4_0^{10}$
$E_7(a_5)$	$2_1^2, 4_0^2, 6_0^2, 8_0^4$	$A_2^2A_1$	$1_0^4, 2_0^4, 3_0^4, 4_0^8$
E_6	$1_0^4, 10_0^2, 16_0^2$	$A_2A_1^3$	$2_1^{16}, 4_0^6$
$E_6(a_1)$	$1_0^2, 5_0^2, 9_0^2, 13_0^2$	D_4	$1_0^8, 2_0^6, 6_0^6$
$E_6(a_3)$	$1_0^4, 5_0^4, 8_0^4$	$D_4(a_1)$	$1_0^8, 4_0^{12}$
D_6	$2_1^3, 6_1, 10_1^3, 14_1$	A_4	$1_0^6, 3_0^2, 5_0^6, 7_0^2$
$D_6(a_1)$	$4_0^2, 8_0^6$	$(A_3A_1)'$	$1_0^4, 2_0^6, 4_0^{10}$
$D_6(a_2)$	$2_1^2, 6_1^6, 8_0^2$	$(A_3A_1)''$	$2_1^8, 4_0^{10}$
A_6	$5_0^2, 7_0^2, 8_0^4$	A_2^2	$1_0^4, 3_0^{12}, 4_0^4$
D_5A_1	$2_1^4, 8_0^6$	$A_2A_1^2$	$1_0^4, 2_0^{12}, 3_0^4, 4_0^4$
$D_5(a_1)A_1$	$2_0^2, 4_0^6, 6_1^2, 8_0^2$	A_1^4	2_1^{28}
A_5A_1	$2_1^2, 6_0^6, 8_0^2$	A_3	$1_0^{12}, 2_0^2, 4_0^{10}$
A_4A_2	$3_0^4, 4_0^4, 7_0^4$	A_2A_1	$1_0^8, 2_0^8, 3_0^8, 4_0^2$
$A_3A_2A_1$	$2_1^4, 4_0^{12}$	$(A_1^3)'$	$1_0^8, 2_0^{24}$
D_5	$1_0^4, 2_0^2, 8_0^6$	$(A_1^3)''$	2_1^{28}
$D_5(a_1)$	$1_0^4, 4_0^6, 6_0^2, 8_0^2$	A_2	$1_0^{20}, 3_0^{12}$
$(A_5)'$	$1_0^4, 6_0^6, 8_0^2$	A_1^2	$1_0^{16}, 2_0^{20}$
$(A_5)''$	$2_1^2, 6_0^6, 8_0^2$	A_1	$1_0^{32}, 2_0^{12}$
$D_4(a_1)A_1$	$2_0^4, 4_0^{12}$		

Table B.6: $G = E_7$ and $V = L(\omega_7)$

Class of u	action on $L_{E_7}(\omega_1)$	Class of u	action on $L_{E_7}(\omega_1)$
$\mathbf{E_7}$	$\mathbf{8_1, 10_1, 16_1, 18_1, 22_1, 26_1, 32_1}$	D_4A_1	$1_0^4, 2_1^{15}, 6_1^{15}, 8_1$
$E_7(a_1)$	$6_1, 14_1, 16_1^7$	A_4A_1	$2_1^6, 3_0^2, 4_1^6, 5_0^4, 6_0^4, 7_0^2, 8_1^4$
$E_7(a_2)$	$2_1, 6_1, 8_1, 10_1^2, 16_1^6$	A_3A_2	$1_0^2, 2_0^2, 3_0^2, 4_1^{30}$
$E_7(a_3)$	$2_1, 8_1^6, 10_1^2, 14_1, 16_1^3$	$(A_3A_2)_2$	$1_0^2, 2_1^2, 3_0^2, 4_1^{30}$
$E_7(a_4)$	$2_1, 4_1, 6_1, 8_1^{15}$	$A_3A_1^2$	$1_0^2, 2_1^{13}, 4_1^{26}$
$E_7(a_5)$	$2_1, 3_0^2, 4_1^4, 6_0^2, 8_1^{12}$	$A_2^2A_1$	$1_0^4, 2_1^{10}, 3_0^4, 4_1^{24}$
E_6	$1_0^2, 6_1, 8_1, 10_0^2, 16_1^6$	$A_2A_1^3$	$2_1^{29}, 3_0^6, 4_1^{14}$
$E_6(a_1)$	$4_1^2, 5_0^2, 8_1^2, 9_0^2, 11_0^2, 13_0^2, 16_1^2$	D_4	$1_0^6, 2_0^{14}, 6_1^{15}, 8_1$
$E_6(a_3)$	$1_0^2, 3_0^2, 4_1^2, 5_0^4, 8_1^{12}$	$D_4(a_1)$	$1_0^6, 3_0^2, 4_1^{30}$
D_6	$1_0^2, 2_1^3, 6_1^3, 8_1, 10_1^4, 14_1^3, 16_1$	A_4	$1_0^8, 3_0^6, 4_1^2, 5_0^8, 7_0^6, 8_1^2$
$D_6(a_1)$	$1_0^2, 4_1, 6_1, 8_1^{15}$	$(A_3A_1)'$	$1_0^4, 2_1^{12}, 4_1^{26}$
$D_6(a_2)$	$1_0^2, 2_1^7, 6_1^6, 8_1^{10}$	$(A_3A_1)''$	$1_0^{14}, 2_1^7, 4_1^{26}$
A_6	$3_0^2, 7_0^2, 8_1^{14}$	A_2^2	$1_0^{16}, 3_0^{12}, 4_1^{20}$
D_5A_1	$2_1^3, 6_1, 8_1^{15}$	$A_2A_1^2$	$1_0^2, 2_1^{28}, 3_0^6, 4_1^{14}$
$D_5(a_1)A_1$	$2_1, 4_1^{14}, 6_1^3, 8_1^7$	A_1^4	$1_0^6, 2_1^{63}$
A_5A_1	$1_0^2, 2_1^7, 6_0^6, 8_1^{10}$	A_3	$1_0^{16}, 2_0^6, 4_1^{26}$
A_4A_2	$1_0^2, 4_1^{12}, 5_0^4, 7_0^2, 8_1^6$	A_2A_1	$1_0^{14}, 2_1^{18}, 3_0^{14}, 4_1^{10}$
$A_3A_2A_1$	$2_1^3, 3_0^2, 4_1^{30}$	$(A_1^3)'$	$1_0^8, 2_1^{62}$
D_5	$1_0^2, 2_0^2, 6_1, 8_1^{15}$	$(A_1^3)''$	$1_0^{26}, 2_1^{53}$
$D_5(a_1)$	$1_0^2, 4_1^{14}, 6_1^3, 8_1^7$	A_2	$1_0^{34}, 3_0^{30}, 4_1^2$
$(A_5)'$	$1_0^4, 2_1^6, 6_0^6, 8_1^{10}$	A_1^2	$1_0^{28}, 2_1^{52}$
$(A_5)''$	$1_0^{14}, 2_1, 6_0^6, 8_1^{10}$	A_1	$1_0^{64}, 2_1^{34}$
$D_4(a_1)A_1$	$1_0^4, 2_1, 3_0^2, 4_1^{30}$		

Table B.7: $G = E_7$ and $V = L(\omega_1)$

Class of u	action on $L_{E_8}(\omega_8)$
E_8	$24_1, 32_1^7$
$E_8(a_1)$	$12_1, 16_1^3, 28_1, 32_1^5$
$E_8(a_2)$	$14_1, 16_1^7, 26_1, 32_1^3$
$E_8(a_3)$	$8_1^4, 14_1, 16_1^3, 22_1^2, 24_0^2, 30_1, 32_1$
$E_8(a_4)$	$12_1^2, 16_1^{14}$
$E_8(a_5)$	$6_1^2, 8_1^6, 14_1^2, 16_1^{10}$
$E_8(a_6)$	$7_0^2, 8_1^{14}, 13_0^2, 16_1^6$
$E_8(a_7)$	$3_0^4, 4_1^{12}, 7_0^4, 8_1^{20}$
$E_8(b_4)$	$1_0^2, 2_1, 6_1, 10_1^2, 14_1^3, 16_1^{11}$
$E_8(b_5)$	$3_0^2, 4_1^2, 6_1, 8_1^3, 10_0^2, 12_0^2, 16_1^{10}$
$E_8(b_6)$	$4_1^8, 7_0^2, 8_1^6, 11_0^4, 12_0^4, 15_0^2, 16_1^2$
E_7	$1_0^2, 2_1^3, 8_1, 10_1^3, 16_1, 18_1^3, 22_1, 26_1^3, 32_1$
$E_7(a_1)$	$1_0^2, 2_1, 6_1, 10_0^2, 14_1^3, 16_1^{11}$
$E_7(a_2)$	$1_0^2, 2_1^6, 6_1, 8_1, 10_1^6, 16_1^{10}$
$E_7(a_3)$	$1_0^2, 2_1^2, 6_0^4, 8_1^{10}, 10_1^2, 14_1^5, 16_1^3$
$E_7(a_4)$	$1_0^4, 2_1, 4_1^5, 6_1, 8_1^{27}$
$E_7(a_5)$	$1_0^2, 2_1^6, 3_0^2, 4_1^8, 6_0^6, 8_1^{20}$
D_7	$2_0^2, 4_1^6, 12_1^9, 16_1^7$
$(D_7(a_1))_2$	$1_0^2, 2_1^2, 6_1^4, 8_1^{10}, 10_1^2, 14_1^5, 16_1^3$
$D_7(a_1)$	$4_1^8, 6_1, 8_1^7, 10_0^2, 12_0^6, 14_1, 16_1^3$
$D_7(a_2)$	$6_1^4, 8_1^{28}$
A_7	$4_0^2, 8_1^{30}$
E_6A_1	$1_0^2, 2_1^6, 6_1, 8_1, 10_0^6, 16_1^{10}$
$E_6(a_1)A_1$	$2_1^4, 4_1^4, 5_0^2, 6_0^2, 8_1^4, 9_0^2, 10_0^2, 11_0^2, 12_0^2, 13_0^2, 14_0^2, 16_1^2$
$E_6(a_3)A_1$	$1_0^2, 2_1^6, 3_0^2, 4_1^6, 5_0^4, 6_0^4, 8_1^{20}$
A_6A_1	$2_1^2, 3_0^2, 4_0^2, 6_0^4, 7_0^2, 8_1^{24}$
$(D_5A_2)_2$	$2_1^3, 4_1^5, 6_1, 8_1^{27}$
D_5A_2	$2_0^2, 3_0^2, 4_1^4, 6_1, 8_1^{27}$
$D_5(a_1)A_2$	$3_0^2, 4_1^{30}, 6_1^3, 8_1^{13}$
A_4A_3	$3_0^2, 4_1^{30}, 6_0^2, 7_0^2, 8_1^{12}$
$A_4A_2A_1$	$1_0^2, 2_1^6, 4_1^{24}, 5_0^4, 6_0^4, 7_0^2, 8_1^{10}$
E_6	$1_0^{14}, 6_1, 8_1, 10_0^6, 16_1^{10}$
$E_6(a_1)$	$1_0^8, 4_1^2, 5_0^6, 8_1^2, 9_0^6, 11_0^2, 13_0^6, 16_1^2$

Table B.8: $G = E_8$ and $V = L(\omega_8)$

Class of u	action on $L_{E_8}(\omega_8)$
$E_6(a_3)$	$1_0^{14}, 3_0^2, 4_1^2, 5_0^{12}, 8_1^{20}$
D_6	$1_0^4, 2_1^{10}, 6_1^5, 8_1, 10_1^{10}, 14_1^5, 16_1$
$D_6(a_1)$	$1_0^4, 2_1, 4_1^5, 6_1, 8_1^{27}$
$D_6(a_2)$	$1_0^4, 2_1^{12}, 6_1^{18}, 8_1^{14}$
A_6	$1_0^4, 3_0^2, 5_0^4, 7_0^6, 8_1^{22}$
D_5A_1	$1_0^2, 2_1^{12}, 6_1, 8_1^{27}$
$D_5(a_1)A_1$	$1_0^2, 2_1^6, 4_1^{26}, 6_1^7, 8_1^{11}$
A_5A_1	$1_0^4, 2_1^{12}, 6_0^{18}, 8_1^{14}$
$(D_4A_2)_2$	$1_0^2, 2_1^6, 4_1^{26}, 6_1^7, 8_1^{11}$
D_4A_2	$2_0^{14}, 3_0^6, 4_1^{14}, 6_1^{15}, 8_1^7$
$D_4(a_1)A_2$	$3_0^8, 4_1^{56}$
A_4A_2	$1_0^6, 3_0^8, 4_1^{20}, 5_0^4, 7_0^{10}, 8_1^6$
$A_4A_1^2$	$2_1^{16}, 3_0^2, 4_1^{14}, 5_0^4, 6_0^{12}, 7_0^2, 8_1^6$
A_3^2	$2_0^4, 4_1^{60}$
$A_3A_2A_1$	$1_0^2, 2_1^{12}, 3_0^2, 4_1^{54}$
$A_2^2A_1^2$	$1_0^4, 2_1^{28}, 3_0^4, 4_1^{44}$
D_5	$1_0^{14}, 2_0^6, 6_1, 8_1^{27}$
$D_5(a_1)$	$1_0^{14}, 4_1^{26}, 6_1^7, 8_1^{11}$
A_5	$1_0^{16}, 2_1^6, 6_0^{18}, 8_1^{14}$
D_4A_1	$1_0^6, 2_1^{36}, 6_1^{27}, 8_1$
$D_4(a_1)A_1$	$1_0^6, 2_1^{10}, 3_0^2, 4_1^{54}$
A_4A_1	$1_0^8, 2_1^{10}, 3_0^6, 4_1^{10}, 5_0^8, 6_0^8, 7_0^6, 8_1^4$
$(A_3A_2)_2$	$1_0^6, 2_1^{10}, 3_0^2, 4_1^{54}$
A_3A_2	$1_0^6, 2_0^6, 3_0^{10}, 4_1^{50}$
$A_3A_1^2$	$1_0^4, 2_1^{30}, 4_1^{46}$
$A_2^2A_1$	$1_0^{16}, 2_1^{18}, 3_0^{12}, 4_1^{40}$
$A_2A_1^3$	$1_0^2, 2_1^{62}, 3_0^6, 4_1^{26}$
D_4	$1_0^{26}, 2_0^{26}, 6_1^{27}, 8_1$
$D_4(a_1)$	$1_0^{26}, 3_0^2, 4_1^{54}$
A_4	$1_0^{24}, 3_0^{10}, 4_1^2, 5_0^{20}, 7_0^{10}, 8_1^2$
A_3A_1	$1_0^{16}, 2_1^{24}, 4_1^{46}$
A_2^2	$1_0^{28}, 3_0^{36}, 4_1^{28}$
$A_2A_1^2$	$1_0^{14}, 2_1^{52}, 3_0^{14}, 4_1^{22}$
A_1^4	$1_0^8, 2_1^{120}$
A_3	$1_0^{44}, 2_0^{10}, 4_1^{46}$
A_2A_1	$1_0^{34}, 2_1^{34}, 3_0^{30}, 4_1^{14}$
A_1^3	$1_0^{28}, 2_1^{110}$
A_2	$1_0^{78}, 3_0^{54}, 4_1^2$
A_1^2	$1_0^{64}, 2_1^{92}$
A_1	$1_0^{132}, 2_1^{58}$

Table B.9: $G = E_8$ and $V = L(\omega_8)$ (continued)

Appendix C

Ordering of the positive roots

Let Φ be an indecomposable root system with base $\Delta = \{\alpha_1, \dots, \alpha_l\}$ and set of positive roots Φ^+. We assume that the simple roots are ordered as in the standard Bourbaki labeling of the Dynkin diagram, see for example [Hum72, 11.4, pg. 58].

In this appendix we give, for Φ of exceptional type, the total ordering $\leq$ on Φ^+ that was defined in Section 2.8 (following [Car72, 2.1]). We recall that for $\alpha, \beta \in \Phi^+$, we set $\alpha \leq \beta$ if and only if one of the following holds:

- $\alpha = \beta$,
- There exists an integer $1 \leq k \leq l$ such that $\beta - \alpha = \sum_{i=1}^{k} c_i \alpha_i$, where $c_i \in \mathbb{Z}$ for all $1 \leq i \leq k$ and $c_k > 0$.

Set $N = |\Phi^+|$. Writing $\Phi^+ = \{\beta_1, \dots, \beta_N\}$, where $\beta_1 < \beta_2 < \cdots < \beta_N$ with respect to the total order $\leq$, we give the expression of β_i for all $1 \leq i \leq N$ as a sum of the simple roots α_i in the tables that follow. In the tables we use the notation $\beta_i = (k_1, k_2, \dots, k_l)$, for $\beta_i = \sum_{j=1}^{l} k_j \alpha_j$.

β_1	$=$	$(1, 0)$
β_2	$=$	$(0, 1)$
β_3	$=$	$(1, 1)$
β_4	$=$	$(2, 1)$
β_5	$=$	$(3, 1)$
β_6	$=$	$(3, 2)$

Table C.1: Ordering of the positive roots of the root system G_2.

β_1	$=$	$(1,0,0,0)$	β_{13}	$=$	$(0,1,2,1)$
β_2	$=$	$(0,1,0,0)$	β_{14}	$=$	$(1,2,2,0)$
β_3	$=$	$(0,0,1,0)$	β_{15}	$=$	$(1,1,2,1)$
β_4	$=$	$(0,0,0,1)$	β_{16}	$=$	$(0,1,2,2)$
β_5	$=$	$(1,1,0,0)$	β_{17}	$=$	$(1,2,2,1)$
β_6	$=$	$(0,1,1,0)$	β_{18}	$=$	$(1,1,2,2)$
β_7	$=$	$(0,0,1,1)$	β_{19}	$=$	$(1,2,3,1)$
β_8	$=$	$(1,1,1,0)$	β_{20}	$=$	$(1,2,2,2)$
β_9	$=$	$(0,1,2,0)$	β_{21}	$=$	$(1,2,3,2)$
β_{10}	$=$	$(0,1,1,1)$	β_{22}	$=$	$(1,2,4,2)$
β_{11}	$=$	$(1,1,2,0)$	β_{23}	$=$	$(1,3,4,2)$
β_{12}	$=$	$(1,1,1,1)$	β_{24}	$=$	$(2,3,4,2)$

Table C.2: Ordering of the positive roots of the root system F_4.

β_1	$=$	$(1,0,0,0,0,0)$	β_{19}	$=$	$(0,1,1,1,1,0)$
β_2	$=$	$(0,1,0,0,0,0)$	β_{20}	$=$	$(0,1,0,1,1,1)$
β_3	$=$	$(0,0,1,0,0,0)$	β_{21}	$=$	$(0,0,1,1,1,1)$
β_4	$=$	$(0,0,0,1,0,0)$	β_{22}	$=$	$(1,1,1,1,1,0)$
β_5	$=$	$(0,0,0,0,1,0)$	β_{23}	$=$	$(1,0,1,1,1,1)$
β_6	$=$	$(0,0,0,0,0,1)$	β_{24}	$=$	$(0,1,1,2,1,0)$
β_7	$=$	$(1,0,1,0,0,0)$	β_{25}	$=$	$(0,1,1,1,1,1)$
β_8	$=$	$(0,1,0,1,0,0)$	β_{26}	$=$	$(1,1,1,2,1,0)$
β_9	$=$	$(0,0,1,1,0,0)$	β_{27}	$=$	$(1,1,1,1,1,1)$
β_{10}	$=$	$(0,0,0,1,1,0)$	β_{28}	$=$	$(0,1,1,2,1,1)$
β_{11}	$=$	$(0,0,0,0,1,1)$	β_{29}	$=$	$(1,1,2,2,1,0)$
β_{12}	$=$	$(1,0,1,1,0,0)$	β_{30}	$=$	$(1,1,1,2,1,1)$
β_{13}	$=$	$(0,1,1,1,0,0)$	β_{31}	$=$	$(0,1,1,2,2,1)$
β_{14}	$=$	$(0,1,0,1,1,0)$	β_{32}	$=$	$(1,1,2,2,1,1)$
β_{15}	$=$	$(0,0,1,1,1,0)$	β_{33}	$=$	$(1,1,1,2,2,1)$
β_{16}	$=$	$(0,0,0,1,1,1)$	β_{34}	$=$	$(1,1,2,2,2,1)$
β_{17}	$=$	$(1,1,1,1,0,0)$	β_{35}	$=$	$(1,1,2,3,2,1)$
β_{18}	$=$	$(1,0,1,1,1,0)$	β_{36}	$=$	$(1,2,2,3,2,1)$

Table C.3: Ordering of the positive roots of the root system E_6.

β_1	$=$	$(1,0,0,0,0,0,0)$	β_{33}	$=$	$(1,1,1,1,1,1,0)$
β_2	$=$	$(0,1,0,0,0,0,0)$	β_{34}	$=$	$(1,0,1,1,1,1,1)$
β_3	$=$	$(0,0,1,0,0,0,0)$	β_{35}	$=$	$(0,1,1,2,1,1,0)$
β_4	$=$	$(0,0,0,1,0,0,0)$	β_{36}	$=$	$(0,1,1,1,1,1,1)$
β_5	$=$	$(0,0,0,0,1,0,0)$	β_{37}	$=$	$(1,1,2,2,1,0,0)$
β_6	$=$	$(0,0,0,0,0,1,0)$	β_{38}	$=$	$(1,1,1,2,1,1,0)$
β_7	$=$	$(0,0,0,0,0,0,1)$	β_{39}	$=$	$(1,1,1,1,1,1,1)$
β_8	$=$	$(1,0,1,0,0,0,0)$	β_{40}	$=$	$(0,1,1,2,2,1,0)$
β_9	$=$	$(0,1,0,1,0,0,0)$	β_{41}	$=$	$(0,1,1,2,1,1,1)$
β_{10}	$=$	$(0,0,1,1,0,0,0)$	β_{42}	$=$	$(1,1,2,2,1,1,0)$
β_{11}	$=$	$(0,0,0,1,1,0,0)$	β_{43}	$=$	$(1,1,1,2,2,1,0)$
β_{12}	$=$	$(0,0,0,0,1,1,0)$	β_{44}	$=$	$(1,1,1,2,1,1,1)$
β_{13}	$=$	$(0,0,0,0,0,1,1)$	β_{45}	$=$	$(0,1,1,2,2,1,1)$
β_{14}	$=$	$(1,0,1,1,0,0,0)$	β_{46}	$=$	$(1,1,2,2,2,1,0)$
β_{15}	$=$	$(0,1,1,1,0,0,0)$	β_{47}	$=$	$(1,1,2,2,1,1,1)$
β_{16}	$=$	$(0,1,0,1,1,0,0)$	β_{48}	$=$	$(1,1,1,2,2,1,1)$
β_{17}	$=$	$(0,0,1,1,1,0,0)$	β_{49}	$=$	$(0,1,1,2,2,2,1)$
β_{18}	$=$	$(0,0,0,1,1,1,0)$	β_{50}	$=$	$(1,1,2,3,2,1,0)$
β_{19}	$=$	$(0,0,0,0,1,1,1)$	β_{51}	$=$	$(1,1,2,2,2,1,1)$
β_{20}	$=$	$(1,1,1,1,0,0,0)$	β_{52}	$=$	$(1,1,1,2,2,2,1)$
β_{21}	$=$	$(1,0,1,1,1,0,0)$	β_{53}	$=$	$(1,2,2,3,2,1,0)$
β_{22}	$=$	$(0,1,1,1,1,0,0)$	β_{54}	$=$	$(1,1,2,3,2,1,1)$
β_{23}	$=$	$(0,1,0,1,1,1,0)$	β_{55}	$=$	$(1,1,2,2,2,2,1)$
β_{24}	$=$	$(0,0,1,1,1,1,0)$	β_{56}	$=$	$(1,2,2,3,2,1,1)$
β_{25}	$=$	$(0,0,0,1,1,1,1)$	β_{57}	$=$	$(1,1,2,3,2,2,1)$
β_{26}	$=$	$(1,1,1,1,1,0,0)$	β_{58}	$=$	$(1,2,2,3,2,2,1)$
β_{27}	$=$	$(1,0,1,1,1,1,0)$	β_{59}	$=$	$(1,1,2,3,3,2,1)$
β_{28}	$=$	$(0,1,1,2,1,0,0)$	β_{60}	$=$	$(1,2,2,3,3,2,1)$
β_{29}	$=$	$(0,1,1,1,1,1,0)$	β_{61}	$=$	$(1,2,2,4,3,2,1)$
β_{30}	$=$	$(0,1,0,1,1,1,1)$	β_{62}	$=$	$(1,2,3,4,3,2,1)$
β_{31}	$=$	$(0,0,1,1,1,1,1)$	β_{63}	$=$	$(2,2,3,4,3,2,1)$
β_{32}	$=$	$(1,1,1,2,1,0,0)$			

Table C.4: Ordering of the positive roots of the root system E_7.

β_1	$=$	$(1,0,0,0,0,0,0,0)$	β_{31}	$=$	$(1,0,1,1,1,1,0,0)$
β_2	$=$	$(0,1,0,0,0,0,0,0)$	β_{32}	$=$	$(0,1,1,2,1,0,0,0)$
β_3	$=$	$(0,0,1,0,0,0,0,0)$	β_{33}	$=$	$(0,1,1,1,1,1,0,0)$
β_4	$=$	$(0,0,0,1,0,0,0,0)$	β_{34}	$=$	$(0,1,0,1,1,1,1,0)$
β_5	$=$	$(0,0,0,0,1,0,0,0)$	β_{35}	$=$	$(0,0,1,1,1,1,1,0)$
β_6	$=$	$(0,0,0,0,0,1,0,0)$	β_{36}	$=$	$(0,0,0,1,1,1,1,1)$
β_7	$=$	$(0,0,0,0,0,0,1,0)$	β_{37}	$=$	$(1,1,1,2,1,0,0,0)$
β_8	$=$	$(0,0,0,0,0,0,0,1)$	β_{38}	$=$	$(1,1,1,1,1,1,0,0)$
β_9	$=$	$(1,0,1,0,0,0,0,0)$	β_{39}	$=$	$(1,0,1,1,1,1,1,0)$
β_{10}	$=$	$(0,1,0,1,0,0,0,0)$	β_{40}	$=$	$(0,1,1,2,1,1,0,0)$
β_{11}	$=$	$(0,0,1,1,0,0,0,0)$	β_{41}	$=$	$(0,1,1,1,1,1,1,0)$
β_{12}	$=$	$(0,0,0,1,1,0,0,0)$	β_{42}	$=$	$(0,1,0,1,1,1,1,1)$
β_{13}	$=$	$(0,0,0,0,1,1,0,0)$	β_{43}	$=$	$(0,0,1,1,1,1,1,1)$
β_{14}	$=$	$(0,0,0,0,0,1,1,0)$	β_{44}	$=$	$(1,1,2,2,1,0,0,0)$
β_{15}	$=$	$(0,0,0,0,0,0,1,1)$	β_{45}	$=$	$(1,1,1,2,1,1,0,0)$
β_{16}	$=$	$(1,0,1,1,0,0,0,0)$	β_{46}	$=$	$(1,1,1,1,1,1,1,0)$
β_{17}	$=$	$(0,1,1,1,0,0,0,0)$	β_{47}	$=$	$(1,0,1,1,1,1,1,1)$
β_{18}	$=$	$(0,1,0,1,1,0,0,0)$	β_{48}	$=$	$(0,1,1,2,2,1,0,0)$
β_{19}	$=$	$(0,0,1,1,1,0,0,0)$	β_{49}	$=$	$(0,1,1,2,1,1,1,0)$
β_{20}	$=$	$(0,0,0,1,1,1,0,0)$	β_{50}	$=$	$(0,1,1,1,1,1,1,1)$
β_{21}	$=$	$(0,0,0,0,1,1,1,0)$	β_{51}	$=$	$(1,1,2,2,1,1,0,0)$
β_{22}	$=$	$(0,0,0,0,0,1,1,1)$	β_{52}	$=$	$(1,1,1,2,2,1,0,0)$
β_{23}	$=$	$(1,1,1,1,0,0,0,0)$	β_{53}	$=$	$(1,1,1,2,1,1,1,0)$
β_{24}	$=$	$(1,0,1,1,1,0,0,0)$	β_{54}	$=$	$(1,1,1,1,1,1,1,1)$
β_{25}	$=$	$(0,1,1,1,1,0,0,0)$	β_{55}	$=$	$(0,1,1,2,2,1,1,0)$
β_{26}	$=$	$(0,1,0,1,1,1,0,0)$	β_{56}	$=$	$(0,1,1,2,1,1,1,1)$
β_{27}	$=$	$(0,0,1,1,1,1,0,0)$	β_{57}	$=$	$(1,1,2,2,2,1,0,0)$
β_{28}	$=$	$(0,0,0,1,1,1,1,0)$	β_{58}	$=$	$(1,1,2,2,1,1,1,0)$
β_{29}	$=$	$(0,0,0,0,1,1,1,1)$	β_{59}	$=$	$(1,1,1,2,2,1,1,0)$
β_{30}	$=$	$(1,1,1,1,1,0,0,0)$	β_{60}	$=$	$(1,1,1,2,1,1,1,1)$

Table C.5: Ordering of the positive roots of the root system E_8.

β_{61}	$=$	$(0,1,1,2,2,2,1,0)$	β_{91}	$=$	$(1,2,2,3,2,2,2,1)$
β_{62}	$=$	$(0,1,1,2,2,1,1,1)$	β_{92}	$=$	$(1,1,2,3,3,2,2,1)$
β_{63}	$=$	$(1,1,2,3,2,1,0,0)$	β_{93}	$=$	$(1,2,3,4,3,2,1,0)$
β_{64}	$=$	$(1,1,2,2,2,1,1,0)$	β_{94}	$=$	$(1,2,2,4,3,2,1,1)$
β_{65}	$=$	$(1,1,2,2,1,1,1,1)$	β_{95}	$=$	$(1,2,2,3,3,2,2,1)$
β_{66}	$=$	$(1,1,1,2,2,2,1,0)$	β_{96}	$=$	$(1,1,2,3,3,3,2,1)$
β_{67}	$=$	$(1,1,1,2,2,1,1,1)$	β_{97}	$=$	$(2,2,3,4,3,2,1,0)$
β_{68}	$=$	$(0,1,1,2,2,2,1,1)$	β_{98}	$=$	$(1,2,3,4,3,2,1,1)$
β_{69}	$=$	$(1,2,2,3,2,1,0,0)$	β_{99}	$=$	$(1,2,2,4,3,2,2,1)$
β_{70}	$=$	$(1,1,2,3,2,1,1,0)$	β_{100}	$=$	$(1,2,2,3,3,3,2,1)$
β_{71}	$=$	$(1,1,2,2,2,2,1,0)$	β_{101}	$=$	$(2,2,3,4,3,2,1,1)$
β_{72}	$=$	$(1,1,2,2,2,1,1,1)$	β_{102}	$=$	$(1,2,3,4,3,2,2,1)$
β_{73}	$=$	$(1,1,1,2,2,2,1,1)$	β_{103}	$=$	$(1,2,2,4,3,3,2,1)$
β_{74}	$=$	$(0,1,1,2,2,2,2,1)$	β_{104}	$=$	$(2,2,3,4,3,2,2,1)$
β_{75}	$=$	$(1,2,2,3,2,1,1,0)$	β_{105}	$=$	$(1,2,3,4,3,3,2,1)$
β_{76}	$=$	$(1,1,2,3,2,2,1,0)$	β_{106}	$=$	$(1,2,2,4,4,3,2,1)$
β_{77}	$=$	$(1,1,2,3,2,1,1,1)$	β_{107}	$=$	$(2,2,3,4,3,3,2,1)$
β_{78}	$=$	$(1,1,2,2,2,2,1,1)$	β_{108}	$=$	$(1,2,3,4,4,3,2,1)$
β_{79}	$=$	$(1,1,1,2,2,2,2,1)$	β_{109}	$=$	$(2,2,3,4,4,3,2,1)$
β_{80}	$=$	$(1,2,2,3,2,2,1,0)$	β_{110}	$=$	$(1,2,3,5,4,3,2,1)$
β_{81}	$=$	$(1,2,2,3,2,1,1,1)$	β_{111}	$=$	$(2,2,3,5,4,3,2,1)$
β_{82}	$=$	$(1,1,2,3,3,2,1,0)$	β_{112}	$=$	$(1,3,3,5,4,3,2,1)$
β_{83}	$=$	$(1,1,2,3,2,2,1,1)$	β_{113}	$=$	$(2,3,3,5,4,3,2,1)$
β_{84}	$=$	$(1,1,2,2,2,2,2,1)$	β_{114}	$=$	$(2,2,4,5,4,3,2,1)$
β_{85}	$=$	$(1,2,2,3,3,2,1,0)$	β_{115}	$=$	$(2,3,4,5,4,3,2,1)$
β_{86}	$=$	$(1,2,2,3,2,2,1,1)$	β_{116}	$=$	$(2,3,4,6,4,3,2,1)$
β_{87}	$=$	$(1,1,2,3,3,2,1,1)$	β_{117}	$=$	$(2,3,4,6,5,3,2,1)$
β_{88}	$=$	$(1,1,2,3,2,2,2,1)$	β_{118}	$=$	$(2,3,4,6,5,4,2,1)$
β_{89}	$=$	$(1,2,2,4,3,2,1,0)$	β_{119}	$=$	$(2,3,4,6,5,4,3,1)$
β_{90}	$=$	$(1,2,2,3,3,2,1,1)$	β_{120}	$=$	$(2,3,4,6,5,4,3,2)$

Table C.6: Ordering of the positive roots of the root system E_8 (continued).

Bibliography

[AJL83] H. H. Andersen, J. Jørgensen, and P. Landrock. The projective indecomposable modules of SL(2, p^n). *Proc. London Math. Soc. (3)*, 46(1):38–52, 1983.

[BC76a] P. Bala and R. W. Carter. Classes of unipotent elements in simple algebraic groups. I. *Math. Proc. Cambridge Philos. Soc.*, 79(3):401–425, 1976.

[BC76b] P. Bala and R. W. Carter. Classes of unipotent elements in simple algebraic groups. II. *Math. Proc. Cambridge Philos. Soc.*, 80(1):1–17, 1976.

[Bar11] M. J. J. Barry. Decomposing tensor products and exterior and symmetric squares. *J. Group Theory*, 14(1):59–82, 2011.

[Bar15] M. J. J. Barry. On a question of Glasby, Praeger, and Xia. *Comm. Algebra*, 43(10):4231–4246, 2015.

[BMR05] M. Bate, B. Martin, and G. Röhrle. A geometric approach to complete reducibility. *Invent. Math.*, 161(1):177–218, 2005.

[Bou59] N. Bourbaki. *Éléments de mathématique. Première partie: Les structures fondamentales de l'analyse. Livre II: Algèbre. Chapitre 9: Formes sesquilinéaires et formes quadratiques.* Actualités Sci. Ind. no. 1272. Hermann, Paris, 1959.

[Bou75] N. Bourbaki. *Éléments de mathématique. Fasc. XXXVIII: Groupes et algèbres de Lie. Chapitre VII: Sous-algèbres de Cartan, éléments réguliers. Chapitre VIII: Algèbres de Lie semi-simples déployées.* Actualités Scientifiques et Industrielles, No. 1364. Hermann, Paris, 1975.

[Bru98] J. Brundan. Dense orbits and double cosets. In *Algebraic groups and their representations (Cambridge, 1997)*, Volume 517 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 259–274. Kluwer Acad. Publ., Dordrecht, 1998.

[Cal02] D. Callan. Jordan and Smith forms of Pascal-related matrices. *ArXiv Mathematics e-prints*, September 2002.

[Car72] R. W. Carter. *Simple groups of Lie type.* John Wiley & Sons, London-New York-Sydney, 1972. Pure and Applied Mathematics, Vol. 28.

[Car85] R. W. Carter. *Finite groups of Lie type.* Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1985. Conjugacy classes and complex characters, A Wiley-Interscience Publication.

[CT17] M. Cavallin and D. M. Testerman. A new family of irreducible subgroups of the orthogonal algebraic groups. preprint, 2017.

[CMT04] A. M. Cohen, S. H. Murray, and D. E. Taylor. Computing in groups of Lie type. *Math. Comp.*, 73(247):1477–1498, 2004.

[Don93] S. Donkin. On tilting modules for algebraic groups. *Math. Z.*, 212(1):39–60, 1993.

[Dye79] R. H. Dye. Interrelations of symplectic and orthogonal groups in characteristic two. *J. Algebra*, 59(1):202–221, 1979.

[Dyn52a] E. B. Dynkin. Maximal subgroups of the classical groups. *Trudy Moskov. Mat. Obšč.*, 1:39–166, 1952.

[Dyn52b] E. B. Dynkin. Semisimple subalgebras of semisimple Lie algebras. *Mat. Sbornik N.S.*, 30(72):349–462 (3 plates), 1952.

[Fei82] W. Feit. *The representation theory of finite groups*, Volume 25 of *North-Holland Mathematical Library.* North-Holland Publishing Co., Amsterdam-New York, 1982.

[Fin47] N. J. Fine. Binomial coefficients modulo a prime. *Amer. Math. Monthly*, 54:589–592, 1947.

[Fon74] P. Fong. On decomposition numbers of J_1 and $R(q)$. In *Symposia Mathematica, Vol. XIII (Convegno di Gruppi e loro Rappresentazioni, INDAM, Rome, 1972)*, pages 415–422. Academic Press, London, 1974.

[For96] B. Ford. Overgroups of irreducible linear groups. I. *J. Algebra*, 181(1):26–69, 1996.

[FH91] W. Fulton and J. Harris. *Representation theory*, Volume 129 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[GAP16] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.8.6*, 2016.

[GN16] S. Garibaldi and D. K. Nakano. Bilinear and quadratic forms on rational modules of split reductive groups. *Canad. J. Math.*, 68(2):395–421, 2016.

[GPX15] S. P. Glasby, C. E. Praeger, and B. Xia. Decomposing modular tensor products: 'Jordan partitions', their parts and p-parts. *Israel J. Math.*, 209(1):215–233, 2015.

[GPX16] S. P. Glasby, C. E. Praeger, and B. Xia. Decomposing modular tensor products, and periodicity of 'Jordan partitions'. *J. Algebra*, 450:570–587, 2016.

[Gow97] R. Gow. Contraction of exterior powers in characteristic 2 and the spin module. *Geom. Dedicata*, 64(3):283–295, 1997.

[Gow98] R. Gow. Construction of $p-1$ irreducible modules with fundamental highest weight for the symplectic group in characteristic p. *J. London Math. Soc. (2)*, 58(3):619–632, 1998.

[GL06] R. Gow and T. J. Laffey. On the decomposition of the exterior square of an indecomposable module of a cyclic p-group. *J. Group Theory*, 9(5):659–672, 2006.

[GW95] R. Gow and W. Willems. Methods to decide if simple self-dual modules over fields of characteristic 2 are of quadratic type. *J. Algebra*, 175(3):1067–1081, 1995.

[GM14] R. Guralnick and G. Malle. Rational rigidity for $E_8(p)$. *Compos. Math.*, 150(10):1679–1702, 2014.

[Hée84] J.-Y. Hée. Groupes de Chevalley et groupes classiques. In *Seminar on finite groups, Vol. II*, Volume 17 of *Publ. Math. Univ. Paris VII*, pages 1–54. Univ. Paris VII, Paris, 1984.

[Hes79] W. H. Hesselink. Nilpotency in classical groups over a field of characteristic 2. *Math. Z.*, 166(2):165–181, 1979.

[HS14] F. Himstedt and P. Symonds. Exterior and symmetric powers of modules for cyclic 2-groups. *J. Algebra*, 410:393–420, 2014.

[Hou03] X. D. Hou. Elementary divisors of tensor products and p-ranks of binomial matrices. *Linear Algebra Appl.*, 374:255–274, 2003.

[Hum72] J. E. Humphreys. *Introduction to Lie algebras and representation theory.* Springer-Verlag, New York-Berlin, 1972. Graduate Texts in Mathematics, Vol. 9.

[Jan73] J. C. Jantzen. Darstellungen halbeinfacher algebraischer Gruppen und zugeordnete kontravariante Formen. *Bonn. Math. Schr.*, (67):v+124, 1973.

[Jan97] J. C. Jantzen. Low-dimensional representations of reductive groups are semisimple. In *Algebraic groups and Lie groups*, Volume 9 of *Austral. Math. Soc. Lect. Ser.*, pages 255–266. Cambridge Univ. Press, Cambridge, 1997.

[Jan03] J. C. Jantzen. *Representations of algebraic groups*, Volume 107 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, second edition, 2003.

[Jan04] J. C. Jantzen. Nilpotent orbits in representation theory. In *Lie theory*, Volume 228 of *Progr. Math.*, pages 1–211. Birkhäuser Boston, Boston, MA, 2004.

[KL90] P. Kleidman and M. Liebeck. *The subgroup structure of the finite classical groups*, Volume 129 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1990.

[Knu97] D. E. Knuth. *The art of computer programming. Vol. 1.* Addison-Wesley, Reading, MA, 1997. Fundamental algorithms, Third edition [of MR0286317].

[Kor17] M. Korhonen. Invariant forms on irreducible modules of simple algebraic groups. *J. Algebra*, 480:385–422, 2017.

[Law95] R. Lawther. Jordan block sizes of unipotent elements in exceptional algebraic groups. *Comm. Algebra*, 23(11):4125–4156, 1995.

[Law98] R. Lawther. Correction to: "Jordan block sizes of unipotent elements in exceptional algebraic groups" [Comm. Algebra **23** (1995), no. 11, 4125–4156; MR1351124 (96h:20084)]. *Comm. Algebra*, 26(8):2709, 1998.

[Law09] R. Lawther. Unipotent classes in maximal subgroups of exceptional algebraic groups. *J. Algebra*, 322(1):270–293, 2009.

[LLS14] R. Lawther, M. W. Liebeck, and G. M. Seitz. Outer unipotent classes in automorphism groups of simple algebraic groups. *Proc. Lond. Math. Soc. (3)*, 109(3):553–595, 2014.

[LT99] R. Lawther and D. M. Testerman. A_1 subgroups of exceptional algebraic groups. *Mem. Amer. Math. Soc.*, 141(674):viii+131, 1999.

[LS94] M. W. Liebeck and G. M. Seitz. Subgroups generated by root elements in groups of Lie type. *Ann. of Math. (2)*, 139(2):293–361, 1994.

[LS96] M. W. Liebeck and G. M. Seitz. Reductive subgroups of exceptional algebraic groups. *Mem. Amer. Math. Soc.*, 121(580):vi+111, 1996.

[LS04] M. W. Liebeck and G. M. Seitz. The maximal subgroups of positive dimension in exceptional algebraic groups. *Mem. Amer. Math. Soc.*, 169(802):vi+227, 2004.

[LS12] M. W. Liebeck and G. M. Seitz. *Unipotent and nilpotent classes in simple algebraic groups and Lie algebras*, Volume 180 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2012.

[LST15] M. W. Liebeck, G. M. Seitz, and D. M. Testerman. Distinguished unipotent elements and multiplicity-free subgroups of simple algebraic groups. *Pacific J. Math.*, 279(1-2):357–382, 2015.

[LT] A. J. Litterick and A. R. Thomas. Complete Reducibility in Good Characteristic. To appear in *Trans. Amer. Math. Soc.*

[Lüb01] F. Lübeck. Small degree representations of finite Chevalley groups in defining characteristic. *LMS J. Comput. Math.*, 4:135–169 (electronic), 2001.

[Lüb17] F. Lübeck. Tables of weight multiplicities. `http://www.math.rwth-aachen.de/~Frank.Luebeck/chev/WMSmall/index.html`, 2017.

[Luc78] E. Lucas. Theorie des Fonctions Numeriques Simplement Periodiques. [Continued]. *Amer. J. Math.*, 1(3):197–240, 1878.

[MT11] G. Malle and D. M. Testerman. *Linear algebraic groups and finite groups of Lie type*, Volume 133 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2011.

[Mat90] O. Mathieu. Filtrations of G-modules. *Ann. Sci. École Norm. Sup. (4)*, 23(4):625–644, 1990.

[McF79] J. D. McFall. How to compute the elementary divisors of the tensor product of two matrices. *Linear and Multilinear Algebra*, 7(3):193–201, 1979.

[McG02] S. McGarraghy. Exterior powers of symmetric bilinear forms. *Algebra Colloq.*, 9(2):197–218, 2002.

[McG05] S. McGarraghy. Symmetric powers of symmetric bilinear forms. *Algebra Colloq.*, 12(1):41–57, 2005.

[McN98] G. J. McNinch. Dimensional criteria for semisimplicity of representations. *Proc. London Math. Soc. (3)*, 76(1):95–149, 1998.

[McN02] G. J. McNinch. Adjoint Jordan Blocks. *ArXiv Mathematics e-prints*, June 2002.

[Mos56] G. D. Mostow. Fully reducible subgroups of algebraic groups. *Amer. J. Math.*, 78:200–221, 1956.

[NP92] P. M. Neumann and C. E. Praeger. A recognition algorithm for special linear groups. *Proc. London Math. Soc. (3)*, 65(3):555–603, 1992.

[Nor95] C. W. Norman. On the Jordan form of the tensor product over fields of prime characteristic. *Linear and Multilinear Algebra*, 38(4):351–371, 1995.

[Pom77] K. Pommerening. Über die unipotenten Klassen reduktiver Gruppen. *J. Algebra*, 49(2):525–536, 1977.

[Pom80] K. Pommerening. Über die unipotenten Klassen reduktiver Gruppen. II. *J. Algebra*, 65(2):373–398, 1980.

[Pre87] A. A. Premet. Weights of infinitesimally irreducible representations of Chevalley groups over a field of prime characteristic. *Mat. Sb. (N.S.)*, 133(175)(2):167–183, 271, 1987.

[Pre03] A. A. Premet. Nilpotent orbits in good characteristic and the Kempf-Rousseau theory. *J. Algebra*, 260(1):338–366, 2003. Special issue celebrating the 80th birthday of Robert Steinberg.

[PS83] A. A. Premet and I. D. Suprunenko. The Weyl modules and the irreducible representations of the symplectic group with the fundamental highest weights. *Comm. Algebra*, 11(12):1309–1342, 1983.

[PST00] R. Proud, J. Saxl, and D. M. Testerman. Subgroups of type A_1 containing a fixed unipotent element in an algebraic group. *J. Algebra*, 231(1):53–66, 2000.

[QSSS76] H. G. Quebbemann, R. Scharlau, W. Scharlau, and M. Schulte. Quadratische Formen in additiven Kategorien. *Bull. Soc. Math. France Suppl. Mem.*, (48):93–101, 1976.

[Ral66] T. Ralley. Decomposition of products of modular representations. *Bull. Amer. Math. Soc.*, 72:1012–1013, 1966.

[Ree57] R. Ree. On some simple groups defined by C. Chevalley. *Trans. Amer. Math. Soc.*, 84:392–400, 1957.

[Ren79] J.-C. Renaud. The decomposition of products in the modular representation ring of a cyclic group of prime power order. *J. Algebra*, 58(1):1–11, 1979.

[Sax98] J. Saxl. Overgroups of special elements in simple algebraic groups and finite groups of Lie type. In *Algebraic groups and their representations (Cambridge, 1997)*, Volume 517 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 291–300. Kluwer Acad. Publ., Dordrecht, 1998.

[SS97] J. Saxl and G. M. Seitz. Subgroups of algebraic groups containing regular unipotent elements. *J. London Math. Soc. (2)*, 55(2):370–386, 1997.

[Sei87] G. M. Seitz. The maximal subgroups of classical algebraic groups. *Mem. Amer. Math. Soc.*, 67(365):iv+286, 1987.

[Sei00] G. M. Seitz. Unipotent elements, tilting modules, and saturation. *Invent. Math.*, 141(3):467–502, 2000.

[Ser94] J.-P. Serre. Sur la semi-simplicité des produits tensoriels de représentations de groupes. *Invent. Math.*, 116(1-3):513–530, 1994.

[Ser97] J.-P. Serre. Semisimplicity and tensor products of group representations: converse theorems. *J. Algebra*, 194(2):496–520, 1997. With an appendix by Walter Feit.

[Ser03] J.-P. Serre. 1998 Moursund Lectures at the University of Oregon. *ArXiv Mathematics e-prints*, May 2003.

[Ser05] J.-P. Serre. Complète réductibilité. *Astérisque*, (299):Exp. No. 932, viii, 195–217, 2005. Séminaire Bourbaki. Vol. 2003/2004.

[Sim13] I. I. Simion. *Centers of Centralizers of Unipotent Elements in Exceptional Algebraic Groups.* PhD thesis, SB, Lausanne, 2013.

[Sin92] P. Sin. On the 1-cohomology of the groups $G_2(2^n)$. *Comm. Algebra*, 20(9):2653–2662, 1992.

[Smi82] S. D. Smith. Irreducible modules and parabolic subgroups. *J. Algebra*, 75(1):286–289, 1982.

[Spa82] N. Spaltenstein. *Classes unipotentes et sous-groupes de Borel*, Volume 946 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin-New York, 1982.

[SS70] T. A. Springer and R. Steinberg. Conjugacy classes. In *Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69)*, Lecture Notes in Mathematics, Vol. 131, pages 167–266. Springer, Berlin, 1970.

[Sri64] B. Srinivasan. The modular representation ring of a cyclic p-group. *Proc. London Math. Soc. (3)*, 14:677–688, 1964.

[Ste65] R. Steinberg. Regular elements of semisimple algebraic groups. *Inst. Hautes Études Sci. Publ. Math.*, (25):49–80, 1965.

[Ste68] R. Steinberg. *Lectures on Chevalley groups*. Yale University, New Haven, Conn., 1968. Notes prepared by John Faulkner and Robert Wilson.

[Ste10] D. I. Stewart. The reductive subgroups of G_2. *J. Group Theory*, 13(1):117–130, 2010.

[Sup95] I. D. Suprunenko. Irreducible representations of simple algebraic groups containing matrices with big Jordan blocks. *Proc. London Math. Soc. (3)*, 71(2):281–332, 1995.

[Sup01] I. D. Suprunenko. On an asymptotic behavior of elements of order p in irreducible representations of the classical algebraic groups with large enough highest weights. *Proc. Amer. Math. Soc.*, 129(9):2581–2589, 2001.

[Sup03] I. D. Suprunenko. The second Jordan block and the recognition of representations. *Dokl. Nats. Akad. Nauk Belarusi*, 47(1):48–52, 125, 2003.

[Sup05] I. D. Suprunenko. Unipotent elements of prime order in representations of exceptional algebraic groups: the second Jordan block. *Dokl. Nats. Akad. Nauk Belarusi*, 49(4):5–9, 124, 2005.

[Sup09] I. D. Suprunenko. The minimal polynomials of unipotent elements in irreducible representations of the classical groups in odd characteristic. *Mem. Amer. Math. Soc.*, 200(939):vi+154, 2009.

[Tes88] D. M. Testerman. Irreducible subgroups of exceptional algebraic groups. *Mem. Amer. Math. Soc.*, 75(390):iv+190, 1988.

[Tes89] D. M. Testerman. A construction of certain maximal subgroups of the algebraic groups E_6 and F_4. *J. Algebra*, 122(2):299–322, 1989.

[Tes95] D. M. Testerman. A_1-type overgroups of elements of order p in semisimple algebraic groups and the associated finite groups. *J. Algebra*, 177(1):34–76, 1995.

[TZ13] D. M. Testerman and A. Zalesski. Irreducibility in algebraic groups and regular unipotent elements. *Proc. Amer. Math. Soc.*, 141(1):13–28, 2013.

[Tsu08] T. Tsujii. A simple proof of Pommerening's theorem. *J. Algebra*, 320(5):2196–2208, 2008.

[vdK01] W. van der Kallen. Steinberg modules and Donkin pairs. *Transform. Groups*, 6(1):87–98, 2001.

[Wey39] H. Weyl. *The Classical Groups. Their Invariants and Representations.* Princeton University Press, Princeton, N.J., 1939.

[Wil76] W. Willems. *Metrische Moduln über Gruppenringen*. PhD thesis, Johannes Gutenberg-Universität, Mainz, 1976.

[Won72] W. J. Wong. Irreducible modular representations of finite Chevalley groups. *J. Algebra*, 20:355–367, 1972.

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